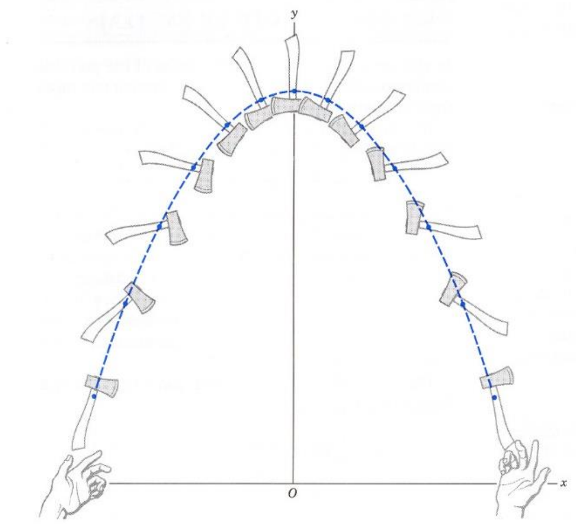
General Physics I

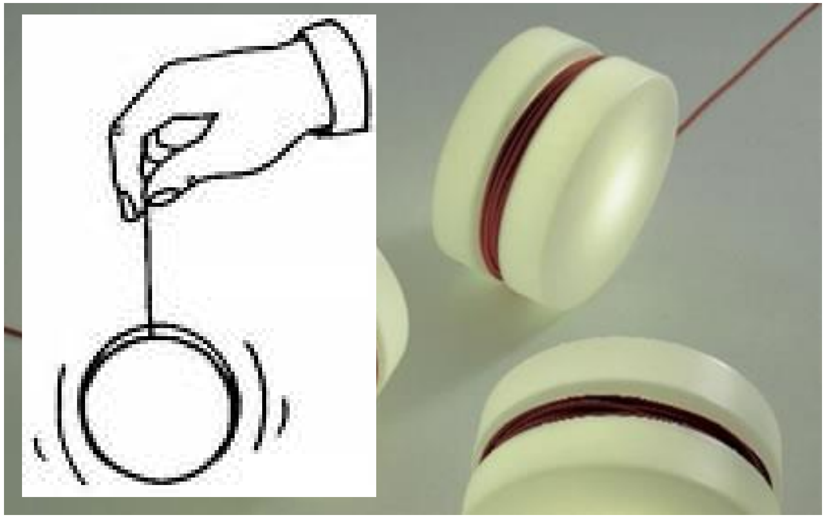
Lecture 10: Rolling Motion and Angular Momentum

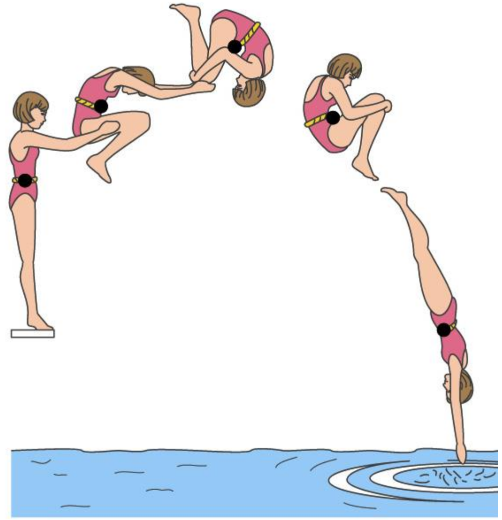
Outline

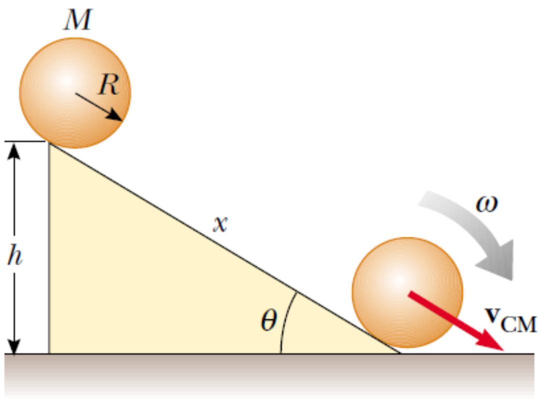
- **Rolling motion of a rigid object: center-of-mass motion + rotation around**
- **Generic rotational motion in the vector language**
  - **Kinematics/Energy/Torque**
- **Angular momentum of**
  - **A particle**
  - **A rotating rigid object**
- **Conservation of angular momentum**

Rotation plus translation motion

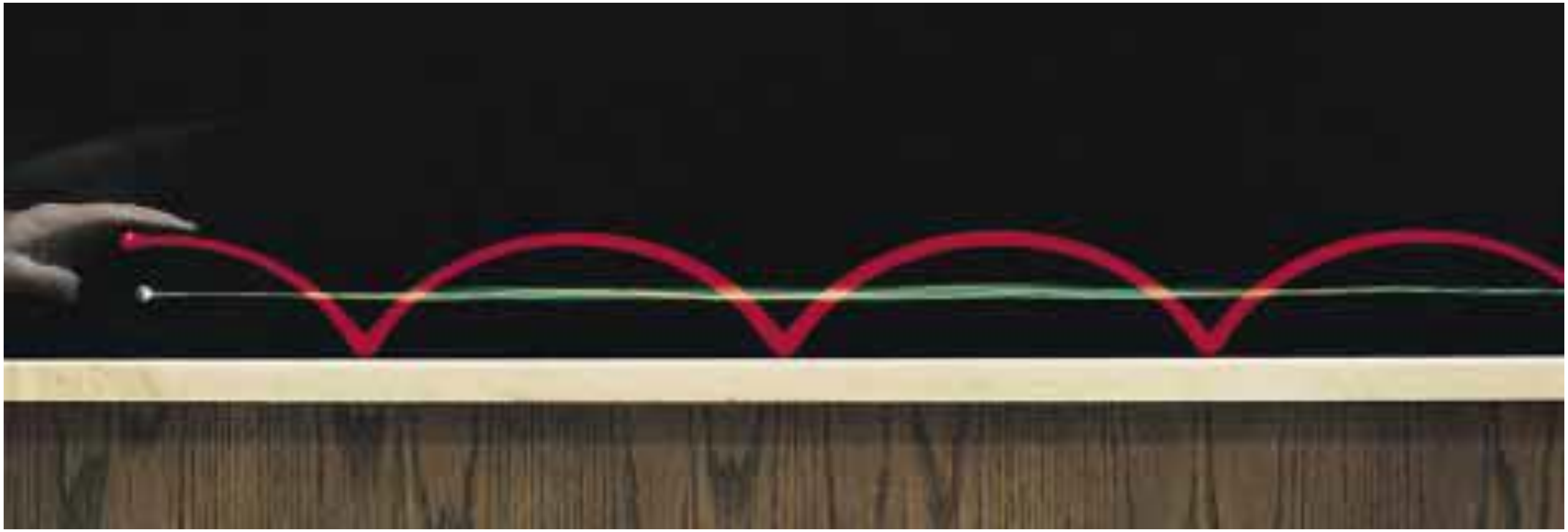






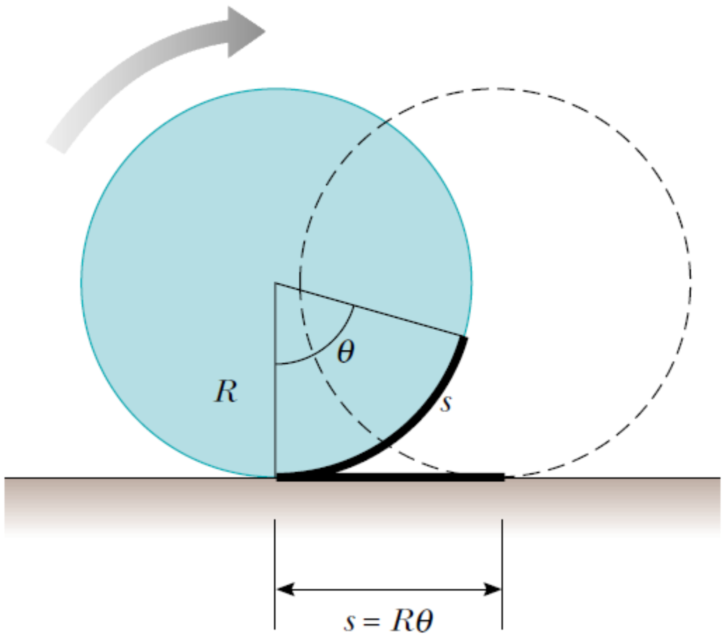


A Cycloid



One light source at the center of a rolling cylinder and another at one point on the rim illustrate the different paths these two points take. The center moves in a straight line (green line), whereas the point on the rim moves in the path called a cycloid (red curve).

Pure Rolling Motion



$$v_{\mathrm{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

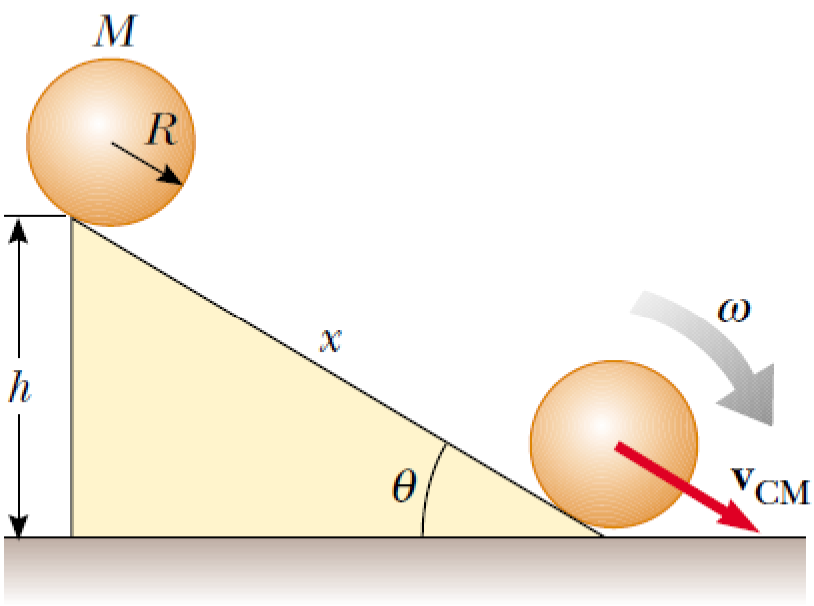
condition for pure rolling motion

Frictional force?

$$a_{\mathrm{CM}} = \frac{dv_{\mathrm{CM}}}{dt} = R\frac{d\omega}{dt} = R\alpha$$

Example: Rolling Sphere

- **For the solid sphere shown below, calculate the linear speed of the center of mass at the bottom of the incline and the magnitude of the linear acceleration of the center of mass.**



- Accelerated rolling motion is possible only if a frictional force is present between the sphere and the incline to produce a net torque about the center of mass.
- Despite the presence of friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant.

Moment of Inertia

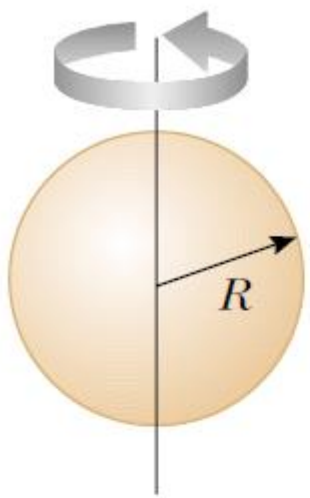
$$K = \tfrac{1}{2} I_{\mathrm{CM}} \left(\frac{v_{\mathrm{CM}}}{R} \right)^2 + \tfrac{1}{2} M {v_{\mathrm{CM}}}^2$$

$$K = \tfrac{1}{2} \left(\frac{I_{\mathrm{CM}}}{R^2} + M \right) {v_{\mathrm{CM}}}^2$$

Now, try it yourself.

Solid sphere

$$I_{\mathrm{CM}} = \frac{2}{5} MR^2$$

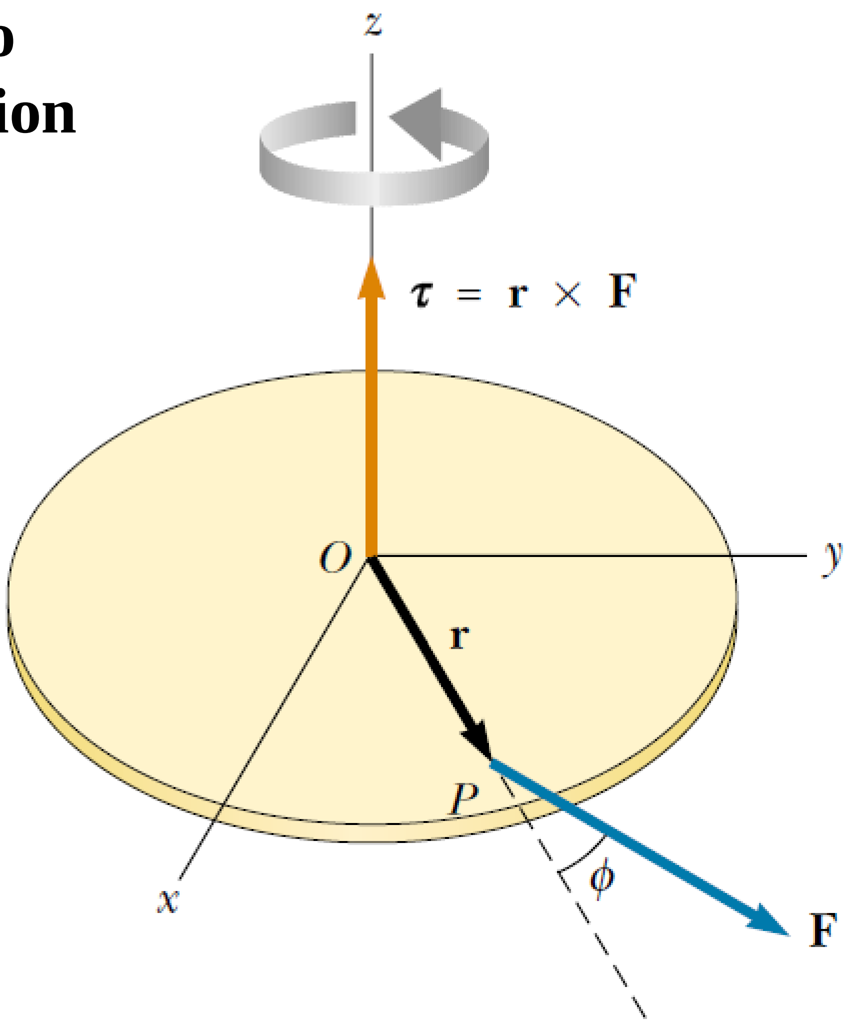


Torque

- **The torque τ involves the two vectors r and F, and its direction is perpendicular to the plane of r and F. We can establish a mathematical relationship between τ, r, and F, using a new mathematical operation called the vector product, or cross product**

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}$$

Will discuss more next week.



Angular Momentum: Motivation

Newton's second law $\Sigma \mathbf{F} = d\mathbf{p}/dt$

$$\Rightarrow \sum \boldsymbol{\tau} = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \cancel{\frac{d\mathbf{r}}{dt} \times \mathbf{p}}$$

$$\vec{v} \times (m\vec{v}) = 0$$

$$\Rightarrow \boxed{\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

(rotational analog)

$$\boxed{\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}}$$

Angular momentum

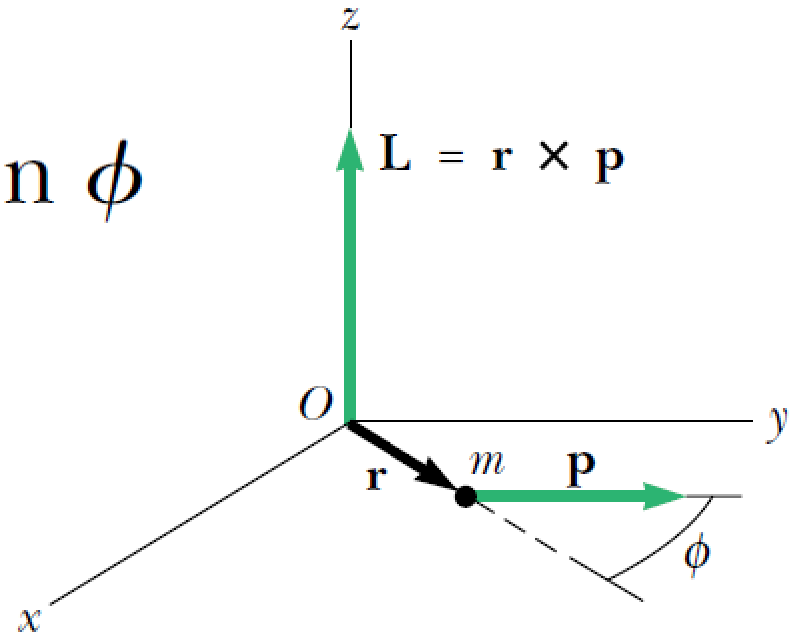
Angular Momentum: Definition

- **The instantaneous angular momentum L of the particle relative to the origin *O* is defined as the cross product of the particle's instantaneous position vector r and its instantaneous linear momentum p.**

Magnitude: $L = mvr\sin\phi$

SI Unit: $\mathrm{kg\cdot m^2/s}$

L is zero when r is parallel or antiparallel to p.



Dependence of the Origin

- **The net torque acting on a particle is equal to the time rate of change of the particle's angular momentum.**

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

- **Both the magnitude and the direction of L depend on the choice of origin.**

- **The expression is valid for any origin fixed in an inertial frame, as long as L and τ are measured about the same origin.**

A System of Particles

- **The time rate of change of the total angular momentum of a system about some origin in an inertial frame equals the net external torque acting on the system about that origin.**

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i$$

$$\sum \boldsymbol{\tau}_{\text{ext}} = \sum_i \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt}\sum_i \mathbf{L}_i = \frac{d\mathbf{L}}{dt}$$

The moment arm *d* from *O* to the line of action of the internal forces is equal for both particles.

Recall $\Sigma \mathbf{F}_{\text{ext}} = d\mathbf{p}/dt$

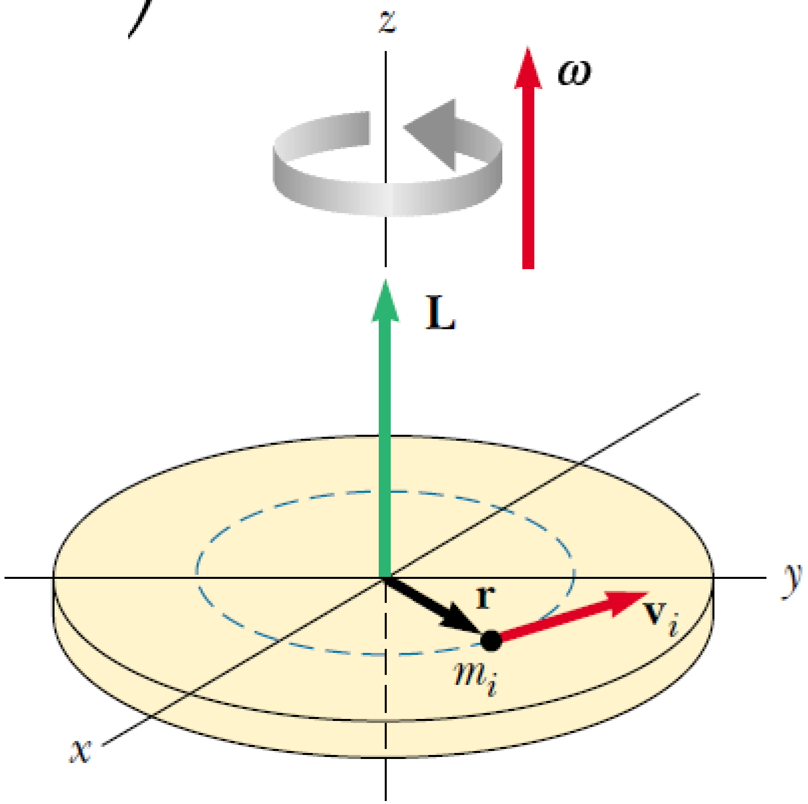
A Rotating Rigid Object

$$L_z = \sum_i m_i r_i^2 \omega = \left(\sum_i m_i r_i^2\right)\omega$$

$$L_z = I\omega$$

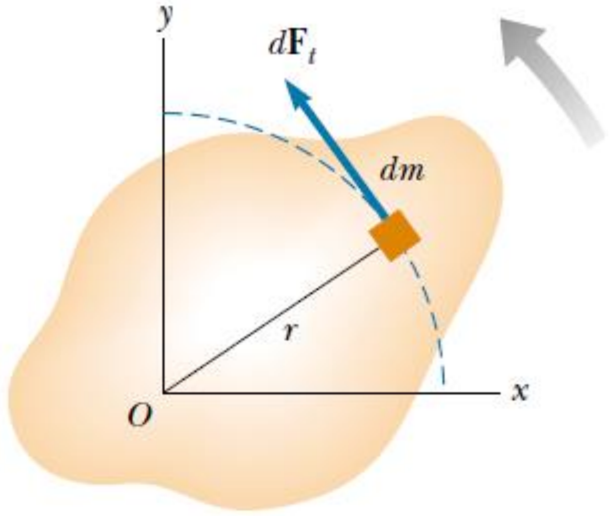
$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

$$\sum \tau_{\text{ext}} = \frac{dL_z}{dt} = I\alpha$$



The Scalar Version

- **Now extend to a rigid object of arbitrary shape rotating about a fixed axis.**



$$dF_t = (dm)\, a_t$$

$$d\boldsymbol{\tau} = r\, dF_t = (r\, dm)\, a_t$$

$$d\boldsymbol{\tau} = (r\, dm)\, r\alpha = (r^2\, dm)\, \alpha$$

$$\sum \boldsymbol{\tau} = \int (r^2\, dm)\, \alpha = \alpha \int r^2\, dm = I\alpha$$

***I*: The moment of inertia**

What's Further?

$$\vec{L} = \sum \vec{r_i} \times m_i \vec{v_i} = \sum \vec{r_i} \times m_i (\vec{\omega} \times \vec{r_i})$$

$$= \sum \left[m_i r_i^2 \vec{\omega} - m_i \vec{r_i} (\vec{r_i} \cdot \vec{\omega}) \right]$$

Identity: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

Let $\vec{r_i} = x_i \hat{x} + y_i \hat{y} + z_i \hat{z}$ $\quad \vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}$

$$L_x = \sum m_i \left(y_i^2 + z_i^2\right) \omega_x - \sum m_i x_i y_i \omega_y - \sum m_i x_i z_i \omega_z$$

$$L_y = -\sum m_i y_i x_i \omega_x + \sum m_i \left(z_i^2 + x_i^2\right) \omega_y - \sum m_i y_i z_i \omega_z$$

$$L_z = -\sum m_i z_i x_i \omega_x - \sum m_i z_i y_i \omega_y + \sum m_i \left(x_i^2 + y_i^2\right) \omega_z$$

Moment of Inertia is a Matrix*

$$L_x = \sum m_i\,(y_i^2 + z_i^2)\omega_x - \sum m_i\,x_i y_i \omega_y - \sum m_i\,x_i z_i \omega_z$$

$$L_y = -\sum m_i\,y_i x_i \omega_x + \sum m_i\,(z_i^2 + x_i^2)\omega_y - \sum m_i\,y_i z_i \omega_z$$

$$L_z = -\sum m_i\,z_i x_i \omega_x - \sum m_i\,z_i y_i \omega_y + \sum m_i\,(x_i^2 + y_i^2)\omega_z$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

dyad product

rank-2 tensor

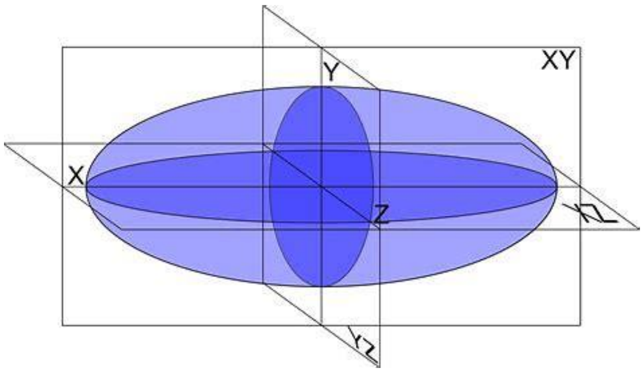
$$\bar{I} = \sum [m_i \vec{r_i} \cdot \vec{r_i} - m_i \vec{r_i}\vec{r_i}]$$

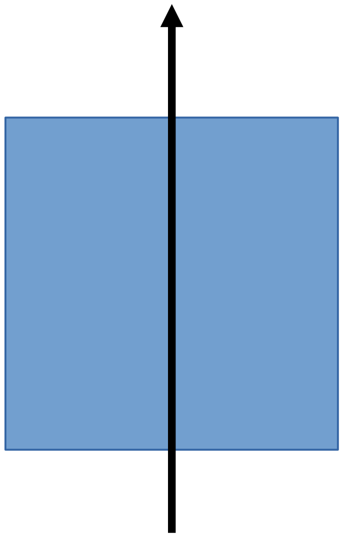
Now you should be ready to advance by yourself.

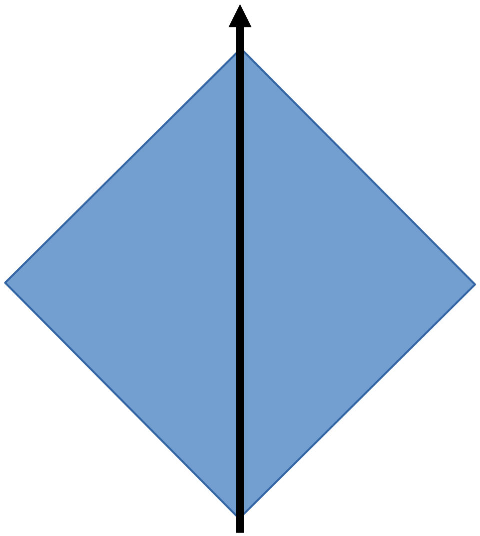
Symmetry Tells Us*

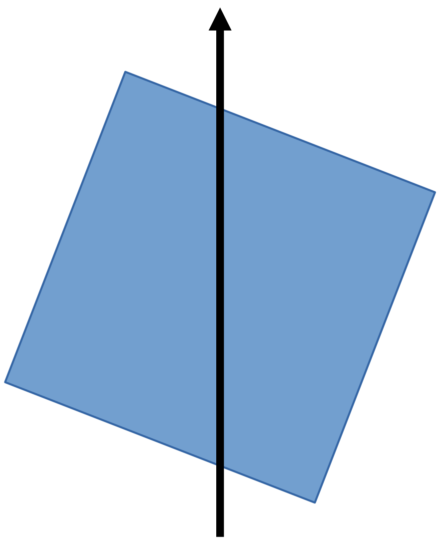
$$\bar{I} = \begin{pmatrix} I_0 & 0 & 0 \\ 0 & I_0 & 0 \\ 0 & 0 & I_0 \end{pmatrix}$$

$$\bar{I} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_0 & 0 \\ 0 & 0 & I_0 \end{pmatrix}$$







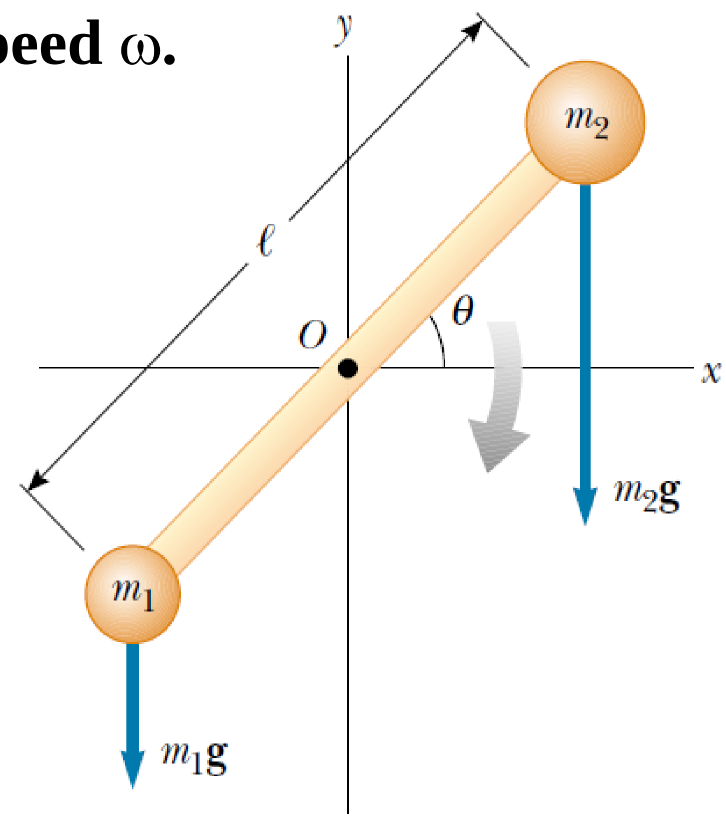


Exercise

A rigid rod of mass M and length l is **pivoted without friction at its center**. Two particles of masses m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed ω.

(a) Find an expression for the magnitude of the angular momentum of the system.

(b) Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizontal.



Solution

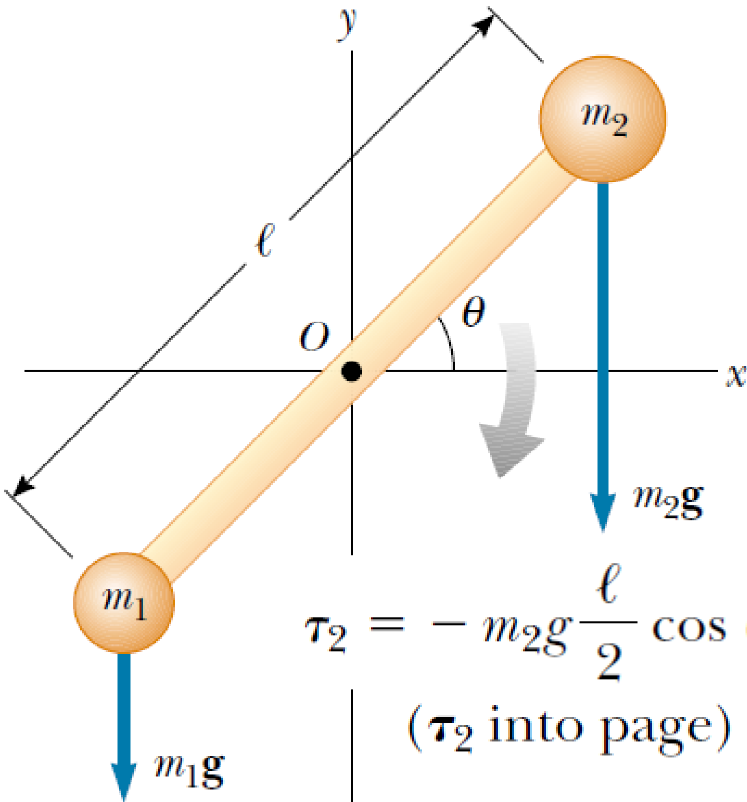
$$I = \frac{1}{12}M\ell^2 + m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2$$

$$= \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)$$

$$L = I\omega = \boxed{\frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega}$$

$$\sum \tau_{\text{ext}} = \tau_1 + \tau_2 = \tfrac{1}{2}(m_1 - m_2)g\ell\cos\theta$$

$$\alpha = \frac{\sum\tau_{\text{ext}}}{I} = \boxed{\frac{2(m_1 - m_2)g\cos\theta}{\ell(M/3 + m_1 + m_2)}}$$



$$\boldsymbol{\tau}_2 = -m_2 g\frac{\ell}{2}\cos\theta$$

($\boldsymbol{\tau}_2$ into page)

$$\boldsymbol{\tau}_1 = m_1 g\frac{\ell}{2}\cos\theta$$

($\boldsymbol{\tau}_1$ out of page)

Conservation of Angular Momentum

- **The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.**

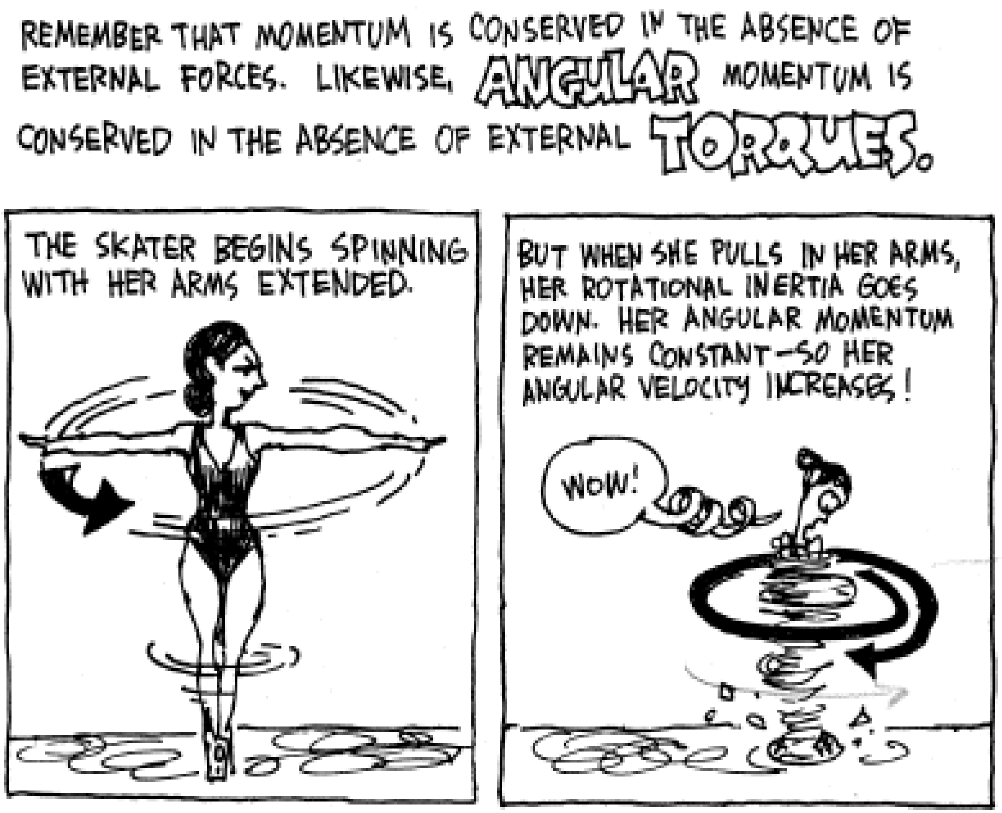
$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt} = 0 \quad \Rightarrow \quad \mathbf{L} = \text{constant}$$

If the system is an object rotating about a fixed axis, such as the *z* axis,

$$I_i\omega_i = I_f\omega_f = \text{constant}$$

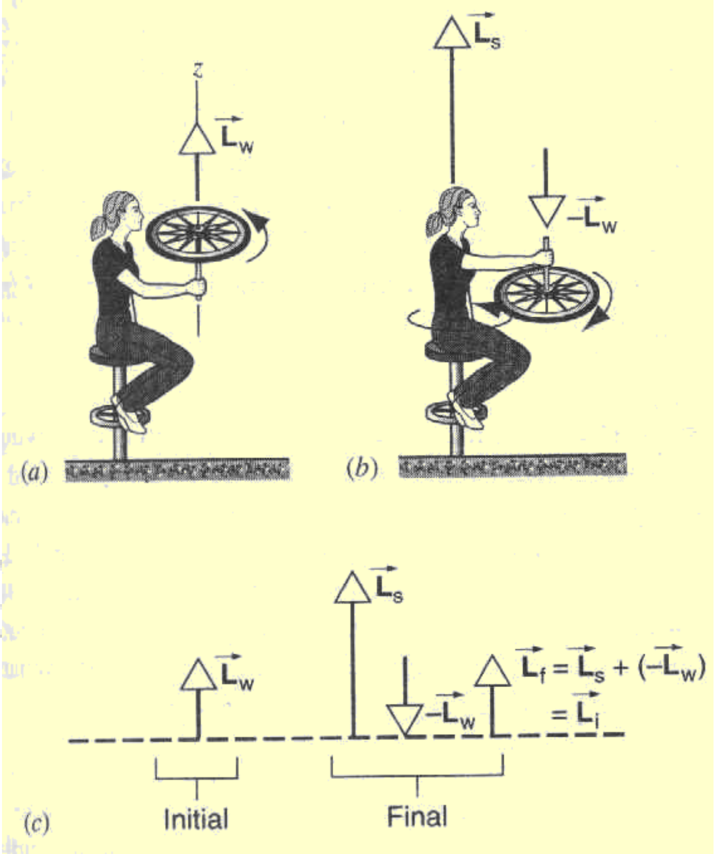
Conservation of Angular Momentum

Figure Skating



Anything wrong?

The rotating bicycle wheel



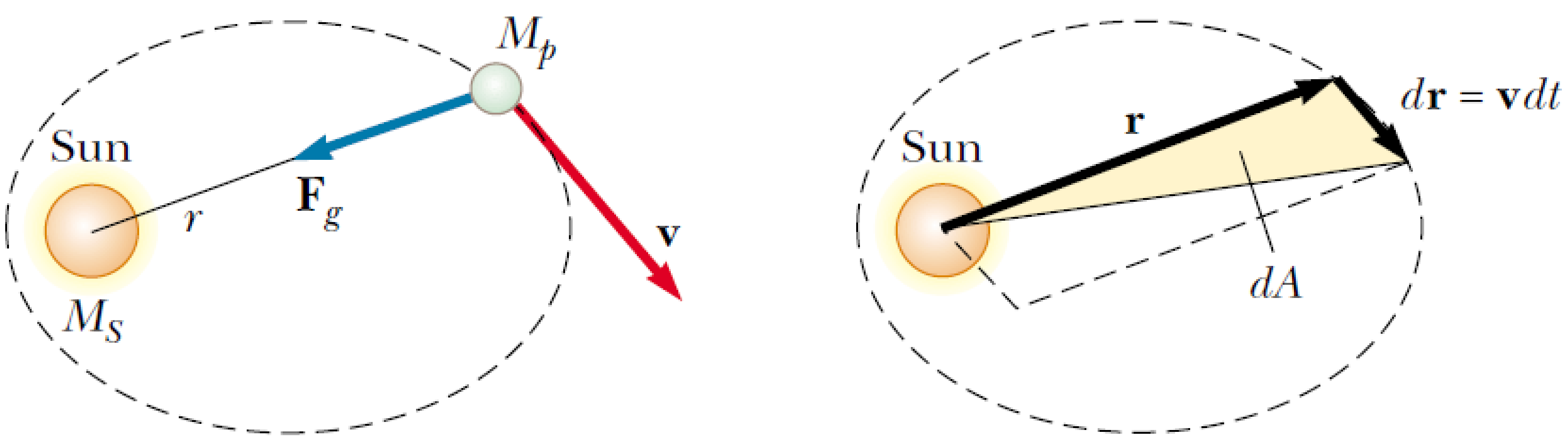
Video—MIT angular momentum

Kepler's Second Law

- **When a force is directed toward or away from a fixed point and is function of *r* only, it is called a central force.**

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F\hat{\mathbf{r}} = 0$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times M_p\mathbf{v} = M_p\mathbf{r} \times \mathbf{v} = \text{constant}$$

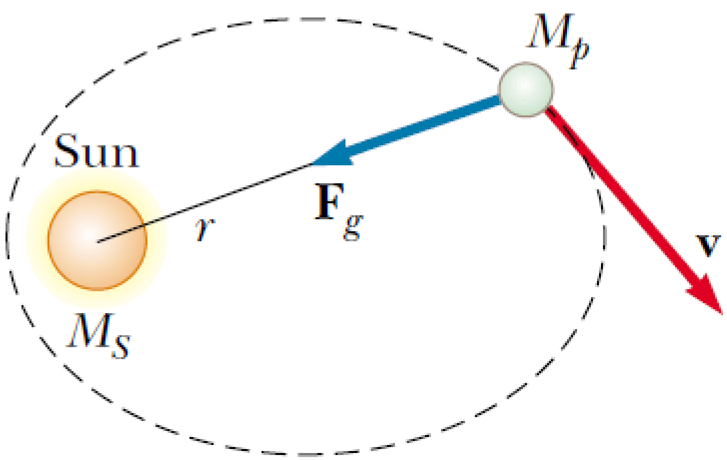


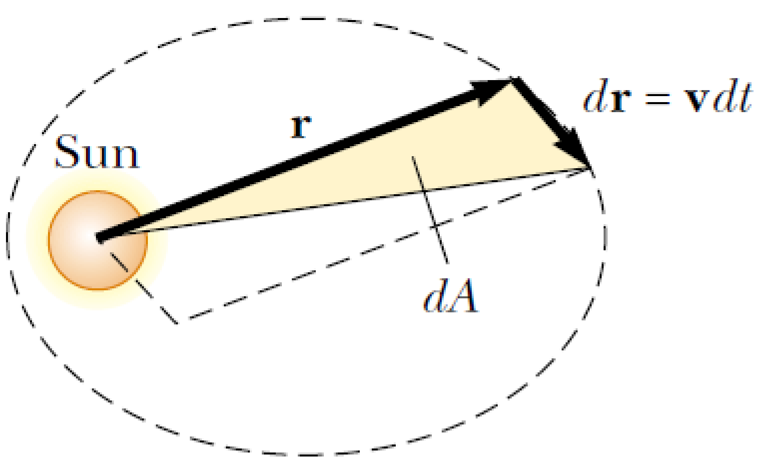
Kepler's Second Law

- **The radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.**

$$dA = \tfrac{1}{2}\left|\mathbf{r} \times d\mathbf{r}\right| = \tfrac{1}{2}\left|\mathbf{r} \times \mathbf{v}\, dt\right| = \frac{L}{2M_p}\, dt$$

$$\boxed{\frac{dA}{dt} = \frac{L}{2M_p} = \text{constant}}$$





Angular Momentum Dynamics of Center of Mass

Define： $\vec{r}' = \vec{r} - \vec{r}_{CM}$

$$M_E = \sum \vec{r}_i \times \vec{f}_{iE} = \sum \vec{r}_i\,' \times \vec{f}_{iE} + \vec{r}_{CM} \times \vec{F}_E$$

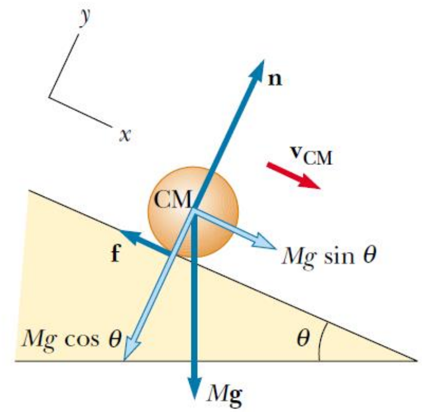
$$\boxed{M_E = M_E' + \vec{r}_{CM} \times \vec{F}_E}$$

$$L = \sum \vec{r}_i \times m_i \vec{v}_i = \sum m_i \left(\vec{r}_i' + \vec{r}_{CM}\right) \times \left(\vec{v}_i' + \vec{V}_{CM}\right)$$

$$= \sum m_i \vec{r}_i' \times \vec{v}_i' + \vec{r}_{CM} \times \cancel{\sum m_i \vec{v}_i'}$$

$$+ \cancel{\sum m_i \vec{r}_i'} \times \vec{V}_{CM} + \sum m_i \vec{r}_{CM} \times \vec{V}_{CM}$$

$$\boxed{L = L_c + \vec{r}_{CM} \times \vec{p}_{CM}}$$



$$\Sigma F_x = Mg \sin\theta - f = Ma_{CM}$$

$$\Sigma F_y = n - Mg\cos\theta = 0$$

$$\tau_{CM} = fR = I_{CM}\,\alpha$$

$$\boxed{\frac{d\vec{p}_{CM}}{dt} = m\vec{a}_{CM} = \vec{F}_E}$$

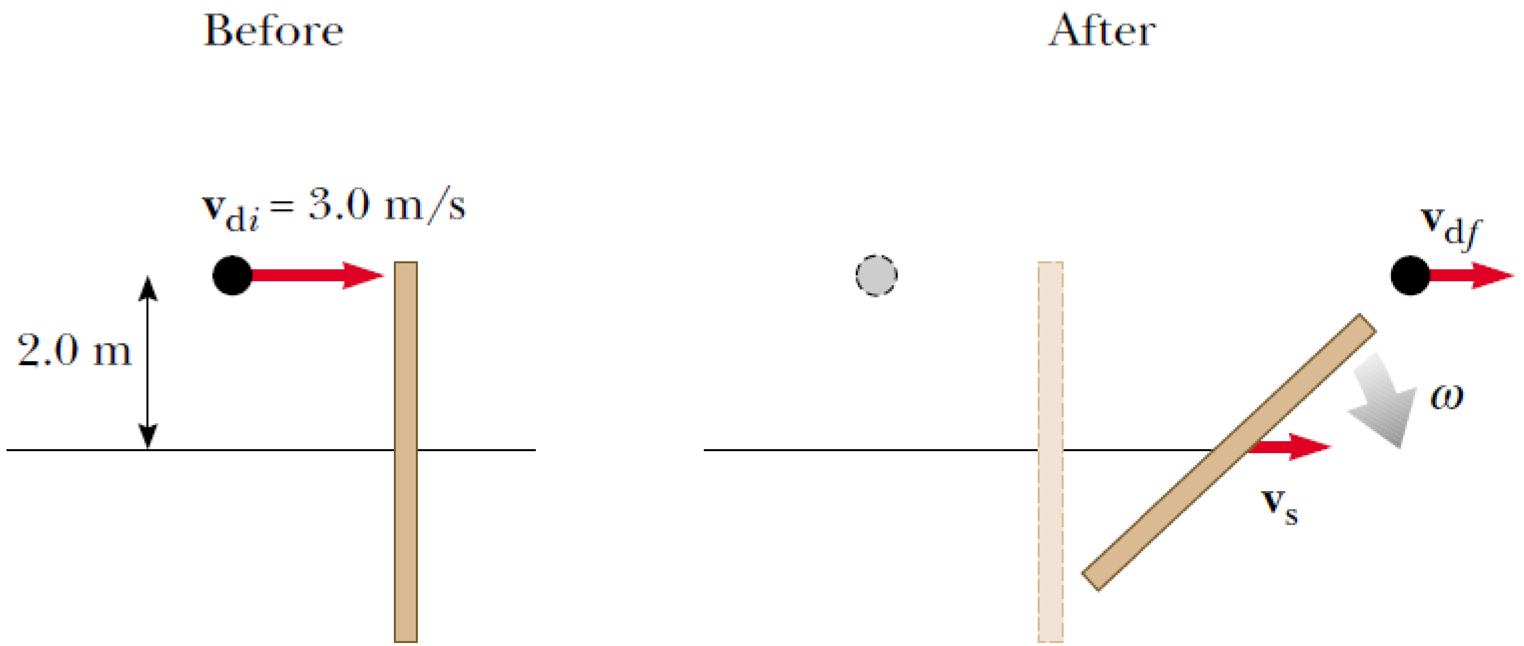
$$\boxed{\frac{d\vec{L}_C}{dt} = \vec{M}_E\,'}$$

Isolated Systems

- **The (mechanical) energy, linear momentum, and angular momentum of an isolated system all remain constant.**

$$\left.\begin{aligned} K_i + U_i &= K_f + U_f \\ \mathbf{p}_i &= \mathbf{p}_f \\ \mathbf{L}_i &= \mathbf{L}_f \end{aligned}\right\} \quad \text{For an isolated system}$$

Example: Ball and Stick

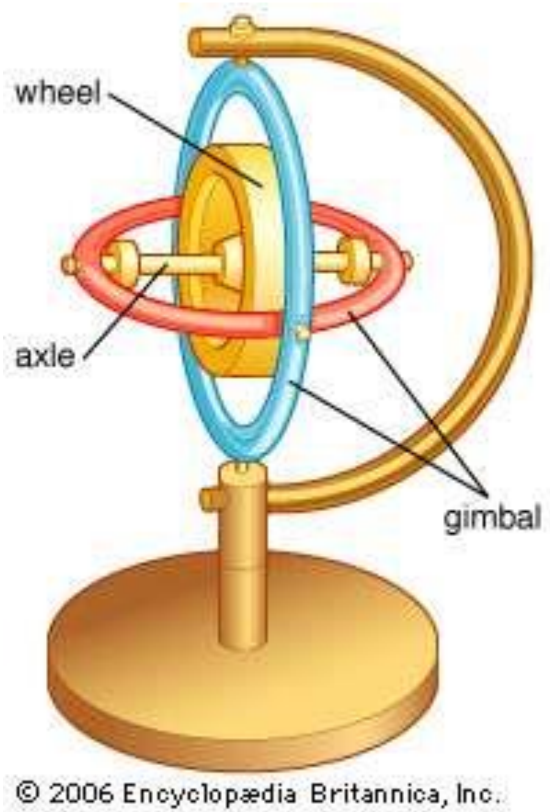


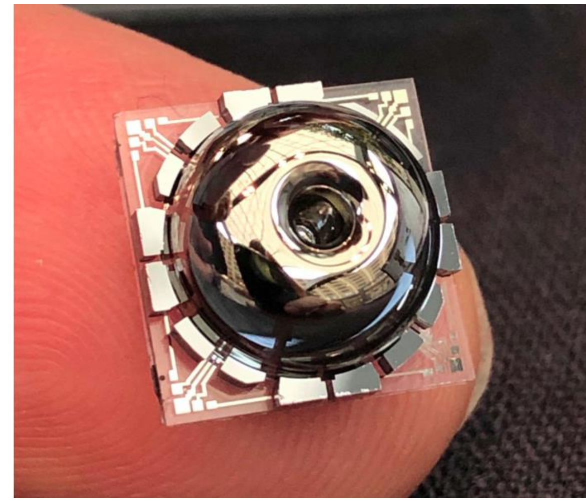
$$m_d v_{di} = m_d v_{df} + m_s v_s$$

$$-r m_d v_{di} = -r m_d v_{df} - I\omega$$

$$\frac{1}{2} m_d v_{di}{}^2 = \frac{1}{2} m_d v_{df}{}^2 + \frac{1}{2} m_s v_s{}^2 + \frac{1}{2} I\omega^2$$

Gyroscope*



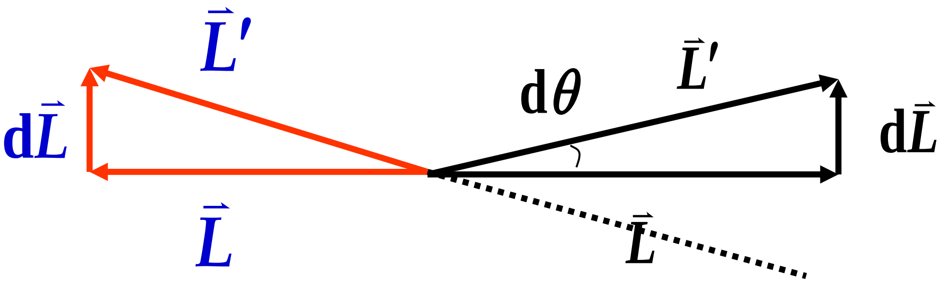


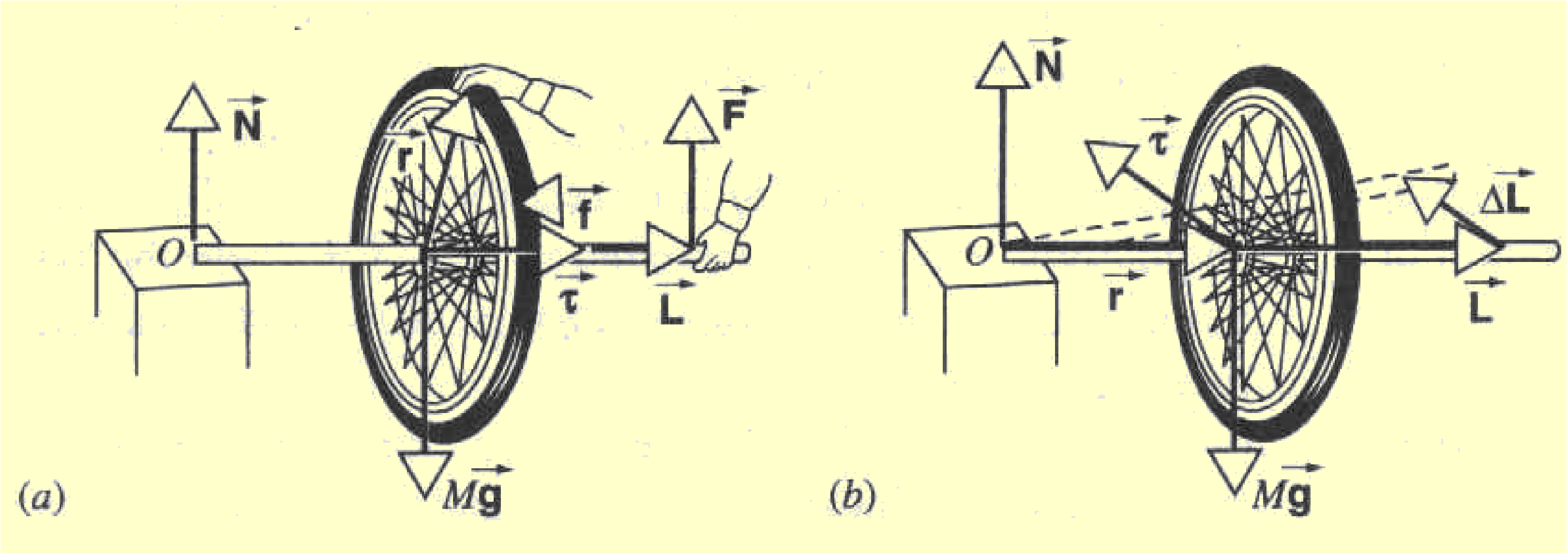
Video—MIT gyroscope

Precession: Rotation of the angular momentum*

$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}_{nex} = \vec{\tau}_{ext}$$

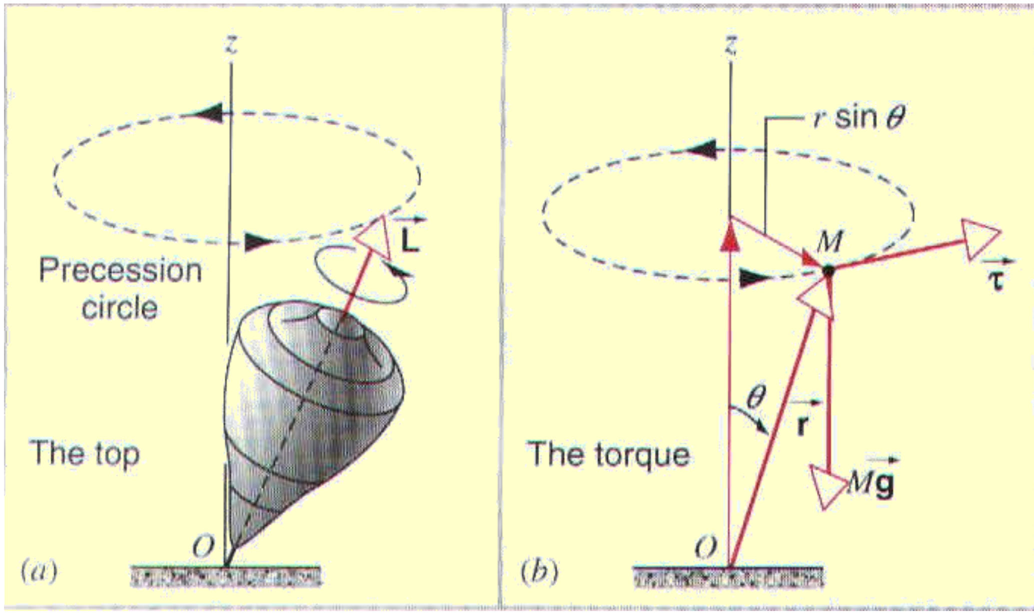
$$\Delta\vec{L} = \vec{\tau}_{ext}\Delta t$$

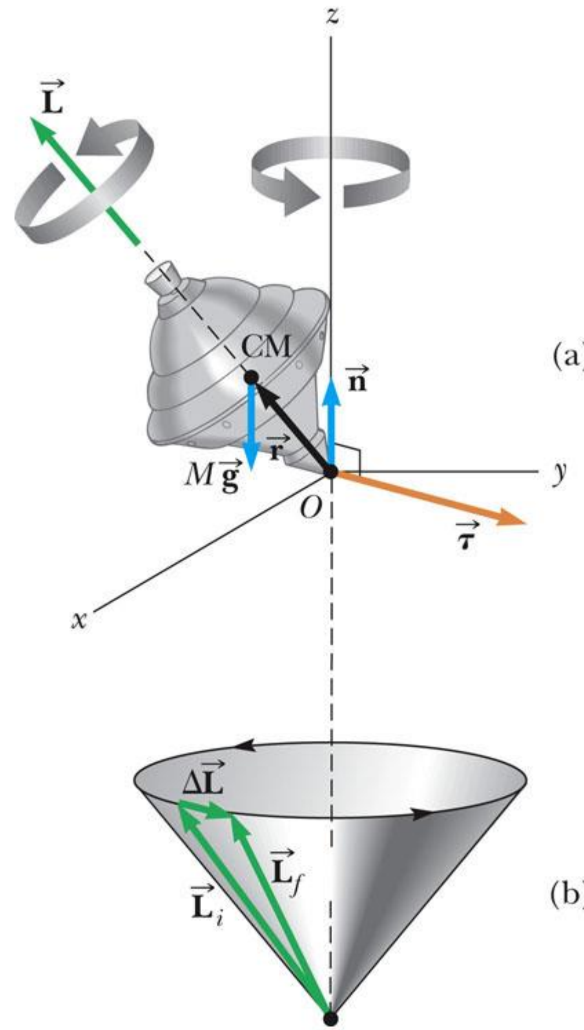




Video—MIT precession

Precession of Gyroscope*





$$\tau = Mgr\sin\theta; \quad \text{and} \quad \vec{\tau} \perp \vec{L}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad d\vec{L} = \vec{L}(t+dt) - \vec{L}(t) = \vec{\tau}dt$$

Precession of Gyroscope*

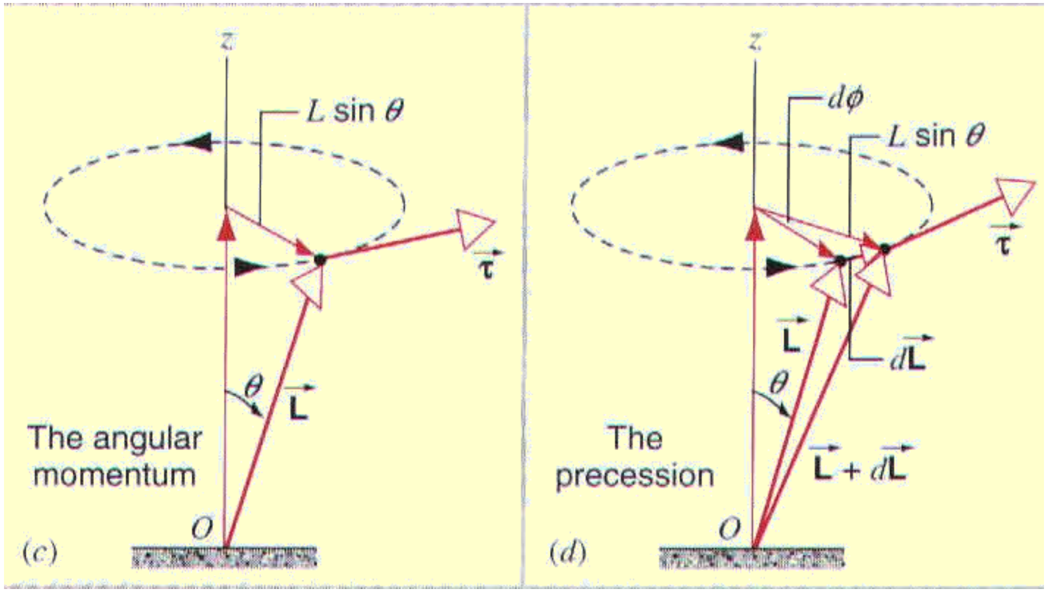
$$\mathrm{d}L = L\sin\theta\,\mathrm{d}\phi$$

$$\mathrm{d}\phi = \frac{\mathrm{d}L}{L\sin\theta} = \frac{\tau\mathrm{d}t}{L\sin\theta}$$

$$\omega_p = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\tau}{L\sin\theta}$$

$$= \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L}$$

$$\vec{\tau} = \vec{\omega}_P \times \vec{L}$$



Applications of the precession:

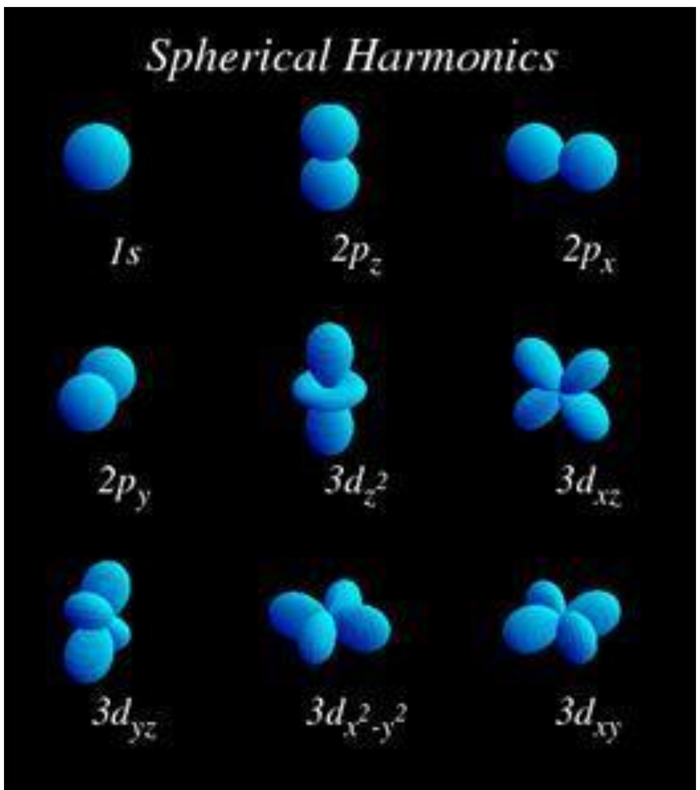
What happen if the direction of spinning is reversed?

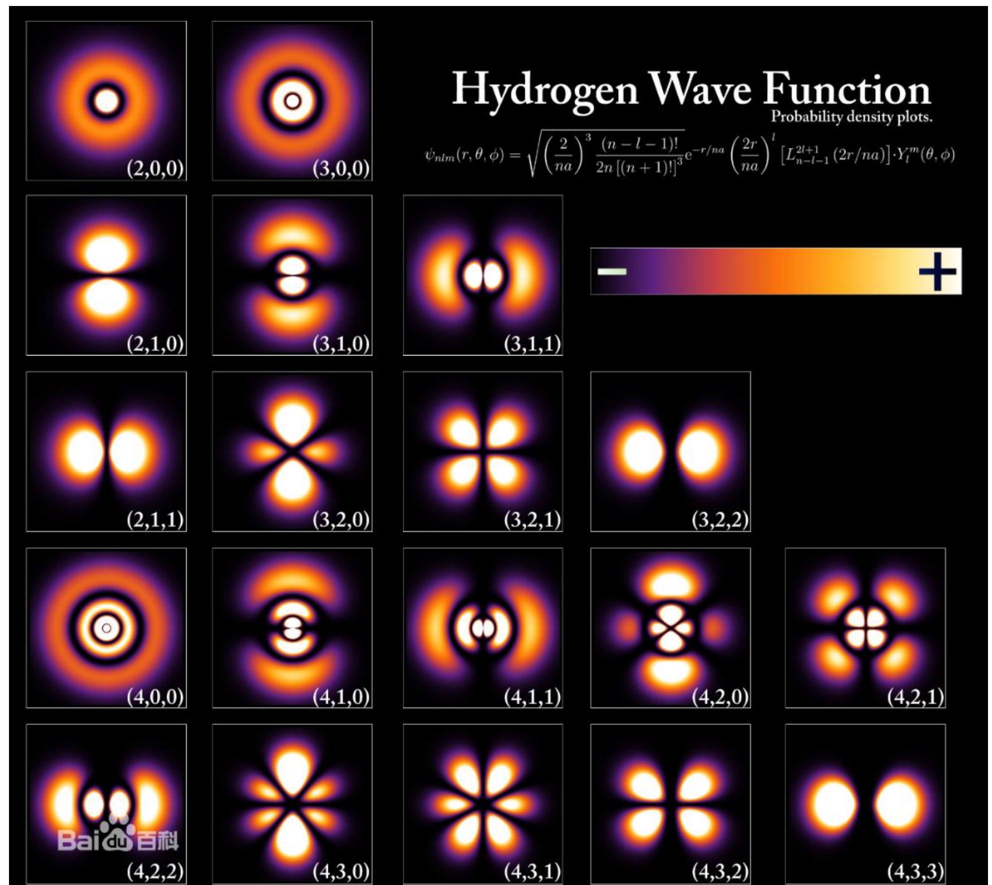
Video—MIT precession of gyroscope

Angular Momentum in atoms*

$$\vec{\mu} = -\frac{e}{2m}\vec{L}$$

$$L = \sqrt{l(l+1)}\,\frac{h}{2\pi} = \sqrt{l(l+1)}\,\hbar \qquad l = 0,1,2,\ldots,(n-1)$$





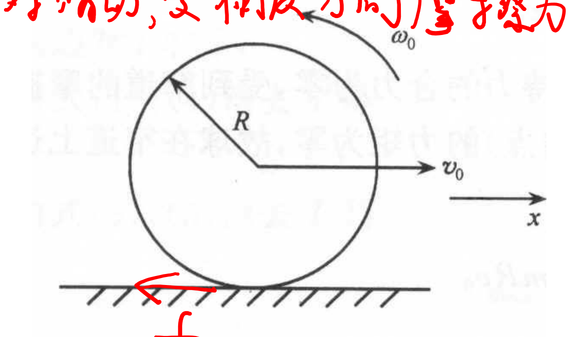
Rotation with Slipping!

【实际问题研究】 一个质量为M，半径为R的轮子以质心速度v_0，角速度ω_0沿水平表面投射，如图所示，ω_0倾向于产生与v_0相反方向地质心速度，轮子与表面间的静摩擦因素为μ.

（1）经多长时间停止滑动;

（2）滑动停止时轮子的质心速度多大?

因为接触点 $V_p = V_0 + \omega_0 R > 0$ 有相对滑动，受相反方向摩擦力 f



$f = -\mu N$ (left-hand)

质心CM： $f = ma = m\frac{dv}{dt}$

$\Rightarrow ft = m\Delta v = m(v' - v_0)$ ①

转动： $fR = I\alpha = I\frac{d\omega}{dt} \Rightarrow fRt = I(-\omega_0 + \omega')$ ②

③ 无滑滚动下 $v' = -\omega' R$ （v'，ω' 二者必有一个反向）

三式联立可求解

$$\begin{cases} \omega' = \frac{1}{3}\left(\omega_0 - 2\frac{v_0}{R}\right) \\ v' = \frac{1}{3}\left(2v_0 - \omega_0 R\right) \\ t = \frac{(v_0 + \omega_0 R)}{3\mu g} \end{cases}$$