General Physics II

Solution #4

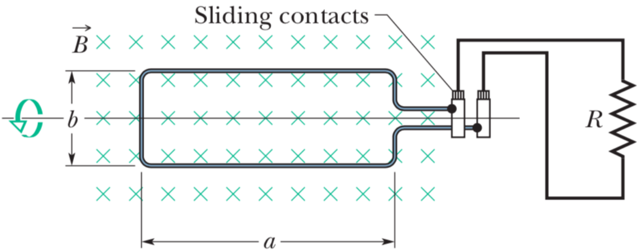
2021/11/10

P4-1. A rectangular coil of N turns and of length a and width b is rotated at frequency f in a uniform magnetic field $\vec{B}$. The coil is connected to co-rotating cylinders, against which metal brushes slide to make contact.
(a) Show that the emf induced in the coil is given (as a function of time t) by

$$\mathcal{E} = 2\pi fNabB \sin(2\pi ft) = \mathcal{E}_0 \sin(2\pi ft).$$

This is the principle of the commercial alternating-current generator.

(b) What value of Nab gives an emf with $\mathcal{E}_0 = 150$ V when the loop is rotated at 60.0 rev/s in a uniform magnetic field of 0.500 T?



Solution:

(a) It should be emphasized that the result, given in terms of $\sin(2\pi ft)$, could as easily be given in terms of $\cos(2\pi ft)$ or even $\cos(2\pi ft + \phi)$ where ϕ is a phase constant. The angular position θ of the rotating coil is measured from some reference line (or plane), and which line one chooses will affect whether the magnetic flux should be written as $BA\cos\theta$, $BA\sin\theta$ or $BA\cos(\theta + \phi)$. Here our choice is such that $\Phi_B = BA\cos\theta$. Since the coil is rotating steadily, θ increases linearly with time. Thus, $\theta = \omega t$ (equivalent to $\theta = 2\pi ft$) if θ is understood to be in radians (and ω would be the angular velocity).

Since the area of the rectangular coil is $A = ab$, Faraday's law leads to

$$\begin{aligned}\mathcal{E} &= -N\frac{d(BA\cos\theta)}{dt} = -NBA\frac{d\cos(2\pi ft)}{dt} \\ &= 2\pi fNabB\sin(2\pi ft)\end{aligned}$$

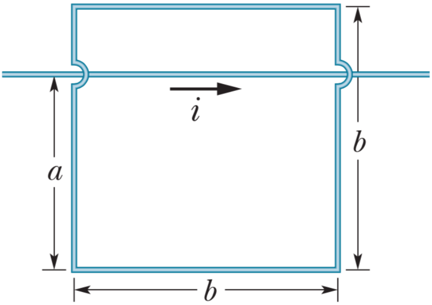
which is the desired result, shown in the problem statement. The second way this is written ($\mathcal{E}_0\sin(2\pi ft)$) is meant to emphasize that the voltage output is sinusoidal (in its time dependence) and has an amplitude of $\mathcal{E}_0 = 2\pi fNabB$.

(b) We solve

$$\mathcal{E}_0 = 2\pi f NabB = 150\ \mathrm{V}$$

when $f = 60.0$ rev/s and $B = 0.500$ T. The three unknowns are N, a, and b which occur in a product; thus, we obtain $Nab = 0.796\ \mathrm{m}^2$.

P4-2. For a wire arrangement, $a = 12.0$ cm and $b = 16.0$ cm. The current in the long straight wire is $i = 4.50t^2 - 10.0t$, where i is in amperes and t is in seconds. (a) Find the emf in the square loop at $t = 3.00$ s. (b) What is the direction of the induced current in the loop?



Solution:

(a) First, we observe that a large portion of the figure contributes flux that "cancels out." The field (due to the current in the long straight wire) through the part of the rectangle above the wire is out of the page (by the right-hand rule) and below the wire it is into the page. Thus, since the height of the part above the wire is $b - a$, then a strip below the wire (where the strip borders the long wire, and extends a distance $b - a$ away from it) has exactly the equal but opposite flux that cancels the contribution from the part above the wire.

Thus, we obtain the non-zero contributions to the flux:

$$\Phi_B = \int B dA = \int_{b-a}^{a} \left(\frac{\mu_0 i}{2\pi r} \right) (b \; dr) = \frac{\mu_0 i b}{2\pi} \ln \left(\frac{a}{b-a} \right) .$$

Faraday's law, then, (with SI units and 3 significant figures understood) leads to

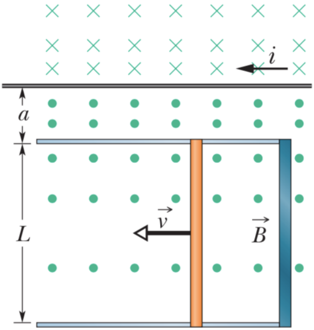
$$\begin{aligned}
\mathcal{E} &= -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \left[\frac{\mu_0 i b}{2\pi} \ln \left(\frac{a}{b-a} \right) \right] \\
&= -\frac{\mu_0 b}{2\pi} \ln \left(\frac{a}{b-a} \right) \frac{di}{dt} \\
&= -\frac{\mu_0 b}{2\pi} \ln \left(\frac{a}{b-a} \right) \frac{d}{dt} \left(\frac{9}{2} t^2 - 10t \right) \\
&= \frac{-\mu_0 b(9t-10)}{2\pi} \ln \left(\frac{a}{b-a} \right) .
\end{aligned}$$

With $a = 0.120$ m and $b = 0.160$ m, then, at $t = 3.00$ s, the magnitude of the emf induced in the rectangular loop is

$$|\mathcal{E}| = \frac{4\pi \times 10^{-7} \cdot 0.160(9 \cdot 3.00 - 10)}{2\pi} \ln \frac{0.12}{0.16 - 0.12}$$
$$= 5.98 \times 10^{-7} \text{ V}.$$

(b) We note that $di/dt > 0$ at $t = 3$ s. From Lenz's law, then, the induced emf (hence, the induced current) in the loop is counterclockwise.

P4-3. A rod of length $L = 10.0$ cm that is forced to move at constant speed $v = 5.00$ m/s along horizontal rails. The rod, rails, and connecting strip at the right form a conducting loop. The rod has resistance $0.400\ \Omega$; the rest of the loop has negligible resistance.



A current $i = 100$ A through the long straight wire at distance $a = 10.0$ mm from the loop sets up a (nonuniform) magnetic field through the loop. Find the (a) emf and (b) current induced in the loop. (c) At what rate is thermal energy generated in the rod? (d) What is the magnitude of the force that must be applied to the rod to make it move at constant speed? (e) At what rate does this force do work on the rod?

Solution:

(a) Letting x be the distance from the right end of the rails to the rod, we find an expression for the magnetic flux through the area enclosed by the rod and rails. The field is $B = \mu_0 i/2\pi r$, where r is the distance from the long straight wire. We consider an infinitesimal horizontal strip of length x and width dr, parallel to the wire and a distance r from it; it has area $A = x\ dr$ and the flux is

$$d\Phi_B = BdA = \frac{\mu_0 i}{2\pi r} x dr.$$

The total flux through the area enclosed by the rod and rails is

$$\Phi_B = \frac{\mu_0 i x}{2\pi} \int_a^{a+L} \frac{dr}{r} = \frac{\mu_0 i x}{2\pi} \ln\left(\frac{a+L}{a}\right).$$

According to Faraday's law the emf induced in the loop is

$$
\begin{aligned}
\varepsilon &= \frac{d\Phi_B}{dt} = \frac{\mu_0 i}{2\pi}\frac{dx}{dt}\ln\left(\frac{a+L}{a}\right) = \frac{\mu_0 i v}{2\pi}\ln\left(\frac{a+L}{a}\right) \\
&= \frac{(4\pi\times 10^{-7})(100)(5.00)}{2\pi}\ln\left(\frac{1.00+10.0}{1.00}\right) \\
&= 2.40\times 10^{-4}\,\mathrm{V}
\end{aligned}
$$

(b) By Ohm's law, the induced current is

$$
i_I = \varepsilon/R = (2.40\times 10^{-4})/(0.400) = 6.00\times 10^{-4}\,\mathrm{A}.
$$

Since the flux is increasing, the magnetic field produced by the induced current must be into the page in the region enclosed by the rod and rails. This means the current is clockwise.

(c) Thermal energy is being generated at the rate

$$P = i_l^2 R = (6.00 \times 10^{-4})^2 (0.400) = 1.44 \times 10^{-7}\,\mathrm{W}.$$

(d) Since the rod moves with constant velocity, the net force on it is zero. The force of the external agent must have the same magnitude as the magnetic force and must be in the opposite direction. The magnitude of the magnetic force on an infinitesimal segment of the rod, with length dr at a distance r from the long straight wire, is

$$dF_B = i_l B\, dr = (\mu_0 i_l i / 2\pi r) dr.$$

We integrate to find the magnitude of the total magnetic force on the rod:

$$
\begin{aligned}
F_B &= \frac{\mu_0 i_l i}{2\pi} \int_a^{a+L} \frac{dr}{r} = \frac{\mu_0 i_l i}{2\pi} \ln\left(\frac{a+L}{a}\right) \\
&= \frac{(4\pi \times 10^{-7})(6.00 \times 10^{-4})(100)}{2\pi} \ln\left(\frac{1.00 + 10.0}{1.00}\right) \\
&= 2.87 \times 10^{-8}\ \mathrm{N}.
\end{aligned}
$$

Since the field is out of the page and the current in the rod is upward in the diagram, the force associated with the magnetic field is toward the right. The external agent must therefore apply a force of 2.87×10^{-8} N, to the left.

(e) The external agent does work at the rate

$$P = Fv = (2.87 \times 10^{-8})(5.00) = 1.44 \times 10^{-7}\mathrm{W}.$$

This is the same as the rate at which thermal energy is generated in the rod. All the energy supplied by the agent is converted to thermal energy.

P4-4. The magnetic field of a cylindrical magnet that has a pole-face diameter of 3.3 cm can be varied sinusoidally between 29.6 T and 30.0 T at a frequency of 15 Hz. (The current in a wire wrapped around a permanent magnet is varied to give this variation in the net field.) At a radial distance of 1.6 cm, what is the amplitude of the electric field induced by the variation?

Solution: The magnetic field B can be expressed as

$$B(t) = B_0 + B_1 \sin(\omega t + \phi_0)$$

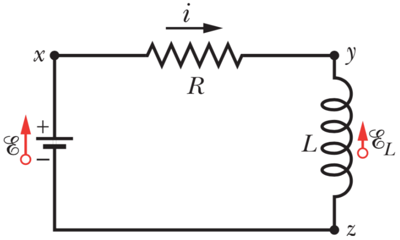
where $B_0 = (30.0 + 29.6)/2 = 29.8$ T and
$B_1 = (30.0 - 29.6)/2 = 0.200$ T.
In cylindrical magnet, $E = \frac{r}{2}\frac{dB}{dt}$, therefore

$$E = \frac{r}{2}\frac{d}{dt}(B_0 + B_1 \sin(\omega t + \phi_0)) = \frac{1}{2}B_1 \omega r \cos(\omega t + \phi_0)$$

We note that $\omega = 2\pi f$ and that the factor in front of the cosine is the maximum value of the field. Consequently,

$$\begin{aligned} E_{\max} &= \frac{1}{2}B_1(2\pi f)r = \frac{1}{2} \cdot 0.200 \cdot 2\pi \cdot 15 \cdot (1.6 \times 10^{-2}) \\ &= 0.15 \ \mathrm{V/m}. \end{aligned}$$

P4-5. For the circuit, assume that $\mathcal{E} = 10.0$ V, $R = 6.70\ \Omega$ and $L = 5.50$ H. The ideal battery is connected at time $t = 0$. (a) How much energy is delivered by the battery during the first 2.00 s? (b) How much of this energy is stored in the magnetic field of the inductor? (c) How much of this energy is dissipated in the resistor?



Solution:

(a) The energy delivered by the battery is the integral

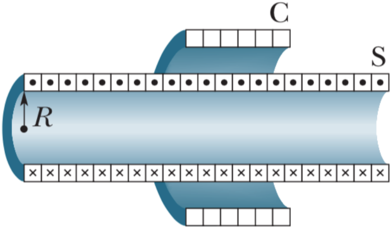
$$\begin{aligned}\int_0^t P_{\text{battery}} dt &= \int_0^t \frac{\varepsilon^2}{R}(1 - e^{-Rt/L})dt \\ &= \frac{\varepsilon^2}{R}\left[t + \frac{L}{R}(e^{-Rt/L} - 1)\right] \\ &= \frac{10.0^2}{6.70}\left[2.00 + \frac{5.50 \cdot (e^{-6.70 \cdot 2.00/5.50} - 1)}{6.70}\right] \\ &= 18.7\text{J}.\end{aligned}$$

(b) The energy stored in the magnetic field is

$$
\begin{aligned}
U_B &= \frac{1}{2}Li^2(t) = \frac{1}{2}L\left(\frac{\varepsilon}{R}\right)^2(1-e^{-Rt/L})^2 \\
&= \frac{1}{2}(5.50)\left(\frac{10.0}{6.70}\right)^2\left[1-e^{-6.70\cdot 2.00/5.50}\right]^2 \\
&= 5.10\ \mathrm{J}.
\end{aligned}
$$

(c) The difference of the previous two results gives the amount "lost" in the resistor: $18.7 - 5.10 = 13.6$ J.

P4-6. A coil C of N turns is placed around a long solenoid S of radius R and n turns per unit length. (a) Show that the mutual inductance for the coil – solenoid combination is given by $M = \mu_0 \pi R^2 nN$. (b) Explain why M does not depend on the shape, size, or possible lack of close packing of the coil.



Solution: (a) The coil-solenoid mutual inductance is

$$M_{cs} = \frac{N\Phi_{cs}}{i_s} = \frac{N(\mu_0 i_s n \pi R^2)}{i_s} = \mu_0 \pi R^2 n N.$$

(b) As long as the magnetic field of the solenoid is entirely contained within the cross section of the coil we have $\Phi_{sc} = B_s A_s = B_s \pi R^2$, regardless of the shape, size, or possible lack of close-packing of the coil.

P4-7. In an oscillating series RLC circuit, show that $\Delta U/U$, the fraction of the energy lost per cycle of oscillation, is given to a close approximation by $2\pi R/\omega L$. The quantity $\omega L/R$ is often called the Q of the circuit (for quality). A high-Q circuit has low resistance and a low fractional energy loss ($= 2\pi/Q$) per cycle.

Solution: Let t be a time at which the capacitor is fully charged in some cycle and let q_{max1} be the charge on the capacitor then. The energy in the capacitor at that time is

$$U(t) = \frac{q_{\mathrm{max1}}^2}{2C} = \frac{Q^2}{2C} e^{-Rt/L}$$

where

$$q_{\mathrm{max1}} = Qe^{-Rt/2L}$$

One period later the charge on the fully charged capacitor is

$$q_{\mathrm{max2}} = Qe^{-R(t+T)/2L}$$

where $T = \frac{2\pi}{\omega'}$, and the energy is

$$U(t+T) = \frac{q_{\mathrm{max2}}^2}{2C} = \frac{Q^2}{2C} e^{-R(t+T)/L}.$$

The fractional loss in energy is

$$\frac{|\Delta U|}{U} = \frac{U(t) - U(t+T)}{U(t)} = 1 - e^{-RT/L}.$$

Assuming that RT/L is very small compared to 1 (which would be the case if the resistance is small), we expand the exponential. The first few terms are:

$$e^{-RT/L} \approx 1 - \frac{RT}{L} + \frac{R^2T^2}{2L^2} + \cdots .$$

If we approximate $\omega \approx \omega'$, then we can write T as $2\pi/\omega$. As a result, we obtain

$$\frac{|\Delta U|}{U} \approx 1 - \left(1 - \frac{RT}{L} + \cdots\right) \approx \frac{RT}{L} = \frac{2\pi R}{\omega L}.$$

P4-8. Assume that an electron of mass m and charge magnitude e moves in a circular orbit of radius r about a nucleus. A uniform magnetic field $\vec{B}$ is then established perpendicular to the plane of the orbit. Assuming also that the radius of the orbit does not change and that the change in the speed of the electron due to field $\vec{B}$ is small, find an expression for the change in the orbital magnetic dipole moment of the electron due to the field.

Solution: An electric field with circular field lines is induced as the magnetic field is turned on. Suppose the magnetic field increases linearly from zero to B in time t. The magnitude of the electric field at the orbit is given by

$$E = \frac{r}{2}\frac{dB}{dt} = \frac{r}{2}\frac{B}{t},$$

where r is the radius of the orbit. The induced electric field is tangent to the orbit and changes the speed of the electron, the change in speed being given by

$$\Delta v = at = \frac{eE}{m_e}t = \frac{erB}{2m_e}.$$

The average current associated with the circulating electron is $i = ev/2\pi r$ and the dipole moment is

$$\mu = Ai = (\pi r^2)(\frac{ev}{2\pi r}) = \frac{1}{2}evr.$$

The change in the dipole moment is

$$\Delta\mu = \frac{1}{2}er\Delta v = \frac{1}{2}er\frac{erB}{2m_e} = \frac{e^2r^2B}{4m_e}.$$