An expressive and optimal pretty printer

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Pretty printers make trade-offs between the *expressiveness* of their pretty printing language, the *optimality* objective that they minimize when choosing between different ways to lay out a document, and the *performance* of their layout algorithm. This paper presents a new pretty printer, Π_e, that is strictly more expressive than all pretty printers in the literature and provably minimizes the optimality objective. Furthermore, the time complexity of Π_e is better than many existing pretty printers. When choosing among different ways to lay out a document, Π_e consults a user-supplied *cost factory*, which determines the optimality objective, giving Π_e a unique degree of flexibility. We use the Lean theorem prover to verify the correctness and optimality of Π_e, and implement Π_e concretely as a pretty printer that we call SnowWhite. To evaluate our pretty printer against others, we develop a formal framework for reasoning about the expressiveness of pretty printing languages, and survey pretty printers in the literature, comparing their expressiveness, optimality, worst-case time complexity, and practical running time. Our evaluation shows that SnowWhite is efficient and effective at producing optimal layouts. SnowWhite has also seen real-world adoption: it serves as a foundation of a code formatter for Racket.

1 INTRODUCTION

General-purpose pretty printers (or, simply, *printers*) are widely used to convert structured data—typically an AST—into human-readable text. Their applications include code reformatting, software reengineering, and synthesized code printing [De Jonge 2002; Prettier 2016; Torlak and Bodik 2014; Yelland 2015]. These printers take as inputs (1) a document in a pretty printing language (*PPL*), which encodes the structured data along with styling choices, and (2) a page width limit. Choices in the document can yield (exponentially) many possible layouts. The task of the printers then is to efficiently choose an *optimal* layout from all possible layouts. Existing printers use a variety of built-in optimality objectives. A good objective reflects the informal notion of "prettiness," such as not having an overflow over the page width limit whenever possible while having as few lines as possible.

Different printers make different trade-offs in the *expressiveness* of the PPL, the *optimality* objective, and the *performance*. With the current landscape of pretty printing, developers who wish to use expressive features from various PPLs, optimize certain objectives, and achieve practical running time are left with two possibilities. They can write an ad-hoc printer, which is time-consuming and error-prone. Alternatively, they can pick an existing general-purpose printer, potentially giving up some desired features. This is further complicated by some common misunderstandings about the three aspects of these existing printers.

This paper presents a printer that we call Π_e. It targets Σ_e, a PPL that is strictly more expressive than all published PPLs. This can be shown via our formal framework for reasoning about the expressiveness of PPLs. Π_e is parameterized by a *cost factory*, which enables developers to specify an optimality objective for Π_e to minimize. The cost factory is versatile. For example, it can express non-linear costs and define concepts such as soft page width limits [Yelland 2016]. As a result, the optimal layout that Π_e chooses can have higher quality compared to existing printers. The time complexity of Π_e is $O(nW^4)$, where n is the size of the document and W is the computation width limit (defined in Section 5). This is better than the time complexity of many printers in the literature, and it is improved to $O(nW^3)$ when Π_e is restricted to process documents in some well-known but less expressive PPLs. We prove the correctness of Π_e in the Lean theorem prover [Moura and Ullrich 2021], ensuring the validity and optimality of the output layout, and demonstrate Π_e's efficiency by evaluating our implementation of Π_e, which we call SnowWhite. We believe these attributes make Π_e not only a good printer by itself, but also a good building block to construct other derived printers.

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```
text "function␣append(first,second,third){"
<> nest 4 (
  let f = text "first␣+" in
  let s = text "second␣+" in
  let t = text "third" in
  nl <> text "return␣" <>
  group (nest 4 (f <> nl <> s <> nl <> t))
) <> nl <> text "}"
```

```
function append(first,second,third){
    return first +
        second +
        third
}
```

```
function append(first,second,third){
    return first + second + third
}
```

(a) A document in the traditional PPL and its corresponding layouts. The **nest** construct increments the current indentation level by some specified amount, causing **nl** (newline) to insert indentation spaces. **<>** is the unaligned concatenation operator, which places the right sub-layout after the left sub-layout on the current indentation level. Lastly, the **group** construct creates two formatting choices: one where the sub-layouts are left alone and one where the sub-layouts are flattened by replacing newlines and indentation spaces due to **nl**s in the group with single spaces.

```
text "function␣append(first,second,third){" <$>
( let f = text "first␣+" in
  let s = text "second␣+" in
  let t = text "third" in
  let sp = text "␣" in
  let indentation = text "␣␣␣␣" in
  let ret = text "return␣" in
  indentation <+>
  (((ret <+> text "(") <$>
    (indentation <+> (f <$> s <$> t)) <$>
    text ")") <|>
   (ret <+> f <+> sp <+> s <+> sp <+> t)))
<$> text "}"
```

```
function append(first,second,third){
    return (
        first +
        second +
        third
    )
}
```

```
function append(first,second,third){
    return first + second + third
}
```

(b) A document in the arbitrary-choice PPL and its corresponding layouts. **<|>** is the arbitrary-choice operator, which per its namesake, creates two formatting choices from layouts by arbitrary sub-document. **<$>** is the vertical concatenation operator, which joins two sub-layouts with a newline. Lastly, **<+>** is the aligned concatenation operator, which joins two sub-layouts horizontally, aligning the whole right sub-layout at the column where it is to be placed in.

Fig. 1. The traditional and arbitrary-choice PPLs, embedded in the host language OCaml. Colored regions in a document and corresponding layouts indicate the correspondence between the colored sub-documents and the colored sub-layouts. We use the **let** construct to make the documents easier to read, even though it is usually not a part of PPLs. Dotted lines illustrate different page width limits at 22 and 36 characters.

A survey of printers in the wild. To evaluate Π_e, we conducted a broad survey of the literature on pretty printing. Most PPLs are embedded in a host programming language, provide a small set of core constructs that allow developers to create a document with text, concatenate documents together, set indentation level, and express formatting choices. High-level constructs can then be built on top of the core constructs. The details of these core constructs can differ from PPL to PPL. We found that there are two main schools of PPLs in the wild, which we call the *traditional* and *arbitrary-choice* PPLs. The traditional PPL centers around manipulation of **nl**s (newlines) and current indentation level, while the arbitrary-choice PPL is characterized by the ability to express arbitrary formatting choices and the use of aligned concatenation to supplant the concept of indentation level. Figure 1 illustrate documents in both PPLs that pretty-print the function definition append in a hypothetical programming language with slightly different styling.

Expressiveness. The literature contains informal claims about the expressiveness of PPLs [Chitil 2005; Podkopaev and Boulytchev 2015; Wadler 2003]. We develop two formal notions of expressiveness: the ability to *express layouts* and the ability to *express features*. The former reflects the functionality of a PPL, while the latter reflects the ease of document construction. Using our framework, we can show that neither the traditional PPL nor the arbitrary-choice PPL is more expressive than the other. For example, the set of layouts in Figure 1b cannot be expressed by any document in the traditional PPL. This is because all layouts due to a particular document in the traditional PPL must be the same modulo whitespace, but one of the layouts in the figure has an extra pair of parentheses.[1] As another example, the document in Figure 1b is awkwardly constructed. It would be more natural to use unaligned concatenation, but the feature cannot be expressed by any combination of features in the arbitrary-choice PPL.[2] To that end, we develop a PPL called Σ_e that is provably strictly more expressive than both the traditional and arbitrary-choice PPLs, facilitating both functionality and ease of document construction.

Optimality. The optimality objective of a printer indicates what it optimizes for when resolving choices. Most printers targeting the traditional PPL minimizes overflow over the page width limit line-by-line, preferring a longer line when there is no overflow. For example, given the document in Figure 1a, the first layout is optimal when the page width limit is 22 (red dotted line), while the second layout is optimal when the page width limit is 36 (green dotted line). Contrary to prior claims [Chitil 2005; Wadler 2003], we discovered that this strategy guarantees neither the absence of overflow whenever possible nor the minimality of the number of lines. In contrast, most printers targeting the arbitrary-choice PPL minimizes the number of lines among layouts with no overflow. However, they *error* when all possible layouts have an overflow, resulting in a poor user experience (e.g., when the width limit is 22 in Figure 1b). Recognizing that unavoidable overflows do occur in practice, we introduce the concept of a *cost factory*. The factory allows developers to choose a desired objective permitted by its interface, including an objective that tolerates overflow gracefully.

Performance. Printing proceeds in two phases: resolving choices and rendering the optimal choice to text (although many printers fuse these two phases together). Time complexity of printers is best measured against the resolving phase[3], and it is usually specified with two parameters: the size of the document n and the width limit W, with the preference that the time complexity be polynomial in W and linear in n. Most printers in the literature leave their time complexity unanalyzed, instead opting to show experimental results that their implementations are efficient in practice. We analyze these printers and demonstrate documents that trigger worse than linear time behavior (in n) on some printers. Further complication arises in printers with the arbitrary choice feature, which gives rise to documents that are structured as DAGs as opposed to trees. We show that many printers that treat the input document as a tree suffer from a combinatorial explosion as the DAG structure is unfolded, resulting in exponential time complexity. With a combination of proof and experimental results, we show that the time complexity of Π_e is linear in the DAG size of the document and that it runs fast in practice.

In summary, this paper makes the following contributions:

- A new PPL called Σ_e that is strictly more expressive than all published PPLs. Constructs in Σ_e are not new, but packaging them all in a single PPL has never been done before.
- A printer Π_e targeting Σ_e that utilizes a *cost factory* to allow a variety of optimality objectives.

[1]Languages such as Python require an extra pair of parentheses around an expression that spans multiple lines [The Python Language Reference 2010]. Similarly, some styles prefer adding an extra comma (also known as trailing comma) when a function call spans multiple lines [ESLint 2014]. Hence, the ability to express layouts with differing content is desirable.
[2]Different programming language styles prefer different concatenation operators. C-like languages heavily use unaligned concatenation, while aligned concatenation has been used for Haskell, Lisp, R, and Julia. However, there are instances where C-like languages would benefit from aligned concatenation, and Haskell would benefit from unaligned concatenation.
[3]This formulation allows us to talk about "linear-time" printers, even though there are, e.g., documents whose size is $O(n)$, but the rendering would result in $O(n^2)$ characters.

Table 1. A comparison of existing printers. n and $\hat{n}$ are the DAG size and tree size of the input document (where $\hat{n}$ in the worst case is exponential in n). W is the width limit.

	Expressiveness		**Optimality**	**Performance**
Printer	Choice	Concatenation	Minimization objective	Time complexity
Oppen [1980]	Group	Unaligned	Lexicographic overflow	$O(n)$
Hughes [1995]	Group	Aligned	Lexicographic overflow	$O(n^2)$
Wadler [2003]	Group	Unaligned	Lexicographic overflow	$O(n^2)$
Leijen [2000]	Group	Both	Lexicographic overflow	$O(n^2)$
Chitil [2005]	Group	Unaligned	Lexicographic overflow	$O(n)$
Kiselyov et al. [2012]	Group	Unaligned	Lexicographic overflow	$O(n)$
Swierstra et al. [1999]	Arbitrary	Aligned	Height†	Exp. in n
Podkopaev and Boulytchev [2015]	Arbitrary	Aligned	Height†	$O(\hat{n}W^4)$
Yelland [2016]	Arbitrary	Aligned	Linear cost	$O(\hat{n}^{3/2})$
Bernardy [2017c]	Arbitrary	Aligned	Height†	$O(nW^6)$
Π_e	Both	Both	Cost (from the cost factory)	$O(nW^4)$
Π_e (aligned only)	Both	Aligned	Cost (from the cost factory)	$O(nW^3)$

† only consider layouts without an overflow over W.

- A proof of correctness for Π_e, formalized in the Lean theorem prover. To our knowledge, this is the first time that a printer has been formally verified.
- A framework to formally reason about the expressiveness of PPLs.
- A survey of printers and an analysis that dispels common misunderstandings about them.
- An implementation of Π_e, SnowWhite, and an evaluation that shows its effectiveness.

The rest of this paper is structured as follows. Section 2 surveys the related work. Section 3 presents the semantics of Σ_e. Section 4 introduces a framework to reason about the expressiveness of PPLs. Section 5 presents Π_e and its analysis. Section 6 discusses SnowWhite, an implementation of Π_e. Section 7 presents an evaluation of SnowWhite that demonstrates its effectiveness. Lastly, Section 8 concludes the paper.

2 RELATED WORK

To understand the trade-off space of printer designs, we conduct a comprehensive analysis of related work in the literature. This section provides our analysis of the printers, grouped by the expressiveness of their public interface.[4] The summary is presented in Table 1. We then compare and contrast our printer Π_e against them.

2.1 Traditional printers

Pretty printing has a long history. Oppen [1980] first introduced a general-purpose printer, written in the imperative style. Oppen pioneered the PPL that we call the traditional PPL, shown in Figure 2a. Instead of representing an input document as a tree, as commonly done in subsequent work, Oppen represents the document as a stream of "instruction tokens." The algorithm's time complexity is $O(n)$, where n is the length of the stream. Furthermore, the algorithm is *bounded*, requiring a limited look-ahead into the stream. As with other printers in the family, the printer greedily minimizes overflow over the page width limit, which neither avoids overflow whenever possible nor minimizes number of lines, as discussed in the paper.

Wadler [2003] designed a printer that targets the traditional PPL. It is used in many real world applications, such as an industrial code formatter [Prettier 2016], and as a basis for much pretty printing research [Chitil 2005; Kiselyov et al. 2012]. The printer aims to be a rewrite of Oppen's

[4]In practice, printers include extensions that increase their expressiveness. A printer may even have different expressiveness across different versions. This section focuses on the core features of these printers.

$d \in \mathcal{D}$ ::=	**text** s	text
|	**nl**	newline
|	d **<>** d	unaligned concatenation
|	**nest** n d	increase indentation level
|	**group** d	create two choices where one flattens layouts

(a) A variant of traditional PPL from Wadler [2003].

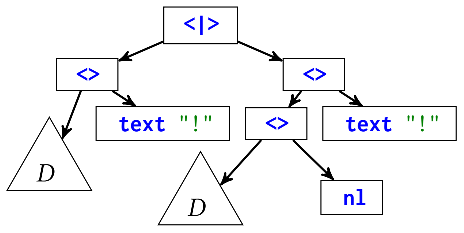
$d \in \mathcal{D}$ ::=	**text** s	text
|	d_a **<+>** d_b	aligned concatenation
|	d_a **<$>** d_b	vertical concatenation
|	d_a **<|>** d_b	create two arbitrary choices

(b) A variant of arbitrary-choice PPL from Podkopaev and Boulytchev [2015].

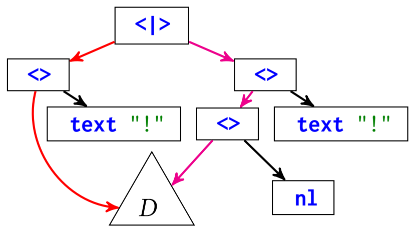
Fig. 2. A comparison between the traditional and arbitrary-choice PPLs. s denotes a string without newline, and n denotes a natural number.

```
let shared := D in (shared <> text "!") <|> ((shared <> nl) <> text "!")
```

(a) A document that encodes (at least) two possible layouts. D is an arbitrary sub-document.



(b) A tree representation of Figure 3a. D contributes to the size twice.



(c) A DAG representation of Figure 3a. D contributes to the size only once.

Fig. 3. An example document that shows the importance of treating document as a DAG rather than a tree. The red and pink paths illustrate that the DAG is properly shared, as will be discussed in Section 5.5.

printer using the functional style employed by Hughes (described later). The printer is claimed [Chitil 2005; Wadler 2003] to produce an output layout that does not exceed the width limit whenever possible, and minimizes the number of lines. However, this is not the case, as shown in Figure 14 (in Appendix A). The time complexity of the printer is claimed to be $O(n)$ where n is the size of document [Wadler 2003], but it is in fact $O(n^2)$ in the worst case, as demonstrated in Figure 15, although this worst case behavior is unlikely to occur in practice.

Chitil [2005] improved Wadler's printer so that it is as efficient as Oppen's, $O(n)$, by using lazy dequeues. Kiselyov et al. [2012] similarly improved Wadler's printer via their generator framework.

Compared to traditional printers, Π_e is more expressive as it allows arbitrary choices and aligned concatenation. Furthermore, Π_e can produce an output layout that minimizes number of lines when the output layout does not exceed the page width limit, and does not exceed the page width limit whenever possible. The tradeoff is that Π_e is less space efficient and slower than traditional printers. The space complexity of traditional printers is sub-linear in the size of document, which was especially important decades ago when memory is scarce. The space complexity of Π_e is $O(nW^3)$ in the worst case (or $O(nW^2)$ when targeting some PPLs). We find that on modern machines, the added memory consumption and performance overhead are rarely an issue in practice (Section 7).

2.2 Arbitrary-choice printers

Azero Alcocer and Swierstra [1998] introduced a printer that supports arbitrary choices with aligned concatenation, starting the line of work that targets the arbitrary-choice PPL, shown in Figure 2b. The printer's optimality objective is to avoid overflow whenever possible and produce a minimal number of lines. However, it does not have the ability to cope with unavoidable overflow. This printer was soon superseded by Swierstra et al. [1999], which improves its performance via heuristics and adds the capability to *share* a sub-document across choices by deeply embedding the (equivalent of the) `let` construct in the PPL. As a result, the later printer can process documents that are structured as DAGs rather than trees, as shown in Figure 3. Nonetheless, the time complexity of both printers is exponential in n [Podkopaev and Boulytchev 2015].

Podkopaev and Boulytchev [2015] improved upon Swierstra et al.'s work by formulating the problem as dynamic programming. The formulation fixes the exponential blowup in the prior work, but treats the document as a tree, making its time complexity $O(\hat{n}W^4)$, where $\hat{n}$ is the tree size of the document, which could be exponentially larger than its DAG size. The paper acknowledges the problem and surmises that memoization may be able to address it.

The paper by Bernardy [2017c] is the main inspiration for our work. The printer uses Pareto frontiers to find an optimal layout. By shallowly embedding the PPL (in Haskell), computations on sub-documents are effectively shared for free. However, as presented in the paper, the printer requires the page width limit to be hard-coded. In the actual implementation [Bernardy 2017b], the page width limit is customizable, accomplished by threading the value through functions. But this change destroys the shared computations, leading to exponential running time. Compared to Podkopaev and Boulytchev [2015]'s work, Bernardy [2017c]'s approach can exploit sparseness to improve practical efficiency, but the use of an inefficient algorithm makes the time complexity of the printer $O(nW^6)$ in the worst case. While the paper does not handle unavoidable overflow, the implementation does by automatically scaling up the page width limit (or equivalently, minimizing the maximum overflow). This, however, allows avoidable overflow elsewhere, as shown in Figure 16 (in Appendix A), which is undesirable. Later on, Bernardy abandoned the arbitrary-choice operator, noting that it could trigger the exponential behavior [Bernardy 2017a].

Yelland [2016] similarly targeted the arbitrary-choice PPL. However, the paper took a very different approach. The core printer restricts the use of aligned concatenation by requiring the left sub-document to be a **`text`** syntactically. This restriction allows the core printer to utilize the concept of "piecewise linear cost function" to seemingly boost the performance. To achieve the expressiveness of the arbitrary-choice PPL, the printer employs rewriting rules to transform the original document into the restricted document. While the printer is careful to avoid exponential blowup by sharing documents in the resulting restricted document, it does not necessarily preserve the sharing structure of the original document, as demonstrated in Figure 17 (in Appendix A). Compound this with the lack of computation width limit, the number of piecewise linear cost functions under consideration could be as large as $O(\hat{n}^{1/2})$, making the time complexity $O(\hat{n}^{3/2})$ in total, as shown in Figure 18. Another aspect to consider is the printer's optimality objective, which is restricted to minimizing a linear combination of quantities like the number of lines and overflow. Hence, the printer will not technically avoid overflow whenever possible (although the overflow coefficient can be made arbitrarily large to arbitrarily discourage overflow). On the other hand, this optimality objective can support unique features, such as incorporating the costs due to multiple soft page width limits.

Compared to arbitrary-choice printers, Π_e is more expressive as it allows unaligned concatenation. Π_e is also asymptotically faster than most arbitrary-choice printers, as it treats a document as a DAG rather than a tree. Like Yelland's printer, for each layout under consideration, Π_e keeps track of two quantities: cost and last line length. This is different from most printers in the family which keep track of three quantities: height, width, and last width. The dimension reduction further makes Π_e more efficient. The concept of cost also allows Π_e to decouple the page width limit and computation width limit, which allows graceful overflow handling. Π_e, unlike Yelland's printer, is parameterized by a cost factory, which supports a variety of optimality objectives without requiring a modification to the core printer. This includes not only the linear optimality objectives that Yelland's printer supports, but also non-linear optimality objectives.

Document	$d \in \mathcal{D}_e$ ::= **`text`** s | **`nl`** | d **`<>`** d | **`nest`** n d | **`align`** d | **`flatten`** d | d **`<|>`** d	String without newline	$s, t, \ldots \in$ Str
		Natural number	$n \in \mathbb{N}$

Fig. 4. Syntax for Σ_e

2.3 Other printers

Hughes [1995] brought a general-purpose printer to the functional world. The printer pioneers using combinators to construct a document, which is now a standard practice. The printer targets a PPL that is neither the traditional nor arbitrary-choice PPL, though the printer is more similar to the traditional printers in how it makes choices greedily, which minimizes neither overflow nor number of lines. The combination of greedy choice making and aligned concatenation makes some documents print very poorly [Bernardy 2017c]. Furthermore, Peyton-Jones [1997] identified quadratic time complexity in the printer.

Leijen [2000] implemented Wadler's printer in Haskell and added support for aligned concatenation via the inclusion of **`align`**, becoming the first printer that supports both aligned and unaligned concatenation. However, similar to Hughes' printer, the printer can produce very poor output [Bernardy 2017c].

3 THE SYNTAX AND SEMANTICS OF Σ_e

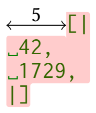
This section presents Σ_e, an expressive PPL. We begin this section by describing *layouts*, which are the textual outputs. Then, we describe the syntax of Σ_e and its informal semantics, which is determined by the evaluation of a document in Σ_e to layouts. Lastly, we formally describe the semantics of Σ_e.

3.1 Layout

A *layout* $l \in \mathcal{L}$ is a textual output. In our work, we represent a layout as a non-empty, finite list of lines (implicitly joined by newlines), where each line is a string without the newline character. This allows us to easily reason about the number of lines and the length of each line.

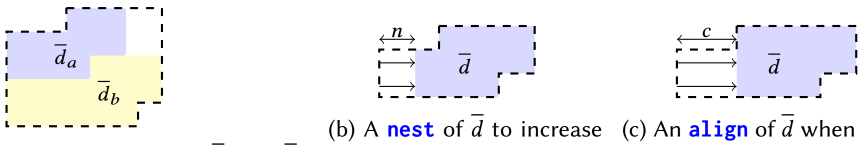
A layout can be *placed* at a column position c. This placing puts the first line of the layout at the column position c, and puts the rest of the lines at the column position 0. While the final layout to present to users is placed at the column position 0, a (sub)layout may be placed at a non-zeroth column position during the rendering process.

Example 3.1. The following illustration displays a layout [`"[|"`, `"␣42,"`, `"␣1729,"`, `"|]"`] placed at the column position 5. The pink area shows the general shape of layout "boundaries", where the first line could be "indented" due to the placing on a non-zeroth column position, and the last line covers its length exactly. The shape will be useful to understand how layouts are composed together.



3.2 The syntax and the informal semantics of Σ_e

Figure 4 shows the syntax of document in Σ_e. Each construct is from either the traditional or the arbitrary-choice PPLs, except the flatten construct **`flatten`** (which is internally used in Wadler [2003]'s printer) and the align construct **`align`** (which is from Leijen [2000]'s printer). Similar to layouts, documents can be placed at a column position. As traditionally done, we for now ignore the choice operator. A document without the arbitrary-choice operator is called a *choiceless document*, which can be *rendered* to a single layout. We denote a choiceless document with $\overline{d} \in \overline{\mathcal{D}_e}$. The informal semantics of choiceless document are as follows:



(a) An unaligned concatenation of $\overline{d}_a$ and $\overline{d}_b$ (b) A **nest** of $\overline{d}$ to increase the indentation level by n (c) An **align** of $\overline{d}$ when placed at the column position c

Fig. 5. Illustrations of constructs in Σ_e. The area with dashed borders is the resulting boundaries.

text s	renders to a layout with a single line s.
nl	normally renders to a layout with two lines. The first line is empty, and the second line consists of i spaces where i is the *indentation level*. **nl** interacts with *flattening*, which reduces it to just a single space.
$\overline{d}_a$ **<>** $\overline{d}_b$	renders to a layout that concatenates the layout of $\overline{d}_a$ and the layout of $\overline{d}_b$ without alignment. The rendering of $\overline{d}_b$ is dependent on the rendering of $\overline{d}_a$, because $\overline{d}_b$ will be placed at a column position after the last character of $\overline{d}_a$. Figure 5a illustrates this.
nest n $\overline{d}$	renders to a layout like $\overline{d}$, but with the indentation level *increased* by n relative to the current indentation level. Figure 5b roughly illustrates this.
align $\overline{d}$	renders to a layout like $\overline{d}$, but with the indentation level *set* (not relatively increased) to the column position that **align** $\overline{d}$ is being placed in. Figure 5c roughly illustrates this.
flatten $\overline{d}$	renders to a layout just like $\overline{d}$, but with all newlines and indentation spaces due to **nl**s flattened to single spaces.

While Figure 5 provides a rough illustration that should be helpful to understand the semantics of choiceless document, it could be misleading.

Example 3.2. The document `text "a" <> (nest 42 (align (text "b" <> nl <> text "c")))` is rendered to `["ab", "␣c"]`. The nesting doesn't visibly increase the indentation level by 42, because the alignment on the inner document overrides the indentation level. This example shows the importance of the indentation level, and why it must be specifically tracked.

This concludes our informal description of how a choiceless document renders to a layout. General documents, by contrast, can contain the arbitrary-choice operator **<|>**, which encodes layouts from two sub-documents into one document. Thus, unlike choiceless documents, which render to a single layout, general documents will *evaluate* to a non-empty, finite set of layouts. Our approach is to first *widen* a document into a set of choiceless documents, then render each choiceless document in the set to produce a set of layouts.

3.3 The formal semantics of Σ_e

The formal semantics of Σ_e, consisting of two relations, is given in Figure 6. The judgment $\langle \overline{d}, c, i, f \rangle \Downarrow_{\mathcal{R}} l$ states that the choiceless document $\overline{d} \in \overline{\mathcal{D}_e}$ placed at column position $c \in \mathbb{N}$ with indentation level $i \in \mathbb{N}$ and flattening mode $f \in \mathbb{B}$, will render to the layout $l \in \mathcal{L}$. The flattening mode f, which indicates whether newlines should be replaced with spaces, can be either on ($\top$) or off ($\bot$). Another judgment $d \Downarrow_{\mathcal{W}} \overline{D}$ states that a document $d \in \mathcal{D}_e$ is widened to a finite, non-empty set of choiceless documents $\overline{D} \in 2^{\overline{\mathcal{D}_e}}$. We sometimes call a combination of c and i (and possibly f) a *printing context*. Now, we elaborate some interesting rules in the figure.

$$\textsc{Text}\ \frac{}{\langle \texttt{text}\ s, c, i, f\rangle \Downarrow_{\mathcal{R}} [s]} \qquad \textsc{Flatten}\ \frac{\langle \overline{d}, c, i, \top\rangle \Downarrow_{\mathcal{R}} [s]}{\langle \texttt{flatten}\ \overline{d}, c, i, f\rangle \Downarrow_{\mathcal{R}} [s]}$$

$$\textsc{LineNoFlatten}\ \frac{}{\langle \texttt{nl}, c, i, \bot\rangle \Downarrow_{\mathcal{R}} [\epsilon, \texttt{"␣"} \times i]} \qquad \textsc{LineFlatten}\ \frac{}{\langle \texttt{nl}, c, i, \top\rangle \Downarrow_{\mathcal{R}} [\texttt{"␣"}]}$$

$$\textsc{ConcatOne}\ \frac{\begin{array}{c}\langle \overline{d}_a, c, i, f\rangle \Downarrow_{\mathcal{R}} [s] \\ \langle \overline{d}_b, c + |s|, i, f\rangle \Downarrow_{\mathcal{R}} [t, t_1, \ldots, t_n]\end{array}}{\langle \overline{d}_a \texttt{ <> } \overline{d}_b, c, i, f\rangle \Downarrow_{\mathcal{R}} [s +\!\!+ t, t_1, \ldots, t_n]} \qquad \textsc{ConcatMult}\ \frac{\begin{array}{c}\langle \overline{d}_a, c, i, f\rangle \Downarrow_{\mathcal{R}} [s_1, \overset{+}{\ldots}, s_n, s] \\ \langle \overline{d}_b, |s|, i, f\rangle \Downarrow_{\mathcal{R}} [t, t_1, \ldots, t_m]\end{array}}{\langle \overline{d}_a \texttt{ <> } \overline{d}_b, c, i, f\rangle \Downarrow_{\mathcal{R}} [s_1, \overset{+}{\ldots}, s_n, s +\!\!+ t, t_1, \ldots, t_m]}$$

$$\textsc{Nest}\ \frac{\langle \overline{d}, c, i + n, f\rangle \Downarrow_{\mathcal{R}} [s_1, \overset{+}{\ldots}, s_m]}{\langle \texttt{nest}\ n\ \overline{d}, c, i, f\rangle \Downarrow_{\mathcal{R}} [s_1, \overset{+}{\ldots}, s_m]} \qquad \textsc{Align}\ \frac{\langle \overline{d}, c, c, f\rangle \Downarrow_{\mathcal{R}} [s_1, \overset{+}{\ldots}, s_n]}{\langle \texttt{align}\ \overline{d}, c, i, f\rangle \Downarrow_{\mathcal{R}} [s_1, \overset{+}{\ldots}, s_n]}$$

$$\textsc{TextWiden}\ \frac{}{\texttt{text}\ s \Downarrow_{\mathcal{W}} \{\texttt{text}\ s\}} \qquad \textsc{LineWiden}\ \frac{}{\texttt{nl} \Downarrow_{\mathcal{W}} \{\texttt{nl}\}}$$

$$\textsc{ConcatWiden}\ \frac{d_a \Downarrow_{\mathcal{W}} \overline{D}_a \quad d_b \Downarrow_{\mathcal{W}} \overline{D}_b}{d_a \texttt{ <> } d_b \Downarrow_{\mathcal{W}} \{\overline{d}_a \texttt{ <> } \overline{d}_b \mid \overline{d}_a \in \overline{D}_a, \overline{d}_b \in \overline{D}_b\}}$$

$$\textsc{NestWiden}\ \frac{d \Downarrow_{\mathcal{W}} \overline{D}}{\texttt{nest}\ n\ d \Downarrow_{\mathcal{W}} \{\texttt{nest}\ n\ \overline{d} \mid \overline{d} \in \overline{D}\}} \qquad \textsc{AlignWiden}\ \frac{d \Downarrow_{\mathcal{W}} \overline{D}}{\texttt{align}\ d \Downarrow_{\mathcal{W}} \{\texttt{align}\ \overline{d} \mid \overline{d} \in \overline{D}\}}$$

$$\textsc{FlattenWiden}\ \frac{d \Downarrow_{\mathcal{W}} \overline{D}}{\texttt{flatten}\ d \Downarrow_{\mathcal{W}} \{\texttt{flatten}\ \overline{d} \mid \overline{d} \in \overline{D}\}} \qquad \textsc{UnionWiden}\ \frac{d_a \Downarrow_{\mathcal{W}} \overline{D}_\alpha \quad d_b \Downarrow_{\mathcal{W}} \overline{D}_\beta}{d_a \texttt{ <|> } d_b \Downarrow_{\mathcal{W}} \overline{D}_\alpha \cup \overline{D}_\beta}$$

Fig. 6. Semantics for Σ_e. "ϵ" is the empty string. "$s \times i$" is the notation for replicating the string s for i times. "$s +\!\!+ t$" is a string concatenation of s and t. Lastly, "$s_1, \ldots, s_n$" and "$s_1, \overset{+}{\ldots}, s_n$" indicate n lines, where $n \geq 0$ and $n \geq 1$ respectively.

Rendering text. The Text rule states that the rendering of a text placement `text` s contains a layout with a single line of the text s. The printing context is completely ignored: indentation level and flattening mode do not affect the rendering of text, and we have already assumed that s will be placed at the column position.

Rendering newline. When the flattening mode is off, the LineNoFlatten rule states that the rendering of `nl` results in a layout with two lines. The first line is empty, while the second line is indented by i spaces. On the other hand, when the flattening mode is on, the LineFlatten rule states that the rendering of the newline results in a layout with a single line of a single space.

Rendering unaligned concatenation. In the rendering of $\overline{d}_a$ `<>` $\overline{d}_b$, we recursively render $\overline{d}_a$ and $\overline{d}_b$, but the rendering of $\overline{d}_b$ is dependent on the rendering of $\overline{d}_a$. Let l_a be the rendering result of $\overline{d}_a$. The ConcatOne rule handles the case where l_a has a single line, and the ConcatMult rule handles the case where l_a has multiple lines.

- If l_a has only a single line s, the column position of $\overline{d}_b$'s rendering needs to be after the string s is placed, i.e. at $c + |s|$. In such case, let l_b be the rendering result of $\overline{d}_b$. The first line of the resulting layout is the concatenation of s and the first line of l_b. The rest of the lines are from the rest of l_b.
- On the other hand, if l_a has multiple lines, the column position of d_b's rendering is simply the column position after the last line is placed. In such case, let l_b be the rendering result of $\overline{d}_b$, the resulting layout contains all but the last line of l_a, a concatenation of the last line of l_a and the first line of l_b, and the rest of l_b.

Widening choice. The UnionWiden rule states that the widening of d_a `<|>` d_b is the union of widen d_a and widen d_b

Both $\Downarrow_{\mathcal{R}}$ and $\Downarrow_{\mathcal{W}}$ are deterministic and total. Thus, we can define $\text{eval}_e(d) = \{l : \langle \overline{d}, 0, 0, \bot\rangle \Downarrow_{\mathcal{R}} l, \overline{d} \in \overline{D}, d \Downarrow_{\mathcal{W}} \overline{D}\}$ as the evaluation function for Σ_e, which consumes a document, widens it, and produces a set of layouts.

4 A FRAMEWORK TO REASON ABOUT EXPRESSIVENESS

In previous sections, we informally made claims about expressiveness of PPLs. This section presents a framework to formally reason about it, based on two notions: *functional completeness* and *definability*. We first define the semantics of the traditional and arbitrary-choice PPLs. Then, we define our framework, and show that Σ_e is strictly more expressive than both the traditional and arbitrary-choice PPLs. Proof sketches of theorems in this section are provided in Appendix B.

4.1 The extended semantics

To reason about the traditional and arbitrary-choice PPLs, we need to precisely define their semantics. To do so, we construct a PPL Σ_{all} that contains all constructs from Σ_e, traditional, and arbitrary-choice PPLs by extending Figure 6 with the rules below (with the straightforward widening rules). Note that we follow Wadler [2003]'s approach by treating **`group`** d as a syntactic sugar for d `<|>` `flatten` d. As `<|>` and `flatten` are already in Σ_{all}, we do not need to adjust anything further.

$$\textsc{VertConcatNoFlatten}\ \frac{\langle \overline{d}_a, c, i, \bot\rangle \Downarrow_{\mathcal{R}} [s_1, \ldots^+, s_n] \quad \langle \overline{d}_b, i, i, \bot\rangle \Downarrow_{\mathcal{R}} [t_1, \ldots^+, t_m]}{\langle \overline{d}_a \texttt{ <\$> } \overline{d}_b, c, i, \bot\rangle \Downarrow_{\mathcal{R}} [s_1, \ldots^+, s_n, t_1, \ldots^+, t_m]}$$

$$\textsc{VertConcatFlatten}\ \frac{\langle \overline{d}_a, c, i, \top\rangle \Downarrow_{\mathcal{R}} [s] \quad \langle \overline{d}_b, c + 1 + |s|, i, \top\rangle \Downarrow_{\mathcal{R}} [t]}{\langle \overline{d}_a \texttt{ <\$> } \overline{d}_b, c, i, \top\rangle \Downarrow_{\mathcal{R}} [s + \texttt{"␣"} + t]}$$

$$\textsc{AlignedConcatOne}\ \frac{\langle \overline{d}_a, c, i, f\rangle \Downarrow_{\mathcal{R}} [s] \quad \langle \overline{d}_b, c + |s|, c + |s|, f\rangle \Downarrow_{\mathcal{R}} [t, t_1, \ldots, t_n]}{\langle \overline{d}_a \texttt{ <+> } \overline{d}_b, c, i, f\rangle \Downarrow_{\mathcal{R}} [s + t, t_1, \ldots, t_n]}$$

$$\textsc{AlignedConcatMult}\ \frac{\langle \overline{d}_a, c, i, f\rangle \Downarrow_{\mathcal{R}} [s_1, \ldots^+, s_n, s] \quad \langle \overline{d}_b, |s|, |s|, f\rangle \Downarrow_{\mathcal{R}} [t, t_1, \ldots, t_m]}{\langle \overline{d}_a \texttt{ <+> } \overline{d}_b, c, i, f\rangle \Downarrow_{\mathcal{R}} [s_1, \ldots^+, s_n, s + t, t_1, \ldots, t_m]}$$

The semantics of the traditional and arbitrary-choice PPLs are then the restricted semantics of Σ_{all} that only allows constructs from the traditional and the arbitrary-choice PPLs, respectively. Throughout this section, we will assume that any PPL is similarly a sublanguage of Σ_{all}, whose semantics is well-defined and consistent with Σ_{all}.

The extended semantics are still deterministic and total. However, it is worth noting that there are many different ways to specify rules in a way that is consistent with the intended semantics of the arbitrary-choice PPL. For instance, an invariant in the arbitrary-choice PPL is that $c = i$ throughout the rendering process. As a result, we could substitute the VertConcatNoFlatten rule with VertConcatNoFlatten*, which is like VertConcatNoFlatten but with a modification to use $\langle \overline{d}_b, c, c, \bot\rangle \Downarrow_{\mathcal{R}} [t_1, \ldots^+, t_m]$ as a premise instead of $\langle \overline{d}_b, i, i, \bot\rangle \Downarrow_{\mathcal{R}} [t_1, \ldots^+, t_m]$, without changing the semantics of the arbitrary-choice PPL. This change could affect the semantics of Σ_{all} and subsequent theorems in this section. We pick VertConcatNoFlatten over VertConcatNoFlatten* because it seemingly integrates better with other features in Σ_{all}.

4.2 Functional completeness

In Section 1, we claimed that the traditional PPL cannot express the two layouts in Figure 1b, as one layout has an extra pair of parentheses. The question that we may want to ask in general then is, given a PPL Σ and a non-empty set of layouts L, is it possible to construct a document in Σ that evaluates to L? This motivates us to define the notion of functional completeness for PPLs.

Definition 4.1. A PPL Σ with an evaluation function $\text{eval}(\cdot)$ is *functionally complete* if for any non-empty set of layouts L, there exists a document d in Σ such that $\text{eval}(d) = L$.

With this definition, we can formally reason about some PPLs that we have previously seen.

Lemma 4.2. *The arbitrary-choice and Σ_e PPLs are functionally complete.*

Lemma 4.3. *The traditional PPL is* not *functionally complete.*

Lemma 4.4. *For each construct* **F** *in* {**text**,**<>**,**nl**,**<|>**}, Σ_e *without* **F** *is not functionally complete.*

If we limit the notion of expressiveness to only functional completeness, then all functionally complete PPLs would be equally expressive. However, intuitively this is clearly not the case. The proof of Lemma 4.2 (in Appendix B) shows that it suffices for a PPL to only have **text**, **<$>**, and **<|>** for functional completeness, yet such a PPL would not be pleasant to use compared to Σ_e, because of the lack of features to, e.g., adjust indentation level. In a sense, functional completeness for PPLs is similar to Turing completeness for programming languages, which similarly does not fully capture expressiveness for programming languages. The next subsection presents a more fine-grained notion of expressiveness, based on the ability to define features.

4.3 Definability

The proof of Lemma 4.2 shows that while Σ_e doesn't have **<$>**, we can simply expand d_a **<$>** d_b to d_a **<> nl <>** d_b, which are in Σ_e, to perform the same functionality. In other words, the construct **<$>** is already *definable* by **<>** and **nl**. Thus, adding **<$>** to Σ_e doesn't increase its expressiveness. In contrast, **<>** is not definable by any combination of features in the arbitrary-choice PPL. To achieve the same functionality of **<>**, it would require a non-local restructuring of the document, making it difficult to construct the document in natural way. In this sense, the inability to define a construct in a PPL means that adding the construct to the PPL increases its expressiveness.

More concretely, consider the arbitrary-choice document in Figure 1b. The document is awkwardly constructed. The `return` keyword must be distributed to combine with a first line of the returned expression, due to the undefinability of unaligned concatenation. This creates a disconnection between the document structure and the underlying AST structure, making it more tedious and error-prone to construct documents. In contrast, the following document is a rewrite of Figure 1b to utilize the full expressiveness of Σ_e in a natural way. The sub-document colored blue fully corresponds to the "returned expression," allowing users to recursively construct documents naturally.

```
text "function␣append(first,second,third){" <> nest 4 (
  let f = text "first␣+" in let s = text "second␣+" in let t = text "third" in
  nl <> text "return␣" <>
  ((text "(" <> (nest 4 (nl <> (f <> nl <> s <> nl <> t))) <> nl <> text ")") <|>
    let sp = text "␣" in (f <> sp <> s <> sp <> t))
) <> nl <> text "}"
```

The notion of definability (also known as expressibility) for programming languages was first developed by Felleisen [1991], and we adapt it for PPLs through a series of definitions as follows:

Definition 4.5. A PPL Σ consists of:

- a set of constructs $\Sigma = \{\mathbf{F}_1, \mathbf{F}_2, \ldots\}$. There could potentially be infinite constructs. Each construct may have different arity and sort, whose arguments may also have different sort.
- a non-empty set of documents $\mathcal{D}$, which are of sort Doc, generated from Σ.
- an evaluation function $\mathrm{eval} : \mathcal{D} \to 2^{\mathcal{L}}$.

Example 4.6. Σ_e contains `nest` $\square_{\mathbb{N}}$ $\square_{\mathrm{Doc}}$, which is a construct with arity 2 of sort Doc. The first argument to `nest` has sort $\mathbb{N}$ and the second argument has sort Doc. Σ_e also contains all natural numbers and strings with no newline, which are constructs with arity 0 of sort $\mathbb{N}$ and Str respectively. The evaluation function for Σ_e is eval_e from Section 3.3.

Henceforth, unless indicated otherwise, $\mathcal{D}_X$ and eval_X are the set of documents and the evaluation function for the PPL Σ_X.

Definition 4.7. A *syntactic abstraction* $\mathbf{M}(\alpha_1, \ldots, \alpha_n)$ of arity n for a PPL Σ is a document in $\Sigma \cup \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1, \ldots, \alpha_n$ are metavariables (nullary constructors) of some sorts. An *instance* $\mathbf{M}(e_1, \ldots, e_n)$ is a document in Σ that substitutes α_i with e_i in $\mathbf{M}(\alpha_1, \ldots, \alpha_n)$ for all $1 \le i \le n$, where e_i and α_i must have a compatible sort.

Example 4.8. $\mathbf{M}(\alpha_1, \alpha_2) = \alpha_1$ `<> nl <>` α_2 is a syntactic abstraction for Σ_e, where α_1 and α_2 have sort Doc. On the other hand, $\mathbf{M}'(\alpha_1) =$ `nest` α_1 `nl <>` α_1 is **not** a syntactic abstraction because the first occurrence of α_1 requires it to have sort $\mathbb{N}$, but the second occurrence requires it to have sort Doc. An instance $\mathbf{M}$(`text "a"`, `text "b"`) is the document `text "a" <> nl <> text "b"`, but $\mathbf{M}$(`text "a"`, 1) is not an instance due to the incompatible sort.

Definition 4.9. Let Σ_{base} be a PPL and $\Sigma_{\mathrm{extended}} = \Sigma_{\mathrm{base}} \cup \{\mathbf{F}\}$ where $\mathbf{F}$ has sort Doc. A *syntactic expansion* $\mathrm{expand}^{\mathbf{M}}_{\mathbf{F}}(d)$ from $\Sigma_{\mathrm{extended}}$ to Σ_{base} is a function from $\mathcal{D}_{\mathrm{extended}}$ to $\mathcal{D}_{\mathrm{base}}$ that replaces every occurrence of $\mathbf{F}(e_1, \ldots, e_n)$ with an instance $\mathbf{M}(e_1, \ldots, e_n)$ in d, where $\mathbf{F}$ and $\mathbf{M}$ must have compatible arity and sort arguments.

Example 4.10. $\mathrm{expand}^{\mathbf{M}}_{\texttt{<\$>}}(\cdot)$ is a syntactic expansion from $\Sigma_e \cup \{$`<$>`$\}$ to Σ_e, where $\mathbf{M}$ is from Example 4.8. Hence, $\mathrm{expand}^{\mathbf{M}}_{\texttt{<\$>}}($`text "a" <$> text "b"`$) =$ `text "a" <> nl <> text "b"`.

We are now ready to define definability.[5]

Definition 4.11. Let Σ_{base} be a PPL and $\Sigma_{\mathrm{extended}} = \Sigma_{\mathrm{base}} \cup \{\mathbf{F}\}$. We say that Σ_{base} can *define* $\mathbf{F}$ (alternatively, $\mathbf{F}$ is *definable* by Σ_{base}) if there exists a syntactic abstraction $\mathbf{M}$ from $\Sigma_{\mathrm{extended}}$ to Σ_{base} such that for every document $d \in \mathcal{D}_{\mathrm{extended}}$, $\mathrm{eval}_{\mathrm{extended}}(d) = \mathrm{eval}_{\mathrm{base}}(\mathrm{expand}^{\mathbf{M}}_{\mathbf{F}}(d))$.

We can now present one of our main results:

THEOREM 4.12. *Σ_e can define every construct in the traditional and arbitrary-choice PPLs.*

Despite the result, we might wonder if Σ_e is actually needed. Could it be that the arbitrary-choice PPL can already define every construct in the traditional PPL? As we forshadowed, the answer to this question is negative. However, we must first develop tools that allow us to answer the question, again following the development in Felleisen's work.

Definition 4.13. A context $C(\alpha)$ for Σ is a unary syntactic abstraction for Σ where α has sort Doc.

Definition 4.14. Given a PPL Σ and a relation $R \subseteq 2^{\mathcal{L}} \times 2^{\mathcal{L}}$, $E^{\Sigma}_R(d_1, d_2)$ is a relation that holds if and only if for all contexts C for Σ, $R(\mathrm{eval}(C(d_1)), \mathrm{eval}(C(d_2)))$ holds[6].

[5]One important distinction of this definition and Felleisen's counterpart is that PPLs are *total*. Hence, observing the termination behavior, as done in Felleisen's work, is not feasible in our formulation.

[6]The relation E^{Σ}_R is a generalization of the operational equivalence relation in Felleisen's work.

Example 4.15. Let width : $\mathcal{L} \longrightarrow \mathbb{N}$ be a function that computes the maximum length across all lines in the input layout, and we lift width to work on any set of layouts. Furthermore, let $R = \{(L_1, L_2) : \mathsf{width}(L_1) = \mathsf{width}(L_2)\}$.

- $E_R^{\Sigma_e}$(`text "a"`, `text "b"`) holds by induction. Intuitively, this is because (1) if we only observe the width, the textual content doesn't matter, and (2) there is no construct in Σ_e that allows us to layout differently in a way that would affect the width based on the textual content.
- On the other hand, $\neg E_R^{\Sigma_e}$(`text "a"`, `text "aa"`). For example, with $C(\alpha) = \alpha$, we have that $\mathsf{width}(\mathsf{eval}_e(C($`text "a"`$))) = \{1\}$, but $\mathsf{width}(\mathsf{eval}_e(C($`text "aa"`$))) = \{2\}$.

The following theorem provides a tool to prove that a construct is not definable in a PPL.

Theorem 4.16. *Given a PPL Σ and a construct* **F**, *if there exists two documents d_1 and d_2 in Σ and a relation R such that $E_R^{\Sigma}(d_1, d_2)$, but $\neg E_R^{\Sigma \cup \{\mathbf{F}\}}(d_1, d_2)$, then* **F** *is not definable in Σ.*

With this tool, we are now able to prove the following:

Theorem 4.17. *The following is true:*

- `<>` *is not definable in the arbitrary-choice PPL.*
- `nest` *is not definable in the arbitrary-choice PPL.*
- `group` *is not definable in the arbitrary-choice PPL.*
- `<+>` *is not definable in the traditional PPL.*

Next, we show a relationship between functional completeness and definability.

Lemma 4.18. *If Σ is not functionally complete, but $\Sigma \cup \{\mathbf{C}\}$ is, then* **C** *is not definable in Σ.*

We now present our last result for this section: Σ_e is *minimal* in a sense that each of its constructs is not definable by Σ_e without it. This is what the design of Σ_e strives to achieve, so that the size of primitive constructs is small.

Theorem 4.19. *For any construct* **F** *of Σ_e,* **F** *is not definable in $\Sigma_e \setminus \{\mathbf{F}\}$.*

5 OUR PRINTER, Π_e

In this section, we describe our printer, Π_e, which targets the PPL Σ_e presented in Section 3. Π_e is parameterized by a *cost factory*, allowing users to customize the optimality objective within the resource budget that they find acceptable. We start with an overview of Π_e. Then, we describe the cost factory interface. Next, we define *measure*, which is an output from the core printer that allows us to record a cost and at the same time avoid a full-blown, expensive rendering. After that, we describe the requirements of the input document structure, which will become important when we analyze the complexity of the printer. Then, we present the Π_e's printing algorithm, which utilizes the cost factory to achieve an optimal and efficient printing. Finally, we show our analysis of Π_e.

5.1 Overview

So far we have defined the *evaluation* of a document, which produces the set of possible layouts. But when we *print* a document, we wish to output only a single, most optimal layout.

A naïve approach is to evaluate the input document, via widening and rendering, to all possible layouts, determine costs of these layouts according to a given optimality objective, and then pick one with the lowest cost. However, this approach is not practical for two reasons. First, widening could produce exponentially many choiceless documents. Second, rendering non-optimal choiceless documents is unnecessary and wasteful.

A better approach would avoid rendering until an optimal choiceless document is identified, and utilize early pruning to reduce the search space. Since we wish to avoid full-blown rendering, we will

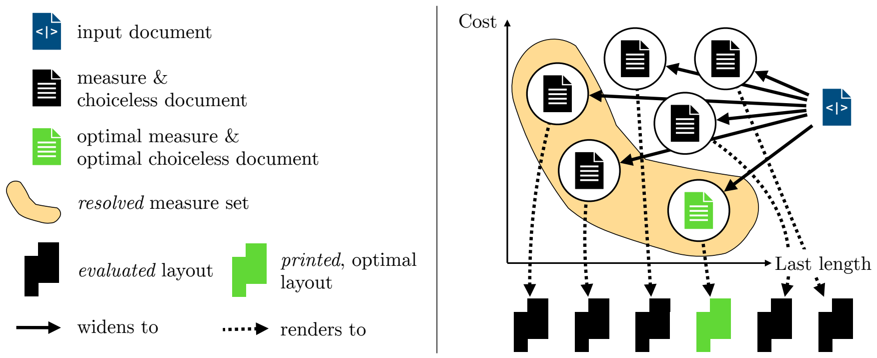

input document
measure & choiceless document
optimal measure & optimal choiceless document
resolved measure set
evaluated layout
printed, optimal layout
widens to
renders to
Cost
Last length

Fig. 7. Π_e resolves the document to a set of measures, and chooses an optimal measure from it to render.

Cost type τ	
$\leq_{\mathcal{F}} : \tau \to \tau \to \mathbb{B}$	$\leq_{\mathcal{F}}$ must be a total ordering (transitive, antisymmetric, and total)
$+_{\mathcal{F}} : \tau \to \tau \to \tau$	$\forall C_1, C_2, C_3, C_4 \in \tau.\ [C_1 \leq_{\mathcal{F}} C_2 \to C_3 \leq_{\mathcal{F}} C_4 \to C_1 +_{\mathcal{F}} C_3 \leq_{\mathcal{F}} C_2 +_{\mathcal{F}} C_4]$
$\text{text}_{\mathcal{F}} : \mathbb{N} \to \mathbb{N} \to \tau$	$\forall c, c', l \in \mathbb{N}.\ [c \leq c' \to \text{text}_{\mathcal{F}}(c, l) \leq_{\mathcal{F}} \text{text}_{\mathcal{F}}(c', l)]$
$\text{newline}_{\mathcal{F}} : \mathbb{N} \to \tau$	$\forall i, i' \in \mathbb{N}.\ [i \leq i' \to \text{newline}_{\mathcal{F}}(i) \leq_{\mathcal{F}} \text{newline}_{\mathcal{F}}(i')]$
$W_{\mathcal{F}} : \mathbb{N}$	$W_{\mathcal{F}}$ has no contract

Fig. 8. The cost factory interface. Users need to supply the cost type τ and implement the operations satisfying the contracts indicated in the interface.

instead operate on *measures* [Bernardy 2017c], which are limited views of layouts. This allows us to record the cost of a layout without expensive rendering. The need to prune early motivates us to devise the notion of *cost factory*, which allows us to incrementally compute costs to be used for pruning decision. At the same time, we design the cost factory to support a variety of optimality objectives.

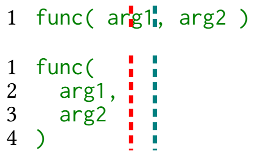
The workflow of Π_e is shown in Figure 7. The printer first resolves choices, with early pruning, to produce a small set of measures that contain the optimal measure. We could then pick the optimal measure from the set and render its choiceless document to produce an optimal layout.

5.2 The cost factory

The cost factory interface is presented in Figure 8. It allows users to define a cost type τ and implement operations on the cost type. $C_1 \leq_{\mathcal{F}} C_2$ tests if the cost C_1 is less than or equal to C_2. $C_1 +_{\mathcal{F}} C_2$ adds the cost C_1 and C_2 together. $\text{text}_{\mathcal{F}}(c, l)$ computes a cost due to a placement of a string without a newline of length l at the column position c. $\text{newline}_{\mathcal{F}}(i)$ is the cost due to a newline with i indentation spaces. The implemented operations must satisfy the indicated contracts in the interface, which are required for the core printer to function correctly. However, in practice, more constraints are recommended so that the printer is not brittle in presence of rewriting rules. We discuss more about this topic in Appendix C. Lastly, $W_{\mathcal{F}}$ is the computation width limit. When printing a document d, Π_e only provides the optimality guarantee among layouts in $\text{eval}_e(d)$ whose column position or indentation level during the printing does not exceeds $W_{\mathcal{F}}$. To illustrate how a cost factory is used in Π_e, we show a printing of the document in Figure 9 with a concrete cost factory.

Example 5.1. Consider an optimality objective that minimizes the sum of *overflows*, which are the numbers of characters that exceed a given *page width limit* w in each line, and *then* minimizes the *height*, which is the total number of newline characters (or equivalently, the number of lines minus one). This objective is thus able to avoid the excessive overflow problem in Bernardy's printer described in Section 2.

```
let args = nl <> text "arg1," <> nl <> text "arg2"
in
text "func(" <>
(group (nest 2 args <> nl)) <>
text ")"
```



```
func( arg1, arg2 )
```

```
func(
  arg1,
  arg2
)
```

Fig. 9. An example document to illustrate how the cost factory computes a cost. The dotted lines show the width limit of 8 and 10.

More formally, given a layout, a cost is a pair of the overflow sum and the height, where the lexicographic order determines which cost is less. With $w = 8$, the first layout in Figure 9 has the cost $(10, 0)$, whereas the second layout has the cost $(0, 3)$. Thus, the second layout is the optimal layout that Π_e should pick.

We implement the optimality objective with the following cost factory $\mathcal{F}$.

$$\tau = \mathbb{N} \times \mathbb{N} \quad \leq_{\mathcal{F}} = \leq_{\text{lex}} \quad (o_a, h_a) +_{\mathcal{F}} (o_b, h_b) = (o_a + o_b, h_a + h_b)$$
$$\text{text}_{\mathcal{F}}(c, l) = (\max(c + l - \max(w, c), 0), 0) \quad \text{newline}_{\mathcal{F}}(i) = (\max(i - w, 0), 1)$$

Each text and newline placement would query the cost factory to compute its cost. For example, `"func("` of length 5 is placed at the column position 0 in both layouts, so the cost is $\text{text}_{\mathcal{F}}(0, 5) = (0, 0)$. In the first layout, `"arg1,"` of length 5 is placed at the column position 6, so the cost is $\text{text}_{\mathcal{F}}(6, 5) = (3, 0)$. On the other hand, in the second layout, `"arg1,"` is placed at the column position 2, so the cost is $\text{text}_{\mathcal{F}}(2, 5) = (0, 0)$. Spaces due to newlines when the flattening mode is on are considered text. $\text{newline}_{\mathcal{F}}$ is never used for the first layout, but the second layout would query $\text{newline}_{\mathcal{F}}$ twice with indentation level 2 and once with indentation level 0. When we sum the costs up using $+_{\mathcal{F}}$, we obtain that the first layout has the cost $(10, 0)$, whereas the second layout has the cost $(0, 3)$ as expected. $\leq_{\mathcal{F}}$ could then be used to conclude that the second layout is optimal.

We have been ignoring the computation width limit $W_{\mathcal{F}}$ until now. When a column position or indentation level exceeds the computation width limit during a printing, the result is *tainted*. For example, with $W_{\mathcal{F}} = 10$, the first layout would be tainted, while the second layout would not. Tainted layouts can usually be discarded right away, except when every possible layout is tainted. In such case, Π_e keeps one tainted layout so that it can still output a layout, but provide no guarantee that the layout will be optimal. The tainting system allows us to bound the computation so that the algorithm is efficient.

The cost factory interface is versatile. The above example shows that Π_e does not need to take a page width limit as an input, because the concept of page width limit can already be defined by users via $\text{text}_{\mathcal{F}}$. It is also possible to, for example, implement the concept of *soft* width limit, compute a linear combination of height and overflow in the style of Yelland [2016], compute a sum of squared overflow, and compute a maximum of overflow. Furthermore, the cost factory plays a central role that enables Π_e to be efficient. In the subsequent subsections, every definition is implicitly parameterized by a cost factory $\mathcal{F}$.

5.3 Measure

As presented earlier, the resolving phase computes *measures*. Each measure is a limited view of a choiceless document rendering that incorporates cost. Presented in Figure 10, a measure consists of five components: length of last line (l), cost (C), choiceless document (d), max column position (x), and max indentation (y). We gray out the last two components because they are ghosted [Owicki and Gries 1976], only needed for the correctness theorem, and not required in the actual implementation.

$$\text{Measure } m \in \mathcal{M} = \langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}$$

$$\begin{array}{llll}
\text{last} : \mathcal{M} \to \mathbb{N} & \text{last}(\langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}) = l & \text{cost} : \mathcal{M} \to \tau & \text{cost}(\langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}) = c \\
\text{doc} : \mathcal{M} \to \overline{\mathcal{D}_e} & \text{doc}(\langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}) = \overline{d} & & \\
\text{maxx} : \mathcal{M} \to \mathbb{N} & \text{maxx}(\langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}) = x & \text{maxy} : \mathcal{M} \to \mathbb{N} & \text{maxy}(\langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}) = y
\end{array}$$

$$\begin{array}{ll}
\circ : \mathcal{M} \to \mathcal{M} \to \mathcal{M} & \langle l_a, C_a, \overline{d}_a, x_a, y_a \rangle_{\mathcal{M}} \circ \langle l_b, C_b, \overline{d}_b, x_b, y_b \rangle_{\mathcal{M}} = \\
& \quad \langle l_b, C_a +_{\mathcal{F}} C_b, \overline{d}_a \mathbin{\texttt{<>}} \overline{d}_b, \max(x_a, x_b), \max(y_a, y_b) \rangle_{\mathcal{M}}
\end{array}$$

$$\begin{array}{ll}
\text{adjustNest} : \mathbb{N} \to \mathcal{M} \to \mathcal{M} & \text{adjustNest}(n, \langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}) = \langle l, C, \texttt{nest}\ n\ \overline{d}, x, y \rangle_{\mathcal{M}} \\
\text{adjustAlign} : \mathbb{N} \to \mathcal{M} \to \mathcal{M} & \text{adjustAlign}(i, \langle l, C, \overline{d}, x, y \rangle_{\mathcal{M}}) = \langle l, C, \texttt{align}\ \overline{d}, x, \max(y, i) \rangle_{\mathcal{M}}
\end{array}$$

$$\preceq : \mathcal{M} \to \mathcal{M} \to \mathbb{B} \qquad \langle l_a, C_a, \overline{d}_a, x, y \rangle_{\mathcal{M}} \preceq \langle l_b, C_b, \overline{d}_b, x, y \rangle_{\mathcal{M}} = l_a \le l_b \land C_a \le_{\mathcal{F}} C_b$$

Fig. 10. Measure and operations on measures

$$\textsc{TextM}\ \frac{}{\langle \texttt{text}\ s, c, i \rangle \Downarrow_{\mathbb{M}} \langle c + |s|, \text{text}_{\mathcal{F}}(c, |s|), \texttt{text}\ s, c + |s|, i \rangle_{\mathcal{M}}}$$

$$\textsc{LineM}\ \frac{}{\langle \texttt{nl}, c, i \rangle \Downarrow_{\mathbb{M}} \langle i, \text{newline}_{\mathcal{F}}(i), \texttt{nl}, \max(c, i), i \rangle_{\mathcal{M}}} \qquad \textsc{ConcatM}\ \frac{\langle \overline{d}_a, c, i \rangle \Downarrow_{\mathbb{M}} m_a \quad \langle \overline{d}_b, \text{last}(m_a), i \rangle \Downarrow_{\mathbb{M}} m_b}{\langle \overline{d}_a \mathbin{\texttt{<>}} \overline{d}_b, c, i \rangle \Downarrow_{\mathbb{M}} m_a \circ m_b}$$

$$\textsc{NestM}\ \frac{\langle \overline{d}, c, i + n \rangle \Downarrow_{\mathbb{M}} m}{\langle \texttt{nest}\ n\ \overline{d}, c, i \rangle \Downarrow_{\mathbb{M}} \text{adjustNest}(n, m)} \qquad \textsc{AlignM}\ \frac{\langle \overline{d}, c, c \rangle \Downarrow_{\mathbb{M}} m}{\langle \texttt{align}\ \overline{d}, c, i \rangle \Downarrow_{\mathbb{M}} \text{adjustAlign}(i, m)}$$

Fig. 11. Measure computation from a choiceless document printing state

Example 5.2. Given $\overline{d}$ = `text "=␣{|" <> nest 1 (nl <> text "1234") <> nl <> text "|}"` being placed at the column position 0, with indentation level 0. The choiceless document would render to a layout `["=␣{|", "␣1234", "|}"]`. With the cost factory in Example 5.1, the cost of the layout is 0. Thus, the measure is $\langle 2, 0, \overline{d}, 5, 1 \rangle_{\mathcal{M}}$.

Figure 11 shows rules that define measure computation. The judgment $\langle d, c, i \rangle \Downarrow_{\mathbb{M}} m$ states that when we compute the measure of $\overline{d} \in \overline{\mathcal{D}_e}$ placed at the column position $c \in \mathbb{N}$ with indentation level $i \in \mathbb{N}$, the resulting measure is $m \in \mathcal{M}$. To simplify the core printer, we (temporarily) remove `flatten` from Σ_e. This allows us to eliminate the flattening mode parameter, which implicitly defaults to $\perp$. Toward the end of this section, we will show how to add support for `flatten` back.

The rules are largely standard. They reflect the actual rendering defined by $\Downarrow_{\mathcal{R}}$, and utilize the cost factory in a straightforward way. The rules use a helper operator function $\circ$ to concatenate two measures, and helper functions adjustNest and adjustAlign to construct a correct measure for `nest` and `align`. These functions are defined in Figure 10. Notably, the LineM rule creates a measure whose maxc is $\max(c, i)$ because before placing the newline, the column position is c, and after placing the newline, the column position is i. The AlignM rule creates a measure whose maxi is $\max(y, i)$ where y is obtained via the recursive computation. This is because the recursive computation discards the current indentation level, so we need to specifically record the information.

$\Downarrow_{\mathbb{M}}$ is deterministic and total. It is also correct with respect to $\Downarrow_{\mathcal{R}}$.

Theorem 5.3. *For any $\overline{d} \in \overline{\mathcal{D}_e}$ and $c, i \in \mathbb{N}$, there exists a cost C and max indentation y such that*

- *if $\langle \overline{d}, c, i, \perp \rangle \Downarrow_{\mathcal{R}} [s]$, then $\langle \overline{d}, c, i \rangle \Downarrow_{\mathbb{M}} \langle c + |s|, C, \overline{d}, c + |s|, y \rangle_{\mathcal{M}}$.*
- *if $\langle \overline{d}, c, i, \perp \rangle \Downarrow_{\mathcal{R}} [s, s_1, \ldots, s_n, t]$, then $\langle \overline{d}, c, i \rangle \Downarrow_{\mathbb{M}} \langle |t|, C, \overline{d}, \max(c + |s|, |s_1|, \ldots, |s_n|, |t|), y \rangle_{\mathcal{M}}$*

So far, we only consider the measure computation for a choiceless document. When we take the choice operator into account, there could be multiple measures under the same printing context.

$$
\begin{array}{llll}
\text{Measure set } S \in \mathcal{S} & ::= & \texttt{Tainted}(\hat{m}) & \text{where } \hat{m} \text{ is a promise that can be forced to a measure} \\
 & | & \texttt{Set}([m_1, \ldots^+, m_n]) & \text{where } \mathsf{last}(m_1) > \ldots > \mathsf{last}(m_n) \text{ and } \forall i \neq j, \neg(m_i \preceq m_j \vee m_j \preceq m_i)
\end{array}
$$

$$
\begin{array}{ll}
\mathsf{taint} : \mathcal{S} \to \mathcal{S} & \mathsf{taint}(\texttt{Tainted}(m)) = \texttt{Tainted}(m) \\
 & \mathsf{taint}(\texttt{Set}([m_0, m_1, \ldots, m_n])) = \texttt{Tainted}(m_0)
\end{array}
$$

$$
\begin{array}{ll}
\mathsf{lift} : \mathcal{S} \to (\mathcal{M} \to \mathcal{M}) \to \mathcal{S} & \mathsf{lift}(\texttt{Tainted}(m), f) = \texttt{Tainted}(f(m)) \\
 & \mathsf{lift}(\texttt{Set}([m_1, \ldots^+, m_n]), f) = \texttt{Set}([f(m_1), \ldots^+, f(m_n)])
\end{array}
$$

$$
\begin{array}{lll}
\mathsf{dedup} : \overrightarrow{\mathcal{M}} \to \overrightarrow{\mathcal{M}} & \mathsf{dedup}([m, m', m_1, \ldots, m_n]) = \mathsf{dedup}([m', m_1, \ldots, m_n]) & \text{if } m' \preceq m \\
 & \mathsf{dedup}([m, m', m_1, \ldots, m_n]) = [m] @ \mathsf{dedup}([m', m_1, \ldots, m_n]) & \text{if } m' \npreceq m \\
 & \mathsf{dedup}([m]) = [m] &
\end{array}
$$

$$
\begin{array}{ll}
\uplus : \mathcal{S} \to \mathcal{S} \to \mathcal{S} & S \uplus \texttt{Tainted}(m) = S \\
 & \texttt{Tainted}(m) \uplus \texttt{Set}([m_1, \ldots^+, m_n]) = \texttt{Set}([m_1, \ldots^+, m_n]) \\
 & \texttt{Set}([m_1, \ldots^+, m_n]) \uplus \texttt{Set}([m'_1, \ldots^+, m'_{n'}]) = \texttt{Set}([m_1, \ldots^+, m_n] \uplus [m'_1, \ldots^+, m'_{n'}])
\end{array}
$$

$$
\begin{array}{ll}
\uplus : \overrightarrow{\mathcal{M}} \to \overrightarrow{\mathcal{M}} \to \overrightarrow{\mathcal{M}} & [\,] \uplus [m_1, \ldots^+, m_n] = [m_1, \ldots^+, m_n] \\
 & [m_1, \ldots^+, m_n] \uplus [\,] = [m_1, \ldots^+, m_n]
\end{array}
$$

$$
[m_0, m_1, \ldots, m_n] \uplus [m'_0, m'_1, \ldots, m'_{n'}] =
\begin{cases}
[m_0, m_1, \ldots, m_n] \uplus [m'_1, \ldots, m'_{n'}] & \text{if } m_0 \preceq m'_0 \\
[m_1, \ldots, m_n] \uplus [m'_0, m'_1, \ldots, m'_{n'}] & \text{if } m'_0 \preceq m_0 \\
[m_0] @ ([m_1, \ldots, m_n] \uplus [m'_0, m'_1, \ldots, m'_{n'}]) & \text{if } \mathsf{last}(m_0) > \mathsf{last}(m'_0) \\
[m'_0] @ ([m_0, m_1, \ldots, m_n] \uplus [m'_1, \ldots, m'_{n'}]) & \text{otherwise}
\end{cases}
$$

Fig. 12. Measure set and the merge operation on measure sets. @ denotes a list concatenation. We treat a promise $\hat{m}$ and a measure m interchangeably, as they can be straightforwardly casted to each other.

The main operation that we can perform on these measures is finding domination $\preceq$, also presented in Figure 10. $m_a \preceq m_b$ when both the cost and the last length of m_a are no worse than those of m_b. The fact that $m_a \preceq m_b$ is useful because it allows us to prune m_b away immediately.

5.4 Measure set

Resolving a document (in a printing context) produces a set of measures. To accommodate taintedness mentioned in Section 5.2, Figure 12 defines a measure set to be either a non-empty `Set` of untainted measures where no measure dominates the other, or a `Tainted` singleton set of a promise $\hat{m}$ that can be forced to a measure. The `Set`, by definition, forms a Pareto frontier. To aid computation, we represent the `Set` with a list ordered by the cost in the strict ascending order (and therefore the last length in the strict descending order). We are able to do so because in a Pareto frontier, all last and cost values must be distinct.

The main operation that we can perform on measure sets is merging two measure sets ($\uplus$), shown in Figure 12, where we prefer a `Set` over a `Tainted`. The merge operation maintains the Pareto frontier invariant, by doing the merge in the style of the merge operation in merge sort, although the Pareto frontier merging can also prune measures away during the operation. One important "quirk" of this merge operation is that it is *left-biased* in presence of taintedness. If two tainted measure sets are merged, the result is always the left one. This means the order of arguments to the merge operation is important, as we will see in the next subsections.

Other operations on measure sets which are used in next subsections are taint, lift, and dedup. taint taints a measure set. When tainting a `Set`, we choose to pick the first measure from the `Set` because it has the lowest cost, which is a greedy heuristic. lift adjusts measures in a measure set. Lastly, dedup prunes measures that are sorted by last in the strict decreasing order and by cost in the non-strict increasing order, so that the result conforms the Pareto frontier invariant.

5.5 The document structure

Section 2.2 has shown that we need to handle document sharing by treating the input document as a DAG. However, documents cannot be arbitrarily shared, as the following example shows:

Example 5.4. The following document $\mathsf{mk}(n)$ has the DAG size of $O(n)$. However, resolving it necessitates $O(2^n)$ units of computation, as the printing contexts are all different. This is bad news because it means resolving could take exponential time in the input size.

```
let rec mk (n : int): doc =
  if n = 0 then text "x" else let shared = mk (n - 1) in shared <> shared
```

However, we argue that the above document is not *properly shared*, because the sub-documents are not shared *across choices*, which is how sharing is employed in practice. The corresponding properly shared document should have $O(2^n)$ DAG size, so $O(2^n)$ units of computation are still linear in the input size. To make this precise, we provide the following definitions:

Definition 5.5. Given a document $d \in \mathcal{D}_e$, $G(d)$ is a DAG rooted at d whose edge in the graph connects a document to its direct subdocuments.

Definition 5.6. A document $d \in \mathcal{D}_e$ is *properly shared* if for any two vertices d_a and d_b in $G(d)$, if p_1 and p_2 are two distinct paths from d_a to d_b, then there exists a common document d' such that (1) d' is a `<|>`; (2) d' occurs in both p_1 and p_2; and (3) d' is not d_b.

Figure 3c shows a properly shared document (assuming that D is properly shared). It illustrates two paths where d_a is the root node, d_b is D, and d' is d_a. In practice, non-properly shared documents can still be processed by Π_e, and in fact can even make resolving (but not rendering) faster when the documents are printed under the same printing context. However, they would be treated as different documents when they are shared in different contexts. For simplicity, we only consider properly shared documents as the input to Π_e in this paper.

5.6 The resolver

We now formally define the core of Π_e, which is the resolver. It is described in Figure 13, which is a fusion of widening in Figure 6 and measure computation in Figure 11, with early pruning inherent in the merge operation and extra bookkeeping for taintedness. The judgment $\langle d, c, i \rangle \Downarrow_{\mathbb{RS}} S$ states that a properly shared document $d \in \mathcal{D}_e$ at a column position $c \in \mathbb{N}$ with an indentation level $i \in \mathbb{N}$ resolves to a measure set S.

Resolving text. If placing the text would exceed $W_{\mathcal{F}}$ or the indentation level is beyond $W_{\mathcal{F}}$, the TextRSTnt rule returns a `Tainted`. Otherwise, the TextRS rule returns a singleton `Set`.

Resolving newline. Resolving a `nl` is similar to resolving a `text`, but we only need to consider the current column position and indentation level, as resolving the newline does not change the column position. The LineRSTnt and LineRS rules cover these two cases.

Resolving nest. Resolving a `nest` is handled by the NestRS rule, which recursively resolves its sub-document, with the indentation level changed. The recursive resolving would determine whether the measure set would be a `Set` or `Tainted`. In all cases, the result is adjusted to construct correct choiceless documents.

Resolving alignment. Resolving an `align` is similar to resolving `nest`. However, because the recursive resolving would discard the current indentation level, which could exceed $W_{\mathcal{F}}$, we need to taint the measure set when the indentation level is beyond $W_{\mathcal{F}}$. The AlignRSTnt rule handles such case, and the AlignRS rule handles other possibilities.

$$\textsc{TextRSSet}\ \frac{c + |s| \le W_{\mathcal{F}} \quad i \le W_{\mathcal{F}} \quad \langle \texttt{text}\ s, c, i\rangle \Downarrow_{\mathbb{M}} m}{\langle \texttt{text}\ s, c, i\rangle \Downarrow_{\mathbb{RS}} \texttt{Set}([m])} \qquad \textsc{LineRSSet}\ \frac{c \le W_{\mathcal{F}} \quad i \le W_{\mathcal{F}} \quad \langle \texttt{nl}, c, i\rangle \Downarrow_{\mathbb{M}} m}{\langle \texttt{nl}, c, i\rangle \Downarrow_{\mathbb{RS}} \texttt{Set}([m])}$$

$$\textsc{TextRSTnt}\ \frac{c + |s| > W_{\mathcal{F}} \lor i > W_{\mathcal{F}} \quad \langle \texttt{text}\ s, c, i\rangle \Downarrow_{\mathbb{M}} m}{\langle \texttt{text}\ s, c, i\rangle \Downarrow_{\mathbb{RS}} \texttt{Tainted}(m)} \qquad \textsc{LineRSTnt}\ \frac{c > W_{\mathcal{F}} \lor i > W_{\mathcal{F}} \quad \langle \texttt{nl}, c, i\rangle \Downarrow_{\mathbb{M}} m}{\langle \texttt{nl}, c, i\rangle \Downarrow_{\mathbb{RS}} \texttt{Tainted}(m)}$$

$$\textsc{NestRS}\ \frac{\langle d, c, i + n\rangle \Downarrow_{\mathbb{RS}} S}{\langle \texttt{nest}\ n\ d, c, i\rangle \Downarrow_{\mathbb{RS}} \mathsf{lift}(S, \mathsf{adjustNest}(n))} \qquad \textsc{AlignRS}\ \frac{i \le W_{\mathcal{F}} \quad \langle d, c, c\rangle \Downarrow_{\mathbb{RS}} S}{\langle \texttt{align}\ d, c, i\rangle \Downarrow_{\mathbb{RS}} \mathsf{lift}(S, \mathsf{adjustAlign}(i))}$$

$$\textsc{UnionRS}\ \frac{\langle d_a, c, i\rangle \Downarrow_{\mathbb{RS}} S_a \quad \langle d_b, c, i\rangle \Downarrow_{\mathbb{RS}} S_b}{\langle d_a\ \texttt{<|>}\ d_b, c, i\rangle \Downarrow_{\mathbb{RS}} S_a \uplus S_b} \qquad \textsc{AlignRSTnt}\ \frac{i > W_{\mathcal{F}} \quad \langle d, c, c\rangle \Downarrow_{\mathbb{RS}} S}{\langle \texttt{align}\ d, c, i\rangle \Downarrow_{\mathbb{RS}} \mathsf{lift}(\mathsf{taint}(S), \mathsf{adjustAlign}(i))}$$

$$\textsc{ConcatRS}\ \frac{\langle d_a, c, i\rangle \Downarrow_{\mathbb{RS}} \texttt{Set}([m_1, \dots, m_n]) \quad \langle m_1, d_b, i\rangle \Downarrow_{\mathbb{RSC}} S_1 \quad \dots \quad \langle m_n, d_b, i\rangle \Downarrow_{\mathbb{RSC}} S_n}{\langle d_a\ \texttt{<>}\ d_b, c, i\rangle \Downarrow_{\mathbb{RS}} S_1 \uplus \dots \uplus S_n}$$

$$\textsc{ConcatRSTnt}\ \frac{\langle d_a, c, i\rangle \Downarrow_{\mathbb{RS}} \texttt{Tainted}(m_a) \quad \langle d_b, \mathsf{last}(m_a), i\rangle \Downarrow_{\mathbb{RS}} S \quad \mathsf{taint}(S) = \texttt{Tainted}(m_b)}{\langle d_a\ \texttt{<>}\ d_b, c, i\rangle \Downarrow_{\mathbb{RS}} \texttt{Tainted}(m_a \circ m_b)}$$

$$\textsc{RSCSet}\ \frac{\langle d_b, \mathsf{last}(m_a), i\rangle \Downarrow_{\mathbb{RS}} \texttt{Set}([m_1, \dots^{+}, m_n])}{\langle m_a, d_b, i\rangle \Downarrow_{\mathbb{RSC}} \texttt{Set}(\texttt{dedup}([m_a \circ m_1, \dots^{+}, m_a \circ m_n]))} \qquad \textsc{RSCTnt}\ \frac{\langle d_b, \mathsf{last}(m_a), i\rangle \Downarrow_{\mathbb{RS}} \texttt{Tainted}(m_b)}{\langle m_a, d_b, i\rangle \Downarrow_{\mathbb{RSC}} \texttt{Tainted}(m_a \circ m_b)}$$

Fig. 13. The resolver

Resolving choice. The UnionRS rule recursively resolves its two sub-documents, and then merges the measure sets. As mentioned in Section 5.4, the merge operation is left-biased. Therefore, the left sub-document will be preferred over the right sub-document if exceeding $W_{\mathcal{F}}$ is unavoidable. It is possible to employ a heuristic to remove this bias, as discussed in Appendix C.

Resolving unaligned concatenation. Resolving a `<>` is done through ConcatRSTnt and ConcatRS rules, which handle the two possibilities of measure set types obtained from the left sub-document's recursive resolving. Notably, the ConcatRS rule employs $\Downarrow_{\mathbb{RSC}}$ to help us concatenates a left measure from the left measure set with a right measure set. The series of merges should be done with the right fold ordering to avoid unnecessary list traversal.

$\Downarrow_{\mathbb{RS}}$ is deterministic and total. This allows us to define the top-level printer as $\Pi_e(d) = l$ where $\langle \mathsf{doc}(m_0), 0, 0, \bot\rangle \Downarrow_{\mathcal{R}} l$ and $\langle d, 0, 0\rangle \Downarrow_{\mathbb{RS}} [m_0, m_1, \dots, m_n]$, which consumes a properly shared document d, resolves it to a set of measures, picks the measure with the lowest cost, and simply renders the associated choiceless document to produce a layout (although our implementation further fuses resolving and rendering together, as described in Appendix C).

While the rules above are enough for correctness, implementing these rules requires further consideration. As we will see in Lemma 5.10, any resolving beyond $W_{\mathcal{F}}$ would eventually result in a tainted measure set. Hence, Π_e should *immediately* delay the computation for any resolving beyond $W_{\mathcal{F}}$. Π_e should also *memoize* the computation, so that on identical document and printing context within $W_{\mathcal{F}}$, the result of the previous computation is reused.

We claim that $\Pi_e(d)$ consumes a properly shared document d in Σ_e and produces an optimal layout among $\mathsf{eval}_e(d)$ within $W_{\mathcal{F}}$. We will prove the claim in the next subsection.

5.7 Correctness of Π_e

$\Downarrow_{\mathbb{P}}$ is correct with respect to $\Downarrow_{\mathbb{M}}$. Two theorems govern the correctness. The first theorem states that the core printer returns a measure set that contains a measure that is no worse than any measure within the computation width limit from all possible measures.

Theorem 5.7 (Optimality). *For any $d \in \mathcal{D}_e$, $c \in \mathbb{N}$, $i \in \mathbb{N}$, if the following conditions hold*

- $\langle d, c, i \rangle \Downarrow_{\mathbb{RS}} S$
- $d \Downarrow_{\mathcal{W}} \overline{D}$
- $\overline{d} \in \overline{D}$
- $\langle \overline{d}, c, i \rangle \Downarrow_{\mathbb{M}} m$
- $\mathsf{maxx}(m) \le W_{\mathcal{F}}$
- $\mathsf{maxy}(m) \le W_{\mathcal{F}}$

then $S = \mathtt{Set}([m_1, \dots^{+}, m_n])$. Furthermore, there exists i such that $m_i \preceq m$.

The second theorem states that measures in the resulting measure set are valid.

Theorem 5.8 (Validity). *For any $d \in \mathcal{D}_e$, $c \in \mathbb{N}$, $i \in \mathbb{N}$ with $d \Downarrow_{\mathcal{W}} \overline{D}$, if $\langle d, c, i \rangle \Downarrow_{\mathbb{RS}}$ $\mathtt{Set}([m_1, \dots^{+}, m_n])$, then for each i, there exists $\overline{d}$ such that $\overline{d} \in \overline{D}$ and $\langle \overline{d}, c, i \rangle \Downarrow_{\mathbb{M}} m_i$. Likewise, if $\langle d, c, i \rangle \Downarrow_{\mathbb{RS}} \mathtt{Tainted}(m_0)$, then there exists $\overline{d}$ such that $\overline{d} \in \overline{D}$ and $\langle \overline{d}, c, i \rangle \Downarrow_{\mathbb{M}} m_0$.*

The correctness of Π_e follows immediately.

While the above theorems guarantee the correctness of the result that the printer produces, it does not concern with efficiency. The following lemmas provide some properties of the printer that allow us to reason about the efficiency.

Lemma 5.9. *For any $d \in \mathcal{D}_e$, $c \le W_{\mathcal{F}}$, $i \le W_{\mathcal{F}}$, if $\langle d, c, i \rangle \Downarrow_{\mathbb{RS}} \mathtt{Set}([m_1, \dots, m_n])$, then $n \le W_{\mathcal{F}} + 1$.*

Lemma 5.10. *For any $d \in \mathcal{D}_e$, if $c > W_{\mathcal{F}}$ or $i > W_{\mathcal{F}}$ and $\langle d, c, i \rangle \Downarrow_{\mathbb{RS}} S$, then S is a $\mathtt{Tainted}$.*

We now informally prove the efficiency of Π_e that we claimed in Section 1. The proof sketches are provided in Appendix B.

Theorem 5.11. *The time complexity of Π_e is $O(nW_{\mathcal{F}}^4)$ where n is the DAG size of the document.*

Theorem 5.12. *If a document d is in the arbitrary-choice PPL, Π_e can print d in $O(nW_{\mathcal{F}}^3)$.*

5.8 Handling flattening

To support **`flatten`**, we make it a function that walks its sub-document and replaces all **`nl`** with **`text`** `"␣"`. The walk is memoized and preserves the original identity of the document whenever possible (i.e. if nothing is flatten in sub-documents, then the document itself is returned unchanged without creating a new document). Thus, each document can be flattened at most once. This flattening creates at most $O(n)$ new documents without destroying the shared structure in the original document. We therefore achieve the functionality of **`flatten`** without affecting the time complexity of the printer.

6 IMPLEMENTATION

We implement Π_e in OCaml and Racket. The printer, which we call SnowWhite, is further refined to be more efficient and practical. Due to the space limit, we describe these refinements in Appendix C. The OCaml SnowWhite, as a reference implementation, is used for comparing against other printers in Section 7. The Racket SnowWhite has even more features, and it has been used to implement the code formatter for the Racket programming language.

SnowWhite provides a pre-defined cost factory that minimizes the sum of squared overflows over the page width limit w (without considering indentation spaces) and then height. The text placement formula is derived from the identity $(a+b)^2 - a^2 = b(2a+b)$ where in each text placement, a is the starting position count past the page width limit and b is the overflow length. With this cost factory and $w = 8$, the first layout in Figure 9 has the cost $(10^2, 0)$ whereas the second layout has the cost $(0, 3)$.

$$\mathrm{text}_{\mathcal{F}}(c, l) = \begin{cases} (b(2a+b), 0) & \text{if } c + l > w \\ (0, 0) & \text{otherwise} \end{cases} \quad \text{where} \quad \begin{aligned} a &= \max(w, c) - w \\ b &= c + l - \max(w, c) \end{aligned}$$

7 EVALUATION

This evaluates the performance and optimality of SnowWhite. The evaluation consists of two parts. First, we compare SnowWhite against Wadler/Leijen [2000] and Bernardy [2017b]'s printers, which are popular practical printers with capabilities from the traditional and arbitrary-choice PPLs. Second, we evaluate the Racket code formatter, which uses SnowWhite as its foundation. The evaluation aims to answer the following questions:

(1) Does SnowWhite run fast in practice?
(2) Does SnowWhite produce pretty layouts in practice?

All experiments are performed on a (non-poisoned) Apple M1 MacBook Pro with 16GB of RAM. We describe the experiments and benchmarks in Section 7.1 and Section 7.2, and discuss the results in Section 7.3.

7.1 Comparison of printers

We compare OCaml SnowWhite against the latest version (1.2.1) of Wadler/Leijen's printer, and the "camera ready version" of Bernardy's printer[7]. This "camera ready version" consists of two printers: the "naïve" variant, which is presented in the paper, and the "practical" implementation, which has more features (such as unavoidable overflow handling) but suffers from the exponential time complexity when the DAG structure unfolds, as discussed in Section 2. We manually remove the capability to customize the width limit from the latter to avoid the issue. Both variants are used for the evaluation, since the naïve variant does not have necessary features for some benchmarks.

SnowWhite is instantiated with the cost factory in Section 6, with the page width limit of 80 (unless indicated otherwise). We run SnowWhite twice with different computation width limits (once with $W_{\mathcal{F}} = 100$, unless indicated otherwise, and once with $W_{\mathcal{F}} = 1000$), in order to observe the effect of the tainting system and how it affects the performance.

The benchmarks (Table 2) are mostly taken from Bernardy [2017c], and we add a few more to test basic constructs. While Leijen's printer is expressive enough to handle all benchmarks (due to the inclusion of **`align`** to support aligned concatenation in addition to constructs from the traditional PPL), Bernardy's printers are not applicable to benchmarks that require constructs from the traditional PPL. Furthermore, Bernardy's naïve printer is not applicable to benchmarks that require extra features like unavoidable overflow handling.

In more detail, the benchmarks test the following kinds of documents:

Concat benchmarks test a long chain of concatenations, which are identified by Peyton-Jones [1997] as a source of quadratic time complexity in Hughes' printer.

FillSep benchmarks test the `fillSep` construct (also known as `fill`), which fills lines with words as long as they fit.

Flatten benchmarks test repeated flattening, as shown in Figure 15.

SExpFull benchmarks are the last two data points from the "full tree" benchmark in Bernardy [2017c]'s paper. They create complete binary trees and print them as S-expressions.

RandFit benchmarks [Bernardy 2017c] are similar to SExpFull, but use random Dyck paths to generate random trees and filter only those that can fit within the page width limit.

RandOver benchmarks are like RandFit with the opposite filtering.

JSON benchmarks are also from Bernardy [2017c]'s paper. They format large JSON files.

[7]We also tried other versions of Bernardy's printer, such as the commit 006fa0e8, which is the version right before the **`<|>`** operator was removed, and supposedly more optimized than the camera ready version. Unfortunately, we find that it has a severe performance deficiency. When attempting to replicate the experiments in Bernardy [2017c], we find that formatting the 10k-line-JSON file takes about 80 seconds, which is much slower than the 145 milliseconds reported in the paper.

Table 2. Comparison between SnowWhite in different configurations and other printers. For each printer and configuration, the first column reports the running time, and the second column reports the line count of the output layout. SnowWhite has an additional third column, where ✓ indicates that the output layout fits $W_{\mathcal{F}}$ and ✗ indicates that the output layout is tainted. "N/A" means the benchmark is not applicable. ⏱ indicates that running the benchmark exceeds the timeout of 60 seconds. "-" means the data is not collected. A grayed row indicates an output mismatch among the printers/configurations. The bolded line count signals that in our manual inspection, the associated layout is the prettiest.

Benchmark	SnowWhite						Wadler/Leijen		Bernardy			
	default $W_{\mathcal{F}}$ (usually 100)			$W_{\mathcal{F}}$ = 1000					Naïve		Practical	
Concat10k	0.89 ms	1	✗	0.91 ms	1	✗	2.05 ms	1	N/A	-	548.14 ms	1
Concat50k	10.58 ms	1	✗	10.55 ms	1	✗	11.55 ms	1	N/A	-	17.18 s	1
FillSep5k	13.51 ms	668	✓	13.71 ms	668	✓	12.34 ms	668	3.32 s	668	⏱	-
FillSep50k	264.34 ms	6834	✓	262.99 ms	6834	✓	73.16 ms	6834	⏱	-	⏱	-
Flatten8k	42.10 ms	7986	✓	47.15 ms	7986	✓	3.33 s	7986	N/A	-	N/A	-
Flatten16k	94.86 ms	15986	✓	92.62 ms	15986	✓	22.14 s	15986	N/A	-	N/A	-
SExpFull15	3.28 s	4107	✓	8.55 s	4107	✓	60.44 ms	4107	678.92 ms	4107	929.26 ms	4107
SExpFull16	5.51 s	8246	✓	21.79 s	8246	✓	98.79 ms	8246	1.35 s	8246	1.83 s	8246
RandFit1k	130.97 ms	**629**	✓	243.59 ms	**629**	✓	15.16 ms	943	51.68 ms	**629**	80.17 ms	**629**
RandFit10k	1.13 s	**7861**	✓	4.66 s	**7861**	✓	42.65	10459	582.83 ms	**7861**	902.99 ms	**7861**
RandOver1k	88.06 ms	**1531**	✗	958.34 ms	**1531**	✓	7.27 ms	1635	N/A	-	73.95 ms	1105
RandOver10k	486.81 ms	**15027**	✗	13.98 s	**15027**	✓	135.75 ms	16015	N/A	-	1.18 s	7953
JSON1k	2.13 ms	564	✓	2.25 ms	564	✓	1.87 ms	564	N/A	-	5.02 ms	564
JSON10k	27.30 ms	5712	✓	25.86 ms	5712	✓	23.06 ms	5712	N/A	-	106.73 ms	5712
JSONW	2.15 ms	**721**	✗	2.20 ms	**721**	✓	8.52 ms	**721**	N/A	-	5.44 ms	709

Table 3. The code formatter benchmarks. The table is in the same format as the SnowWhite column in Table 2.

Benchmark	$W_{\mathcal{F}}$ = 100			$W_{\mathcal{F}}$ = 1000		
`class-internal`	651 ms	5750	✗	665 ms	**5749**	✓
`xform`	796 ms	5154	✗	819 ms	5154	✓

Benchmark	$W_{\mathcal{F}}$ = 100			$W_{\mathcal{F}}$ = 1000		
`list`	98 ms	993	✓	90 ms	993	✓
`hash`	8 ms	83	✓	9 ms	83	✓

JSONW benchmark is the same as JSON1k but with a page width limit of 50 instead of 80, and we further adjust SnowWhite's default $W_{\mathcal{F}}$ from 100 to 60 to test the tainting system.

7.2 Racket code formatter

We evaluate the effectiveness of a Racket code formatter that uses the Racket SnowWhite as its foundation. Racket [Felleisen et al. 2018] is a programmable programming language. Its main syntax is S-expression, but this can be customized via its `#lang` protocol to read an arbitrary syntax. Even in the S-expression syntax, users can define custom forms via the macro system. Our long-term plan for the code formatter is to make it extensible to support any syntax and custom forms. SnowWhite is thus a natural choice as a foundational printer, due to its expressiveness.

The code formatter currently supports only S-expression formatting. However, the task is already challenging. While the S-expression syntax may look simple and uniform, Racket users employ a variety of styles for different forms to make them look distinctive in order to improve readability. Each function application, for example, has three possible styles (while most languages have two function application styles). The search space of the code formatter is thus quite large.

The benchmarks (Table 3) consist of files of different sizes from the Racket language codebase[8]. `class-internal` and `xform` are the two largest files. We use the code formatter to format these files with the page width limit of 80. We run the code formatter twice, once with $W_{\mathcal{F}} = 100$ and once with $W_{\mathcal{F}} = 1000$.

[8]https://github.com/racket/racket/tree/master/racket/collects at commit 4f1a2bd4

7.3 Results

Performance. The benchmarking results in Table 2 and Table 3 show that overall, SnowWhite is sufficiently fast in practice. While not the fastest, it can process large, practical workloads `class-internal` and `xform` under a second. Furthermore, it provides a performance guarantee even on tricky inputs. The same is not true for other printers. The Flatten benchmarks work very poorly for Wadler's printer, and the FillSep benchmarks work very poorly for Bernardy's printer. Interestingly, Bernardy's naïve printer is faster than its practical variant, even though the latter is more optimized; this is due to the extra features that the practical printer needs to support. SnowWhite, by contrast, is set to support these features from the start.

We note two interesting observations on SnowWhite. First, it performs poorly on SExpFull relative to other printers. This is due to the memory pressure from memoization. Better engineering effort may be able to alleviate this issue. Second, although the time complexity of Π_e is $O(nW_{\mathcal{F}}^4)$, this worst case behavior happens only if Pareto frontiers are always full. In practice, this is not the case[9], as evidenced by the fact that increasing $W_{\mathcal{F}}$ tenfold does not multiply the running time by 10^4. On the contrary, increasing $W_{\mathcal{F}}$ does not affect the running time at all on most benchmarks.

Optimality. We find that SnowWhite is the prettiest compared to others, offering high quality output when we use the cost factory described in Section 6. Table 2 shows (via line count) that the output layouts in many benchmarks agree in all printers. The exceptions are RandFit, RandOver, and JSONW benchmarks. Upon manual inspection, we find that the layouts produced by SnowWhite are better. JSONW and RandOver are cases where there is an unavoidable overflow, causing Bernardy's printer to overflow more than necessary. Figure 16 (in Appendix A) shows a concrete example of this problem. RandFit and RandOver are cases where the greedy minimization and the **`align`** construct in Leijen's printer interact poorly, as discussed and illustrated in Bernardy [2017c].

It should also be noted that neither Leijen's nor Bernardy's printers support custom optimality objectives, as their optimality objectives are integral to their algorithms. SnowWhite, in contrast, allows users to customize optimality objective via the cost factory.

Lastly, we evaluate the effectiveness of the tainting system. For almost every benchmark that gets a tainted layout (✗) with the default $W_{\mathcal{F}}$, we find that using $W_{\mathcal{F}} = 1000$ in an attempt to avoid taintedness[10] yields the same result, confirming the optimality of the output layout. The only exception is the `class-internal` benchmark in Table 3, for which the output layouts are different in one line and otherwise identical, because the greedy heuristic in the `taint` operation prunes the optimal choice away. This demonstrates that despite being tainted, and thus no longer guaranteed to be optimal, the output layout is still reasonable (at least with respect to the cost factory that we employ and the heuristic to avoid bias described in Appendix C).

8 CONCLUSION

We have described Π_e, an expressive printer that supports a variety of optimality objectives and is practically efficient. We developed a framework for reasoning about the expressiveness of PPLs, and we used this framework to guide the design of the PPL that Π_e targets. By surveying existing pretty printers, we have shown that Π_e is well-placed in the design space of printers. Π_e is proven correct in the Lean theorem prover and implemented as a practical printer SnowWhite, which powers a real-world code formatter for the Racket programming language. Our results show that Π_e is both pretty and fast.

[9]This observation also applies to Bernardy's printers, which are also based on Pareto frontiers.

[10]Therefore, the Concat benchmarks do not count, since they are still tainted afterwards. The benchmarks are not interesting anyway, since there is no choice in the documents, so the output layouts are always optimal.

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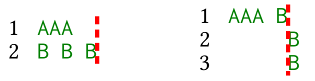
A AN ANALYSIS OF PRINTERS

```
group (text "AAA" <> nl) <>
nest 5 (group (text "B" <> nl <>
               text "B" <> nl <> text "B"))
```



```
1  AAA
2  B B B
```

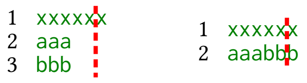
```
1  AAA B
2        B
3        B
```

Fig. 14. A document in the traditional PPL and two of their corresponding layouts. Under the width limit of 5, the first layout is optimal—it does not overflow and occupies minimal number of lines. In contrast, the second layout, which is produced by Wadler's printer, overflows and does not occupy minimal number of lines.

```
let rec quadratic (n : int): doc =
  if n = 0 then text "line"
  else group (quadratic (n - 1) <> nl <> text "line")
```

Fig. 15. The function quadratic generates a document of size $O(n)$ that Wadler's algorithm takes $O(n^2)$ to print at any width limit, due to repeated flattening.

```
text "xxxxxx" <$>
((text "aaa" <+> text "bbb") <|>
 (text "aaa" <$> text "bbb"))
```



```
1  xxxxxx
2  aaa
3  bbb
```

```
1  xxxxxx
2  aaabbb
```

Fig. 16. A document in the arbitrary-choice PPL and two of their corresponding layouts. Under the width limit of 5, the first layout minimally overflows. In contrast, the second layout, which is produced by Bernardy's practical implementation, overflows than necessary.

```
let rec mk (n : int): doc =
  if n = 0 then text "X" <|> text "XX"
  else let subdoc = mk (n - 1) in (chr n <> subdoc <> chr n) <|> subdoc
```

(a) The function mk generates a document whose DAG size is $O(n)$. $\text{chr}(n)$ denotes a **text** whose content is a string of length one that contains the nth character.

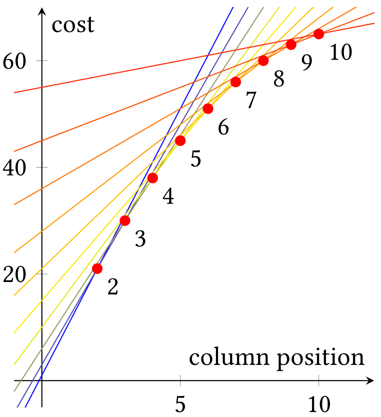
$$\begin{aligned}
C'[D_n, Z_{[]}] &= C'[\text{chr}(n) \mathbin{\texttt{<>}} D_{n-1} \mathbin{\texttt{<>}} \text{chr}(n), Z_{[]}] \mathbin{\texttt{<|>}} C'[D_{n-1}, Z_{[]}] \\
&= C'[\text{chr}(n), C'[D_{n-1}, C'[\text{chr}(n), Z_{[]}]]] \mathbin{\texttt{<|>}} C'[D_{n-1}, Z_{[]}] \\
&= C'[\text{chr}(n), C'[D_{n-1}, Z_{[n]}]] \mathbin{\texttt{<|>}} C'[D_{n-1}, Z_{[]}] \\
C'[D_{n-1}, Z_{[]}] &= C'[\text{chr}(n-1), C'[D_{n-2}, Z_{[n-1]}]] \mathbin{\texttt{<|>}} C'[D_{n-2}, Z_{[]}] \\
C'[D_{n-1}, Z_{[n]}] &= C'[\text{chr}(n-1), C'[D_{n-2}, C'[\text{chr}(n-1), Z_{[n]}]]] \mathbin{\texttt{<|>}} C'[D_{n-2}, Z_{[n]}] \\
&= C'[\text{chr}(n-1), C'[D_{n-2}, Z_{[n-1,n]}]] \mathbin{\texttt{<|>}} C'[D_{n-2}, Z_{[n]}]
\end{aligned}$$

(b) Let D_n denote $\text{mk}(n)$. Yelland's C' function would transform the original document D_n into a restricted document where every aligned concatenation has a **text** as its left subdocument. However, the above derivation shows that the transformation has a combinatorial explosion. Define $Z_{[]}$ to be $\blacksquare$ in Yelland's paper and $Z_{[x,x_1,\ldots,x_n]}$ to be $C'[\text{chr}(x), Z_{[x_1,\ldots,x_n]}]$. The derivation shows that D_{n-k} is recursively transformed in 2^k different contexts.

Fig. 17. A family of documents that illustrates how the transformation C' in Yelland's algorithm does not necessarily preserve the sharing structure in the original document.

```
(* make an empty document of size n; n >= 1 *)
let rec make_dummy (n : int): doc =
  if n = 1 then text ""
  else text "" <+> make_dummy (n - 1)

(* make n lines; n >= 1 *)
let rec make_lines (n : int): doc =
  if n = 1 then text ""
  else text "" <$> make_lines (n - 1)

(* nth triangle number *)
let tri (n : int): int = n * (n + 1) / 2

let make_choices (k : int): doc =
  let rec loop (i : int): doc =
    let doc =
      (make_lines i) <+>
       text (String.make (tri (k - i + 1)) 'a')
    in if i = 1 then doc else doc <|> loop (i - 1)
  in loop k

let rec example (k : int): doc =
  let dummy = make_dummy (k * k) in
  let giant = make_choices k in
  dummy <+> giant
```

(a) The function example produces a document that triggers the worst-case time complexity of Yelland's algorithm (that we are aware of). For a fixed k, giant is a document with k choices, where the i-th choice has i lines and $\mathrm{tri}(k - i + 1)$ characters (tri is the triangle number function). Thus, its document tree size is $O(k^2)$. By concatenating giant with dummy, which is an "empty" document of size $O(k^2)$, the total document tree size is still $O(k^2)$. giant is designed so that it has k segmented linear cost functions. Thus, the aligned concatenation of dummy and giant takes $O(k^3)$. By normalizing the document size to $\hat{n}$, we obtain that the time complexity of the printer is $O(\hat{n}^{3/2})$.



(b) A plot of the piecewise linear cost function (lines along the red dots) for giant in Figure 18a with $k = 10$. The x-axis is column positions that giant will be placed. The y-axis is costs due to the placement. The plot consists of $O(k)$ segmented cost functions, where each segment is a linear function. For simplicity, we assume that (1) the page width limit is 0; (2) there is no cost for newlines; and (3) the cost for each character past the page width limit is 1. Let $\overline{d}_i$ be the i-th choice in giant. The cost function for $\overline{d}_i$ then is $C_{\overline{d}_i}(c) = ic + \mathrm{tri}(k - i + 1)$. These cost functions intersect at $c = 2, \ldots, k$. Thus, the cost function for giant is unable to prune any segments away.

Fig. 18. In Yelland's algorithm, every choiceless document (in the arbitrary-choice PPL) $\overline{d}$ has an associated *piecewise linear* cost function $C_{\overline{d}}$, where $C_{\overline{d}}(c)$ determines the cost of placing $\overline{d}$ at the column position c. A general document d similarly has an associated piecewise linear cost function C_d, which takes the minimum of the cost functions from all choiceless documents d generates. The algorithm appears to be efficient at first glance, since taking the minimum can prune away many segmented linear cost functions. However, we are able to construct a document giant of size $O(\hat{n})$ whose cost function has $O(\sqrt{\hat{n}})$ segmented linear cost functions, where $\hat{n}$ is the tree size of the document. As the time complexity of the printer is $O(\hat{n}M)$ where M is the maximum number of piecewise linear cost functions in a cost function, we obtain $O(\hat{n}^{3/2})$.

B SELECTED PROOF SKETCHES

Lemma 4.2. *The arbitrary-choice and Σ_e PPLs are functionally complete.*

Proof sketch. For the arbitrary-choice PPL with the evaluation function $\mathrm{eval}(\cdot)$, let L be any non-empty set of layouts. For each $l_i \in L$ where $l_i = [s^i_1, \ldots, s^i_{|l_i|}]$, we construct d_i to be **text** s^i_1 **<$>** ... **<$>** **text** $s^i_{|l_i|}$. Finally, we construct d to be d_1 **<|>** ... **<|>** $d_{|L|}$. We can see that $\mathrm{eval}(d) = L$. The proof for Σ_e PPL is similar, but we would use **nl** and **<>** instead of **<$>**. □

Lemma 4.3. *The traditional PPL is* not *functionally complete.*

Proof sketch. It is not possible to construct a document in the traditional PPL that evaluates to the set of layouts $E = \{[\texttt{"a"}], [\texttt{"b"}]\}$. To see why, let rmspace : $\mathcal{L} \to$ Str be a function that joins all lines in a layout into a single line, with all whitespaces removed, and we lift rmspace to work on a set of layouts (i.e., $\text{rmspace}(L) = \{\text{rmspace}(l) : l \in L\}$. Let $\text{eval}(\cdot)$ be the evaluation function for the traditional PPL. We can prove by induction that $\text{rmspace}(\text{eval}(d))$ is a singleton set for any document d. In other words, all layouts in $\text{eval}(d)$ are the same, modulo whitespaces. However, $\text{rmspace}(E) = \{\texttt{"a"}, \texttt{"b"}\}$, which is not a singleton set. Hence, by congruence, no document can render to E.

Note that there are other sets of layouts that are the same modulo whitespaces, but can't be evaluated to by the traditional PPL. This is due to how the `group` construct is the only way to express choices, but a choice that it creates is very simple: collapsing everything into a single line. Therefore, *synchronized* differences of spacing across multiple lines can't be accounted for. □

Lemma 4.4. *For each construct* **F** *in* {`text`, `<>`, `nl`, `<|>`}, Σ_e *without* **F** *is not functionally complete.*

Proof sketch. It is not possible to construct a document in each language in question that evaluates to the following set of layouts

Σ_e *without* `text`. $\{[\texttt{"a"}]\}$, because all we can produce is whitespaces.

Σ_e *without* `<>`. $\{[\texttt{"a"}, \texttt{"b"}, \texttt{"c"}]\}$, because all we can produce is at most two lines.

Σ_e *without* `nl`. $\{[\texttt{"a"}, \texttt{"b"}]\}$, because all we can produce is a single line.

Σ_e *without* `<|>`. $\{[\texttt{"a"}], [\texttt{"b"}]\}$, because all we can produce is a single layout. □

Theorem 4.12. Σ_e *can define every construct in the traditional and arbitrary-choice PPLs.*

Proof sketch. Following syntactic abstractions can be used to define the constructs:

- `group` is definable by $\mathbf{M}(\alpha_1) = \alpha_1$ `<|>` `flatten` α_1
- `<$>` is definable by $\mathbf{M}(\alpha_1, \alpha_2) = \alpha_1$ `<>` `nl` `<>` α_2.
- `<+>` is definable by $\mathbf{M}(\alpha_1, \alpha_2) = \alpha_1$ `<>` `align` α_2.

The rest of the constructs is already in Σ_e. □

Theorem 4.16. *Given a PPL* Σ *and a construct* **F**, *if there exists two documents* d_1 *and* d_2 *in* Σ *and a relation* R *such that* $E^{\Sigma}_{R}(d_1, d_2)$, *but* $\neg E^{\Sigma\cup\{\mathbf{F}\}}_{R}(d_1, d_2)$, *then* **F** *is not definable in* Σ.

Proof sketch. Let $\text{eval}_a(\cdot)$ and $\text{eval}_b(\cdot)$ denote the evaluation functions for Σ and $\Sigma \cup \{\mathbf{F}\}$, respectively. We prove the contraposition. Assuming that **F** is definable in Σ, we need to prove that for any d_1, d_2, and R, $E^{\Sigma}_{R}(d_1, d_2)$ implies $E^{\Sigma\cup\{\mathbf{F}\}}_{R}(d_1, d_2)$. Let d_1, d_2, and R be arbitrary. We suppose that for all context C in Σ, $R(\text{eval}_a(C(d_1)), \text{eval}_a(C(d_2)))$ holds, and need to prove that for all context C in $\Sigma \cup \{\mathbf{F}\}$, $R(\text{eval}_b(C(d_1)), \text{eval}_b(C(d_2)))$ holds.

Let C be a context in $\Sigma \cup \{\mathbf{F}\}$. Because **F** is definable in Σ, we can perform a syntactic expansion on C to obtain a context C^* in Σ such that $\text{eval}_a(C^*(d)) = \text{eval}_b(C(d))$ for all document d in Σ. Hence, it suffices to prove that $R(\text{eval}_a(C^*(d_1)), \text{eval}_a(C^*(d_2)))$ holds, but this is our hypothesis (instantiated with C^*). □

Theorem 4.17. *The following is true:*

- `<>` *is not definable in the arbitrary-choice PPL.*
- `nest` *is not definable in the arbitrary-choice PPL.*
- `group` *is not definable in the arbitrary-choice PPL.*
- `<+>` *is not definable in the traditional PPL.*

Proof sketch. In each proof, we need to show that **F** is not definable in Σ, where **F** and Σ are the construct and the PPL in question. We do so by providing a counterexample (which is initially discovered by using Rosette [Porncharoenwase et al. 2022; Torlak and Bodik 2014]), which consists of documents d_1 and d_2, and the relation R. By induction, it can be shown that $E_R^{\Sigma}(d_1, d_2)$. We will further provide a counterexample context to show that $\neg E_R^{\Sigma \cup \{\mathbf{F}\}}(d_1, d_2)$. By Theorem 4.16, this suffices to show that **F** is not definable in Σ.

`<>` is not definable in the arbitrary-choice PPL. Given width from Example 4.8, the counterexample is d_1 = `text "a" <$> text "bb"`, d_2 = `text "aa" <$> text "bb"`, and $R = \{(L_a, L_b) : \text{width}(L_a) = \text{width}(L_b)\}$. In particular, with $C(\alpha)$ = `text "c" <>` α, we have that $\text{width}(\text{eval}(d_1)) = \{2\}$, but $\text{width}(\text{eval}(d_2)) = \{3\}$.

`nest` is not definable in the arbitrary-choice PPL. Given width from Example 4.8, the counterexample is d_1 = `text "bb" <$> text "a"`, d_2 = `text "cc" <$> text "bb" <$> text "a"`, and $R = \{(L_a, L_b) : \text{width}(L_a) = \text{width}(L_b)\}$. In particular, with $C(\alpha)$ = `nest 1` α, we have that $\text{width}(\text{eval}(d_1)) = \{2\}$, but $\text{width}(\text{eval}(d_2)) = \{3\}$.

`group` is not definable in the arbitrary-choice PPL. Let $\text{maxa} : \mathcal{L} \to \mathbb{N}$ finds the maximum number of the character "a" in lines of the layout, and we lift maxa to work on a set of layouts. The counterexample is d_1 = `text "a" <$> text "a"`, d_2 = `text "a" <$> text "a" <$> text "a"`, and $R = \{(L_a, L_b) : \text{maxa}(L_a) = \text{maxa}(L_b)\}$. In particular, with $C(\alpha)$ = `group` α, we have that $\text{maxa}(\text{eval}(d_1)) = \{1, 2\}$, but $\text{maxa}(\text{eval}(d_2)) = \{1, 3\}$.

`<+>` is not definable in the traditional PPL. Let $\text{spaces} : \mathcal{L} \to \mathbb{N}$ counts the number of spaces in a layout (not counting newlines), and we lift spaces to work on a set of layouts. The counterexample is d_1 = `text "a"`, d_2 = `text "aa"`, and $R = \{(L_a, L_b) : \text{spaces}(L_a) = \text{spaces}(L_b)\}$. In particular, with $C(\alpha) = \alpha$ `<+> (text "b" <> nl <> text "c")`, we have that that $\text{spaces}(\text{eval}(d_1)) = \{1\}$, but $\text{spaces}(\text{eval}(d_2)) = \{2\}$. □

Lemma 4.18. *If Σ is not functionally complete, but $\Sigma \cup \{\mathbf{C}\}$ is, then* **C** *is not definable in Σ.*

Proof sketch. Because Σ is not functionally complete, there is a set of layouts L^* that can't be evaluated to by any document in Σ. Since $\Sigma \cup \{\mathbf{C}\}$ is functionally complete, there is a document d^* (which necessarily contains **C**) that evaluates to L^*. Let d_1 and d_2 be any document in Σ, and $R = (2^{\mathcal{L}} \times 2^{\mathcal{L}}) \setminus \{(L^*, L^*)\}$. Then $E_R^{\Sigma}(d_1, d_2)$ holds trivially. However, with $C(\alpha) = d^*$, we have that $\neg E_R^{\Sigma \cup \{\mathbf{C}\}}(d_1, d_2)$. This concludes the proof that **C** is not definable in Σ. □

Theorem 4.19. *For any construct* **F** *of Σ_e,* **F** *is not definable in $\Sigma_e \setminus \{\mathbf{F}\}$.*

Proof sketch. The proofs for `text`, `nl`, `<>`, and `<|>` are applications of Lemma 4.2, Lemma 4.4, and Lemma 4.18. The proofs for `nest`, `flatten`, and `align` are just like how we proved Theorem 4.17 for `nest`, `group`, and `<+>`. □

Theorem 5.11. *The time complexity of Π_e is $O(nW_{\mathcal{F}}^4)$ where n is the DAG size of the document.*

Proof sketch. The most expensive operation in the printer is concatenation (via ConcatRSSet). The operation resolves the left sub-document, resulting in a measure set whose size is at most $W_{\mathcal{F}}$ according to Lemma 5.9. It then resolves the right sub-document in at most $W_{\mathcal{F}}$ different contexts. Thus, there are at most $W_{\mathcal{F}}^2$ different measures from the right sub-document that the printer needs to concatenate and prune.

Consider $\langle d, c, i \rangle \Downarrow_{\mathbb{RS}} S$. d can range over n different values. c and i can range over $W_{\mathcal{F}}$ different values that are under $W_{\mathcal{F}}$. Hence, there are $O(nW_{\mathcal{F}}^2)$ different contexts under the computation width limit. Multiplying this with the maximum units of computation in the previous paragraph, we obtain that the time complexity due to resolving within $W_{\mathcal{F}}$ is $O(nW_{\mathcal{F}}^4)$, assuming that the resolver reuses memoized measure set under the same context.

When d is printed beyond $W_{\mathcal{F}}$, however, it can be fully resolved for at most once, because:

(1) While we would resolve both sub-documents of choice nodes, they would be all tainted, due to Lemma 5.10. Because all tainted measure sets are promises, all computations are delayed. The merge operation then chooses only one tainted measure set as the result, discarding the other one.
(2) The document is properly shared, so under a path a document is encountered at most once.

As a result, the time complexity due to printing over $W_{\mathcal{F}}$ is simply $O(n)$. Combining both parts, we obtain that the time complexity of Π_e is $O(nW_{\mathcal{F}}^4)$. □

Theorem 5.12. *If a document d is in the arbitrary-choice PPL, Π_e can print d in $O(nW_{\mathcal{F}}^3)$.*

Proof sketch. In the arbitrary-choice PPL, $c = i$ is (mostly) maintained throughout the printing. Hence, there is one less dimension to consider, leading to the time complexity of $O(nW_{\mathcal{F}}^3)$. □

C DISCUSSION

In this section, we broadly discuss the design of our work.

C.1 Additional constructs

SnowWhite supports additional constructs `fail` and `flat`. The Racket SnowWhite further supports additional constructs `full` and `cost`. These constructs are out of scope for the paper, and we leave their formalization as future work.

Failure. The `fail` construct widens to the empty set, thus introducing the possibility that a printing could fail. Furthermore, it is the identity for the operation `<|>`. `fail` makes Σ_e more expressive because it is impossible to make Σ_e evaluate to the empty set. In this sense, it could be said that Σ_e is not truly "functionally complete," but Σ_e with `fail` is. Supporting `fail` can be done via rewriting rules: every document with `fail` can be normalize to a semantically-equivalent document without `fail`, or to a single `fail`. Hence, there is no need to modify the core printer to add the construct. However, the construct presents a challenge for us to state the correctness of the printer, because there is no longer a one-to-one mapping between widened choiceless document and evaluated layout.

Flatness. `fail` paves way to support another construct `flat` d, which is also implemented in Bernardy [2017b]'s practical printer as *single line*[11]. The construct adds a constraint to d to eliminate layouts with multiple lines. This is especially very useful as an addition to the arbitrary-choice PPL, since it can eliminate the "ladder" layouts that result from multiple aligned concatenations. Π_e can support `flat` in the same way it supports `flatten` (Section 5.8), although we would replace `nl` with `fail` instead of a single space. Note that `flat` interacts with `flatten` in the intended way. For example, `flat (flatten (text "\n"))` evaluates to {"␣"}, but `flatten (flat (text "\n"))` evaluates to the empty set.

Fullness. `full` d marks d as *full*, which means there must be no more text afterward in the same line. The construct is especially useful for code formatters for languages with inline comments, as it is illegal to collapse a piece of code after a comment. A simpler variant of `full` is also implemented in Yelland's printer for the R code formatter [Yelland 2015]. Unlike `fail` and `flat`, where we can

[11]This is one of the extra features that is supported in Bernardy's practical printer but not by the naïve printer.

workaround to avoid the core printer modification, `full` requires involved changes to the core printer.

- The measure set definition is now required to recognize the empty set (where we prefer a tainted measure set over an empty set).
- The resolver would consume additional two boolean arguments, which indicate the fullness status before and after the document.
- Merging two tainted measure sets must now keep both tainted measure sets, and we may need to try both if the first one resolves to the empty set.

To keep the time complexity of the printer $O(nW_{\mathcal{F}}^4)$, we rely on the fact that "emptiness" in resolving (that is, resolving to the empty set) is independent from column positions and indentation levels. Thus, even though we now need to try many tainted measure sets, a document can be tried at most four times (one for each before and after fullness statuses), which bounds the time complexity.

Cost. `cost` C d adds a cost C to measures due to d. This construct is not expressive in the traditional sense, as it does not affect layout results. However, it allows us to make *weighted* choices, so that we can prefer one style over another when all else is equal. Due to the flexibility of the cost factory, it is even possible to make multidimensional weights.

C.2 Safety

As shown in the proof of Lemma 4.3, the traditional PPL is not functionally complete because all layouts must have the same content, modulo whitespaces. While this property is restrictive for many tasks as elaborated in the paper, it does provide a sort of safety guarantee that the layouts will not be wildly different. `<|>`, however, allows us to violate this property. In fact, some arbitrary-choice printers (e.g. a prototype of Bernardy's printer [Bernardy 2015]) *intend* that `<|>` should be restricted to maintain the property. Similarly, the inclusion of `fail`, `flat`, or `full` makes it possible to evaluate to an empty set, but the PPLs without the essence of `fail` provide a safety guarantee that an evaluation will never result in an empty set. Generally, the more expressive a language is, the more properties it will break, and the more burden will be put on the users to carefully use the constructs.

We argue that the spirit of these safety properties can still be accomplished in PPLs with a functionally complete core. One possible approach is similar to Wadler's treatment of `<|>` and `group`: define high-level, "safe" constructs with just enough expressiveness to solve a domain-specific task, based on the core, "unsafe" constructs, and then hide these "unsafe" constructs away from the external interface. For example, one may hide `<|>`, and instead provide groupParen(d) = `(text "(" <>` d `<> text ")") <|> flatten` d, which evaluates to either d with parentheses wrapped around, or the flattened d. The language as defined by the external interface is no longer functionally complete, but enjoys the property that all layouts are the same modulo whitespaces and parentheses. Another possible approach is to export the core, "unsafe" constructs, but perform a static analysis to make sure that the document satisfies the intended safety property.

In any case, the expressive core constructs are what enable the advanced features that may be required by tasks. Thus, our view is that an expressive printer is the key. We should start with an expressive albeit unsafe printer, rather than a safe but non-expressive one.

C.3 Memoization

While memoization is important to guarantee that Π_e will not take exponential time, it is also the performance bottleneck when the input document is large, due to too much memory allocation. In SnowWhite, we employ a heuristic to reduce memory allocation by adding a metadata *memoization weight* to each document node, which counts how long memoization has not been performed on descendant nodes. When the weight reaches a limit (set to 6 in our implementation), we perform memoization on the node, and reset the weight to 0. This can significantly speeds up the performance of SnowWhite in some large documents.

C.4 Fusing resolving and rendering

One optimization in SnowWhite is to fuse resolving a document to a measure set and rendering of a choiceless document to a layout together. This is done by replacing the doc component in a measure with a *token function*, which consumes a list of rendered tokens *after* the document is placed, and returns a new list of rendered tokens. A similar technique was employed by Podkopaev and Boulytchev [2015]. For example, the token function on printing a `nl` at the indentation level i would be toks $\mapsto$ ["\n", "␣" $\times\ i$]@ toks. As another example, the token function on printing a `<>` would be toks $\mapsto f_1(f_2(\text{toks}))$, where f_1 and f_2 are the token functions from the left and right measures to be concatenated together. The printer would then pick a measure with the lowest cost, and invoke the corresponding token function with an empty list to obtain the full list of rendered tokens, which could then be displayed to users.

C.5 Handling bias in presence of taintedness

In Section 5, we see that the merge operation and thus the `<|>` operator is left-biased in presence of taintedness. When exceeding $W_{\mathcal{F}}$ is unavoidable, all text could be put in one line in the worst case if all left sub-documents use the "horizontal styling"! The proper solution is to increase $W_{\mathcal{F}}$. However, SnowWhite also implements a heuristic to infer a sub-document with the "vertical styling." The heuristic adds a metadata *estimated number of lines* (estl) to each document node. For example, we would have that estl(`nl`) = 1, estl(`text` s) = 0, estl(d_a `<>` d_b) = estl(d_a) + estl(d_b), and estl(d_a `<|>` d_b) = max(estl(d_a), estl(d_b)). SnowWhite then uses a document with a larger estl as the left sub-document in choice documents.

C.6 Rewriting rules and cost factory

Our cost factory is minimally specified and only concerns with the correctness of the core printer. Therefore, if one wishes to employ rewriting rules to transform a document to another semantically-equivalent document, one must make sure that the specified cost factory admits the rewriting rules. For example, if we want to rewrite a document D `<> text ""` to simply D for any D, it would require, at the very least, that the $+_{\mathcal{F}}$ operation has an identity element, and that the cost of `text ""` is the identity element. Associativity and commutativity of $+_{\mathcal{F}}$ are another examples of properties that ordinary cost factories should satisfy.

C.7 Rewriting rules and partial evaluation

Similar to how we can perform partial evaluation in programming languages, we can also perform partial evaluation in PPL using rewriting rules. For example, a concatenation of two `text` can be partially evaluated to a single `text` right away. However, this partial evaluation must be done with care to still preserve the sharing structure, since unconstrained rewriting may unfold the DAG structure into a tree. It is also worth noting that the partial evaluation may not necessarily preserve the semantics in presence of taintedness. For example, one may want to reduce a `nest` n `(text` s`)` to `text` s for any n and s, but when $n > W_{\mathcal{F}}$, the document will definitely resolves to a tainted measure set, while the partially evaluated one does not necessarily[12].

[12]One may argue, however, that this semantic change is acceptable, because the change is for the better.