Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

Chapter Summary

- Sets（集合）
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions（函数）
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences（序列）and Summations（求和）
  - Types of Sequences
  - Summation Formulae
- Set Cardinality（基数）
  - Countable Sets
- Matrices
  - Matrix Arithmetic

Sets

Section 2.1

Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

$S = \{a,b,c,d\} = \{b,c,a,d\}$

- Each distinct object is either a member or not; listing more than once does not change the set.

$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$S = \{a,b,c,d, \ldots,z \}$

Roster Method

- Set of all vowels in the English alphabet:

$V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

$O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

$S = \{1,2,3,\ldots,99\}$

- Set of all integers less than 0:

$S = \{\ldots, -3,-2,-1\}$

Some Important Sets

$\mathbf{N}$ = *natural numbers* = {0,1,2,3….}

$\mathbf{Z}$ = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z^+}$ = *positive integers* = {1,2,3,…..}

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R^+}$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers*.

$\mathbf{Q}$ = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

    $S = \{x \mid x \text{ is a positive integer less than } 100\}$

    $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

    $O = \{x \in \mathbf{Z^+} \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

    $S = \{x \mid P(x)\}$

- Example: $S = \{x \mid \text{Prime}(x)\}$
- Positive rational numbers:

    $\mathbf{Q^+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

$[a,b] = \{x \mid a \le x \le b\}$
$[a,b) = \{x \mid a \le x < b\}$
$(a,b] = \{x \mid a < x \le b\}$
$(a,b) = \{x \mid a < x < b\}$

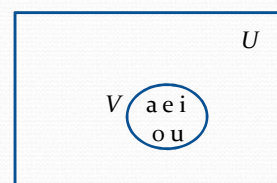*closed interval* [a,b]
open interval   (a,b)

Universal Set and Empty Set

- The *universal set*（全集）*U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram
（文氏图）

John Venn (1834-1923)
Cambridge, UK

Russell's Paradox（罗素悖论）

- Let *S* be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is *S* a member of itself?"
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Some things to remember

- Sets can be elements of sets.

  {{1,2,3},*a*, {*b,c*}}

  {**N,Z,Q,R**}
- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x (x \in A \leftrightarrow x \in B)$
- We write *A = B* if *A* and *B* are equal sets.

   $\{1,3,5\} = \{3, 5, 1\}$

   $\{1,5,5,5,3,3,1\} = \{1,3,5\}$

Subsets

Definition: The set *A* is a *subset*（子集） of *B*, if and only if every element of *A* is also an element of *B*.

- The notation $A \subseteq B$ is used to indicate that *A* is a subset of the set *B*.
- $A \subseteq B$ holds if and only if $\forall x (x \in A \rightarrow x \in B)$ is true.
   1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set *S*.
   2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set *S*.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:
1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff
$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to
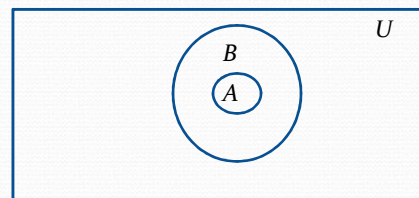$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset*（真子集） of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \to x \in B) \land \exists x(x \in B \land x \notin A)$$

is true.

Venn Diagram

Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted P(A), is called the *power set*（幂集） of A.

Example: If A = {a,b} then
$$\mathcal{P}(A) = \{\varnothing, \{a\},\{b\},\{a,b\}\}$$

- If a set has n elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

- The *ordered n-tuple* $(a_1,a_2,.....,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*（序偶）.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

René Descartes
(1596-1650)

Cartesian Product

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b)|a \in A \wedge b \in B\}$$

Example:
$A = \{a,b\}$ $B = \{1,2,3\}$
$A \times B = \{(a,1),(a,2),(a,3),\ (b,1),(b,2),(b,3)\}$

- **Definition**: A subset *R* of the Cartesian product $A \times B$ is called a *relation*（关系） from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered *n*-tuples $(a_1, a_2, \ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n)|a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C = \{(0,1,0),\ (0,1,1),\ (0,1,2),(0,2,0),\ (0,2,1),\ (0,2,2),(1,1,0),\ (1,1,1),\ (1,1,2),\ (1,2,0),\ (1,2,1),\ (1,1,2)\}$

Truth Sets of Quantifiers

- Given a predicate *P* and a domain *D*, we define the *truth set*（真值集） of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Example**: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "|*x*| = 1" is the set {-1,1}

Homework

第7版：Sec. 2.1 11, 18, 22, 24, 32(a),(c)
第8版： Sec. 2.1 13, 20, 24, 26, 34(a),(c)

Set Operations

Section 2.2

Section Summary

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Boolean Algebra

- Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*. This is discussed in Chapter 12.
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
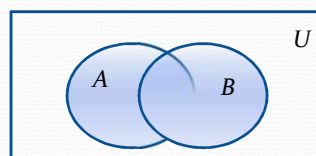- As always there must be a universal set *U*. All sets are assumed to be subsets of *U*.

Union

- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A* ∪ *B*, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is   {1,2,3} ∪ {3, 4, 5}?

Venn Diagram for *A* ∪ *B*

Solution: {1,2,3,4,5}

Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is
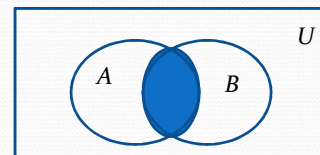$$\{x | x \in A \land x \in B\}$$
- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?
  - Solution: $\{3\}$

  Venn Diagram for $A \cap B$
- **Example:** What is?
  $\{1,2,3\} \cap \{4,5,6\}$?
  Solution: $\emptyset$

Complement

Definition: If *A* is a set, then the complement of the *A* (with respect to *U*), denoted by $\bar{A}$ is the set $U - A$
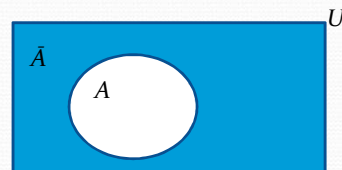$$\bar{A} = \{x \in U \mid x \notin A\}$$
(The complement of A is sometimes denoted by A^c.)

Example: If *U* is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$
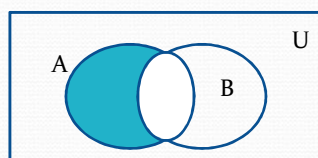
Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement

Difference

- **Definition**: Let *A* and *B* be sets. The *difference* of *A* and *B*, denoted by *A* – *B*, is the set containing the elements of *A* that are not in *B*. The difference of *A* and *B* is also called the complement of *B* with respect to *A*.
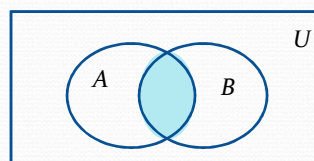
$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion（容斥原理）
$|A \cup B| = |A| + |B| - |A \cap B|$

Venn Diagram for *A, B, A ∩ B, A ∪ B*

- **Example**: Let *A* be the math majors in your class and *B* be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of *n* sets, where *n* is a positive integer.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$
   Solution: $\{1,2,3,4,5,6,7,8\}$
2. $A \cap B$
   Solution: $\{4,5\}$
3. $\bar{A}$
   Solution: $\{0,6,7,8,9,10\}$
4. $\bar{B}$
   Solution: $\{0,1,2,3,9,10\}$
5. $A - B$
   Solution: $\{1,2,3\}$
6. $B - A$
   Solution: $\{6,7,8\}$

Symmetric Difference (*optional*)

Definition: The *symmetric difference*（对称差） of **A** and **B**, denoted by $A \oplus B$ is the set
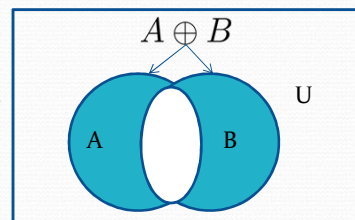
$$(A - B) \cup (B - A)$$

Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is:

- **Solution**: $\{1,2,3,6,7,8\}$

$A \oplus B$

A   B   U

Venn Diagram

Set Identities

- Identity laws（恒等律）

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws（支配律）

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws（幂等律）

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law（补律）

$$\overline{(\overline{A})} = A$$

Set Identities

- Commutative laws（交换律）

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws（结合律）

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws（分配律）

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities

- De Morgan's laws（德摩根律）

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws（吸收律）

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws（互补律）

$$A \cup \overline{A} = U \qquad\qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

◄

Set-Builder Notation: Second De Morgan Law

$$
\begin{aligned}
\overline{A \cap B} \;=\;& \{x \mid x \notin A \cap B\} && \text{by defn. of complement} \\
=\;& \{x \mid \neg(x \in (A \cap B))\} && \text{by defn. of does not belong symbol} \\
=\;& \{x \mid \neg(x \in A \wedge x \in B\} && \text{by defn. of intersection} \\
=\;& \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by 1st De Morgan law} \\
& && \text{for Prop Logic} \\
=\;& \{x \mid x \notin A \vee x \notin B\} && \text{by defn. of not belong symbol} \\
=\;& \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by defn. of complement} \\
=\;& \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by defn. of union} \\
=\;& \overline{A} \cup \overline{B} && \text{by meaning of notation}
\end{aligned}
$$

◄

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- Let $A_1, A_2, ..., A_n$ be an indexed collection of sets. We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

- For $i = 1,2,...$, let $A_i = \{i, i + 1, i + 2, ...\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

Computer Representation of Set

Using bit strings to represent sets.

(1) Specify an arbitrary ordering of the elements of *U*,

for instance $a_1, a_2, \cdots, a_n$

(2) Represent a subset *A* of *U* with the bit string of length *n*, where the *i*th bit is 1 if a_i belongs to *A* and is 0 if a_i does not belong to *A*.

Union: bitwise OR
Intersection: bitwise AND

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〖**Example** 〗 Let *U* ={1, 2, 3, 4, 5,6,7,8,9}, *A*={1, 2, 3, 4, 5}, *B*= (1, 3, 5, 7, 9). Use bit strings to find the difference of *A* and *B*.

solution:
The bit string for the set *A*: 11 1110 000
The bit string for the set *B*: 10 1010 101

$A - B = A \cap \overline{B}$

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Homework

第7版： Sec. 2.2 17, 48, 57(c)
第8版： Sec. 2.2 19, 54, 63(c)