Inductors and Inductance

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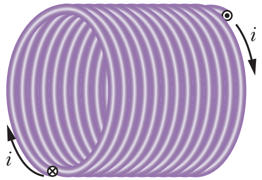
Lecture 12

Capacitor vs Inductor

- A **capacitor** can be used to produce a desired electric field. Our basic type of capacitor is the parallel-plate arrangement.
- Similarly, an **inductor** can be used to produce a desired magnetic field. Our basic type of inductor is a long solenoid (more specifically, a short length near the middle of a long solenoid, to avoid any fringing effects).



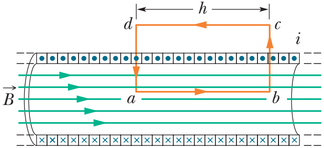
Outline

- Inductors and Inductance
- Self-Induction
- RL Circuits
- Energy Stored in a Magnetic Field
- Mutual Induction

Revisiting Solenoid

- What is the direction of B of a very long solenoid?
- According to Ampere's law,

$$Bh = \mu_0 inh,$$



where n is the number of turns per unit length of the solenoid.

- The magnetic flux through the cross-sectional area A is

$$\Phi_B = BA = \mu_0 inA,$$

which motivates us to pay attention to the property Φ_B / i.

Inductors

- The solenoid is an inductor, through whose central region a current i produces a magnetic flux Φ_B. The **inductance** of the inductor is then defined in terms of that flux as

$$L = N\Phi_B/i,$$

  where N is the number of turns.
- The inductance L is thus a measure of the **magnetic flux linkage** ($N\Phi_B$) produced by the inductor per unit of current.

- Because the SI unit of magnetic flux is the tesla–square meter, the SI unit of inductance is the tesla–square meter per ampere ($\mathrm{T{\cdot}m^2/A}$). We call this the henry (H), after American physicist Joseph Henry, the codiscoverer of the law of induction and a contemporary of Faraday. Thus,

$$1 \text{ henry} = 1 \text{ H} = 1 \text{ T} \cdot \text{m}^2/\text{A}.$$

- In this lecture, we assume that all inductors, no matter what their geometric arrangement, have no magnetic materials such as iron in their vicinity. Such materials would distort the magnetic field of an inductor.

Inductance of a Solenoid

- For the solenoid, we then have

$$L = \frac{N\Phi_B}{i} = \frac{(n\ell)(\mu_0 inA)}{i} = \mu_0 n^2 \ell A.$$

- Thus, the inductance per unit length near the center of a long solenoid is

$$\frac{L}{\ell} = \mu_0 n^2 A.$$

- Inductance, like capacitance, depends only on the geometry of the device. The dependence on the square of the number of turns per unit length is to be expected.

- If the length ℓ of a solenoid is very much longer than its radius, then, to a good approximation, its inductance is

$$L = \mu_0 n^2 \ell A = N^2 (\mu_0 A/\ell).$$

- This approximation neglects the spreading of the magnetic field lines near the ends of the solenoid, just as the parallel-plate capacitor formula

$$C = \epsilon_0 A/d$$

neglects the fringing of the electric field lines near the edges of the capacitor plates.

Self-Induction

- If two coils — which we can now call inductors — are near each other, a current i in one coil produces a magnetic flux Φ_B through the second coil.
- We have seen that if we change this flux by changing the current, an induced emf appears in the second coil according to Faraday's law.

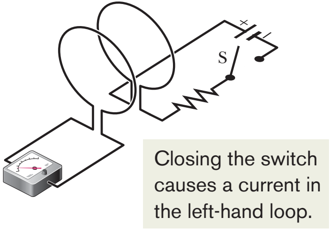


Figure 1: Two coils (or inductors) close to each other but not touching.

- In fact, an induced emf appears in the first coil as well.

- This process is called **self-induction**, and the emf that appears is called a **self-induced emf**. It obeys Faraday's law of induction just as other induced emfs do.
- According to the definition of inductance, $N\Phi_B = Li$.
- Faraday's law tells us that

$$\mathcal{E}_L = -\frac{d(N\Phi_B)}{dt} = -L\frac{di}{dt}.$$

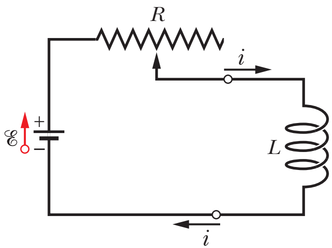
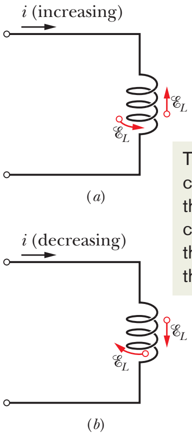


Figure 2: If the current in a coil is changed by varying the contact position on a variable resistor, a self-induced emf $\mathcal{E}_L$ will appear in the coil.

- Thus, in any inductor (such as a coil, a solenoid, or a toroid) a self-induced emf appears whenever the current changes with time.
- The direction of a self-induced emf obeys Lenz's law. The minus sign indicates that – as the law states – the self-induced emf $\mathcal{E}_L$ has the orientation such that it opposes the change in current.



The changing current changes the flux, which creates an emf that opposes the change.

- When a self-induced emf is produced in the inductor, we cannot define an electric potential within the inductor itself, where the flux is changing.
- However, potentials can still be defined at points of the circuit that are not within the inductor—points where the electric fields are due to charge distributions and their associated electric potentials.
- We can define a **self-induced potential difference** V_L across an inductor (between its terminals, which we assume to be outside the region of changing flux).
- For an **ideal inductor** (its wire has negligible resistance), the magnitude of V_L is equal to the magnitude of the self-induced emf $\mathcal{E}_L$.

RL Circuits

- If we introduce an emf $\mathcal{E}$ into a single-loop circuit containing a resistor R and an inductor L, a self-induced emf $\mathcal{E}_L$ appears in the circuit, which opposes the rise of the current, i.e., it opposes the battery emf $\mathcal{E}$ in polarity.
- A long time later, the inductor acts like ordinary connecting wire.

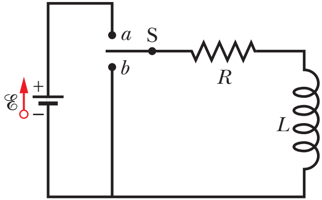


Figure 3: When switch S is closed on a, the current rises and approaches a limiting value $\mathcal{E}/R$.

- To analyze the situation quantitatively, we apply the loop rule

$$-iR - L\frac{di}{dt} + \mathcal{E} = 0,$$

or

$$L\frac{di}{dt} + Ri = \mathcal{E}.$$

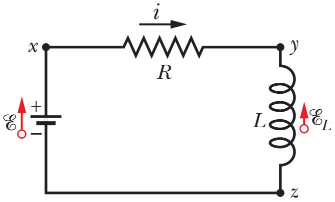


Figure 4: We apply the loop rule for the circuit clockwise, starting at x.

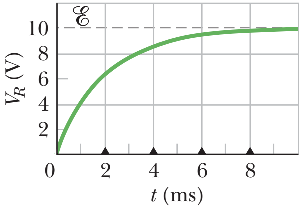
- Dimension analysis of the left side leads to a time scale

$$\tau_L = L/R,$$

which is known as the **inductive time constant**.

- The time constant is the time it takes the current in the circuit to reach about 63% ($= 1 - e^{-1}$) of its final equilibrium value $\mathcal{E}/R$.
- The solution of the current, as a function of time, is

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right).$$



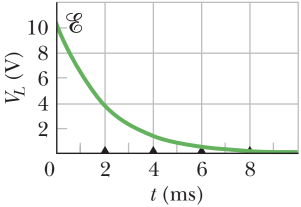


Figure 5: The potential difference across R and L.

Quiz 12-1: RL Circuit

Energy Stored in a Magnetic Field

- When we pull two charged particles of opposite signs away from each other, we say that the resulting *electric potential energy is stored in the electric field of the particles* and, quantitatively,

$$U = \frac{q^2}{2C} = \frac{1}{2}CV^2.$$

  We get it back from the field by letting the particles move closer together again.
- In the same way we say *energy is stored in a magnetic field*, but now we deal with current instead of electric charges.

- According to the loop rule, we find

$$\mathcal{E} = L\frac{di}{dt} + Ri.$$

- If we multiply each side by i, we obtain

$$\mathcal{E}i = Li\frac{di}{dt} + i^2 R.$$

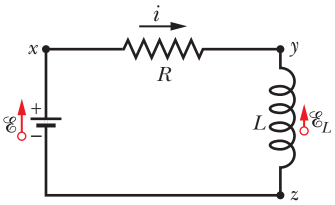


Figure 6: Consider a source of emf $\mathcal{E}$ connected to a resistor R and an inductor L.

- The equation can be interpreted in terms of the work done by the battery and the resulting energy transfers.

- $(\mathcal{E}\, dq)/dt = \mathcal{E}\, i$ represents the rate at which the emf device delivers energy, by doing work on charge passing through the battery, to the rest of the circuit.
- $i^2 R$ represents the rate at which energy appears as thermal energy in the resistor.
- Energy that is delivered to the circuit but does not appear as thermal energy must, by the conservation-of-energy hypothesis, be stored in the magnetic field of the inductor with the rate

$$\frac{dU_B}{dt} = Li\frac{di}{dt} = \frac{d}{dt}\left(\frac{1}{2}Li^2\right).$$

- Therefore, the total energy stored by an inductor L carrying a current i is

$$U_B = \frac{1}{2} L i^2.$$

- Note the similarity in form between this expression for the energy stored in a magnetic field and the expression for the energy stored in an electric field by a capacitor with capacitance C and charge q:

$$U_E = \frac{q^2}{2C}.$$

Energy Density of a Magnetic Field

- The energy U_B stored by a length h of an ideal solenoid must lie entirely within volume Ah because the magnetic field outside such a solenoid is approximately zero.
- Moreover, the stored energy must be uniformly distributed within the solenoid because of the translational symmetry.

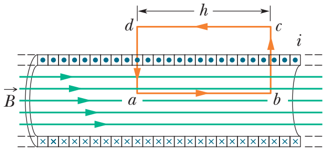


Figure 7: An ideal solenoid of cross-sectional area A carrying current i. Ampere's law tells us $B = \mu_0 in$.

- The energy stored per unit volume of the field is

$$u_B = \frac{U_B}{Ah} = \frac{Li^2}{2Ah} = \frac{L}{h}\frac{i^2}{2A}.$$

- Substituting for L/h by the inductance per unit length of the solenoid

$$L/h = \mu_0 n^2 A,$$

where n is the number of turns per unit length, we find the energy density to be (notice $B = \mu_0 i n$)

$$u_B = \frac{1}{2}\mu_0 n^2 i^2 = \frac{B^2}{2\mu_0}.$$

- Even though we derived the magnetic energy density by considering the special case of a solenoid, the expression holds for all magnetic fields, no matter how they are generated.
- The equation

$$u_B = \frac{B^2}{2\mu_0}$$

resembles that for the energy density in an electric field,

$$u_E = \frac{1}{2}\epsilon_0 E^2.$$

- Note that both u_B and u_E are proportional to the square of the appropriate field magnitude, B or E.

Mutual Induction

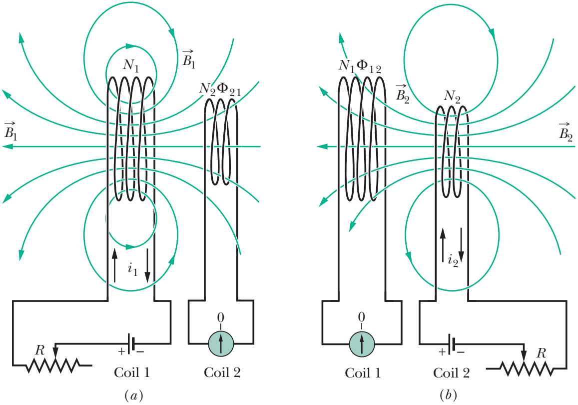
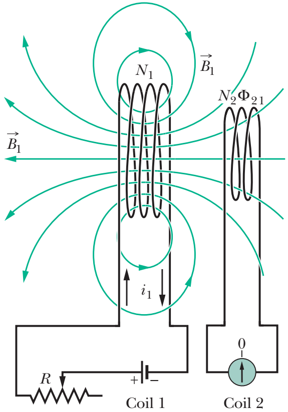


Figure 8: To distinguish from self-induction, we define **mutual induction** as the flux-induced interaction of two coils.

- Consider two circular close-packed coils near each other and sharing a common central axis. The current in coil 1 creates a magnetic field $\vec{B}_1$.
- The flux Φ_{21} through coil 2 associated with the current in coil 1 links the N_2 turns of coil 2.
- We define the **mutual inductance** M_{21} of coil 2 with respect to coil 1 as

$$M_{21} = \frac{N_2 \Phi_{21}}{i_1}.$$



- According to Faraday's law, the emf in coil 2

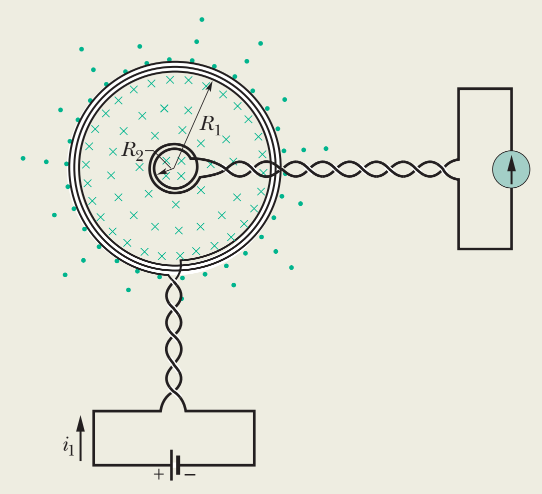
$$\mathcal{E}_{21} = -\frac{d(N_2\Phi_{21})}{dt} = -M_{21}\frac{di_1}{dt}.$$

- M_{21} is a purely *geometrical* quantity, having to do with the sizes, shapes, and relative positions of the two coils.
- If we switch the role of coil 1 and 2, we obtain

$$\mathcal{E}_{12} = -M_{12}\frac{di_2}{dt}.$$

- It can be proved that $M_{21} = M_{12}$ *regardless of the shapes and positions of the coils.*

Mutual Inductance of Two Parallel Coils



First, we calculate the flux linkage in the smaller coil 2, which is small enough for us to assume that the magnetic field through it due to the larger coil is approximately uniform.

- The flux through coil 2 due to the coil 1 is approximately uniform, such that

$$\Phi_{21} = A_2 B_1.$$

- Biot-Savart law tells us, for each loop,

$$\frac{B_1}{N_1} = \frac{\mu_0}{4\pi} \frac{i_1(2\pi R_1)}{R_1^2} = \frac{\mu_0 i_1}{2R_1}.$$

- To find M we calculate the flux linkage $N_2\Phi_{21}$ in the smaller coil:

$$M = \frac{N_2\Phi_{21}}{i_1} = \frac{\mu_0 N_1 N_2(\pi R_2^2)}{2R_1}.$$

- What we have calculated is M_{21}, the mutual inductance of coil 2 with respect to coil 1.
- Can you show that we can also calculate the mutual inductance M_{12} of coil 1 with respect to coil 2?
- For that, we need to figure out the flux linkage in the larger coil 1 due to the coil 2.
- Note that we can model coil 2 as a magnetic dipole. We only have a simple expression for the magnetic field not very close to coil 2, and you expect (don't you?)

$$\Phi_{12} = \frac{\mu_0 N_2 i_2 (\pi R_2^2)}{2R_1} = \frac{\mu_0 \mu_2}{2R_1} = \int_{R_1}^{\infty} \frac{\mu_0 \mu_2}{4\pi r^3}(2\pi r)dr,$$

where $\mu_2 = N_2 i_2 (\pi R_2^2)$, such that $M_{12} = M_{21} = M$.

Summary

- Concepts: inductance L, magnetic flux linkage $N\Phi_B$, inductive time constant τ_L
- Useful formulas:

$$L = \frac{N\Phi_B}{i} \qquad L = \mu_0 n^2 \ell A$$

$$U_B = \frac{1}{2} L i^2 \qquad u_B = \frac{B^2}{2\mu_0}$$

Can you write down the corresponding formulas in the context of capacitance?

- Comparison: RL circuit and RC circuit

$$\tau_C = RC, \quad \tau_L = L/R$$

- Unification: Self-inductance L (of a single circuit) and mutual inductance $M_{12} = M_{21}$ (of two circuits)

$$\mathcal{E}_{11} = -\frac{d(N_1 \Phi_{11})}{dt} = -L_1 \frac{di_1}{dt}$$

$$\mathcal{E}_{21} = -\frac{d(N_2 \Phi_{21})}{dt} = -M_{21} \frac{di_1}{dt}$$

Do you see a (symmetric) inductance matrix here?

Reading

Halliday, Resnick & Krane:

- Chapter 36: Inductance