DC Circuits

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Lecture 7

Outline

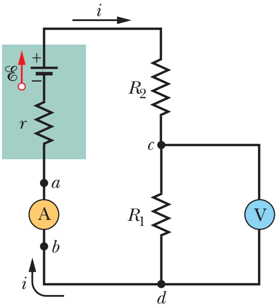
- Concepts in an Electric Circuit
- Energy Conservation in an Electric Circuit
- RC Circuits
- Capacitor with a Dielectric

Electric Circuit

- An **electric circuit** is a path through which charge can flow. If we want to make charge carriers flow through a resistor, we must establish a potential difference between the ends of the device.
- To produce a steady flow of charge, we need an **emf device**, which maintains a potential difference between a pair of terminals and supplies the energy for the motion of charge carriers via the work it does.
- A **battery** is such a device in which electric forces from electrochemical reactions can move internal charge.

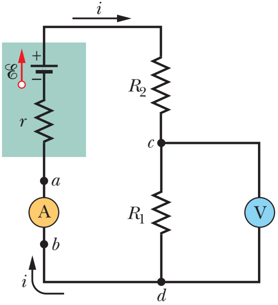
The Ammeter and the Voltmeter

- An instrument used to measure currents is called an **ammeter**. To measure the current in a wire, we usually have to break or cut the wire and insert the ammeter so that the current to be measured passes through the meter.



- It is essential that the resistance R_A of the ammeter be *very much smaller* than other resistances in the circuit. Otherwise, the very presence of the meter will change the current to be measured.

- A meter used to measure potential differences is called a **voltmeter**. To find the potential difference between any two points in the circuit, the voltmeter terminals are connected between those points.
- It is essential that the resistance R_V of a voltmeter be *very much larger* than the resistance of any circuit element across which the voltmeter is connected. Otherwise, the meter alters the potential difference that is to be measured.



Grounding a Circuit

- Grounding a circuit usually means connecting the circuit to a conducting path to Earth's surface (actually to the electrically conducting moist dirt and rock below ground).
- Such a connection means only that the potential is defined to be zero at the grounding point in the circuit.

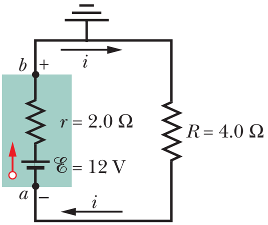


Figure 1: Ground is taken to be zero potential.

Work and Emf

- We can represent the emf of the device with an arrow that points from the negative terminal toward the positive terminal. A small circle on the tail of the emf arrow distinguishes it from the arrows that indicate current direction.
- The emf device keeps the positive terminal (labeled $+$) at a higher electric potential than the negative terminal (labeled $-$).

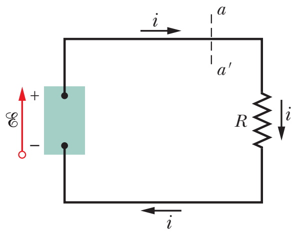


Figure 2: A simple electric circuit.

- When an emf device is connected to a circuit, its internal chemistry causes a net flow of positive charge carriers from the negative terminal to the positive terminal, in the direction of the emf arrow.
- This motion is just the opposite of what the electric field between the terminals would cause the charge carriers to do. Thus, there must be some source of energy within the device, enabling it to do work on the charges by forcing them to move as they do.
- The energy source may be chemical, as in a *fuel cell*. It may involve mechanical forces, as in an *electric generator*. The Sun may supply it, as in a *solar cell*.

- We define the **emf** of the emf device as the work per unit charge that the device does in moving charge from its low-potential terminal to its high-potential terminal:

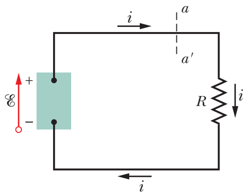
$$\mathcal{E} = \frac{W}{q},$$

whose SI unit is the joule per coulomb, i.e., the volt.

  - An **ideal emf device** has no internal resistance to the internal movement of charge from terminal to terminal.
  - A **real emf device** has internal resistance to the internal movement of charge. When that device has current through it, the potential difference between its terminals differs from its emf.

Energy Conservation in a Simple Circuit

- Consider the simple circuit with an ideal emf device and a resistor. The emf device maintains a potential difference of magnitude V across its own terminals. A steady current i is produced in the circuit.
- The amount of charge dq that moves through the emf device in time interval dt is equal to idt. So its generated electric potential energy is



$$dU = dq\mathcal{E} = idt\mathcal{E}.$$

- As an electron moves through a resistor at constant drift speed, its average kinetic energy remains constant and its lost electric potential energy appears as thermal energy in the resistor and the surroundings.
- This energy transfer is due to collisions between the electron and the molecules of the resistor, which leads to an increase in the temperature of the resistor lattice.
- For a resistor with resistance $R = V/i$, the rate of electrical energy dissipation is

$$P = iV = i^2R = V^2/R.$$

- The unit of power is the volt-ampere (V·A). We can write it as

$$1\ \mathrm{V} \cdot \mathrm{A} = \left(1\ \frac{\mathrm{J}}{\mathrm{C}}\right)\left(1\ \frac{\mathrm{C}}{\mathrm{s}}\right) = 1\ \mathrm{W}.$$

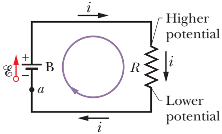
- The principle of conservation of energy tells us that the total rate of energy change for the single-loop circuit is

$$\frac{dU}{dt} = i\mathcal{E} - i^2 R = 0.$$

- Therefore, we have $\mathcal{E} - iR = 0$, which can generalized to any loop of a circuit.

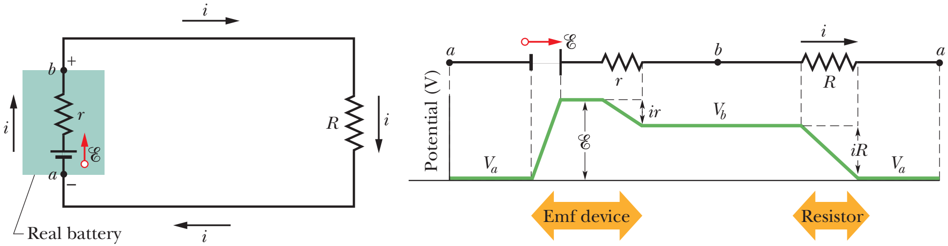
- This is **Kirchhoff's loop rule**: The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.

The battery drives current through the resistor, from high potential to low potential.



- For a move through a resistance in the direction of the current, the change in potential is $-iR$; in the opposite direction it is $+iR$.
- For a move through an ideal emf device in the direction of the emf arrow, the change in potential is $+\mathcal{E}$; in the opposite direction it is $-\mathcal{E}$.

Example: Internal Resistance

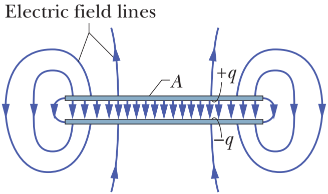


- If we apply the loop rule clockwise, we have $\mathcal{E} - ir - iR = 0$. Solving for the current, we find

$$i = \frac{\mathcal{E}}{R + r}.$$

Energy Stored in a Capacitor

- To charge up a capacitor, we remove electrons from the positive plate and carry them to the negative plate.
- In doing so we fight against the electric field, which is pulling electrons back toward the positive conductor and pushing them away from the negative one.



- Therefore, work must be done by an external agent to charge a capacitor. Actually, a battery does all this for us, at the expense of its stored chemical energy.

- We can imagine doing the work ourselves by transferring electrons from one plate to the other, one by one. As the charges build, so does the electric field between the plates, which opposes the continued transfer.
- Suppose that, at a given instant, a charge q' has been transferred from one plate of a capacitor to the other. The potential difference V' between the plates at that instant will be q'/C. If an extra increment of charge dq' is then transferred, the increment of work required will be

$$dW = V'dq' = (q'/C)dq'.$$

- The work required to bring the total capacitor charge up to a final value q is

$$W = \int dW = \int_0^q (q'/C)dq' = q^2/(2C).$$

- This work is stored as potential energy U in the capacitor, so that

$$U = \frac{q^2}{2C} = \frac{1}{2}CV^2.$$

RC Circuits

- Here we begin a discussion of time-varying currents in an RC series circuit consisting of a capacitance C, an ideal battery of emf $\mathcal{E}$, and a resistance R.

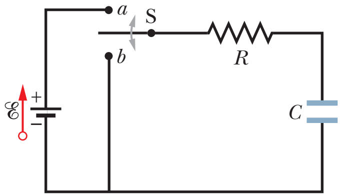
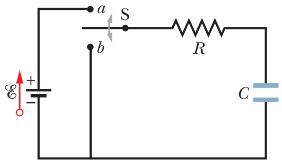


Figure 3: When switch S is closed on a, the capacitor is charged through the resistor. When the switch is afterward closed on b, the capacitor discharges through the resistor.

Charging a Capacitor

- As the circuit is complete on a, charge begins to flow between capacitor plates and battery terminals.
- The potential energy change of the capacitor is



$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{q^2}{2C}\right) = \frac{q}{C}\frac{dq}{dt}.$$

- By the conservation of energy, the change comes from the energy generated by the battery and the energy dissipated by the resistor, i.e.,

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} = i\mathcal{E} - i^2 R.$$

- Noting $i = dq/dt$, we find

$$R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E},$$

which can also be obtained by the loop rule. As we move down through the capacitor, there is a potential drop q/C.

- We first identify the characteristic time scale, or the **capacitive time constant** of the RC circuit. By dimension analysis, we can write

$$\tau = RC.$$

That is, the dimensions of $R dq/dt$ and q/C are the same.
- One can verify that the solution for q is

$$q = C\mathcal{E}(1 - e^{-t/\tau}).$$

- The equilibrium (final) charge on the then fully charged capacitor is equal to $q_0 = C\mathcal{E}$.

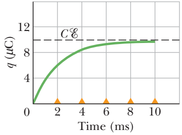
- Therefore, the current decreases in time as

$$i = \frac{dq}{dt} = \left(\frac{\mathcal{E}}{R}\right) e^{-t/\tau},$$

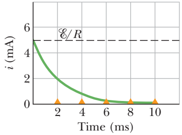
and the potential difference across the capacitor increases as

$$V_C = \frac{q}{C} = \mathcal{E}(1 - e^{-t/\tau})$$

The capacitor's charge grows as the resistor's current dies out.



(a)



(b)

- When that potential difference equals the potential difference across the battery, the current is zero.

Discharging a Capacitor

- Assume the capacitor is fully charged to a potential $V_0 = q_0/C$. At a new time $t = 0$, switch S is thrown from a to b and the capacitor discharges through R.
- Once again, the potential energy change of the capacitor is

$$\frac{d}{dt}\left(\frac{q^2}{2C}\right) = \frac{q}{C}\frac{dq}{dt}.$$

- By the conservation of energy, the energy change is dissipated by the resistor as i^2R, where $i = dq/dt$.

- Therefore, the differential equation describing q is

$$R\frac{dq}{dt} + \frac{q}{C} = 0.$$

- Recall that the charge on the fully charged capacitor is

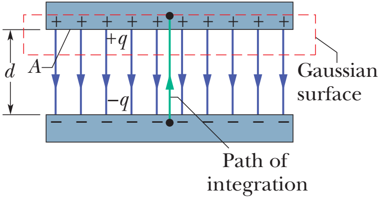
$$q_0 = C\mathcal{E}.$$

- The solution for q is $q = q_0 e^{-t/\tau}$. Thus, we find

$$i = \frac{dq}{dt} = -\left(\frac{q_0}{\tau}\right) e^{-t/\tau} = -\left(\frac{\mathcal{E}}{R}\right) e^{-t/\tau}.$$

Energy Density

- In a parallel-plate capacitor, neglecting fringing, the electric field has the same value at all points between the plates.
- Consider two parallel-plate capacitors that are identical except that capacitor 1 has twice the plate separation d of capacitor 2.



- Capacitor 1 has twice the volume (Ad) between its plates and also half the capacitance ($\epsilon_0 A/d$) of capacitor 2.

- If both capacitors have the same charge q, the electric fields between their plates are identical, but capacitor 1 has twice the stored potential energy of capacitor 2, as

$$U = \frac{q^2}{2C} = \frac{q^2 d}{2\epsilon_0 A} \propto d.$$

- Thus, of two otherwise identical capacitors with the same charge and same electric field, the one with twice the volume between its plates has twice the stored potential energy.

- Explicitly, with $C = \epsilon_0 A/d$ and $V = Ed$, we find

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\epsilon_0 E^2 (Ad).$$

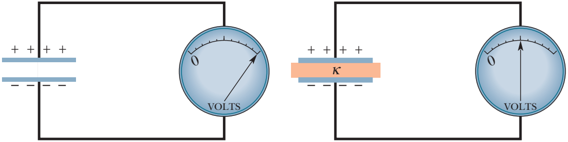
- The energy density u, i.e., the potential energy per unit volume between the plates, is thus uniform:

$$u = \frac{1}{2}\epsilon_0 E^2.$$

- This result holds for any electric field. Therefore, *the potential energy of a charged capacitor may be viewed as being stored in the electric field between its plates*.

Capacitor with a Dielectric

- A **dielectric** is an insulating material, which can be used to fill the space between the plates of a capacitor.



q = a constant

- For fixed charge, the effect of a dielectric is to reduce the potential difference between the plates. See Appendix 7B for microscopic mechanisms.

- Michael Faraday found that, correspondingly, the capacitance *increased* by a numerical factor κ, which is called the **dielectric constant** of the insulating material.
- In fact, in a region completely filled by a dielectric material of dielectric constant κ, all electrostatic equations containing the permittivity constant ϵ_0 are to be modified by replacing ϵ_0 with $\epsilon = \kappa\epsilon_0$, the permittivity of the material.
- For example, for a point charge inside a dielectric,

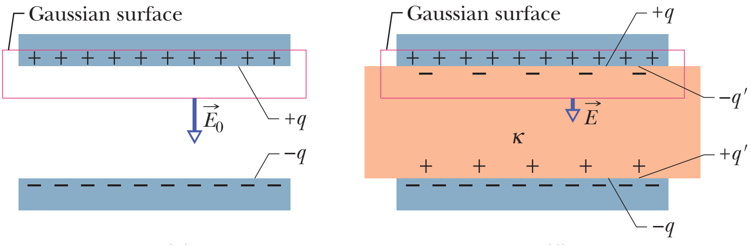
$$E = \frac{1}{4\pi\epsilon}\frac{q}{r^2} = \frac{1}{4\pi\kappa\epsilon_0}\frac{q}{r^2}.$$

- The dielectric constant of a vacuum is unity by definition. The dielectric constant of air is only slightly greater than unity.
- If the applied electric field is sufficiently large (beyond the **dielectric strength**), the dielectric material will break down and form a conducting path between the plates.

Material	Dielectric Constant κ	Dielectric Strength (kV/mm)
Air (1 atm)	1.00054	3
Polystyrene	2.6	24
Paper	3.5	16
Transformer oil	4.5	
Pyrex	4.7	14
Ruby mica	5.4	
Porcelain	6.5	
Silicon	12	
Germanium	16	
Ethanol	25	
Water (20°C)	80.4	
Water (25°C)	78.5	
Titania ceramic	130	
Strontium titanate	310	8

For a vacuum, κ = unity.

Dielectrics and Gauss' Law



- The effect of the dielectric is to weaken the original field $E_0 = q/(\epsilon_0 A)$ by a factor of κ; so we may write

$$E = \frac{q}{\kappa\epsilon_0 A} = \frac{E_0}{\kappa} = \frac{q}{\kappa\epsilon_0 A} = \frac{q/\kappa}{\epsilon_0 A}.$$

- Therefore, the induced surface charge q' satisfies $q - q' = q/\kappa$, so

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q - q' = q/\kappa.$$

This way, we still use the original Gauss' law in vacuum.
- Alternatively, we can recast Gauss' law as

$$\epsilon_0 \oint \kappa\vec{E} \cdot d\vec{A} = \oint \vec{D} \cdot d\vec{A} = q,$$

where we define the **electric displacement** $\vec{D} \equiv \kappa\epsilon_0\vec{E}$. We modify the Gauss' law for the dielectric, but the charge q enclosed by the Gaussian surface is only the *free charge*.

Summary

- Concepts: emf $\mathcal{E}$, power P, dielectric constant κ, electric displacement $\vec{D}$

$$\mathcal{E} = \frac{W}{q}$$

$$P = iV$$

$$P = i^2 R = \frac{V^2}{R}$$

- Energy (density):

$$u = \frac{1}{2}\epsilon_0 E^2 \qquad\qquad U = \frac{q^2}{2C} = \frac{1}{2}CV^2$$

- Gauss' law with a dielectric (with $\vec{D} \equiv \kappa\epsilon_0\vec{E}$)

$$\oint \vec{D}\cdot d\vec{A} = \epsilon_0 \oint \kappa\vec{E}\cdot d\vec{A} = q$$

- Circuit analysis: loop rule, junction rule (see Appendix 7A for a discussion)

Reading

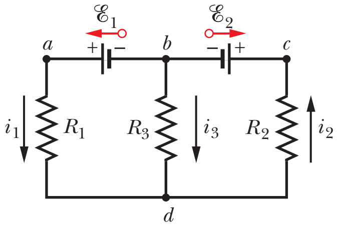
Halliday, Resnick & Krane:

- Chapter 30: Capacitance
- Chapter 31: DC Circuits

Appendix 7A: Current in a Multiloop Circuit

- Basic tools for solving complex circuits are the *loop rule* (based on the conservation of energy) and the *junction rule* (based on the conservation of charge).
- **Kirchhoff's junction rule**: The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.
  - $i_1 + i_3 = i_2$

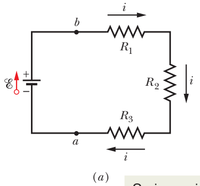
The current into the junction must equal the current out (charge is conserved).



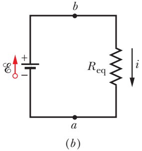
Resistances in Series

- Resistances connected in series can be replaced with an equivalent resistance R_{eq} that has the same current i and the same total potential difference V as the actual resistances.

$$\mathcal{E} - iR_1 - iR_2 - iR_3 = 0$$
$$\mathcal{E} - iR_{\mathrm{eq}} = 0$$



Series resistors and their equivalent have the same current ("ser-i").



- The extension to n resistances is

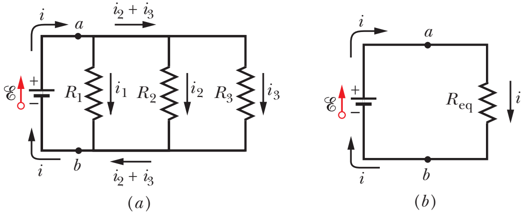
$$R_{\mathrm{eq}} = \sum_{j=1}^{n} R_j.$$

Resistances in Parallel

- When a potential difference V is applied across resistances connected in parallel, the resistances all have that same potential difference V.
- Resistances connected in parallel can be replaced with an equivalent resistance R_{eq}:

$$\frac{1}{R_{\mathrm{eq}}} = \sum_{j=1}^{n} \frac{1}{R_j}$$

Parallel resistors and their equivalent have the same potential difference ("par-V").



Capacitors in Series

- When several capacitors connected *in series*, the capacitors have identical charge q.
- The applied potential difference V is equal to the *sum of the potential differences* across all the capacitors:

$$V = \sum_{j=1}^{n} V_j.$$

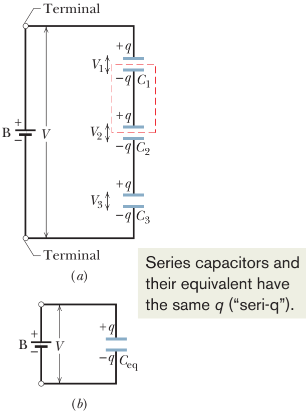
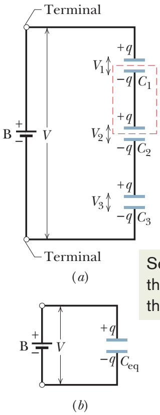


Figure 4: Capacitors in series.

- Capacitors that are connected in series can be replaced with an equivalent capacitor that has the same charge q and the same total potential difference V as the actual series capacitors.

$$\frac{1}{C_{\mathrm{eq}}} = \sum_{j=1}^{n} \frac{1}{C_j}.$$



Series capacitors and their equivalent have the same q ("seri-q").

Capacitors in Parallel

- When a potential difference V is applied across several capacitors connected *in parallel*, it is applied across each capacitor.
- The total charge q stored on the capacitors is the *sum of the charges* stored on all the capacitors:

$$q = \sum_j q_j.$$

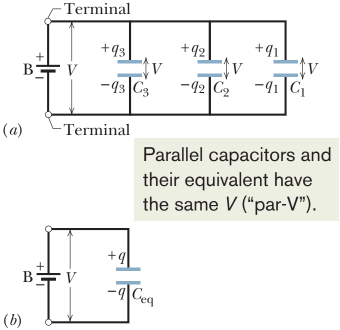
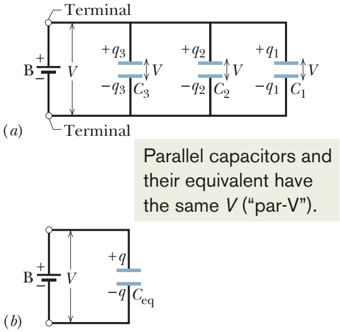


Figure 5: Capacitors in parallel.

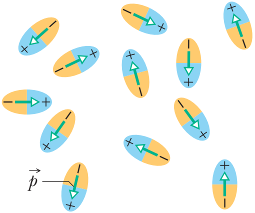
- Capacitors connected in parallel can be replaced with an equivalent capacitor that has the same total charge q and the same potential difference V as the actual capacitors.

$$C_{\text{eq}} = \sum_{j=1}^{n} C_j.$$

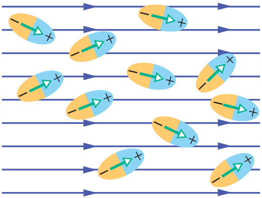


Appendix 7B: An Atomic View of Dielectrics

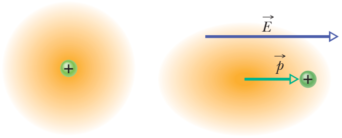
- Type I: *Polar dielectrics*. The molecules of some dielectrics, like water, have permanent electric dipole moments.
- In such materials, the electric dipoles tend to line up with an external electric field. The alignment of the electric dipoles produces an electric field that is directed opposite the applied field and is smaller in magnitude.



(a)

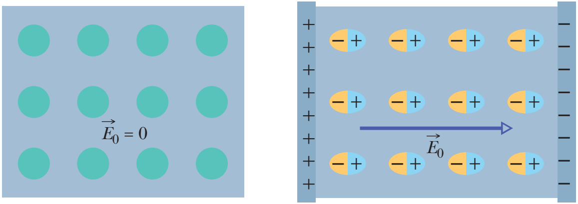

(b)

- Type II: *Nonpolar dielectrics*. Nonpolar molecules acquire dipole moments by induction when placed in an external electric field.



- In a nonpolar molecule and in every isolated atom, the centers of the positive and negative charges coincide and thus no dipole moment is set up.
- However, in an external electric field, the field distorts (or **polarizes**) the electron orbits and separates the centers of positive and negative charge, setting up a dipole moment $\vec{p}$ that points in the direction of the field.

- In a nonpolar dielectric slab, polarization produces positive charge on one face of the slab (due to the positive ends of dipoles there) and negative charge on the opposite face (due to the negative ends of dipoles there).



- The slab as a whole remains electrically neutral and, within the slab, there is no excess charge in any volume element.

- The induced surface charges on the faces produce an electric field $\vec{E}'$ in the direction opposite that of the applied electric field $\vec{E}_0$.
- The resultant field $\vec{E}$ inside the dielectric (the vector sum of fields $\vec{E}_0$ and $\vec{E}'$) has the direction of $\vec{E}_0$ but is smaller in magnitude.

- Therefore, the effect of both polar and nonpolar dielectrics is to weaken any applied field within them, as between the plates of a capacitor.