The Electrical Properties of Conductors

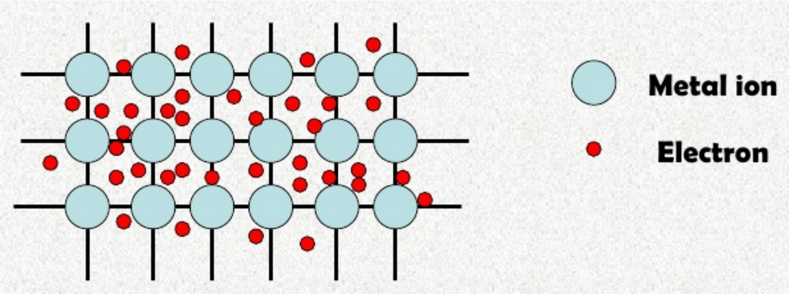
Xin Lu/Gentaro Watanabe

Lecture 5

An Analogy

- The electrical difference between a good conductor and a good insulator is as vast as the mechanical difference between a liquid and a solid.
- Both properties depend on the *mobility* of microscopic particles.
  - Electrical case: electrons or ions.
  - Mechanical case: atoms or molecules.
- When atoms of a conductor like copper come together to form the solid, some of their electrons become free to wander about within the solid, leaving behind positively charged atoms (positive ions).

The Structure of Metals



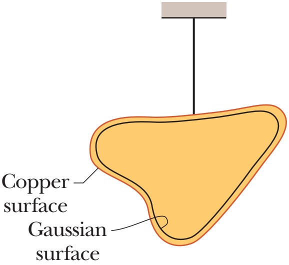
- The structure of a metal consists of a lattice of bound metal cations with a sea of electrons. The electrons are not bound and can move throughout the structure – hence metals are good conductors of electricity.

Outline

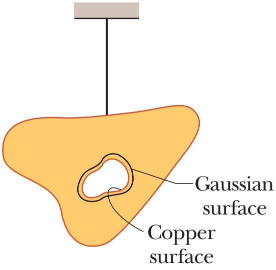
- Electrostatic Properties of Conductors
  - Electric Charge
  - Electric Field
  - Electric Potential
- The Method of Images

A Charged Isolated Conductor

- Consider an isolated conductor with excess charge. *In electrostatic equilibrium, the electric field $\vec{E}$ inside the isolated conductor must be zero.* Otherwise, the field would exert forces on the conduction electrons, and thus generate perpetual current in an isolated conductor.
- Gauss' law then tells us that the net charge inside the Gaussian surface must also be zero; the excess charge must lie on the actual surface of the conductor.

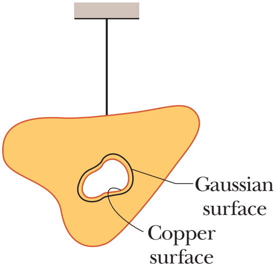


- For the same conductor with a cavity, is there excess charge on the cavity wall?
- We draw a Gaussian surface surrounding the cavity, close to its surface but inside the conducting body.



- Because $\vec{E} = 0$ inside the conductor, there can be no flux through this new Gaussian surface, hence no net charge inside by Gauss' law. Therefore, there is no net charge on the cavity walls; all the excess charge remains on the outer surface of the conductor.

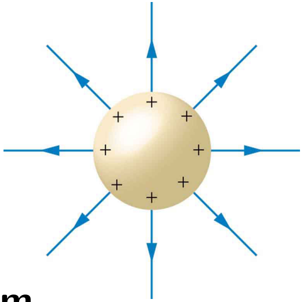
- Is the electric field (zero inside the conductor) set up by the charge on the surface, or by the conductor itself?
- The cavity can be enlarged until it consumes the entire conductor, leaving only the charges. The electric field would not change at all; so the electric field is set up by the charges and not by the conductor.



- This finishes our argument that excess charge on an isolated conductor moves entirely to the conductor's surface.

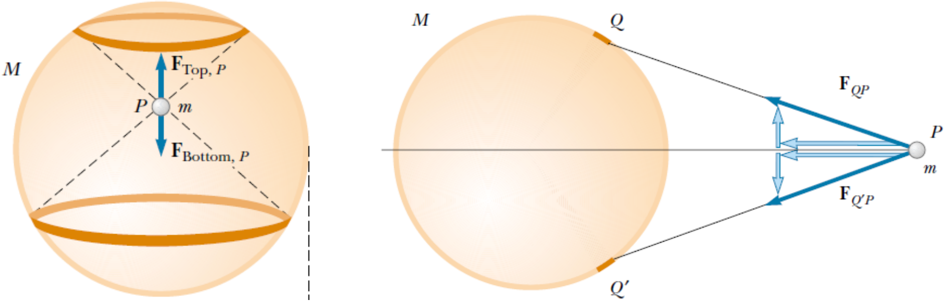
Electric Field Outside Spherical Conductors

- If the conductor is spherical, the system has spherical symmetry. Therefore, *charge distributes itself uniformly*.
- If we enclose the conductor in a concentric Gaussian sphere, Gauss' law tells us that the electric field outside the surface of the conductor looks as if all the excess charge on the shell were concentrated at its center.



- This is also known as the **shell theorem**.

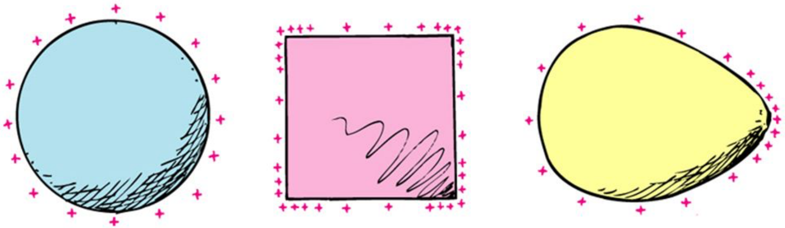
Comparison with the Gravitational Force



- **Case 1**. If a particle of mass m is located inside a spherical shell of mass M, the gravitational force acting on it is zero.
- **Case 2**. If the particle is located outside the shell, the shell attracts the particle as though the shell's mass were concentrated at its center.

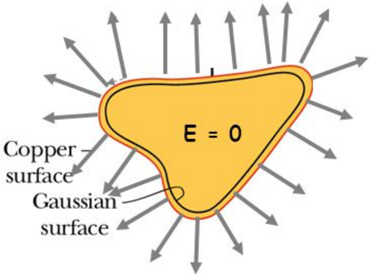
Electric Field Outside Isolated Conductors

- The surface charge density σ varies, however, over the surface of any nonspherical conductor.
- Generally, this variation makes the determination of the electric field set up by the surface charges very difficult.



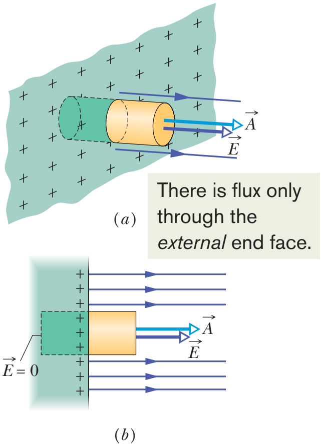
- Nevertheless, the electric field just outside the surface of a conductor can be determined using Gauss' law.

- What is the direction of $\vec{E}$ just outside the conductor?
- The electric field $\vec{E}$ at and just outside the conductor's surface must also be perpendicular to that surface. Why?
    - If it were not, then it would cause surface charge to move.
    - But, there must be no current in *electrostatic equilibrium*.



- Consider a section of the surface that is small enough to be treated as being flat.

- Imagine a tiny cylindrical Gaussian surface to be partially embedded in the section: One end cap is fully inside the conductor, the other is fully outside, and the cylinder is perpendicular to the conductor's surface.

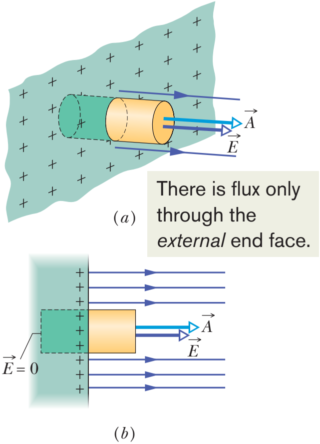


- The only flux through the Gaussian surface is that through the external end cap. Gauss' law becomes

$$\epsilon_0 EA = \sigma A,$$

from which we obtain

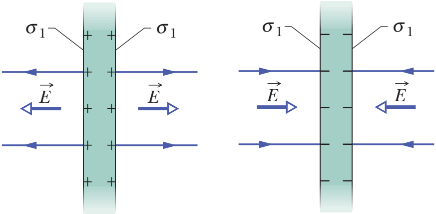
$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}.$$

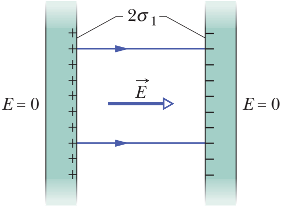


- Recall that for a thin, infinite, nonconducting sheet with a uniform surface charge density σ, we have $E = \sigma/(2\epsilon_0)$.

Parallel Plates

- For a single plate, all the excess charge spreads out on the two faces of the plate with a uniform surface charge density in the absence of an external field.

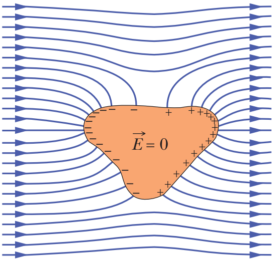




- For two parallel plates, the excess charge on one plate attracts the excess charge on the other plate, and *all the excess charge moves onto the inner faces of the plates*.

Electric Potential of Isolated Conductors

- An excess charge placed on an isolated conductor will distribute itself on the surface of that conductor so that *all points of the conductor – whether on the surface or inside – come to the same potential*.



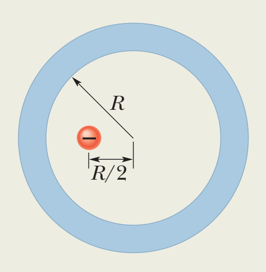
- This is true even if the conductor has an internal cavity and even if that cavity contains a net charge.

- In a thunderstorm, it is wise to enclose yourself in a cavity inside a conducting shell, where the electric field is guaranteed to be zero. A car (*not* convertible or plastic) is almost ideal.

Figure 1: Not wise staying inside a LEGO® car during a thunderstorm.

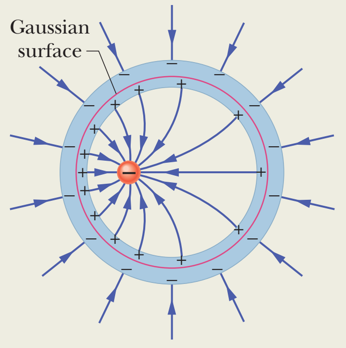
Charge Inside a Spherical Metal Shell

- A particle with a charge of $-Q$ is located at a distance $R/2$ from the center of an electrically neutral, spherical metal shell of inner radius R.

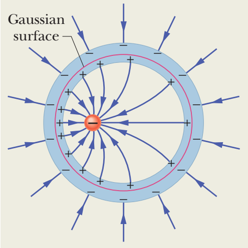


- What are the (induced) charges on its inner and outer surfaces?
- Are those charges uniformly distributed?
- What is the field pattern inside and outside the shell?

- Consider a spherical Gaussian surface within the metal, just outside the inner wall of the shell.
- The electric field must be zero on the Gaussian surface inside the metal. This means that the electric flux through the Gaussian surface must also be zero.
- Gauss' law then tells us that the net charge enclosed by the Gaussian surface must be zero.



- Therefore, a total charge Q must lie on the inner wall of the shell in order that the net enclosed charge be zero.
- Since the particle is off-center, the distribution of positive charge is skewed. But it cannot produce an electric field in the shell to affect the distribution of charge on the outer wall.



- Because the shell is electrically neutral, a total charge $-Q$ leave the inner wall and move to the outer wall. There they must spread out uniformly. In fact, this would be true no matter where inside the shell the particle happened to be located.

Can We Quantify the Charge Distribution?

- Suppose a point charge q is held a distance d above an infinite grounded conducting plane.
- How can we possibly calculate the potential, when we do not know how induced charge is distributed?

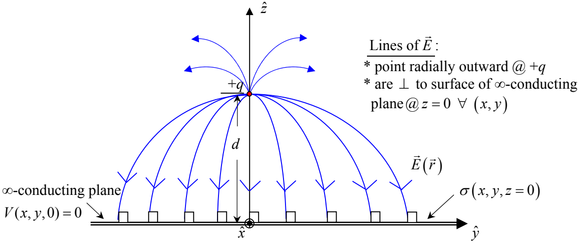
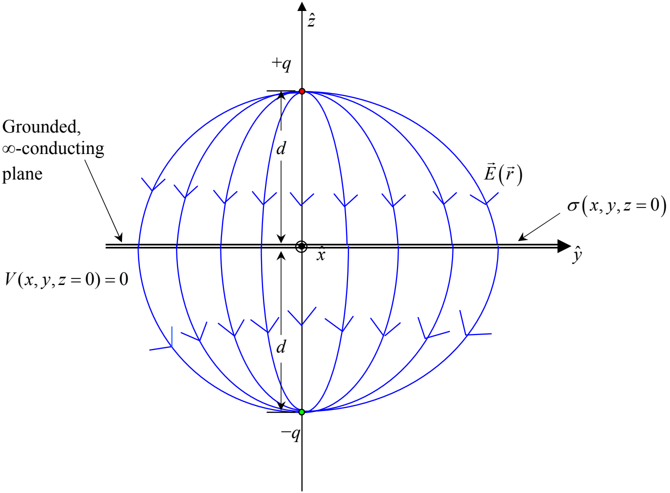


Figure 2: The total potential is due in part to q, and in part to charge $\sigma(x, y, z = 0)$ induced on the nearby surface of the conductor.

- *Hint: Let us look at an apparently different problem.*



The Method of Images

- Evidently, the second problem happens to produce exactly the same potential as the first problem, in the upper region.
- It turns out that there is a uniqueness theorem that guarantees the correctness of this solution.
- This is an example of the **method of images**. The auxiliary charge in the lower region ($z < 0$) is the image of the original charge in the "mirror" that comprises the grounded conducting plane.
- Other configurations of charges and grounded conductors can be treated similarly. *Invent a set of charges inside the conductor so that the boundary condition is met.*

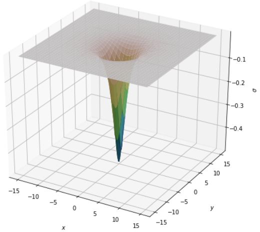
- *Solution*: We can write down the potential for a system of two point charges, $+q$ at $(0, 0, d)$ and $-q$ at $(0, 0, -d)$, without the conducting plane,

$$V = \frac{q/4\pi\epsilon_0}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q/4\pi\epsilon_0}{\sqrt{x^2 + y^2 + (z+d)^2}}.$$

- We assert this is the solution for the upper half region.
  - We can check that $V(x, y, z = 0) = 0$, and $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$.
  - At $z = 0$, $\vec{E} = -\nabla V$ is perpendicular to the xy plane, just as it would need to be with the grounded plane.
  - The only charge in the region $z > 0$ is q at $(0, 0, d)$.

Back to the Original Problem

- According to Gauss' law, the surface charge density σ induced on the conductor is (verify, please!)



$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} = -\frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}.$$

- The total charge induced on the $z = 0$ plane is $-q$. Why?
- Note that all electric field lines end at the metal surface. This is different from those of a single charge.

- Show that the energy for bringing q from infinity is

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d},$$

which is half of the energy for bringing two point charges to a distance $2d$ with no conductor. Why?

- We will discuss the energy stored in an electric field in the context of capacitors.

Summary

- Inside a conductor, $\vec{E} = 0$ and $\rho = 0$ in electrostatic equilibrium. Any net charge resides on the surface.
- Just outside a conductor, $\vec{E} = (\sigma/\epsilon_0)\hat{n}$ is perpendicular to the surface.
- All points of the conductor – whether on the surface or inside – come to the same potential.

Reading

Halliday, Resnick & Krane:

- Chapter 29: The Electrical Properties of Materials