FEniCS Course

Lecture 8: The Stokes problem

Contributors
André Massing, Kent-Andre Mardal

The Stokes equations

$$\begin{aligned}
-\Delta u + \nabla p &= f && \text{in } \Omega && \text{Momentum equation} \\
\nabla \cdot u &= 0 && \text{in } \Omega && \text{Continuity equation} \\
u &= g_D && \text{on } \partial\Omega_D && \\
\frac{\partial u}{\partial n} - pn &= g_N && \text{on } \partial\Omega_N &&
\end{aligned}$$

- u is the fluid velocity and p is the pressure
- f is a given body force per unit volume
- g_{D} is a given boundary flow
- g_{N} is a given function for the natural boundary condition

Variational problem

Multiply the momentum equation by a test function v and integrate by parts:

$$\int_\Omega \nabla u : \nabla v \,\mathrm{d}x - \int_\Omega p \nabla \cdot v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\partial\Omega_N} g_N \cdot v \,\mathrm{d}s$$

Short-hand notation:

$$\underbrace{\langle \nabla u, \nabla v\rangle}_{a(u,v)} \underbrace{-\langle p, \nabla \cdot v\rangle}_{b(v,p)} = \underbrace{\langle f, v\rangle + \langle g_N, v\rangle_{\partial\Omega_N}}_{L(v)}$$

Multiply the continuity equation by a test function q:

$$\underbrace{\pm\langle \nabla \cdot u, q\rangle}_{b(u,q)} = 0$$

Definitions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are meaningful if $u \in H^1(\Omega)$ and $p \in L^2(\Omega)$

Saddle point formulation of the Stokes problem

The Stokes problem is an example of a saddle point problem:
Find $(u, p) \in V \times Q$ such that for all $(v, q) \in \widehat{V} \times \widehat{Q}$

$$\begin{aligned} a(u, v) + b(v, p) &= L(v) \\ b(u, q) \qquad\qquad &= 0 \end{aligned}$$

Sum up: $A(u, p; v, q) := a(u, v) + b(v, p) + b(u, q) = L(v)$
Mixed spaces:

$$\begin{aligned} V &= [H^1_{g_D,\Gamma_D}(\Omega)]^d & \qquad \widehat{V} &= [H^1_{0,\Gamma_D}(\Omega)]^d \\ Q &= L^2(\Omega) & \qquad \widehat{Q} &= L^2(\Omega) \end{aligned}$$

The inf-sup condition

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geqslant C$$

is critical for the unique solvability of the saddle point problem

Discrete variational problem

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in \widehat{V_h} \times \widehat{Q_h}$

$A_h(u_h, p_h; v_h, q_h) := a_h(u_h, v_h) + b_h(v_h, p_h) + b_h(u_h, q_h) = L_h(v_h)$

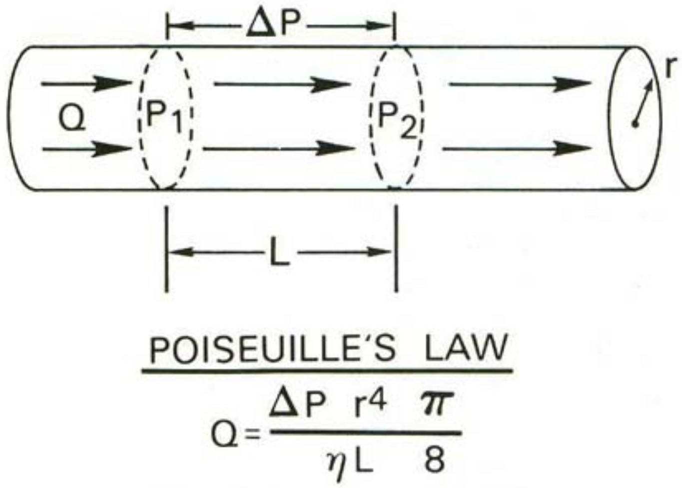
A stable mixed element $V_h \times Q_h \subset V \times Q$ should satisfy a uniform inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geqslant c_b$$

with c_b independent of the mesh $\mathcal{T}_h$!

$\Rightarrow$ The right "mixture" of elements is critical for stability and convergence!

The famous Poiseuille flow



Poiseuille flow with $\mathbb{P}_2 - \mathbb{P}_1$ elements

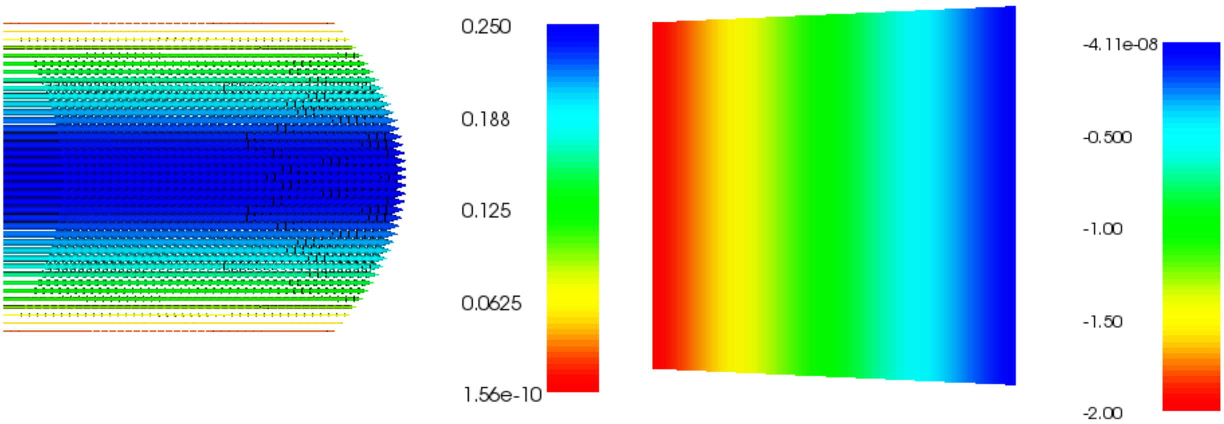


Figure: Illustration of Poiseuille flow in 2D as computed with $\mathbb{P}_1 - \mathbb{P}_1$ elements in FEniCS. Left image shows the velocity vectors while the right image shows the pressure. Both velocity and pressure are correct up to round-off error.

Poiseuille flow with $\mathbb{P}_1 - \mathbb{P}_1$ elements

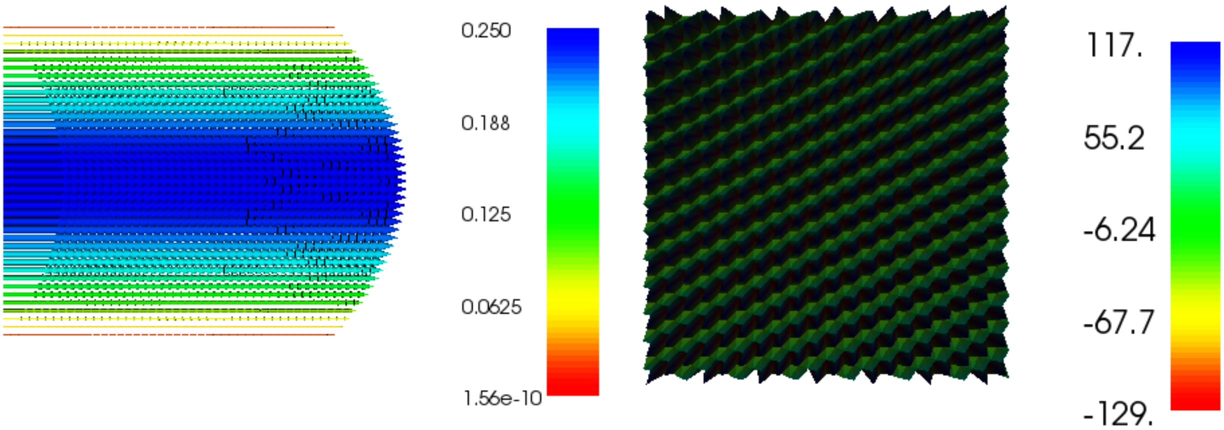
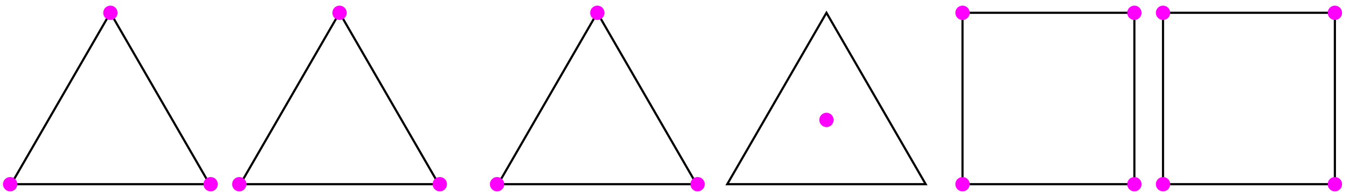


Figure: Illustration of Poiseuille flow in 2D as computed with $\mathbb{P}_1 - \mathbb{P}_1$ elements in FEniCS. Left image shows the velocity vectors while the right image shows the pressure. The velocity is correct but the pressure is *not*. The $\mathbb{P}_1 - \mathbb{P}_1$ discretization violates the inf-sup condition.

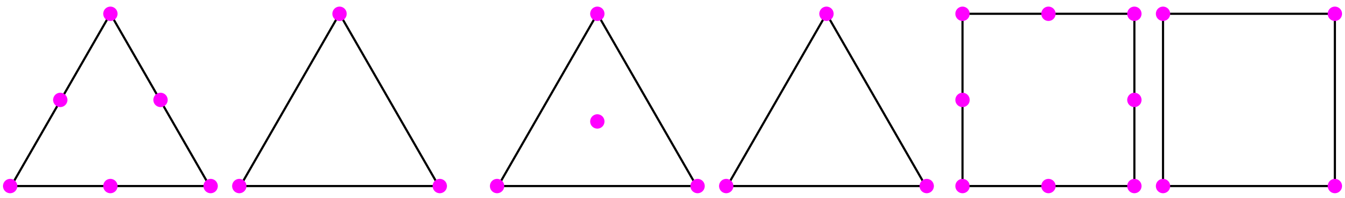
Unstable and stable Stokes elements

Unstable elements



$\mathbb{P}_1/\,\mathbb{P}_1$ $\quad$ $\mathbb{P}_1/\,\mathbb{P}_0$ $\quad$ $\mathbb{P}_1/\,\mathbb{P}_1$

Stable elements



$\mathbb{P}_2/\,\mathbb{P}_1$ $\quad$ $\mathbb{P}_1^b/\,\mathbb{P}_1$ $\quad$ $\mathbb{Q}_2/\,\mathbb{Q}_1$

Taylor-Hood elements: $\mathbb{P}_{k+1}/\,\mathbb{P}_k$, $\mathbb{Q}_{k+1}/\,\mathbb{Q}_k$ for $k \geqslant 1$
Mini-element: $\mathbb{P}_1^b/\,\mathbb{P}_1$

The Stokes problem is a saddle point problem

Saddle point problem

Given bilinear forms and linear forms

- $a(\cdot,\cdot) : V \times V \to \mathbb{R}$
- $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$
- $L(\cdot) : V \to \mathbb{R}$

Find $(u, p) \in V \times Q$ such that for all $(v, q) \in \widehat{V} \times \widehat{Q}$

$$\begin{aligned} a(u, v) + b(v, p) &= L(v) \\ b(u, q) \qquad\qquad &= 0 \end{aligned} \tag{1}$$

Operator formulation

Define operators (Riesz representation theorem)

- $\langle Aw, v\rangle_{V',V} = a(w, v) \qquad \forall\ (w, v) \in V \times V$
- $\langle Bv, q\rangle_{Q',Q} = b(v, q) \qquad \forall\ (v, q) \in V \times Q$

$$\begin{aligned} Au + B^\top p &= L(v) \\ Bu \qquad &= 0 \end{aligned}$$

Existence, uniqueness and stability: The continuous case

- Continuity of A and B

$$\begin{aligned} a(w,v) &\leqslant C_a \|w\|_V \|v\|_V \quad \forall\,(w,v) \in V \times V \\ b(v,q) &\leqslant C_b \|v\|_V \|q\|_Q \quad \forall\,(v,q) \in V \times Q \end{aligned}$$

- Coercivity of A on $\ker B$:

$$c_a \|v\|_V \leqslant a(v,v) \quad \forall\, v \in \ker B$$

- Inf-sup condition:

$$\inf_{q\in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geqslant c_b$$

- Compatibility condition for g

Then there exists a unique $(u,p) \in V \times Q$ solving (SPP), satisfying

$$\begin{aligned} \|u\|_V &\leqslant \frac{1}{c_a} \|f\|_{V'} \big(+ \frac{c_a + C_a}{c_b} \|g\|_{Q'} \big) \\ \|p\|_{Q'} &\leqslant \frac{1}{c_b} \big((1 + \frac{C_a}{c_a}) \|f\|_{V'} + \frac{C_a(c_a + C_a)}{c_a c_b} \|g\|_{Q'} \big) \end{aligned}$$

Existence, uniqueness and stability: The discrete case

- Continuity of A_h and B_h

$$
\begin{aligned}
a(w_h, v_h) &\leqslant C_a \|w_h\|_V \|v_h\|_V \quad \forall\, (w_h, v_h) \in V_h \times V_h \\
b(v_h, q_h) &\leqslant C_b \|v_h\|_V \|q_h\|_Q \quad \forall\, (v_h, q_h) \in V_h \times Q_h
\end{aligned}
$$

- Coercivity of A_h on $\ker B_h$:

$$
c_a \|v_h\|_V \leqslant a(v_h, v_h) \quad \forall\, v_h \in \ker B
$$

- Inf-sup condition: There is a mesh-independent constant c_b s.t.

$$
\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geqslant c_b
$$

- Compatibility condition for g

Then there exists a unique $(u_h, p_h) \in V_h \times Q_h$ solving (SPP), satisfying

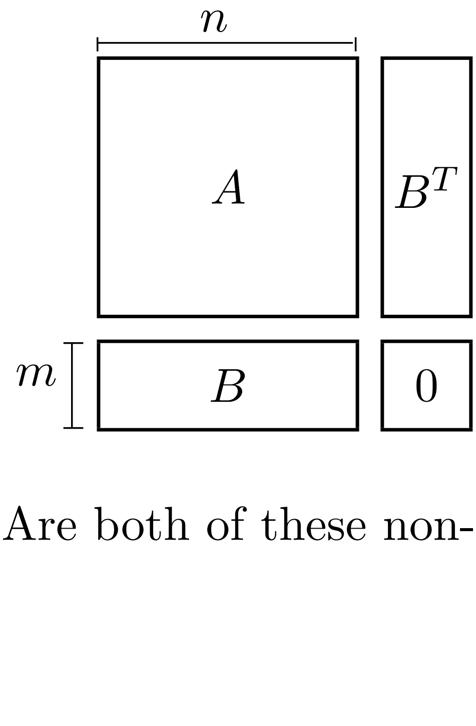
$$
\begin{aligned}
\|u_h\|_V &\leqslant \frac{1}{c_a} \|f\|_{V_h'} \big(+ \frac{c_a + C_a}{c_b} \|g\|_{Q_h'}\big) \\
\|p_h\|_{Q_h'} &\leqslant \frac{1}{c_b} \big((1 + \frac{C_a}{c_a}) \|f\|_{V_h'} + \frac{C_a(c_a + C_a)}{c_a c_b} \|g\|_{Q_h'}\big)
\end{aligned}
$$

The Brezzi conditions, linear algebra point of view

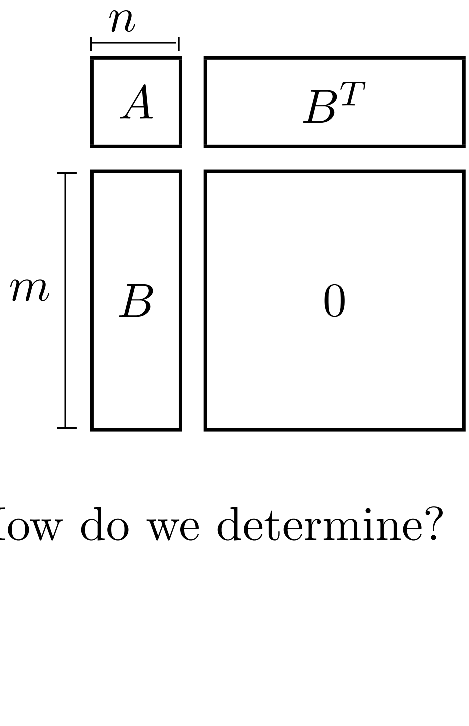
Letting $u_h = \sum_{i=1}^{n} u_i N_i$, $p_h = \sum_{i=1}^{m} p_i L_i$, $v_h = N_j$, and $q_h = L_j$ we obtain a linear system on the form

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{2}$$

The question is what the system looks like. Two alternatives:



or



Are both of these non-singular? How do we determine?

The Brezzi conditions, linear algebra point of view, cont'd.

- Continuity and coersivity of A_h ensures that A_h is non-singular
- Continuity and inf-sup condition of B_h ensures that B_h is non-singular

How come we need inf-sup condition on B_h ? The coersivity condition seems easier to deal with!

The Brezzi conditions, linear algebra point of view, cont'd.

- Continuity and coersivity of A_h ensures that A_h is non-singular
- Continuity and inf-sup condition of B_h ensures that B_h is non-singular

Remember that B_h is a rectangular matrix! The inf-sup condition corresponds to coersivity of $B_h B_h^T$, where B_h is a discrete divergence and B_h^T the discrete gradient.

We also remark that the coersivity and inf-sup conditions are not inherited from the continuous operators while continuity is inherited (for conforming methods).

Abstract error estimate for saddle point problems

$$\|u - u_h\|_V \leqslant \big(1 + \frac{C_a}{c_{a,h}}\big) \inf_{v_h \in} \|u - v_h\|_V + \frac{C_b}{c_{a,h}} \inf_{q_h \in Q_h} \|p - q_h\|_Q$$

$$\begin{aligned}\|p - p_h\|_Q \leqslant & \frac{C_a}{c_{b,h}}\big(1 + \frac{C_a}{c_{a,h}}\big) \inf_{v_h \in} \|u - v_h\|_V \\ & + \big(1 + \frac{C_b}{c_{b,h}} + \frac{C_a C_b}{c_{a,h} c_{b,h}}\big) \inf_{q_h \in Q_h} \|p - q_h\|_Q\end{aligned}$$

Abstract error estimate for saddle point problems for $\mathbb{P}_k - \mathbb{P}_l$ discretizations

$$\|u - u_h\|_1 + \|p - p_h\|_0 \leqslant Ch^k \|u\|_{k+1} + Dh^{l+1} \|p\|_{l+1}$$

Here, $\| \cdot \|_r$ is the norm of the Sobolev space H^r, i.e., a norm containing r derivatives.
Note that $k = l + 1$ will give result in the simplifed estimate

$$\|u - u_h\|_1 + \|p - p_h\|_0 \leqslant Ch^k (\|u\|_{k+1} + \|p\|_k)$$

Taylor–Hood, Crouzeix-Raviart elements are examples of such elements.

Useful FEniCS tools (I)

Mixed elements:

```
V = VectorFunctionSpace(mesh, "Lagrange", 2)
Q = FunctionSpace(mesh, "Lagrange", 1)
W = V*Q
```

Defining functions, test and trial functions:

```
up = Function(W)
(u,p) = split(up)
```

Shortcut:

```
(u, p) = Functions(W)
# similar for test and trial functions
(u, p) = TrialFunctions(W)
(v, q) = TestFunctions(W)
```

Useful FEniCS tools (II)

Access subspaces:

```
W.sub(0) #corresponds to V
W.sub(1) #corresponds to Q
```

Splitting solution into components:

```
w = Function(W)
solve(a == L, w, bcs)
(u, p) = w.split()
```

Rectangle mesh:

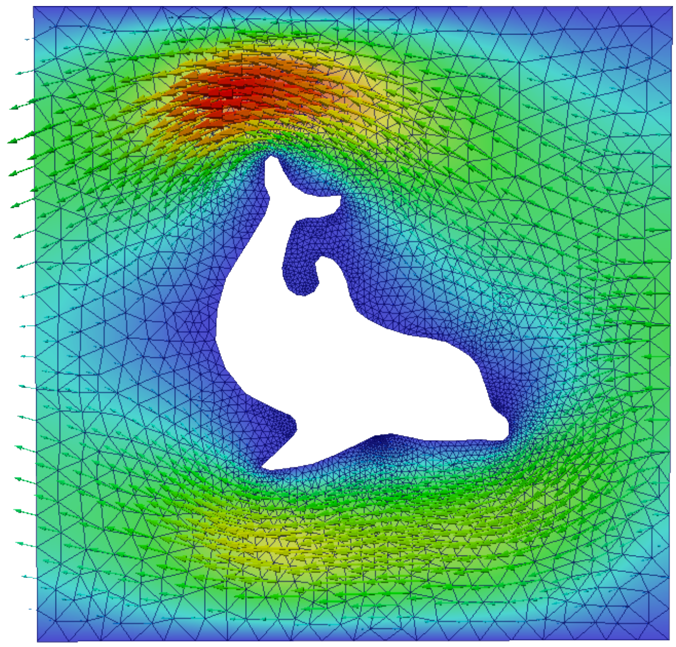
```
mesh = RectangleMesh(0.0, 0.0, 5.0, 1.0, 50, 10)
```

```
h = CellSize(mesh)
```

The FEniCS challenge!

Compute the Stokes flow around a swimming dolphin!

- Set a no-slip boundary condition on the upper and lower channel walls and around the dolphin
- Set $u = (-\sin(\pi y), 0)$ on the right inflow boundary
- Impose $p = 0$ on the left outflow boundary
- Implement a scheme based on Taylor–Hood elements
- Implement a scheme based on the stabilized $\mathbb{P}_2/\mathbb{P}_2$ elements with a stabilization parameter β. What happens if you reduce the size of β?



Exercise: check the convergence of a known scheme

We had the error estimate:

$$\|u - u_h\|_1 + \|p - p_h\|_0 \leqslant Ch^k \|u\|_{k+1} + Dh^{l+1} \|p\|_{l+1}$$

Check if this is correct by manufacturing a right-hand side (and bc) from a known solution. Assume that $u = \nabla \times sin(\pi xy)$ and $p = sin(2\pi x)$ and compute the right-hand side as $f = -\Delta u + \nabla p$. The $\nabla\times$ operator is defined as $(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$. Compute numerical solutions u_h and p_h on refinements of the unit square and check it the error estimate is valid. Use Taylor-Hood $(\mathbb{P}_2 - \mathbb{P}_1)$ as well as $(\mathbb{P}_1 - \mathbb{P}_1)$.