CHAPTER 5
Induction and Recursion

SECTION 5.1 Mathematical Induction

Important note about notation for proofs by mathematical induction: *In performing the inductive step, it really does not matter what letter we use. We see in the text the proof of* $P(k) \to P(k+1)$*; but it would be just as valid to prove* $P(n) \to P(n+1)$*, since the* k *in the first case and the* n *in the second case are just dummy variables. We will use both notations in this* Guide*; in particular, we will use* k *for the first few exercises but often use* n *afterwards.*

2. We can prove this by mathematical induction. Let $P(n)$ be the statement that the golfer plays hole n. We want to prove that $P(n)$ is true for all positive integers n. For the basis step, we are told that $P(1)$ is true. For the inductive step, we are told that $P(k)$ implies $P(k+1)$ for each $k \ge 1$. Therefore by the principle of mathematical induction, $P(n)$ is true for all positive integers n.

4. a) Plugging in $n = 1$ we have that $P(1)$ is the statement $1^3 = [1 \cdot (1+1)/2]^2$.

b) Both sides of $P(1)$ shown in part **(a)** equal 1.

c) The inductive hypothesis is the statement that

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 .$$

d) For the inductive step, we want to show for each $k \ge 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can prove

$$[1^3 + 2^3 + \cdots + k^3] + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 .$$

e) Replacing the quantity in brackets on the left-hand side of part **(d)** by what it equals by virtue of the inductive hypothesis, we have

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + k + 1\right) = (k+1)^2\left(\frac{k^2+4k+4}{4}\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2 ,$$

as desired.

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

6. The basis step is clear, since $1 \cdot 1! = 2! - 1$. Assuming the inductive hypothesis, we then have

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)!(1 + k + 1) - 1 = (k+2)! - 1 , \end{aligned}$$

as desired.

8. The proposition to be proved is $P(n)$:

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^n = \frac{1 - (-7)^{n+1}}{4}.$$

In order to prove this for all integers $n \ge 0$, we first prove the basis step $P(0)$ and then prove the inductive step, that $P(k)$ implies $P(k+1)$. Now in $P(0)$, the left-hand side has just one term, namely 2, and the right-hand side is $(1-(-7)^1)/4 = 8/4 = 2$. Since $2 = 2$, we have verified that $P(0)$ is true. For the inductive step, we *assume* that $P(k)$ is true (i.e., the displayed equation above), and derive from it the truth of $P(k+1)$, which is the equation

$$2 - 2\cdot 7 + 2\cdot 7^2 - \cdots + 2\cdot(-7)^k + 2\cdot(-7)^{k+1} = \frac{1-(-7)^{(k+1)+1}}{4}.$$

To prove an equation like this, it is usually best to start with the more complicated side and manipulate it until we arrive at the other side. In this case we start on the left. Note that all but the last term constitute precisely the left-hand side of $P(k)$, and therefore by the inductive hypothesis, we can replace it by the right-hand side of $P(k)$. The rest is algebra:

$$\begin{aligned}[2 - 2\cdot 7 + 2\cdot 7^2 - \cdots + 2\cdot(-7)^k] + 2\cdot(-7)^{k+1} &= \frac{1-(-7)^{k+1}}{4} + 2\cdot(-7)^{k+1} \\ &= \frac{1-(-7)^{k+1} + 8\cdot(-7)^{k+1}}{4} \\ &= \frac{1+7\cdot(-7)^{k+1}}{4} \\ &= \frac{1-(-7)\cdot(-7)^{k+1}}{4} \\ &= \frac{1-(-7)^{(k+1)+1}}{4}.\end{aligned}$$

10. **a)** By computing the first few sums and getting the answers $1/2$, $2/3$, and $3/4$, we guess that the sum is $n/(n+1)$.

b) We prove this by induction. It is clear for $n=1$, since there is just one term, $1/2$. Suppose that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

We want to show that

$$\left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Starting from the left, we replace the quantity in brackets by $k/(k+1)$ (by the inductive hypothesis), and then do the algebra

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2},$$

yielding the desired expression.

12. We proceed by mathematical induction. The basis step ($n=0$) is the statement that $(-1/2)^0 = (2+1)/(3\cdot 1)$, which is the true statement that $1 = 1$. Assume the inductive hypothesis, that

$$\sum_{j=0}^{k} \left(-\frac{1}{2}\right)^j = \frac{2^{k+1}+(-1)^k}{3\cdot 2^k}.$$

We want to prove that

$$\sum_{j=0}^{k+1} \left(-\frac{1}{2}\right)^j = \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}.$$

Split the summation into two parts, apply the inductive hypothesis, and do the algebra:

$$\begin{aligned}\sum_{j=0}^{k+1}\left(-\frac{1}{2}\right)^j &= \sum_{j=0}^{k}\left(-\frac{1}{2}\right)^j + \left(-\frac{1}{2}\right)^{k+1}\\ &= \frac{2^{k+1}+(-1)^k}{3\cdot 2^k} + \frac{(-1)^{k+1}}{2^{k+1}}\\ &= \frac{2^{k+2}+2(-1)^k}{3\cdot 2^{k+1}} + \frac{3(-1)^{k+1}}{3\cdot 2^{k+1}}\\ &= \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}\,.\end{aligned}$$

For the last step, we used the fact that $2(-1)^k = -2(-1)^{k+1}$.

14. We proceed by induction. Notice that the letter k has been used in this problem as the dummy index of summation, so we cannot use it as the variable for the inductive step. We will use n instead. For the basis step we have $1\cdot 2^1 = (1-1)2^{1+1}+2$, which is the true statement $2=2$. We assume the inductive hypothesis, that

$$\sum_{k=1}^{n} k\cdot 2^k = (n-1)2^{n+1}+2\,,$$

and try to prove that

$$\sum_{k=1}^{n+1} k\cdot 2^k = n\cdot 2^{n+2}+2\,.$$

Splitting the left-hand side into its first n terms followed by its last term and invoking the inductive hypothesis, we have

$$\sum_{k=1}^{n+1} k\cdot 2^k = \left(\sum_{k=1}^{n} k\cdot 2^k\right) + (n+1)2^{n+1} = (n-1)2^{n+1}+2+(n+1)2^{n+1} = 2n\cdot 2^{n+1}+2 = n\cdot 2^{n+2}+2\,,$$

as desired.

16. The basis step reduces to $6=6$. Assuming the inductive hypothesis we have

$$\begin{aligned}&1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)\\ &\quad= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)\\ &\quad= (k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)\\ &\quad= \frac{(k+1)(k+2)(k+3)(k+4)}{4}\,.\end{aligned}$$

18. **a)** Plugging in $n=2$, we see that $P(2)$ is the statement $2! < 2^2$.
b) Since $2! = 2$, this is the true statement $2 < 4$.
c) The inductive hypothesis is the statement that $k! < k^k$.
d) For the inductive step, we want to show for each $k \ge 2$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can prove that $(k+1)! < (k+1)^{k+1}$.
e) $(k+1)! = (k+1)k! < (k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}$
f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n greater than 1.

20. The basis step is $n=7$, and indeed $3^7 < 7!$, since $2187 < 5040$. Assume the statement for k. Then $3^{k+1} = 3\cdot 3^k < (k+1)\cdot 3^k < (k+1)\cdot k! = (k+1)!$, the statement for $k+1$.

22. A little computation convinces us that the answer is that $n^2 \le n!$ for $n = 0,\ 1$, and all $n \ge 4$. (Clearly the inequality does not hold for $n = 2$ or $n = 3$.) We will prove by mathematical induction that the inequality holds for all $n \ge 4$. The basis step is clear, since $16 \le 24$. Now suppose that $n^2 \le n!$ for a given $n \ge 4$. We must show that $(n+1)^2 \le (n+1)!$. Expanding the left-hand side, applying the inductive hypothesis, and then invoking some valid bounds shows this:

$$\begin{aligned} n^2 + 2n + 1 &\le n! + 2n + 1 \\ &\le n! + 2n + n = n! + 3n \\ &\le n! + n \cdot n \le n! + n \cdot n! \\ &= (n+1)n! = (n+1)! \end{aligned}$$

24. The basis step is clear, since $1/2 \le 1/2$. We assume the inductive hypothesis (the inequality shown in the exercise) and want to prove the similar inequality for $n+1$. We proceed as follows, using the trick of writing $1/(2(n+1))$ in terms of $1/(2n)$ so that we can invoke the inductive hypothesis:

$$\begin{aligned} \frac{1}{2(n+1)} &= \frac{1}{2n} \cdot \frac{2n}{2(n+1)} \\ &\le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1))}{2 \cdot 4 \cdots 2n} \cdot \frac{2n}{2(n+1)} \\ &\le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1))}{2 \cdot 4 \cdots 2n} \cdot \frac{2n+1}{2(n+1)} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdots 2n \cdot 2(n+1)} \end{aligned}$$

26. One can get to the proof of this by doing some algebraic tinkering. It turns out to be easier to think about the given statement as $na^{n-1}(a-b) \ge a^n - b^n$. The basis step ($n = 1$) is the true statement that $a - b \ge a - b$. Assume the inductive hypothesis, that $ka^{k-1}(a-b) \ge a^k - b^k$; we must show that $(k+1)a^k(a-b) \ge a^{k+1} - b^{k+1}$. We have

$$\begin{aligned} (k+1)a^k(a-b) &= k \cdot a \cdot a^{k-1}(a-b) + a^k(a-b) \\ &\ge a(a^k - b^k) + a^k(a-b) \\ &= a^{k+1} - ab^k + a^{k+1} - ba^k\,. \end{aligned}$$

To complete the proof we want to show that $a^{k+1} - ab^k + a^{k+1} - ba^k \ge a^{k+1} - b^{k+1}$. This inequality is equivalent to $a^{k+1} - ab^k - ba^k + b^{k+1} \ge 0$, which factors into $(a^k - b^k)(a-b) \ge 0$, and this is true, because we are given that $a > b$.

28. The base case is $n = 3$. We check that $4^2 - 7 \cdot 4 + 12 = 0$ is nonnegative. Next suppose that $n^2 - 7n + 12 \ge 0$; we must show that $(n+1)^2 - 7(n+1) + 12 \ge 0$. Expanding the left-hand side, we obtain $n^2 + 2n + 1 - 7n - 7 + 12 = (n^2 - 7n + 12) + (2n - 6)$. The first of the parenthesized expressions is nonnegative by the inductive hypothesis; the second is clearly also nonnegative by the assumption that n is at least 3. Therefore their sum is nonnegative, and the inductive step is complete.

30. The statement is true for $n = 1$, since $H_1 = 1 = 2 \cdot 1 - 1$. Assume the inductive hypothesis, that the statement is true for n. Then on the one hand we have

$$\begin{aligned} H_1 + H_2 + \cdots + H_n + H_{n+1} &= (n+1)H_n - n + H_{n+1} \\ &= (n+1)H_n - n + H_n + \frac{1}{n+1} \\ &= (n+2)H_n - n + \frac{1}{n+1}\,, \end{aligned}$$

and on the other hand

$$\begin{aligned}(n+2)H_{n+1}-(n+1) &= (n+2)\left(H_n+\frac{1}{n+1}\right)-(n+1)\\ &= (n+2)H_n+\frac{n+2}{n+1}-(n+1)\\ &= (n+2)H_n+1+\frac{1}{n+1}-n-1\\ &= (n+2)H_n-n+\frac{1}{n+1}\,.\end{aligned}$$

That these two expressions are equal was precisely what we had to prove.

32. The statement is true for the base case, $n=0$, since $3\mid 0$. Suppose that $3\mid(k^3+2k)$. We must show that $3\mid\big((k+1)^3+2(k+1)\big)$. If we expand the expression in question, we obtain $k^3+3k^2+3k+1+2k+2=(k^3+2k)+3(k^2+k+1)$. By the inductive hypothesis, 3 divides k^3+2k, and certainly 3 divides $3(k^2+k+1)$, so 3 divides their sum, and we are done.

34. The statement is true for the base case, $n=0$, since $6\mid 0$. Suppose that $6\mid(n^3-n)$. We must show that $6\mid\big((n+1)^3-(n+1)\big)$. If we expand the expression in question, we obtain $n^3+3n^2+3n+1-n-1=(n^3-n)+3n(n+1)$. By the inductive hypothesis, 6 divides the first term, n^3-n. Furthermore clearly 3 divides the second term, and the second term is also even, since one of n and $n+1$ is even; therefore 6 divides the second term as well. This tells us that 6 divides the given expression, as desired. (Note that here we have, as promised, used n as the dummy variable in the inductive step, rather than k.)

36. It is not easy to stumble upon the trick needed in the inductive step in this exercise, so do not feel bad if you did not find it. The form is straightforward. For the basis step ($n=1$), we simply observe that $4^{1+1}+5^{2\cdot 1-1}=16+5=21$, which is divisible by 21. Then we assume the inductive hypothesis, that $4^{n+1}+5^{2n-1}$ is divisible by 21, and let us look at the expression when $n+1$ is plugged in for n. We want somehow to manipulate it so that the expression for n appears. We have

$$\begin{aligned}4^{(n+1)+1}+5^{2(n+1)-1} &= 4\cdot 4^{n+1}+25\cdot 5^{2n-1}\\ &= 4\cdot 4^{n+1}+(4+21)\cdot 5^{2n-1}\\ &= 4(4^{n+1}+5^{2n-1})+21\cdot 5^{2n-1}\,.\end{aligned}$$

Looking at the last line, we see that the expression in parentheses is divisible by 21 by the inductive hypothesis, and obviously the second term is divisible by 21, so the entire quantity is divisible by 21, as desired.

38. The basis step is trivial, as usual: $A_1\subseteq B_1$ implies that $\bigcup_{j=1}^{1}A_j\subseteq\bigcup_{j=1}^{1}B_j$ because the union of one set is itself. Assume the inductive hypothesis that if $A_j\subseteq B_j$ for $j=1,2,\ldots,k$, then $\bigcup_{j=1}^{k}A_j\subseteq\bigcup_{j=1}^{k}B_j$. We want to show that if $A_j\subseteq B_j$ for $j=1,2,\ldots,k+1$, then $\bigcup_{j=1}^{k+1}A_j\subseteq\bigcup_{j=1}^{k+1}B_j$. To show that one set is a subset of another we show that an arbitrary element of the first set must be an element of the second set. So let $x\in\bigcup_{j=1}^{k+1}A_j=\left(\bigcup_{j=1}^{k}A_j\right)\cup A_{k+1}$. Either $x\in\bigcup_{j=1}^{k}A_j$ or $x\in A_{k+1}$. In the first case we know by the inductive hypothesis that $x\in\bigcup_{j=1}^{k}B_j$; in the second case, we know from the given fact that $A_{k+1}\subseteq B_{k+1}$ that $x\in B_{k+1}$. Therefore in either case $x\in\left(\bigcup_{j=1}^{k}B_j\right)\cup B_{k+1}=\bigcup_{j=1}^{k+1}B_j$.

This is really easier to do directly than by using the principle of mathematical induction. For a noninductive proof, suppose that $x\in\bigcup_{j=1}^{n}A_j$. Then $x\in A_j$ for some j between 1 and n, inclusive. Since $A_j\subseteq B_j$, we know that $x\in B_j$. Therefore by definition, $x\in\bigcup_{j=1}^{n}B_j$.

40. If $n=1$ there is nothing to prove, and the $n=2$ case is the distributive law (see Table 1 in Section 2.2). Those take care of the basis step. For the inductive step, assume that

$$(A_1\cap A_2\cap\cdots\cap A_n)\cup B=(A_1\cup B)\cap(A_2\cup B)\cap\cdots\cap(A_n\cup B)\,;$$

we must show that

$$(A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B) \cap (A_{n+1} \cup B)\,.$$

We have

$$\begin{aligned}(A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \cup B &= ((A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}) \cup B \\ &= ((A_1 \cap A_2 \cap \cdots \cap A_n) \cup B) \cap (A_{n+1} \cup B) \\ &= (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B) \cap (A_{n+1} \cup B)\,.\end{aligned}$$

The second line follows from the distributive law, and the third line follows from the inductive hypothesis.

42. If $n = 1$ there is nothing to prove, and the $n = 2$ case says that $(A_1 \cap \overline{B}) \cap (A_2 \cap \overline{B}) = (A_1 \cap A_2) \cap \overline{B}$, which is certainly true, since an element is in each side if and only if it is in all three of the sets A_1, A_2, and $\overline{B}$. Those take care of the basis step. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B\,;$$

we must show that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B\,.$$

We have

$$\begin{aligned}&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\ &\qquad = ((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)) \cap (A_{n+1} - B) \\ &\qquad = ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1}) - B) \\ &\qquad = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B\,.\end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

44. If $n = 1$ there is nothing to prove, and the $n = 2$ case says that $(A_1 \cap \overline{B}) \cup (A_2 \cap \overline{B}) = (A_1 \cup A_2) \cap \overline{B}$, which is the distributive law (see Table 1 in Section 2.2). Those take care of the basis step. For the inductive step, assume that

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B\,;$$

we must show that

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) \cup (A_{n+1} - B) = (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) - B\,.$$

We have

$$\begin{aligned}&(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) \cup (A_{n+1} - B) \\ &\qquad = ((A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B)) \cup (A_{n+1} - B) \\ &\qquad = ((A_1 \cup A_2 \cup \cdots \cup A_n) - B) \cup (A_{n+1}) - B) \\ &\qquad = (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) - B\,.\end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

46. This proof will be similar to the proof in Example 10. The basis step is clear, since for $n = 3$, the set has exactly one subset containing exactly three elements, and $3(3 - 1)(3 - 2)/6 = 1$. Assume the inductive hypothesis, that a set with n elements has $n(n - 1)(n - 2)/6$ subsets with exactly three elements; we want to prove that a set S with $n + 1$ elements has $(n + 1)n(n - 1)/6$ subsets with exactly three elements. Fix an element a in S, and let T be the set of elements of S other than a. There are two varieties of subsets of S containing exactly three elements. First there are those that do not contain a. These are precisely the three-element subsets of T, and by the inductive hypothesis, there are $n(n - 1)(n - 2)/6$ of them. Second, there are those that contain a together with two elements of T. Therefore there are just as many of these subsets as there are two-element subsets of T. By Exercise 45, there are exactly $n(n-1)/2$ such subsets of T; therefore there are also $n(n-1)/2$ three-element subsets of S containing a. Thus the total number of subsets of S containing exactly three elements is $(n(n - 1)(n - 2)/6) + n(n - 1)/2$, which simplifies algebraically to $(n + 1)n(n - 1)/6$, as desired.

48. We will show that any minimum placement of towers can be transformed into the placement produced by the algorithm. Although it does not strictly have the form of a proof by mathematical induction, the spirit is the same. Let $s_1 < s_2 < \cdots < s_k$ be an optimal locations of the towers (i.e., so as to minimize k), and let $t_1 < t_2 < \cdots < t_l$ be the locations produced by the algorithm from Exercise 47. In order to serve the first building, we must have $s_1 \leq x_1 + 1 = t_1$. If $s_1 \neq t_1$, then we can move the first tower in the optimal solution to position t_1 without losing cell service for any building. Therefore we can assume that $s_1 = t_1$. Let x_j be smallest location of a building out of range of the tower at s_1; thus $x_j > s_1 + 1$. In order to serve that building there must be a tower s_i such that $s_i \leq x_j + 1 = t_2$. If $i > 2$, then towers at positions s_2 through s_{i-1} are not needed, a contradiction. As before, it then follows that we can move the second tower from s_2 to t_2. We continue in this manner for all the towers in the given minimum solution; thus $k = l$. This proves that the algorithm produces a minimum solution.

50. When $n = 1$ the left-hand side is 1, and the right-hand side is $(1 + \frac{1}{2})^2/2 = 9/8$. Thus the basis step was wrong.

52. We prove by mathematical induction that a function $f : A \to \{1, 2, \ldots, n\}$ where $|A| > n$ cannot be one-to-one. For the basis step, $n = 1$ and $|A| > 1$. Let x and y be distinct elements of A. Because the codomain has only one element, we must have $f(x) = f(y)$, so by definition f is not one-to-one. Assume the inductive hypothesis that no function from any A to $\{1, 2, \ldots, n\}$ with $|A| > n$ is one-to-one, and let f be a function from A to $\{1, 2, \ldots, n, n+1\}$, where $|A| > n + 1$. There are three cases. If $n + 1$ is not in the range of f, then the inductive hypothesis tells us that f is not one-to-one. If $f(x) = n + 1$ for more than one value of $x \in A$, then by definition f is not one-to-one. The only other case has $f(a) = n + 1$ for exactly one element $a \in A$. Let $A' = A - \{a\}$, and consider the function f' defined as f restricted to A'. Since $|A'| > n$, by the inductive hypothesis f' is not one-to-one, and therefore neither is f.

54. The base case is $n = 1$. If we are given a set of two elements from $\{1, 2\}$, then indeed one of them divides the other. Assume the inductive hypothesis, and consider a set A of $n + 2$ elements from $\{1, 2, \ldots, 2n, 2n + 1, 2n+2\}$. We must show that at least one of these elements divides another. If as many as $n+1$ of the elements of A are less than $2n + 1$, then the desired conclusion follows immediately from the inductive hypothesis. Therefore we can assume that both $2n + 1$ and $2n + 2$ are in A, together with n smaller elements. If $n + 1$ is one of these smaller elements, then we are done, since $n + 1 \mid 2n + 2$. So we can assume that $n + 1 \notin A$. Now apply the inductive hypothesis to $B = A - \{2n + 1, 2n + 2\} \cup \{n + 1\}$. Since B is a collection of $n + 1$ numbers from $\{1, 2, \ldots, 2n\}$, the inductive hypothesis guarantees that one element of B divides another. If $n + 1$ is not one of these two numbers, then we are done. So we can assume that $n + 1$ is one of these two numbers. Certainly $n + 1$ can't be the divisor, since its smallest multiple is too big to be in B, so there is some $k \in B$ that divides $n+1$. But now k and $2n+2$ are numbers in A, with k dividing $n+2$, and we are done. An alternative proof of this theorem is given in Example 11 of Section 6.2.

56. There is nothing to prove in the base case, $n = 1$, since $\mathbf{A} = \mathbf{A}$. For the inductive step we just invoke the inductive hypothesis and the definition of matrix multiplication:

$$\begin{aligned}\mathbf{A}^{n+1} = \mathbf{A}\mathbf{A}^n &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} \\ &= \begin{bmatrix} a \cdot a^n + 0 \cdot 0 & a \cdot 0 + 0 \cdot b^n \\ 0 \cdot a^n + b \cdot 0 & 0 \cdot 0 + b \cdot b^n \end{bmatrix} = \begin{bmatrix} a^{n+1} & 0 \\ 0 & b^{n+1} \end{bmatrix}\end{aligned}$$

58. The basis step is trivial, since we are already given that $\mathbf{AB} = \mathbf{BA}$. Next we assume the inductive hypothesis, that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$, and try to prove that $\mathbf{AB}^{n+1} = \mathbf{B}^{n+1}\mathbf{A}$. We calculate as follows: $\mathbf{AB}^{n+1} = \mathbf{AB}^n\mathbf{B} = \mathbf{B}^n\mathbf{AB} = \mathbf{B}^n\mathbf{BA} = \mathbf{B}^{n+1}\mathbf{A}$. Note that we used the definition of matrix powers (that $\mathbf{B}^{n+1} = \mathbf{B}^n\mathbf{B}$), the inductive hypothesis, and the basis step.

60. This is identical to Exercise 43, with $\vee$ replacing $\cup$, $\wedge$ replacing $\cap$, and $\neg$ replacing complementation. The basis step is trivial, since it merely says that $\neg p_1$ is equivalent to itself. Assuming the inductive hypothesis, we look at $\neg(p_1 \vee p_2 \vee \cdots \vee p_n \vee p_{n+1})$. By De Morgan's law (grouping all but the last term together) this is the same $\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \wedge \neg p_{n+1}$. But by the inductive hypothesis, this equals, $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n \wedge \neg p_{n+1}$, as desired.

62. The statement is true for $n = 1$, since 1 line separates the plane into 2 regions, and $(1^2 + 1 + 2)/2 = 2$. Assume the inductive hypothesis, that n lines of the given type separate the plane into $(n^2 + n + 2)/2$ regions. Consider an arrangement of $n+1$ lines. Remove the last line. Then there are $(n^2 + n + 2)/2$ regions by the inductive hypothesis. Now we put the last line back in, drawing it slowly, and see what happens to the regions. As we come in "from infinity," the line separates one infinite region into two (one on each side of it); this separation is complete as soon as the line hits one of the first n lines. Then, as we continue drawing from this first point of intersection to the second, the line again separates one region into two. We continue in this way. Every time we come to another point of intersection between the line we are drawing and the figure already present, we lop off another additional region. Furthermore, once we leave the last point of intersection and draw our line off to infinity again, we separate another region into two. Therefore the number of additional regions we formed is equal to the number of points of intersection plus one. Now there are n points of intersection, since our line must intersect each of the other lines in a distinct point (this is where the geometric assumptions get used). Therefore this arrangement has $n+1$ more points of intersection than the arrangement of n lines, namely $\big((n^2 + n + 2)/2\big) + (n+1)$, which, after a bit of algebra, reduces to $\big((n+1)^2 + (n+1) + 2\big)/2$, exactly as desired.

64. For the base case $n = 1$ there is nothing to prove. Assume the inductive hypothesis, and suppose that we are given $p \mid a_1 a_2 \cdots a_n a_{n+1}$. We must show that $p \mid a_i$ for some i. Let us look at $\gcd(p, a_1 a_2 \cdots a_n)$. Since the only divisors of p are 1 and p, this is either 1 or p. If it is 1, then by Lemma 2 in Section 4.3, we have $p \mid a_{n+1}$ (here $a = p$, $b = a_1 a_2 \cdots a_n$, and $c = a_{n+1}$), as desired. On the other hand, if the greatest common divisor is p, this means that $p \mid a_1 a_2 \cdots a_n$. Now by the inductive hypothesis, $p \mid a_i$ for some $i \le n$, again as desired.

66. Suppose that a statement $\forall n P(n)$ has been proved by this method. Let S be the set of counterexamples to P, i.e., let $S = \{\, n \mid \neg P(n) \,\}$. We will show that $S = \emptyset$. If $S \neq \emptyset$, then let n be the minimum element of S (which exists by the well-ordering property). Clearly $n \neq 1$ and $n \neq 2$, by the basis steps of our proof method. But since n is the least element of S and $n \ge 3$, we know that $P(n-1)$ and $P(n-2)$ are true. Therefore by the inductive step of our proof method, we know that $P(n)$ is also true. This contradicts the choice of n. Therefore $S = \emptyset$, as desired.

68. The basis step is $n = 1$ and $n = 2$. If there is one guest present, then he or she is vacuously a celebrity, and no questions are needed; this is consistent with the value of $3(n-1)$. If there are two guests, then it is certainly true that we can determine who the celebrity is (or determine that neither of them is) with three questions. In fact, two questions suffice (ask each one if he or she knows the other). Assume the inductive hypothesis that if there are k guests present ($k \ge 2$), then we can determine whether there is a celebrity with at most $3(k-1)$ questions. We want to prove the statement for $k+1$, namely, if there are $k+1$ at the party, then we can find the celebrity (or determine that there is none) using $3k$ questions. Let Alex and Britney be two of the guests. Ask Alex whether he knows Britney. If he says yes, then we know that he is not a celebrity. If he says no, then we know that Britney is not a celebrity. Without loss of generality, assume that we have eliminated Alex as a possible celebrity. Now invoke the inductive hypothesis on the k guests excluding Alex, asking $3(k-1)$ questions. If there is no celebrity, then we know that there is no celebrity at our party. If there is, suppose that it is person x (who might be Britney or might be someone else). We then ask two more questions to determine whether x is in fact a celebrity; namely ask Alex whether he knows x, and ask x whether s/he knows Alex. Based on the answers, we will now know whether x is a celebrity for the whole party or there is no celebrity present. We have asked a total of at most $1+3(k-1)+2=3k$ questions. Note that in fact we did a little better than $3(n-1)$; because only two questions were needed for $n=2$, only $3(n-1)-1=3n-4$ questions are needed in the general case for $n \geq 2$.

70. We prove this by mathematical induction. The basis step, $G(4)=2\cdot 4-4=4$ was proved in Exercise 69. For the inductive step, suppose that when there are k callers, $2k-4$ calls suffice; we must show that when there are $k+1$ callers, $2(k+1)-4$ calls suffice, that is, two more calls. It is clear from the hint how to proceed. For the first extra call, have the $(k+1)^{\text{st}}$ person exchange information with the k^{th} person. Then use $2k-4$ calls for the first k people to exchange information. At that point, each of them knows all the gossip. Finally, have the $(k+1)^{\text{st}}$ person again call the k^{th} person, at which point he will learn the rest of the gossip.

72. We follow the hint. If the statement is true for some value of n, then it is also true for all smaller values of n, because we can use the same arrangement among those smaller numbers. Thus is suffices to prove the statement when n is a power of 2. We use mathematical induction to prove the result for 2^k. If $k=0$ or $k=1$, there is nothing to prove. Notice that the arrangement 1324 works for $k=2$. Assume that we can arrange the positive integers from 1 to 2^k so that the average of any two of these numbers never appears between them. Arrange the numbers from 1 to 2^{k+1} by taking the given arrangement of 2^k numbers, replacing each number by its double, and then following this sequence with the sequence of 2^k numbers obtained from these 2^k even numbers by subtracting 1. Thus for $k=3$ we use the sequence 1324 to form the sequence 26481537. This clearly is a list of the numbers from 1 to 2^{k+1}. The average of an odd number and an even number is not an integer, so it suffices to shows that the average of two even numbers and the average of two odd numbers in our list never appears between the numbers being averaged. If the average of two even numbers, say $2a$ and $2b$, whose average is $a+b$, appears between the numbers being averaged, then by the way we constructed the sequence, there would have been a similar violation in the 2^k list, namely, $(a+b)/2$ would have appeared between a and b. Similarly, if the average of two odd numbers, say $2c-1$ and $2d-1$, whose average is $c+d-1$, appears between the numbers being averaged, then there would have been a similar violation in the 2^k list, namely, $(c+d)/2$ would have appeared between c and d.

74. **a)** The basis step works, because for $n=1$ the statement $1/2 < 1/\sqrt{3}$ is true. The inductive step would require proving that

$$\frac{1}{\sqrt{3n}} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3(n+1)}}\,.$$

Squaring both sides and clearing fractions, we see that this is equivalent to $4n^2+4n+1 < 4n^2+4n$, which of course is not true.

b) The basis step works, because the statement $3/8 < 1/\sqrt{7}$ is true. The inductive step this time requires proving that

$$\frac{1}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3(n+1)+1}}\,.$$

A little algebraic manipulation shows that this is equivalent to

$$12n^3+28n^2+19n+4 < 12n^3+28n^2+20n+4\,,$$

which is true.

76. The upper left 4×4 quarter of the figure given in the solution to Exercise 77 gives such a tiling.

78. **a)** Every $3 \times 2k$ board can be covered in an obvious way: put two pieces together to form a 3×2 rectangle, then lay the rectangles edge to edge. In particular, for all $n \geq 1$ the 3×2^n rectangle can be covered.

b) This is similar to part **(a)**. For all $k \geq 1$ it is easy to cover the $6 \times 2k$ board, using two coverings of the $3 \times 2k$ board from part **(a)**, laid side by side.

c) A little trial and error shows that the $3^1 \times 3^1$ board cannot be covered. Therefore not all such boards can be covered.

d) All boards of this shape can be covered for $n \geq 1$, using reasoning similar to parts **(a)** and **(b)**.

80. This is too complicated to discuss here. For a solution, see the article by I. P. Chu and R. Johnsonbaugh, "Tiling Deficient Boards with Trominoes," *Mathematics Magazine* **59** (1986) 34–40. (Notice the variation in the spelling of this made-up word.)

82. In order to explain this argument, we label the squares in the 5×5 checkerboard 11, 12, ..., 15, 21, ..., 25, ..., 51, ..., 55, where the first digit stands for the row number and the second digit stands for the column number. Also, in order to talk about the right triomino (L-shaped tile), think of it positioned to look like the letter L; then we call the square on top the head, the square in the lower right the tail, and the square in the corner the corner. We claim that the board with square 12 removed cannot be tiled. First note that in order to cover square 11, the position of one piece is fixed. Next we consider how to cover square 13. There are three possibilities. If we put a head there, then we are forced to put the corner of another piece in square 15. If we put a corner there, then we are forced to put the tail of another piece in 15, and if we put a tail there, then square 15 cannot be covered at all. So we conclude that squares 13, 14, 15, 23, 24, and 25 will have to be covered by two more pieces. By symmetry, the same argument shows that two more pieces must cover squares 31, 41, 51, 32, 42, and 52. This much has been forced, and now we are left with the 3×3 square in the lower left part of the checkerboard to cover with three more pieces. If we put a corner in 33, then we immediately run into an impasse in trying to cover 53 and 35. If we put a head in 33, then 53 cannot be covered; and if we put a tail in 33, then 35 cannot be covered. So we have reached a contradiction, and the desired covering does not exist.

SECTION 5.2 Strong Induction and Well-Ordering

Important note about notation for proofs by mathematical induction: *In performing the inductive step, it really does not matter what letter we use. We see in the text the proof of* $(\forall j{\leq}k\, P(j)) \rightarrow P(k+1)$*; but it would be just as valid to prove* $(\forall j{\leq}n\, P(j)) \rightarrow P(n+1)$*, since the* k *in the first case and the* n *in the second case are just dummy variables. Furthermore, we could also take the inductive hypothesis to be* $\forall j{<}n\, P(j)$ *and then prove* $P(n)$*. We will use all three notations in this* Guide.

2. Let $P(n)$ be the statement that the n^{th} domino falls. We want to prove that $P(n)$ is true for all positive integers n. For the basis step we note that the given conditions tell us that $P(1)$, $P(2)$, and $P(3)$ are true. For the inductive step, fix $k \geq 3$ and assume that $P(j)$ is true for all $j \leq k$. We want to show that $P(k+1)$ is true. Since $k \geq 3$, $k-2$ is a positive integer less than or equal to k, so by the inductive hypothesis we know that $P(k-2)$ is true. That is, we know that the $(k-2)^{\text{nd}}$ domino falls. We were told that "when a domino falls, the domino three farther down in the arrangement also falls," so we know that the domino in position $(k-2)+3 = k+1$ falls. This is $P(k+1)$.

Note that we didn't use strong induction exactly as stated in the text. Instead, we considered all the cases $n = 1$, $n = 2$, and $n = 3$ as part of the basis step. We could have more formally included $n = 2$ and $n = 3$ in the inductive step as a special case. Writing our proof this way, the basis step is just to note that the first domino falls, so $P(1)$ is true. For the inductive step, if $k = 1$ or $k = 2$, then we are already told that the second and third domino fall, so $P(k+1)$ is true in those cases. If $k > 2$, then the inductive hypothesis tells us that the $(k-2)^{\text{nd}}$ domino falls, so the domino in position $(k-1)+2 = k+1$ falls.

4. a) $P(18)$ is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps. $P(19)$ is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp. $P(20)$ is true, because we can form 20 cents of postage with five 4-cent stamps. $P(21)$ is true, because we can form 20 cents of postage with three 7-cent stamps.

b) The inductive hypothesis is the statement that using just 4-cent and 7-cent stamps we can form j cents postage for all j with $18 \le j \le k$, where we assume that $k \ge 21$.

c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 4-cent and 7-cent stamps.

d) We want to form $k+1$ cents of postage. Since $k \ge 21$, we know that $P(k-3)$ is true, that is, that we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.

e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

6. a) We can form the following amounts of postage as indicated: $3 = 3$, $6 = 3+3$, $9 = 3+3+3$, $10 = 10$, $12 = 3+3+3+3$, $13 = 10+3$, $15 = 3+3+3+3+3$, $16 = 10+3+3$, $18 = 3+3+3+3+3+3$, $19 = 10+3+3+3$, $20 = 10+10$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form all amounts of postage greater than or equal to 18 cents using just 3-cent and 10-cent stamps.

b) Let $P(n)$ be the statement that we can form n cents of postage using just 3-cent and 10-cent stamps. We want to prove that $P(n)$ is true for all $n \ge 18$. The basis step, $n = 18$, is handled above. Assume that we can form k cents of postage (the inductive hypothesis); we will show how to form $k+1$ cents of postage. If the k cents included two 10-cent stamps, then replace them by seven 3-cent stamps ($7 \cdot 3 = 2 \cdot 10 + 1$). Otherwise, k cents was formed either from just 3-cent stamps, or from one 10-cent stamp and $k - 10$ cents in 3-cent stamps. Because $k \ge 18$, there must be at least three 3-cent stamps involved in either case. Replace three 3-cent stamps by one 10-cent stamp, and we have formed $k+1$ cents in postage ($10 = 3 \cdot 3 + 1$).

c) $P(n)$ is the same as in part **(b)**. To prove that $P(n)$ is true for all $n \ge 18$, we note for the basis step that from part **(a)**, $P(n)$ is true for $n = 18, 19, 20$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $18 \le j \le k$, where k is a fixed integer greater than or equal to 20. We want to show that $P(k+1)$ is true. Because $k - 2 \ge 18$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired. In this proof our inductive hypothesis included all values between 18 and k inclusive, and that enabled us to jump back three steps to a value for which we knew how to form the desired postage.

8. Since both 25 and 40 are multiples of 5, we cannot form any amount that is not a multiple of 5. So let's determine for which values of n we can form $5n$ dollars using these gift certificates, the first of which provides 5 copies of \$5, and the second of which provides 8 copies. We can achieve the following values of n: $5 = 5$, $8 = 8$, $10 = 5+5$, $13 = 8+5$, $15 = 5+5+5$, $16 = 8+8$, $18 = 8+5+5$, $20 = 5+5+5+5$, $21 = 8+8+5$, $23 = 8+5+5+5$, $24 = 8+8+8$, $25 = 5+5+5+5+5$, $26 = 8+8+5+5$, $28 = 8+5+5+5+5$, $29 = 8+8+8+5$, $30 = 5+5+5+5+5+5$, $31 = 8+8+5+5+5$, $32 = 8+8+8+8$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form total amounts of the form $5n$ for all $n \geq 28$ using these gift certificates. (In other words, \$135 is the largest multiple of \$5 that we cannot achieve.)

To prove this by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using just 25-dollar and 40-dollar certificates. We want to prove that $P(n)$ is true for all $n \geq 28$. From our work above, we know that $P(n)$ is true for $n = 28, 29, 30, 31, 32$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $28 \leq j \leq k$, where k is a fixed integer greater than or equal to 32. We want to show that $P(k+1)$ is true. Because $k - 4 \geq 28$, we know that $P(k-4)$ is true, that is, that we can form $5(k-4)$ dollars. Add one more \$25-dollar certificate, and we have formed $5(k+1)$ dollars, as desired.

10. We claim that it takes exactly $n-1$ breaks to separate a bar (or any connected piece of a bar obtained by horizontal or vertical breaks) into n pieces. We use strong induction. If $n = 1$, this is trivially true (one piece, no breaks). Assume the strong inductive hypothesis, that the statement is true for breaking into k or fewer pieces, and consider the task of obtaining $k+1$ pieces. We must show that it takes exactly k breaks. The process must start with a break, leaving two smaller pieces. We can view the rest of the process as breaking one of these pieces into $i+1$ pieces and breaking the other piece into $k-i$ pieces, for some i between 0 and $k-1$, inclusive. By the inductive hypothesis it will take exactly i breaks to handle the first piece and $k-i-1$ breaks to handle the second piece. Therefore the total number of breaks will be $1 + i + (k - i - 1) = k$, as desired.

12. The basis step is to note that $1 = 2^0$. Notice for subsequent steps that $2 = 2^1$, $3 = 2^1 + 2^0$, $4 = 2^2$, $5 = 2^2 + 2^0$, and so on. Indeed this is simply the representation of a number in binary form (base two). Assume the inductive hypothesis, that every positive integer up to k can be written as a sum of distinct powers of 2. We must show that $k+1$ can be written as a sum of distinct powers of 2. If $k+1$ is odd, then k is even, so 2^0 was not part of the sum for k. Therefore the sum for $k+1$ is the same as the sum for k with the extra term 2^0 added. If $k+1$ is even, then $(k+1)/2$ is a positive integer, so by the inductive hypothesis $(k+1)/2$ can be written as a sum of distinct powers of 2. Increasing each exponent by 1 doubles the value and gives us the desired sum for $k+1$.

14. We prove this using strong induction. It is clearly true for $n = 1$, because no splits are performed, so the sum computed is 0, which equals $n(n-1)/2$ when $n = 1$. Assume the strong inductive hypothesis, and suppose that our first splitting is into piles of i stones and $n-i$ stones, where i is a positive integer less than n. This gives a product $i(n-i)$. The rest of the products will be obtained from splitting the piles thus formed, and so by the inductive hypothesis, the sum of the products will be $i(i-1)/2 + (n-i)(n-i-1)/2$. So we must show that

$$i(n-i) + \frac{i(i-1)}{2} + \frac{(n-i)(n-i-1)}{2} = \frac{n(n-1)}{2}$$

no matter what i is. This follows by elementary algebra, and our proof is complete.

16. We follow the hint to show that there is a winning strategy for the first player in Chomp played on a $2 \times n$ board that starts by removing the rightmost cookie in the bottom row. Note that this leaves a board with n cookies in the top row and $n-1$ cookies in the bottom row. It suffices to prove by strong induction on n that a player presented with such a board will lose if his opponent plays properly. We do this by showing how the opponent can return the board to this form following any nonfatal move this player might make. The basis step is $n = 1$, and in that case only the poisoned cookie remains, so the player loses. Assume the inductive hypothesis (that the statement is true for all smaller values of n). If the player chooses a nonpoisoned cookie in the top row, then that leaves another board with two rows of equal length, so again the opponent chooses the rightmost cookie in the bottom row, and we are back to the hopeless situation, for some board with fewer than n cookies in the top row. If the player chooses the cookie in the m^{th} column from the left in the bottom row (where necessarily $m < n$), then the opponent chooses the cookie in the $(m+1)^{\text{st}}$ column from the left in the top row, and once again we are back to the hopeless situation, with m cookies in the top row.

18. We prove something slightly stronger: If a convex n-gon whose vertices are labeled consecutively as v_m, v_{m+1}, ..., v_{m+n-1} is triangulated, then the triangles can be numbered from m to $m+n-3$ so that v_i is a vertex of triangle i for $i = m, m+1, \ldots, m+n-3$. (The statement we are asked to prove is the case $m = 1$.) The basis step is $n = 3$, and there is nothing to prove. For the inductive step, assume the inductive hypothesis that the statement is true for polygons with fewer than n vertices, and consider any triangulation of a convex n-gon whose vertices are labeled consecutively as v_m, v_{m+1}, ..., v_{m+n-1}. One of the diagonals in the triangulation must have either v_{m+n-1} or v_{m+n-2} as an endpoint (otherwise, the region containing v_{m+n-1} would not be a triangle). So there are two cases. If the triangulation uses diagonal $v_k v_{m+n-1}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-1} as v_{k+1} in the polygon that contains v_m. This gives us the desired numbering of the triangles, with numbers v_m through v_{k-1} in the first polygon and numbers v_k through v_{m+n-3} in the second polygon. If the triangulation uses diagonal $v_k v_{m+n-2}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-2} as v_{k+1} and v_{m+n-1} as v_{k+2} in the polygon that contains v_{m+n-1}, and renumbering all the vertices by adding 1 to their indices in the other polygon. This gives us the desired numbering of the triangles, with numbers v_m through v_k in the first polygon and numbers v_{k+1} through v_{m+n-3} in the second polygon. Note that we did not need the convexity of our polygons.

20. The proof takes several pages and can be found in an article entitled “Polygons Have Ears” by Gary H. Meisters in *The American Mathematical Monthly* **82** (1975) 648–651.

22. The basis step for this induction is no problem, because for $n = 3$, there can be no diagonals and therefore there are two vertices that are not endpoints of the diagonals. (Note, though, that $Q(3)$ is not true.) For $n = 4$, there can be at most one diagonal, and the two vertices that are not its endpoints satisfy the requirements for both $P(4)$ and $Q(4)$. We look at the inductive steps.

a) The proof would presumably try to go something like this. Given a polygon with its set of nonintersecting diagonals, think of one of those diagonals as splitting the polygon into two polygons, each of which then has a set of nonintersecting diagonals. By the inductive hypothesis, each of the two polygons has at least two vertices that are not endpoints of any of these diagonals. We would hope that these two vertices would be the vertices we want. However, one or both of them in each case might actually be endpoints of that separating diagonal, which is a side, not a diagonal, of the smaller polygons. Therefore we have no guarantee that *any* of the points we found do what we want them to do in the original polygon.

b) As in part **(a)**, given a polygon with its set of nonintersecting diagonals, think of one of those diagonals—let’s call it uv—as splitting the polygon into two polygons, each of which then has a set of nonintersecting diagonals. By the inductive hypothesis, each of the two polygons has at least two nonadjacent vertices that are not endpoints of any of these diagonals. Furthermore, the two vertices in each case cannot both be u and v, because u and v are adjacent. Therefore there is a vertex w in one of the smaller polygons and a vertex x in the other that differ from u and v and are not endpoints of any of the diagonals. Clearly w and x do what we want them to do in the original polygon—they are not adjacent and they are not the endpoints of any of the diagonals.

24. Call a suitee w and a suitor m “possible” for each other if there exists a stable assignment in which m and w are paired. We will prove that if a suitee w rejects a suitor m, then w is impossible for m. Since the suitors propose in their preference order, the desired conclusion follows. The proof is by induction on the round in which the rejection happens. We will let m be Bob and w be Alice in our discussion. If it is the first round, then say that Bob and Ted both propose to Alice (necessarily the first choice of each of them), and Alice rejects Bob because she prefers Ted. There can be no stable assignment in which Bob is paired with Alice, because then Alice and Ted would form an unstable pair (Alice prefers Ted to Bob, and Ted prefers Alice to everyone else so in particular prefers her to his mate). So assume the inductive hypothesis, that every suitor who has been rejected so far is impossible for every suitee who has rejected him. At this point Bob proposes to Alice and Alice rejects him in favor of, say, Ted. The reason that Ted has proposed to Alice is that she is his favorite among everyone who has not already rejected him; but by the inductive hypothesis, all the suitees who have rejected him are impossible for him. But now there can be no stable assignment in which Bob and Alice are paired, because such an assignment would again leave Alice and Ted unhappy—Alice because she prefers Ted to Bob, and Ted because he prefers Alice to the person he ended up with (remember that by the inductive hypothesis, he cannot have ended up with anyone he prefers to Alice). This completes the inductive step.

For more information, see the seminal article on this topic ("College Admissions and the Stability of Marriage" by David Gale and Lloyd S. Shapley in *The American Mathematical Monthly* **69** (1962) 9–15) or a definitive book (*The Stable Marriage Problem: Structure and Algorithms* by Dan Gusfield and Robert W. Irving (MIT Press, 1989)).

26. **a)** Clearly these conditions tell us that $P(n)$ is true for the even values of n, namely, 0, 2, 4, 6, 8, $\ldots$. Also, it is clear that there is no way to be sure that $P(n)$ is true for other values of n.

b) Clearly these conditions tell us that $P(n)$ is true for the values of n that are multiples of 3, namely, 0, 3, 6, 9, 12, $\ldots$. Also, it is clear that there is no way to be sure that $P(n)$ is true for other values of n.

c) These conditions are sufficient to prove by induction that $P(n)$ is true for all nonnegative integers n.

d) We immediately know that $P(0)$, $P(2)$, and $P(3)$ are true, and clearly there is no way to be sure that $P(1)$ is true. Once we have $P(2)$ and $P(3)$, the inductive step $P(n) \to P(n+2)$ gives us the truth of $P(n)$ for all $n \geq 2$.

28. We prove by strong induction on n that $P(n)$ is true for all $n \geq b$. The basis step is $n = b$, which is true by the given conditions. For the inductive step, fix an integer $k \geq b$ and assume the inductive hypothesis that if $P(j)$ is true for all j with $b \leq j \leq k$, then $P(k+1)$ is true. There are two cases. If $k+1 \leq b+j$, then $P(k+1)$ is true by the given conditions. On the other hand, if $k+1 > b+j$, then the given conditional statement has its antecedent true by the inductive hypothesis and so again $P(k+1)$ follows.

30. The flaw comes in the inductive step, where we are implicitly assuming that $k \geq 1$ in order to talk about a^{k-1} in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). Our basis step was $n = 0$, so we are not justified in assuming that $k \geq 1$ when we try to prove the statement for $k+1$ in the inductive step. Indeed, it is precisely at $n = 1$ that the proposition breaks down.

32. The proof is invalid for $k = 4$. We cannot increase the postage from 4 cents to 5 cents by either of the replacements indicated, because there is no 3-cent stamp present and there is only one 4-cent stamp present. There is also a minor flaw in the inductive step, because the condition that $j \geq 3$ is not mentioned.

34. We use the technique from part **(b)** of Exercise 33. We are thinking of k as fixed and using induction on n. If $n = 1$, then the sum contains just one term, which is just $k!$, and the right-hand side is also $k!$, so the proposition is true in this case. Next we assume the inductive hypothesis,

$$\sum_{j=1}^{n} j(j+1)(j+2)\cdots(j+k-1) = \frac{n(n+1)(n+2)\cdots(n+k)}{k+1},$$

and prove the statement for $n+1$, namely,

$$\sum_{j=1}^{n+1} j(j+1)(j+2)\cdots(j+k-1) = \frac{(n+1)(n+2)\cdots(n+k)(n+k+1)}{k+1}.$$

We have

$$\begin{aligned}
\sum_{j=1}^{n+1} j(j+1)(j+2)\cdots(j+k-1) &= \left(\sum_{j=1}^{n} j(j+1)(j+2)\cdots(j+k-1)\right) + (n+1)(n+2)\cdots(n+k)\\
&= \frac{n(n+1)(n+2)\cdots(n+k)}{k+1} + (n+1)(n+2)\cdots(n+k)\\
&= (n+1)(n+2)\cdots(n+k)\left(\frac{n}{k+1}+1\right)\\
&= (n+1)(n+2)\cdots(n+k)\cdot\frac{n+k+1}{k+1},
\end{aligned}$$

as desired.

36. **a)** That S is nonempty is trivial, since letting $s=1$ and $t=1$ gives $a+b$, which is certainly a positive integer in S.

b) The well-ordering property asserts that every nonempty set of positive integers has a least element. Since we just showed that S is a nonempty set of positive integers, it has a least element, which we will call c.

c) If d is a divisor of a and of b, then it is also a divisor of as and bt, and hence of their sum. Since c is such a sum, d is a divisor of c.

d) This is the hard part. By symmetry it is enough to show one of these, say that $c\,|\,a$. Assume (for a proof by contradiction) that $c \not| a$. Then by the division algorithm (Section 4.1), we can write $a = qc + r$, where $0 < r < c$. Now $c = as + bt$ (for appropriate choices of s and t), since $c \in S$, so we can compute that $r = a - qc = a - q(as+bt) = a(1-qs) + b(-qt)$. This expresses the positive integer r as a linear combination with integer coefficients of a and b and hence tells us that $r \in S$. But since $r < c$, this contradicts the choice of c. Therefore our assumption that $c \not| a$ is wrong, and $c\,|\,a$, as desired.

e) We claim that the c found in this exercise is the greatest common divisor of a and b. Certainly by part **(d)** it is a common divisor of a and b. On the other hand, part **(c)** tells us that every common divisor of a and b is a divisor of (and therefore no greater than) c. Thus c is a greatest common divisor of a and b. Of course the greatest common divisor is unique, since one cannot have two numbers, each of which is greater than the other.

38. In Exercise 46 of Section 1.8, we found a closed path that snakes its way around an 8×8 checkerboard to cover all the squares, and using that we were able to prove that when one black and one white square are removed, the remaining board can be covered with dominoes. The same reasoning works for any size board, so it suffices to show that any board with an even number of squares has such a snaking path. Note that a board with an even number of squares must have either an even number of rows or an even number of columns, so without loss of generality, assume that it has an even number of rows, say $2n$ rows and m columns. Number the squares in the usual manner, so that the first row contains squares 1 to m from left to right, the second row contains squares $m+1$ to $2m$ from left to right, and so on, with the final row containing squares $(2n-1)m+1$ to $2nm$ from left to right.

We will prove the stronger statement that any such board contains a path that includes the top row traversed from left to right. The basis step is $n=1$, and in that case the path is simply 1, 2, $\ldots$, m, $2m$, $2m-1$, $\ldots$, $m+1$, 1. Assume the inductive hypothesis and consider a board with $2n+2$ rows. By the inductive hypothesis, the board obtained by deleting the top two rows has a closed path that includes its top row from left to right (i.e., $2m+1,\ 2m+2,\ \ldots,\ 3m$). Replace this subsequence by $2m+1,\ m+1,\ 1,\ 2,\ \ldots,\ m,\ 2m,\ 2m-1,\ \ldots,\ m+2,\ 2m+2,\ \ldots,\ 3m$, and we have the desired path.

40. If $x < y$ then $y-x$ is a positive real number, and its reciprocal $1/(y-x)$ is a positive real number, so we can choose a positive integer $A > 1/(y-x)$. (Technically this is the Archimedean property of the real numbers; see Appendix 1.) Now look at $\lfloor x \rfloor + (j/A)$ for positive integers j. Each of these is a rational number. Choose j to be the least positive integer such that this number is greater than x. Such a j exists by the well-ordering property, since clearly if j is large enough, then $\lfloor x \rfloor + (j/A)$ exceeds x. (Note that $j=0$ results in a value not greater than x.) So we have $r = \lfloor x \rfloor + (j/A) > x$ but $\lfloor x \rfloor + ((j-1)/A) = r - (1/A) \le x$. From this last inequality, substituting $y-x$ for $1/A$ (which only makes the left-hand side smaller) we have $r-(y-x) < x$, whence $r < y$, as desired.

42. The strong induction principle clearly implies ordinary induction, for if one has shown that $P(k) \to P(k+1)$, then it automatically follows that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$; in other words, strong induction can always be invoked whenever ordinary induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose that $Q(k)$ is true, i.e., $P(1) \wedge \cdots \wedge P(k)$ is true. By the proof of $\forall n P(n)$ by strong induction it follow that $P(k+1)$ is true. But $Q(k) \wedge P(k+1)$ is just $Q(k+1)$. Thus we have proved $\forall n Q(n)$ by ordinary induction.

SECTION 5.3 Recursive Definitions and Structural Induction

2. **a)** $f(1) = -2f(0) = -2 \cdot 3 = -6$, $f(2) = -2f(1) = -2 \cdot (-6) = 12$, $f(3) = -2f(2) = -2 \cdot 12 = -24$, $f(4) = -2f(3) = -2 \cdot (-24) = 48$, $f(5) = -2f(4) = -2 \cdot 48 = -96$

b) $f(1) = 3f(0)+7 = 3 \cdot 3 + 7 = 16$, $f(2) = 3f(1)+7 = 3 \cdot 16 + 7 = 55$, $f(3) = 3f(2)+7 = 3 \cdot 55 + 7 = 172$, $f(4) = 3f(3)+7 = 3 \cdot 172 + 7 = 523$, $f(5) = 3f(4)+7 = 3 \cdot 523 + 7 = 1576$

c) $f(1) = f(0)^2 - 2f(0) - 2 = 3^2 - 2 \cdot 3 - 2 = 1$, $f(2) = f(1)^2 - 2f(1) - 2 = 1^2 - 2 \cdot 1 - 2 = -3$, $f(3) = f(2)^2 - 2f(2) - 2 = (-3)^2 - 2 \cdot (-3) - 2 = 13$, $f(4) = f(3)^2 - 2f(3) - 2 = 13^2 - 2 \cdot 13 - 2 = 141$, $f(5) = f(4)^2 - 2f(4) - 2 = 141^2 - 2 \cdot 141 - 2 = 19{,}597$

d) First note that $f(1) = 3^{f(0)/3} = 3^{3/3} = 3 = f(0)$. In the same manner, $f(n) = 3$ for all n.

4. **a)** $f(2) = f(1) - f(0) = 1 - 1 = 0$, $f(3) = f(2) - f(1) = 0 - 1 = -1$, $f(4) = f(3) - f(2) = -1 - 0 = -1$, $f(5) = f(4) - f(3) = -1 - 1 = 0$

b) Clearly $f(n) = 1$ for all n, since $1 \cdot 1 = 1$.

c) $f(2) = f(1)^2 + f(0)^3 = 1^2 + 1^3 = 2$, $f(3) = f(2)^2 + f(1)^3 = 2^2 + 1^3 = 5$, $f(4) = f(3)^2 + f(2)^3 = 5^2 + 2^3 = 33$, $f(5) = f(4)^2 + f(3)^3 = 33^2 + 5^3 = 1214$

d) Clearly $f(n) = 1$ for all n, since $1/1 = 1$.

6. **a)** This is valid, since we are provided with the value at $n=0$, and each subsequent value is determined by the previous one. Since all that changes from one value to the next is the sign, we conjecture that $f(n) = (-1)^n$. This is true for $n=0$, since $(-1)^0 = 1$. If it is true for $n=k$, then we have $f(k+1) = -f(k+1-1) = -f(k) = -(-1)^k$ by the inductive hypothesis, whence $f(k+1) = (-1)^{k+1}$.

b) This is valid, since we are provided with the values at $n = 0,\ 1,$ and 2, and each subsequent value is determined by the value that occurred three steps previously. We compute the first several terms of the sequence: $1,\ 0,\ 2,\ 2,\ 0,\ 4,\ 4,\ 0,\ 8,\ \ldots$. We conjecture the formula $f(n) = 2^{n/3}$ when $n \equiv 0 \pmod 3$,

$f(n)=0$ when $n\equiv 1 \pmod 3$, $f(n)=2^{(n+1)/3}$ when $n\equiv 2 \pmod 3$. To prove this, first note that in the base cases we have $f(0)=1=2^{0/3}$, $f(1)=0$, and $f(2)=2=2^{(2+1)/3}$. Assume the inductive hypothesis that the formula is valid for smaller inputs. Then for $n\equiv 0 \pmod 3$ we have $f(n)=2f(n-3)=2\cdot 2^{(n-3)/3}=2\cdot 2^{n/3}\cdot 2^{-1}=2^{n/3}$, as desired. For $n\equiv 1 \pmod 3$ we have $f(n)=2f(n-3)=2\cdot 0=0$, as desired. And for $n\equiv 2 \pmod 3$ we have $f(n)=2f(n-3)=2\cdot 2^{(n-3+1)/3}=2\cdot 2^{(n+1)/3}\cdot 2^{-1}=2^{(n+1)/3}$, as desired.

c) This is invalid. We are told that $f(2)$ is defined in terms of $f(3)$, but $f(3)$ has not been defined.

d) This is invalid, because the value at $n=1$ is defined in two conflicting ways—first as $f(1)=1$ and then as $f(1)=2f(1-1)=2f(0)=2\cdot 0=0$.

e) This appears syntactically to be not valid, since we have conflicting instruction for odd $n\geq 3$. On the one hand $f(3)=f(2)$, but on the other hand $f(3)=2f(1)$. However, we notice that $f(1)=f(0)=2$ and $f(2)=2f(0)=4$, so these apparently conflicting rules tell us that $f(3)=4$ on the one hand and $f(3)=2\cdot 2=4$ on the other hand. Thus we got the same answer either way. Let us show that in fact this definition is valid because the rules coincide.

We compute the first several terms of the sequence: $2, 2, 4, 4, 8, 8, \ldots$. We conjecture the formula $f(n)=2^{\lceil (n+1)/2\rceil}$. To prove this inductively, note first that $f(0)=2=2^{\lceil (0+1)/2\rceil}$. For larger values we have for n odd using the first part of the recursive step that $f(n)=f(n-1)=2^{\lceil (n-1+1)/2\rceil}=2^{\lceil n/2\rceil}=2^{\lceil (n+1)/2\rceil}$, since $n/2$ is not an integer. For $n\geq 2$, whether even or odd, using the second part of the recursive step we have $f(n)=2f(n-2)=2\cdot 2^{\lceil (n-2+1)/2\rceil}=2\cdot 2^{\lceil (n+1)/2\rceil-1}=2\cdot 2^{\lceil (n+1)/2\rceil}\cdot 2^{-1}=2^{\lceil (n+1)/2\rceil}$, as desired.

8. Many answers are possible.

a) Each term is 4 more than the term before it. We can therefore define the sequence by $a_1=2$ and $a_{n+1}=a_n+4$ for all $n\geq 1$.

b) We note that the terms alternate: $0, 2, 0, 2$, and so on. Thus we could define the sequence by $a_1=0$, $a_2=2$, and $a_n=a_{n-2}$ for all $n\geq 3$.

c) The sequence starts out $2, 6, 12, 20, 30$, and so on. The differences between successive terms are $4, 6, 8, 10$, and so on. Thus the n^{th} term is $2n$ greater than the term preceding it; in symbols: $a_n=a_{n-1}+2n$. Together with the initial condition $a_1=2$, this defines the sequence recursively.

d) The sequence starts out $1, 4, 9, 16, 25$, and so on. The differences between successive terms are $3, 5, 7, 9$, and so on—the odd numbers. Thus the n^{th} term is $2n-1$ greater than the term preceding it; in symbols: $a_n=a_{n-1}+2n-1$. Together with the initial condition $a_1=1$, this defines the sequence recursively.

10. The base case is that $S_m(0)=m$. The recursive part is that $S_m(n+1)$ is the successor of $S_m(n)$ (i.e., the integer that follows $S_m(n)$, namely $S_m(n)+1$).

12. The basis step ($n=1$) is clear, since $f_1^2=f_1f_2=1$. Assume the inductive hypothesis. Then $f_1^2+f_2^2+\cdots+f_n^2+f_{n+1}^2=f_nf_{n+1}+f_{n+1}^2=f_{n+1}(f_n+f_{n+1})=f_{n+1}f_{n+2}$, as desired.

14. The basis step ($n=1$) is clear, since $f_2f_0-f_1^2=1\cdot 0-1^2=-1=(-1)^1$. Assume the inductive hypothesis. Then we have

$$\begin{aligned}
f_{n+2}f_n-f_{n+1}^2 &= (f_{n+1}+f_n)f_n-f_{n+1}^2\\
&= f_{n+1}f_n+f_n^2-f_{n+1}^2\\
&= -f_{n+1}(f_{n+1}-f_n)+f_n^2\\
&= -f_{n+1}f_{n-1}+f_n^2\\
&= -(f_{n+1}f_{n-1}-f_n^2)\\
&= -(-1)^n=(-1)^{n+1}.
\end{aligned}$$

16. The basis step ($n = 1$) is clear, since $f_0 - f_1 + f_2 = 0 - 1 + 1 = 0$, and $f_1 - 1 = 0$ as well. Assume the inductive hypothesis. Then we have (substituting using the defining relation for the Fibonacci sequence where appropriate)

$$\begin{aligned} f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} - f_{2n+1} + f_{2n+2} &= f_{2n-1} - 1 - f_{2n+1} + f_{2n+2} \\ &= f_{2n-1} - 1 + f_{2n} \\ &= f_{2n+1} - 1 \\ &= f_{2(n+1)-1} - 1\,. \end{aligned}$$

18. We prove this by induction on n. Clearly $\mathbf{A}^1 = \mathbf{A} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix}$. Assume the inductive hypothesis. Then

$$\mathbf{A}^{n+1} = \mathbf{A}\mathbf{A}^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+1} + f_n & f_n + f_{n-1} \\ f_{n+1} & f_n \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix},$$

as desired.

20. The max or min of one number is itself; $\max(a_1, a_2) = a_1$ if $a_1 \geq a_2$ and a_2 if $a_1 < a_2$, whereas $\min(a_1, a_2) = a_2$ if $a_1 \geq a_2$ and a_1 if $a_1 < a_2$; and for $n \geq 2$,

$$\max(a_1, a_2, \ldots, a_{n+1}) = \max(\max(a_1, a_2, \ldots, a_n), a_{n+1})$$

and

$$\min(a_1, a_2, \ldots, a_{n+1}) = \min(\min(a_1, a_2, \ldots, a_n), a_{n+1})\,.$$

22. Clearly only positive integers can be in S, since 1 is a positive integer, and the sum of two positive integers is again a positive integer. To see that all positive integers are in S, we proceed by induction. Obviously $1 \in S$. Assuming that $n \in S$, we get that $n + 1$ is in S by applying the recursive part of the definition with $s = n$ and $t = 1$. Thus S is precisely the set of positive integers.

24. **a)** Odd integers are obtained from other odd integers by adding 2. Thus we can define this set S as follows: $1 \in S$; and if $n \in S$, then $n + 2 \in S$.

b) Powers of 3 are obtained from other powers of 3 by multiplying by 3. Thus we can define this set S as follows: $3 \in S$ (this is 3^1, the power of 3 using the smallest positive integer exponent); and if $n \in S$, then $3n \in S$.

c) There are several ways to do this. One that is suggested by Horner's method is as follows. We will assume that the variable for these polynomials is the letter x. All integers are in S (this base case gives us all the constant polynomials); if $p(x) \in S$ and n is any integer, then $xp(x) + n$ is in S. Another method constructs the polynomials term by term. Its base case is to let 0 be in S; and its inductive step is to say that if $p(x) \in S$, c is an integer, and n is a nonnegative integer, then $p(x) + cx^n$ is in S.

26. **a)** If we apply each of the recursive step rules to the only element given in the basis step, we see that $(2,3)$ and $(3,2)$ are in S. If we apply the recursive step to these we add $(4,6)$, $(5,5)$, and $(6,4)$. The next round gives us $(6,9)$, $(7,8)$, $(8,7)$, and $(9,6)$. A fourth set of applications adds $(8,12)$, $(9,11)$, $(10,10)$, $(11,9)$, and $(12,8)$; and a fifth set of applications adds $(10,15)$, $(11,14)$, $(12,13)$, $(13,12)$, $(14,11)$, and $(15,10)$.

b) Let $P(n)$ be the statement that $5 \mid a + b$ whenever $(a,b) \in S$ is obtained by n applications of the recursive step. For the basis step, $P(0)$ is true, since the only element of S obtained with no applications of the recursive step is $(0,0)$, and indeed $5 \mid 0 + 0$. Assume the strong inductive hypothesis that $5 \mid a + b$ whenever $(a,b) \in S$ is obtained by k or fewer applications of the recursive step, and consider an element obtained with

$k+1$ applications of the recursive step. Since the final application of the recursive step to an element (a,b) must be applied to an element obtained with fewer applications of the recursive step, we know that $5 \mid a+b$. So we just need to check that this inequality implies $5 \mid a+2+b+3$ and $5 \mid a+3+b+2$. But this is clear, since each is equivalent to $5 \mid a+b+5$, and 5 divides both $a+b$ and 5.

c) This holds for the basis step, since $5 \mid 0+0$. If this holds for (a,b), then it also holds for the elements obtained from (a,b) in the recursive step by the same argument as in part **(b)**.

28. **a)** The simplest elements of S are $(1,2)$ and $(2,1)$. That is the basis step. To get new elements of S from old ones, we need to maintain the parity of the sum, so we either increase the first coordinate by 2, increase the second coordinate by 2, or increase each coordinate by 1. Thus our recursive step is that if $(a,b) \in S$, then $(a+2,b) \in S$, $(a,b+2) \in S$, and $(a+1,b+1) \in S$.

b) The statement here is that b is a multiple of a. One approach is to have an infinite number of base cases to take care of the fact that every element is a multiple of itself. So we have $(n,n) \in S$ for all $n \in \mathbf{Z}^+$. If one objects to having an infinite number of base cases, then we can start with $(1,1) \in S$ and a recursive rule that if $(a,a) \in S$, then $(a+1,a+1) \in S$. Larger multiples of a can be obtained by adding a to a known multiple of a, so our recursive step is that if $(a,b) \in S$, then $(a,a+b) \in S$.

c) The smallest pairs in which the sum of the coordinates is a multiple of 3 are $(1,2)$ and $(2,1)$. So our basis step is $(1,2) \in S$ and $(2,1) \in S$. If we start with a point for which the sum of the coordinates is a multiple of 3 and want to maintain this divisibility condition, then we can add 3 to the first coordinate, or add 3 to the second coordinate, or add 1 to the one of the coordinates and 2 to the other. Thus our recursive step is that if $(a,b) \in S$, then $(a+3,b) \in S$, $(a,b+3) \in S$, $(a+1,b+2) \in S$, and $(a+2,b+1) \in S$.

30. Since we are concerned only with the substrings 01 and 10, all we care about are the changes from 0 to 1 or 1 to 0 as we move from left to right through the string. For example, we view 0011110110100 as a block of 0's followed by a block of 1's followed by a block of 0's followed by a block of 1's followed by a block of 0's followed by a block of 1's followed by a block of 0's. There is one occurrence of 01 or 10 at the start of each block other than the first, and the occurrences alternate between 01 and 10. If the string has an odd number of blocks (or the string is empty), then there will be an equal number of 01's and 10's. If the string has an even number of blocks, then the string will have one more 01 than 10 if the first block is 0's, and one more 10 than 01 if the first block is 1's. (One could also give an inductive proof, based on the length of the string, but a stronger statement is needed: that if the string ends in a 1 then 01 occurs at most one more time than 10, but that if the string ends in a 0, then 01 occurs at most as often as 10.)

32. **a)** $ones(\lambda) = 0$ and $ones(wx) = x + ones(w)$, where w is a bit string and x is a bit (viewed as an integer when being added)

b) The basis step is when $t=\lambda$, in which case we have $ones(s\lambda) = ones(s) = ones(s)+0 = ones(s)+ones(\lambda)$. For the inductive step, write $t = wx$, where w is a bit string and x is a bit. Then we have $ones(s(wx)) = ones((sw)x) = x + ones(sw)$ by the recursive definition, which is $x + ones(s) + ones(w)$ by the inductive hypothesis, which is $ones(s) + (x + ones(w))$ by commutativity and associativity of addition, which finally equals $ones(s) + ones(wx)$ by the recursive definition.

34. **a)** 1010 **b)** 1 1011 **c)** 1110 1001 0001

36. We induct on w_2. The basis step is $(w_1\lambda)^R = w_1^R = \lambda w_1^R = \lambda^R w_1^R$. For the inductive step, assume that $w_2 = w_3x$, where w_3 is a string of length one less than the length of w_2, and x is a symbol (the last symbol of w_2). Then we have $(w_1w_2)^R = (w_1w_3x)^R = x(w_1w_3)^R$ (by the recursive definition given in the solution to Exercise 35). This in turn equals $xw_3^R w_1^R$ by the inductive hypothesis, which is $(w_3x)^R w_1^R$ (again by the definition). Finally, this equals $w_2^R w_1^R$, as desired.

38. There are two types of palindromes, so we need two base cases, namely λ is a palindrome, and x is a palindrome for every symbol x. The recursive step is that if α is a palindrome and x is a symbol, then $x\alpha x$ is a palindrome.

40. The key fact here is that if a bit string of length greater than 1 has more 0's than 1's, then either it is the concatenation of two such strings, or else it is the concatenation of two such strings with one 1 inserted either before the first, between them, or after the last. This can be proved by looking at the running count of the excess of 0's over 1's as we read the string from left to right. Therefore one recursive definition is that 0 is in the set, and if x and y are in the set, then so are xy, $1xy$, $x1y$, and $xy1$.

42. Recall from Exercise 37 the recursive definition of the i^{th} power of a string. We also will use the result of Exercise 36 and the following lemma: $w^{i+1} = w^i w$ for all $i \geq 0$, which is clear (or can be proved by induction on i, using the associativity of concatenation).

Now to prove that $(w^R)^i = (w^i)^R$, we use induction on i. It is clear for $i = 0$, since $(w^R)^0 = \lambda = \lambda^R = (w^i)^R$. Assuming the inductive hypothesis, we have $(w^R)^{i+1} = w^R(w^R)^i = w^R(w^i)^R = (w^i w)^R = (w^{i+1})^R$, as desired.

44. For the basis step we have the tree consisting of just the root, so there is one leaf and there are no internal vertices, and $l(T) = i(T) + 1$ holds. For the recursive step, assume that this relationship holds for T_1 and T_2, and consider the tree with a new root, whose children are the roots of T_1 and T_2. The new root is an internal vertex of T, and every internal vertex in T_1 or T_2 is an internal vertex of T, so $i(T) = i(T_1) + i(T_2) + 1$. Similarly, the leaves of T_1 and T_2 are the leaves of T, so $l(T) = l(T_1) + l(T_2)$. Thus we have $l(T) = l(T_1) + l(T_2) = i(T_1) + 1 + i(T_2) + 1$ by the inductive hypothesis, which equals $(i(T_1) + i(T_2) + 1) + 1 = i(T) + 1$, as desired.

46. The basis step requires that we show that this formula holds when $(m, n) = (1, 1)$. The induction step requires that we show that if the formula holds for all pairs smaller than (m, n) in the lexicographic ordering of $\mathbf{Z}^+ \times \mathbf{Z}^+$, then it also holds for (m, n). For the basis step we have $a_{1,1} = 5 = 2(1+1)+1$. For the inductive step, assume that $a_{m',n'} = 2(m'+n')+1$ whenever (m', n') is less than (m, n) in the lexicographic ordering of $\mathbf{Z}^+ \times \mathbf{Z}^+$. By the recursive definition, if $n = 1$ then $a_{m,n} = a_{m-1,n}+2$; since $(m-1, n)$ is smaller than (m, n), the induction hypothesis tells us that $a_{m-1,n} = 2(m-1+n)+1$, so $a_{m,n} = 2(m-1+n)+1+2 = 2(m+n)+1$, as desired. Now suppose that $n > 1$, so $a_{m,n} = a_{m,n-1} + 2$. Again we have $a_{m,n-1} = 2(m + n - 1) + 1$, so $a_{m,n} = 2(m + n - 1) + 1 + 2 = 2(m + n) + 1$, and the proof is complete.

48. **a)** $A(1, 0) = 0$ by the second line of the definition.
b) $A(0, 1) = 2$ by the first line of the definition.
c) $A(1, 1) = 2$ by the third line of the definition.
d) $A(2, 2) = A(1, A(2, 1)) = A(1, 2) = A(0, A(1, 1)) = A(0, 2) = 4$

50. We prove this by induction on n. It is clear for $n = 1$, since $A(1, 1) = 2 = 2^1$. Assume that $A(1, n) = 2^n$. Then $A(1, n + 1) = A(0, A(1, n)) = A(0, 2^n) = 2 \cdot 2^n = 2^{n+1}$, as desired.

52. This is impossible to compute, if by compute we mean write down a nice numeral for the answer. As explained in the solution to Exercise 51, one can show by induction that $A(2, n)$ is equal to $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$, with n 2's in the tower. To compute $A(3, 4)$ we use the definition to write $A(3, 4) = A(2, A(3, 3))$. We saw in the solution to Exercise 51, however, that $A(3, 3) = 65536$, so $A(3, 4) = A(2, 65536)$. Thus $A(3, 4)$ is a tower of 2's with 65536 2's in the tower. There is no nicer way to write or describe this number—it is too big.

54. We use a double induction here, inducting first on m and then on n. The outside base case is $m=0$ (with n arbitrary). Then $A(m,n)=2n$ for all n. Also $A(m+1,n)=2n$ for $n=0$ and $n=1$, and $2n \geq 2n$ in those cases; and $A(m+1,n)=2^n$ for all $n>1$ (by Exercise 50), and in those cases $2^n \geq 2n$ as well. Now we assume the inductive hypothesis, that $A(m+1,t) \geq A(m,t)$ for all t. We will show by induction on n that $A(m+2,n) \geq A(m+1,n)$. For $n=0$ this reduces to $0 \geq 0$, and for $n=1$ it reduces to $2 \geq 2$. Assume the inner inductive hypothesis, that $A(m+2,n) \geq A(m+1,n)$. Then

$$\begin{aligned} A(m+2,n+1) &= A(m+1,A(m+2,n)) \\ &\geq A(m+1,A(m+1,n)) \quad \text{(using the inductive hypothesis and Exercise 53)} \\ &\geq A(m,A(m+1,n)) \quad \text{(by the inductive hypothesis on } m\text{)} \\ &= A(m+1,n+1)\,. \end{aligned}$$

56. Let $P(n)$ be the statement "F is well-defined at n." Then $P(0)$ is true, since $F(0)$ is specified. Assume that $P(n)$ is true. Then F is also well-defined at $n+1$, since $F(n+1)$ is given in terms of $F(n)$. Therefore by mathematical induction, $P(n)$ is true for all n, i.e., F is well-defined as a function on the set of all nonnegative integers.

58. **a)** This would be a proper definition if the recursive part were stated to hold for $n \geq 2$. As it stands, however, $F(1)$ is ambiguous, and $F(0)$ is undefined.

b) This definition makes no sense as it stands; $F(3)$ is not defined, since $F(0)$ isn't. Also, $F(2)$ is ambiguous.

c) For $n=3$, the recursive part makes no sense, since we would have to know $F(3/2)$. Also, $F(2)$ is ambiguous.

d) The definition is ambiguous about $n=1$, since both the second clause and the third clause seem to apply. This would be a valid definition if the third clause applied only to odd $n \geq 3$.

e) We note that $F(1)$ is defined explicitly, $F(2)$ is defined in terms of $F(1)$, $F(4)$ is defined in terms of $F(2)$, and $F(3)$ is defined in terms of $F(8)$, which is defined in terms of $F(4)$. So far, so good. However, let us see what the definition says to do with $F(5)$:

$$F(5)=F(14)=1+F(7)=1+F(20)=1+1+F(10)=1+1+1+F(5)\,.$$

This not only leaves us begging the question as to what $F(5)$ is, but is a contradiction, since $0 \neq 3$. (If we replace "$3n-1$" by "$3n+1$" in this problem, then it is an unsolved problem—the Collatz conjecture—as to whether F is well-defined; see Example 23 in Section 1.8.)

60. In each case we will apply the definition. Note that $\log^{(1)} n = \log n$ (for $n>0$). Similarly, $\log^{(2)} n = \log(\log n)$ as long as it is defined (which is when $n>1$), $\log^{(3)} n = \log(\log(\log n))$ as long as it is defined (which is when $n>2$), and so on. Normally the parentheses are understood and omitted.

a) $\log^{(2)} 16 = \log\log 16 = \log 4 = 2$, since $2^4=16$ and $2^2=4$

b) $\log^{(3)} 256 = \log\log\log 256 = \log\log 8 = \log 3 \approx 1.585$

c) $\log^{(3)} 2^{65536} = \log\log\log 2^{65536} = \log\log 65536 = \log 16 = 4$

d) $\log^{(4)} 2^{2^{65536}} = \log\log\log\log 2^{2^{65536}} = \log\log\log 2^{65536} = 4$ by part **(c)**

62. Note that $\log^{(1)} 2 = 1$, $\log^{(2)} 2^2 = 1$, $\log^{(3)} 2^{2^2} = 1$, $\log^{(4)} 2^{2^{2^2}} = 1$, and so on. In general $\log^{(k)} n = 1$ when n is a tower of k 2s; once n exceeds a tower of k 2s, $\log^{(k)} n > 1$. Therefore the largest n such that $\log^* n = k$ is a tower of k 2s. Here $k=5$, so the answer is $2^{2^{2^{2^2}}} = 2^{65536}$. This number overflows most calculators. In order to determine the number of decimal digits it has, we recall that the number of decimal digits of a positive integer x is $\lfloor \log_{10} x \rfloor + 1$. Therefore the number of decimal digits of 2^{65536} is $\lfloor \log_{10} 2^{65536} \rfloor + 1 = \lfloor 65536 \log_{10} 2 \rfloor + 1 = 19{,}729$.

64. Each application of the function f divides its argument by 2. Therefore iterating this function k times (which is what $f^{(k)}$ does) has the effect of dividing by 2^k. Therefore $f^{(k)}(n) = n/2^k$. Now $f_1^*(n)$ is the smallest k such that $f^{(k)}(n) \le 1$, that is, $n/2^k \le 1$. Solving this for k easily yields $k \ge \log n$, where logarithm is taken to the base 2. Thus $f_1^*(n) = \lceil \log n \rceil$ (we need to take the ceiling function because k must be an integer).

SECTION 5.4 Recursive Algorithms

2. First, we use the recursive step to write $6! = 6 \cdot 5!$. We then use the recursive step repeatedly to write $5! = 5 \cdot 4!$, $4! = 4 \cdot 3!$, $3! = 3 \cdot 2!$, $2! = 2 \cdot 1!$, and $1! = 1 \cdot 0!$. Inserting the value of $0! = 1$, and working back through the steps, we see that $1! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 = 6$, $4! = 4 \cdot 3! = 4 \cdot 6 = 24$, $5! = 5 \cdot 4! = 5 \cdot 24 = 120$, and $6! = 6 \cdot 5! = 6 \cdot 120 = 720$.

4. First, because $n = 10$ is even, we use the **else if** clause to see that

$$mpower(2, 10, 7) = mpower(2, 5, 7)^2 \ \mathbf{mod}\ 7 .$$

We next use the **else** clause to see that

$$mpower(2, 5, 7) = (mpower(2, 2, 7)^2 \ \mathbf{mod}\ 7 \cdot 2 \ \mathbf{mod}\ 7) \ \mathbf{mod}\ 7 .$$

Then we use the **else if** clause again to see that

$$mpower(2, 2, 7) = mpower(2, 1, 7)^2 \ \mathbf{mod}\ 7 .$$

Using the **else** clause again, we have

$$mpower(2, 1, 7) = (mpower(2, 0, 7)^2 \ \mathbf{mod}\ 7 \cdot 2 \ \mathbf{mod}\ 7) \ \mathbf{mod}\ 7 .$$

Finally, using the **if** clause, we see that $mpower(2, 0, 7) = 1$. Now we work backward: $mpower(2, 1, 7) = (1^2 \ \mathbf{mod}\ 7 \cdot 2 \ \mathbf{mod}\ 7) \ \mathbf{mod}\ 7 = 2$, $mpower(2, 2, 7) = 2^2 \ \mathbf{mod}\ 7 = 4$, $mpower(2, 5, 7) = (4^2 \ \mathbf{mod}\ 7 \cdot 2 \ \mathbf{mod}\ 7) \ \mathbf{mod}\ 7 = 4$, and finally $mpower(2, 10, 7) = 4^2 \ \mathbf{mod}\ 7 = 2$. We conclude that $2^{10} \ \mathbf{mod}\ 7 = 2$.

6. With this input, the algorithm uses the **else** clause to find that $\gcd(12, 17) = \gcd(17 \ \mathbf{mod}\ 12, 12) = \gcd(5, 12)$. It uses this clause again to find that $\gcd(5, 12) = \gcd(12 \ \mathbf{mod}\ 5, 5) = \gcd(2, 5)$, then to get $\gcd(2, 5) = \gcd(5 \ \mathbf{mod}\ 2, 2) = \gcd(1, 2)$, and once more to get $\gcd(1, 2) = \gcd(2 \ \mathbf{mod}\ 1, 1) = \gcd(0, 1)$. Finally, to find $\gcd(0, 1)$ it uses the first step with $a = 0$ to find that $\gcd(0, 1) = 1$. Consequently, the algorithm finds that $\gcd(12, 17) = 1$.

8. The sum of the first n positive integers is the sum of the first $n - 1$ positive integers plus n. This trivial observation leads to the recursive algorithm shown here.

procedure *sum of first*(n : positive integer)
if $n = 1$ **then return** 1
else return *sum of first*$(n - 1) + n$

10. The recursive algorithm works by comparing the last element with the maximum of all but the last. We assume that the input is given as a sequence.

procedure $max(a_1, a_2, \ldots, a_n$: integers)
if $n = 1$ **then return** a_1
else
    $m := max(a_1, a_2, \ldots, a_{n-1})$
    if $m > a_n$ **then return** m
    else return a_n

12. This is the inefficient method.

> **procedure** *power*$(x, n, m$: positive integers)
> **if** $n = 1$ **then return** x **mod** m
> **else return** $\big(x \cdot power(x, n-1, m)\big)$ **mod** m

14. This is actually quite subtle. The recursive algorithm will need to keep track not only of what the mode actually is, but also of how often the mode appears. We will describe this algorithm in words, rather than in pseudocode. The input is a list $a_1, a_2, \ldots, a_n$ of integers. Call this list L. If $n = 1$ (the base case), then the output is that the mode is a_1 and it appears 1 time. For the recursive case ($n > 1$), form a new list L' by deleting from L the term a_n and all terms in L equal to a_n. Let k be the number of terms deleted. If $k = n$ (in other words, if L' is the empty list), then the output is that the mode is a_n and it appears n times. Otherwise, apply the algorithm recursively to L', obtaining a mode m, which appears t times. Now if $t \geq k$, then the output is that the mode is m and it appears t times; otherwise the output is that the mode is a_n and it appears k times.

16. The sum of the first one positive integer is 1, and that is the answer the recursive algorithm gives when $n = 1$, so the basis step is correct. Now assume that the algorithm works correctly for $n = k$. If $n = k + 1$, then the **else** clause of the algorithm is executed, and $k + 1$ is added to the (assumed correct) sum of the first k positive integers. Thus the algorithm correctly finds the sum of the first $k + 1$ positive integers.

18. We use mathematical induction on n. If $n = 0$, we know that $0! = 1$ by definition, so the **if** clause handles this basis step correctly. Now fix $k \geq 0$ and assume the inductive hypothesis—that the algorithm correctly computes $k!$. Consider what happens with input $k + 1$. Since $k + 1 > 0$, the **else** clause is executed, and the answer is whatever the algorithm gives as output for input k, which by inductive hypothesis is $k!$, multiplied by $k + 1$. But by definition, $k! \cdot (k + 1) = (k + 1)!$, so the algorithm works correctly on input $k + 1$.

20. Our induction is on the value of y. When $y = 0$, the product $xy = 0$, and the algorithm correctly returns that value. Assume that the algorithm works correctly for smaller values of y, and consider its performance on y. If y is even (and necessarily at least 2), then the algorithm computes 2 times the product of x and $y/2$. Since it does the product correctly (by the inductive hypothesis), this equals $2(x \cdot y/2)$, which equals xy by the commutativity and associativity of multiplication. Similarly, when y is odd, the algorithm computes 2 times the product of x and $(y - 1)/2$ and then adds x. Since it does the product correctly (by the inductive hypothesis), this equals $2(x \cdot (y - 1)/2) + x$, which equals $xy - x + x = xy$, again by the rules of algebra.

22. The largest in a list of one integer is that one integer, and that is the answer the recursive algorithm gives when $n = 1$, so the basis step is correct. Now assume that the algorithm works correctly for $n = k$. If $n = k + 1$, then the **else** clause of the algorithm is executed. First, by the inductive hypothesis, the algorithm correctly sets m to be the largest among the first k integers in the list. Next it returns as the answer either that value or the $(k + 1)$st element, whichever is larger. This is clearly the largest element in the entire list. Thus the algorithm correctly finds the maximum of a given list of integers.

24. We use the hint.

> **procedure** *twopower*$(n$: positive integer, a : real number)
> **if** $n = 1$ **then return** a^2
> **else return** $twopower(n - 1, a)^2$

26. We use the idea in Exercise 24, together with the fact that $a^n = (a^{n/2})^2$ if n is even, and $a^n = a \cdot (a^{(n-1)/2})^2$ if n is odd, to obtain the following recursive algorithm. In essence we are using the binary expansion of n implicitly.

procedure *fastpower*(n : positive integer, a : real number)
if $n = 1$ **then return** a
else if n is even **then return** *fastpower*$(n/2, a)^2$
else return $a \cdot$ *fastpower*$((n-1)/2, a)^2$

28. To compute f_7, Algorithm 7 requires $f_8 - 1 = 20$ additions, and Algorithm 8 requires $7 - 1 = 6$ additions.

30. This is essentially just Algorithm 8, with a different operation and different initial conditions.

procedure *iterative*(n : nonnegative integer)
if $n = 0$ **then** $y := 1$
else
  $x := 1$
  $y := 2$
  for $i := 1$ **to** $n - 1$
    $z := x \cdot y$
    $x := y$
    $y := z$
return y {the n^{th} term of the sequence}

32. This is very similar to the recursive procedure for computing the Fibonacci numbers. Note that we can combine the three base cases (stopping rules) into one.

procedure *sequence*(n : nonnegative integer)
if $n < 3$ **then return** $n + 1$
else return *sequence*$(n-1)$ + *sequence*$(n-2)$ + *sequence*$(n-3)$

34. The iterative algorithm is much more efficient here. If we compute with the recursive algorithm, we end up computing the small values (early terms in the sequence) over and over and over again (try it for $n = 5$).

36. We obtain the answer by computing $P(m, m)$, where P is the following procedure, which we obtain simply by copying the recursive definition from Exercise 47 in Section 5.3 into an algorithm.

procedure $P(m, n$: positive integers)
if $m = 1$ **then return** 1
else if $n = 1$ **then return** 1
else if $m < n$ **then return** $P(m, m)$
else if $m = n$ **then return** $1 + P(m, m-1)$
else return $P(m, n-1) + P(m-n, n)$

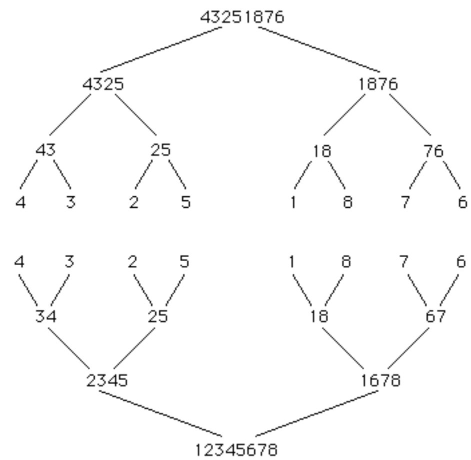
38. The following algorithm practically writes itself.

procedure *power*(w : bit string, i : nonnegative integer)
if $i = 0$ **then return** λ
else return w concatenated with *power*$(w, i-1)$

40. If $i = 0$, then by definition w^i is no copies of w, so it is correct to output the empty string. Inductively, if the algorithm correctly returns the i^{th} power of w, then it correctly returns the $(i+1)^{\text{st}}$ power of w by concatenating one more copy of w.

42. If $n = 3$, then the polygon is already triangulated. Otherwise, by Lemma 1 in Section 5.2, the polygon has a diagonal; draw it. This diagonal splits the polygon into two polygons, each of which has fewer than n vertices. Recursively apply this algorithm to triangulate each of these polygons. The result is a triangulation of the original polygon.

44. The procedure is the same as that given in the solution to Example 9. We will show the tree and inverted tree that indicate how the sequence is taken apart and put back together.



46. From the analysis given before the statement of Lemma 1, it follows that the number of comparisons is $m+n-r$, where the lists have m and n elements, respectively, and r is the number of elements remaining in one list at the point the other list is exhausted. In this exercise $m=n=5$, so the answer is always $10-r$.

a) The answer is $10-1=9$, since the second list has only 1 element when the first list has been emptied.

b) The answer is $10-5=5$, since the second list has 5 elements when the first list has been emptied.

c) The answer is $10-2=8$, since the second list has 2 elements when the first list has been emptied.

48. In each case we need to show that a certain number of comparisons is necessary in the worst case, and then we need to give an algorithm that does the merging with this many comparisons.

a) There are 5 possible outcomes (the element of the first list can be greater than 0, 1, 2, 3, or 4 elements of the second list). Therefore by decision tree theory (see Section 11.2), at least $\lceil \log 5 \rceil = 3$ comparisons are needed. We can achieve this with a binary search: first compare the element of the first list to the second element of the second, and then at most two comparisons are needed to find the correct place for this element.

b) Algorithm 10 merges the lists with 5 comparisons. We must show that 5 are needed in the worst case. Naively applying decision tree theory does not help, since $\lceil \log 15 \rceil = 4$ (there are $C(5+2-1,2)=15$ ways to choose the places among the second list for the elements of the first list to go). Instead, suppose that the lists are a_1, a_2 and b_1, b_2, b_3, b_4, in order. Then without loss of generality assume that the first comparison is a_1 against b_i. If $i \geq 2$ and $a_1 < b_i$, then there are at least 9 outcomes still possible, requiring $\lceil \log 9 \rceil = 4$ more comparisons. If $i=1$ and $a_1 > b_1$, then there are 10 outcomes, again requiring 4 more comparisons.

c) There are $C(5+3-1,3)=35$ outcomes, so at least $\lceil \log 35 \rceil = 6$ comparisons are needed. On the other hand Algorithm 10 uses only 6 comparisons.

d) There are $C(5+4-1,4)=70$ outcomes, so at least $\lceil \log 70 \rceil = 7$ comparisons are needed. On the other hand Algorithm 10 uses only 7 comparisons.

50. On the first pass, we separate the list into two lists, the first being all the elements less than 3 (namely 1 and 2), and the second being all the elements greater than 3, namely $5,7,8,9,4,6$ (in that order). As soon as each of these two lists is sorted (recursively) by quick sort, we are done. We show the entire process in the following sequence of list. The numbers in parentheses are the numbers that are correctly placed by the algorithm on the current level of recursion, and the brackets are those elements that were correctly placed previously. Five levels of recursion are required. 12(3)578946, (1)2[3]4(5)7896, [1](2)[3](4)[5]6(7)89, [1][2][3][4][5](6)[7](8)9, [1][2][3][4][5][6][7][8](9)

52. In practice, this algorithm is coded differently from what we show here, requiring more comparisons but being more efficient because the data structures are simpler (and the sorting is done in place). We denote the list $a_1, a_2, \ldots, a_n$ by a, with similar notations for the other lists. Also, rather than putting a_1 at the end of the first sublist, we put it between the two sublists and do not have to deal with it in either sublist.

```
procedure quick(a_1, a_2, ..., a_n)
b := the empty list
c := the empty list
temp := a_1
for i := 2 to n
        if a_i < a_1 then adjoin a_i to the end of list b
        else adjoin a_i to the end of list c
{notation: m = length(b) and k = length(c)}
if m ≠ 0 then quick(b_1, b_2, ..., b_m)
if k ≠ 0 then quick(c_1, c_2, ..., c_k)
{now put the sorted lists back into a}
for i := 1 to m
        a_i := b_i
a_{m+1} := temp
for i := 1 to k
        a_{m+i+1} := c_i
{the list a is now sorted}
```

54. In the best case, the initial split will require 3 comparisons and result in sublists of length 1 and 2 still to be sorted. These require 0 and 1 comparisons, respectively, and the list has been sorted. Therefore the answer is $3 + 0 + 1 = 4$.

SECTION 5.5 Program Correctness

2. There are two cases. If $x \geq 0$ initially, then nothing is executed, so $x \geq 0$ at the end. If $x < 0$ initially, then x is set equal to 0, so $x = 0$ at the end; hence again $x \geq 0$ at the end.

4. There are three cases. If $x < y$ initially, then min is set equal to x, so $(x \leq y \wedge min = x)$ is true. If $x = y$ initially, then min is set equal to y (which equals x), so again $(x \leq y \wedge min = x)$ is true. Finally, if $x > y$ initially, then min is set equal to y, so $(x > y \wedge min = y)$ is true. Hence in all cases the disjunction $(x \leq y \wedge min = x) \vee (x > y \wedge min = y)$ is true.

6. There are three cases. If $x < 0$, then y is set equal to $-2|x|/x = (-2)(-x)/x = 2$. If $x > 0$, then y is set equal to $2|x|/x = 2x/x = 2$. If $x = 0$, then y is set equal to 2. Hence in all cases $y = 2$ at the termination of this program.

8. We prove that Algorithm 8 in Section 5.4 is correct. It is clearly correct if $n = 0$ or $n = 1$, so we assume that $n \geq 2$. Then the program terminates when the **for** loop terminates, so we concentrate our attention on that loop. Before the loop begins, we have $x = 0$ and $y = 1$. Let the loop invariant p be "$(x = f_{i-1} \wedge y = f_i) \vee (i \text{ is undefined} \wedge x = f_0 \wedge y = f_1)$." This is true at the beginning of the loop, since i is undefined and $f_0 = 0$ and $f_1 = 1$. What we must show now is $p \wedge (1 \leq i < n)\{S\}p$. If $p \wedge (1 \leq i < n)$, then $x = f_{i-1}$ and $y = f_i$. Hence z becomes f_{i+1} by the definition of the Fibonacci sequence. Now x becomes y, namely f_i, and y becomes z, namely f_{i+1}, and i is incremented. Hence for this new (defined) i, $x = f_{i-1}$ and $y = f_i$, as desired. We therefore conclude that upon termination $x = f_{i-1} \wedge y = f_i \wedge i = n$; hence $y = f_n$, as desired.

10. We must show that if p_0 is true before S is executed, then q is true afterwards. Suppose that p_0 is true before S is executed. By the given conditional statement, we know that p_1 is also true. Therefore, since $p_1\{S\}q$, we conclude that q is true after S is executed, as desired.

12. Suppose that the initial assertion is true before the program begins, so that a and d are positive integers. Consider the following loop invariant p: "$a = dq + r$ and $r \geq 0$." This is true before the loop starts, since the equation then states $a = d \cdot 0 + a$, and we are told that a (which equals r at this point) is a positive integer, hence greater than or equal to 0. Now we must show that if p is true and $r \geq d$ before some pass through the loop, then it remains true after the pass. Certainly we still have $r \geq 0$, since all that happened to r was the subtraction of d, and $r \geq d$ to begin this pass. Furthermore, let q' denote the new value of q and r' the new value of r. Then $dq' + r' = d(q+1) + (r-d) = dq + d + r - d = dq + r = a$, as desired. Furthermore, the loop terminates eventually, since one cannot repeated subtract the positive integer d from the positive integer r without r eventually becoming less than d. When the loop terminates, the loop invariant p must still be true, and the condition $r \geq d$ must be false—i.e., $r < d$ must be true. But this is precisely the desired final assertion.

SUPPLEMENTARY EXERCISES FOR CHAPTER 5

2. The proposition is true for $n = 1$, since $1^3 + 3^3 = 28 = 1(1+1)^2(2 \cdot 1^2 + 4 \cdot 1 + 1)$. Assume the inductive hypothesis. Then

$$\begin{aligned} 1^3 + 3^3 + \cdots + (2n+1)^3 + (2n+3)^3 &= (n+1)^2(2n^2 + 4n + 1) + (2n+3)^3 \\ &= 2n^4 + 8n^3 + 11n^2 + 6n + 1 + 8n^3 + 36n^2 + 54n + 27 \\ &= 2n^4 + 16n^3 + 47n^2 + 60n + 28 \\ &= (n+2)^2(2n^2 + 8n + 7) \\ &= (n+2)^2(2(n+1)^2 + 4(n+1) + 1)\,. \end{aligned}$$

4. Our proof is by induction, it being trivial for $n = 1$, since $1/3 = 1/3$. Under the inductive hypothesis

$$\begin{aligned} \frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{1}{2n+1}\left(n + \frac{1}{2n+3}\right) \\ &= \frac{1}{2n+1}\left(\frac{2n^2+3n+1}{2n+3}\right) \\ &= \frac{1}{2n+1}\left(\frac{(2n+1)(n+1)}{2n+3}\right) = \frac{n+1}{2n+3}\,, \end{aligned}$$

as desired.

6. We prove this statement by induction. The base case is $n = 5$, and indeed $5^2 + 5 = 30 < 32 = 2^5$. Assuming the inductive hypothesis, we have $(n+1)^2 + (n+1) = n^2 + 3n + 2 < n^2 + 4n < n^2 + n^2 = 2n^2 < 2(n^2 + n)$, which is less than $2 \cdot 2^n$ by the inductive hypothesis, and this equals 2^{n+1}, as desired.

8. We can let $N = 16$. We prove that $n^4 < 2^n$ for all $n > N$. The base case is $n = 17$, when $17^4 = 83521 < 131072 = 2^{17}$. Assuming the inductive hypothesis, we have $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 < n^4 + 4n^3 + 6n^3 + 4n^3 + 2n^3 = n^4 + 16n^3 < n^4 + n^4 = 2n^4$, which is less than $2 \cdot 2^n$ by the inductive hypothesis, and this equals 2^{n+1}, as desired.

10. If $n = 0$ (base case), then the expression equals $0 + 1 + 8 = 9$, which is divisible by 9. Assume that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9. We must show that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is also divisible by 9. The difference of these two expressions is $(n+3)^3 - n^3 = 9n^2 + 27n + 27 = 9(n^2 + 3n + 3)$, a multiple of 9. Therefore since the first expression is divisible by 9, so is the second.

12. We want to prove that 64 divides $9^{n+1} + 56n + 55$ for every positive integer n. For $n = 1$ the expression equals $192 = 64 \cdot 3$. Assume the inductive hypothesis that $64 \mid 9^{n+1} + 56n + 55$ and consider $9^{n+2} + 56(n+1) + 55$. We have $9^{n+2} + 56(n+1) + 55 = 9(9^{n+1} + 56n + 55) - 8 \cdot 56n + 56 - 8 \cdot 55 = 9(9^{n+1} + 56n + 55) - 64 \cdot 7n - 6 \cdot 64$. The first term is divisible by 64 by the inductive hypothesis, and the second and third terms are patently divisible by 64, so our proof by mathematical induction is complete.

14. The two parts are nearly identical, so we do only part **(a)**. Part **(b)** is proved in the same way, substituting multiplication for addition throughout. The basis step is the tautology that if $a_1 \equiv b_1 \pmod{m}$, then $a_1 \equiv b_1 \pmod{m}$. Assume the inductive hypothesis. This tells us that $\sum_{j=1}^{n} a_j \equiv \sum_{j=1}^{n} b_j \pmod{m}$. Combining this fact with the fact that $a_{n+1} \equiv b_{n+1} \pmod{m}$, we obtain the desired congruence, $\sum_{j=1}^{n+1} a_j \equiv \sum_{j=1}^{n+1} b_j \pmod{m}$ from Theorem 5 in Section 4.1.

16. After some computation we conjecture that $n + 6 < (n^2 - 8n)/16$ for all $n \geq 28$. (We find that it is not true for smaller values of n.) For the basis step we have $28 + 6 = 34$ and $(28^2 - 8 \cdot 28)/16 = 35$, so the statement is true. Assume that the statement is true for $n = k$. Then since $k > 27$ we have

$$\begin{aligned} \frac{(k+1)^2 - 8(k+1)}{16} = \frac{k^2 - 8k}{16} + \frac{2k - 7}{16} &> k + 6 + \frac{2k-7}{16} \quad \text{by the inductive hypothesis} \\ &> k + 6 + \frac{2 \cdot 27 - 7}{16} > k + 6 + 2.9 > (k+1) + 6, \end{aligned}$$

as desired.

18. When $n = 1$, we are looking for the derivative of $g(x) = e^{cx}$, which is ce^{cx} by the chain rule, so the statement is true for $n = 1$. Assume that the statement is true for $n = k$, that is, the kth derivative is given by $g^{(k)} = c^k e^{cx}$. Differentiating by the chain rule again (and remembering that c^k is constant) gives us the $(k+1)$st derivative: $g^{(k+1)} = c \cdot c^k e^{cx} = c^{k+1} e^{cx}$, as desired.

20. We look at the first few Fibonacci numbers to see if there is a pattern (all congruences are modulo 3): $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3 \equiv 0$, $f_5 = 5 \equiv 2$, $f_6 = 8 \equiv 2$, $f_7 = 13 \equiv 1$, $f_8 = 21 \equiv 0$, $f_9 = 34 \equiv 1$. We may not see a pattern yet, but note that f_8 and f_9 are the same, modulo 3, as f_0 and f_1. Therefore the sequence must continue to repeat from this point, since the recursive definition gives f_n just in terms of n_{n-1} and f_{n-2}. In particular, $f_{10} \equiv f_2 = 1$, $f_{11} \equiv f_3 = 2$, and so on. Since the pattern has period 8, we can formulate our conjecture as follows:

$$\begin{gathered} f_n \equiv 0 \pmod{3} \text{ if } n \equiv 0 \text{ or } 4 \pmod{8} \\ f_n \equiv 1 \pmod{3} \text{ if } n \equiv 1,\ 2, \text{ or } 7 \pmod{8} \\ f_n \equiv 2 \pmod{3} \text{ if } n \equiv 3,\ 5, \text{ or } 6 \pmod{8} \end{gathered}$$

To prove this by mathematical induction is tedious. There are two base cases, $n = 0$ and $n = 1$. The conjecture is certainly true in each of them, since $0 \equiv 0 \pmod{8}$ and $f_0 \equiv 0 \pmod{3}$, and $1 \equiv 1 \pmod{8}$ and $f_0 \equiv 1 \pmod{3}$. So we assume the inductive hypothesis and consider a given $n + 1$. There are eight cases to consider, depending on the value of $(n+1)$ **mod** 8. We will carry out one of them; the other seven cases are similar. If $n + 1 \equiv 5 \pmod{8}$, for example, then $n - 1$ and n are congruent to 3 and 4 modulo 8, respectively. By the inductive hypothesis, $f_{n-1} \equiv 2 \pmod{3}$ and $f_n \equiv 0 \pmod{3}$. Therefore f_{n+1}, which is the sum of these two numbers, is equivalent to $2 + 0$, or 2, modulo 3, as desired.

22. There are two base cases: for $n = 0$ we have $f_0 + f_2 = 0 + 1 = 1 = l_1$, and $f_1 + f_3 = 1 + 2 = 3 = l_2$, as desired. Assume the inductive hypothesis, that $f_k + f_{k+2} = l_{k+1}$ for all $k \leq n$ (we are using strong induction here). Then $f_{n+1} + f_{n+3} = f_n + f_{n-1} + f_{n+2} + f_{n+1} = (f_n + f_{n+2}) + (f_{n-1} + f_{n+1}) = l_{n+1} + l_n$ by the inductive hypothesis (with $k = n$ and $k = n - 1$). This last expression equals $l_{n+2} = l_{(n+1)+1}$, however, by the definition of the Lucas numbers, as desired.

24. We follow the hint. Starting with the trivial identity

$$\frac{m+n-1}{n} = \frac{m-1}{n} + 1$$

and multiplying both sides by

$$\frac{m(m+1)\cdots(m+n-2)}{(n-1)!}$$

we obtain the identity given in the hint:

$$\frac{m(m+1)\cdots(m+n-1)}{n!} = \frac{(m-1)m(m+1)\cdots(m+n-2)}{n!} + \frac{m(m+1)\cdots(m+n-2)}{(n-1)!}$$

Now we want to show that the product of any n consecutive positive integers is divisible by $n!$. We prove this by induction on n. The case $n = 1$ is clear, since every integer is divisible by $1!$. Assume the inductive hypothesis, that the statement is true for $n-1$. To prove the statement for n, now, we will give a proof using induction on the starting point of the sequence of n consecutive positive integers. Call this starting point m. The basis step, $m = 1$, is again clear, since the product of the first n positive integers *is* $n!$. Assume the inductive hypothesis that the statement is true for $m-1$. Note that we have two inductive hypotheses active here: the statement is true for $n-1$, and the statement is true also for $m-1$ and n. We are trying to prove the statement true for m and n. At this point we simply stare at the identity given above. The first term on the right-hand side is an integer by the inductive hypothesis about $m-1$ and n. The second term on the right-hand side is an integer by the inductive hypothesis about $n-1$. Therefore the expression is an integer. But the statement that the left-hand side is an integer is precisely what we wanted—that the product of the n positive integers starting with m is divisible by $n!$.

26. The algebra gets very messy here, but the ideas are not advanced. We will use the following standard trigonometric identity, which is proved using the standard formulae for the sine and cosine of sums and differences:

$$\cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

The proof of the identity in this exercise is by induction, of course. The basis step ($n = 1$) is the true statement that

$$\cos x = \frac{\cos x \sin(x/2)}{\sin(x/2)}.$$

Assume the inductive hypothesis:

$$\sum_{j=1}^{n} \cos jx = \frac{\cos((n+1)x/2)\sin(nx/2)}{\sin(x/2)}$$

Now it is clear that the inductive step is equivalent to showing that adding the $(n+1)^{\text{th}}$ term in the sum to the expression on the right-hand side of the last displayed equation yields the same expression with $n+1$ substituted for n. In other words, we must show that

$$\cos(n+1)x + \frac{\cos((n+1)x/2)\sin(nx/2)}{\sin(x/2)} = \frac{\cos((n+2)x/2)\sin((n+1)x/2)}{\sin(x/2)},$$

which can be rewritten without fractions as

$$\sin(x/2)\cos(n+1)x + \cos((n+1)x/2)\sin(nx/2) = \cos((n+2)x/2)\sin((n+1)x/2).$$

But this follows after a little calculation using the trigonometric identity displayed at the beginning of this solution, since both sides equal

$$\frac{\sin((2n+3)x/2) - \sin(x/2)}{2}.$$

28. We compute a few terms to get a feel for what is going on: $x_1 = \sqrt{6} \approx 2.45$, $x_2 = \sqrt{\sqrt{6}+6} \approx 2.91$, $x_3 \approx 2.98$, and so on. The values seem to be approaching 3 from below in an increasing manner.

a) Clearly $x_0 < x_1$. Assume that $x_{k-1} < x_k$. Then $x_k = \sqrt{x_{k-1}+6} < \sqrt{x_k+6} = x_{k+1}$, and the inductive step is proved.

b) Since $\sqrt{6} < \sqrt{9} = 3$, the basis step is proved. Assume that $x_k < 3$. Then $x_{k+1} = \sqrt{x_k+6} < \sqrt{3+6} = 3$, and the inductive step is proved.

c) By a result from mathematical analysis, an increasing bounded sequence converges to a limit. If we call this limit L, then we must have $L = \sqrt{L+6}$, by letting $n \to \infty$ in the defining equation. Solving this equation for L yields $L = 3$. (The root $L = -2$ is extraneous, since L is positive.)

30. We first prove that such an expression exists. The basis step will handle all $n < b$. These cases are clear, because we can take $k = 0$ and $a_0 = n$. Assume the inductive hypothesis, that we can express all nonnegative integers less than n in this way, and consider an arbitrary $n \geq b$. By the division algorithm (Theorem 2 in Section 4.1), we can write n as $q \cdot b + r$, where $0 \leq r < b$. By the inductive hypothesis, we can write q as $a_k b^k + a_{k-1}b^{k-1} + \cdots + a_1 b + a_0$. This means that $n = (a_k b^k + a_{k-1}b^{k-1} + \cdots + a_1 b + a_0) \cdot b + r = a_k b^{k+1} + a_{k-1}b^k + \cdots + a_1 b^2 + a_0 b + r$, and this is in the desired form.

For uniqueness, suppose that $a_k b^k + a_{k-1}b^{k-1} + \cdots + a_1 b + a_0 = c_k b^k + c_{k-1}b^{k-1} + \cdots + c_1 b + c_0$, where we have added initial terms with zero coefficients if necessary so that each side has the same number of terms; thus we have $0 \leq a_i < b$ and $0 \leq c_i < b$ for all i. Subtracting the second expansion from both sides gives us $(a_k - c_k)b^k + (a_{k-1} - c_{k-1})b^{k-1} + \cdots + (a_1 - c_1)b + (a_0 - c_0) = 0$. If the two expressions are different, then there is a smallest integer j such that $a_j \neq c_j$; that means that $a_i = c_i$ for $i = 0, 1, \ldots, j-1$. Hence

$$b^j\big((a_k - c_k)b^{k-j} + (a_{k-1} - c_{k-1})b^{k-j-1} + \cdots + (a_{j+1} - c_{j+1})b + (a_j - c_j)\big) = 0,$$

so

$$(a_k - c_k)b^{k-j} + (a_{k-1} - c_{k-1})b^{k-j-1} + \cdots + (a_{j+1} - c_{j+1})b + (a_j - c_j) = 0.$$

Solving for $a_j - c_j$ we have

$$\begin{aligned} a_j - c_j &= (c_k - a_k)b^{k-j} + (c_{k-1} - a_{k-1})b^{k-j-1} + \cdots + (c_{j+1} - a_{j+1})b \\ &= b\big((c_k - a_k)b^{k-j-1} + (c_{k-1} - a_{k-1})b^{k-j-2} + \cdots + (c_{j+1} - a_{j+1})\big). \end{aligned}$$

But this means that b divides $a_j - c_j$. Because both a_j and c_j are between 0 and $b-1$, inclusive, this is possible only if $a_j = b_j$, a contradiction. Thus the expression is unique.

32. For simplicity we will suppress the arguments ("(x)")and just write f' for the derivative of f. We also assume, of course, that denominators are not zero. If $n = 1$ there is nothing to prove, and the $n = 2$ case is just an application of the product rule:

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}.$$

Assume the inductive hypothesis and consider the situation for $n+1$:

$$\begin{aligned} \frac{(f_1 f_2 \cdots f_n f_{n+1})'}{f_1 f_2 \cdots f_n f_{n+1}} &= \frac{(f_1 f_2 \cdots f_n)' f_{n+1} + (f_1 f_2 \cdots f_n) f_{n+1}'}{(f_1 f_2 \cdots f_n) f_{n+1}} \\ &= \frac{(f_1 f_2 \cdots f_n)'}{(f_1 f_2 \cdots f_n)} + \frac{f_{n+1}'}{f_{n+1}} \\ &= \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_n'}{f_n} + \frac{f_{n+1}'}{f_{n+1}}. \end{aligned}$$

The first line followed from the product rule, the second line was algebra, and the third line followed from the inductive hypothesis.

34. Call a coloring proper if no two regions that have an edge in common have a common color. For the basis step we can produce a proper coloring if there is only one line by coloring the half of the plane on one side of the line red and the other half blue. Assume that a proper coloring is possible with k lines. If we have $k+1$ lines, remove one of the lines, properly color the configuration produced by the remaining lines, and then put the last line back. Reverse all the colors on one side of the last line. The resulting coloring will be proper.

36. It will be convenient to clear fractions by multiplying both sides by the product of all the x_s's; this makes the desired inequality

$$(x_1^2+1)(x_2^2+1)\cdots(x_n^2+1) \ge (x_1x_2+1)(x_2x_3+1)\cdots(x_{n-1}x_n+1)(x_nx_1+1)\,.$$

The basis step is

$$(x_1^2+1)(x_2^2+1) \ge (x_1x_2+1)(x_2x_1+1)\,.$$

which after algebraic simplification and factoring becomes $(x_1-x_2)^2 \ge 0$ and therefore is correct. For the inductive step, we assume that the inequality is true for n and hope to prove

$$(x_1^2+1)(x_2^2+1)\cdots(x_n^2+1)(x_{n+1}^2+1) \ge (x_1x_2+1)(x_2x_3+1)\cdots(x_{n-1}x_n+1)(x_nx_{n+1}+1)(x_{n+1}x_1+1)\,.$$

Because of the cyclic form of this inequality, we can without loss of generality assume that x_{n+1} is the largest (or tied for the largest) of all the given numbers. By the inductive hypothesis we have

$$(x_1^2+1)(x_2^2+1)\cdots(x_n^2+1)(x_{n+1}^2+1) \ge (x_1x_2+1)(x_2x_3+1)\cdots(x_{n-1}x_n+1)(x_nx_1+1)(x_{n+1}^2+1)\,,$$

so it suffices to show that

$$(x_nx_1+1)(x_{n+1}^2+1) \ge (x_nx_{n+1}+1)(x_{n+1}x_1+1)\,.$$

But after algebraic simplification and factoring, this becomes $(x_{n+1}-x_1)(x_{n+1}-x_n) \ge 0$, which is true by our assumption that x_{n+1} is the largest number in the list.

38. (It will be helpful for the reader to draw a diagram to help in following this proof.) We use induction on n, the number of cities, the result being trivial if $n=1$ or $n=2$. Assume the inductive hypothesis and suppose that we have a country with $k+1$ cities, labeled c_1 through c_{k+1}. Remove c_{k+1} and apply the inductive hypothesis to find a city c that can be reached either directly or with one intermediate stop from each of the other cities among c_1 through c_k. If the one-way road leads from c_{k+1} to c, then we are done, so we can assume that the road leads from c to c_{k+1}. If there are any one-way roads from c_{k+1} to a city with a one-way road to c, then we are also done, so we can assume that each road between c_{k+1} and a city with a one-way road to c leads from such a city to c_{k+1}. Thus c and all the cities with a one-way road to c have a direct road to c_{k+1}. All the remaining cities must have a one-way road from them to a city with a one-way road to c (that was part of the definition of c), and so they have paths of length 2 to c_{k+1}, via some such city. Therefore c_{k+1} satisfies the conditions of the problem, and the proof is complete.

40. We have to assume from the statement of the problem that all the cars get are equally efficient in terms of miles per gallon. We proceed by induction on n, the number of cars in the group. If $n=1$, then the one car has enough fuel to complete the lap. Assume the inductive hypothesis that the statement is true for a group of k cars, and suppose we have a group of $k+1$ cars. It helps to think of the cars as stationary, not moving yet. We claim that at least one car c in the group has enough fuel to reach the next car in the group. If this were not so, then the total amount of fuel in all the cars combined would not cover the full lap (think of each car as traveling as far as it can on its own fuel). So now pretend that the car d just ahead of car c is not present, and instead the fuel in that car is in c's tank. By the inductive hypothesis (we still have the same total amount of fuel), some car in this situation can complete a lap by obtaining fuel from other cars as it travels around the track. Then this same car can complete the lap in the actual situation, because if and when it needs to move from the location of car c to the location of the car d, the amount of fuel it has available without d's fuel that we are pretending c already has will be sufficient for it to reach d, at which time this extra fuel becomes available (because this car made it to c's location and car c has enough fuel to reach d's location).

42. The basis step is $n = 3$. Because the hypotenuse is the longest side of a right triangle, $c > a$ and $c > b$. Therefore

$$c^3 = c \cdot c^2 = c(a^2 + b^2) = c \cdot a^2 + c \cdot b^2 > a \cdot a^2 + b \cdot b^2 = a^3 + b^3 .$$

For the inductive step,

$$c^{k+1} = c \cdot c^k > c(a^k + b^k) = c \cdot a^k + c \cdot b^k > a \cdot a^k + b \cdot b^k = a^{k+1} + b^{k+1} .$$

One can also give a noninductive proof much along the same lines:

$$c^n = c^2 \cdot c^{n-2} = (a^2 + b^2) \cdot c^{n-2} = a^2 \cdot c^{n-2} + b^2 \cdot c^{n-2} > a^2 \cdot a^{n-2} + b^2 \cdot b^{n-2} = a^n + b^n$$

44. **a)** The basis step is to prove the statement that this algorithm terminates for all fractions of the form $1/q$. Since this fraction is already a unit fraction, there is nothing more to prove.
b) For the inductive step, assume that the algorithm terminates for all proper positive fractions with numerators smaller than p, suppose that we are starting with the proper positive fraction p/q, and suppose that the algorithm selects $1/n$ as the first step in the algorithm. Note that necessarily $n > 1$. Therefore we can write $p/q = p'/q' + 1/n$. If $p/q = 1/n$, we are done, so assume that $p/q > 1/n$. By finding a common denominator and subtracting, we see that we can take $p' = np - q$ and $q' = nq$. We claim that $p' < p$, which algebraically is easily seen to be equivalent to $p/q < 1/(n-1)$, and this is true by the choice of n such that $1/n$ is the largest unit fraction not exceeding p/q. Therefore by the inductive hypothesis we can write p'/q' as the sum of distinct unit fractions with increasing denominators, and thereby have written p/q as the sum of unit fractions. The only thing left to check is that $p'/q' < 1/n$, so that the algorithm will not try to choose $1/n$ again for p'/q'. But if this were not the case, then $p/q \geq 2/n$, and combining this with the inequality $p/q < 1/(n-1)$ given above, we would have $2/n < 1/(n-1)$, which would mean that $n = 1$, a contradiction.

46. What we really need to show is that the definition "terminates" for every n. It is conceivable that trying to apply the definition gets us into some kind of infinite loop, using the second line; we need to show that this is not the case. We will give a very strange kind of proof by mathematical induction. First, following the hint, we will show that the definition tells us that $M(n) = 91$ for all positive integers $n \leq 101$. We do this by backwards induction, starting with $n = 101$ and going down toward $n = 1$. There are 11 base cases: $n = 101$, 100, 99, $\ldots$, 91. The first line of the definition tells us immediately that $M(101) = 101 - 10 = 91$. To compute $M(100)$ we have

$$\begin{aligned} M(100) &= M(M(100 + 11)) = M(M(111)) \\ &= M(111 - 10) = M(101) = 91 . \end{aligned}$$

The last equality came from the fact that we had already computed $M(101)$. Similarly,

$$\begin{aligned} M(99) &= M(M(99 + 11)) = M(M(110)) \\ &= M(110 - 10) = M(100) = 91 , \end{aligned}$$

and so on down to

$$\begin{aligned} M(91) &= M(M(91 + 11)) = M(M(102)) \\ &= M(102 - 10) = M(92) = 91 . \end{aligned}$$

In each case the final equality comes from the previously computed value. Now assume the inductive hypothesis, that $M(k) = 91$ for all k from $n+1$ through 101 (i.e., if $n+1 \le k \le 101$); we must prove that $M(n) = 91$, where n is some fixed positive integer less than 91. To compute $M(n)$, we have

$$M(n) = M(M(n+11)) = M(91) = 91$$

where the next to last equality comes from the fact that $n+11$ is between $n+1$ and 101. Thus we have proved that $M(n) = 91$ for all $n \le 101$. The first line of the definition takes care of values of n greater than 101, so the entire function is well-defined.

48. We proceed by induction on n. The case $n = 2$ is just the definition of symmetric difference. Assume that the statement is true for $n-1$; we must show that it is true for n. By definition $R_n = R_{n-1} \oplus A_n$. We must show that an element x is in R_n if and only if it belongs to an odd number of the sets A_1, A_2, $\ldots$, A_n. The inductive hypothesis tells us that x is in R_{n-1} if and only if x belongs to an odd number of the sets A_1, A_2, $\ldots$, A_{n-1}. There are four cases. Suppose first that $x \in R_{n-1}$ and $x \in A_n$. Then x belongs to an odd number of the sets A_1, A_2, $\ldots$, A_{n-1} and therefore belongs to an even number of the sets A_1, A_2, $\ldots$, A_n; thus $x \notin R_n$, which is correct by the definition of $\oplus$. Next suppose that $x \in R_{n-1}$ and $x \notin A_n$. Then x belongs to an odd number of the sets A_1, A_2, $\ldots$, A_{n-1} and therefore belongs to an odd number of the sets A_1, A_2, $\ldots$, A_n; thus $x \in R_n$, which is again correct by the definition of $\oplus$. For the third case, suppose that $x \notin R_{n-1}$ and $x \in A_n$. Then x belongs to an even number of the sets A_1, A_2, $\ldots$, A_{n-1} and therefore belongs to an odd number of the sets A_1, A_2, $\ldots$, A_n; thus $x \in R_n$, which is again correct by the definition of $\oplus$. The last case ($x \notin R_{n-1}$ and $x \notin A_n$) is similar.

50. This problem is similar to and uses the result of Exercise 62 in Section 5.1. The lemma we need is that if there are n planes meeting the stated conditions, then adding one more plane, which intersects the original figure in the manner described, results in the addition of $(n^2+n+2)/2$ new regions. The reason for this is that the pattern formed on the new plane by all the lines of intersection of this plane with the planes already present has, by Exercise 62 in Section 5.1, $(n^2+n+2)/2$ regions; and each of these two-dimensional regions separates the three-dimensional region through which it passes into two three-dimensional regions. Therefore the proof by induction of the present exercise reduces to noting that one plane separates space into $(1^3+5\cdot 1+6)/6 = 2$ regions, and verifying the algebraic identity

$$\frac{n^3+5n+6}{6} + \frac{n^2+n+2}{2} = \frac{(n+1)^3+5(n+1)+6}{6}\,.$$

52. **a)** This set is not well ordered, since the set itself has no least element (the negative integers get smaller and smaller).

b) This set is well ordered—the problem inherent in part **(a)** is not present here because the entire set has -99 as its least element. Every subset also has a least element.

c) This set is not well ordered. The entire set, for example, has no least element, since the numbers of the form $1/n$ for n a positive integer get smaller and smaller.

d) This set is well ordered. The situation is analogous to part **(b)**.

54. In the preamble to Exercise 42 in Section 4.3, an algorithm was described for writing the greatest common divisor of two positive integers as a linear combination of these two integer (see also Theorem 6 and Example 17 in that section). We can use that algorithm, together with the result of Exercise 53, to solve this problem. For $n = 1$ there is nothing to do, since $a_1 = a_1$, and we already have an algorithm for $n = 2$. For $n > 2$, we can write $\gcd(a_{n-1}, a_n)$ as a linear combination of a_{n-1} and a_n, say as

$$\gcd(a_{n-1}, a_n) = c_{n-1}a_{n-1} + c_n a_n\,.$$

Then we apply the algorithm recursively to the numbers a_1, a_2, ..., a_{n-2}, $\gcd(a_{n-1}, a_n)$. This gives us the following equation:

$$\gcd(a_1, a_2, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n)) = c_1 a_1 + c_2 a_2 + \cdots + c_{n-2} a_{n-2} + Q \cdot \gcd(a_{n-1}, a_n)$$

Plugging in from the previous display, we have the desired linear combination:

$$\begin{aligned}\gcd(a_1, a_2, \ldots, a_n) &= \gcd(a_1, a_2, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n)) \\ &= c_1 a_1 + c_2 a_2 + \cdots + c_{n-2} a_{n-2} + Q(c_{n-1} a_{n-1} + c_n a_n) \\ &= c_1 a_1 + c_2 a_2 + \cdots + c_{n-2} a_{n-2} + Q c_{n-1} a_{n-1} + Q c_n a_n\end{aligned}$$

56. The following definition works. The empty string is in the set, and if x and y are in the set, then so are xy, $1x00$, $00x1$, and $0x1y0$. One way to see this is to think of graphing, for a string in this set, the quantity (number of 0's) $- 2 \cdot$ (number of 1's) as a function of the position in the string. This graph must start and end at the horizontal axis. If it contains another point on the axis, then we can split the string into xy where x and y are both in the set. If the graph stays above the axis, then the string must be of the form $00x1$, and if it stays below the axis, then it must be of the form $1x00$. The only other case is that in which the graph crosses the axis at a 1 in the string, without landing on the axis. In this case, the string must look like $0x1y0$.

58. **a)** The set contains three strings of length 3, and each of them gives us four more strings of length 6, using the fourth through seventh rules, except that there is a bit of overlap, so that in fact there are only 13 strings in all. The strings are *abc*, *bac*, *acb*, *abcabc*, *ababcc*, *aabcbc*, *abcbac*, *abbacc*, *abacbc*, *bacabc*, *abcacb*, *aacbbc*, and *acbabc*.

b) We prove this by induction on the length of the string. The basis step is vacuously true, since there are no strings in the set of length 0 (and it is trivially true anyway, since 0 is a multiple of 3). Assume the inductive hypothesis that the statement is true for shorter strings, and let y be a string in S. If $y \in S$ by one of the first three rules, then y has length 3. If $y \in S$ by one of the last four rules, then the length of y is equal to 3 plus the length of x. By the inductive hypothesis, the length of x is a multiple of 3, so the length of y is also a multiple of 3.

60. By applying the recursive rules we get the following list: $((()))$, $(()())$, $()()()$, $()(())$, $(())()$.

62. We use induction on the length of the string x of balanced parentheses. If $x = \lambda$, then the statement is true since $0 = 0$. Otherwise $x = (a)$ or $x = ab$, where a and b are shorter balanced strings of parentheses. In the first case, the number of parentheses of each type in x is one more than the corresponding number in a, so by the inductive hypothesis these numbers are equal. In the second case, the number of parentheses of each type in x is the sum of the corresponding numbers in a and b, so again by the inductive hypothesis these numbers are equal.

64. We prove the "only if" part by induction on the length of the balanced string w. If $w = \lambda$, then there is nothing to prove. If $w = (x)$, then we have by the inductive hypothesis that $N(x) = 0$ and that $N(a) \geq 0$ if a is a prefix of x. Then $N(w) = 1 + 0 + (-1) = 0$; and $N(b) \geq 1 \geq 0$ if b is a nonempty prefix of w, since $b = (a$. If $w = xy$, then we have by the inductive hypothesis that $N(x) = N(y) = 0$; and $N(a) \geq 0$ if a is a prefix of x or y. Then $N(w) = 0 + 0 = 0$; and $N(b) \geq 0$ if b is a prefix of w, since either b is a prefix of x or $b = xa$ where a is a prefix of y.

We also prove the "if" part by induction on the length of the string w. Suppose that w satisfies the condition. If $w = \lambda$, then $w \in B$. Otherwise w must begin with a parenthesis, and it must be a left parenthesis, since otherwise the prefix of length 1 would give us $N\big()\big) = -1$. Now there are two cases: either $w = ab$, where $N(a) = N(b) = 0$ and $a \neq \lambda \neq b$, or not. If so, then a and b are balanced strings of parentheses by the inductive hypothesis (noting that prefixes of a are prefixes of w, and prefixes of b are a followed by prefixes of w), so w is balanced by the recursive definition of the set of balanced strings. In the other case, $N(u) \geq 1$ for all nonempty prefixes u of w, other than w itself. Thus w must end with a right parenthesis to make $N(w) = 0$. So $w = (x)$, and $N(x) = 0$. Furthermore $N(u) \geq 0$ for every prefix u of x, since if $N(u)$ dipped to -1, then $N\big((u\big) = 0$ and we would be in the first case. Therefore by the inductive hypothesis x is balanced, and so by the definition of balanced strings w is balanced, as desired.

66. We copy the definition into an algorithm.

> **procedure** $gcd(a, b:$ nonnegative integers, not both zero$)$
> **if** $a > b$ **then return** $gcd(b, a)$
> **else if** $a = 0$ **then return** b
> **else if** a and b are even **then return** $2 \cdot gcd(a/2, b/2)$
> **else if** a is even and b is odd **then return** $gcd(a/2, b)$
> **else return** $gcd(a, b - a)$

68. To prove that a recursive program is correct, we need to check that it works correctly for the base case, and that it works correctly for the inductive step under the inductive assumption that it works correctly on its recursive call. To apply this rule of inference to Algorithm 1 in Section 5.4, we reason as follows. The base case is $n = 1$. In that case the **then** clause is executed, and not the **else** clause, and so the procedure gives the correct value, namely 1. Now assume that the procedure works correctly for $n-1$, and we want to show that it gives the correct value for the input n, where $n > 1$. In this case, the **else** clause is executed, and not the **then** clause, so the procedure gives us n times whatever the procedure gives for input $n-1$. By the inductive hypothesis, we know that this latter value is $(n-1)!$. Therefore the procedure gives $n \cdot (n-1)!$, which by definition is equal to $n!$, exactly as we wished.

70. We apply the definition:

$$\begin{aligned}
a(0) &= 0 \\
a(1) &= 1 - a(a(0)) = 1 - a(0) = 1 - 0 = 1 \\
a(2) &= 2 - a(a(1)) = 2 - a(1) = 2 - 1 = 1 \\
a(3) &= 3 - a(a(2)) = 3 - a(1) = 3 - 1 = 2 \\
a(4) &= 4 - a(a(3)) = 4 - a(2) = 4 - 1 = 3 \\
a(5) &= 5 - a(a(4)) = 5 - a(3) = 5 - 2 = 3 \\
a(6) &= 6 - a(a(5)) = 6 - a(3) = 6 - 2 = 4 \\
a(7) &= 7 - a(a(6)) = 7 - a(4) = 7 - 3 = 4 \\
a(8) &= 8 - a(a(7)) = 8 - a(4) = 8 - 3 = 5 \\
a(9) &= 9 - a(a(8)) = 9 - a(5) = 9 - 3 = 6
\end{aligned}$$

72. We follow the hint. First note that by algebra, $\mu^2 = 1 - \mu$, and that $\mu \approx 0.618$. Therefore we have $(\mu n - \lfloor \mu n \rfloor) + (\mu^2 n - \lfloor \mu^2 n \rfloor) = \mu n - \lfloor \mu n \rfloor + (1-\mu)n - \lfloor (1-\mu)n \rfloor = \mu n - \lfloor \mu n \rfloor + n - \mu n - \lfloor n - \mu n \rfloor = \mu n - \lfloor \mu n \rfloor + n - \mu n - n - \lfloor -\mu n \rfloor = -\lfloor \mu n \rfloor - (-\lceil \mu n \rceil) = -\lfloor \mu n \rfloor + \lceil \mu n \rceil = 1$, since μn is irrational and therefore not an integer. (We used here some of the properties of the floor and ceiling function from Table 1 in Section 2.3.) Next, continuing with the hint, suppose that $0 \leq \alpha < 1 - \mu$, and consider $\lfloor (1+\mu)(1-\alpha) \rfloor + \lfloor \alpha + \mu \rfloor$. The second floor term is 0, since $\alpha < 1 - \mu$. The product $(1+\mu)(1-\alpha)$ is greater than $(1+\mu)\mu = \mu + \mu^2 = 1$ and less than $(1 + 1 - \alpha)(1-\alpha) < 2 \cdot 1 = 2$, so the whole sum equals 1, as desired. For the other case, suppose that $1 - \mu < \alpha < 1$, and again consider $\lfloor (1+\mu)(1-\alpha) \rfloor + \lfloor \alpha + \mu \rfloor$. Here $\alpha + \mu$ is between 1 and 2, and $(1+\mu)(1-\alpha) < 1$, so again the sum is 1.

The rest of the proof is pretty messy algebra. Since we already know from Exercise 71 that the function $a(n)$ is well-defined by the recurrence $a(n) = n - a(a(n-1))$ for all $n \geq 1$ and initial condition $a(0) = 0$, it suffices to prove that $\lfloor (n+1)\mu \rfloor$ satisfies these equations. It clearly satisfies the second, since $0 < \mu < 1$. Thus we must show that $\lfloor (n+1)\mu \rfloor = n - \lfloor (\lfloor n\mu \rfloor + 1)\mu \rfloor$ for all $n \geq 1$. Let $\alpha = n\mu - \lfloor n\mu \rfloor$; then $0 \leq \alpha < 1$, and $\alpha \neq 1 - \mu$, since μ is irrational. First consider $\lfloor (\lfloor n\mu \rfloor + 1)\mu \rfloor$. It equals $\lfloor \mu(1 + \mu n - \alpha) \rfloor = \lfloor \mu + \mu^2 n - \alpha\mu \rfloor = \lfloor \mu + 1 - \alpha + \lfloor \mu^2 n \rfloor - \alpha\mu \rfloor$ by the first fact proved above. Since $\lfloor \mu^2 n \rfloor$ is an integer, this equals $\lfloor \mu^2 n \rfloor + \lfloor \mu + 1 - \alpha - \alpha\mu \rfloor = \lfloor \mu^2 n \rfloor + \lfloor (1+\mu)(1-\alpha) \rfloor = \mu^2 n - 1 + \alpha + \lfloor (1+\mu)(1-\alpha) \rfloor$. Next consider $\lfloor (n+1)\mu \rfloor$. It equals $\lfloor \mu n + \mu \rfloor = \lfloor \lfloor \mu n \rfloor + \alpha + \mu \rfloor = \lfloor \mu n \rfloor + \lfloor \alpha + \mu \rfloor = \mu n - \alpha + \lfloor \alpha + \mu \rfloor$. Putting these together we have $\lfloor (\lfloor n\mu \rfloor + 1)\mu \rfloor + \lfloor (n+1)\mu \rfloor - n = \mu^2 n - 1 + \alpha + \lfloor (1+\mu)(1-\alpha) \rfloor + \mu n - \alpha + \lfloor \alpha + \mu \rfloor - n = (\mu^2 + \mu - 1)n - 1 + \lfloor (1+\mu)(1-\alpha) \rfloor + \lfloor \alpha + \mu \rfloor$, which equals $0 - 1 + 1 = 0$ by the definition of μ and the second fact proved above. This is equivalent to what we wanted.

74. a) We apply the definition:

$$
\begin{aligned}
&a(0) = 0\\
&a(1) = 1 - a(a(a(0))) = 1 - a(a(0)) = 1 - a(0) = 1 - 0 = 1\\
&a(2) = 2 - a(a(a(1))) = 2 - a(a(1)) = 2 - a(1) = 2 - 1 = 1\\
&a(3) = 3 - a(a(a(2))) = 3 - a(a(1)) = 3 - a(1) = 3 - 1 = 2\\
&a(4) = 4 - a(a(a(3))) = 4 - a(a(2)) = 4 - a(1) = 4 - 1 = 3\\
&a(5) = 5 - a(a(a(4))) = 5 - a(a(3)) = 5 - a(2) = 5 - 1 = 4\\
&a(6) = 6 - a(a(a(5))) = 6 - a(a(4)) = 6 - a(3) = 6 - 2 = 4\\
&a(7) = 7 - a(a(a(6))) = 7 - a(a(4)) = 7 - a(3) = 7 - 2 = 5\\
&a(8) = 8 - a(a(a(7))) = 8 - a(a(5)) = 8 - a(4) = 8 - 3 = 5\\
&a(9) = 9 - a(a(a(8))) = 9 - a(a(5)) = 9 - a(4) = 9 - 3 = 6
\end{aligned}
$$

b) We apply the definition:

$$
\begin{aligned}
&a(0) = 0\\
&a(1) = 1 - a(a(a(a(0)))) = 1 - a(a(a(0))) = 1 - a(a(0)) = 1 - a(0) = 1 - 0 = 1\\
&a(2) = 2 - a(a(a(a(1)))) = 2 - a(a(a(1))) = 2 - a(a(1)) = 2 - a(1) = 2 - 1 = 1\\
&a(3) = 3 - a(a(a(a(2)))) = 3 - a(a(a(1))) = 3 - a(a(1)) = 3 - a(1) = 3 - 1 = 2\\
&a(4) = 4 - a(a(a(a(3)))) = 4 - a(a(a(2))) = 4 - a(a(1)) = 4 - a(1) = 4 - 1 = 3\\
&a(5) = 5 - a(a(a(a(4)))) = 5 - a(a(a(3))) = 5 - a(a(2)) = 5 - a(1) = 5 - 1 = 4\\
&a(6) = 6 - a(a(a(a(5)))) = 6 - a(a(a(4))) = 6 - a(a(3)) = 6 - a(2) = 6 - 1 = 5\\
&a(7) = 7 - a(a(a(a(6)))) = 7 - a(a(a(5))) = 7 - a(a(4)) = 7 - a(3) = 7 - 2 = 5\\
&a(8) = 8 - a(a(a(a(7)))) = 8 - a(a(a(5))) = 8 - a(a(4)) = 8 - a(3) = 8 - 2 = 6\\
&a(9) = 9 - a(a(a(a(8)))) = 9 - a(a(a(6))) = 9 - a(a(5)) = 9 - a(4) = 9 - 3 = 6
\end{aligned}
$$

c) We apply the definition:

$$\begin{aligned}
a(1) &= 1\\
a(2) &= 1\\
a(3) &= a(3-a(2)) + a(3-a(1)) = a(3-1) + a(3-1) = a(2) + a(2) = 1+1 = 2\\
a(4) &= a(4-a(3)) + a(4-a(2)) = a(4-2) + a(4-1) = a(2) + a(3) = 1+2 = 3\\
a(5) &= a(5-a(4)) + a(5-a(3)) = a(5-3) + a(5-2) = a(2) + a(3) = 1+2 = 3\\
a(6) &= a(6-a(5)) + a(6-a(4)) = a(6-3) + a(6-3) = a(3) + a(3) = 2+2 = 4\\
a(7) &= a(7-a(6)) + a(7-a(5)) = a(7-4) + a(7-3) = a(3) + a(4) = 2+3 = 5\\
a(8) &= a(8-a(7)) + a(8-a(6)) = a(8-5) + a(8-4) = a(3) + a(4) = 2+3 = 5\\
a(9) &= a(9-a(8)) + a(9-a(7)) = a(9-5) + a(9-5) = a(4) + a(4) = 3+3 = 6\\
a(10) &= a(10-a(9)) + a(10-a(8)) = a(10-6) + a(10-5) = a(4) + a(5) = 3+3 = 6
\end{aligned}$$

76. The first term a_1 tells how many 1's there are. If $a_1 \geq 2$, then the sequence would not be nondecreasing, since a 1 would follow this 2. Therefore $a_1 = 1$. This tells us that there is one 1, so the next term must be at least 2. By the same reasoning as before, a_2 can't be 3 or larger, so $a_2 = 2$. This tells us that there are two 2's, and they must all come together since the sequence is nondecreasing. So $a_3 = 2$ as well. But now we know that there are two 3's, and of course they must come next. We continue in this way and obtain the first 20 terms:

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8$$