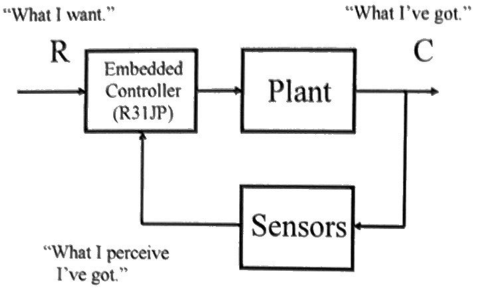
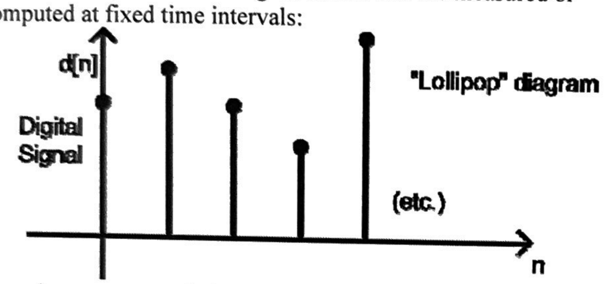
Today: Implement a digital, discrete-time feedback loop using the R31JP:

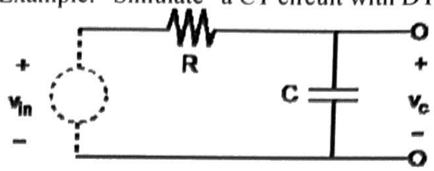


Here's the problem: The plant is often described by CT (continuous time) differential equations. The R31JP lives in a DT (discrete time) world with signal values that are measured or computed at fixed time intervals:



How do we connect the DT and CT worlds, and how do we make a model for the "whole" system?

Example: "Simulate" a CT circuit with DT computer equations



$$\frac{dv_c}{dt} = \frac{v_{in} - v_c}{RC}$$

we know that if $v_{in}(t) = u(t)$,

$$v_c(t) = \left[1 - e^{-t/RC}\right]u(t)$$

But suppose we didn't know the answer and wanted the computer to find it for us. One (poor ☺) approach is a DT approximation with Forward Euler.

$$\frac{dv_c}{dt} \approx \frac{v_c(t+\Delta t) - v_c(t)}{\Delta t} = \frac{v_c[n+1] - v_c[n]}{\Delta t}$$

where $v_c[n] = v_c(n\Delta t)$

so
$$\frac{v_c[n+1] - v_c[n]}{\Delta t} = \frac{v_{in}[n] - v_c[n]}{RC}$$

or
$$v_c[n+1] = \left[1 - \frac{\Delta t}{RC}\right]v_c[n] + \frac{\Delta t}{RC}v_{in}[n]$$

or
$$v_c[n+1] + \left[\frac{\Delta t}{RC} - 1\right]v_c[n] = \frac{\Delta t}{RC}v_{in}[n]$$

Solve this difference equation by finding homogeneous and particular solutions, as for differential equations.

$$v_c[n+1] + \left[\frac{\Delta t}{RC} - 1\right]v_c[n] = \frac{\Delta t}{RC}v_{in}[n]$$

assume unit step input, zero initial conditions

homogeneous

$$v_c[n+1] + \left[\frac{\Delta t}{RC} - 1\right]v_c[n] = 0$$

guess $v_c[n] = z^n \cdot A$

$$Az^{n+1} + \left[\frac{\Delta t}{RC} - 1\right]Az^n = 0$$

$$z = \left[1 - \frac{\Delta t}{RC}\right]$$

$$v_c[n] = A\left[1 - \frac{\Delta t}{RC}\right]^n$$

particular

$$v_c[n+1] + \left[\frac{\Delta t}{RC} - 1\right]v_c[n] = \frac{\Delta t}{RC}v_{in}[n]$$

guess $v_c[n] = C$, a constant

$$C + \left[\frac{\Delta t}{RC} - 1\right]C = \frac{\Delta t}{RC}$$

or $C = 1$

$$v_c[n] = 1$$

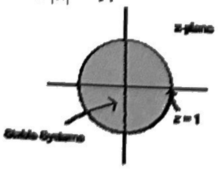
$$v_c[n] = 1 + A\left[1 - \frac{\Delta t}{RC}\right]^n$$

A= -1 to match initial condition

So $v_c[n] = 1 - \left[1 - \frac{\Delta t}{RC}\right]^n$ (DT)

(In CT, the actual capacitor voltage is: $v_c(t) = \left[1 - e^{-t/\tau}\right]u(t)$)

For stability we must have $|z| < 1$!



Aside:
Notice that transfer fxns. Still work in DT.

Before, $s \Leftrightarrow \frac{d}{dt}$

Now, $z \Leftrightarrow$ a shift!

Example: $$v_c[n+1] + \left[\frac{\Delta t}{RC} - 1\right]v_c[n] = \frac{\Delta t}{RC}v_{in}[n]$$

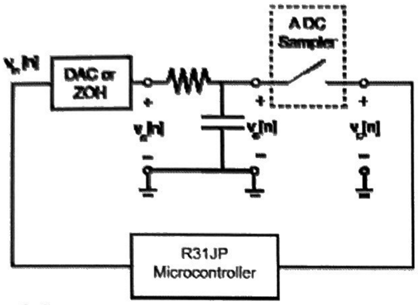
z-shift! $$\left[z + \left(\frac{\Delta t}{RC} - 1\right)\right]v_c(z) = \frac{\Delta t}{RC}v_{in}(z)$$

Transfer fxn: $$\frac{v_c(z)}{v_{in}(z)} = \frac{\frac{\Delta t}{RC}}{z + \left(\frac{\Delta t}{RC} - 1\right)} = H(z)$$

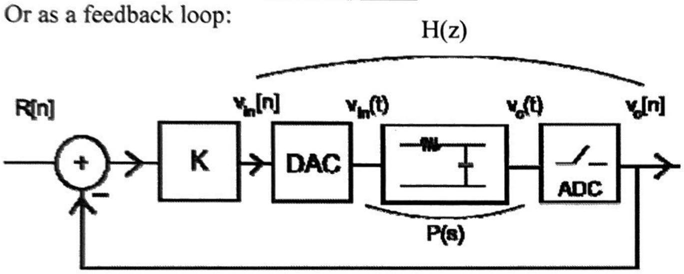
and: $$z = \left(1 - \frac{\Delta t}{RC}\right)$$

The z "pole" is the solution of the homogeneous response. Must be inside the unit circle for stability.

Back to the R31JP: Problem → feedback loop is mixed DT & CT!
Example:



Or as a feedback loop:



Strange!

$v_c(t)$ & $v_{in}(t)$ could be transferred and related by a "Laplace" style s-type transfer fxn.

$v_c[n]$ & $v_{in}[n]$ could be related by z-transform transfer fxn.

Neat! 1 common representation, all in DT or all in CT, to describe feedback loop.

How is H(z) related to P(s)?
Step invariant transformation! Sample the CT step response!

A unit step (DT) in $v_{in}[n]$ gives a unit step (CT) in $v_{in}(t)$.
Find H(z):

1. First, compute CT step response:

$$v_{in}(t) = u(t) \Rightarrow v_c(t) = \left(1 - e^{-t/\tau}\right)u(t)$$

2. Sample the CT step response:

$$v_c(nT) = \left(1 - e^{-nT/\tau}\right)u(nT)$$

or $\quad v_c[n] = \left[1 - \lambda^n\right]u[n]$ where $\lambda = e^{-T/\tau}$

3. Find a difference equation whose unit step response is the same as in step 2.

If you know 6.003, take z-transform:

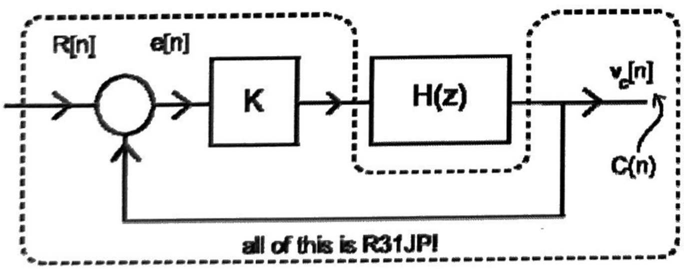
$$v_c(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-\lambda z^{-1}} = \frac{(1-\lambda)z^{-1}}{(1-z^{-1})(1-\lambda z^{-1})}$$

so $\quad \dfrac{v_c(z)}{u(z)} = \dfrac{v_c(z)}{v_{in}(z)} = \dfrac{z^{-1}(1-\lambda)}{(1-\lambda z^{-1})} = H(z)$

or $\quad v_c[n] - \lambda v_c[n-1] = (1-\lambda)v_{in}[n-1]$

4. If you haven't already done so, take z-transform!

Now, the whole loop is in "DT"



People have tabulated how $P(s) \Leftrightarrow H(z)$!
Makes life easy!

Now:

$$\frac{C}{R}(z) = \frac{\dfrac{Kz^{-1}(1-\lambda)}{1-\lambda z^{-1}}}{1 + \dfrac{Kz^{-1}(1-\lambda)}{1-\lambda z^{-1}}} = \frac{Kz^{-1}(1-\lambda)}{1-\lambda z^{-1} + K(1-\lambda)z^{-1}}$$

$$= \frac{Kz^{-1}(1-\lambda)}{1+\left[K(1-\lambda)-\lambda\right]z^{-1}}$$

Design K to give desired performance!

$q \Leftrightarrow z$

$a \leftrightarrow \frac{1}{\tau}$

$h \leftrightarrow \Delta t$
(T)
Sample Interval

TABLE 3.1 Sampling of a continuous time system, $G(s)$.

The table gives the zero-order-hold equivalent of the continuous time system, $G(s)$, preceded by a zero-order hold. The sampled system is described by its pulse transfer operator. For second-order systems the pulse-transfer operator is given in terms of the coefficients of

$$H(q) = \frac{b_1 q + b_2}{q^2 + a_1 q + a_2}$$

$G(s)$	$H(q)$ or the coefficients in $H(q)$
$\frac{1}{s}$	$\frac{h}{q-1}$
$\frac{1}{s^2}$	$\frac{h^2(q+1)}{2(q-1)^2}$
e^{-sh}	q^{-1}
$\frac{a}{s+a}$	$\frac{1-\exp(-ah)}{q-\exp(-ah)}$
$\frac{a}{s(s+a)}$	$b_1 = \frac{1}{a}(ah - 1 + e^{-ah})$ $b_2 = \frac{1}{a}(1 - e^{-ah} - ahe^{-ah})$ $a_1 = -(1 + e^{-ah})$ $a_2 = e^{-ah}$
$\frac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah)$ $b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_1 = -2e^{-ah}$ $a_2 = e^{-2ah}$
$\frac{ab}{(s+a)(s+b)}$	$b_1 = \frac{b(1-e^{-ah}) - a(1-e^{-bh})}{b-a}$ $b_2 = \frac{a(1-e^{-bh})e^{-ah} - b(1-e^{-ah})e^{-bh}}{b-a}$ $a_1 = -(e^{-ah} + e^{-bh})$ $a_2 = e^{-(a+b)h}$
$\frac{(s+c)}{(s+a)(s+b)}$	$b_1 = \frac{e^{-bh} - e^{-ah} + (1-e^{-bh})c/b - (1-e^{-ah})c/a}{b-a}$ $b_2 = \frac{c}{ab}e^{-(a+b)h} + \frac{b-c}{b(a-b)}e^{-ah} + \frac{c-a}{a(a-b)}e^{-bh}$ $a_1 = -e^{-ah} - e^{-bh}$ $a_2 = e^{-(a+b)h}$
$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha\left(\beta + \frac{\zeta\omega_0}{\omega}\gamma\right)$ $\omega = \omega_0\sqrt{1-\zeta^2}$ $\zeta < 1$ $b_2 = \alpha^2 + \alpha\left(\frac{\zeta\omega_0}{\omega}\gamma - \beta\right)$ $\alpha = e^{-\zeta\omega_0 h}$ $a_1 = -2\alpha\beta$ $\beta = \cos(\omega h)$ $a_2 = \alpha^2$ $\gamma = \sin(\omega h)$
$\frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = \frac{1}{\omega}e^{-\zeta\omega_0 h}\sin(\omega h)$ $b_2 = -b_1$ $\omega = \omega_0\sqrt{1-\zeta^2}$ $a_1 = -2e^{-\zeta\omega_0 h}\cos(\omega h)$ $a_2 = e^{-2\zeta\omega_0 h}$