1. **A point charge in front of a conducting sphere (25 points).**

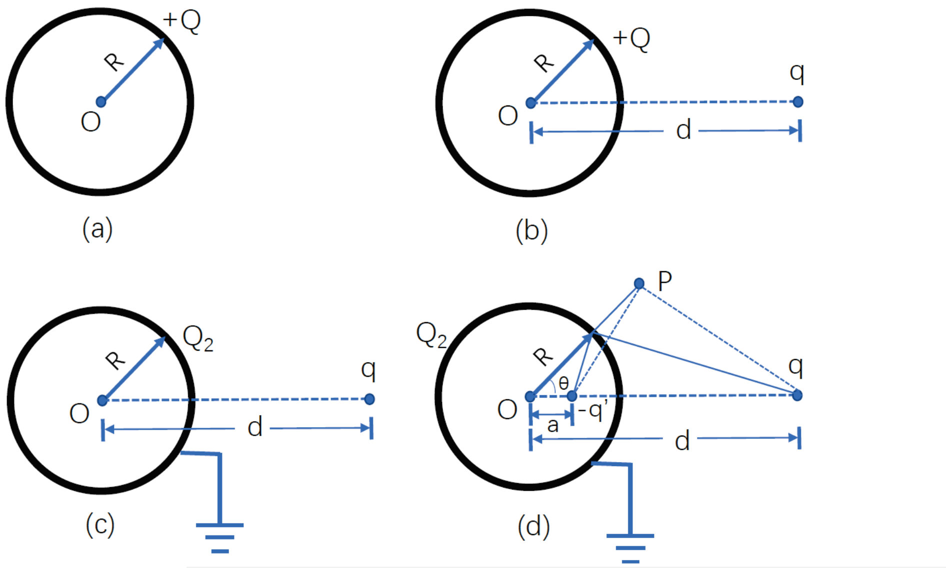

Figure 1: A point charge in front of a conducting sphere.

(a) A conducting sphere has a radius of R, and charges +Q uniformly distributed on the surface. Evaluate the electric field E_1 and potential V_1 inside and outside the sphere;

Solution:

Applying Gauss' Law:

Case 1: Inside the sphere:

$$\oint \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E = \frac{1}{\epsilon_0} \sum q = 0$$

thus

$$\mathbf{E} = 0$$

and

$$V_1 = V_R = \frac{Q}{4\pi\epsilon_0 R}$$

Case 2: Outside the sphere:

$$\oint \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E = \frac{1}{\epsilon_0} \sum q = \frac{Q}{\epsilon_0}$$

thus

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

and

$$V_1 = \int_r^\infty \mathbf{E} \cdot d\mathbf{r} = \frac{Q}{4\pi\epsilon_0 r}$$

(b) A point charge +q is newly located at a distance d from the center of the conducting sphere ($d > R$), which will cause a redistribution of the charges on the surface of the conducting sphere. Evaluate the electric field E_2 and potential V_2 at the center of the sphere;

Solution:

Inside the sphere, $\mathbf{E}_2 = 0$, thus V_2 is the same everywhere. Therefore

$$V_2 = \int \frac{dQ}{4\pi\epsilon_0 R} + \frac{q}{4\pi\epsilon_0 d} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{R} + \frac{q}{d}\right)$$

(c) Now the conducting sphere is grounded and some charges flow into the ground with residual unknown charges of Q_2. Evaluate the total amount of charges flowing from the sphere into the ground $Q - Q_2$;

Solution:

$$V_3 = 0 = \int \frac{dq'}{4\pi\epsilon_0 R} + \frac{q}{4\pi\epsilon_0 d} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_2}{R} + \frac{q}{d}\right)$$

Thus

$$Q_2 = -\frac{qR}{d}$$

Therefore

$$Q - Q_2 = Q + \frac{qR}{d}$$

(d) With the conducting sphere grounded, the charges Q_2 are not uniformly distributed on the surface. Applying the mirror image method, evaluate the electric potential V_3 outside the conducting sphere at point P.
Hint: Assume a negative point-charge $-q'$ located in the line of OD with a distance of a from O, it should satisfy the combined electric potential from q and $-q'$ on the sphere surface should be zero.

Solution: In the mirror image method, the boundary condition $V_R = 0$ should be satisfied. Therefore

$$\begin{aligned}
V(r = R, \theta) &= \frac{q}{4\pi\epsilon_0 r_2} + \frac{-q'}{4\pi\epsilon_0 r_1} \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{d^2 + R^2 - 2dR\cos\theta}} - \frac{q'}{\sqrt{a^2 + R^2 - 2aR\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q/\sqrt{dR}}{\sqrt{d/R + R/d - 2\cos\theta}} - \frac{q'/\sqrt{aR}}{\sqrt{a/R + R/a - 2\cos\theta}}\right] \\
&= 0
\end{aligned}$$

This requires:

$$\frac{q}{\sqrt{dR}} = \frac{q'}{\sqrt{aR}}$$

and

$$\frac{d}{R} = \frac{R}{a}$$

Thus

$$q' = \frac{R}{d}q$$

and

$$a = \frac{R^2}{d}$$

So

$$V_3(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{d^2+r^2-2rd\cos\theta}} - \frac{qR/d}{\sqrt{R^4/d^2+r^2-2R^2r\cos\theta/d}}\right]$$

<u>Note:</u> If you calculate in cartesian coordinate system, it is also fine.

2. **Lossy Spherical Capacitor (25 points).** A spherical capacitor has internal radius a and external radius b. Inside the capacitor, the space is filled with a lossy dielectric medium of relative dielectric permittivity ϵ_r and conductivity σ. At time $t = 0$, the charge of the capacitor is Q_0.

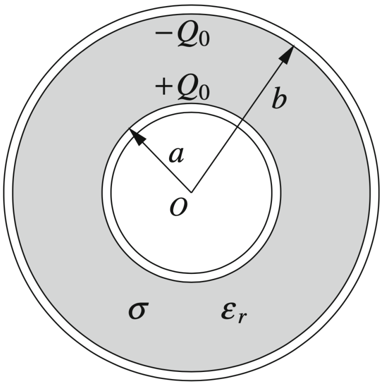

Figure 2: Spherical Capacitor

(a) Evaluate the capacitance C, and the resistance R between the two plates.

Solution:

Assuming the capacitor carries charge q. Using Gaussian's Law, it's fairly easy to determine that:

$$\oint \mathbf{D} \cdot d\mathbf{S} = q$$

yielding $\mathbf{D} = \frac{1}{4\pi}\frac{q}{r^2}\hat{r}$, thus

$$\mathbf{E} = \frac{1}{4\pi\epsilon_r\epsilon_0}\frac{q}{r^2}\hat{r}$$

Therefore the voltage drop is:

$$U = \int_b^a \mathbf{E} \cdot d\mathbf{r} = \frac{1}{4\pi\epsilon_r\epsilon_0}q\left(\frac{1}{a} - \frac{1}{b}\right)$$

This leads to

$$C = \frac{q}{U} = 4\pi\epsilon_r\epsilon_0\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$$

For resistivity, we can regard it composed of slices of spherical shells in serial. Each slice of spherical shell contributes:

$$dR = \rho\frac{dr}{S(r)} = \frac{1}{\sigma}\frac{dr}{4\pi r^2}$$

Thus

$$R = \int_a^b dR = \int_a^b \frac{1}{\sigma}\frac{dr}{4\pi r^2} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

(b) Evaluate the time-dependent charge $Q(t)$ on the plate.

Solution:

At any moment t, assuming there is charge q carried by the capacitor, and an current i between the plates. Using energy conservation, we have

$$\frac{d}{dt}\left(\frac{q^2}{2C}\right) + i^2 R = 0$$

Using $i = \frac{dq}{dt}$, we have

$$\frac{q}{C} + R\frac{dq}{dt} = 0$$

Thus, the solution is

$$Q(t) = q = Q_0 e^{-t/\tau}$$

with $\tau = RC = \frac{\epsilon_r \epsilon_0}{\sigma}$.

(c) Evaluate the power dissipated by Joule heating inside the capacitor.

Solution:

Using above result, $i = \frac{dq}{dt} = -\frac{Q_0}{\tau} e^{-t/\tau}$ Thus, the disipatted *power* is

$$P = i^2 R = \frac{Q_0^2}{\tau^2} R e^{-2t/\tau}$$

with $\tau = RC = \frac{\epsilon_r \epsilon_0}{\sigma}$.

Note: Power, not the total energy.

3. **Magnetic field of a rotating conducting sphere with charge Q (25 points).**

A conducting sphere has a radius of R, and a total charge $+Q$ uniformly distributed on the surface. The sphere rotates around the z-axis through its center with angular velocity ω and suppose that the charge distribution does not change. In such a case, the current is circulating around the z-axis, yielding a magnetic field.

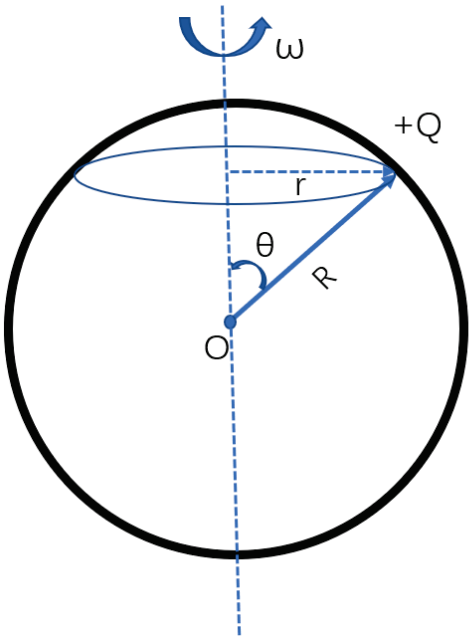

Figure 3: A rotating conducting sphere with charge Q.

(a) Calculate the charge density $\sigma(\theta) = \frac{dq(\theta)}{d\theta}$, signaling the amount of charges for the ring between θ and $\theta + d\theta$, and current density $J(\theta) = \frac{\sigma(\theta)}{\Delta T}$, where θ is the angle from its north pole and $\Delta T = \frac{2\pi}{\omega}$;

Solution: The amount of charge on the ring between θ and $\theta + d\theta$:

$$dq = \int_0^{2\pi} \sigma R^2 \sin\theta d\theta d\phi = \frac{Q}{4\pi R^2} 2\pi R^2 \sin\theta d\theta = \frac{Q \sin\theta d\theta}{2}$$

Thus

$$\sigma(\theta) = \frac{dq}{d\theta} = \frac{Q\sin\theta}{2}$$

and

$$J(\theta) = \frac{\sigma(\theta)}{\Delta T} = \frac{Q\sin\theta/2}{2\pi/\omega} = \frac{Q\omega\sin\theta}{4\pi}$$

(b) Calculate the magnetic dipole moment μ for the rotating sphere, where the magnetic dipole moment $d\mu = J(\theta)A(\theta)d\theta$ for the ring between θ and $\theta + d\theta$ and $A(\theta)$ is the area of the ring;

Solution:

$$\mu = \int d\mu = \int_0^\pi J(\theta)A(\theta)d\theta = \int_0^\pi \frac{Q\omega \sin\theta}{4\pi}\pi R^2 \sin^2\theta d\theta = \frac{Q\omega R^2}{3}$$

(c) Applying the Biot-Savart law to evaluate the magnetic field B at the center O of the sphere.

Solution:

From the Biot-Savart's Law, the field at the center due to part of the ring between θ and $\theta + d\theta$ is:

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{J(\theta)d\theta d\mathbf{l} \times \hat{R}}{R^2}$$

where $d\mathbf{l}$ is a slice of the ring with azimuthal angle between ϕ and $\phi + d\phi$. $\mathbf{R}$ is the vector from center O to the slice of ring. Thus $d\mathbf{l} \perp \hat{R}$. Due to the rotational symmetry, the effective field is along the z-axis.

$$B = \int dB \sin\theta = \int_0^\pi \frac{\mu_0}{4\pi}\frac{J(\theta)d\theta 2\pi R\sin\theta}{R^2}\sin\theta = \frac{\mu_0}{4\pi}\int_0^\pi d\theta \frac{Q\omega \sin^3\theta}{2R} = \frac{\mu_0}{6\pi}\frac{Q\omega}{R} = \frac{\mu_0\mu}{2\pi R^3}$$

4. **Faraday Disk (25 points).**

A perfect conducting disk, of radius a and thickness $h \ll a$, rotates at constant angular velocity ω, in the presence of a uniform and constant magnetic field $\mathbf{B}$ parallel to ω.

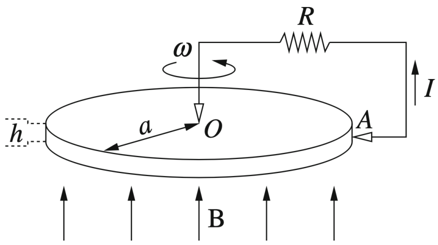

Figure 4: Faraday Disk

(a) Evaluate the electric field $\mathbf{E}$ in the disk in steady state conditions (the circuit is disconnected in this case), as well as the corresponding potential drop between the center and the boundary of the disk.

Solution: In steady state, for any charge on the disk, the Lorentz force and electronic force must cancel each other, this leads to:

$$q\mathbf{E} = q\mathbf{v} \times \mathbf{B}$$

For a point on the disk, located r away from the center O, its velocity is

$$v = \omega r$$

its direction is tangential to the circle, which is perpendicular to $\mathbf{B}$. Therefore, the electric field $\mathbf{E} = -B\omega r\hat{r}$. $-\hat{r}$ means its direction is radial but pointing to the center.

Note: Please be noted that $\mathbf{E}$ is not uniform!!!

Therefore, the potential drop is

$$V = \int_0^a \mathbf{E} \cdot d\mathbf{r} = \frac{1}{2}B\omega a^2$$

Note: You can also construct a closed loop and use Faraday's law to get the potential drop V. But you have to be very careful to get $\mathbf{E}$ in this case.

(b) Now consider a closed circuit by connecting the center of the disk to a point of the circumference by brush contacts, as shown in the figure. Let R be the total resistance of the resulting circuit. Calculate the external torque needed to keep the disk in rotation at constant angular speed.

Solution:

To maintain the constant angular speed, the torque power has to compensate for the loss caused by Ohm's heat, which is:

$$P = i^2 R = \frac{V^2}{R} = \frac{\omega^2 B^2 a^4}{4R}$$

The torque power is

$$P = \tau\omega$$

thus

$$\tau = \frac{B^2 a^4 \omega}{4R}$$

Note: It is not advisable to directly calculate force using $i d\mathbf{l} \times \mathbf{B}$ and integrate, since it involves the assumption of current distribution.

(c) If the torque is removed at $t = 0$, when the disk is rotating at ω_0, evaluate the time-dependent angular speed $\omega(t)$.

Solution:

Using energy conservation, we have:

$$\frac{d}{dt}\left(\frac{1}{2} I \omega^2\right) + i^2 R = 0$$

Thus

$$\frac{d\omega}{dt} + \frac{B^2 a^4}{4IR}\omega = 0$$

Its solution is

$$\omega = \omega_0 e^{-t/\tau}$$

with

$$\tau = \frac{4IR}{B^2 a^4}$$

Note: Using $I\alpha = \tau$ is also fine.