MEK3230 - Fluidmekanikk Høst 2017

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2 Korte oppgaver

2.1 Bølge ringer

Strekning: $s = 10\ m$

Hastighet på bølgetoppen: $c = 1\ \frac{m}{s}$

Hastighet på bølgegruppen: $c_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c$

$c_g = \frac{1}{2}\cdot 1\ \frac{m}{s} = \underline{\frac{1}{2}\ \frac{m}{s}}$ (Dypt vann)

Tiden det tar: $t = \frac{s}{v} = \frac{s}{c_g} = \frac{10\ m}{\frac{1}{2}\ \frac{m}{s}} = \underline{\underline{20\ s}}$

2.2 Inkompressibilitet

Hvis luft hadde vært eksakt inkompressibilt så ville vi hørt lyden før vi hadde sett lynet. Men hvis vi nærmer oss til nesten eksakt så ville vi sett og hørt lynet som tidrg.

2.3 Kule som synker

$$\nabla p = \mu \nabla^2 u$$

$$\nabla p \sim \frac{P}{L}, \quad \nabla^2 u \sim \frac{U}{L^2}$$

$$\frac{P}{L} = \frac{\mu U}{L^2} \rightarrow P = \frac{\mu U}{L}$$

$$\tau = \mu \frac{\partial \vec{u}}{\partial \vec{n}} = \mu \frac{\partial u}{\partial x} \sim \mu \frac{U}{L}$$

motstandskraften / drag: $A = 4\pi R^2 \sim L^2$

$$F = \int (P + \tau) \cdot A = \int 2\mu \frac{U}{L} \cdot L^2 \sim \underline{\underline{\int \mu U L}}$$

2.4 Krypstrømning

$F_1 = F_2$ pga stokes ligning er reversibel.

Når du roterer så får du $-u$ og $-F$

Som igjen er det samme som u og F

3 Lange oppgaver

3.1 Spredning av en kopp med sirup på en plate

1)
$$x: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$z: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \underbrace{\cancel{\nu \frac{\partial^2 w}{\partial z^2}}}_{\rightarrow 0} - g$$

vertikal friksjonskrefter << tyngdekraft

$$p = -\rho g z + C$$

$$p_1 = -(\rho - \Delta\rho) g z + C$$

$$p_1(z = H_d) = p_0 \Rightarrow C = p_0 + (\rho - \Delta\rho) g H_d$$

$$\underline{p_1 = g(\rho - \Delta\rho)(H_d - z) + p_0}$$

$$p_2 = -\rho g z + C$$

$$p_1(z = h) = p_2(z = h)$$

$$-\rho g z + C = g(\rho - \Delta\rho)(H_d - h) + p_0$$

$$\Rightarrow C = g(\rho - \Delta\rho)(H_d - h) + \rho g h + p_0$$

$$\underline{p_2 = -\rho g z + g(\rho - \Delta\rho)(H_d - h) + \rho g h + p_0}$$

2) Antar følgende:

- $\frac{h_0}{L} \ll 1 \to \left(\frac{h_0}{L}\right)^2 Re \ll 1 \to$ Lubrikasjons teori
- Stasjonær $\frac{\partial \bar{u}}{\partial t} = 0$ (inkompressibelt)
- w e mindre enn horisontal hastighet av en faktor $\frac{h_0}{L}$, følger det at $\frac{\partial p}{\partial z}$ e mye mindre enn $\frac{\partial p}{\partial x}$.

Da ender jeg med Lubrikasjons ligningen:

$$x: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$z: \quad 0 = -\frac{1}{\rho}\underbrace{\cancel{\frac{\partial p}{\partial z}}}_{\to 0} + \nu \underbrace{\cancel{\frac{\partial^2 w}{\partial z^2}}}_{\to 0} - g$$

<u>Skalering:</u>

$$\frac{1}{\cancel{\rho}}\frac{\partial p}{\partial x} = \frac{\mu}{\cancel{\rho}}\frac{\partial^2 u}{\partial z^2} \to \frac{P}{L} = \mu \frac{U}{h_0^2}$$

$$\underline{\underline{P = \mu \frac{UL}{h_0^2}}}$$

3) Randbetingelser / Grenseverdiger

1) $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

2) $\frac{\partial P}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}$ (Lubrikasjon)

3) $u(z=0) = 0$
4) $w(z=0) = 0$ } No-slip (på veggene)

5) $w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}$; $z=h$ (overflaten)

6) $\left.\frac{\partial u}{\partial z}\right|_{z=h} = 0$

Integrerer 1: $\int_0^{h(x,t)} \frac{\partial w}{\partial z}\, dz = -\int_0^{h(x,t)} \frac{\partial u}{\partial x}\, dz$

V.s: $[w(z)]_0^{h(x,t)} = w(h) - w(0)$ 4) og 5)
$\Rightarrow w(h) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}$

H.s: $-\int_0^{h(x,t)} \frac{\partial u}{\partial x}\, dz$ Bruker Leibenz integrasjons regel:

$$\left(\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,y)\, dy = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}\, dy + f(x,b)\frac{\partial b}{\partial x} - \underbrace{f(x,a)\frac{\partial a}{\partial x}}_{\to\, u(z=0)=0}\right)$$

$$-\int_0^{h(x,t)} \frac{\partial u}{\partial x}\, dz = u(x,h)\frac{\partial h}{\partial x} - \frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dz$$

$$\frac{\partial h}{\partial t} + \cancel{u\frac{\partial h}{\partial x}} = \cancel{u\frac{\partial h}{\partial x}} - \frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dz$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\underbrace{\int_0^{h(x,t)} u\, dz}_{\text{fluksen } q} = 0$$

Integrer 2 : $\dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{\mu}\dfrac{\partial P}{\partial z} \Rightarrow \dfrac{\partial u}{\partial z} = \dfrac{1}{\mu}\dfrac{\partial P}{\partial x} z + C_1$

6) ingen skjær ved overflaten: $C_1 = -\dfrac{1}{\mu}\dfrac{\partial P}{\partial x} h$ ($z = h$ ved overflaten)

$$\frac{\partial u}{\partial z} = \frac{1}{\mu}\frac{\partial P}{\partial x}(z-h)$$

$$u(z) = \frac{1}{\mu}\frac{\partial P}{\partial x}\left(\frac{z^2}{2} - zh\right) + C_2$$

4): $u(z=0) = 0 \Rightarrow C_2 = 0$

$$u(z) = \frac{1}{\mu}\frac{\partial P}{\partial x}\left(\frac{z^2}{2} - zh\right)$$

$$q: \int_0^{h(x,t)} u\, dz = \frac{1}{\mu}\frac{\partial P}{\partial x}\left[\frac{z^3}{6} - \frac{z^2 h}{2}\right]_0^{h(x,t)}$$

$$= \frac{1}{\mu}\frac{\partial P}{\partial x}\left[\frac{h^3}{6} - \frac{3h^3}{6}\right] = -\frac{1}{\mu}\frac{\partial P}{\partial x}\frac{h^3}{3} \qquad \left[\frac{\partial P}{\partial x} = g\Delta\rho\frac{\partial h}{\partial x}\right]$$

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(\frac{h^3}{3\mu}\frac{\partial P}{\partial x}\right) = 0$$

$$\Rightarrow \underline{\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(\frac{g h^3 \Delta\rho}{3\mu}\frac{\partial h}{\partial x}\right)} = 0$$

4) Dimensjonsløs / Skalering av (1) $(z \cdot h_0 = H \cdot h_0)$

$$\frac{\partial (H \cdot h_0) \cdot (g \Delta\rho h_0^3)}{\partial (T \cdot (3L^2 \mu))} - \frac{\partial}{\partial (X \cdot L)}\left(\frac{g (H \cdot h_0)^3 \Delta\rho}{3\mu} \frac{\partial (H \cdot h_0)}{\partial (X \cdot L)}\right) = 0$$

$$g \frac{\Delta\rho h_0^4}{3L^2 \mu}\left(\frac{\partial H}{\partial T} - \frac{\partial}{\partial X}\left(H^3 \frac{\partial H}{\partial X}\right)\right) = 0$$ Deler på $g \frac{\Delta\rho h_0^4}{3L^2\mu}$

$$\underline{\frac{\partial H}{\partial T} - \frac{\partial}{\partial X}\left(H^3 \frac{\partial H}{\partial X}\right) = 0}$$

<u>Skalering</u>:

$$\frac{H}{T} - \frac{H^4}{X^2} = 0 \qquad \frac{\frac{q}{x}}{T} = \frac{\left(\frac{q}{x}\right)^4}{x^2}$$

$$\frac{H}{T} - \frac{H^4}{\left(\frac{q}{H}\right)^2} = 0 \qquad \frac{q}{xT} = \frac{q^4}{x^6}$$

$$\frac{H}{T} - \frac{H^6}{q^2} = 0 \qquad x^5 \simeq T$$

$$X \simeq T^{\frac{1}{5}}$$

$$\frac{q}{T} = H^5 \Rightarrow H \simeq T^{-\frac{1}{5}} \qquad \Rightarrow \underline{\alpha = \frac{1}{5}}$$

$$\Rightarrow \underline{\beta = -\frac{1}{5}}$$

Bruker at arealet $q = X \cdot H$ finner β ved å ta $X = \frac{q}{H}$ og α ved å ta $H = \frac{q}{X}$

5

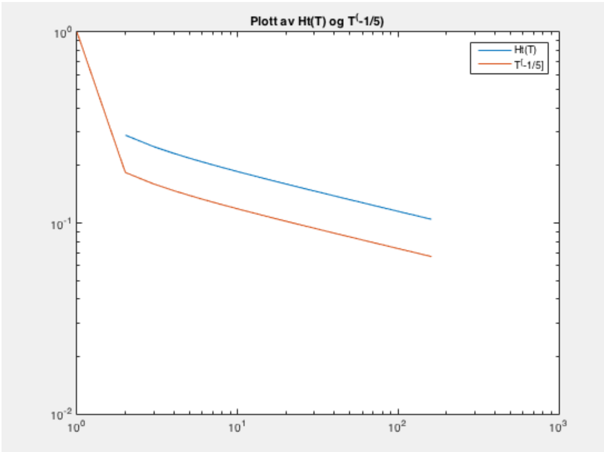

Figure 1: Plotting av$H_T(T)$ og $T^{-\frac{1}{5}}$

Ser at β stemmer ganske greit med $H_T(T)$, er like fra start til slutt av $H_T(T)$. $H_T(T)$ er max H i hver vektor.

6

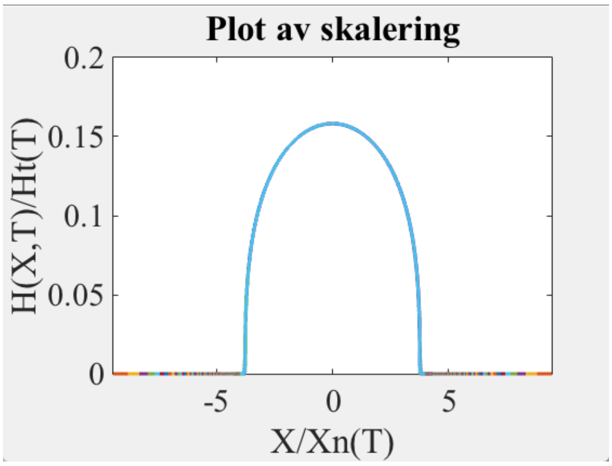

Figure 2: Plotting av stabile med k = 1.1^2 og ustabile k = 0.9^2, med Bo = 1

Ser at plotting av den skalerte er oppå hverandre hele veien, dette sier at det er ganske bra stimulering.

7) $t \to \infty$ har sirupen spredt seg til en tynn flate. Snur platen opp-ned og ser hva som skjer.

Ny trykkledd: $p^* = -p - \gamma \frac{\partial^2 h}{\partial x^2}$, Bruker (1)

$$\frac{g h_0^4 \Delta\rho}{3L^2\mu} \frac{\partial H}{\partial T} + \frac{\partial}{\partial X}\left[H^3 \cdot \frac{h_0^3}{3\mu L}\left(\frac{\partial\left(-p - \gamma \frac{\partial^2 H \cdot h_0}{\partial X^2 L^2}\right)}{\partial X}\right)\right] = 0$$

$$\frac{g h_0^4 \Delta\rho}{\cancel{3}L^2\mu} \frac{\partial H}{\partial T} + \frac{\partial}{\partial X}\left[H^3\left(\frac{g h_0^4 \Delta\rho}{\cancel{3}\mu L^2}\frac{\partial H}{\partial X} + \frac{\gamma h_0^4}{3\mu L^3}\frac{\partial^3 H}{\partial X^3}\right)\right]$$

$$\frac{\partial H}{\partial T} + \frac{\partial}{\partial X}\left[H^3\left(\frac{\partial H}{\partial X} + \frac{1}{B_0}\frac{\partial^3 H}{\partial X^3}\right)\right]$$

$$\frac{1}{B_0} = \frac{\gamma}{g\Delta\rho L^2} \Rightarrow B_0 = g\frac{\Delta\rho}{\gamma}L^2 = \left[\frac{\cancel{m^3}\cdot \cancel{kg}\cdot \cancel{s^2}}{\cancel{m^3}\cdot \cancel{kg}\cdot \cancel{s^2}} = \frac{1}{1}\right]$$

Linearisering:

$$\epsilon\frac{\partial \hat{H}}{\partial T} + \frac{\partial}{\partial X}\left[(H_0 + \epsilon\hat{H})^3\left(\epsilon\frac{\partial \hat{H}}{\partial X} + \frac{\epsilon}{B_0}\frac{\partial^3 H}{\partial X^3}\right)\right]$$

$$\epsilon\frac{\partial \hat{H}}{\partial T} + \frac{\partial}{\partial X}\left[\left(H_0^3 + \cancel{3H_0^2\epsilon\hat{H}}\right)\left(\epsilon\frac{\partial \hat{H}}{\partial X} + \frac{\epsilon}{B_0}\frac{\partial^3 \hat{H}}{\partial X^3}\right)\right] = 0$$

Deler på ϵ:

$$\frac{\partial \hat{H}}{\partial T} + \frac{\partial}{\partial X}\left[h_0^3\left(\frac{\partial \hat{H}}{\partial X} + \frac{1}{B_0}\frac{\partial^3 \hat{H}}{\partial X^3}\right)\right] = 0$$

8 $\hat{H}(x,t) = e^{i(kX - \omega T)}$

$$\frac{\partial \hat{H}}{\partial T} + \frac{\partial}{\partial X}\left[h_0^3\left(\frac{\partial \hat{H}}{\partial X} + \frac{1}{B_0}\frac{\partial^3 \hat{H}}{\partial X^3}\right)\right] = 0$$

Setter inn $\hat{H}(x,t)$:

$$-i\omega e^{i(kX-\omega T)} + \frac{\partial}{\partial X}\left[h_0^3\left(ik e^{i(kX-\omega T)} - \frac{ik^3}{B_0} e^{i(kX-\omega T)}\right)\right]$$

$$-i\omega \cancel{e^{i(kX-\omega T)}} + h_0^3\left(-k^2 \cancel{e^{i(kX-\omega T)}} + \frac{k^4}{B_0} \cancel{e^{i(kX-\omega T)}}\right)$$

$$-i\omega + h_0^3\left(-k^2 + \frac{k^4}{B_0}\right) = 0$$

$$i\omega = h_0^3\left(\frac{k^4}{B_0} - k^2\right) \rightarrow \omega = \frac{h_0^3\left(\frac{k^4}{B_0} - k^2\right)}{i}$$

Setter ω inn i $\hat{H}$:

$$\hat{H}(x,t) = e^{i\left(kX - \frac{h_0^3\left(\frac{k^4}{B_0} - k^2\right)}{i}T\right)} = e^{ikX} e^{-h_0^3\left(\frac{k^4}{B_0} - k^2\right)T}$$

har at $e^{\sigma_i X} e^{\sigma_r T}$ der σ_r avgjør om stabilitet

Vår tilfelle: $\sigma_r = -h_0^3\left(\frac{k^4}{B_0} - k^2\right)$

$\sigma_r > 0$ (ustabil) $\rightarrow K < \sqrt{B_0}$

$\sigma_r < 0$ (stabil) $\rightarrow K > \sqrt{B_0}$

$\sigma_r = 0$ (nøytral) $\rightarrow K = \sqrt{B_0}$

8 $\hat{H}(x,t) = e^{i(kX - \omega T)}$

$$\frac{\partial \hat{H}}{\partial T} + \frac{\partial}{\partial X}\left[h_0^3\left(\frac{\partial \hat{H}}{\partial X} + \frac{1}{B_0}\frac{\partial^3 \hat{H}}{\partial X^3}\right)\right] = 0$$

Setter inn $\hat{H}(x,t)$:

$$-i\omega e^{i(kX-\omega T)} + \frac{\partial}{\partial X}\left[h_0^3\left(ik e^{i(kX-\omega T)} - \frac{ik^3}{B_0} e^{i(kX-\omega T)}\right)\right]$$

$$-i\omega \cancel{e^{i(kX-\omega T)}} + h_0^3\left(-k^2 \cancel{e^{i(kX-\omega T)}} + \frac{k^4}{B_0}\cancel{e^{i(kX-\omega T)}}\right)$$

$$-i\omega + h_0^3\left(-k^2 + \frac{k^4}{B_0}\right) = 0$$

$$i\omega = h_0^3\left(\frac{k^4}{B_0} - k^2\right) \rightarrow \omega = \frac{h_0^3\left(\frac{k^4}{B_0} - k^2\right)}{i}$$

Setter ω inn i $\hat{H}$:

$$\hat{H}(x,t) = e^{i\left(kX - \frac{h_0^3\left(\frac{k^4}{B_0} - k^2\right)}{i}T\right)} = e^{ikX} e^{-h_0^3\left(\frac{k^4}{B_0} - k^2\right)T}$$

har at $e^{\sigma_i X} e^{\sigma_r T}$ der σ_r avgjør om stabilitet

Vår tilfelle: $\sigma_r = -h_0^3\left(\frac{k^4}{B_0} - k^2\right)$

$\sigma_r > 0$ (ustabil) $\rightarrow$ $K < \sqrt{B_0}$

$\sigma_r < 0$ (stabil) $\rightarrow$ $K > \sqrt{B_0}$

$\sigma_r = 0$ (nøytral) $\rightarrow$ $K = \sqrt{B_0}$

9

Videre har jeg lagt ved 5 plot: 2 ustabile, 2 stabile og 1 der det består av en stabil og en ikke stabil. Alle har T = 600. Ut fra figurene ser jeg at den ustabile vil gå oppover mot et større talle enn den stabile, den stabile vil være ”stabilt” bortover.

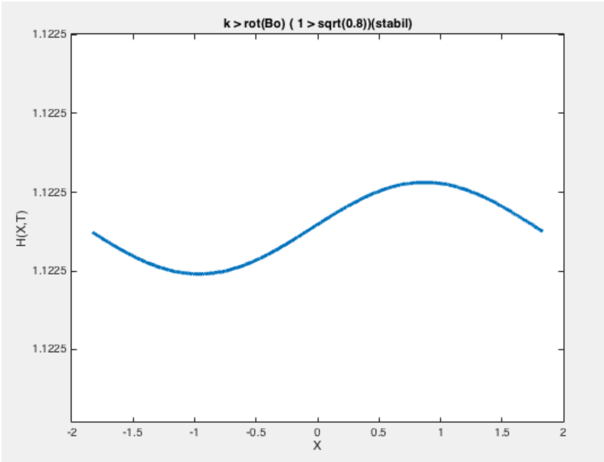

Figure 3: Stabile med k > $\sqrt{Bo}$

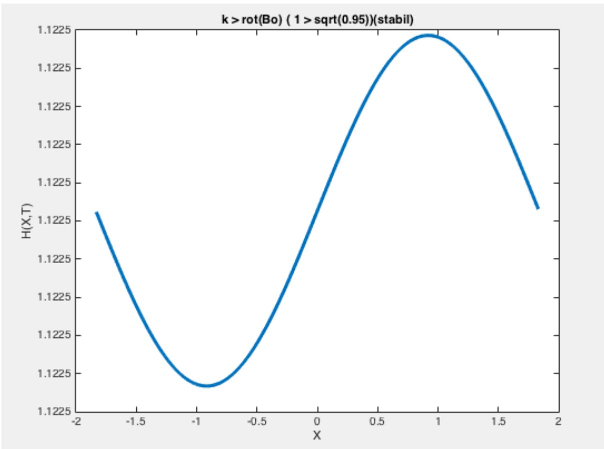

Figure 4: Stabile med k > $\sqrt{Bo}$

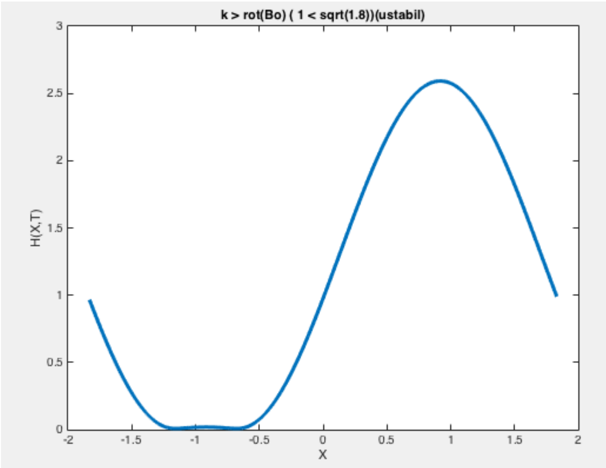

Figure 5: Ustabile med k < $\sqrt{Bo}$

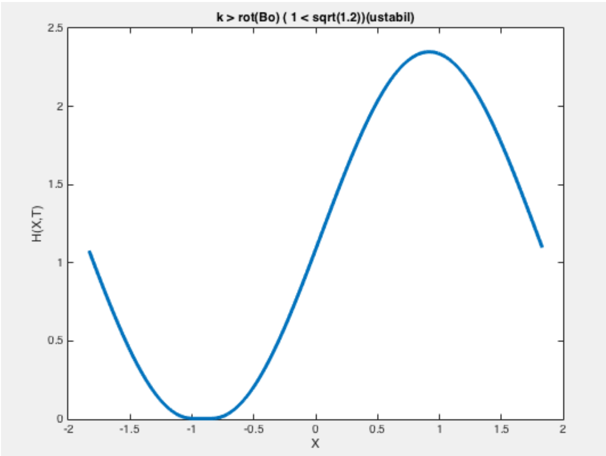

Figure 6: Ustabile med k < $\sqrt{Bo}$

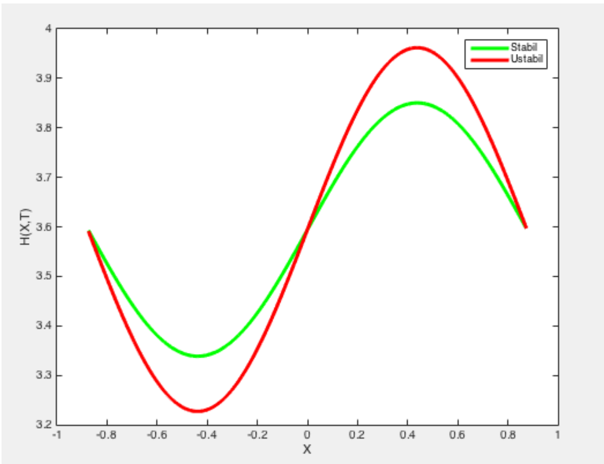

Figure 7: Plotting av stabile med k = 1.1^2 og ustabile k = 0.9^2, med Bo = 1

3.2 Bølge oppgave

1) Betingelsen for at vi kan beskrive strømningen i væskelagene via hastighetspotensialet $\varphi(x,t)$: $\nabla \times \bar{v} = 0$ virvelfritt og ingen friksjon

2) Randbetingelser, linearisert rundt $z=0$

(K1) $\frac{D\eta}{dt} = w$ for $z=\eta$ (kinematisk)

Linearisert: $\frac{\partial \eta}{\partial t} = w \left(\Rightarrow \frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z} \right)$ ved $z=0$

(D1) $\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\eta = f(t)$ for $z=\eta$ (Dynamisk)

Linearisert: $\frac{\partial \varphi}{\partial t} + g\eta = 0$ ved $z=0$

(G1) $w = 0$ ved $z = \pm H$

3) Antar løsning: $\varphi = A \cosh(B) \cos(C)$

A, B og C er ikke (nødvendigvis) konst.

fra (9.14) får vi $\varphi_1 = -\frac{ac}{\sinh(kH)} \cosh k(z+H) \cos k(x-ct)$

med $c^2 = \frac{g}{k} \tanh(kH)$

$\varphi_2 = -\frac{ac}{\sinh(kH)} \cosh k(z-H) \cos k(x-ct)$

Sjekker det ved hjelp av randbetingelser:

(K1): $\frac{\partial \varphi_1}{\partial z} = -\frac{cak}{\sinh(kH)} \cdot \cos k(x-ct) \sinh k(z+H)$

$$\left.\frac{\partial \varphi_1}{\partial z}\right|_{z=0} = -cak\cos(x-ct)$$

(D1): $\frac{\partial \varphi_1}{\partial t} = -\frac{ac^2k}{\sinh(kH)} \cdot \cosh k(z+H) \sin k(x-ct)$

$$\left.\frac{\partial \varphi_1}{\partial t}\right|_{z=0} = -\frac{ac^2 \sin k(x-ct)}{\tanh(kH)}$$

$(z=0)$ $g\eta = g\,a\,\sin k(x-ct) = \frac{c^2 ak \sin k(x-ct)}{\tanh(kH)} = \frac{\partial \varphi}{\partial t}$

$\Rightarrow$ $c^2 = \frac{g}{k}\tanh(kH)$ $\left(z=0 \rightarrow \frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z}\right)$

Fra φ_1 ser vi at:

$A = -\frac{ac}{\sinh(kH)}$, $B = k(z+h)$, $C = k(x-ct)$

L)

har $c^2 = \frac{g}{k}\tanh(kH)$

$\rightarrow$ $c = \sqrt{\frac{g}{k}\tanh(kH)}$

Siden vi er på dypt vann $c_g = \frac{1}{2}c$

Dermed har vi $\underline{c_g = \sqrt{\frac{g}{4k}\tanh(kH)}}$

(Vi får φ_1 og en φ_2 etter å ha regnet ut for meg at (K1) og (D1) stemmer for begge)