7.4

证明内积的正整数幂函数：

$$K(x,z)=(x\cdot z)^p$$

是正定核函数，这里p是正整数，$x,z\in R^n$。

> **定理7.5（正定核的充要条件）** 设$K:\mathcal{X}\times\mathcal{X}\rightarrow R$是对称函数，则$K(x,z)$为正定核函数的充要条件是对任意$x_i\in\mathcal{X},i=1,2,\cdots,m$，$K(x,z)$对应的Gram矩阵：
>
> $$K=[K(x_i,x_j)]_{m\times m}$$
>
> 是半正定矩阵。

> **定义7.6（核函数）** 设$\mathcal{X}$是输入空间（欧式空间R^n的子集或离散集合），又设$\mathcal{H}$为特征空间（希尔伯特空间），如果存在一个从$\mathcal{X}$到$\mathcal{H}$的映射
>
> $$\phi(x):\mathcal{X}\rightarrow\mathcal{H}$$
>
> 使得对所有$x,z\in\mathcal{X}$，函数$K(x,z)$满足条件
>
> $$K(x,z)=\phi(x)\cdot\phi(z)$$
>
> 则称$K(x,z)$为核函数，$\phi(x)$为映射函数，式中$\phi(x)\cdot\phi(z)$为$\phi(x)$和$\phi(z)$的内积。

1. 当$p=1$时，$K(x,z)=x\cdot z=\Phi(x)\cdot\Phi(z)$，其中$\Phi(x)=x$，所以$K(x,z)$是正定核函数
2. 若$p=k(k>=1)$时 $K(x,z)=(x\cdot z)^k$是正定核函数，则存在一个输入空间为R^n，R^n到$\mathcal{H}$的映射 $\Phi_k(x)$，使得对所有$x,z\in R^n$，函数$K(x,z)=(x\cdot z)^k$满足条件$K(x,z)=\Phi_k(x)\cdot\Phi_k(z)$，其中$\Phi_k(x)$为映射函数

设$\Phi_k(x)=(f_1(x),f_2(x),\cdots,f_m(x))^T$，其中$x=(x^{(1)},x^{(2)},\cdots,x^{(n)})^T$，$m$为映射后的维度

当$p=k+1$时，$K(x,z)=(x\cdot z)^{k+1}$，可得

$$\begin{aligned}
K(x,z)&=(x\cdot z)^{k+1}\\
&=(x\cdot z)^k(x\cdot z)\\
&=(\phi_k(x)\cdot\phi_k(z))(x\cdot z)\\
&=(f_1(x)f_1(z)+f_2(x)f_2(z)+\cdots+f_m(x)f_m(z))(x^{(1)}z^{(1)}+x^{(2)}z^{(2)}+\cdots+x^{(n)}z^{(n)})\\
&=f_1(x)f_1(z)(x^{(1)}z^{(1)}+x^{(2)}z^{(2)}+\cdots+x^{(n)}z^{(n)})\\
&\quad+f_2(x)f_2(z)(x^{(1)}z^{(1)}+x^{(2)}z^{(2)}+\cdots+x^{(n)}z^{(n)})+\cdots\\
&\quad+f_m(x)f_m(z)(x^{(1)}z^{(1)}+x^{(2)}z^{(2)}+\cdots+x^{(n)}z^{(n)})\\
&=(f_1(x)x^{(1)})(f_1(z)z^{(1)})+(f_1(x)x^{(2)})(f_1(z)z^{(2)})+\cdots+(f_1(x)x^{(n)})(f_1(z)z^{(n)})\\
&+(f_2(x)x^{(1)})(f_2(z)z^{(1)})+(f_2(x)x^{(2)})(f_2(z)z^{(2)})+\cdots+(f_2(x)x^{(n)})(f_2(z)z^{(n)})\\
&+\cdots\\
&+(f_m(x)x^{(1)})(f_m(z)z^{(1)})+(f_m(x)x^{(2)})(f_m(z)z^{(2)})+\cdots+(f_m(x)x^{(n)})(f_m(z)z^{(n)})
\end{aligned}$$

$\therefore$

$$\phi_{k+1}(x)=(f_1(x)x^{(1)},f_1(x)x^{(2)},\cdots,f_1(x)x^{(n)},f_2(x)x^{(1)},f_2(x)x^{(2)},\cdots,f_2(x)x^{(n)},f_m(x)x^{(1)},\cdots,f_m(x)x^{(n)})^T$$

$\therefore K(x,z)=(x\cdot z)^{k+1}$是正定核函数

由数学归纳法可知，$K(x,z)=(x\cdot z)^{p}$是正定核函数

习题2

已知错分误差$R(f)$定义如下,$Y=\{-1,1\}$,求 $R(f)$的级小值点.

Definition 9.1 Let ρ be a probability distribution on $Z := X \times Y$. The *misclassificatio*
$\mathcal{R}(f)$ for a classifier $f : X \to Y$ is defined to be the probability of a wrong prediction, i.e., the
measure of the event $\{f(x) \neq y\}$,

$$\mathcal{R}(f) := \underset{z\in Z}{\operatorname{Prob}}\{f(x)\neq y\} = \int_X \underset{y\in Y}{\operatorname{Prob}}(y\neq f(x)\mid x)d\rho_X. \tag{9.1}$$

$$\begin{aligned}
R(f) &= \int_X Prob_{y\in Y}(y=1,f(x)=-1|x)d\rho(x)+\int_X Prob_{y\in Y}(y=-1,f(x)=1|x)d\rho(x)\\
&= \int_X (Prob_{y\in Y}(y=1|x)(1-0.5(f(x)+1))+Prob_{y\in Y}(y=-1|x)(0.5(f(x)+1)))d\rho(x)\\
&= \frac{1}{2}\int_X ((Prob_{y\in Y}(y=1|x)+Prob_{y\in Y}(y=-1|x))+f(x)+(Prob_{y\in Y}(y=-1|x)-Prob_{y\in Y}(y=1|x)))d\rho(x)\\
&= \frac{1}{2}\int_X (1+f(x)(Prob_{y\in Y}(y=-1|x)-Prob_{y\in Y}(y=1|x)))d\rho(x)
\end{aligned}$$

$\therefore$ $R(f)$的极小值点为

$$f_{min}(x)=\begin{cases}1 & \text{if } Prob_{y\in Y}(y=-1|x)-Prob_{y\in Y}(y=1|x)\leq 0\\ -1 & \text{if } Prob_{y\in Y}(y=-1|x)-Prob_{y\in Y}(y=1|x)>0\end{cases}$$