General Physics I

Lecture 9: Vector Cross Product; Coriolis Force

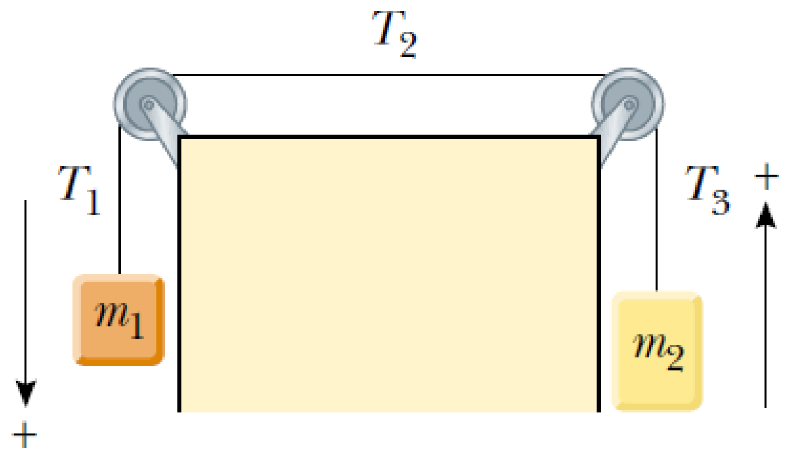
Outline

- Examples of the rotation of a rigid object about a fixed axis
  - Force/torque point of view
  - Energy point of view
- Vector cross product
  - Necessity of the new math
  - Definition and examples
  - Coriolis effect

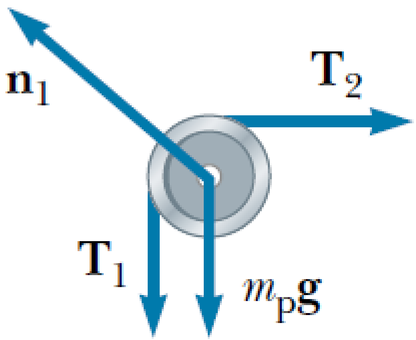
Example: Atwood's Machine

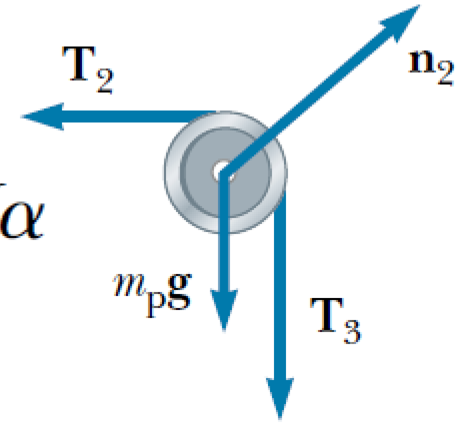
- Two blocks having masses m_1 and m_2 are connected to each other by a light cord that passes over two identical, frictionless pulleys, each having a moment of inertia I and radius R. Find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord.

(Assume no slipping between cord and pulleys.)



Example: Atwood's Machine

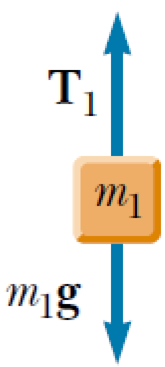




$$(T_1 - T_2)R = I\alpha$$

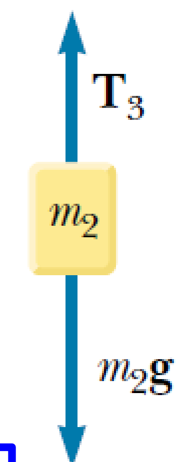
$$(T_2 - T_3)R = I\alpha$$

$$\boxed{(T_1 - T_3)R = 2I\alpha}$$



$$\boxed{\alpha = a/R}$$

$$m_1 g - T_1 = m_1 a$$

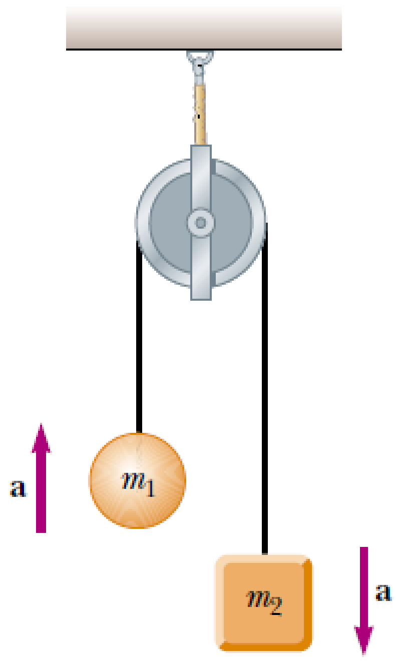


$$T_3 - m_2 g = m_2 a$$

$$\boxed{T_1 - T_3 = (m_1 - m_2)g - (m_1 + m_2)a}$$

Example: Atwood's Machine

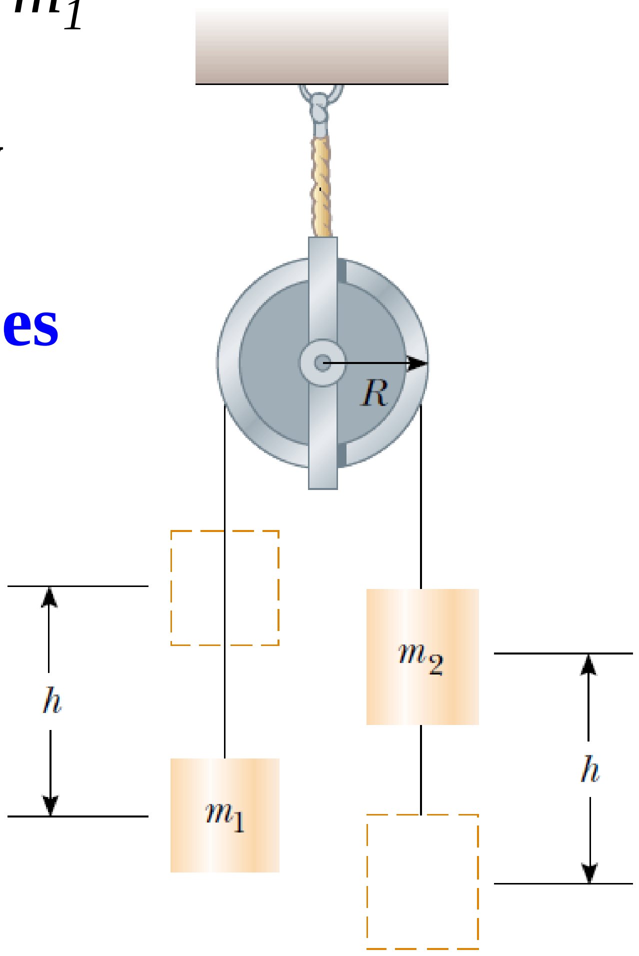
$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2\frac{I}{R^2}}$$



- **Discussion:**
  - **Equal mass: At equilibrium.**
  - **Unequal mass: Normally we assume a direction for the acceleration. If the result is negative, the real acceleration is in the opposite direction.**
  - ***I* = 0: Goes back to the same old Newton's laws.**

Example: Connected Cylinders

- Consider two cylinders having masses m_1 and m_2, where $m_1 \neq m_2$, connected by a string passing over a pulley. The pulley has a radius R and **moment of inertia** I about its axis of rotation. **The string does not slip on the pulley**, and the system is released from rest. Find the **linear speeds** of the cylinders after cylinder 2 descends through a distance h, and the **angular speed** of the pulley at this time.



Detailed Analysis

- We are now able to account for the effect of a massive pulley. Because the string does not slip, the pulley rotates.
- We neglect friction in the axle about which the pulley rotates for the following reason:
  - Because the axle's radius is small relative to that of the pulley, the frictional torque is much smaller than the torque applied by the two cylinders, provided that their masses are quite different.
- Mechanical energy is constant; hence, the increase in the system's kinetic energy (the system being the two cylinders, the pulley, and the Earth) equals the decrease in its potential energy.

Example: Connected Cylinders

$$\Delta K + \Delta U_1 + \Delta U_2 = 0$$

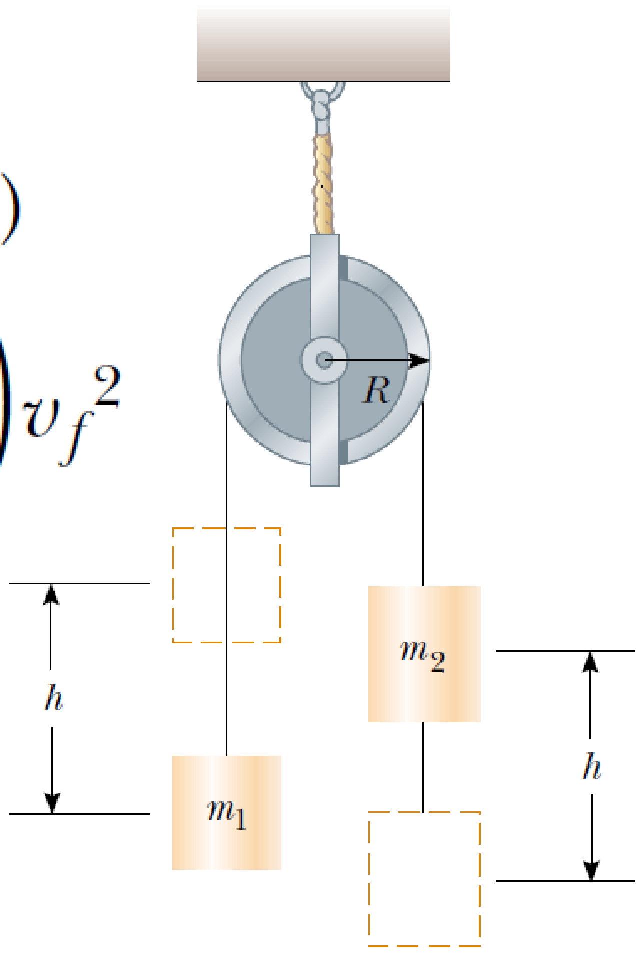
$$\Delta K = (\tfrac{1}{2}m_1 {v_f}^2 + \tfrac{1}{2}m_2 {v_f}^2 + \tfrac{1}{2}I{\omega_f}^2)$$

$v_f = R\omega_f$

$$\Delta K = \tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right){v_f}^2$$

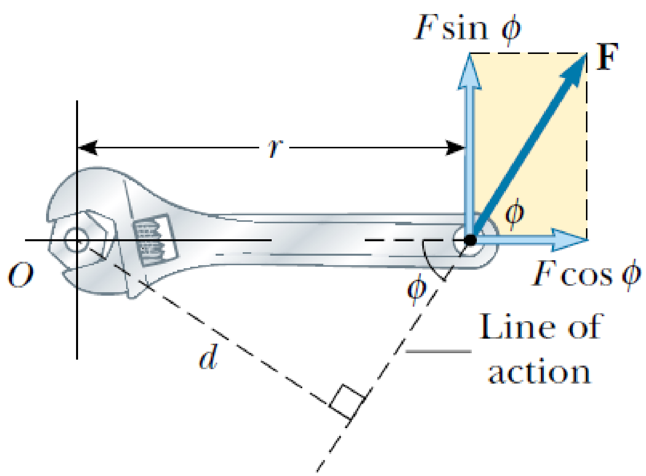
$$\Delta U_1 = m_1 g h \qquad \Delta U_2 = -m_2 g h$$

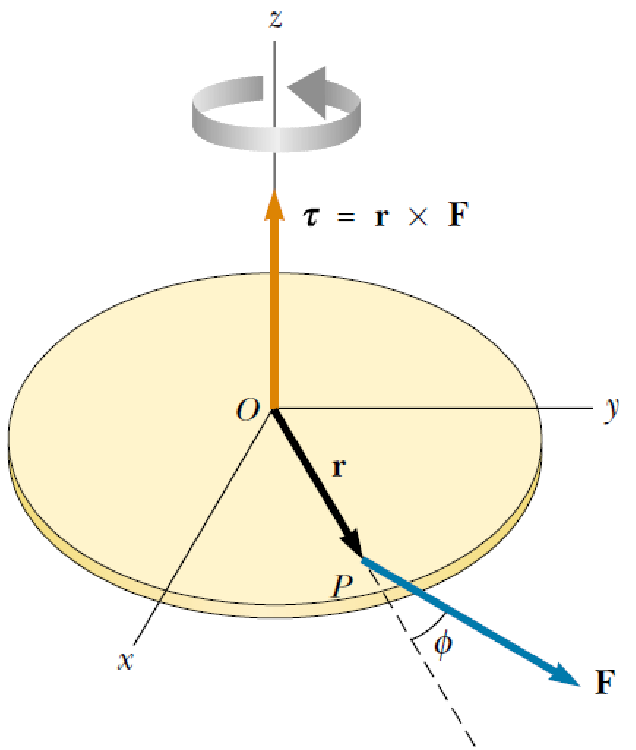
$$v_f = \left[\frac{2(m_2 - m_1)gh}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}\right]^{1/2}$$



Brainstorming I

- **How to understand** $\tau = rf \sin\phi$**?**
  - **Force is a vector.**
  - **Position is a vector.**
  - **Torque is also a vector.**

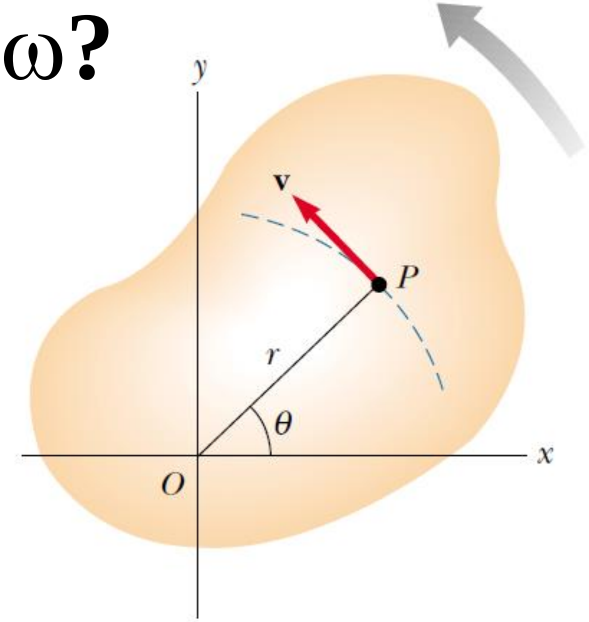




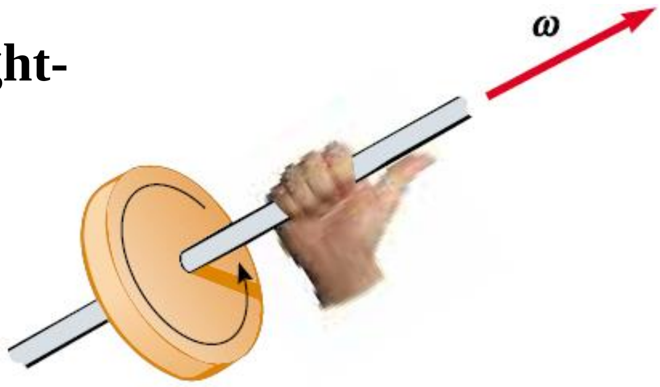
$$\tau = I\alpha$$

Brainstorming II

- **How to understand** v = rω**?**
  - **Velocity is a vector.**
  - **Position is a vector.**
  - **Angular velocity is also a vector.**



Remember right-hand rule?



$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

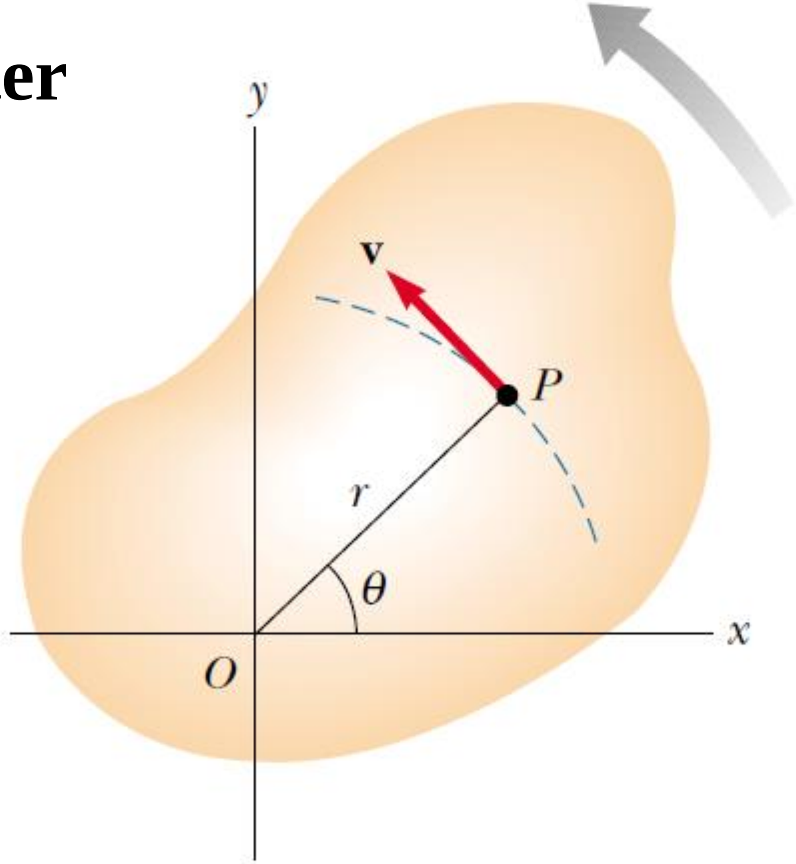
Angular vs Linear Speed

- when a rigid object rotates about a fixed axis, every particle of the object moves in a circle whose center is the axis of rotation.

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\Rightarrow \quad \boxed{v = r\omega}$$

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

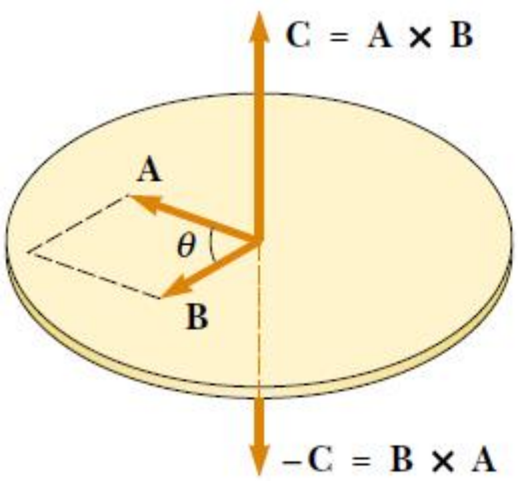


The Vector Product

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$C \equiv AB \sin\,\theta$$

The direction of C is perpendicular to the plane formed by A and B , and the best way to determine this direction is to use the right-hand rule.



Right-hand rule

Properties of Vector Product

- Unlike the scalar product, the vector product is not commutative. Instead, the order in which the two vectors are multiplied in a cross product is important:

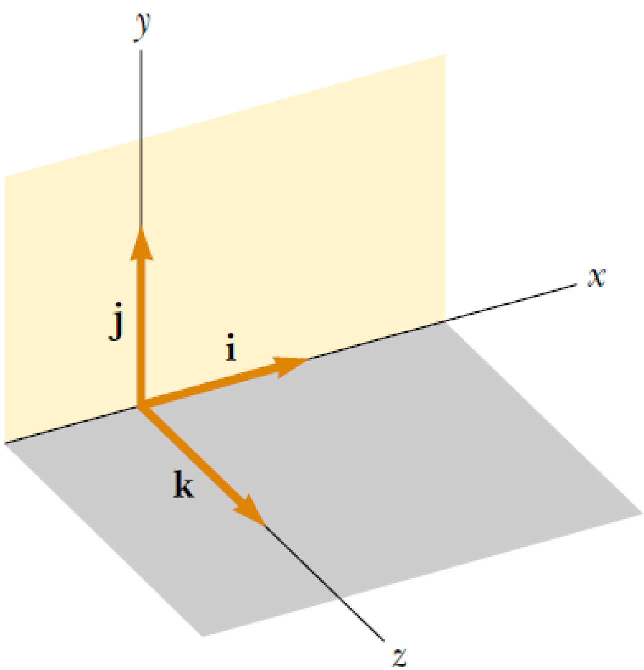
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

- The vector product obeys the distributive law:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

Orthogonal Unit Vectors

- **If A is parallel or antiparallel to B, then A × B = 0; therefore, it follows that A × A = 0.**
- **If A is perpendicular to B, then |A × B|** = *AB*



$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}$$

Generic Results

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$A_x\hat{i} \times B_y\hat{j} = A_x B_y\hat{k}$$

$$A_y\hat{j} \times B_x\hat{i} = -A_y B_x\hat{k}$$

$$\vec{C} = \vec{A} \times \vec{B} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

$$= \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \hat{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Derivative

- The derivative of the cross product with respect to some variable (such as *t*) is

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B}$$

- where it is important to preserve the multiplicative order of A and B

Acceleration: Vector Version

For rotation of a rigid object about a fixed axis

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

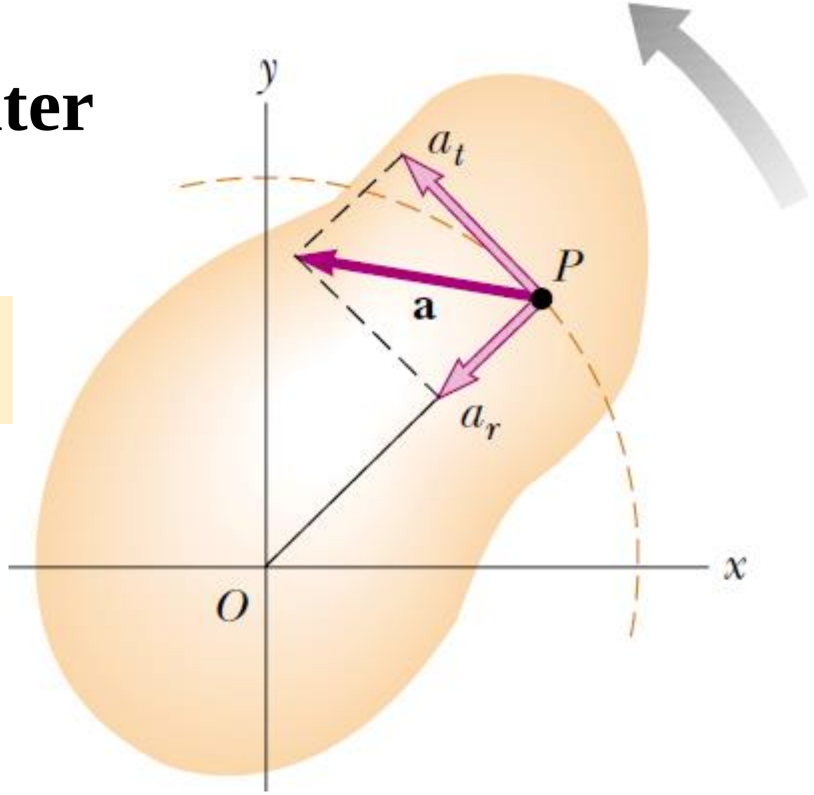
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

The vector formulas have no requirement that **r** is perpendicular to the axis of rotation.

Angular vs Linear Acceleration

- **when a rigid object rotates about a fixed axis, every particle of the object moves in a circle whose center is the axis of rotation.**



$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \Rightarrow a_t = r\alpha$$

$$a_r = \frac{v^2}{r} = r\omega^2$$

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$$

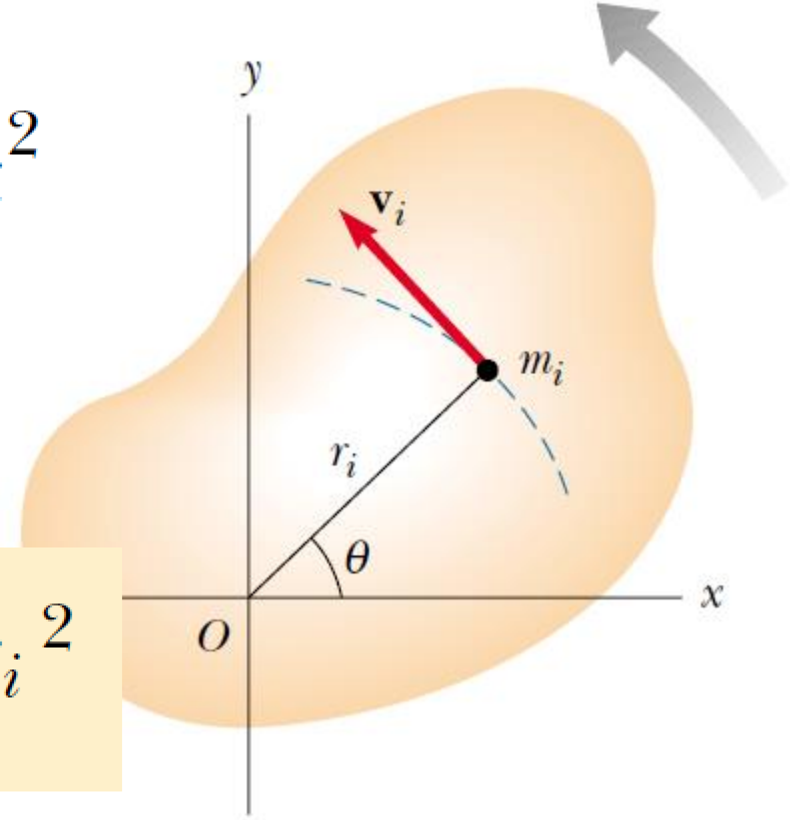
$$\Rightarrow a = \sqrt{{a_t}^2 + {a_r}^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Rotational Kinetic Energy

- **The total rotational kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles**

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} \sum_i m_i r_i^2 \omega^2$$



$$K_R = \frac{1}{2} I \omega^2$$

$$I \equiv \sum_i m_i r_i^2$$

Rotational Kinetic Energy

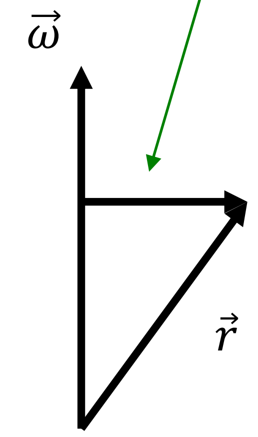
$$K = \frac{1}{2}\sum m_i\,|\vec{\omega} \times \vec{r_i}|^2$$

$$= \frac{1}{2}\sum m_i\,(\vec{\omega} \times \vec{r_i}) \cdot (\vec{\omega} \times \vec{r_i})$$

$$= \frac{1}{2}\sum m_i\left[\omega^2 r_i^2 - \omega^2(\hat{\omega} \cdot \vec{r_i})^2\right]$$

$$= \frac{1}{2}\left(\sum m_i\, r_{i,\perp}^2\right)\omega^2 = \frac{1}{2} I\omega^2$$

$$\vec{r}_\perp = \vec{r} - (\vec{r} \cdot \hat{\omega})\hat{\omega}$$

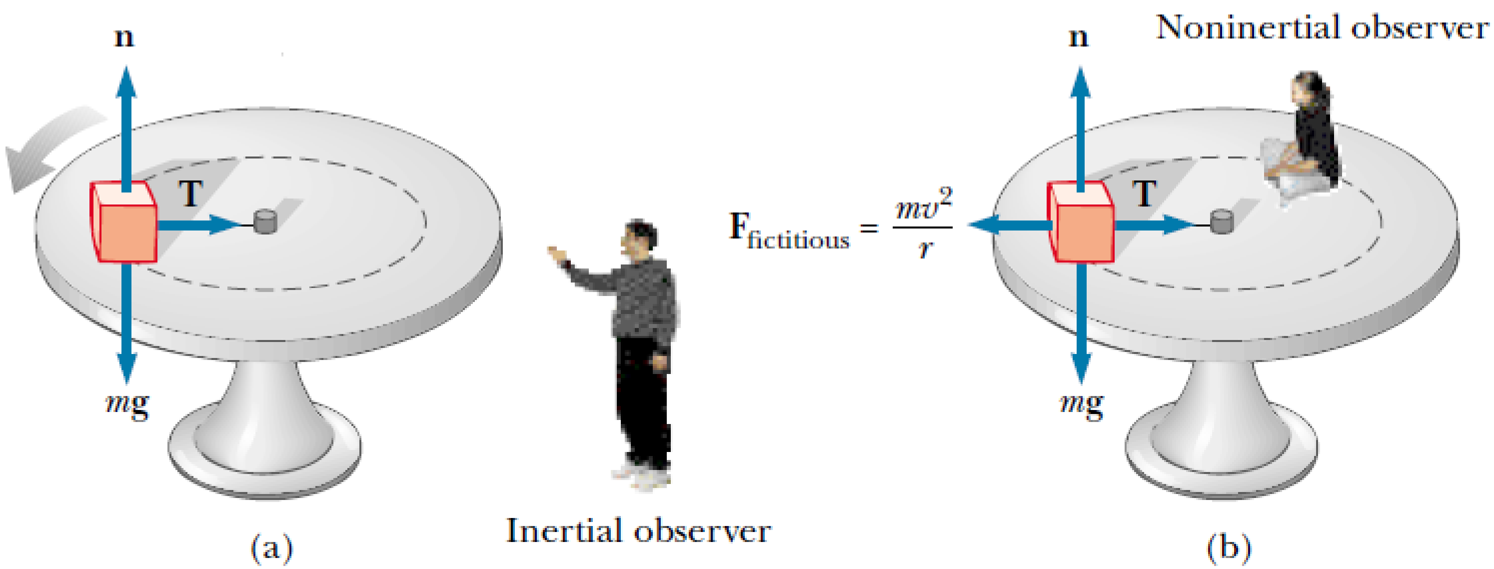


Identity: $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

Typhoon (Low Pressure)

Rotation: Clockwise in the southern hemisphere and counterclockwise in the northern hemisphere.

In a Rotating System



Fictitious centrifugal force

Newton's Second Law:

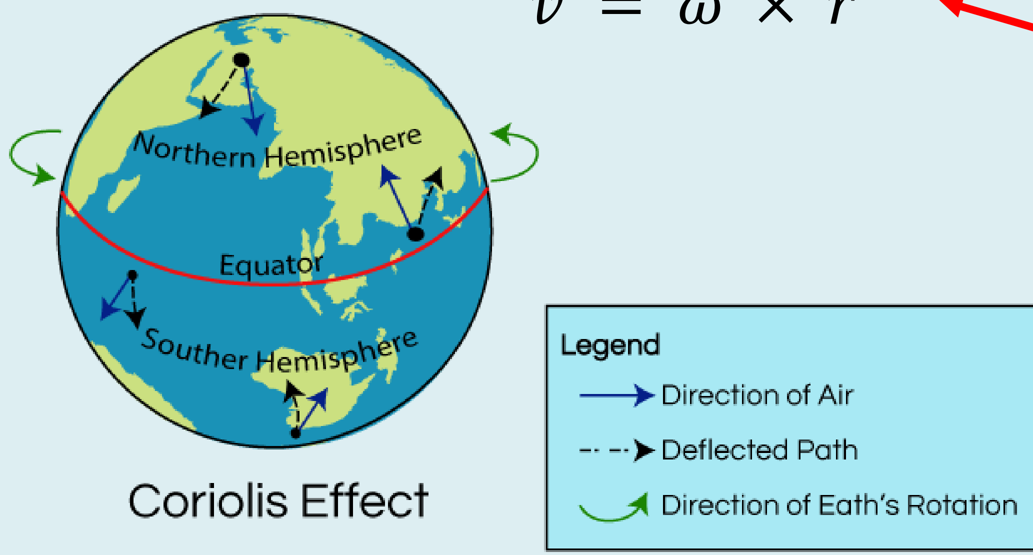
$$T = m\frac{v^2}{r}$$

$$T - m\frac{v^2}{r} = 0$$

Coriolis Effect

- **The Coriolis effect is a deflection of moving objects when the motion is described relative to a rotating reference frame.**

$$\vec{v} = \vec{\omega} \times \vec{r}$$



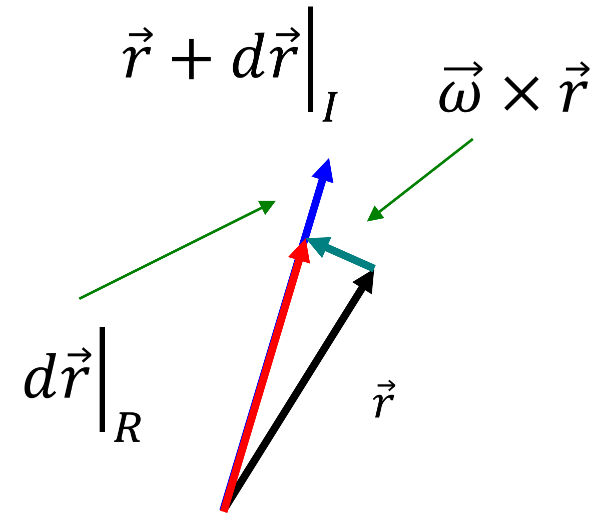
for point object at rest in the rotational frame

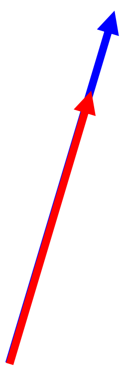
What if it is moving?

Graphic Illustration

- **In the inertia frame of reference**

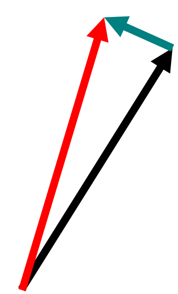
$$\left.\frac{d\vec{r}}{dt}\right|_I = \left.\frac{d\vec{r}}{dt}\right|_R + \vec{\omega} \times \vec{r}$$





change of the displacement within the rotational frame

change of the displacement due to the rotation of the reference frame



Mathematical Origin

This is generically true for any vector u.

$$\left.\frac{d\vec{u}}{dt}\right|_I = \left.\frac{d\vec{u}}{dt}\right|_R + \vec{\omega} \times \vec{u}$$

That is, we need to take into account the change of directions of the axes.

x: $$\frac{d(u_x\hat{i})}{dt} = \frac{du_x}{dt}\hat{i} + u_x\frac{d\hat{i}}{dt} = \frac{du_x}{dt}\hat{i} + \vec{\omega} \times (u_x\hat{i})$$

y: $$\frac{d(u_y\hat{j})}{dt} = \frac{du_y}{dt}\hat{j} + u_y\frac{d\hat{j}}{dt} = \frac{du_y}{dt}\hat{j} + \vec{\omega} \times (u_y\hat{j})$$

z: $$\frac{d(u_z\hat{k})}{dt} = \frac{du_z}{dt}\hat{k} + u_z\frac{d\hat{k}}{dt} = \frac{du_z}{dt}\hat{k} + \vec{\omega} \times (u_z\hat{k})$$

Operator Formalism

One can recast the reference frame change

$$\left.\frac{d\vec{u}}{dt}\right|_I = \left.\frac{d\vec{u}}{dt}\right|_R + \vec{\omega} \times \vec{u}$$

by an operator form

$$\left.\frac{d\vec{u}}{dt}\right|_I = \bar{O}\vec{u} \equiv \left[\left.\frac{d}{dt}\right|_R + \vec{\omega} \times\right]\vec{u}$$

So

$$\bar{O}^2\vec{u} = \left[\left.\frac{d}{dt}\right|_R + \vec{\omega} \times\right]^2 \vec{u}$$

$$= \left.\frac{d^2\vec{u}}{dt^2}\right|_R + 2\vec{\omega} \times \left.\frac{d\vec{u}}{dt}\right|_R + \vec{\omega} \times (\vec{\omega} \times \vec{u})$$

Acceleration as the 2nd Derivative

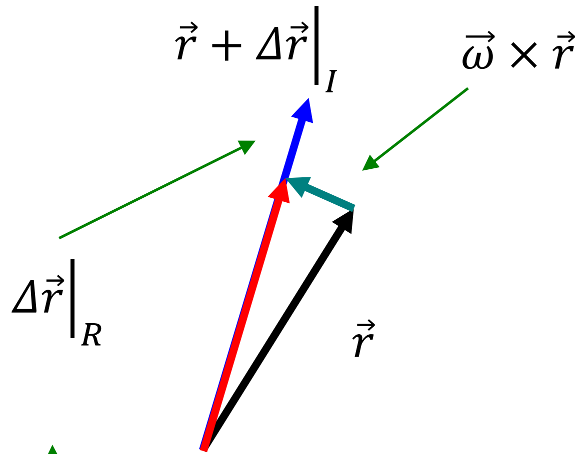
- **In the inertia frame of reference**

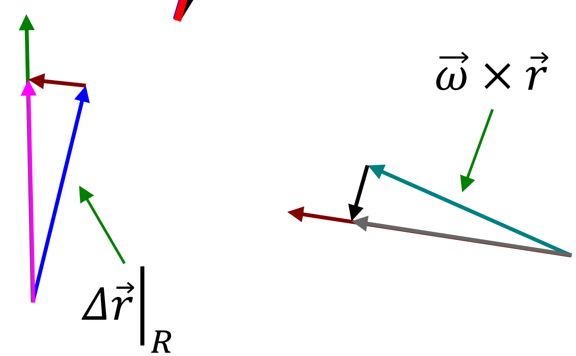
$$v\Big|_I = \frac{d\vec{r}}{dt}\Big|_I = \frac{d\vec{r}}{dt}\Big|_R + \vec{\omega} \times \vec{r}$$

$$\frac{d\vec{v}}{dt}\Big|_I = \frac{d\vec{v}}{dt}\Big|_R + \vec{\omega} \times \vec{v}$$

$$\frac{d^2\vec{r}}{dt^2}\Big|_I = \left[\frac{d}{dt}\Big|_R + \vec{\omega} \times\right]^2 \vec{r}$$

$$= \frac{d^2\vec{r}}{dt^2}\Big|_R + 2\vec{\omega} \times \frac{d\vec{r}}{dt}\Big|_R + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$





"Force" in the Rotating Frame

- **In the rotating frame of reference, "total force"**

Coriolis force **centrifugal force**

$$F_{net} = m\left.\frac{d^2\vec{r}}{dt^2}\right|_R$$

$$= m\left.\frac{d^2\vec{r}}{dt^2}\right|_I - 2m\vec{\omega}\times\left.\frac{d\vec{r}}{dt}\right|_R - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

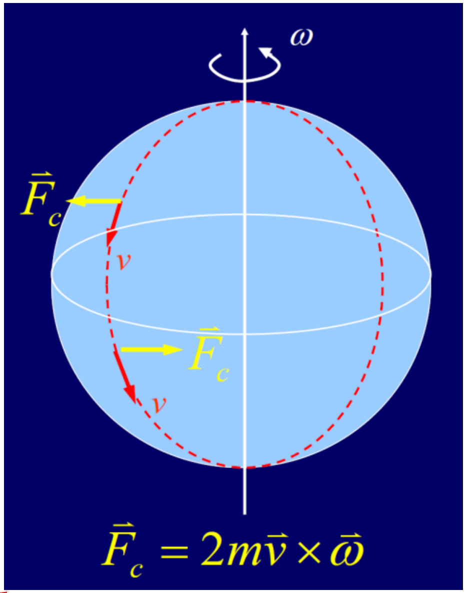
$$= F - 2m\vec{\omega}\times\overrightarrow{v_R} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

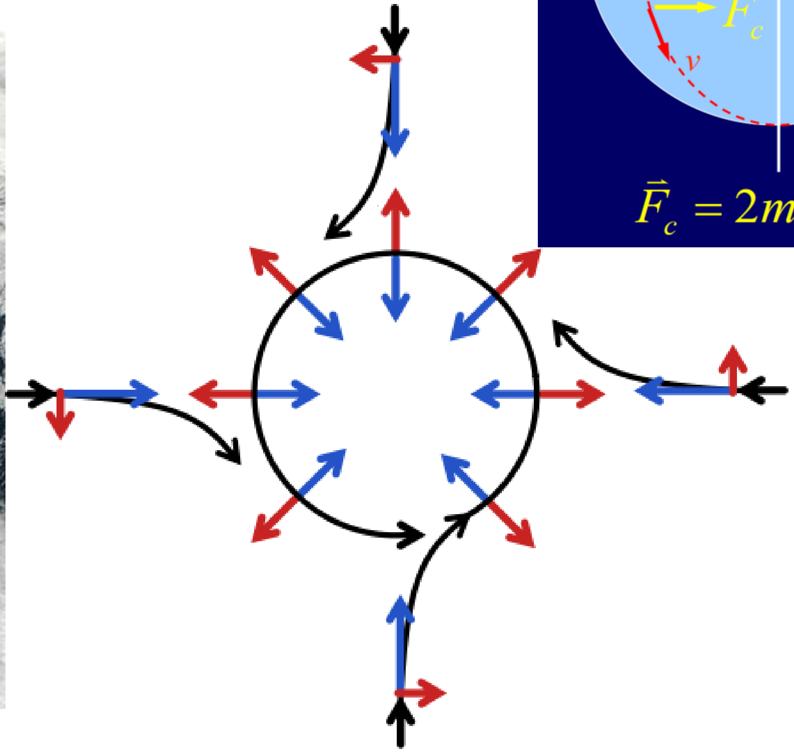
$$= F - 2m\vec{\omega}\times\overrightarrow{v_R} + m\omega^2\vec{r} \qquad for\ \vec{r}\perp\vec{\omega}$$

Identity: $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

Manifestation of Coriolis

- This low-pressure system (in Northern hemisphere) spins counterclockwise due to balance between the Coriolis force and the pressure gradient force.

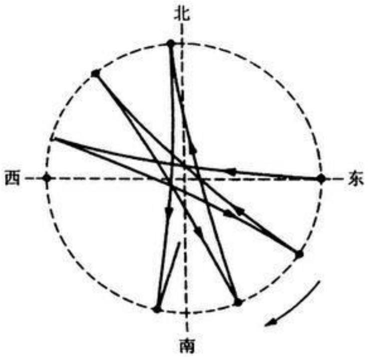




Manifestation of Coriolis*

Famous Qianjiang tide

Foucault Pendulum(1851)



$$\Omega=\omega\sin\psi$$

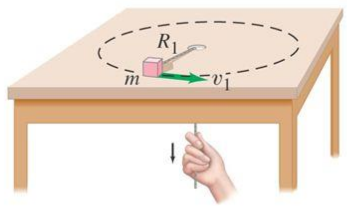
Warning

- The treatment of the Coriolis force is to study the motion of an object in the rotating frame of reference (typically on another rigid object, like the earth, which rotates with a constant angular velocity).
- This is different from studying the rotation of a rigid object in the inertia frame of reference, which is the main subject of our interest.
- Once again, stay in the inertial frame of reference unless you cannot.

Example 11-1: Object rotating on a string of changing length.

Initially, the mass revolves with a speed v_1 = 2.4 m/s in a circle of radius R_1 = 0.80 m.

The string is then pulled slowly through the hole so that the radius is reduced to R_2 = 0.48 m. What is the speed, v_2, of the mass now?



$$\vec{v} = \frac{d\vec{r}}{dt} = v_r \hat{u}_r + \omega r \hat{u}_\theta$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v_r \hat{u}_r + \omega r \hat{u}_\theta)$$

$$= \left(\frac{\mathrm{d}v_r}{dt} - \omega^2 r\right)\hat{u}_r + \left(2\omega v_r + \frac{d\omega}{dt} r\right)\hat{u}_\theta$$

$$2\omega v_r + \frac{d\omega}{dt} r = 0 \leftrightarrow 2\omega \frac{dr}{dt} + \frac{d\omega}{dt} r = 0 \Rightarrow \frac{d\omega}{\omega} = -\frac{2dr}{r}$$

$$\ln\omega \Big|_{\omega_0}^{\omega_1} = -2 \ln r \Big|_{r_0}^{r_1}$$

$$\omega_0 r_0^2 = \omega_1 r_1^2 = c$$

Conservation of Angular Momentum

$$L = m\omega_0 r_0^2 = C$$