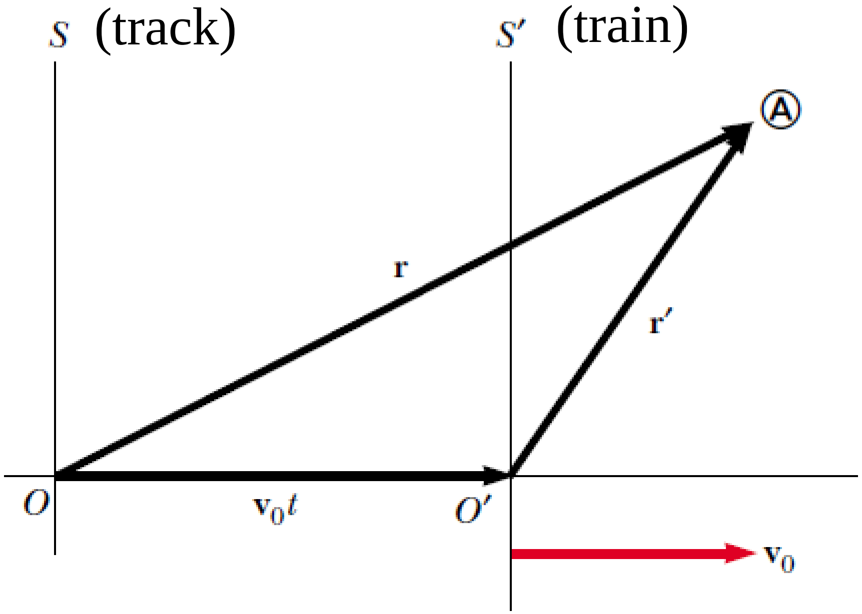
General Physics I

Lecture 20: Lorentz Transformation

Outline

- Lorentz transformation
- The invariant interval
- Minkowski diagram; Geometrical meaning of Lorentz transformation

Galilean Transformation



- If two events happen at the same place *O'* in the train frame, then in the track frame the distance |*OO'*| between them is equal to the time between them, multiplied by the speed v_0 of the train along the tracks. Familiar? (Relative velocity v_0 of frames)

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t$$

$$|OO'| = v_0 t$$

- $t' = t$: Asymmetry in space and time.

Galilean Transformation

- **If two events happen at the same place *O'* in the train frame, then in the track frame the distance |*OO'*| between them is equal to the time between them, multiplied by the speed v_0 of the train along the tracks.** **Familiar? (Relative velocity v_0 of frames)**

$$|OO'| = v_0 t$$

Now, try to exchange time and space….

Beyond Galilean Transformation

- If two events happen at the same place O' in the train frame, then in the track frame the distance $|OO'|$ between them is equal to the time between them, multiplied by the speed v_0 of the train along the tracks. Familiar? (Relative velocity v_0 of frames)

- If two events happen at the same time $t' = 0$ in the train frame, then in the track frame the time T_F between them is equal to the distance D_F between them, multiplied by the speed v_0/c^2 of the train along the tracks. Familiar? Peculiar?

$$|OO'| = v_0 t$$

$$T_F = D_F \, v_0 / c^2$$

- If we choose proper units such that c = 1, space and time are interchangeable.

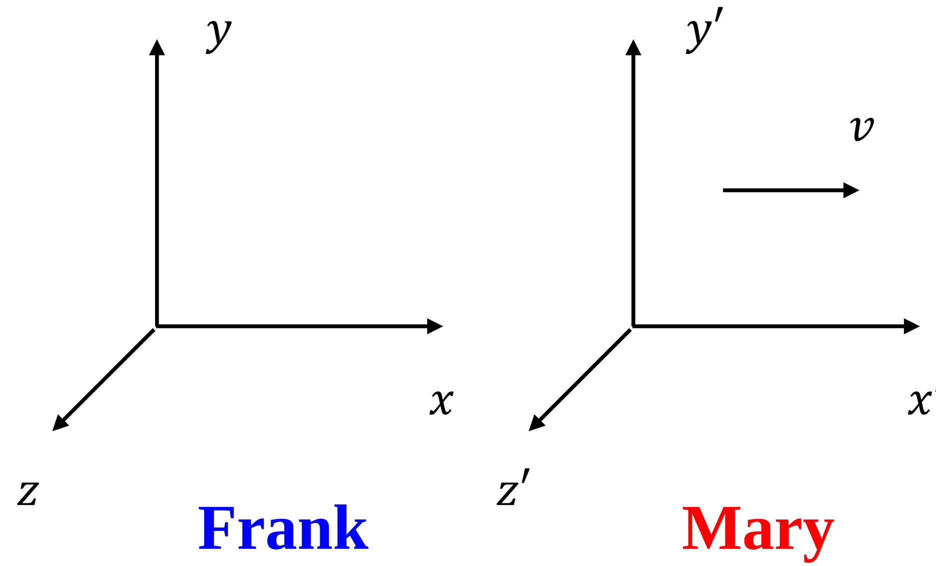
Lorentz Transformation

Expect to find a linear map

$$t = at' + bx'$$

$$x = ct' + dx'$$

where a, b, c, and d depend on v.



How to determine a, b, c, and d?

The Galilean transformation:

$$t = t' \qquad x = x' + vt'$$

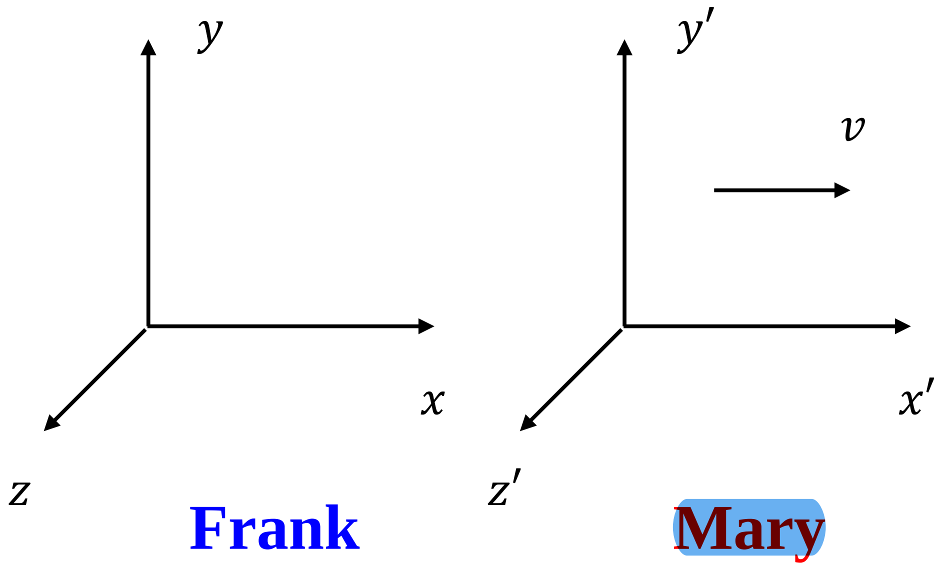
Lorentz Transformation

Expect to find a linear map

$$t = at' + bx'$$

$$x = ct' + dx'$$

where a, b, c, and d depend on v.



1. Relative speed *v*: For the origin in Frank's frame,

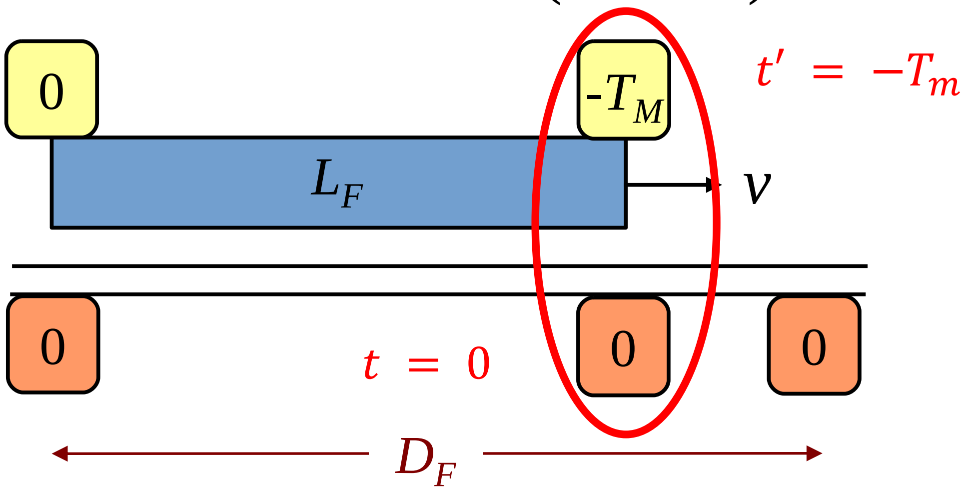
$$x = 0 \qquad x' = -vt'$$

$$\Rightarrow \quad c/d = v$$

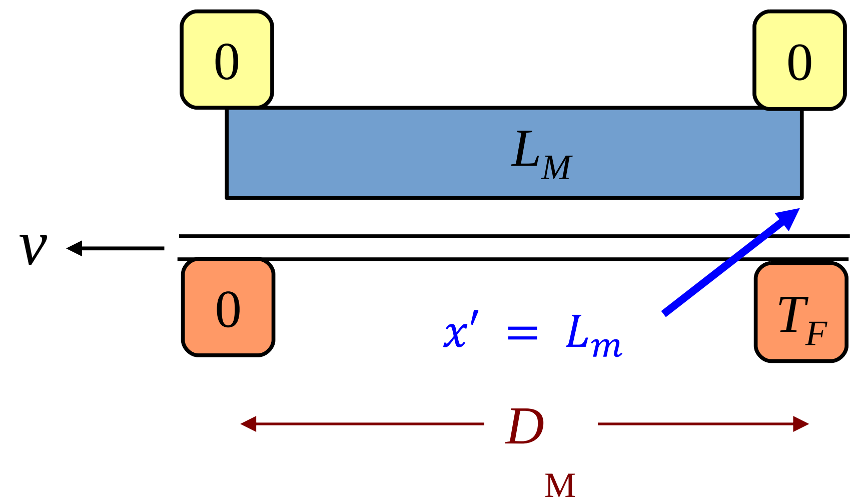
$$x = d(x' + vt')$$

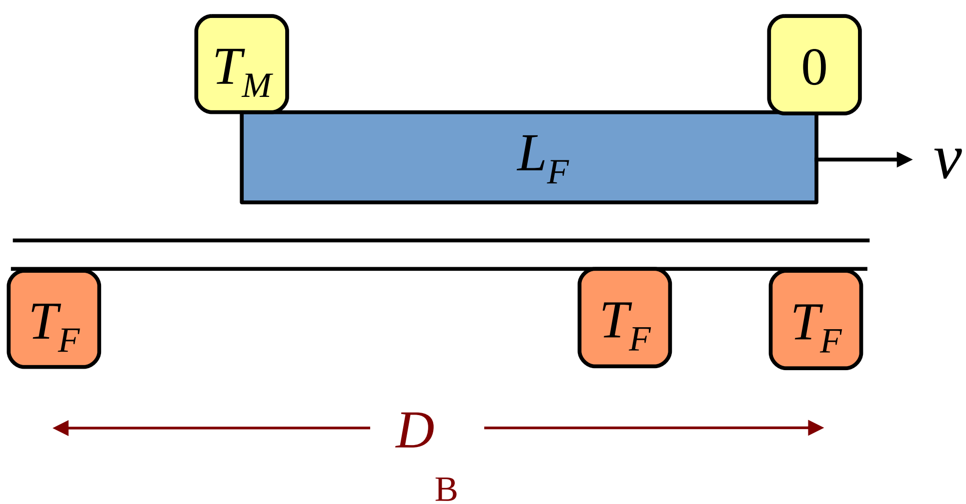
The Tale of Two Frames

Track Frame (Frank's)



Train Frame (Mary's)





$$T_F = D_F\, v/c^2$$

By the principle of relativity,

$$(-T_M) = -L_M\, v/c^2$$

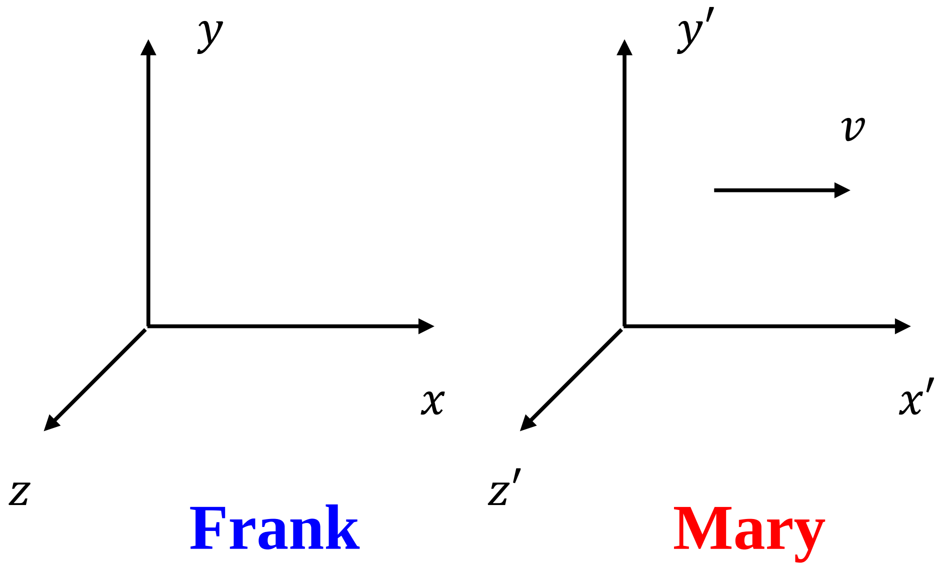
Lorentz Transformation

Expect to find a linear map

$$t = at' + bx'$$

$$x = ct' + dx'$$

where a, b, c, and d depend on v.



2. Rear clock ahead

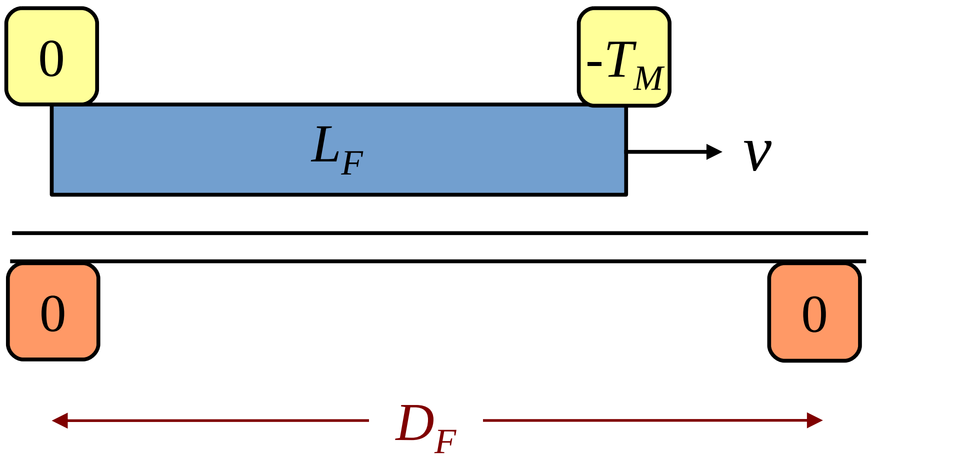
$$t = 0 \qquad t' = -x' v/c^2$$

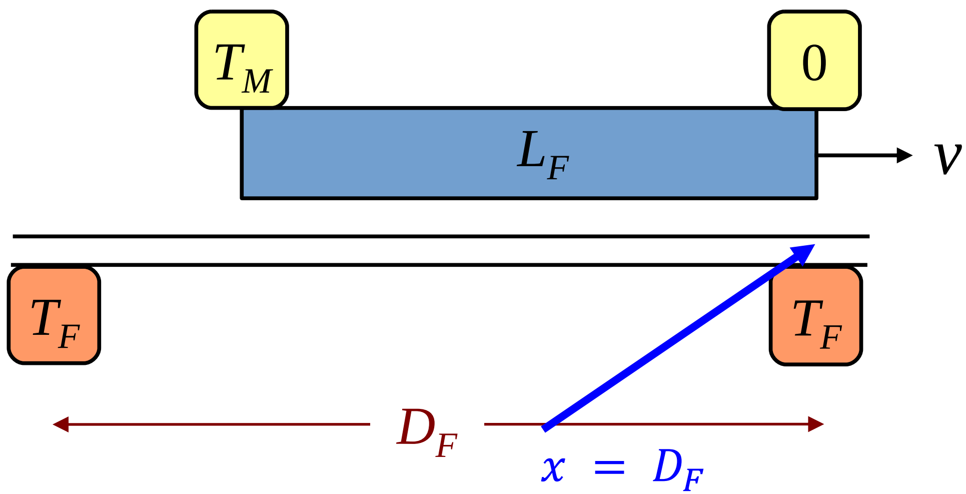
$$b/a = v/c^2$$

$$t = a(t' + x' v/c^2)$$

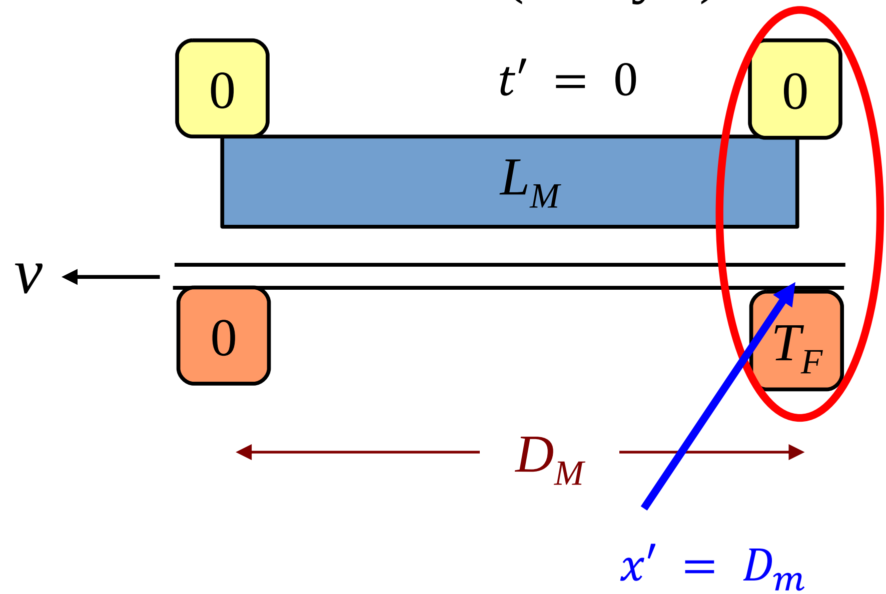
Length Contraction

Track Frame (Frank's)





Train Frame (Mary's)



$$L_M = D_M = sD_F$$

where $s = \sqrt{1 - v^2/c^2}$

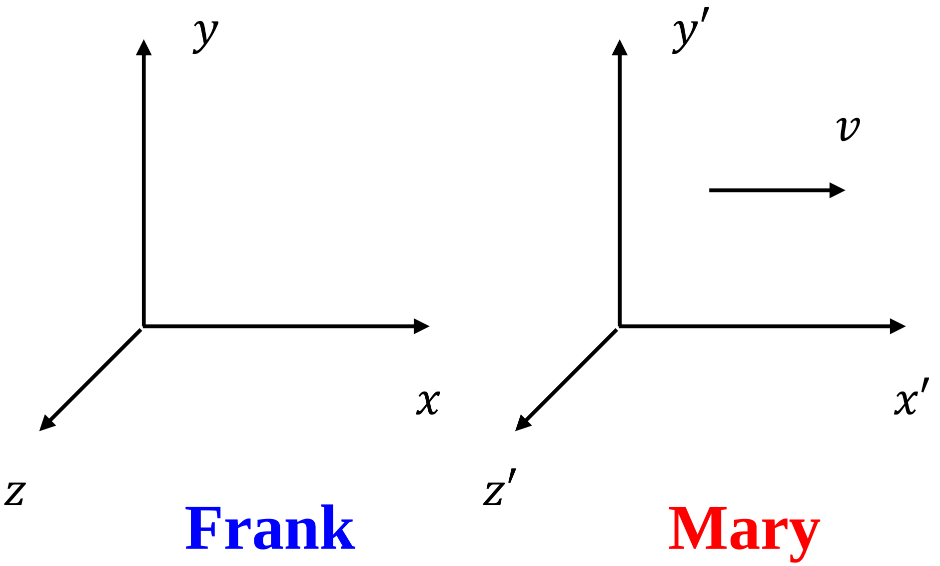
Lorentz Transformation

Expect to find a linear map

$$t = at' + bx'$$

$$x = ct' + dx'$$

where a, b, c, and d depend on v.

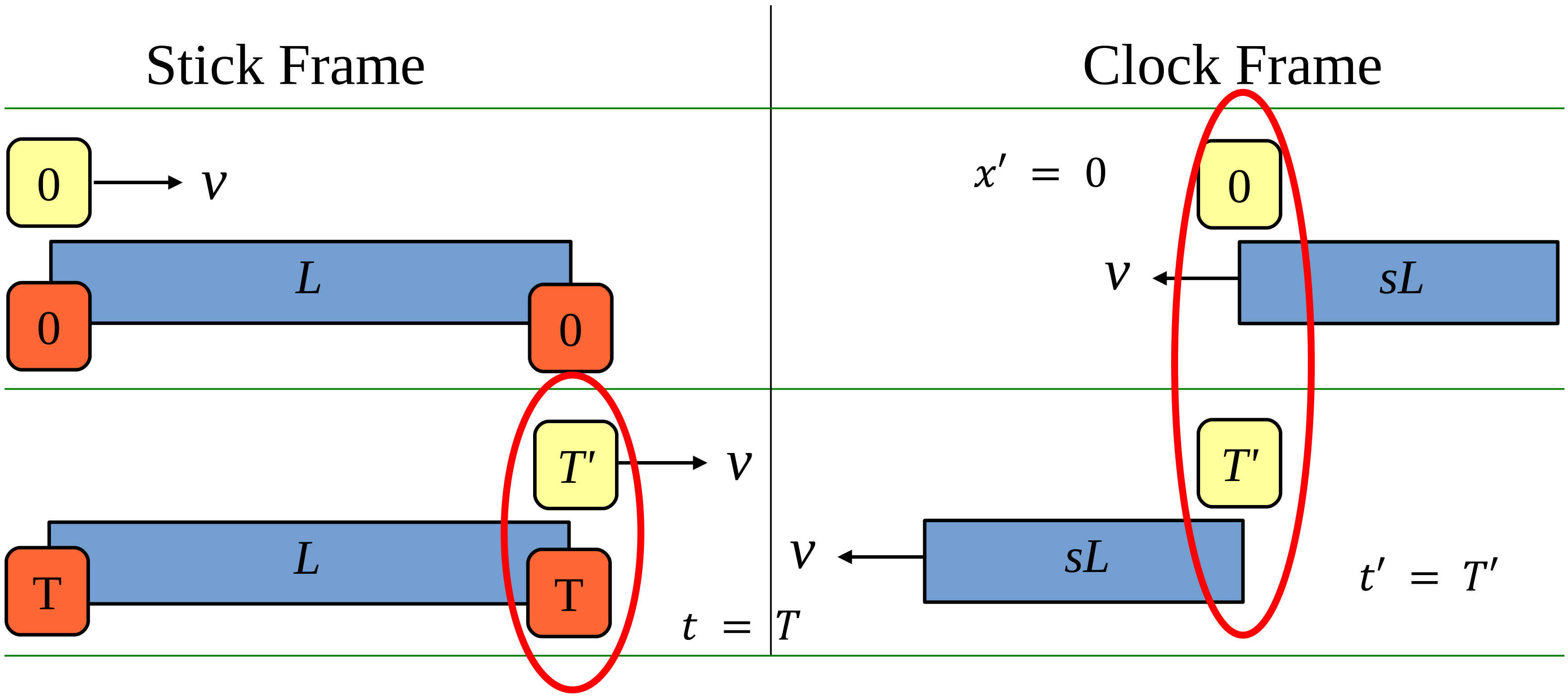


3. Length contraction

$$t' = 0 \qquad x' = sx$$

$$d = 1/s$$

Time Dilation



In the stick frame (at rest),

$T = L/v$ for clock to arrive at the right end.

In the clock frame, $T' = s(L/v)$ (time dilation)

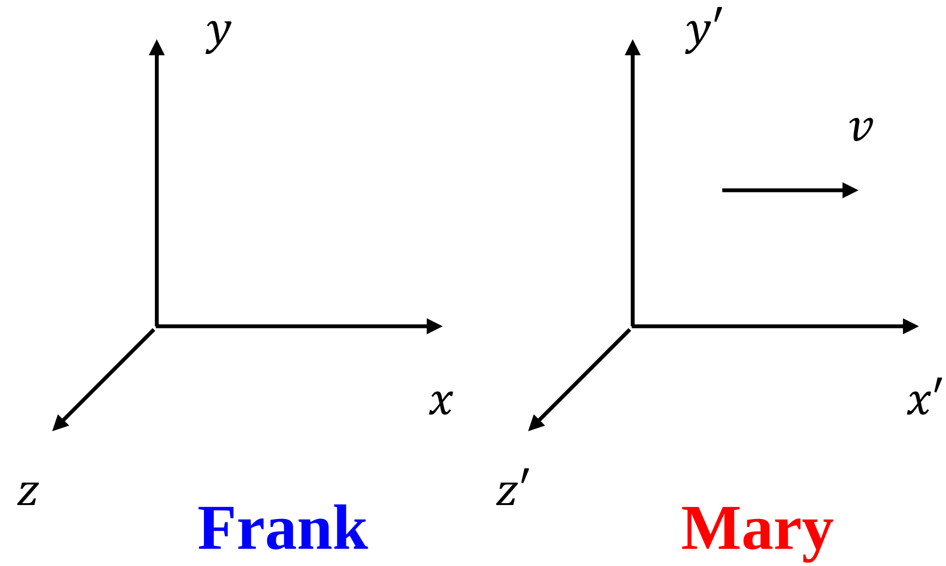
Lorentz Transformation

Expect to find a linear map

$$t = at' + bx'$$

$$x = ct' + dx'$$

where a, b, c, and d depend on v.



4. Time dilation

$$x' = 0 \qquad t' = st$$

$$\Rightarrow \qquad a = 1/s$$

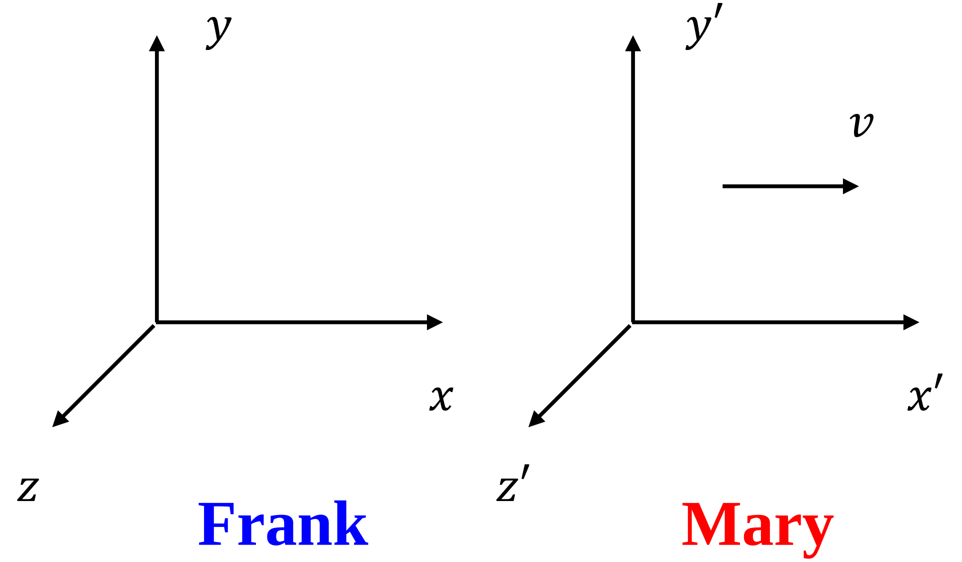
Lorentz Transformation

Expect to find a linear map

$$t = at' + bx'$$

$$x = ct' + dx'$$

where a, b, c, and d depend on v.



Relative speed *v*: $c/d = v$

Rear clock ahead: $b/a = v/c^2$

Length contraction: $d = 1/s$

Time dilation: $a = 1/s$

$$\Rightarrow$$

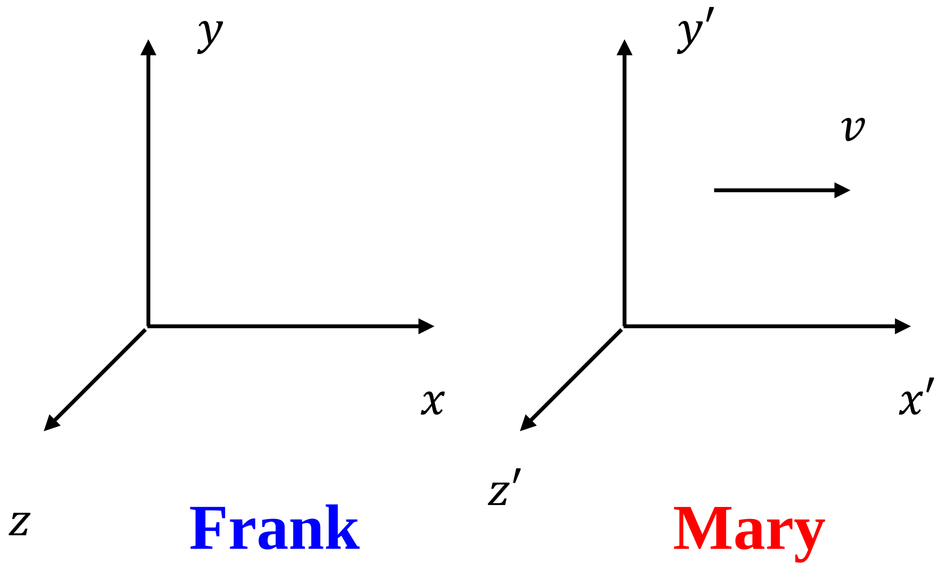
$$t = \frac{t' + vx'/c^2}{s}$$

$$x = \frac{vt' + x'}{s}$$

Lorentz Transformation

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$



Solve for t', x':

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

When $v \ll c$

$$t' = t$$

$$x' = x - vt$$

inverse transformation: v → -v

recover the Galilean transformation.

Lorentz Transformation

The complete transformation

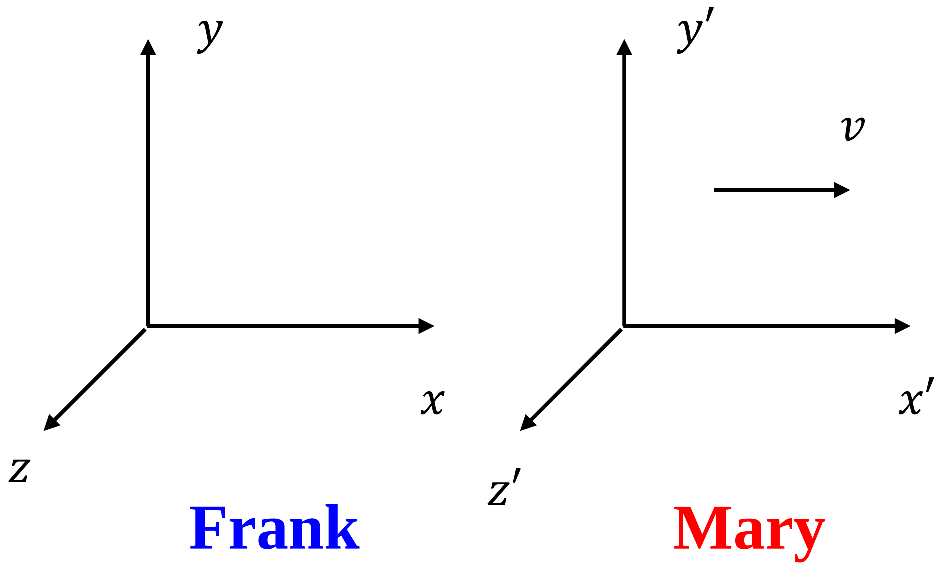
1) $t' = \gamma(t - vx/c^2)$

2) $x' = \gamma(x - vt)$

3) $y' = y$

4) $z' = z$

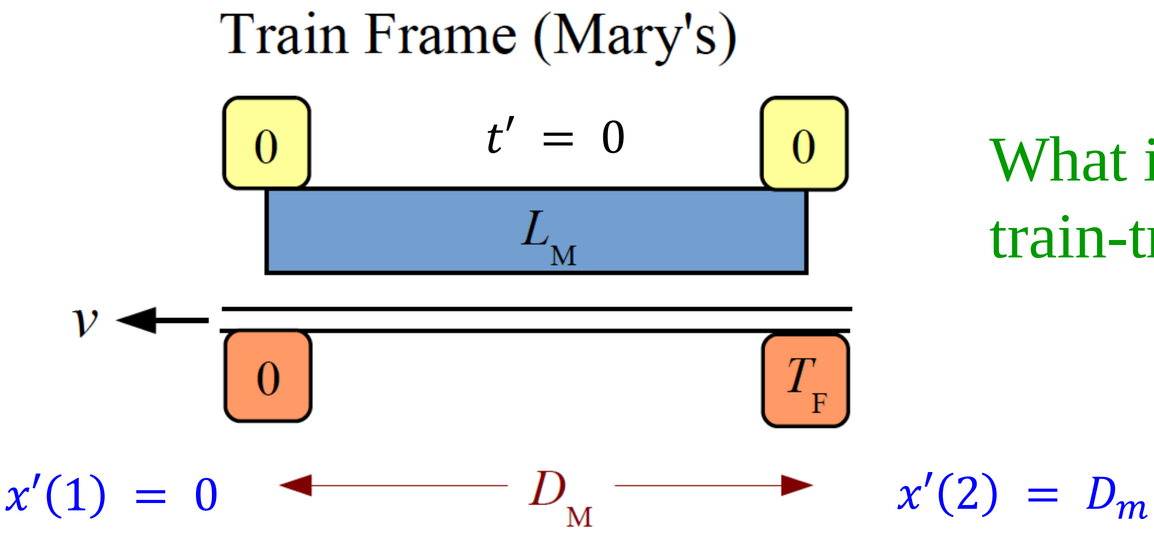
where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$



We went all the way from the postulates to the fundamental effects, and finally to the Lorentz transformation. One may also start from the postulates to derive the Lorentz transformation first, then to the relativistic effects. See, e.g., Classical Dynamics of Particles and Systems by S. T. Thornton and J. B. Marion.

Example: Length Contraction

- **Consider a rod of length l lying along the x-axis of an inertial frame K. An observer in system K' moving with uniform speed v along the x-axis measures the length of the rod in the observer's own coordinate system by determining at a given instant of time t' the difference in the coordinates of the ends of the rod, $x'(2) - x'(1)$.**

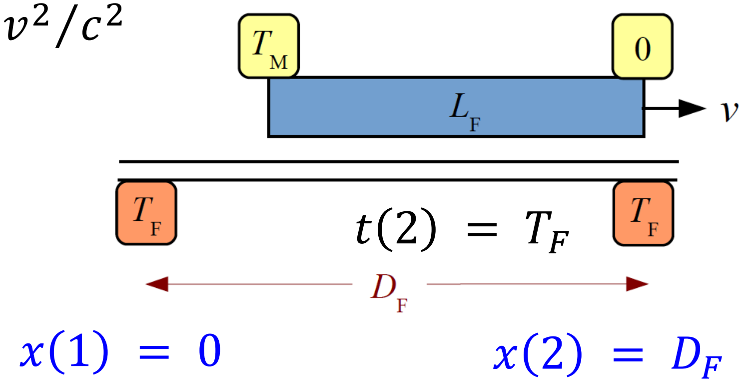


What is the "rod" in our train-track problem?

Example: Length Contraction

- **According to Lorentz transformation (Eq. 2)**

$$x'(2) - x'(1) = \frac{[x(2) - x(1)] - v[t(2) - t(1)]}{\sqrt{1 - v^2/c^2}}$$



Can you translate the equation to the language of our train-track problem?

- **where** $x(2) - x(1) = l$**. Because** $t'(2) = t'(1)$**, Eq. 1 leads to**

$$t(2) - t(1) = [x(2) - x(1)]\, v/c^2$$

Translate to: $T_F = D_F v/c^2$

Example: Length Contraction

- The length l' as measured in the K' system is therefore

$$l' = x'(2) - x'(1) = [x(2) - x(1)]\sqrt{1 - v^2/c^2} = sl$$

- where $s = \sqrt{1 - v^2/c^2}$
- Therefore, to the observer in K', objects in K appear contracted.
- Similarly, to a stationary observer in K, objects in K' also appear contracted.
- Put together, to an observer in motion relative to an object, the dimensions of objects are contracted by a factor s.

The Invariant Interval

- **Under Galilean transformation**
  - **the distance between two events is invariant, and**
  - **Newton's laws are invariant.**

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

- **Under Lorentz transformation, we postulate**
  - **the interval between events is invariant, and**
  - **we will consider dynamics in the coming lecture.**

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

- **to be proved on the next slide**

Proof of the Invariant Interval

- **For simplicity we drop Δ and ignore *y* and *z* directions.**

According to the (inverse) Lorentz transformation

$$c^2t^2 - x^2 = \frac{c^2(t' + vx'/c^2)^2}{1 - v^2/c^2} - \frac{(x' + vt')^2}{1 - v^2/c^2}$$

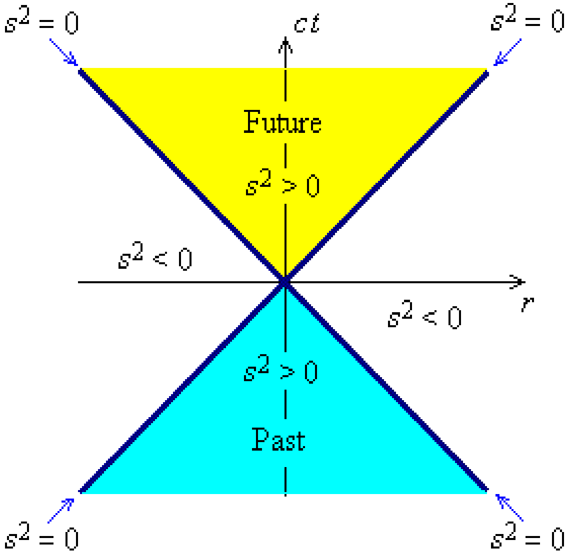
$$= \frac{t'^2(c^2 - v^2) - x'^2(1 - v^2/c^2)}{1 - v^2/c^2}$$

$$= c^2t'^2 - x'^2$$

$$\equiv s^2$$

Timelike, Spacelike & Lightlike

- **Timelike separation: $s^2 > 0$**
  - **It is possible to find a frame in which the two events happen at the same place. s/c is called the proper time.**
  - **It is possible for a particle to travel from one event to the other.**

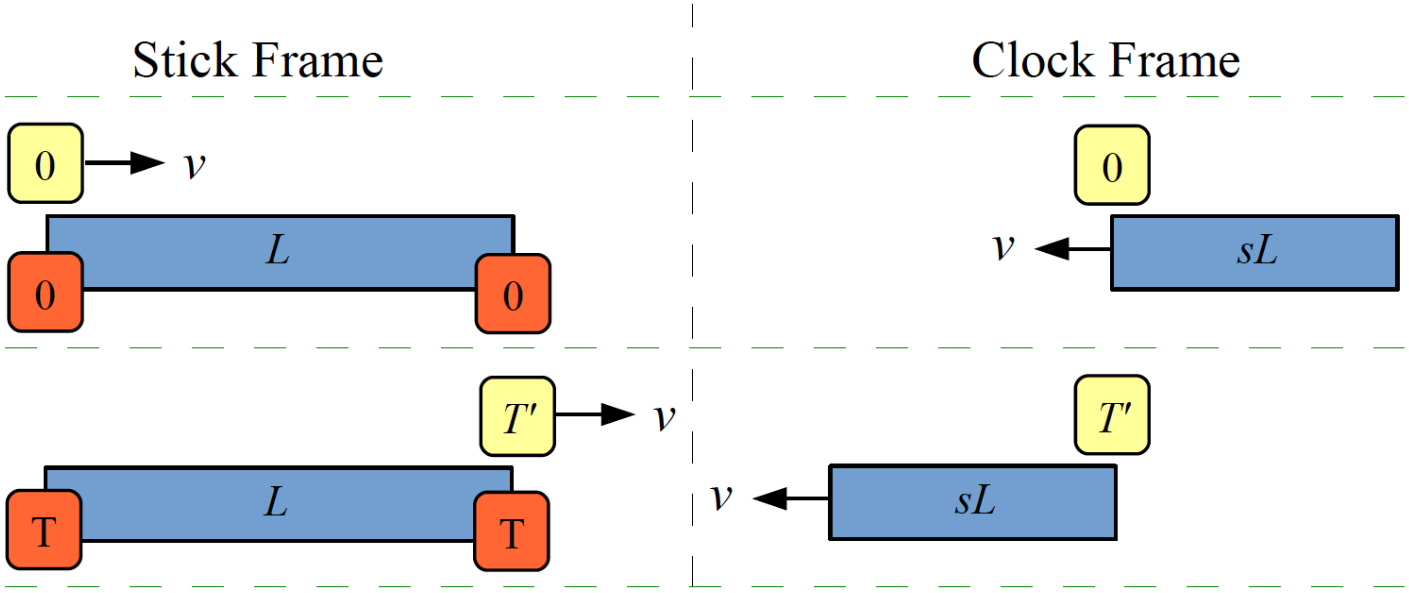


- **Spacelike separation: $s^2 < 0$**
  - **It is possible to find a frame in which the two events happen at the same time. $|s|$ is called the proper distance, or proper length.**

- **Lightlike separation: $s^2 = 0$**
  - **In every frame a photon emitted at one of the events will arrive at the other.**

Ex: Time Dilation

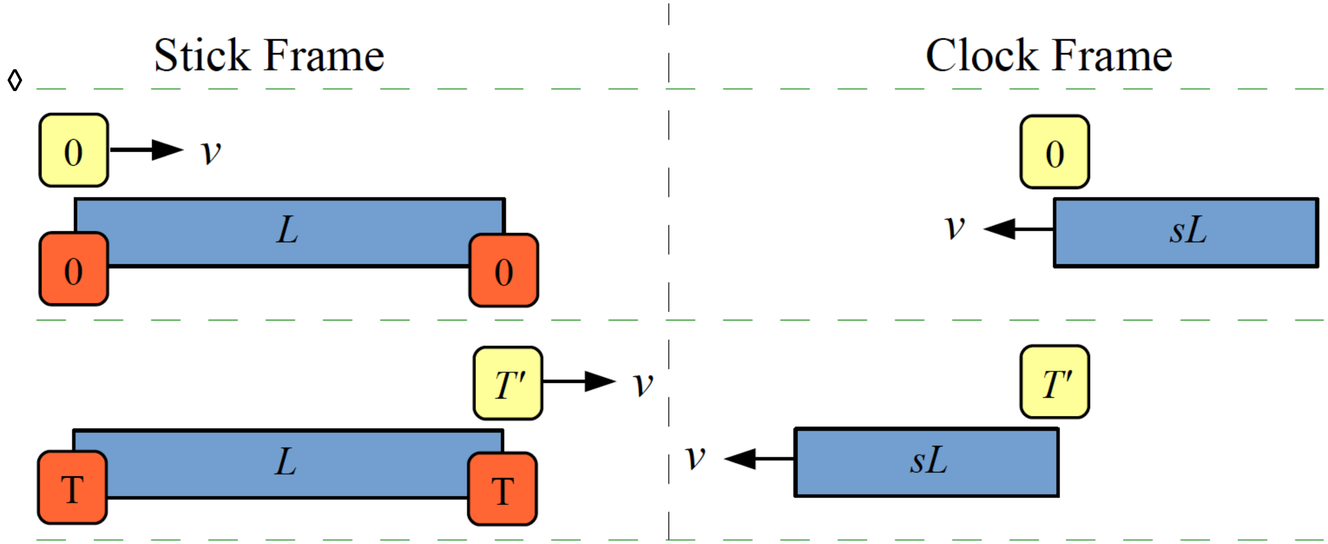
- **Let frame K' move at speed v with respect to frame K. Consider two events at the origin of K', separated by time t'. The separation between the events is**



- **in K: $(x, t) = (vt, t)$ and in K': $(x', t') = (0, t')$.**

Ex: Time Dilation

- **Again, in K: $(x, t) = (vt, t)$ and in K': $(x', t') = (0, t')$.**



- **The invariant interval implies**

$$c^2 t'^2 - 0 = c^2 t^2 - v^2 t^2 \quad \Rightarrow \quad t = \frac{t'}{\sqrt{1 - v^2/c^2}}$$

Note that $x' = 0$.

Lorentz Transformation

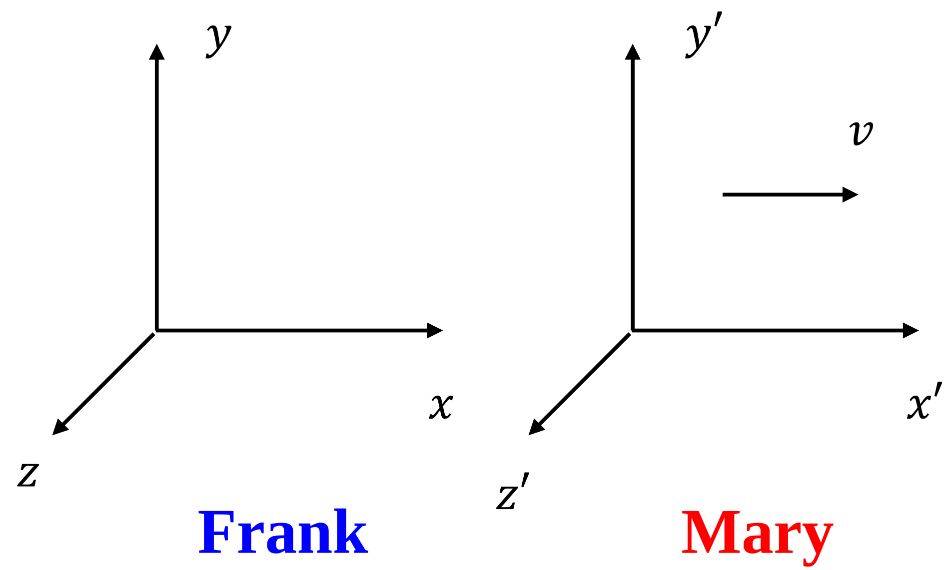
The complete transformation

1) $t = \gamma(t' + x'v/c^2)$

2) $x = \gamma(x' + vt')$

3) $y = y'$

4) $z = z'$



Or,

1) $ct = \gamma[(ct') + \beta x']$

2) $x = \gamma[\beta(ct') + x']$

3) $y = y'$

4) $z = z'$

$$\beta = v/c$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Space-Time Squeezing in

$$(ct', x') = (1,0) \rightarrow (ct, x) = (\gamma, \gamma\beta)$$

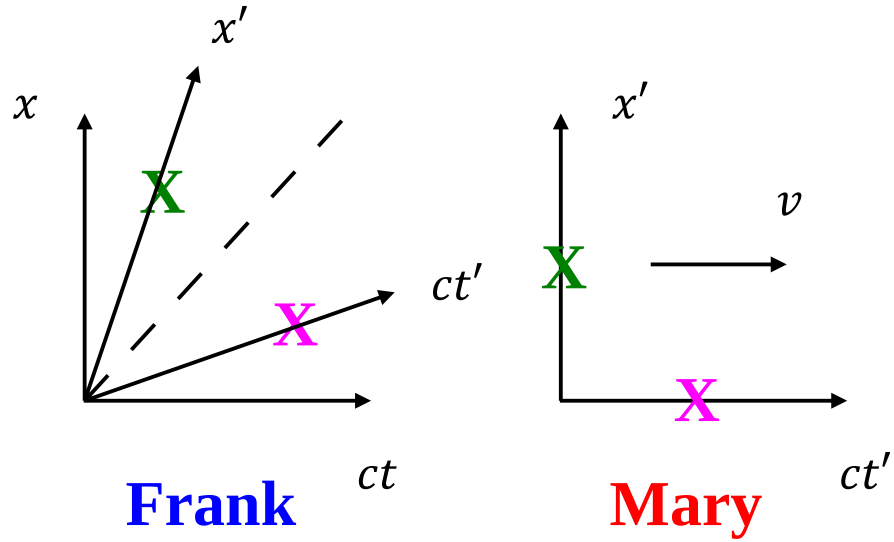
$$(ct', x') = (0,1) \rightarrow (ct, x) = (\gamma\beta, \gamma)$$

Unit lengths for the ct' and x' axes change to

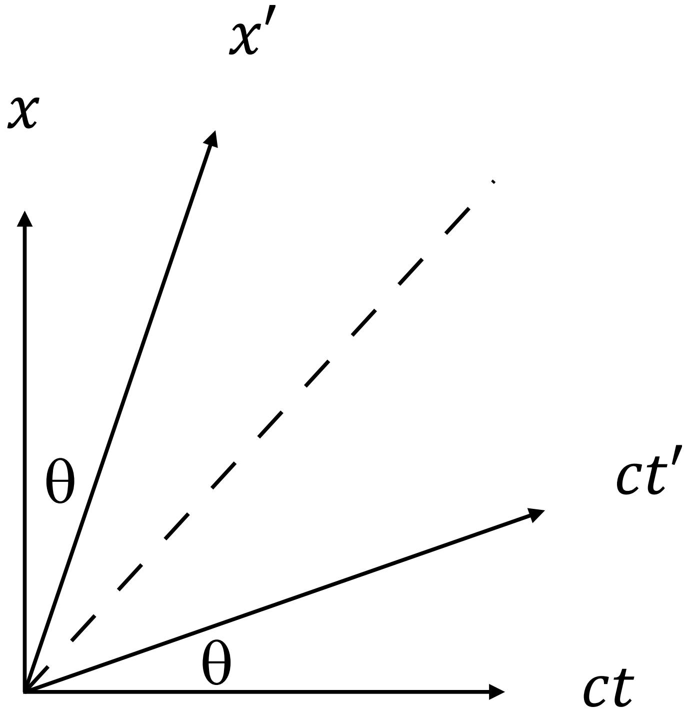
$$\gamma\sqrt{1+\beta^2} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

$$= \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$



Minkowski Diagram



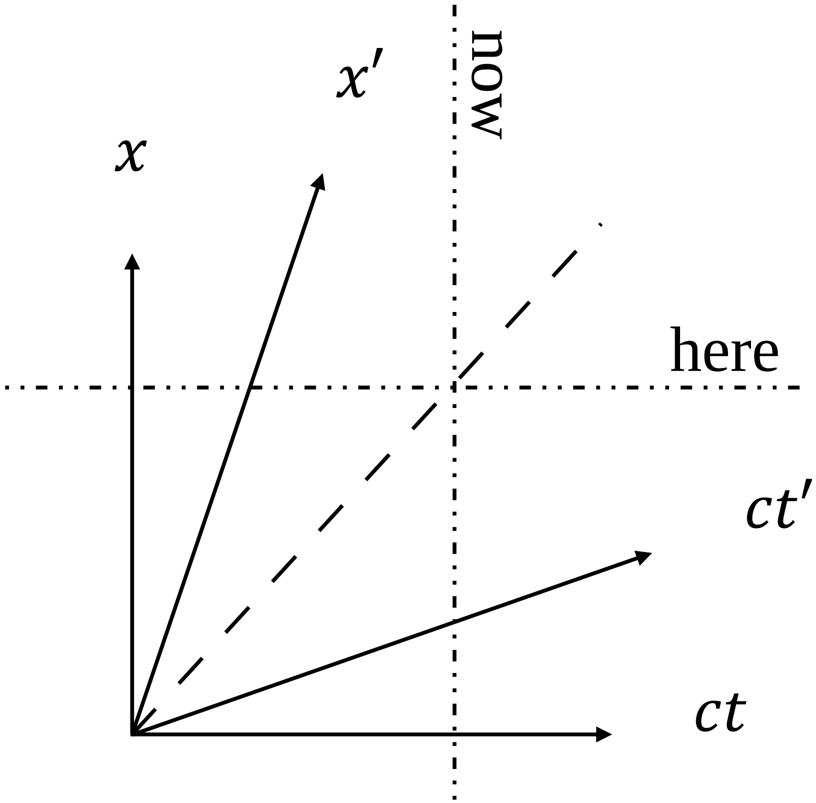
$$tan\theta \ = \ \beta$$

What does “now” mean?

What does “here” mean?

Well, they depend on the choice of frame.

Minkowski Diagram

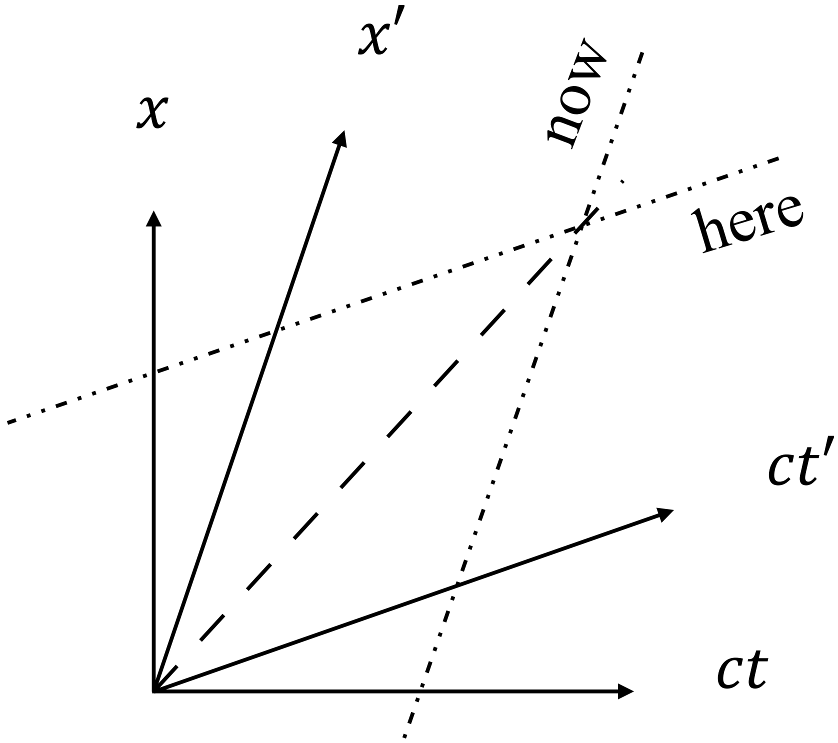


What does “now” mean?

What does “here” mean?

In the rest frame K, now means t = const, and here means x = const.

Minkowski Diagram

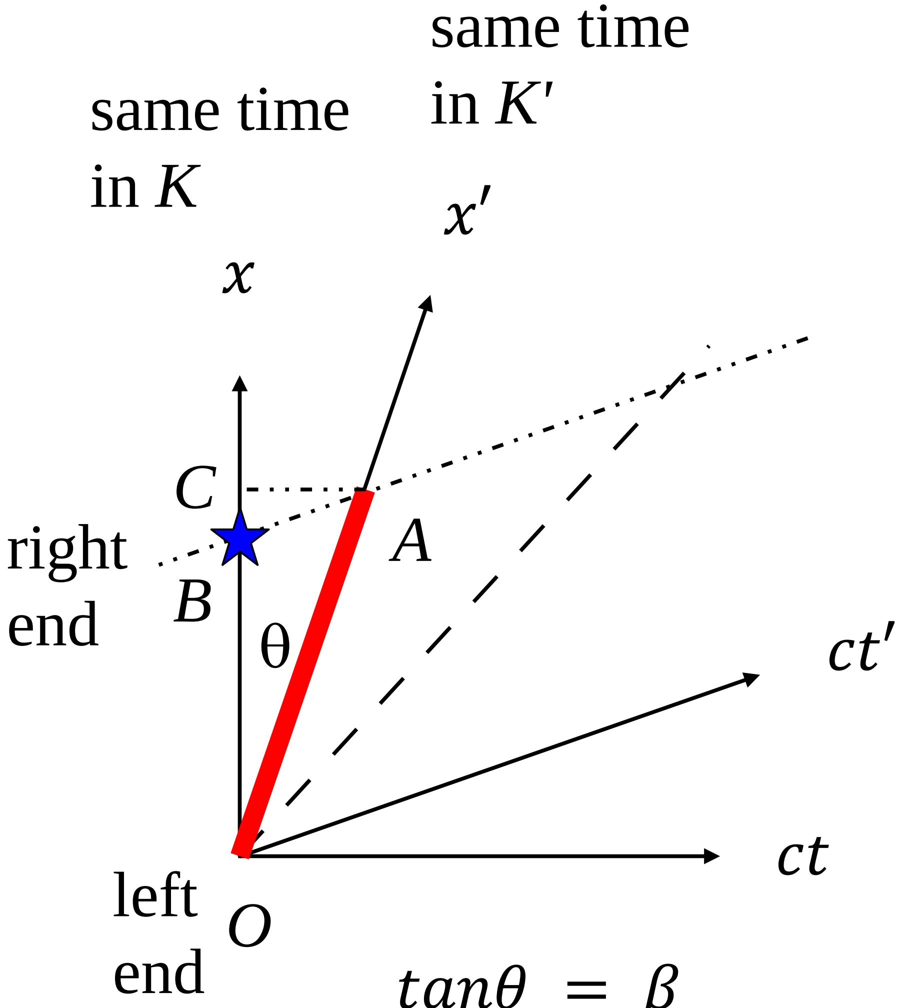


What does “now” mean?

What does “here” mean?

In the moving frame K', now means t' = const, and here means x' = const.

Length Contraction, Again



$$tan\theta = \beta$$

Suppose the length of the stick is 1 in the K' frame. This means the length of OA is

$$\overline{OA} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

The length measured by Frank in the K frame is the length of OB.

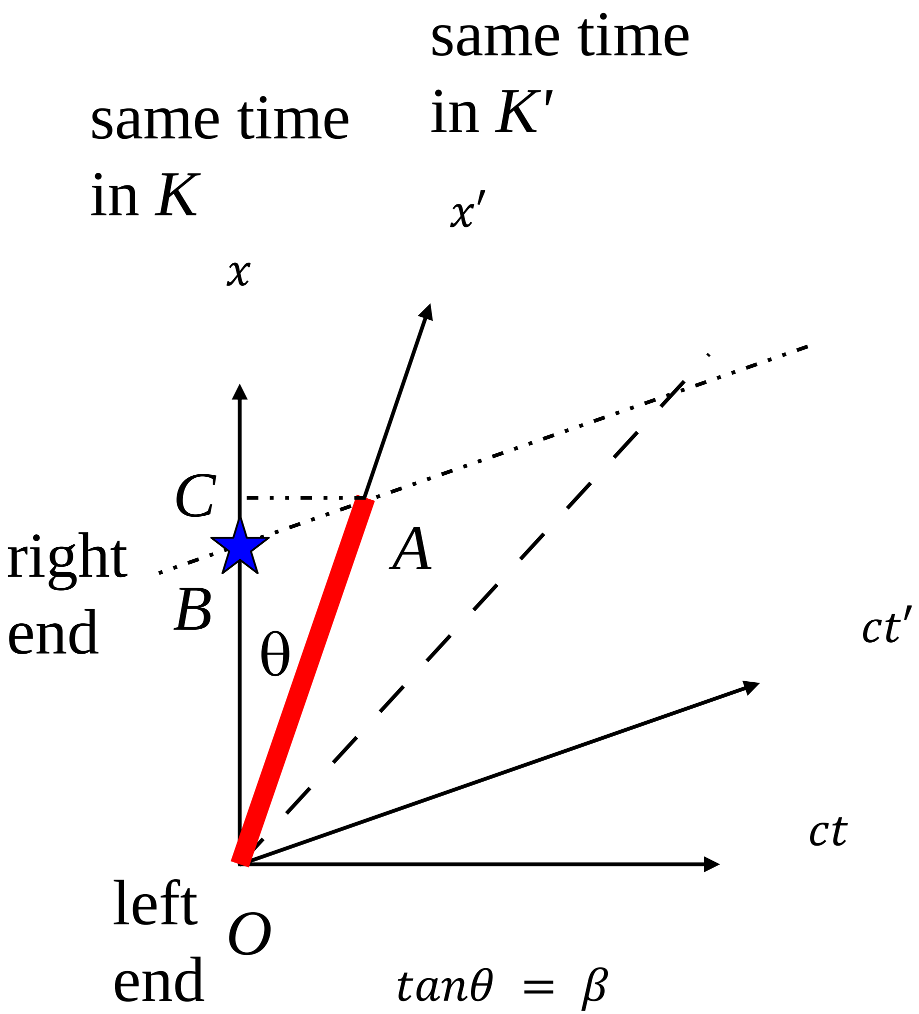
$$\overline{OC} = \overline{OA}cos\theta$$

$$\overline{BC} = \overline{AC}tan\theta$$

$$\overline{AC} = \overline{OA}sin\theta$$

$$\overline{OB} = ?$$

Length Contraction, Again



$tan\theta = \beta$

Therefore, the length of the stick measured in the *K* frame is

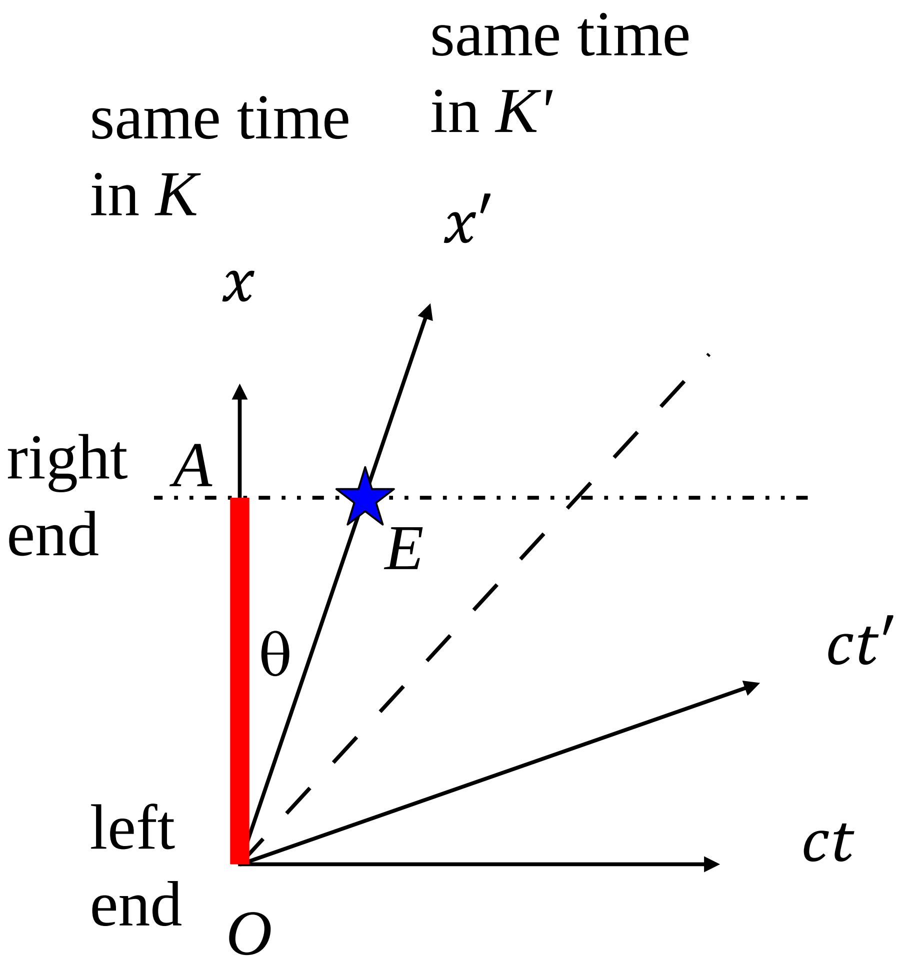
$$\overline{OB} = \overline{OC} - \overline{BC}$$

$$= \sqrt{1 - \beta^2}$$

which is the standard length contraction result. Note that you can figure out

$$cos\theta = \frac{1}{\sqrt{1 + \beta^2}}$$

Length Contraction, Again



$$tan\theta = \beta$$

Suppose the length of the stick is 1 in the K frame, i.e.

$$\overline{OA} = 1$$

Therefore,

$$\overline{OE} = \overline{OA}/cos\,\theta$$

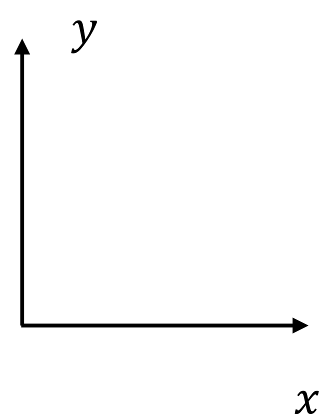
$$= \sqrt{1 + \beta^2}$$

The length measured by Mary in the K' frame is in units of the x' axis, or

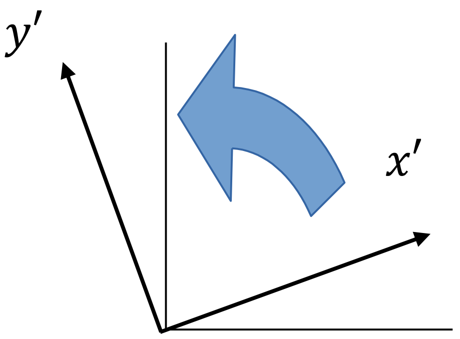
$$\frac{\overline{OE}}{\sqrt{1+\beta^2}/\sqrt{1-\beta^2}} = \sqrt{1-\beta^2}$$

Rotation in Space

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$



Frank



Rose

Show that if we define

$$tanh\phi = \beta = v/c$$

The Lorentz transformations in the matrix form become

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} cosh\phi & sinh\phi \\ sinh\phi & cosh\phi \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

Exercise

Consider a spaceship leaving the Earth at a coasting speed of 0.8*c*, where *c* is the speed of light in vacuum. The spaceship will travel to another planet 4 light years away and return immediately with the same speed after it arrives. Three years, in the Earth frame, after the spaceship was launched, a light signal is sent from the Earth to the traveling spaceship. Suppose a clock on the spaceship is set to zero when it departs. When the spaceship receives the signal, what time is it on the clock (in units of years)? Neglect the time it takes the spaceship to turn around.