Diffraction

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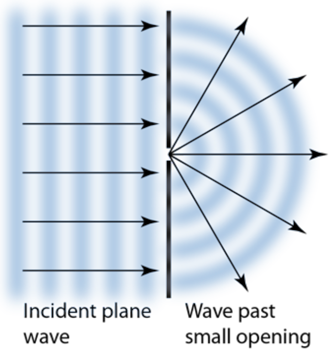
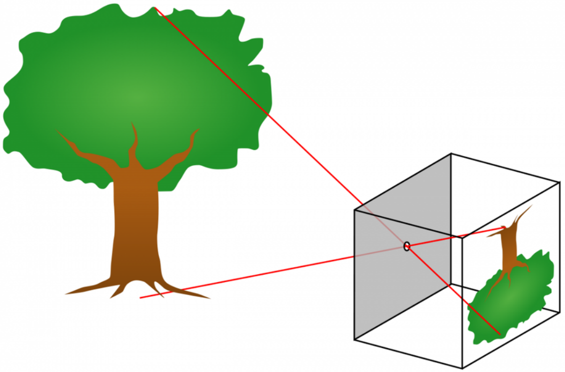
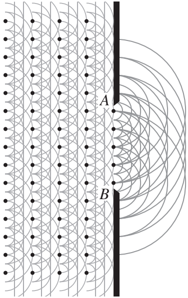
Lecture 18

Motivation



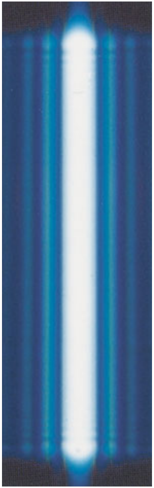


- Fresnel further developed Huygens' principle that every unobstructed point of a wavefront, at a given instant, serves as a source of spherical secondary wavelets (with the same frequency as that of the primary wave).
- He proposed that the amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering their amplitudes and relative phases).
- Notice the wavelength is smaller than the opening in the figure.



- In this lecture, our focus is to understand the **diffraction** of light as it passes through a narrow slit or past either a narrow obstacle or an edge.

- We studied the interference of two light beams in Young's double-slit experiment. In fact, diffraction through a single slit (with a non-negligible width) is more complicated, because the light also interferes with itself and produces an interference pattern.

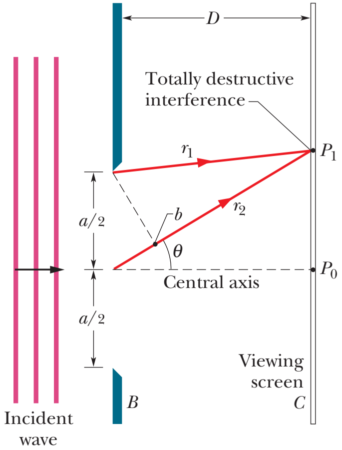


Outline

- Single-Slit Diffraction
- Intensity in Single-Slit Diffraction
- Fourier Methods in Diffraction Theory
- Diffraction by a Circular Aperture

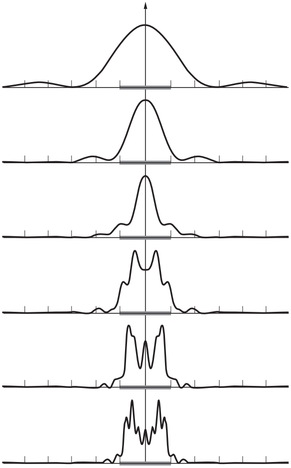
Single-Slit Diffraction

- We consider the diffraction pattern of plane waves of light of wavelength λ that are diffracted by a single long, narrow slit of width a in an otherwise opaque screen B.
- Waves from different points within the slit undergo interference and produce a diffraction pattern of bright and dark fringes on screen C.



Fraunhofer vs Fresnel Diffraction

- At large screen distance ($D > a^2/\lambda$), the shape of the projected pattern is independent of D. This is **Fraunhofer** or **far-field diffraction**.
- At small D, however, both the size and shape of the diffraction pattern changes with the distance. This phenomenon is known as **Fresnel** or **near-field diffraction**.



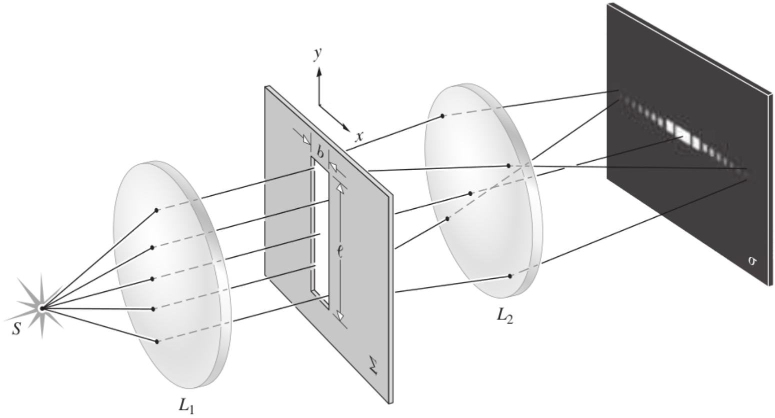


Figure 1: Single-slit Fraunhofer diffraction using lenses so that the source and fringe pattern can both be at convenient distances from the aperture.

A Revisit to Young's Interference Experiment

- Using complex amplitudes,

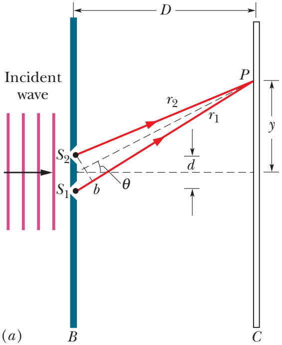
$$\begin{aligned}\tilde{E} &= E_0 e^{-i\omega t}\left(e^{ikr_2} + e^{ikr_1}\right) \\ &= E_0 e^{-i\omega t} e^{ikr_2}\left(1 + e^{ikd\sin\theta}\right).\end{aligned}$$

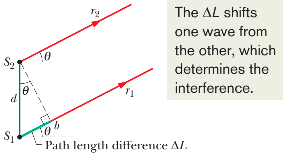
- Condition for dark fringes:

$$kd\sin\theta = 2m\pi + \pi, \text{ or}$$

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda.$$

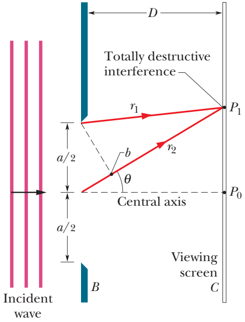
- What is the phasor picture?





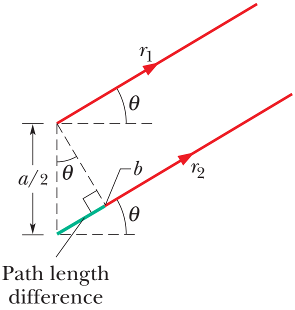
Locations of the Dark Fringes

- First, let us divide the slit into two zones of equal widths $a/2$. We extend to P_1 a light ray r_1 from the top point of the top zone and a light ray r_2 from the top point of the bottom zone.
- To find the dark fringes, we want the wavelets along these two rays to cancel each other when they arrive at P_1.
- Then any similar pairing of rays from the two zones will give cancellation at the same point P_1 (note $D \gg a$).



- The wavelets of the pair of rays r_1 and r_2 are in phase within the slit because they originate from the same wavefront passing through the slit, along the width of the slit.
- They are out of phase by $\lambda/2$ when they reach P_1.
- If we arrange the screen separation D to be much larger than the slit width a, we have, approximately,

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}.$$

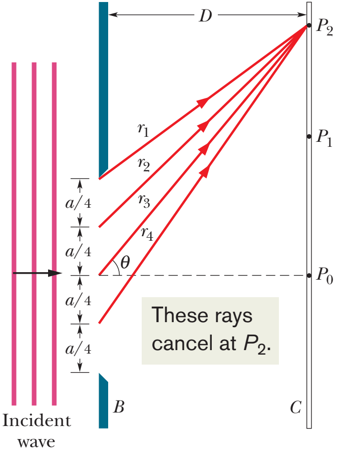


- The angle θ of the first dark fringe above and (by symmetry) below the central axis is, therefore, determined by

$$\sin\theta = \frac{\lambda}{a}.$$

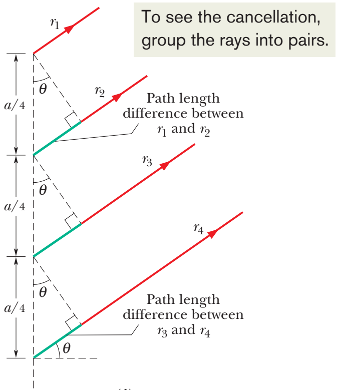
- If we narrow the slit such that $a = \lambda$, the angle θ at which the first dark fringes appear increases to 90°. Since the first dark fringes mark the two edges of the central bright fringe, this means that bright fringe must then cover the entire viewing screen.

- To find the second dark fringes above and below the central axis, we now divide the slit into four zones of equal widths $a/4$.
- To produce the second dark fringe at P_2, the path length difference between r_1 and r_2, that between r_2 and r_3, and that between r_3 and r_4 must all be equal to $\lambda/2$.



- For $D \gg a$, we can approximate these four rays as being parallel, at angle θ to the central axis.
- The path length difference for any two rays that originate at corresponding points in two adjacent zones is $(a/4)\sin\theta$. Since in each such case the path length difference is equal to $\lambda/2$, we have

$$\frac{a}{4}\sin\theta = \frac{\lambda}{2}.$$



(b)

- We could now continue to locate dark fringes in the diffraction pattern by splitting up the slit into more zones of equal width.
- We would find that the dark fringes above and below the central axis can be located with the general equation

$$a \sin\theta = m\lambda,$$

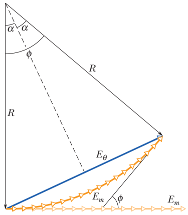
  for $m = 1, 2, 3, \ldots$.
- In other words, in a single-slit diffraction experiment, dark fringes are produced where the path length differences ($a \sin\theta$) between the top and bottom rays are equal to λ, 2λ, 3λ, ...

Electric Field and Intensity

- To find an expression for the intensity at an arbitrary point P on the viewing screen, corresponding to a particular small angle θ, we need to divide the slit into N zones of equal widths $\Delta x = a/N$ small enough that we can assume each zone acts as a source of Huygens wavelets.
- We then add the phasors for the wavelets, which form a geometric series (notice $r_{i+1} - r_i = \Delta x \sin\theta$):

$$\begin{aligned}\tilde{E}_\theta = & \frac{E_0}{N} e^{-i\omega t} e^{ikr_1} \\ & \times \left[1 + e^{ik(r_2 - r_1)} + e^{ik(r_3 - r_1)} + \cdots + e^{ik(r_N - r_1)}\right].\end{aligned}$$

- The arc of phasors represents the wavelets that reach an arbitrary point P on the viewing screen, corresponding to a small angle θ. The amplitude E_θ of the resultant wave at P is the vector sum of these phasors. In the limit of $N \to \infty$, the arc of phasors approaches the arc of a circle.
- We use a graphic construction:



$$\frac{I(\theta)}{I_{\max}} = \frac{E_\theta^2}{E_m^2} = \left[\frac{\sin\alpha}{\alpha}\right]^2,$$

where $\alpha = (\pi/\lambda) a \sin\theta$.

- Notice $\phi = 2\alpha$ is the phase difference between the rays from the top and bottom of the entire slit.

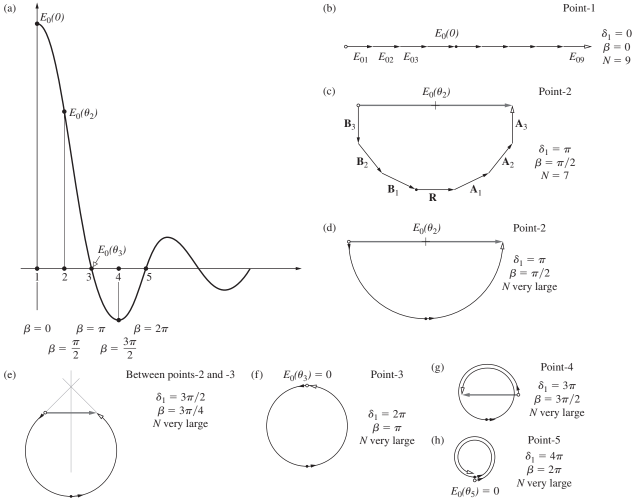
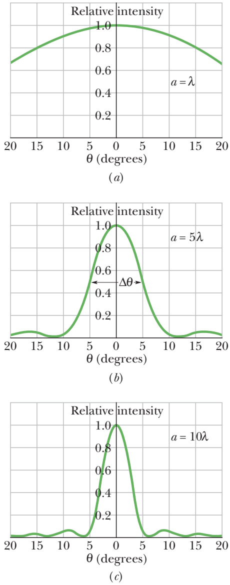


Figure 2: Electric field for single-slit Fraunhofer diffraction. Notice the notation changes $\beta \equiv \alpha$ and $\delta_1 \equiv \phi = 2\alpha$.

- The right figure shows plots of the intensity of a single-slit diffraction pattern for three slit widths: $a = \lambda$, $a = 5\lambda$, $a = 10\lambda$.
- As the slit width increases (relative to the wavelength), how do the following quantities vary?
  - The width of the central diffraction maximum, i.e., the central hill-like region of the graphs.
  - The width and relative height of the secondary maxima.
- What happens in the limit when $a \gg \lambda$?



From Sum to Integral

- Before we leave, let us revisit the geometric series of phasers (notice $r_{i+1} - r_i = \Delta x \sin\theta$):

$$
\begin{aligned}
\tilde{E}_\theta &= \frac{E_0}{N} e^{-i\omega t}\left[e^{ikr_1} + e^{ikr_2} + e^{ikr_3} + \cdots + e^{ikr_N}\right] \\
&= \frac{E_0 \Delta x}{a} e^{-i\omega t} \sum_i e^{ikr_i} \\
&\xrightarrow{N\to\infty} E_0 e^{-i\omega t} \frac{1}{a} \int_0^a e^{ik(r_1 + x\sin\theta)} dx \\
&\sim \int_0^a e^{ik_x x} dx \sim \int_{-a/2}^{a/2} e^{ik_x x} dx,
\end{aligned}
$$

where $k_x = k\sin\theta$.

Fourier Transform

- The Fourier transform of a one-dimensional function $f(x)$ is defined as

$$F(k_x) = \int_{-\infty}^{\infty} f(x) e^{ik_x x} dx,$$

whose inverse transform is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x) e^{-ik_x x} dk_x.$$

- Fourier-transform theory provides a particularly beautiful insight into the mechanism of Fraunhofer diffraction and double-slit interference.

Revisiting the Single Slit

- Consider the long slit in the y-direction, illuminated by a plane wave. Assuming that there are no phase or amplitude variations across the aperture, the one-dimensional **aperture function** has the form of a square pulse:

$$E_{\mathrm{sq}}(x) = \begin{cases} E_0 & |x| \le a/2, \\ 0 & |x| > a/2. \end{cases}$$

- The Fourier transform of $E_{\mathrm{sq}}(x)$ is

$$\tilde{E}_{\mathrm{sq}}(k_x) = \int_{-\infty}^{\infty} E_{\mathrm{sq}}(x) e^{ik_x x} dx = E_0 \int_{x=-a/2}^{a/2} e^{ik_x x} dx.$$

- The integral can be evaluated to be

$$\tilde{E}_{\mathrm{sq}}(k_x) = E_0 a \frac{\sin\alpha}{\alpha},$$

where $\alpha = k_x(a/2) = k\sin\theta(a/2)$.

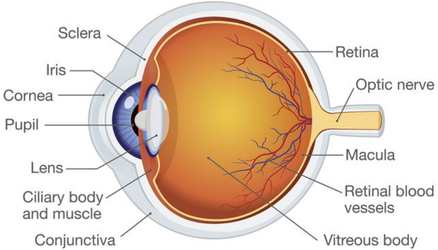
- This leads to the result we have derived:

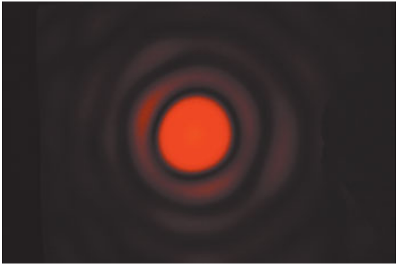
$$I_{\mathrm{sq}}(\theta) = I_{\mathrm{max}}\left[\frac{\sin\alpha}{\alpha}\right]^2.$$

- The key message is that *the field distribution in the Fraunhofer diffraction pattern is the Fourier transform of the field distribution across the aperture*.

Diffraction by a Circular Aperture

- Consider diffraction by a circular aperture of diameter d — that is, a circular opening, such as a circular lens, through which light can pass. The image is not a point, as geometrical optics would suggest, but a circular disk surrounded by several progressively fainter secondary rings.





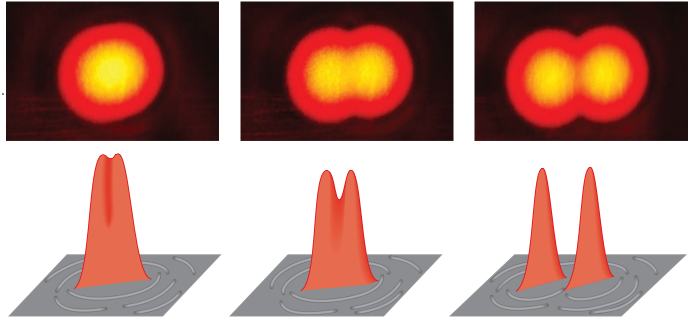
- We are essentially collecting only a fraction of the incident wavefront and therefore cannot hope to form a perfect image.
- The image is related to the Fourier transform of a disk and is known as the **Airy pattern**.
- The analysis of such patterns shows that the first minimum for the diffraction pattern of a circular aperture of diameter a is located by

$$\sin\theta = 1.22\frac{\lambda}{a},$$

in contrast to $\sin\theta = \lambda/a$ in the slit case.

Resolvability

- The fact that lens images are diffraction patterns is important when we wish to resolve (distinguish) two distant point objects whose angular separation is small.



(*a*) (*b*) (*c*)

- Two objects cannot be distinguished from a single point object, if their diffraction patterns (mainly their central maxima) overlap.
- **Rayleigh's criterion** for resolvability states that the two point objects are barely resolved if their angular separation is such that the central maximum of the diffraction pattern of one source is centered on the first minimum of the diffraction pattern of the other, i.e.,

$$\theta_{\mathrm{R}} = \sin^{-1} \frac{1.22\lambda}{a} \approx 1.22\frac{\lambda}{a}.$$

Figure 3: A picture dotified by GIMP. With the viewer very close to the screen, the dots are visible. At sufficiently large viewing distances, the dots are irresolvable and thus blend.

Pointillism – Maximilien Luce

Summary

- How to describe the single-slit diffraction pattern? Where are the locations of the dark fringes?
- What are the similarities between interference and diffraction? What are the differences?
- Why do you think the human visual resolvability is related to the diffraction of a circular aperture? How? (*Search for the structure of eye on the internet.*)
- In the context of single-slit diffraction, understand how one can use either phasor addition or Fourier transform to obtain the Fraunhofer diffraction pattern.

Reading

Halliday, Resnick & Krane:

- Chapter 42: Diffraction