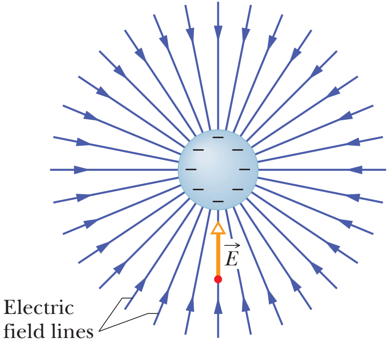
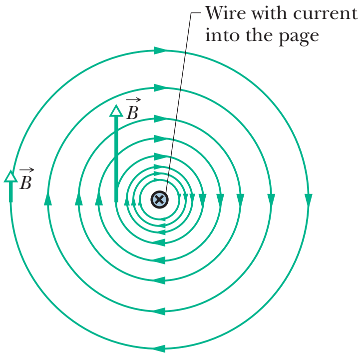
Faraday's Law of Induction

Xin Lu/Gentaro Watanabe

Lecture 11

Magnetostatics vs Electrostatics





$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Toward the Unification of E&M

- We have understood well that, surprisingly or not, an electric current produces a magnetic field.
- Perhaps even more surprising was the discovery of the reverse effect: A magnetic field can produce an electric field that can drive a current.
- This link between a magnetic field and the electric field it induces is now called **Faraday's law of induction**.
- The bidirectional connection between electric fields and magnetic fields eventually led Maxwell to the *unification of electricity and magnetism*.

Outline

- Faraday's Law of Induction
- Lenz's Law
- Induction and Energy Transfers
- Induced Electric Fields
- Faraday's Law in the Differential Form

First Experiment

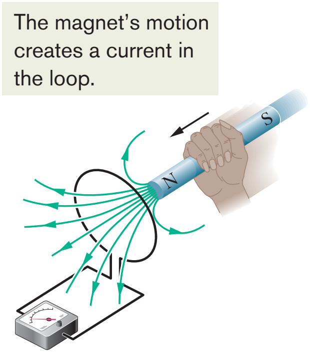


Figure 1: A conducting loop connected to a sensitive ammeter.

- If we move a bar magnet toward the loop, a current suddenly appears in the circuit.
- The current disappears when the magnet stops.
- If we then move the magnet away, a current again suddenly appears, but now in the opposite direction.

1. A current appears only if there is relative motion between the loop and the magnet (one must move relative to the other); the current disappears when the relative motion between them ceases.
2. Faster motion produces a greater current.
3. If moving the magnet's north pole toward the loop causes, say, clockwise current, then moving the north pole away causes counterclockwise current. Moving the south pole toward or away from the loop also causes currents, but in the reversed directions.

- The current produced in the loop is called an **induced current**; the work done per unit charge to produce that current (to move the conduction electrons that constitute the current) is called an **induced emf**; and the process of producing the current and emf is called **induction**.
- The induced emf and induced current in the experiment are apparently caused when something changes — but what is that "something"?
    - Magnetic field? Or changing magnetic field?
    - Along the loop? Or enclosed by the loop?
    - How do we quantify?

Second Experiment

- If we close switch S, to turn on a current in the right-hand loop, the meter suddenly and briefly registers an induced current in the left-hand loop.
- If we then open the switch, another sudden and brief induced current appears in the left-hand loop *in the opposite direction*.

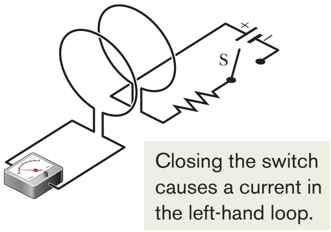


Figure 2: Two conducting loops close to each other but not touching.

- We get an induced current only when the current in the right-hand loop is changing and not when it is constant (even if it is large).

Faraday's Law of Induction

- Faraday realized that an emf and a current can be induced in a loop, as in the two experiments, by changing the amount of magnetic field passing through the loop.
- He further realized that the "amount of magnetic field" can be visualized in terms of the magnetic field lines passing through the loop.
- Faraday's law of induction, stated in terms of the experiments, is that an emf is induced in a close loop when *the number of magnetic field lines that pass through the loop is changing*.

Quantitative Treatment

- First, we define the **magnetic flux** through the loop as

$$\Phi_B = \int \vec{B} \cdot d\vec{A},$$

  where the integration is over the area A enclosed by the loop.
- Quantitatively, Faraday's law of induction states that *the magnitude of the emf $\mathcal{E}$ induced in a conducting loop is equal to the rate at which the magnetic flux Φ_B through that loop changes with time*, i.e.,

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

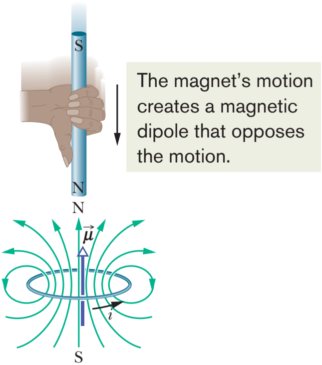
- For a coil of N turns, the total induced emf is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}.$$

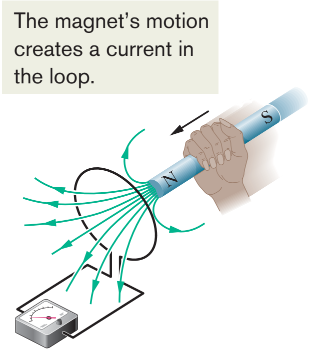
- The magnetic flux through a coil can be changed by
  - the change of the strength of $\vec{B}$ or the angle of $\vec{B}$, or
  - the change of the total area of the coil or the portion of that area that lies within the magnetic field.
- To understand the minus sign, we move on to Lenz's rule for determining the direction of an induced current in a loop.

Lenz's Law

- An induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current.
- The direction of an induced emf is that of the induced current.

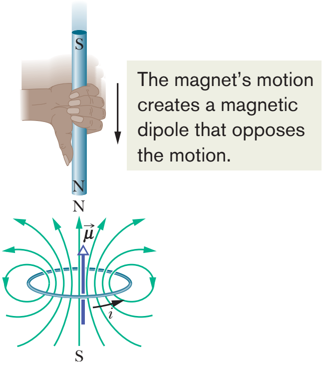


1. Opposition to Flux Change



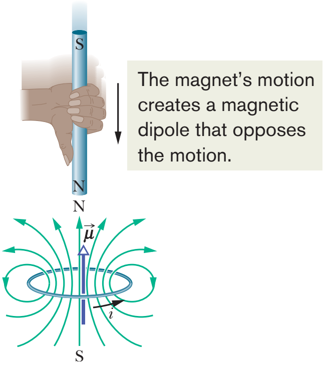
- As the north pole of the magnet approaches the loop with its magnetic field $\vec{B}$ directed leftward, the flux through the loop increases.
- To oppose the flux increase, the induced current i must set up its own field $\vec{B}_{\rm ind}$ directed rightward inside the loop.
- The curled-straight right-hand rule then tells us that i must be clockwise.

2. Opposition to Pole Movement



- The approach of the magnet's north pole increases the magnetic flux through the loop and thereby induces a current in the loop.
- The loop then acts as a magnetic dipole with a south pole and a north pole, and that its magnetic dipole moment $\vec{\mu}$ is directed from south to north.

2. Opposition to Pole Movement



- To oppose the magnetic flux increase being caused by the approaching magnet, the loop's north pole (and thus $\vec{\mu}$) must face toward the approaching north pole so as to repel it.
- Then the curled-straight right-hand rule for $\vec{\mu}$ tells us that the current induced in the loop must be counterclockwise.

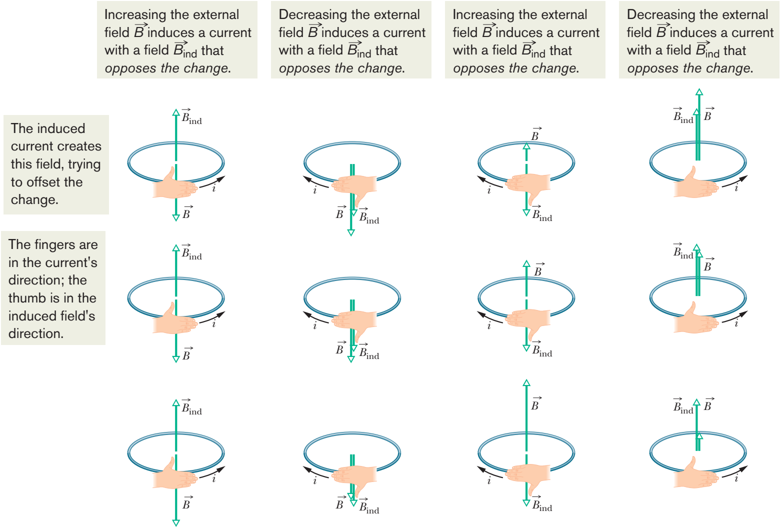


Figure 3: $\vec{B}_{\text{ind}}$ is not always opposite $\vec{B}$.

Induction and Energy Transfers

- When you move the magnet toward or away from the loop, a magnetic force resists the motion, so Lenz's law requires your applied force to do positive work.
- At the same time, thermal energy is produced in the material of the loop because of the material's electrical resistance to the induced current.
- The energy you transfer to the closed loop-magnet system via your applied force ends up in this thermal energy. (For now, we neglect energy that is radiated away from the loop as electromagnetic waves during the induction.)

- We wish to find an expression for your pulling force F in terms of magnetic field B, and the resistance R and the dimension L of the loop.
- The magnitude of the flux through the loop is

$$\Phi_B = BLx.$$

- Faraday's law (with the minus sign dropped) says

$$\mathcal{E} = \frac{d\Phi_B}{dt} = BL\frac{dx}{dt} = BLv.$$

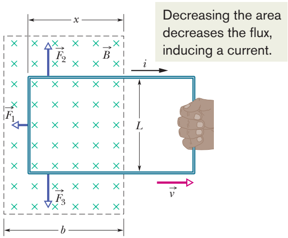
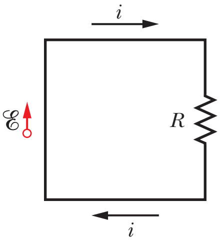


Figure 4: A rectangular loop of wire of width L has one end in a uniform external perpendicular magnetic field. The loop is pulled to the right at a constant velocity $\vec{v}$.

- The direction of the induced current i can be determined by Lenz's law. For the effective circuit, we can write



$$i = \frac{\mathcal{E}}{R} = \frac{BLv}{R}.$$

- In the magnetic field, the current-carrying loop segments are subject to deflecting forces

$$\vec{F}_d = i\vec{L} \times \vec{B}.$$

- Due to symmetry, $\vec{F}_2 = -\vec{F}_3$, so $F = F_1 = iLB$.

- The force to pull the loop at a constant velocity is thus

$$F = iLB = \frac{B^2L^2v}{R},$$

and the rate at which you do work to pull the loop is

$$P = Fv = \frac{B^2L^2v^2}{R}.$$

- On the other hand, the rate at which thermal energy appears in the loop as you pull it along at constant speed is

$$P = i^2R = \left(\frac{BLv}{R}\right)^2 R = \frac{B^2L^2v^2}{R},$$

which is exactly the rate you do work on the loop.

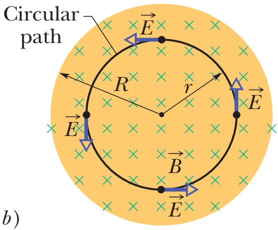
Induced Electric Fields

- When there is a physical conducting loop (say a copper ring) in an external magnetic field with steadily increasing strength, Faraday's law says there is an induced emf and thus an induced current in the loop.
- A step forward is to ask what if there is no such copper ring.
- In the presence of the copper ring, there is an emf, thus a current. There must be an electric field along the ring. Once we remove the ring, is the electric field still there?

- The electric field $\vec{E}$ must have been induced by the changing magnetic flux, and is just as real as an electric field produced by static charge; either field will exert a force $q_0\vec{E}$ on a test particle of charge q_0.
- In other words, we cannot distinguish the electric fields of different origins.
- If this is acceptable, we are led to the inspiring statement: *A changing magnetic field produces an electric field.* The electric field would appear even if the changing magnetic field were in a vacuum.

A Reformulation of Faraday's Law

- Consider a particle of charge q_0 moving around a circular path in a changing magnetic field. The work done on it in one revolution by the induced electric field is $W = q_0\mathcal{E}$.



(b)

- From mechanical point of view, the work is

$$W = \oint \vec{F} \cdot d\vec{s} = q_0 \oint \vec{E} \cdot d\vec{s}.$$

- Thus, we find that

$$\mathcal{E} = \frac{W}{q_0} = \oint \vec{E} \cdot d\vec{s}.$$

Faraday's Law Again

- We find that an induced emf can be defined without the need of a current or particle: An induced emf is the sum—via integration—of quantities $\vec{E} \cdot d\vec{s}$ around a closed path, where $\vec{E}$ is the electric field induced by a changing magnetic flux and $d\vec{s}$ is a differential length vector along the path.
- Now, we can rewrite Faraday's law as

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}.$$

Note that the choice of loop is arbitrary.

- We can convert it to differential form by applying the Stokes' theorem (or the fundamental theorem for curls)

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{A} = \oint_P \vec{v} \cdot d\vec{s}.$$

- The differential form of Faraday's law is, thus,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

  that is, precisely, *a changing magnetic field produces an electric field.*
- Here we use the partial derivative of $\vec{B}$, which may vary in space.

Electric Potential, No Longer Meaningful

- We have learned in electrostatics that $\vec{E}$ is a special kind of vector, whose curl is always zero,

$$\nabla \times \vec{E} = -\nabla \times (\nabla V) = 0.$$

- This is no longer true in electrodynamics. In fact, electric potential has no meaning for electric fields that are produced by induction, because, in general,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \neq 0,$$

so it is meaningless to define

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}.$$

Summary

- Faraday's law of induction

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Reading

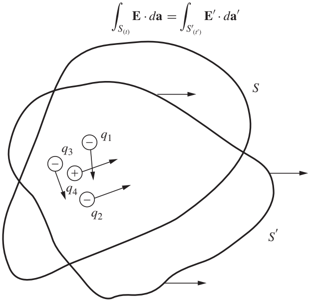
Halliday, Resnick & Krane:

- Chapter 34: Faraday's Law of Induction

Appendix 11A: Inertial Frames of Reference

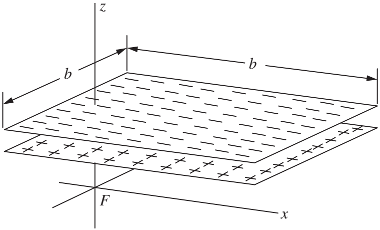
- We have learned the special theory of relativity. In particular, Einstein argued that in electromagnetism as well as in mechanics, phenomena has no properties corresponding to the concept of absolute rest.
- On the other hand, the force on a particle carrying charge q is given by
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$
in an electric field $\vec{E}$ and a magnetic field $\vec{B}$.
- It is remarkable that the magnetic part of the force is proportional to v. In which inertial frame, then?

- The first question to ask is whether the charge of a particle can be changed by the motion of the particle.
- Fortunately, experiments tell us the answer is no, although we seldom think how remarkable and fundamental this answer is.
- In fact, we have relativistic **invariance of charge**, which means that if we look at some charged matter from any frame of reference, we shall measure exactly the same amount of charge.



Appendix 11B: E-Field from Relative Motion

- We have to wait a few more lectures before we can understand the Lorentz transformation of the electromagnetic force. But we can discuss a simpler case here, namely, the electric field measured in different frames of reference.
- We consider two stationary sheets of charge of uniform density σ and $-\sigma$ in a certain inertial frame F.



- In a different frame F' that moves with velocity $\vec{v}$, with respect to F, what is the electric field?
- Suppose $\vec{v} = v\hat{x}$. The x' dimension is contracted from b to sb, where $s = \sqrt{1-(v/c)^2}$. So the charge density measured in F' must be $\sigma' = \sigma/s$.
- Gauss' law tells us that

$$E'_z = \frac{\sigma'}{\epsilon_0} = \frac{1}{s}\frac{\sigma}{\epsilon_0} = \frac{E_z}{s} = \gamma E_z,$$

where $\gamma = 1/s$.

- Now rotate the charge sheets such that they are oriented perpendicular to the x axis. The surface charge density $\sigma' = \sigma$, so

$$E'_x = \frac{\sigma'}{\epsilon_0} = \frac{\sigma}{\epsilon_0} = E_x.$$

- We can generalize our conclusion such that for fields that arise from charges stationary in frame F, we have

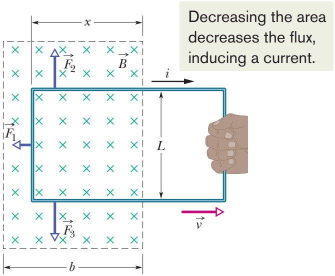
$$E'_{\parallel} = E_{\parallel}$$

$$E'_{\perp} = \gamma E_{\perp}$$

in frame F' that move with velocity $\vec{v}$ relative to F.

Appendix 11B: Induction and Relative Motion

- Motional emf is determined by the velocity of the object moving through the magnetic field, it clearly depends on the reference frame of the observer.



- For the observer S fixed with respect to the magnet, the induced emf in the moving loop is given by

$$\mathcal{E} = BLv = \oint (\vec{v} \times \vec{B}) \cdot d\vec{s} = \oint \vec{E} \cdot d\vec{s}.$$

- For the observer S' at rest with respect to the loop, charge drifts around the loop. Since there is no magnetic force, S' must postulate that an electric field $\vec{E}'$ is induced by the action of the moving magnet, which produces the induced emf is given by

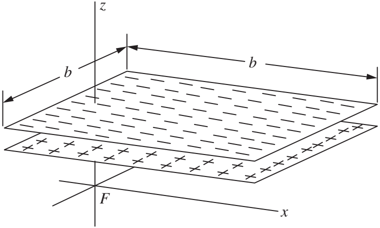
$$\mathcal{E}' = \oint \vec{E}' \cdot d\vec{s}'.$$

- The drift field $\vec{E}'$ observed by S' should be related to $\vec{E}$ observed by S through $\vec{E}' = \gamma \vec{E}$, which is perpendicular to $\vec{v}$.
- Therefore, in the inertial frame F' of the observer S',

$$\vec{E}' = \gamma \vec{v} \times \vec{B}.$$

Appendix 11C: B-Field from Relative Motion

- Consider two stationary sheets of charge of uniform density σ and $-\sigma$ in a certain inertial frame F. The electric field between the sheets is $E = \sigma/\epsilon_0$.



- In a different frame F' that moves with velocity $\vec{v}$ (perpendicular to $\vec{E}$) with respect to F, we have learned that the charge density must be $\sigma' = \gamma\sigma$ and

$$E' = \frac{\sigma'}{\epsilon_0} = \gamma\frac{\sigma}{\epsilon_0} = \gamma E.$$

- In frame F' we have two oppositely charged sheets moving at speed v, or, equivalently, two current sheets with current density

$$J_s' = \sigma' v$$

in opposite directions, where $\sigma' = \gamma\sigma$ (density increase due to length contraction).
- The magnetic fields of the two sheets add (in the region in between the sheets) to give $B' = \mu_0 J_s'$, so we have (notice, amazingly, $\mu_0\epsilon_0 = 1/c^2$)

$$B' = \mu_0 J_s' = \mu_0\sigma' v = \mu_0\gamma\sigma v = \mu_0\gamma\epsilon_0 E v = \frac{\gamma v E}{c^2}.$$

- The message is that in the theory of relativity $\vec{E}$ and $\vec{B}$ are mixed together when we go from one inertial frame to another.
- However, $\vec{E}$ and $\vec{B}$ certainly do not transform like a 4-vector; altogether, they have 6 components.
- In fact, the components of $\vec{E}$ and $\vec{B}$ form an *antisymmetric, second rank tensor*, which is an object of two indices and transforms with two Lorentz transformation matrices.