IN5270 Mandatory exercise 3

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Exercise 2: Compute the deflection of a cable with sine functions

Studying a cable hanging with length L and with significant tension T has a deflection w(x) governed by:

$$Tw''(x) = l(x)$$

where l(x) the vertical load per unit length. The cable is fixed at x = 0 and x = L so the boundary conditions become w(0) = w(L) = 0. The deflection w is positive upwards, and l is positive when it acts downwards.
we assume l = constant, the solution expected to be symmetric around x = L/2. it means that $w(x_0 - h) = w(x_0 + h)$ if x_0 is the symmetric point. then we have $w'(x_0) = lim_{h\to 0}(w(x_0 - h) - w(x_0 + h))/(2h) = 0$. We seek w(x) in [0,L/2] with boundary conditions w(0) = 0 and w'(L/2) = 0.
The problem can be scaled with dimensionless : $\bar{x} = \frac{x}{L/2}$,$\bar{u} = \frac{w}{w_c}$, inserting in the problem for w:

$$\frac{4Tw_c}{L^2}\frac{d^2\bar{u}}{d\bar{x}^2} = l(= constant)$$

choosing w_c such that $|d^2\bar{u}/d\bar{x}^2| = 1$. then $w_c = \frac{1}{4}lL^2/T$, then the problem for the scaled vertical deflection u becomes:

$$u'' = 1. \quad x\epsilon(0,1), \quad u(0) = 0, u'(0) = 0$$

a)

The exact solution to deflection u:

$$u'' = 1 \to u' = x + A \to u = \frac{x^2}{2} + Ax + B$$
$$0 = u(0) = B, \quad 0 = u'(1) = 1 + A \to A = -1$$
$$u = \frac{x^2}{2} - x = \frac{x(x-2)}{2}$$

or run the function 'exact_solver()' in the python program were i solve this with sympy

b)

The function space is spanned by $\psi_i = sin((2i+1)\pi x/2)$, i =0,...,N. both boundary conditions holds: $\psi_i(0) = 0, \psi_i'(1) = 0$ such that $u = \sum_j c_i \phi_j$.
Now I use Galerkin and a least squares method to find the coefficients c_j in $u = \sum_j c_i \phi_j$ and how fast the coefficients decrease in magnitude by looking at c_j/c_{j-1}. And the error in the maximum deflection at x = 1 when only one basis function is used (N=0).
Finding the residual R:

$$R = u''_{exact} - u'' = 1 - u'' = 1 - \frac{d^2u}{dx^2} = \sum_j c_j \frac{d^2\psi_j(x)}{dx^2}$$
$$\frac{d^2\psi_j(x)}{dx^2} = -(\frac{\pi}{2})^2(i+\frac{1}{2})^2 sin(\frac{(2i+1)\pi x}{2})$$
$$R = 1 + \sum_j c_j(\frac{\pi}{2})^2(i+\frac{1}{2})^2 sin(\frac{(2i+1)\pi x}{2})$$

Galerkin method:

$$(R,v) = 0 \rightarrow (1-u'',v) = (1,v) - (u'',v) = 0 \rightarrow (1,v) = (u'',v)$$
$$\int_0^1 u''v = -(u',v') \rightarrow (u',v') = -(1,v)$$

were v = ψ_i, then we have that $A_{ji} = (\psi_j, \psi_i)$ and $b_i = -(1,\psi_i)$, if j $\neq$ i we have $A_{ji} = 0$, and when j = j I get c_i:

$$c_i = \frac{b_i}{A_{ii}} = -\frac{16}{\pi^3(2i+1)}$$

Least squares method:

$$(R, \frac{\partial R}{\partial c_i}) = 0$$
$$\sum_j^N c_j(\psi_j'', \psi_i'') = (-1, \psi_i'')$$
$$\sum_j^N c_j A_{ji} = b_i$$

same as in the Galerkin method we get:

$$c_i = \frac{b_i}{A_{ii}} = -\frac{16}{\pi^3(2i+1)}$$

written i python as function Galerkin and Least_squares.
Looking at who fast the coefficients decrease: if the coefficients decreases we need that $c_j < c_{j-1}$ and the c_j/c_{j-1} will be less then 1. added the function coeff_dec(method) to

calculate those.
Error in the maximum deflection at x = 1:

$$u_e(1) = \frac{1(1-2)}{2} = -\frac{1}{2}$$

$$c_0 = \frac{-16}{\pi^3} \quad , \quad \psi_0 = sin(\frac{\pi x}{2}), \quad u(x) = \frac{-16}{\pi^3} sin(\frac{\pi x}{2})$$

$$u(1) = -\frac{16}{\pi^3}$$

The Error is than:

$$e = 1 - \frac{u_e}{u} = -\frac{-\frac{1}{2}}{-\frac{16}{\pi^3}} = 1 - 0.968 = 0.031 = 3.1\%$$

c)

Visualize the solutions in b) for N=0,1,20: function visualize1()

d)

With $\psi_i = sin((i+1)\frac{\pi x}{2})$, using Galerkin we get:

$$A_{ij} = (\psi_i', \psi_j') = \int_0^1 (i+1)(j+1)(\frac{\pi}{2})^2 \cos((i+1)\frac{\pi x}{2}) \cos((j+1)\frac{\pi x}{2}) dx$$

$$b_i = \int_0^1 (-1) \sin((i+1)\frac{\pi x}{2}) = -\int_0^1 \sin((i+1)\frac{\pi x}{2})$$

I think that the additional basis functions will improve the solution because we get more points.

e)

Now we drop the symmetry condition at x = 1 and extend the domain to [0,2] such that it covers the entire (scaled) physical cable:

$$u'' = 1, \quad x\epsilon(0,2), u(0) = u(2) = 0$$

Visualize for N=0,1,20: function visualize2()

Exercise 5: Compute the deflection of a cable with 2 P1 elements

Solving the problem for u in Exercise 2: Compute the deflection of a cable with sine functions using two P1 linear elements. Incorporate the condition u(0)=0 by two methods:
1) Excluding the unknown at x=0
2) Keeping the unknown at x=0, but modifying the linear system.
finite element mesh with Ne cells, all with length h, and number the cells from left to right. Choosing P1 elements, there are two nodes per cell, and the coordinates of the nodes become:

$$x_i = ih, \quad h = \frac{L}{N}, \quad i = 0, ..., N_n - 1 = N_e$$

Any node i is associated with a finite element basis function $\psi_i(x)$.

$$\psi_i(x) = \begin{cases} 0 & \text{x} < \text{x}_i \\ (\text{x - x}_{i-1})/h & \text{x}_{i-1} \le x < x_i \\ 1 \text{ - (x - x}_i)/h & \text{x}_i \le x < x_{i+1} \\ 0 & \text{x} \ge x_{i+1} \end{cases}$$

1) Excluding the unknown at x=0

Want an approcimation on the form $u_1 = c_0\psi_1 + c_1\psi_2$. The element Matrix $A^{(e)}$ and element vector $b^{(e)}$:

$$A^{(e)} = \frac{1}{h}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad , \quad b^{(e)} = -h\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\frac{1}{h}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = -h\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

And we get $u_1 = -3h^2\psi_1 - 4h^2\psi_2$

2) Keeping the unknown at x=0

Want an approcimation on the form $u_2 = c_0\psi_1 + c_1\psi_2 + c_2\psi_3$. The element Matrix $A^{(e)}$ and element vector $b^{(e)}$:

$$A^{(e)} = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad , \quad b^{(e)} = -h\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = -h\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

And we get $u_2 = -\frac{4h^2}{2}\psi_1 - \frac{h^2}{3}\psi_2 + 0 * \psi_3 \rightarrow u = -\frac{4h^2}{2}\psi_1 - \frac{h^2}{3}\psi_2$.
Need to choose h to find that u_1 and u_2 are good approximation to the exact solution.