Pricing Related Goods

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December 27, 2020

Be forewarned that while I think I have some pretty numerical results, the analytical side is still a bit rough. For instance I would love to be able to prove the concavity of the profit equations, but perhaps another day. A Java implementation of the profit optimization can be found here `https://github.com/sophist0/opt_profit`

Two Good Model

Assume we have two goods labeled G_1 and G_2 where G_2 is a sequel to G_1. For instance G_1 is Home Alone and G_2 is Home Alone 2. These have a potential audience A. The fraction of the audience that wants to good G_1 first is $f_{0,1}$. The faction that wants to see movie M_2 first is $f_{0,2}$. The fraction of viewers that see G_1 first and then want to see G_2 is $f_{1,2}$ and the fraction of viewers that see G_2 first and then want see G_1 is $f_{2,1}$. These fractions are subject to the following restriction $0 \leq f_{i,j} \leq 1$ for $i \in [0, 1, 2]$ and $j \in [1, 2]$. Additionally no sum of fractions leaving an audience node may be greater than 1 which in this case implies that

$$0 \leq f_{0,1} + f_{0,2} \leq 1 \tag{1}$$

These fractions $f_{i,j}$ are often empirically available to online marketplaces such as YouTube Movies.

The possible orders in which the audience can purchase G_1 and G_2 can be represented as a graph.

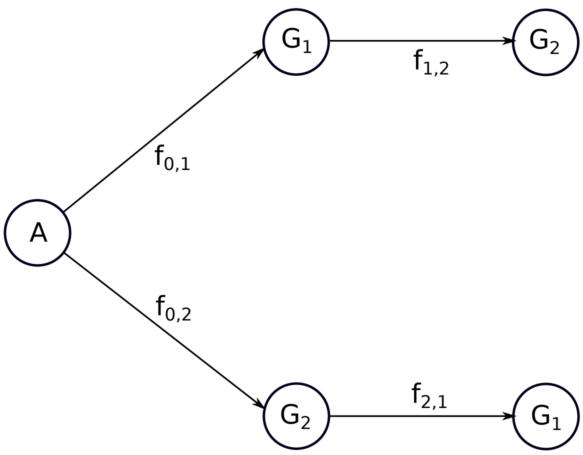

Figure 1: Two Good Purchase Order Model

To this model we can add the price p_i of each good. But before doing so lets normalize the min and max pricing of the goods such that at $p_i = 0$ everyone who wants good G_i purchases it and at $p_i = 1$ no one who wants good G_i purchases it. As with the fractions $f_{i,j}$ the actual min and max prices of a good can be estimated empirically, but here I simply assume they are known. Define p_i as having domain $p_i \in [0, 1]$. Finally I assume that every member of the audience is subject to the same linear price sensitivity function $s_i = (1 - p_i)$ so at $p_i = 1/2$ half the members of the audience who want to purchase good G_i do so. I have no reason to think that this price sensitivity function reflects actual consumer behavior, I chose it for its simplicity. The resulting two good pricing model is given below,

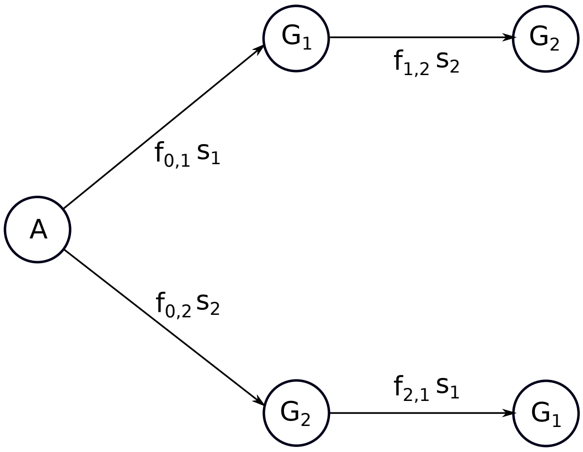

Figure 2: Two Good Pricing Model

This model can be collapsed to the following graph,

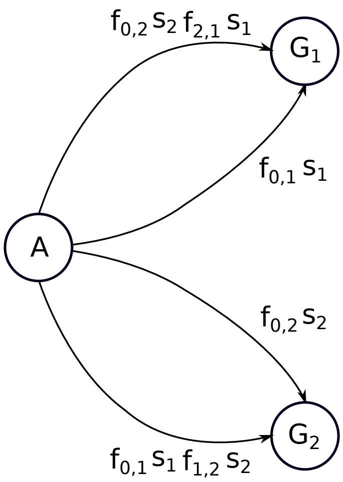

Figure 3: Collapsed Two Good Pricing Model

Two Good Profit Equation

If the audience size is N its clear from the model above that the profit earned by both goods can be computed as

$$\begin{aligned}
profit &:= f(p_1, p2) && (2)\\
&= N[(f_{0,1}s_1 + f_{0,2}s_2f_{2,1}s_1)p_1 + (f_{0,2}s_2 + f_{0,1}s_1f_{1,2}s_2)p_2] && (3)\\
&\propto (f_{0,1}s_1 + f_{0,2}s_2f_{2,1}s_1)p_1 + (f_{0,2}s_2 + f_{0,1}s_1f_{1,2}s_2)p_2 && (4)\\
&= [(f_{0,1}(1-p_1) + f_{0,2}f_{2,1}(1-p_1)(1-p_2)]p_1 + [(f_{0,2}(1-p_2) + f_{0,1}f_{1,3}(1-p_1)(1-p_2))]p_2 && (5)\\
&= (p_1 - p_1^2)(f_{0,1} + f_{0,2}f_{2,1}(1-p_2)) + (p_2 - p_2^2)(f_{0,2} + f_{0,1}f_{1,2}(1-p_1)) && (6)
\end{aligned}$$

Lets simplify the notation using the following mapping.

$$p_1 \to x,\ p_2 \to y,\ f_{0,1} \to a,\ f_{0,2} \to b,\ f_{0,1}f_{1,2} \to c,\ f_{0,2}f_{2,1} \to d \tag{7}$$

Under this mapping the equation simplifies to

$$g(x,y) = \frac{f(x,y)}{N} = (x - x^2)(a + d(1-y)) + (y - y^2)(b + c(1-x)) \tag{8}$$

Taking the first and second partial derivates of $g(x, y)$ gives

$$\frac{\partial}{\partial x} g(x, y) = (1 - 2x)(a + d(1 - y)) - c(y - y^2) \tag{9}$$

$$\frac{\partial}{\partial y} g(x, y) = (1 - 2y)(b + c(1 - x)) - d(x - x^2) \tag{10}$$

$$\frac{\partial^2}{\partial x^2} g(x, y) = -2(a + d(1 - y)) \tag{11}$$

$$\frac{\partial^2}{\partial y^2} g(x, y) = -2(a + d(1 - x)) \tag{12}$$

$$\frac{\partial^2}{\partial x \partial y} g(x, y) = 2d\left(x - \frac{1}{2}\right) - 2c\left(y - \frac{1}{2}\right) \tag{13}$$

$$\frac{\partial^2}{\partial y \partial x} g(x, y) = 2d\left(x - \frac{1}{2}\right) - 2c\left(y - \frac{1}{2}\right) \tag{14}$$

Therefore the Hessian matrix of $g(x, y)$ is

$$Hg(x, y) = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2[a + d(1 - y)] & 2[d(x - 1/2) - c(y - 1/2)] \\ 2[d(x - 1/2) - c(y - 1/2)] & -2[b + c(1 - x)] \end{bmatrix} \tag{15}$$

Results for the Two Good Profit Equation

The squared terms in the derivative of $g(x, y)$ make finding a closed form solution for the critical points of $g(x, y)$ difficult in terms of arbitrary parameterizations of $f_{0,1}$, $f_{0,2}$, $f_{1,2}$, and $f_{2,1}$. Therefore before solving for these points numerically I set the values of these parameters as follows,

$$f_{0,1} = \frac{9}{10},\ f_{0,2} = \frac{1}{10},\ f_{1,2} = \frac{3}{4},\ f_{2,1} = \frac{1}{4} \tag{16}$$

Practically these fractions could be found empirically by setting $p_1 = 0$ and $p_2 = 0$. Using these values under the mapping defined above gives

$$a = \frac{9}{10},\ b = \frac{1}{10},\ c = \frac{27}{40},\ d = \frac{1}{40} \tag{17}$$

Plugging these values into the partial derivatives of $g(x, y)$ gives

$$\frac{\partial}{\partial x} g(x, y) = (1 - 2x)\left(\frac{9}{10} + \frac{1}{40}(1 - y)\right) - \frac{27}{40}(y - y^2) = 0 \tag{18}$$

$$\frac{\partial}{\partial y} g(x, y) = (1 - 2y)\left(\frac{1}{10} + \frac{27}{40}(1 - x)\right) - \frac{1}{40}(x - x^2) = 0 \tag{19}$$

Solving this system of equations with Sympy's nsolve and initial points $x, y = 1/2$, I found the critical point for $g(x, y)$ is $x^* = 0.408$ and $y^* = 0.494$ rounding to the thousandths place.

Plugging the values into the Hessian matrix we can determine if this solution a saddle point or a local maxima using the partial derivative test.

$$Hg(x^*, y^*) = \begin{bmatrix} -1.8253 & 0.0035 \\ 0.0035 & -0.9992 \end{bmatrix} \tag{20}$$

The computing eigenvalues of $Hg(x^*, y^*)$ with numpy gives $\lambda_1 = -1.8253$ and $\lambda_2 = -0.9992$. Since all the eigenvalues of the Hessian are negative, the Hessian is a negative definite matrix. Therefore $g(x, y)$ achieves a local maxima at point $(0.408, 0.494)$.

Is this solution better than if we fixed $p_1 = p_2$ in the Two Good Pricing Model above? If we set $x = y$, $g(x, y)$ reduces to

$$g(x) = x^3(c + d) - x^2(a + b + 2(c + d)) + x(a + b + c + d) \tag{21}$$

and the derivatives of $g(x)$ are

$$\frac{d}{dx}g(x) = 3x^2(c + d) - 2x(a + b + 2(c + d)) + (a + b + c + d) \tag{22}$$

$$\frac{d^2}{dx^2}g(x) = 6x(c + d) - 2(a + b + 2(c + d)) \tag{23}$$

Setting the first derivative of $g(x)$ to zero and solving for the two critical points gives $x^* \in 0.438, 1.848$. Since by definition $0 \leq x < 1$ it follows that $x^* = 0.438$.

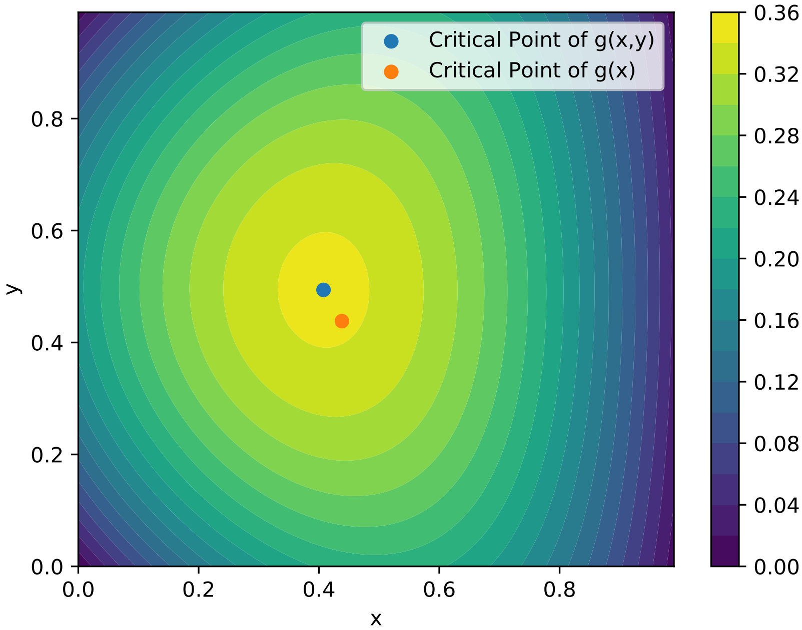

Figure 4: Contour Plot of $g(x, y)$

Now looking at the second derivative of $g(x)$ at x^* we find that

$$\frac{d^2}{dx^2}g(x^*) = -0.560 < 0 \tag{24}$$

Therefore $g(x^*)$ is a local maxima by the second derivative test and the only maxima in the domain of x.

What is the difference in profit achieved using $g(x, y)$ vs $g(x)$ to optimize pricing? Well

$$g(x^*) = 0.34299 \tag{25}$$

$$g(x^*, y^*) = 0.34532 \tag{26}$$

$$\frac{g(x^*, y^*)}{g(x^*)} = 1.00678 \tag{27}$$

or about a 0.68% increase as a result of price optimizing via $g(x, y)$. But recall this is the normalized profit per potential audience member. If the maximum price anyone is willing to pay for G_1 and G_2 is $p_{max} = \$20$ while the minimum is $p_{min} = \$0$, this sets $p_1 = \$8.16$, $p_2 = \$9.88$ under $g(x, y)$ and $p_1 = p_2 = \$8.76$ under $g(x)$. Now if the audience size is 10^8 the difference in profit using $g(x, y)$ to set prices vs $g(x)$ is $\$13,560,000$

which is not trivial.

n Good Model

From the small example given in Fig. 2 its clear that to compute the profit over a set of goods purchased in sequence requires looking at every permutation of that sequence. The figure below gives a tree representing these permutations for a sequence of n goods.

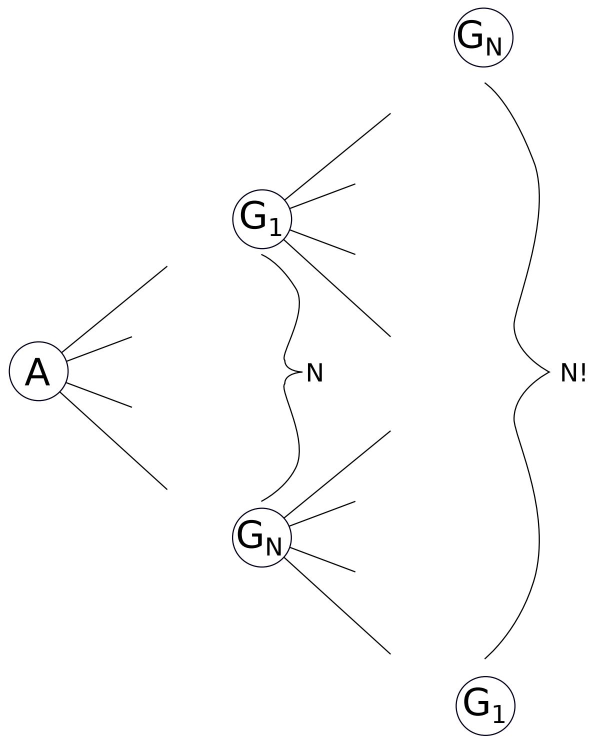

Figure 5: n Good Model

This tree has the following statistics.

- The n goods have $n!$ permutations and therefore this tree has $n!$ leaves.
- The number of edges in the tree can be computed as

$$\begin{aligned} edges &= n + n(n-1) + n(n-1)(n-2) + \dots && (28) \\ &= \sum_{i=0}^{n-1} \left(\prod_{j=0}^{i} (n-j) \right) && (29) \end{aligned}$$

- There are $2\binom{n}{2} + n$ unique parameters representing fractions $f_{i,j}$ where the final term captures the fraction of the audience that wants to purchase each good first i.e. fractions $f_{0,j}$.

If the graph of the n Good Model is collapsed as we collapsed the graph of the Two Good Model the result is n graphs of the form below each with $n!$ weighted edges from the audience to a good G_l.

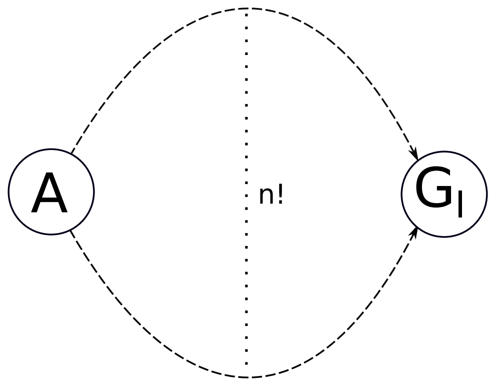

Figure 6: Collapsed n Goods Model

Each of these $n!$ edges is weighted by the product of between 1 and n terms $f_{x,y}(1-p_j)$ depending on where in the permutation G_l is and each of these weighted edges is multiplied by p_l the price of good G_l giving equation for calculating the profit of good G_l along the k_{th} edge the form,

$$g_{l,k}(p_j,\dots,p_k) = f_{x,y}(1-p_j)\cdot f_{q,r}(1-p_l)\cdots f_{w,z}(1-p_k)p_l \tag{30}$$

where the total profit is

$$profit = f(p_1,\dots,p_n) = \sum_{l=1}^{n} g_l(p_1,\dots,p_n) = \sum_{l=1}^{n}\sum_{k=1}^{n!} g_{l,k}(p_j,\dots,p_k) \tag{31}$$

Taking the second partial derivatives of $g_{l,k}(p_1,\dots,p_n)$ gives

$$\frac{\partial^2}{\partial p_l^2} g_{l,k}(p_j,\dots,p_k) = -2f_{x,y}(1-p_j)\cdots f_{u,v}(1-p_{l-1})f_{q,r}f_{e,o}(1-p_{l+1})\cdots f_{w,z}(1-p_k) \tag{32}$$

$$\frac{\partial^2}{\partial p_i \partial p_j} g_l(p_j,\dots,p_k) = 0 \qquad \text{for } i \neq l \tag{33}$$

$$\frac{\partial^2}{\partial p_l \partial p_j} g_{l,k}(p_j,\dots,p_k) = -f_{x,y}f_{q,r}(1-2p_l)\cdots f_{w,z}(1-p_k) \tag{34}$$

To simplify notation lets define this second partial derivatives of $g_l(p_j,\dots,p_k)$ and $g_{l,k}(p_j,\dots,p_k)$ as

$$g''_{l,k}(l,j) := \frac{\partial^2}{\partial p_l \partial p_j} g_{l,k}(p_j,\dots,p_k) \tag{35}$$

$$g''_l(l,j) := \frac{\partial^2}{\partial p_l \partial p_j} g_l(p_j,\dots,p_k) \tag{36}$$

where $l,j \in [1,\dots,n]$. It follows from the sum rule for derivatives that

$$g''_l(l,j) = \sum_{k=1}^{n!} g''_{l,k}(l,j) \tag{37}$$

Notice that the functions $g''_l(l,j)$ are the l,j entries of the Hessian matrix for the profit function $g(p_1,\dots,p_n)$ such that

$$Hg(p_1,\dots,p_n) = \begin{bmatrix} g''_1(1,1) & g''_1(1,2) & \dots & g''_1(1,n) \\ g''_2(2,1) & g''_2(2,2) & \dots & g''_2(2,n) \\ \vdots & \vdots & \dots & \vdots \\ g''_n(n,1) & g''_n(n,2) & \dots & g''_n(n,n) \end{bmatrix} \tag{38}$$

Which in theory could be computed for given a set of prices and audience fraction parameters for an arbitrary n. This Hessian matrix could then be used to determine if the price point is a local maxima.

The Profit Algorithm

In terms of computing profit over all permutations of a length n sequence, the graph representation is irrelevant. The profit can be directly computed from the permutations. The algorithm I give below while inefficient is straight forward.

If $\mathcal{G}$ is the set of goods with arbitrary initial ordering $\boldsymbol{\sigma}_0 \equiv [1, \ldots, n]$ where $n = |\mathcal{G}|$, and $\mathbf{p} = [0, p_1, \ldots, p_n]$ is the prices of these goods with the 0 added to simplify indexing. The matrix $\mathbf{F} = [i, j] \in \mathbf{M}_{n+1}$ has entries which if $i = 0$ is the fraction of individuals in the audience or if $i > 0$ the faction of the audience who have purchased goods $\boldsymbol{\sigma}_{x,1}$ to $\boldsymbol{\sigma}_{x,i}$ and would purchase good $\boldsymbol{\sigma}_{x,j}$ if $p_j = 0$ where $\boldsymbol{\sigma}_{x,j}$ is the j_{th} good in the x_{th} permutation.

The set of permutations of the goods in set $\mathcal{G}$ is denoted $\mathcal{P}_n$ where $|\mathcal{P}_n| = n!$. Additionally if $a \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^l$ let $\oplus$ denote the concatenation of an element with a vector such that $a \oplus \mathbf{v} = \mathbf{w}$ where $\mathbf{w}_0 = a$ and $\mathbf{w} \in \mathbb{R}^{l+1}$. Finally let $|\mathbf{w}|$ denote the length of $\mathbf{w}$. Below is the algorithm used to compute the profit.

```
profit = 0
for σ_i ∈ P_n do
    c = 1                                   ▷ Current fraction of the audience
    b = 0                                   ▷ Profit over the current permutation
    π = 0 ⊕ σ_i
    for j ∈ [0, ..., |π| − 1] do
        l = π_j
        m = π_{j+1}
        c = c(1 − p_m)F_{l,m}
        b = b + c p_m
    end for
    profit = profit + b
end for
```

Using stochastic gradient ascent (SGA) to optimize the Two Good Model with the parameters f_1, f_2, f_3, f_4 given and computing the gradient of profit using the algorithm above as a function of the prices $\mathbf{p}$. The SGA algorithm initiates all prices to 1/2, the learning rate is $\nu = 0.01$, and it iterates 1000 times, resulting in $profit = 0.34529$ for $x = 0.403$, $y = 0.489$ which is the same to the thousandths place as the local maxima of $g(x, y)$ found above.

Results for the n Good Profit Equation

The SGA optimization algorithm can be used to optimize larger sets of goods and is limited only by the difficulty of computing the profit over the $n!$ permutations of the goods. This difficulty can be mitigated if its assumed most these permutations do not occur. I think the number of permutations that are likely to occur can be quantified using typical sets. But I have not done any pruning of the permutation space here. If we assume the fractions of the audience who would like to purchase the n goods are defined as follows.

$$f_{0,i} = 2^{-i} \tag{39}$$

$$f_{i,i+1} = \frac{9}{10} \text{ for } i > 0 \tag{40}$$

$$f_{i,j} = \frac{1}{100} \text{ for } j \neq i+1 \tag{41}$$

We can look at how the optimal prices vary with n as computed by the SGA given $\nu = 0.005$ in the table below,

n	p_1^*	p_2^*	p_3^*	p_4^*	p_5^*	p_6^*	$g(p_1, \ldots p_n)$	$g(p)$	$g(p_1, \ldots p_n)/g(p)$
1	0.495						0.1250	0.1250	1.000000
2	0.384	0.494					0.2504	0.2472	1.012944
3	0.411	0.419	0.494				0.5598	0.5572	1.004666
4	0.443	0.438	0.444	0.494			1.6751	1.6736	1.000896
5	0.460	0.457	0.456	0.460	0.494		6.6159	6.6145	1.000211
6	0.468	0.467	0.466	0.466	0.469	0.494	32.7161	32.7143	1.000055

It is interesting that the last price is relatively stable and proceeding prices are nearly symmetric. It is also interesting that the ratio of profit under independent pricing and identical pricing decreases with n increasing. Both of these phenomina are likely due to the parameters $f_{i,j}$ I have choosen rather than anything inherent to the structure of optimally pricing related goods. But to have any confidence in this statement would require additional exploration.