Predicates and Quantifiers

Section 1.4

Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier （全称量词）
  - Existential Quantifier （存在量词）
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic
- Logic Programming (*optional*)

Propositional Logic Not Enough

- If we have:
  "All men are mortal."
  "Socrates is a man."
- Does it follow that "Socrates is mortal?"
- Can't be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.
- Later we'll see how to draw inferences.

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: $P(x)$, $M(x)$
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).
- The statement $P(x)$ is said to be the value of the propositional function P at x.
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:
  P(-3) is false.
  P(0) is false.
  P(3) is true.
- Often the domain is denoted by U. So in this example U is the integers.

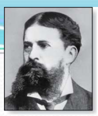
Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:
  R(2,-1,5)
  Solution: F
  R(3,4,7)
  Solution: T
  R(x, 3, z)
  Solution: Not a Proposition
- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:
  Q(2,-1,3)
  Solution: T
  Q(3,4,7)
  Solution: F
  Q(x, 3, z)
  Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes "$x > 0$," find these truth values:
  - $P(3) \vee P(-1)$ **Solution**: T
  - $P(3) \wedge P(-1)$ **Solution**: F
  - $P(3) \rightarrow P(-1)$ **Solution**: F
  - $P(-1) \rightarrow P(3)$ **Solution**: T
- Expressions with variables are not propositions and therefore do not have truth values. For example,
  - $P(3) \wedge P(y)$
  - $P(x) \rightarrow P(y)$
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers



Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier*, "For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, $P(x)$" or "For every x, $P(x)$"

  Examples:

  1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
  2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
  3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Domain (domain of discourse / universe of discourse): range of the possible values of the variable x

- An element for which $P(x)$ is false is called a counterexample（反例） of $\forall x\, P(x)$
- Example:

  Let $P(x)$ be the statement "$x<2$." In the domain of all real numbers, $x=3$ is a counterexample for $\forall x\, P(x)$.

- Many ways to express universal quantification:
  - For all
  - For every
  - All of
  - For each
  - Given any
  - For arbitrary
  - For any

- Examples:

3. What is the truth value of $\forall\, x\, P(x)$, where $P(x)$ is the statement "x<3" and the domain is $\{1,2,3\}$?

Solution:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

Because $P(3)$, which is the statement "3<3," is false, it follows that $\forall\, x\, P(x)$ is false.

- Remark: Given the domain as $\{x_1, x_2, \cdots, x_n\}$,

$$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

【Definition】A existential quantification of $P(x)$, denoted by $\exists\, x\, P(x)$, is the statement "There exists an element x in the domain such that $P(x)$. "

- $\exists$: existential quantifier
- Other expressions:

  For some x $P(x)$

  There is an x such that $P(x)$

  There is at least one x such that $P(x)$

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- Examples:

2. What is the truth value of ∃ x $P(x)$, where $P(x)$ is the statement "x<3" and the domain is $\{1,2,3\}$?

Solution:

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

Because $P(1)$, which is the statement "1<3," is true, it follows that ∃ x $P(x)$ is true.

- Remark: Given the domain as $\{x_1, x_2, \cdots, x_n\}$,

$$\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

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Uniqueness Quantifier（唯一性量词）

- ∃!x $P(x)$ means that $P(x)$ is true for <u>one and only one</u> x in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then ∃!x $P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then ∃!x $P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique x such that $P(x)$ can be expressed as:

$$\exists x\ (P(x) \wedge \forall y\ (P(y) \rightarrow y = x))$$

Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- Examples:
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Binding Variables

- Bound variable: a variable is bound if it is known or quantified.
- Free variable: a variable neither quantified nor specified with a value
- *All the variables in a propositional function must be quantified or set equal to a particular value to turn it into a proposition.*
- Scope（作用域） of a quantifier: the part of a logical expression to which the quantifier is applied
- Examples

  $\exists x\, (x+y)=1$

  $\exists x\, (P(x) \wedge Q(x)) \vee \forall x\, R(x)$

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Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 1: But if *U* is all people, then translate as

$\exists x\, (S(x) \wedge J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Returning to the Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain as all people.
- The two premises are: $\forall x (Man(x) \rightarrow Mortal(x))$
  $Man(Socrates)$
- The conclusion is: $Mortal(Socrates)$
- Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- Example: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Logical Equivalences Involving Quantifiers

x is not occurring in A.

(1) $\forall x P(x) \vee A \equiv \forall x (P(x) \vee A)$

(2) $\forall x P(x) \wedge A \equiv \forall x (P(x) \wedge A)$

(3) $\exists x P(x) \vee A \equiv \exists x (P(x) \vee A)$

(4) $\exists x P(x) \wedge A \equiv \exists x (P(x) \wedge A)$

(5) $\forall x (A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$

Proof:

$$\begin{aligned} \forall x (A \rightarrow P(x)) &\equiv \forall x (\neg A \vee P(x)) \\ &\equiv \neg A \vee \forall x P(x) \\ &\equiv A \rightarrow \forall x P(x) \end{aligned}$$

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Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.
- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

  Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."
- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

Translating from English into Logical Expressions

- Goal: To produce a logical expression that is simple and can be easily used in subsequent reasoning.
- Steps:
  - Clearly identify the appropriate quantifier(s)
  - Introduce variable(s) and predicate(s)
  - Translate using quantifiers, predicates, and logical operators

There can be many ways to translate a particular sentence.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x\ (S(x) \wedge M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x\ (S(x) \rightarrow (M(x) \vee C(x)))$$

Example

- $C(x)$: x is a CS student, $E(x)$: x is a Math student, $S(x)$: x is a smart student, and the domain consists of all students in our class

  1) Everyone is a CS student.

  $$\forall x\ C(x)$$

  2) Nobody is a Math student.

  $$\forall x \neg E(x) \quad \text{or} \quad \neg \exists x\ E(x)$$

  3) All CS students are smart students.

  $$\forall x\ (C(x) \rightarrow S(x))$$

  4) Some CS students are smart students.

  $$\exists x\ (C(x) \wedge S(x))$$

Example

- $C(x)$: x is a CS student, $E(x)$: x is an Math student, $S(x)$: x is a smart student, and the domain consists of all students in our class

  5) No CS student is an Math student.

  - If x is a CS student, then that student is not a Math student.

    $\forall x\, (C(x) \to \neg\, E(x))$

  - There does not exist a CS student who is also a Math student.

    $\neg\, \exists x\, [C(x) \wedge E(x)]$

  6) If any Math student is a smart student then he is also a CS student.

  $\forall x\, ((E(x) \wedge S(x)) \to C(x))$

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Some Fun with Translating from English into Logical Expressions

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  Translate "Everything is a fleegle"

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"Nothing is a snurd."

Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x \neg S(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"All fleegles are snurds."

Solution: $\forall x\, (F(x) \rightarrow S(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\, (F(x) \land T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"No snurd is a thingamabob."

Solution: $\neg\exists x\, (S(x) \land T(x))$ What is this equivalent to?

Solution: $\forall x\, (\neg S(x) \lor \neg T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "If any fleegle is a snurd then it is also a thingamabob."

  Solution: $\forall x\, ((F(x) \wedge S(x)) \rightarrow T(x))$

System Specification Example

- Predicate logic is used for specifying properties that systems must satisfy.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."
- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message m is larger than y megabytes."
  - Let $C(m)$ denote "Mail message m will be compressed."
  - Let $A(u)$ represent "User u is active."
  - Let $S(n, x)$ represent "Network link n is state x".
- Now we have: $\forall m(L(m,1) \rightarrow C(m))$

  $\exists u\, A(u) \rightarrow \exists n\, S(n, available)$

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

- The first two are called *premises* and the third is called the *conclusion*.
  1. "All lions are fierce."
  2. "Some lions do not drink coffee."
  3. "Some fierce creatures do not drink coffee."
- Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.
  1. $\forall x\, (P(x) \rightarrow Q(x))$
  2. $\exists x\, (P(x) \wedge \neg R(x))$
  3. $\exists x\, (Q(x) \wedge \neg R(x))$
- Later we will see how to prove that the conclusion follows from the premises.

Some Predicate Calculus Definitions (*optional*)

- An assertion involving predicates and quantifiers is *valid* if it is true
  - for all domains
  - every propositional function substituted for the predicates in the assertion.

  Example: $\forall x \neg S(x) \leftrightarrow \neg \exists x S(x)$
- An assertion involving predicates is *satisfiable* if it is true
  - for some domains
  - some propositional functions that can be substituted for the predicates in the assertion.

  Otherwise it is *unsatisfiable*.

  Example: $\forall x (F(x) \leftrightarrow T(x))$ not valid but satisfiable

  Example: $\forall x (F(x) \wedge \neg F(x))$ unsatisfiable

More Predicate Calculus Definitions (*optional*)

- The *scope* of a quantifier is the part of an assertion in which variables are bound by the quantifier.

Example: $\forall x(F(x) \vee S(x))$ *x* has wide scope

Example: $\forall x(F(x)) \vee \forall y(S(y))$ *x* has narrow scope

Logic Programming (optional)

- Prolog (from *Pro*gramming in *Log*ic) is a programming language developed in the 1970s by researchers in artificial intelligence (AI).
- Prolog programs include *Prolog facts* and *Prolog rules*.
- As an example of a set of Prolog facts consider the following:

```
instructor(chan, math273).
instructor(patel, ee222).
instructor(grossman, cs301).
enrolled(kevin, math273).
enrolled(juna, ee222).
enrolled(juana, cs301).
enrolled(kiko, math273).
enrolled(kiko, cs301).
```

- Here the predicates *instructor(p,c)* and *enrolled(s,c)* represent that professor *p* is the instructor of course *c* and that student *s* is enrolled in course *c*.

Logic Programming (cont)

- In Prolog, names beginning with an uppercase letter are variables.
- If we have apredicate *teaches(p,s)* representing "professor *p* teaches student *s*," we can write the rule:

```
teaches(P,S) :- instructor(P,C), enrolled(S,C).
```

- This Prolog rule can be viewed as equivalent to the following statement in logic (using our conventions for logical statements).

$$\forall p\ \forall c\ \forall s(I(p,c) \wedge E(s,c)) \rightarrow T(p,s))$$

Logic Programming (cont)

- Prolog programs are loaded into a *Prolog interpreter*. The interpreter receives *queries* and returns answers using the Prolog program.
- For example, using our program, the following query may be given:

```
?enrolled(kevin,math273).
```

- Prolog produces the response:

```
yes
```

- Note that the ? is the prompt given by the Prolog interpreter indicating that it is ready to receive a query.

Logic Programming (cont)

- The query:

```
?enrolled(X,math273).
```

  produces the response:

```
X = kevin;
X = kiko;
no
```

- The query:

```
?teaches(X,juana).
```

  produces the response:

```
X = patel;
X = grossman;
no
```

The Prolog interpreter tries to find an instantiation for X. It does so and returns `X = kevin.` Then the user types the `;` indicating a request for another answer. When Prolog is unable to find another answer it returns `no`.

Logic Programming (cont)

- The query:

```
?teaches(chan,X).
```

  produces the response:

```
X = kevin;
X = kiko;
no
```

- A number of very good Prolog texts are available. *Learn Prolog Now!* is one such text with a free online version at http://www.learnprolognow.org/
- There is much more to Prolog and to the entire field of logic programming.

Homework

- 第7版： Section 1.4 6(c,d,e,f), 9(b,d), 20(e), 24(b,d), 36， 40(b), 44, 49(a)
- 第8版：Section 1.4 6(c,d,e,f), 9(b,d), 20(e), 24(b,d), 36， 42(b), 46, 51(a)

Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- 【Definition】 Two quantifiers are nested if one is within the scope of the other.
- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $\forall x \exists y(x + y = 0)$

  where the domains of x and y are the real numbers.
- We can also think of nested propositional functions:

  $\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

    $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is true, but $\exists y \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,
Define $P(x,y) : x \cdot y = 0$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\, P(x,y)$
   Answer: True
4. $\exists x\, \exists\, y\, P(x,y)$
   Answer: True

Questions on Order of Quantifiers

Example 2: Let U be the positive real numbers,
Define $P(x,y) : x \,/\, y = 1$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\, P(x,y)$
   Answer: False
4. $\exists x\, \exists\, y\, P(x,y)$
   Answer: True

Quantifications of Two Variables

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	**$P(x, y)$ is true for every pair x, y.**	**There is a pair x, y for which $P(x, y)$ is false.**
$\forall x \exists y P(x, y)$	**For every x there is a y for which $P(x, y)$ is true.**	**There is an x for which $P(x, y)$ is false for every y.**
$\exists x \forall y P(x, y)$	**There is an x for which $P(x, y)$ is true for every y.**	**For every x there is a y for which $P(x, y)$ is false.**
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	**There is a pair x, y for which $P(x, y)$ is true.**	**$P(x, y)$ is false for every pair x, y.**

Translating from Nested Quantifiers into English

- Examples

1. Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$ into English, where $C(x)$ is "x has a computer," $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students at ZJU.

Solution:

For every student x at ZJU, x has a computer or there is a student y such that y has a computer and x and y are friends.

$\Updownarrow$

Every student at ZJU has a computer or has a friend who has a computer.

Translating from Nested Quantifiers into English

- Examples

2. Translate the following statement into English, where $F(x,y)$ means x and y are friends and the domain for x, y, and z consists of all students at ZJU. $\exists x \forall y \forall z(((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)))$

Solution:

There is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends.

⇳

There is a student none of whose friends are also friends with each other.

Translating English into Logical Expressions

- Examples

1. Express the statement "Everyone has exactly one best friend" as a logical expression with a domain consisting of all people.

Solution:

Rewrite the original statement as

"For every person x , $\boxed{x \text{ has exactly one best friend.}}$"

⇩

"There is a person y who is the best friend of x, and furthermore, that for every person z, if z is not y, then z is not the best friend of x."

Let $B(x, y)$ be the statement "y is the best friend of x."

$$\forall x \exists y \forall z(B(x, y) \wedge ((z \neq y) \rightarrow \neg B(x, z)))$$

Translating English into Logical Expressions

- Examples

2. Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression with a domain consisting of all people.

Solution:

Rewrite the original statement as

"For every person x, if x is female and x is a parent, then there exists a person y such that x is the mother of y."

Let $F(x)$: "x is female,"

$P(x)$: "x is a parent,"

$M(x, y)$: "x is the mother of y."

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$$
$$\forall x \exists y((F(x) \wedge P(x)) \rightarrow M(x, y))$$

Quantifiers with Restricted Domains

- Example: What do the statements $\forall x<0\ (x^2>0)$, $\exists y>0\ (y^2=2)$ mean, where the domain in each case consists of the real numbers?

Solution:

$$\forall x<0\ (x^2>0)$$

For every real number x with $x<0$, $x^2>0$

$$\forall x\ (x<0 \rightarrow x^2>0)$$

$$\exists y>0\ (y^2=2)$$

There exists a real number y with $y>0$ such that $y^2=2$

$$\exists y(y>0 \wedge y^2=2)$$

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Translating English into Logical Expressions

- Examples

3. Express the definition of a limit using quantifiers.

Solution:

The definition of the limit $\lim_{x \to a} f(x) = L$ is:

"For every real number ε>0 there exists a real number δ>0 such that $|f(x) - L| < \varepsilon$ *wherever* $|x - a| < \delta$ "

Assume the domain for the variables ε, δ, and x consist of all real numbers, we have

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon)$$

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables *x* and *y*, and specify the domain, to obtain:

   "For all positive integers *x* and *y*, *x* + *y* is positive."

3. The result is:

   $$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w\ \forall a\ \exists f\ (P(w,f) \land Q(f,a))$$

Calculus in Logic (*optional*)

Example: Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.

Solution: Recall the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is "For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$."

Using quantifiers:

$$\forall \epsilon \exists \delta \forall x (0 < |\ x - a\ | < \delta \rightarrow |\ f(x) - L\ | < \epsilon)$$

Where the domain for the variables ε and δ consists of all positive real numbers and the domain for x consists of all real numbers.

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$
2. $\forall w \neg \forall a \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \lor \neg Q(f,a))$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Negating Nested Quantifiers

- Examples:

1. Express the negation of the statement $\forall x \exists y\ (xy=1)$ so that no negation precedes a quantifiers.

Solution:

$$\neg(\forall x \exists y\ (xy=1))$$
$$\equiv \exists x(\neg \exists y\ (xy=1))$$
$$\equiv \exists x \forall y\ (\neg\ (xy=1))$$
$$\equiv \exists x \forall y\ (xy \neq 1)$$

Return to Calculus and Logic (*Opt*)

Example : Recall the logical expression developed in the calculus example three slides back. Use quantifiers and predicates to express that $\lim_{x \to a} f(x)$ does not exist.

1. We need to say that for all real numbers L, $\lim_{x \to a} f(x) \neq L$

2. The result from the previous example can be negated to yield:
$$\neg \forall \epsilon \exists \delta \forall x (0 < |\ x - a\ | < \delta \to |\ f(x) - L\ | < \epsilon)$$

3. Now we can repeatedly apply the rules for negating quantified expressions:

$$\neg \forall \epsilon \exists \delta \forall x (0 < |\ x - a\ | < \delta \to |\ f(x) - L\ | < \epsilon)$$
$$\equiv \exists \epsilon \neg \exists \delta \forall x (0 < |\ x - a\ | < \delta \to |\ f(x) - L\ | < \epsilon)$$
$$\equiv \exists \epsilon \forall \delta \neg \forall x (0 < |\ x - a\ | < \delta \to |\ f(x) - L\ | < \epsilon)$$
$$\equiv \exists \epsilon \forall \delta \exists x \neg (0 < |\ x - a\ | < \delta \to |\ f(x) - L\ | < \epsilon)$$
$$\equiv \exists \epsilon \forall \delta \exists x \neg (0 < |\ x - a\ | < \delta \wedge |\ f(x) - L\ | \geq \epsilon)$$

The last step uses the equivalence $\neg(p \to q) \equiv p \wedge \neg q$

Calculus in Predicate Logic (*optional*)

4. Therefore, to say that $\lim_{x\to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x\to a} f(x) \neq L$ can be expressed as:

   $$\forall L \exists \epsilon \forall \delta \exists x \neg (0 < |\, x - a \,| < \delta \wedge |\, f(x) - L \,| \geq \epsilon)$$

   Remember that ε and δ range over all positive real numbers and x over all real numbers.

5. Translating back into English we have, for every real number L, there is a real number ε > 0, such that for every real number δ > 0, there exists a real number x such that $0 < |\, x - a \,| < \delta$ and $|f(x) - L| \geq \varepsilon$.

Some Questions about Quantifiers

- Can you switch the order of quantifiers?
  - Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

    Solution: Yes! The left and the right side will always have the same truth value. The order in which *x* and *y* are picked does not matter.
  - Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

    Solution: No! The left and the right side may have different truth values for some propositional functions for *P*. Try "x + y = 0" for *P(x,y)* with *U* being the integers. The order in which the values of *x* and *y* are picked does matter.
- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence? $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

    Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.
  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

    Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.

Homework

- 第7版第8版一样：Sec. 1.5 6(e, f), 12(d, h, k, n), 14(c, d, e, f), 24(a, d), 34，32(d), 38(b, d), 42