Magnetic Field of a Current

Xin Lu/Gentaro Watanabe

Lecture 9

Outline

- Magnetic Field Due to a Current (vs. Coulomb's law)
  - A Long Straight Wire
  - Force Between Two Parallel Wires
- Ampere's Law (vs. Gauss' law)
  - Solenoids
  - Toroids
- Gauss' Law for Magnetism

Magnetic Field Due to a Current

- *Stationary charge* produce electric fields that are constant in time; hence the term **electrostatics**.
- *Steady currents* produce magnetic fields that are constant in time; the theory of steady current is called **magnetostatics**.
- The key difference is that whereas a charge element dq producing an electric field is a *scalar*, a current-length element $id\vec{s}$ producing a magnetic field is a *vector*, being the product of a scalar and a vector.

- Experiments found that the field $d\vec{B}$ produced at point P at distance r by a current-length element $id\vec{s}$ turns out to be an *inverse-square law*

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{id\vec{s} \times \vec{r}}{r^3},$$

where the constant $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ is called the **permeability constant**.

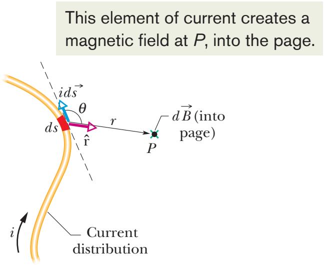


Figure 1: Illustration of the **Biot-Savart law**. *If current in a wire is either directly toward or directly away from P, the magnetic field at P from the current is zero.*

Example: A Long Straight Wire

- Before we calculate, we assert that the field strength depends only on the current i and the perpendicular distance R of the point from the wire. Why?
- We shall prove that

$$B = \frac{\mu_0 i}{2\pi R},$$

and its direction follows a right-hand rule.

The magnetic field vector at any point is tangent to a circle.

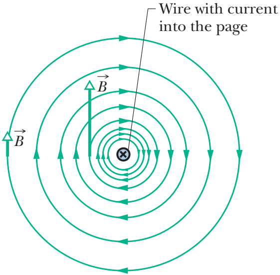
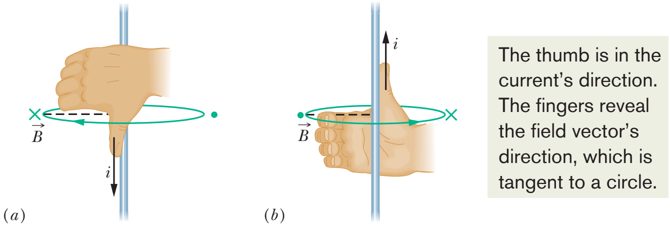
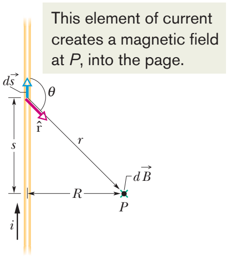


Figure 2: The magnetic field lines form concentric circles.

- *Curled–straight right-hand rule*: Grasp the element in your right hand with your extended thumb pointing in the direction of the current. Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.



- According to the Biot-Savart law,



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\vec{s}\times\vec{r}}{r^3}$$
$$= \frac{\mu_0}{4\pi}\frac{id\vec{s}\times\vec{R}}{r^3}.$$

- Integrating dB from $s=-\infty$ to ∞, we have

$$B = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{iRds}{r^3} = \frac{\mu_0 i}{4\pi R}\left[\int_{-\infty}^{\infty}\frac{R^2 ds}{r^3}\right],$$

where the integral in the bracket is dimensionless.

- Notice $\sin\theta = R/r$ and $\cos\theta = -s/r$.
- We also have $dr/ds = s/r$, so

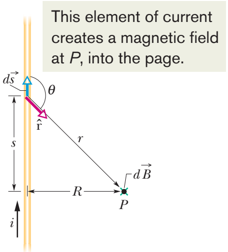
$$\cos\theta d\theta = d(\sin\theta) = -\frac{R}{r^2}\frac{dr}{ds}ds = -\frac{R}{r^2}\frac{s}{r}ds = \cos\theta\frac{Rds}{r^2}.$$

You can also obtain this by geometry (Try!). Therefore,

$$\int_{-\infty}^{\infty}\frac{R^2ds}{r^3} = \int_0^{\pi}\sin\theta d\theta = 2.$$

- Finally, we have

$$B = \frac{\mu_0 i}{2\pi R}.$$



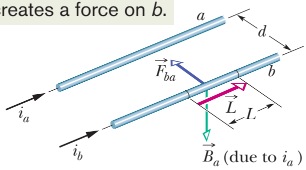
Force Between Two Parallel Wires

- The current in wire *a* produces a magnetic field

$$B_a = \frac{\mu_0 i_a}{2\pi d},$$

at wire *b*, pointing down.

The field due to *a* at the position of *b* creates a force on *b*.



- The force on a length L of wire b due to $\vec{B}_a$ is

$$F_{ba} = |i_b \vec{L} \times \vec{B}_a| = \frac{\mu_0 L i_a i_b}{2\pi d},$$

where $\vec{L}$ is the length vector of the wire.

- To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.
- Parallel currents attract each other, and antiparallel currents repel each other.
- The **ampere** is that constant current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in vacuum, would produce on each of these conductors a force of magnitude 2×10^{-7} newton per meter of wire length.

Electric Field vs Magnetic Field

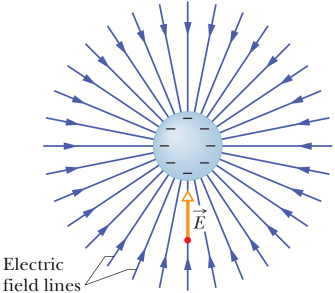
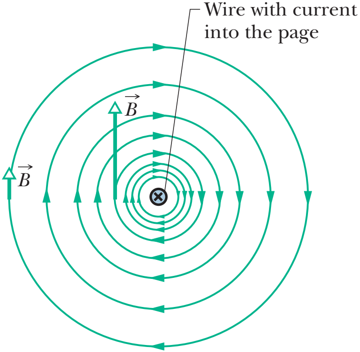


Figure 3: ▲ Electric field lines near a negatively charged sphere.
▶ Magnetic field lines form concentric circles around the wire.

The magnetic field vector at any point is tangent to a circle.



Magnetic Field Circulation

- The magnetic field that circles around a long straight wire gets weaker (precisely as $1/r$) as we get farther from the wire.
- We can define a circulation of the magnetic field as

$$\text{Circulation} = \oint \vec{B} \cdot d\vec{s},$$

for a closed path, or an **Amperian loop**.

- For the long wire the circulation is independent of the path around the wire. As we move farther from the wire, the path gets longer but the field gets weaker.

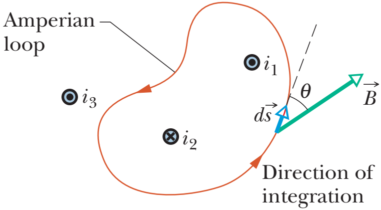
Ampere's Law

- Ampere's law relates steady electric currents to their circulating magnetic field in the generic case:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}},$$

where i_{enc} is the net current encircled by the closed loop.

Only the currents encircled by the loop are used in Ampere's law.



- Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

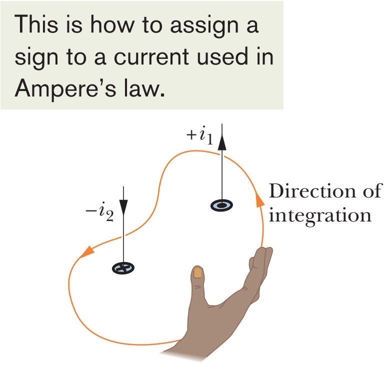


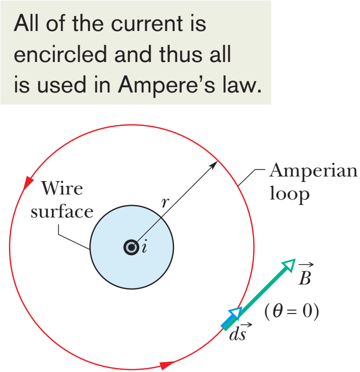
Figure 4: Net current $i_{\text{enc}} = i_1 - i_2$.

Outside a Long Straight Wire

- In many cases when we apply Ampere's law, symmetry does allow us to simplify and solve the integral, hence to find the magnetic field.
- A long straight wire has *cylindrical symmetry*; so has $\vec{B}$ about the wire.
- Ampere's law tells us

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_0 i,$$

$$\text{or, } B = \frac{\mu_0 i}{2\pi r}.$$



Inside a Long Straight Wire

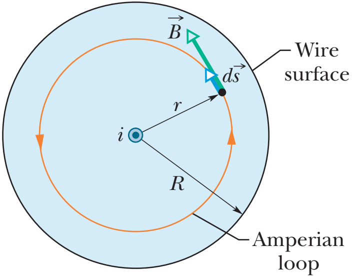
- Suppose that the current is uniformly distributed over the cross section of the wire, $\vec{B}$ produced by the current must be *cylindrically symmetrical*.
- For a concentric Amperian loop inside the wire,

$$i_{\text{enc}} = i\frac{\pi r^2}{\pi R^2}.$$

- Ampere's law now leads to

$$B = \frac{\mu_0 i_{\text{enc}}}{2\pi r} = \frac{\mu_0 i}{2\pi R^2} r.$$

Only the current encircled by the loop is used in Ampere's law.



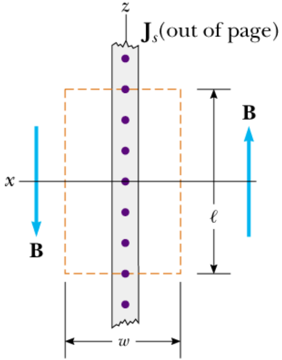
A Sheet of Moving Charge

- Consider an infinite flat sheet of current density J_s in the y-direction. Symmetry tells us that magnetic field will be in the directions indicated in the figure.
- Ampere's law can be applied to the rectangular path:

$$\oint \vec{B} \cdot d\vec{s} = 2B\ell = \mu_0(J_s\ell),$$

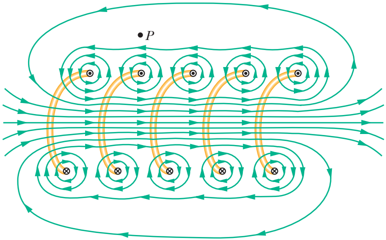
or

$$B = \frac{\mu_0 J_s}{2}.$$



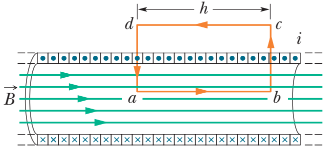
Magnetic Field of a Solenoid

- A solenoid is a long, tightly wound helical coil of wire. We assume that the length of the solenoid is much greater than the diameter.
- The solenoid's magnetic field is the vector sum of the fields produced by the individual turns (windings) that make up the solenoid.



- In the limiting case of an ideal solenoid, which is infinitely long and consists of tightly packed (close-packed) turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis. The magnetic field outside the solenoid is zero.
- The direction of the magnetic field along the solenoid axis is given by a curled-straight right-hand rule: Grasp the solenoid with your right hand so that your fingers follow the direction of the current in the windings; your extended right thumb then points in the direction of the axial magnetic field.

- Take the Amperian loop *abcda*. Applying Ampere's law to the ideal solenoid, we find



$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\mathrm{enc}}.$$

- The loop integral can be separated into four segments, and the only nonzero contribution is

$$\int_a^b \vec{B} \cdot d\vec{s} = Bh.$$

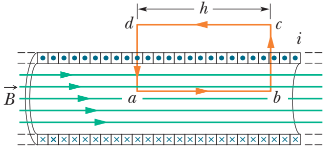
Along the other segments, $\vec{B}$ either is perpendicular to $d\vec{s}$ or is zero.

- Let n be the number of turns per unit length of the solenoid; then the loop encloses nh turns and

$$i_{\mathrm{enc}} = i(nh).$$

- Ampere's law then gives us

$$Bh = \mu_0 inh,$$

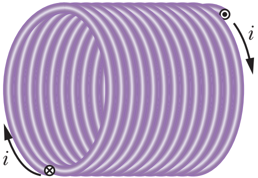


or

$$B = \mu_0 in.$$

- It holds quite well for actual solenoids if we apply it only at interior points and well away from the solenoid ends.

- The magnetic field magnitude B within a solenoid does not depend on the diameter or the length of the solenoid.
- B is uniform over the solenoidal cross section.



- A solenoid thus provides a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor provides a practical way to set up a known uniform electric field.

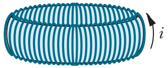
Magnetic Field of a Toroid

- A **toroid** is a solenoid that has been curved until its two ends meet, forming a hollow donut.
- Symmetry dictates that the lines of $\vec{B}$ form concentric circles inside the toroid.
- Ampere's law yields

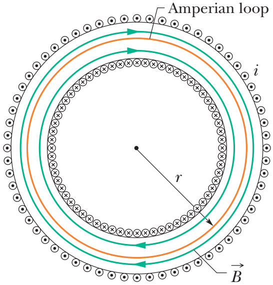
$$(B)(2\pi r) = \mu_0 i N,$$

or,

$$B = \mu_0 i \frac{N}{2\pi r}.$$



(a)



(b)

- In contrast to the situation for a solenoid, B is not constant over the cross section of a toroid.
- One can show, with Ampere's law, that $B = 0$ for points outside an ideal toroid (as if the toroid were made from an ideal solenoid).
- The direction of the magnetic field within a toroid follows from our curled – straight right-hand rule: Grasp the toroid with the fingers of your right hand curled in the direction of the current in the windings; your extended right thumb points in the direction of the magnetic field.

The Curl of $\vec{B}$

- According to the Stokes' theorem or the fundamental theorem for curls (Appendix 9B),

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{A} = \oint_P \vec{v} \cdot d\vec{s}.$$

- Apply the theorem to Ampere's law ($\vec{v} \equiv \vec{B}$), we have, for an arbitrary surface S,

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{A} = \oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} = \mu_0 \int_S \vec{J} \cdot d\vec{A},$$

where $\vec{J}$ is the current density. Hence,

$$\nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r}).$$

The Divergence of $\vec{B}$

- For volume currents, the Biot-Savart law becomes

$$\vec{B}(x,y,z) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(x',y',z') \times \vec{r}}{r^3} dx'dy'dz',$$

where the length element $id\vec{s}$ is replace by the volume element $JdV' \equiv \vec{J}(x',y',z')dx'dy'dz'$ and

$$\vec{r} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}.$$

- Applying the divergence, we obtain

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{\vec{J} \times \vec{r}}{r^3}\right) dV'.$$

- Since the divergence does not apply to J, which does not depend on (x, y, z), we can rewrite

$$\nabla \cdot \vec{B} = -\frac{\mu_0}{4\pi} \int J \cdot \left(\nabla \times \frac{\vec{r}}{r^3} \right) dV'.$$

- Notice the form $\vec{r}/r^3 = -\nabla(1/r)$ is nothing but the electric field of a point charge ($q = 4\pi\epsilon_0$); it does not *twists around*; it only *spreads out*. Its curl is zero (as known in electrostatics).
- Therefore, we conclude

$$\nabla \cdot \vec{B} = 0.$$

Gauss' Law for Magnetism

- The integral form of $\nabla \cdot \vec{B} = 0$ can be obtained by constructing a closed Gaussian surface:

$$\oint \vec{B} \cdot d\vec{A} = \int (\nabla \cdot \vec{B}) dV = 0.$$

  In the first equality we have used the fundamental theorem of divergences.
- The law asserts that the net magnetic flux Φ_B through any closed Gaussian surface is zero. This is a formal way of saying that magnetic monopoles do not exist. The simplest magnetic structure that can exist is a magnetic dipole.

Summary

- The Biot-Savart law vs Coulomb's law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i d\vec{s} \times \hat{r}}{r^2} \qquad d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq\hat{r}}{r^2}$$

- Ampere's law vs Gauss' law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}, \quad \text{or } \nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\epsilon_0}, \quad \text{or } \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

- Gauss' law for magnetism

$$\nabla \cdot \vec{B} = 0, \quad \oint \vec{B} \cdot d\vec{A} = 0$$

- Recall the corresponding pictures and derive the following formulas.

$$B = \mu_0 i \frac{1}{2\pi r} \qquad B = \mu_0 i \frac{r}{2\pi R^2}$$

$$B = \mu_0 i n \qquad B = \mu_0 i \frac{N}{2\pi r}$$

Reading

Halliday, Resnick & Krane:

- Chapter 33: The Magnetic Field of a Current

Appendix 9A: Geometrical Meaning of Curl

- The curl of a vector field is a measure of the field's tendency to circulate about a point.

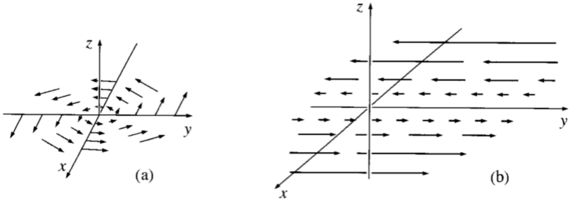
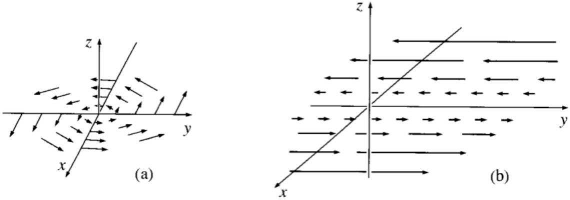


Figure 5: Functions with nonzero curl.

- A common misconception is that the curl of a vector field is non-zero wherever the field appears to curve. However, the curl depends not only on the curvature of the lines but also on the strength of the field.



- At a point with nonzero curl in a fluid, a small floating object will start to rotate.

- Vector fields with zero curl at all points are called irrotational.

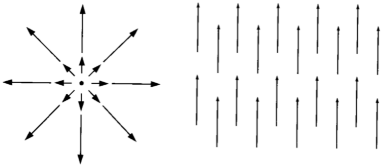


Figure 6: The diverging flow lines and the uniform flow would not cause an object to rotate, meaning that the curl is zero.

Appendix 9B: Stokes' Theorem

- **Stokes' theorem**, also known as the **fundamental theorem for curls**, states that

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{A} = \oint_P \vec{v} \cdot d\vec{s}.$$

- The curl of a vector $\vec{v}$ measures how it *curls around* the point in question. Stokes' theorem means that we can determine the *total amount of swirl* (the flux of the curl through surface S) just as well by going around the edge and finding how much the flow is following the boundary P.

- Applying Stokes' theorem to $\vec{E}$, we find that the line integral of $\vec{E}$ around any closed loop is zero, because $\nabla \times \vec{E} = 0$.
- The fact that the line integral of $\vec{E}$ is independent of path justifies why we can define electric potential V as

$$V(\vec{r}) \equiv -\int_O^{\vec{r}} \vec{E} \cdot d\vec{s},$$

  because V is then conservative.
- Here O is the reference point at which $V = 0$. The choice of O is arbitrary; changing references points only amounts to a constant in V and does not affect $\vec{E}$, or the gradient of V.