Lab 2: Mine Crafting

Investigations of the Vertical Mines at different locations on the Earth and Moon

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I. Introduction

In preparation for measuring the vertical depth of a shaft by dropping a 1 kg test mass, my team and I have conducted thorough investigations on physical influences that might affect the time it takes the mass to hit the bottom, such as density, forces, and variable acceleration. It is crucial that we understand what different forces are at play at different locations of the mines. We have quantified the forces, position, and velocity in the variations of the mine setup which will allow for the most precise measurement of time it takes the mass to hit the bottom.

II. Calculation of fall time (including drag and variable g)

The first calculation my team modeled was the fall time for a 4 km deep shaft, taking into account drag and a variable gravitational acceleration. A projectile experiencing constant gravitational force and drag force is described by the following differential equation (1):

$$\frac{d^2y}{dt^2} = -g + \alpha\left|\frac{dy}{dt}\right|^{\gamma} \quad (1)$$

where t is time in seconds, y is height in meters, g is the gravitational acceleration in m/s^2, γ is the speed dependence of the drag, and α is the drag coefficient in units of s$^{\gamma-2}$/m$^{\gamma-1}$. It can be equivalently represented with the system of equations (2), where v is velocity in m/s:

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = -g + \alpha|v|^{\gamma} \quad (2)$$

My team modeled the position and velocity of the mass in free fall in the shaft in Eq.2 with no drag and found a time of 28.6 seconds to reach the bottom of the well. We compared this to the analytic kinematic solution to equation (2b) and got the exact same value up to the precision of the numerical solver: $t = \sqrt{\frac{y}{0.5g}}$ (2b), where y is the height, and g = 9.81 m/s^2.

We then assumed that g is a function of the distance *r* from the center of the Earth with the following equation (3): $$g(r) = g_0\left(\frac{r}{R_{earth}}\right) \quad (3)$$

where g_0 is the gravity at the surface of 9.81 m/s^2, R_{earth} is the radius of the Earth of 6378.1 km.

It is important to consider the variable g case because incorporating a height-dependent g decreases the fall time. Without height dependence, it takes 28.6 seconds to reach the bottom of the well, but with time-dependence, it takes 28.5 seconds, decreasing fall time by 0.3%. This is because as the object falls, the acceleration increases a very small amount, so it ends up falling faster as it gets deeper, taking less time to get to the bottom of the well. We also computed the position and velocity of the 1 kg mass including drag. Using prior knowledge of terminal velocity, or the max speed achieved by an object falling through the air, we empirically found the value of α = 0.004 m^{-1} in order to get a terminal velocity of 50 m/s, and chose γ = 2. We found the mass had a fall time of 84.3 seconds with drag. Including drag is important because it increases the fall time, making the object fall "slower" as seen in Fig.1, so it takes longer to reach the bottom of the well.

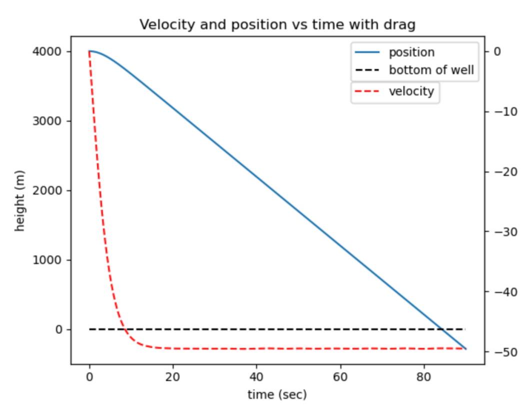

Fig. 1: Numerical solution of velocity and position to Eq.2 with variable g (Eq.3) and drag as a function of height above the bottom of the shaft

Because drag and variable gravitational acceleration can affect the fall time of the object in the shaft, my team found it vital that we include those complexities when calculating the depth of the mine shaft. However, for the finite distance of the 4 km shaft, drag is much more important to take into account when measuring the vertical depth of the mine shaft because it triples the fall time.

III. Feasibility of depth measurement approach (including Coriolis forces)

The next calculation my team modeled was the Coriolis force on the mass as it falls down the shaft. Coriolis force is an apparent force that deflects moving objects as a result of Earth's rotation. The Coriolis force $\vec{F}_c$ is given by the following equation (4):

$$\vec{F}_c = -2m(\vec{\Omega} \times \vec{v}) \quad (4)$$

where $\vec{\Omega}$ is the Earth's rotational velocity of $7.272*10^{-5}$ rad/sec, $\vec{v}$ is the velocity in m/s, and m is the mass of the object of 1 kg.

We simulated dropping the test mass from the center of the shaft and found that the test mass collides with the wall, because it takes 21.9 sec to hit the wall at 1645.6 m, while it takes 28.5 sec for the test mass to reach the bottom. With drag, the mass hits the wall after 29.7 sec at 2703.2 m, while it takes 84.3 sec to hit the bottom. We found that with or without drag, the test mass hit the side of the wall first before reaching the bottom. Because of this, we do not recommend proceeding with the depth measurement technique. It is critical to understand how the Coriolis force affects the fall time, especially because our company operates a vertical mine at the equator. Otherwise we risk measuring the wrong fall time and wrong depth.

IV. Calculation of crossing times for homogeneous and non-homogeneous earth

The last calculation my team modeled was the crossing times, or the time it takes for the mass to reach the other side of the Earth in an infinitely deep mineshaft, for a homogeneous and non-homogeneous Earth. We are interested in expanding our operation to create the first tunnel through the center of the Moon, so we first modelled the tunnel on Earth. For the homogeneous Earth, we modelled the infinite shaft from pole to pole to avoid the Coriolis force and neglected drag. We found that the mass oscillates in our infinite tunnel model as seen in Fig. 2. We numerically calculated using a coupled differential equation solver that the mass takes 1266.6 sec to reach the center of the Earth with a maximum speed of 7910.1 m/s, and it reaches the other side of the Earth after 2482.1 sec. Next, my team considered the orbital period, where we assumed the object was on a circular orbit in centripetal balance with the gravitational force using the following equations (5):

$$\frac{v^2}{R} = \frac{GM}{R^2}, T = \frac{4\pi R}{v} \quad (5)$$

where v is the orbital velocity in m/s, R is the radius in m, M is the mass of the Earth of $5.972*10^{24}$ kg, G is the gravitational constant of $6.6743*10^{-11}$ m^3 kg^{-1} s^{-2}, and T is the period in seconds. For the homogenous Earth, we empirically determined that the orbital period

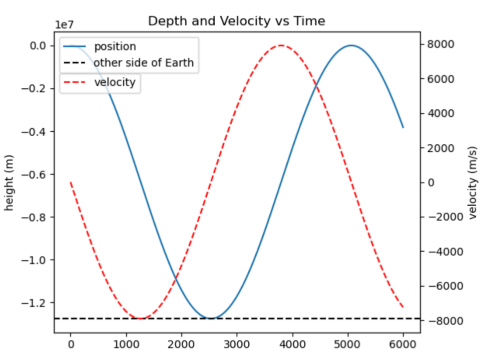

Fig.2: Numerical solution of velocity and height to Eq.2 with variable g for a tunnel through the Earth as a function of depth below the surface of the Earth

was 2 times greater than the crossing time. Now, for the non-homogenous Earth, we modelled density ρ as a function of distance from center of the Earth *r* using the following equation (6):

$$\rho(r) = \rho_n \left(1 - \frac{r^2}{R_{earth}^{\ 2}}\right)^n$$

where ρ_n is a normalizing constant, and *n* is an exponent. We used the mass of the Earth to normalize density with the following equation (7):

$$M_{earth} = 4\pi \int_0^{R_{earth}} \rho(r) r^2 dr \quad (7)$$

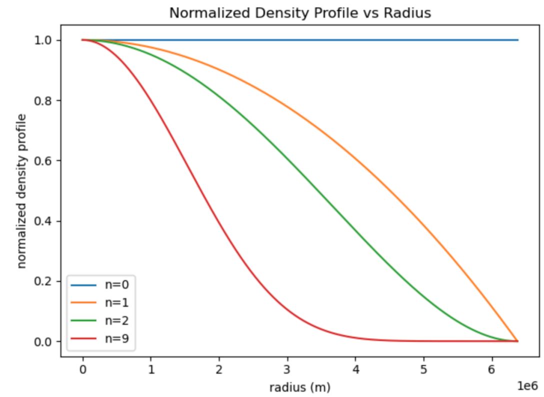

Fig.3: Numerical solution to Eq.6 for n=0,1,2,9, normalizing Pn with Eqs.7,8 as a function of distance from the center of the Earth

where M_{earth} is the mass of the Earth. From Fig.3, we observe that n = 0 corresponds to the homogenous Earth scenario with a crossing time of 2534.1 sec, while n = 9 corresponds to mass being concentrated in the center with a crossing time of 1888.1 sec. This greater density concentration causes a 25% decrease in crossing time, due to the n = 9 case having a greater gravitational force closer to the center of the Earth, where the force is given by the following equation (8):

$$F_{grav} = -\frac{4\pi G m}{r^2} \int_0^r \rho(r) r^2 dr$$

where m is the test mass and *r* is the distance from the center of the Earth.

To conclude our study, we repeated the above process for variable gravity in Eq (3) using the radius and mass of the moon, which are 1738.1 km and $7.35 \cdot 10^{22}$ kg respectively. We found the trans-lunar and trans-planetary crossing times to be 3249.6 sec and 2482.1 sec respectively. Then, using Eqs (5), and the mass-density relationship, we derived the following equation (9):

$$T = 2\pi\sqrt{\frac{R^3}{GM}} = 2\pi\sqrt{\frac{3R^3}{4\pi\rho M}} = \sqrt{\frac{3\pi}{G\rho}} \approx 2t_{crossing} \Rightarrow t_{crossing} \, \alpha \, \frac{1}{\sqrt{\rho}} \Rightarrow \boxed{\frac{t_1}{t_2} \alpha \sqrt{\frac{\rho_2}{\rho_1}}} \quad (9)$$

We found this relationship to be true for the densities of the Earth and Moon within a margin of error of 0.1 sec. This proportion is important in predicting future crossing times based on density, which can be helpful when we build tunnels on other planets.

V. Discussion and Future Work

Through these calculations, my team is optimistic about the accuracy of the depth measurement technique. However, we must continue iterating on our models to make sure that all approximations are taken into account to avoid any catastrophic events. While creating our models in Section II – III, we approximated the Earth and Moon as perfect spheres with smooth density curves, which eliminated the geological inhomogeneities that can cause differences in densities and forces. The infinite tunnel model also had many simplifying assumptions like drag and the Coriolis effect that we must understand if we continue to use the model. Newer models should take these issues into consideration for more accurate measurements, like a wider shaft for the Coriolis effect. My team along with others will continue iterating our models of the depth measurement technique to contribute to the efficiency and safety of our mining operation.