

Relations

Chapter 9

Chapter Summary

- Relations and Their Properties
- Representing Relations
- Closures of Relations
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

Section 9.1

Section Summary

- Relations and Functions
- Properties of Relations
 - Reflexive Relations
 - Symmetric and Antisymmetric Relations
 - Transitive Relations
- Combining Relations

Social Relationships

- There are many kinds of relationships in the world:
- Relative: Relationship by blood or by a common ancestor.
- Friendship: boyfriend and girlfriend
- Relations between Teachers and students
- Relations between bosses and employees

Social Relationships

- Relations between war and peace
- Relations between city and village
- Relations between God and mankind
- Relations between mankind and their environment
- Relations between obama and osama (bin laden)
- And so on...

Abstract Relationships

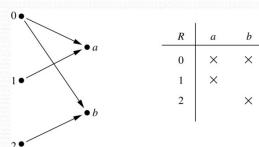
- The question is how to represent relationship in mathematical methods
- N-ary relationships (complex): relationships among many objects.
- But most of the relationship can be formalized in the idea of binary relation.
- Binary relation is the simplest relation, it is what we will study in this course.

Binary Relations

Definition: A binary relation R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B .
- We can represent relations from a set A to a set B graphically or using a table:



Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A .

Binary Relation on a Set

Definition: A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:

- Suppose that $A = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A .
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary Relation on a Set (cont.)

Question: How many relations are there on a set A ?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A .

Binary Relations on a Set (cont.)

Example: Consider these relations on the set of integers:

$$\begin{aligned} R_1 &= \{(a,b) \mid a \leq b\}, & R_4 &= \{(a,b) \mid a = b\}, \\ R_2 &= \{(a,b) \mid a > b\}, & R_5 &= \{(a,b) \mid a = b + 1\}, \\ R_3 &= \{(a,b) \mid a = b \text{ or } a = -b\}, & R_6 &= \{(a,b) \mid a + b \leq 3\}. \end{aligned}$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

$(1,1)$, $(1,2)$, $(2,1)$, $(1,-1)$, and $(2,2)$?

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1, R_3, R_4 , and R_6 ; $(1,2)$ is in R_1 and R_6 ; $(2,1)$ is in R_2, R_5 , and R_6 ; $(1,-1)$ is in R_2, R_3 , and R_6 ; $(2,2)$ is in R_1, R_3 , and R_4 .

Reflexive (自反) Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x[x \in U \rightarrow (x,x) \in R]$$

Example: The following relations on the integers are reflexive:

$$\begin{aligned} R_1 &= \{(a,b) \mid a \leq b\}, \\ R_3 &= \{(a,b) \mid a = b \text{ or } a = -b\}, \\ R_4 &= \{(a,b) \mid a = b\}. \end{aligned}$$

The following relations are not reflexive:

$$\begin{aligned} R_2 &= \{(a,b) \mid a > b\} \text{ (note that } 3 \not> 3), \\ R_5 &= \{(a,b) \mid a = b + 1\} \text{ (note that } 3 \neq 3 + 1), \\ R_6 &= \{(a,b) \mid a + b \leq 3\} \text{ (note that } 4 + 4 \not\leq 3). \end{aligned}$$

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y [(x,y) \in R \rightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

The following are not symmetric:

$$R_1 = \{(a,b) \mid a \leq b\} \text{ (note that } 3 \leq 4, \text{ but } 4 \not\leq 3),$$

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 4 > 3, \text{ but } 3 \not> 4),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1).$$

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*.

Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y [(x,y) \in R \wedge (y,x) \in R \rightarrow x = y]$$

- Example:** The following relations on the integers are antisymmetric:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_5 = \{(a,b) \mid a = b + 1\}.$$

The following relations are not antisymmetric:

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$$

(note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

$$R_6 = \{(a,b) \mid a + b \leq 3\} \text{ (note that both } (1,2) \text{ and } (2,1) \text{ belong to } R_6).$$

For any integer, if $a \leq b$ and $a \leq b$, then $a = b$.

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z [(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R]$$

- Example:** The following relations on the integers are transitive:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

For every integer, $a \leq b$
and $b \leq c$, then $a \leq c$.

The following are not transitive:

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that both } (3,2) \text{ and } (4,3) \text{ belong to } R_5, \text{ but not } (3,3)),$$

$$R_6 = \{(a,b) \mid a + b \leq 3\} \text{ (note that both } (2,1) \text{ and } (1,2) \text{ belong to } R_6, \text{ but not } (2,2)).$$

Question:

Symmetric, transitive \Rightarrow reflexive ?

$$\left. \begin{array}{l} (a,b) \in R \\ R \text{ is symmetric} \end{array} \right\} \Rightarrow \left. \begin{array}{l} (b,a) \in R \\ R \text{ is transitive} \end{array} \right\} \Rightarrow (a,a) \in R$$

This argument makes an assumption that $\forall a \exists b (a,b) \in R$

Therefore, symmetry and transitivity are not enough to infer reflexivity

Combining Relations

- Given two relations R_1 and R_2 , we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

- Example:** Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \quad R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

Composition

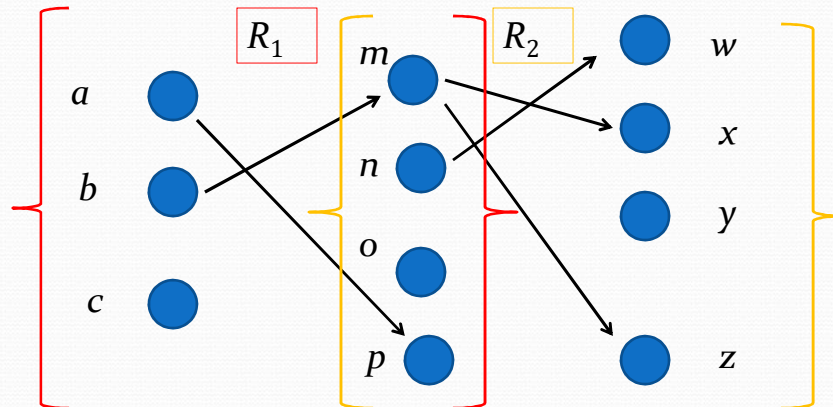
Definition: Suppose

- R_1 is a relation from a set A to a set B .
- R_2 is a relation from B to a set C .

Then the *composition* (or *composite*) of R_2 with R_1 , is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2 , then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b, x), (b, z)\}$$

relational composition

- Let M be the relation “is mother of”
- Let F be the relation “is father of”
- What is $M \circ F$?
 - If $(a, b) \in F$, then a is the father of b
 - If $(b, c) \in M$, then b is the mother of c
 - Thus, $M \circ F$ denotes the relation “maternal grandfather”
- What is $F \circ M$?
 - If $(a, b) \in M$, then a is the mother of b
 - If $(b, c) \in F$, then b is the father of c
 - Thus, $F \circ M$ denotes the relation “paternal grandmother”
- What is $M \circ M$?
 - If $(a, b) \in M$, then a is the mother of b
 - If $(b, c) \in M$, then b is the mother of c
 - Thus, $M \circ M$ denotes the relation “maternal grandmother”
- Note that M and F are not transitive relations!!!

Powers of a Relation

Definition: Let R be a binary relation on A . Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$
- Example: $R = \{(1,1), (2,1), (3,2), (4,3)\}$

Find R^2 , R^3 , and R^4

$$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

$$R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

Example

- $R = \{(a,b), \text{a is parent of b or vice versa}\}$
- $R^2 = \{(a,b), \text{a is grandparent of b or vice versa}\}$
- N-generations blood relationship: if $(a,b) \in R^n$, we say a and b have n -generations blood relationship

Theorem 1

- Then relation R on a set A is transitive if and only if $R^n \subseteq R$. ($n=1,2,3,\dots$)
- If part: $R^n \subseteq R$, $R^2 \subseteq R$. if $(a,b) \in R$ and $(b,c) \in R$ for any $a,b,c \in A$, then $(a,c) \in R^2$, hence, $(a,c) \in R$, R is transitive.
- Only if part: if R is transitive, $(a,c) \in R^2$, then there exist $b \in A$ such that $(a,b) \in R$ and $(b,c) \in R$. Hence $(a,c) \in R$
- This implies that $R^2 \subseteq R$

Cont...

- Further more, $R^3 = R^2 \circ R \subseteq R \circ R = R^2 \subseteq R$
- Then for any $n=1,2,3,\dots$
- $R^n = R^{n-1} \circ R \subseteq \dots \subseteq R \circ R = R^2 \subseteq R$
- **Inverse Relation:** Let R be a relation from set A to set B , the inverse of R is a relation from B to A such that :
- $R^{-1} = \{(a,b) | (b,a) \in R\}$

Homework

- 第七版 Sec. 9.1 7(a,c,h), 26, 32, 47, 51
- 第八版 Sec. 9.1 7(a,c,h), 26, 32, 49, 53

Representing Relations

Section 9.3

Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$.
 - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j .

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1), (3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Examples of Representing Relations Using Matrices (cont.)

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

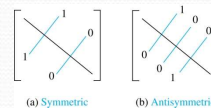
$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Matrices of Relations on Sets

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.



- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

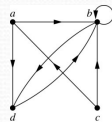
Solution: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b) , and the vertex b is called the *terminal vertex* of this edge.

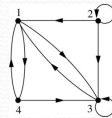
- An edge of the form (a,a) is called a *loop*.

Example 7: A drawing of the directed graph with vertices a, b, c , and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) , and (d, b) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?

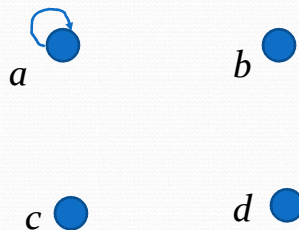


Solution: The ordered pairs in the relation are $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, and $(4, 3)$

Determining which Properties a Relation has from its Digraph

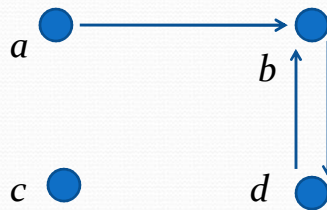
- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x) .
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z) .

Determining which Properties a Relation has from its Digraph – Example 1



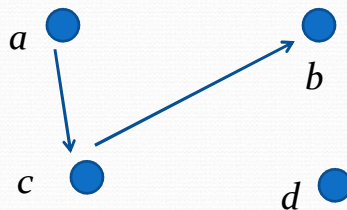
- *Reflexive*? No, not every vertex has a loop
- *Symmetric*? Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric*? Yes (trivially), there is no edge from one vertex to another
- *Transitive*? Yes, (trivially) since there is no edge from one vertex to another

Determining which Properties a Relation has from its Digraph – Example 2



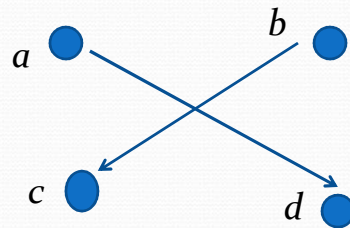
- *Reflexive?* No, there are no loops
- *Symmetric?* No, there is an edge from a to b , but not from b to a
- *Antisymmetric?* No, there is an edge from d to b and b to d
- *Transitive?* No, there are edges from a to c and from c to b , but there is no edge from a to d

Determining which Properties a Relation has from its Digraph – Example 3



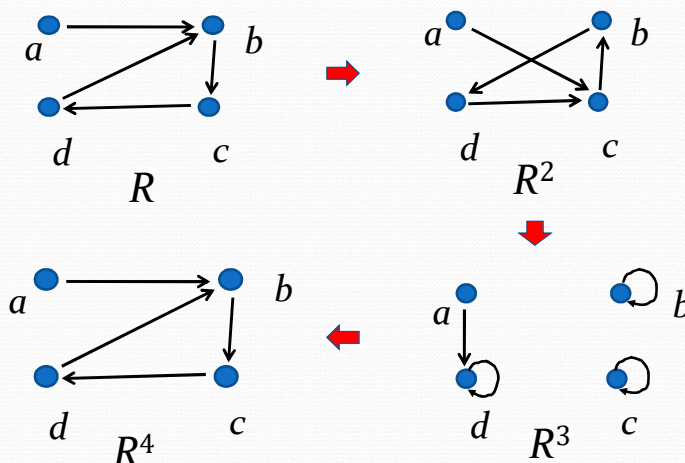
- Reflexive?* No, there are no loops
- Symmetric?* No, for example, there is no edge from c to a
- Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- Transitive?* No, there is no edge from a to b

Determining which Properties a Relation has from its Digraph – Example 4



- *Reflexive*? No, there are no loops
- *Symmetric*? No, for example, there is no edge from d to a
- *Antisymmetric*? Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive*? Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

Inverse relation

$$R = \{(a, b) \mid a \in A, b \in B, aRb\}$$

The **inverse relation** from B to A : $R^{-1}(R^c)$

$$\{(b, a) \mid (a, b) \in R, a \in A, b \in B\}$$

Question:

How to get R^{-1} ?

(1) Using the definition directly

For example, $R = \{(a, b) \mid a \mid b, a, b \in \mathbb{Z}^+\}$
 $R^{-1} = \{(a, b) \mid b \mid a, a, b \in \mathbb{Z}^+\}$

(2) Reverse all the arcs in the digraph representation of R

(3) Take the transpose M_R^T of the connection matrix M_R of R .

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The properties of relation operations

Suppose that R, S are the relations from A to B , T is the relation from B to C , P is the relation from C to D , then

(1) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof:

$$\forall (x, y) \in (R \cup S)^{-1}$$

$$\Leftrightarrow (y, x) \in R \cup S$$

$$\Leftrightarrow (y, x) \in R \text{ or } (y, x) \in S$$

$$\Leftrightarrow (x, y) \in R^{-1} \text{ or } (x, y) \in S^{-1}$$

$$\Leftrightarrow (x, y) \in R^{-1} \cup S^{-1}$$

The properties of relation operations

Suppose that R, S are the relations from A to B , T is the relation from B to C , P is the relation from C to D , then

$$(1) (R \cup S)^{-1} = R^{-1} \cup S^{-1}$$

$$(2) (R \cap S)^{-1} = R^{-1} \cap S^{-1}$$

$$(3) (\bar{R})^{-1} = \overline{R^{-1}}$$

$$(4) (R - S)^{-1} = R^{-1} - S^{-1}$$

$$(5) (A \times B)^{-1} = B \times A$$

Proof:

$$\forall (x, y) \in (A \times B)^{-1}$$

$$\Leftrightarrow (y, x) \in A \times B$$

$$\Leftrightarrow (x, y) \in B \times A$$

The properties of relation operations

Suppose that R, S are the relations from A to B , T is the relation from B to C , P is the relation from C to D , then

$$(1) (R \cup S)^{-1} = R^{-1} \cup S^{-1}$$

$$(2) (R \cap S)^{-1} = R^{-1} \cap S^{-1}$$

$$(3) (\bar{R})^{-1} = \overline{R^{-1}}$$

$$(4) (R - S)^{-1} = R^{-1} - S^{-1}$$

$$(5) (A \times B)^{-1} = B \times A$$

$$(6) \bar{R} = A \times B - R$$

$$(7) (S \circ T)^{-1} = T^{-1} \circ S^{-1}$$

$$(8) (R \circ T) \circ P = R \circ (T \circ P)$$

$$(9) (R \cup S) \circ T = R \circ T \cup S \circ T$$

Homework

Sec. 9.3 13,14,31