Problem 3.60 Let $A = [2,5]$, $B = [3,7]$, $C = [1,3]$, $D = [2,4]$. Compute each of the following.

(a) $(A \cap B) * (C \cap D)$

$(A \cap B) = \{ \}$
$(C \cap B) = \{ \}$
$(A \cap B) * (C \cap D) = \{ \}$

(b) $(A * C) \cap (B * D)$

$(A * C) = \{[2,2],[2,4],[5,2],[5,4]\}$
$(B * D) = \{[3,2],[3,4],[7,2],[7,4]\}$
$(A * C) \cap (B * D) = \{ \}$

(e) $A * (B \cap C)$

$A = [2,5]$
$(B \cap C) = \{3\}$
$A * (B \cap C) = \{[2,3],[5,3]\}$

(f) $(A * B) \cap (A * C)$

$(A * B) = \{[2,3],[2,7],[5,3],[5,7]\}$
$(A * C) = \{[2,1],[2,3],[5,1],[5,3]\}$
$(A * B) \cap (A * C) = \{[2,3],[5,3]\}$

$\blacksquare$

Problem 3.61 Let A, B, C and D be sets. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) $(A \cap B) * (C \cap D) = (A * C) \cap (B * D)$: True
Let (x, y) be an element in $(A \cap B) * (C \cap D)$. This means that x is in A and B, and that y is in C and D.
Therefore, (x, y) is also in $A * C$ and $B * D$. Hence, (x, y) is in the intersection of $A * C$ and $B * D$, which is $(A * C) \cap (B * D)$.
Now we have to prove that $(A * C) \cap (B * D) = (A \cap B) * (C \cap D)$.

Let (x, y) be an element in $(A * C) \cap (B * D)$. Then, x is in A and y is in C, and x is also in B and y is in D.
Therefore, x is in A and B, and y is in C and D. This means that (x, y) is in the product of $(A \cap B)$ and $C \cap D$, which is $(A \cap B) * (C \cap D)$, meaning our statement is true.

(c) $A * (B \cap C) = (A * B) \cap (A * C)$: True

Proof : we need to show that each element in$A * (B \cap C)$ is also in $(A * B) \cap (A * C)$, and vice versa.

Let (x, y) be an element in $A * (B \cap C)$. Then, x is in A, and y is in both B and C. Therefore, (x, y) is in $A * B$ and $A * C$. Hence, (x, y) is in the intersection of A×B and A×C, which is $(A * B) \cap (A * C)$.

Conversely, let (x, y) be an element in $(A * B) \cap (A * C)$. Then, x is in A, and y is in both B and C. Therefore, y is in the intersection of B and C, which is $B \cap C$.

Hence, (x, y) is in the product of A and $(B \cap C)$, which is $A * (B \cap C)$.

(e) $A * (A * B) = (A * B) \setminus (A * C)$: False
Counterexample: $A = \{1\}, B = \{2\}, andC = \{3\}$

■

Problem 3.62 If A and B are sets, conjecture a way to rewrite $(A * B)^C$ in a way that involves A^C and B^C and then prove your conjecture.

■

Problem 4.4 For all $n \in N, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Base Case $(n = 1)$:
$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$
$1 = 1$
Now we have proven that the base case, $(n = 1)$, is true
inductive step:
next we assume that the statement is true for $n = k$
$\sum_{i=1}^{k} = \frac{k(k+1)}{2}$
we must then prove that the statement is true for $n = k + 1$, or $\frac{k+1(k+2)}{2}$

$$
\begin{aligned}
\sum_{i=1}^{k+1} i = & \quad \sum_{i=1}^{k} i + k + 1 \\
= & \quad \sum_{i=1}^{k} i + k + 1 \\
= & \quad \frac{k(k+1)}{2} + (k+1) \\
= & \quad \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
= & \quad \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

Now that we have proven that the statement is true for $k+1$ the statement is proven using induction.

■

Problem 4.5 For all $n \in N$, 3 divides $4^n - 1$.

This problem can be stated as $3|4^n - 1 = (4^n - 1 = 3i)$ for some $i \in \mathbb{Z}$
Base case ($n = 1$):
$4^1 - 1 = 3i$
$3 = 3i$
$i = 1$
there exists an i such that 3 times i is equal to $4^1 - 1$ and therefore we have proven our base case.

inductive step: Now assuming that the statement is true for all $n = k$ we must prove it is true for all $n = k + 1$

$$
\begin{aligned}
4^k - 1 &= 3i \\
4^k &= 3i + 1 \\
4^k * 4 &= (3i + 1)4 \\
4^{k+1} &= 12i + 4 \\
4^{k+1} - 1 &= 12i + 3 \\
4^{k+1} - 1 &= 3(4i + 1) \\
4^{k+1} - 1 &= 3j \text{ for some } j \in Z
\end{aligned}
$$

now we have proven that the statement is true for $n = k+1$ and therefore the statement is proven using induction.

■

Problem 4.6 For all $n \in N$, 6 divides $n^3 - n$.

the statement can be expressed as $n^3 - n = 6i$ for some $i \in \mathbb{Z}$ Base case $(n = 0)$:
$0^3 - 0 = 6i$
$i = 0$
now we have proven that our statement is true for our base case
inductive step: now assuming that the statement is true for all $n = k$ we must prove that it is true for $n = k + 1$

$$\begin{aligned}
&k^3 - k = 6i\\
&k^3 - k + 3k^2 + 3k = 6i + 3k^2 + 3k\\
&k^3 + 3k^2 + 3k - k = 6i + 3k^2 + 3k\\
&k^3 + 3k^2 + 3k - k + 1 - (k + 1) = 6i + 3k^2 + 2k\\
&(k + 1)^3 - (k + 1) = 6i + 3k^2 + 2k
\end{aligned}$$

now we have proven that the statement is true for $n = k+1$ and our statement is proven using induction.

■

Problem 4.7 Let $p_1, p_2, ..., p_n$ be n distinct points arranged on a circle. Then the number of line segments joining all pairs of points is $\frac{n^2-n}{2}$.

■

Problem 4.8 Consider a grid of square that is 2^n squares wide by 2^n squares long, where $n \in N$. One of the squares has been cut out, but you do not know which one! You have a bunch of L-shapes made up of 3 squares. Prove that you can perfectly cover this chess-board with the L-shapes (with no overlap) for any $n \in N$. Figure 4.1 depicts one possible covering of the case involving $n = 2$ and a fixed cut-out square.

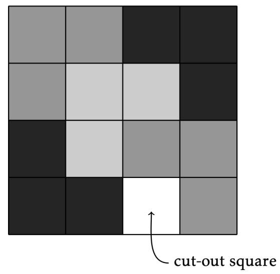

Figure 4.1: One possible covering for the case involving $n = 2$ for Problem 4.8.

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