General Physics I

Lecture 8: Rotation of a Rigid Object About a Fixed Axis

New Territory

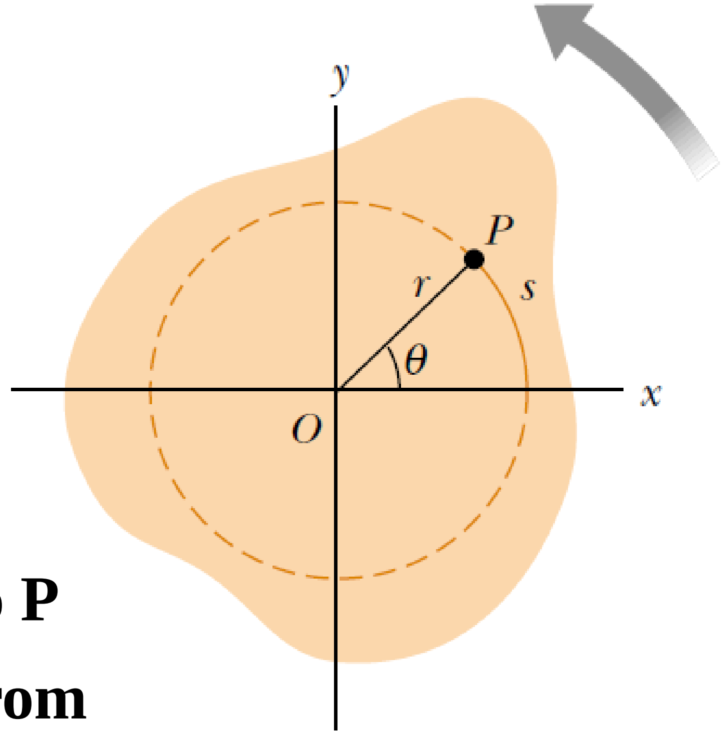
- **Object**
  - **In the past, point particle (no rotation, no vibration, ...)**
  - **Now, extended but rigid object: an object that is nondeformable – i.e., it is an object in which the separations between all parts of particles remain constant.**
- **Motion**
  - **In the past, linear acceleration, linear velocity, …**
  - **Now, the rotation of a rigid body about a fixed axis (pure rotational motion)**

Outline

- Angular displacement, velocity, and acceleration
- Rotational kinematics (with constant angular acceleration)
- Relations between angular and linear quantities
- Torque and its connection to angular acceleration
- Work, power, and energy in rotational motion

Parametrization

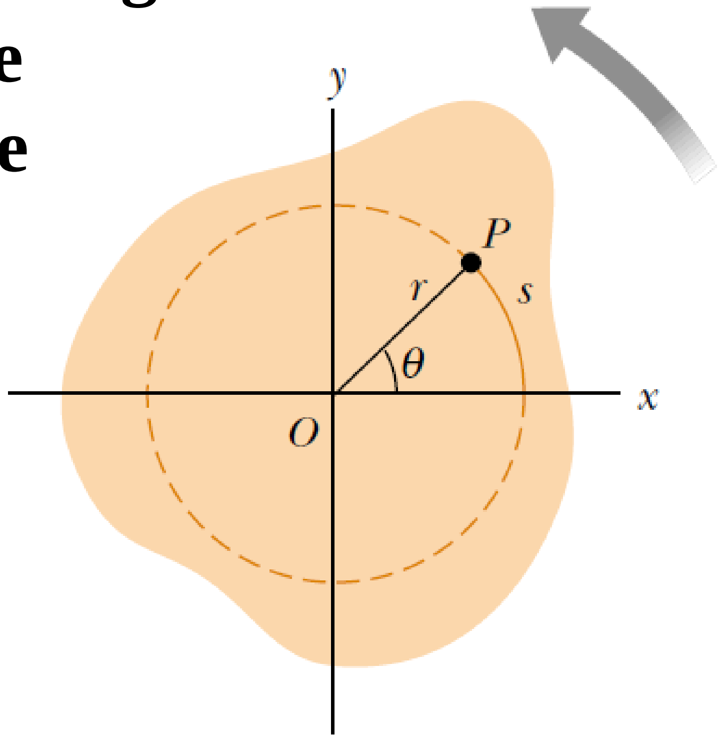
- A planar (flat), rigid object of arbitrary shape confined to the *xy* plane and rotating about a fixed axis through *O*. The axis is perpendicular to the plane of the figure, and *O* is the origin of an *xy* coordinate system.



- Represent the position of P with its polar coordinates (r, θ)
  - r: the distance from the origin to P
  - θ: measured counterclockwise from the positive x axis (or any other preferred direction).

Arc Length and Radian

- **As the particle moves along the circle from the positive *x* axis to *P*, it moves through an arc of length *s*, which is related to the angular position θ through the relationship**



$$s = r\theta \quad \textbf{or} \quad \theta = \frac{s}{r}$$

$$2\pi\,(rad) = 360^{o}\,(deg)$$

One radian is the angle subtended by an arc length equal to the radius of the arc.

Angular Displacement and Speed

- **Angular displacement**

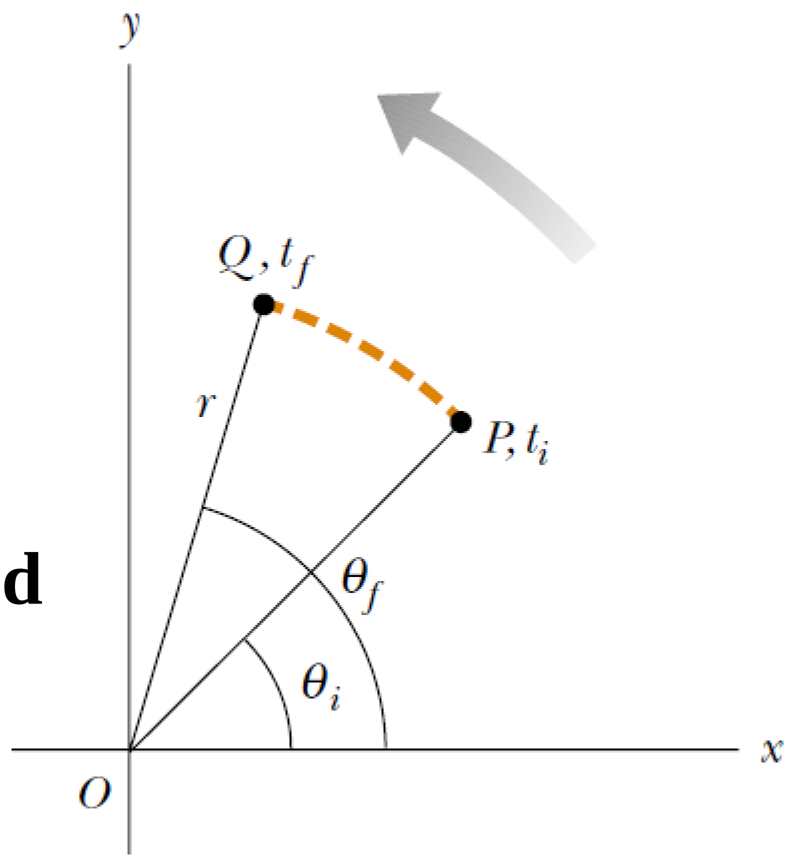
$$\Delta\theta = \theta_f - \theta_i$$

- **Average angular speed**

$$\overline{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

- **Instantaneous angular speed**

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



Units: rad/s or simply s^{-1}; Direction: positive for counterclockwise motion

Angular Acceleration

- **Average angular acceleration**

$$\overline{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

- **Instantaneous angular acceleration**

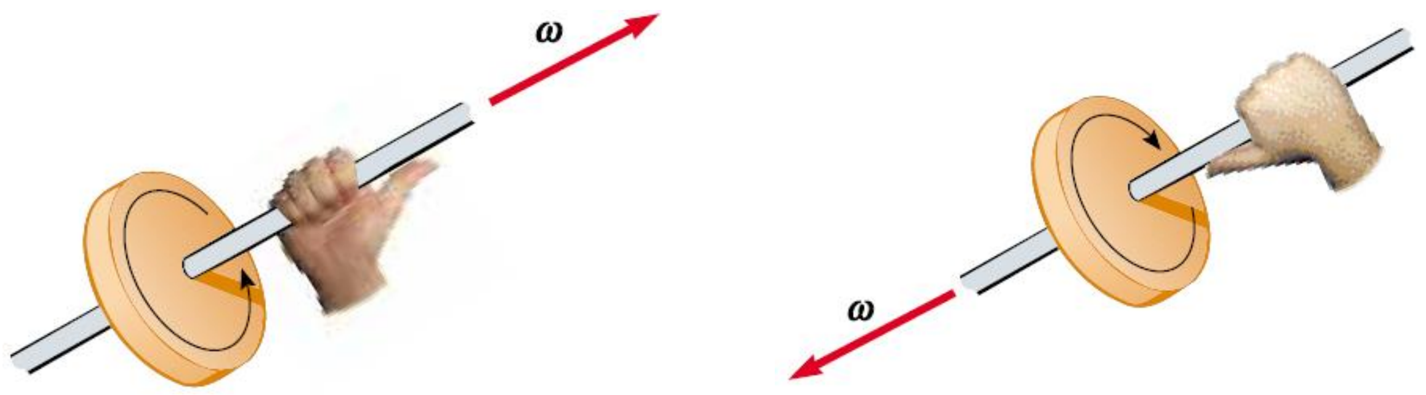
$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

- **When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and the same angular acceleration.**

Units: rad/s^2 or simply s^{-2}; Direction: positive when the rate of counterclockwise rotation is increasing.

Right-Hand Rule

- **For rotation about a fixed axis, the only direction that uniquely specifies the rotational motion is the direction along the axis of rotation. Therefore, the directions of ω and α are along this axis.**



Why do we need the right-hand rule? Isn't clockwise good enough?

Rotational Kinematics

- **Under constant angular acceleration**

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

Linear Motion

$$v_f = v_i + at$$

$$x_f = x_i + v_i t + \frac{1}{2} at^2$$

$$v_f^2 = v_i^2 + 2a(x_f - x_i)$$

Of the same form as those for linear motion under constant linear acceleration with the substitutions

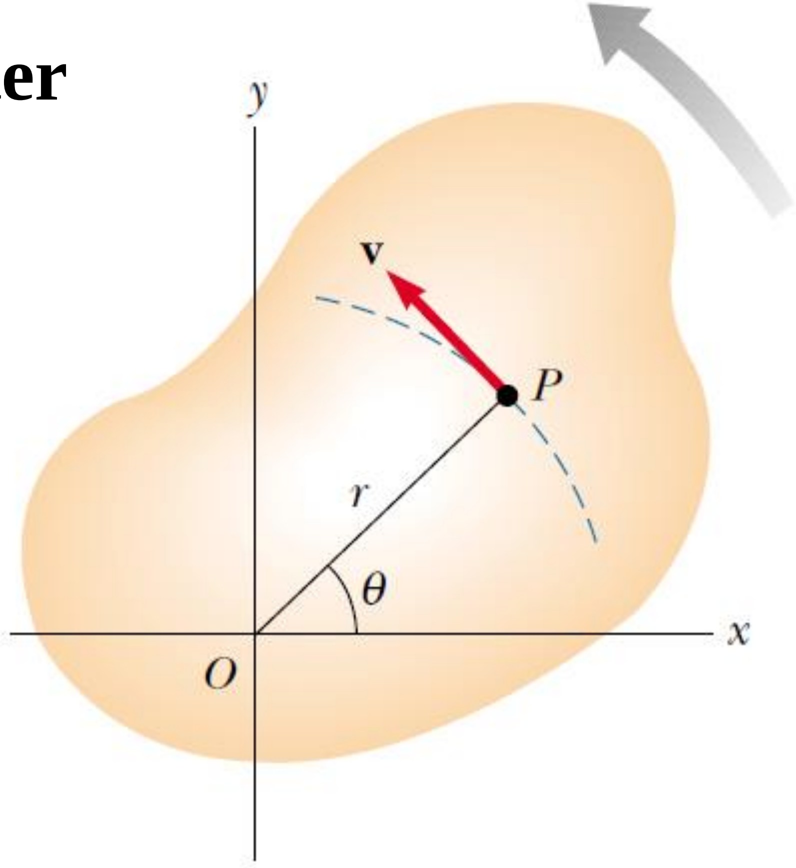
$$x \rightarrow \theta,\ v \rightarrow \omega,\ \text{and}\ \ a \rightarrow \alpha$$

Angular vs Linear Speed

- when a rigid object rotates about a fixed axis, every particle of the object moves in a circle whose center is the axis of rotation.

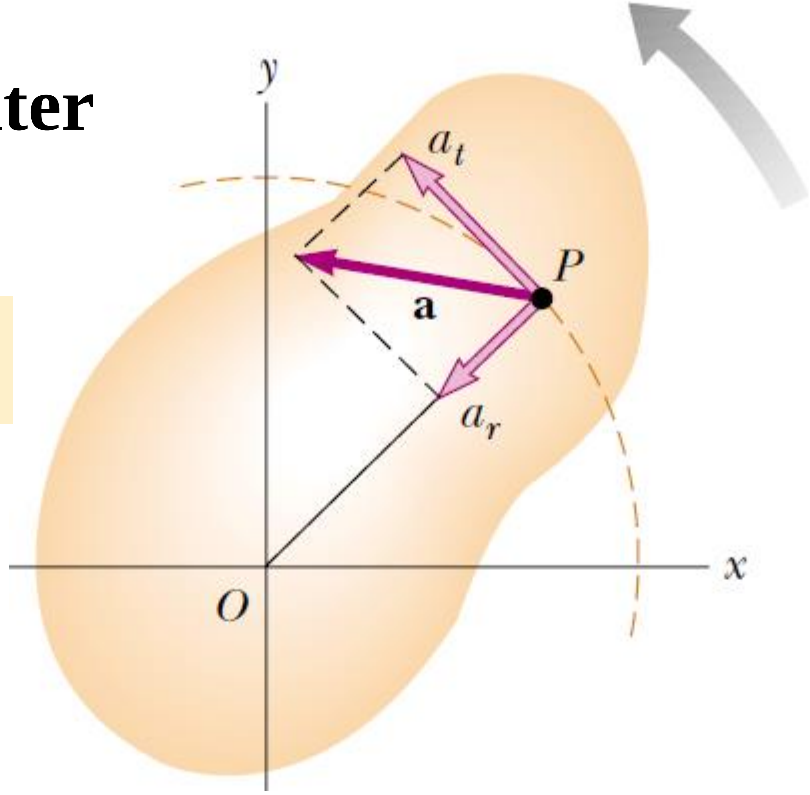
$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\Rightarrow \quad \boxed{v = r\omega}$$



Angular vs Linear Acceleration

- when a rigid object rotates about a fixed axis, every particle of the object moves in a circle whose center is the axis of rotation.



$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \Rightarrow \boxed{a_t = r\alpha}$$

$$\boxed{a_r = \frac{v^2}{r} = r\omega^2}$$

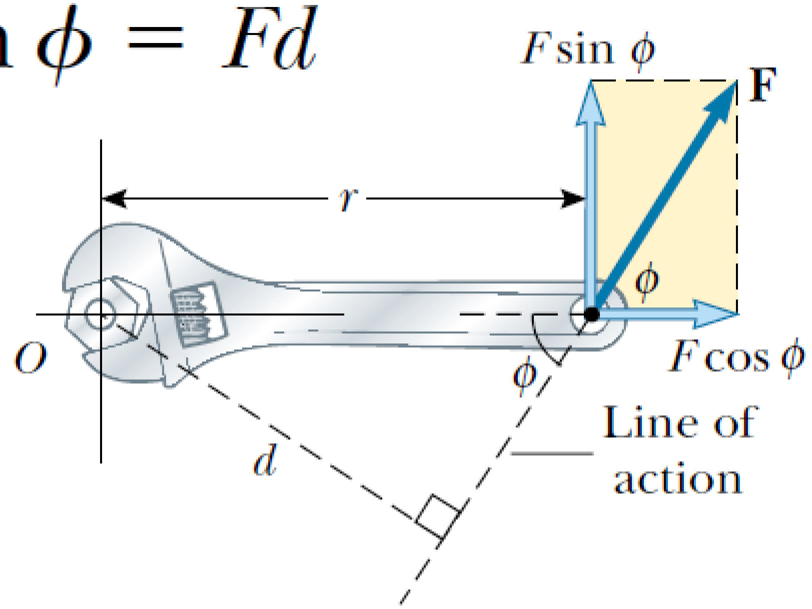
$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$$

$$\Rightarrow a = \sqrt{{a_t}^2 + {a_r}^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Torque

- The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque**

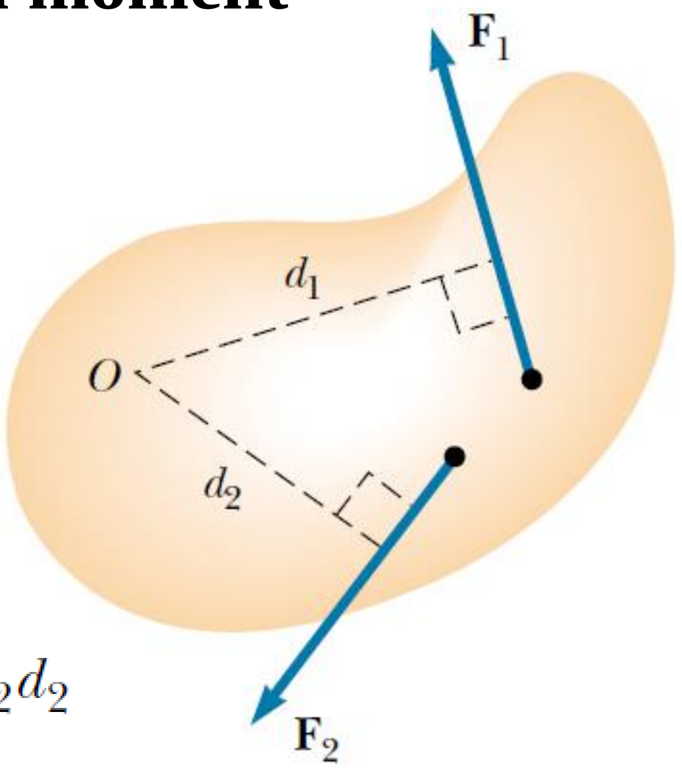
$$\tau \equiv rF\sin\phi = Fd$$



- r: the distance between the pivot point and the point of application of F.
- d: the perpendicular distance from the pivot point to the line of action of F.
    - This quantity d is called the **moment arm** of F.

Notes on Torque

- **Torque is defined only when a reference axis is specified.** **Torque is the product of a force and the moment arm of that force, and moment arm is defined only in terms of an axis of rotation.**



- **If two or more forces are acting on a rigid object, each tends to produce rotation about the pivot at O.**

$$\sum \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$

Torque and Angular Acceleration

- **An instructive example: a particle rotating about some fixed point under the influence of an external force.**

$$F_t = ma_t$$

I: The moment of inertia

$$\tau = F_t r = (ma_t) r$$

$$a_t = r\alpha \Rightarrow \tau = (mr\alpha) r = (mr^2)\alpha$$

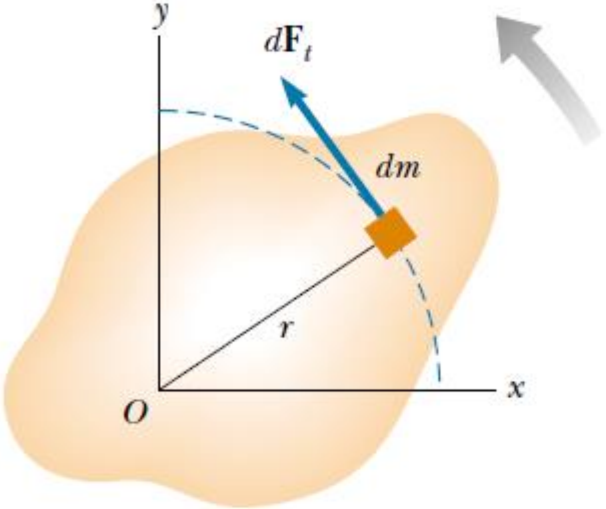
$$\tau = I\alpha$$

The rotational analog of Newton's second law of motion.

The torque acting on the particle is proportional to its angular acceleration.

Torque and Angular Acceleration

- **Now extend to a rigid object of arbitrary shape rotating about a fixed axis.**



$$dF_t = (dm)\,a_t$$

$$d\boldsymbol{\tau} = r\,dF_t = (r\,dm)\,a_t$$

$$d\boldsymbol{\tau} = (r\,dm)\,r\alpha = (r^2\,dm)\,\alpha$$

$$\sum \tau = \int (r^2\,dm)\,\alpha = \alpha \int r^2\,dm = I\alpha$$

I**: The moment of inertia**

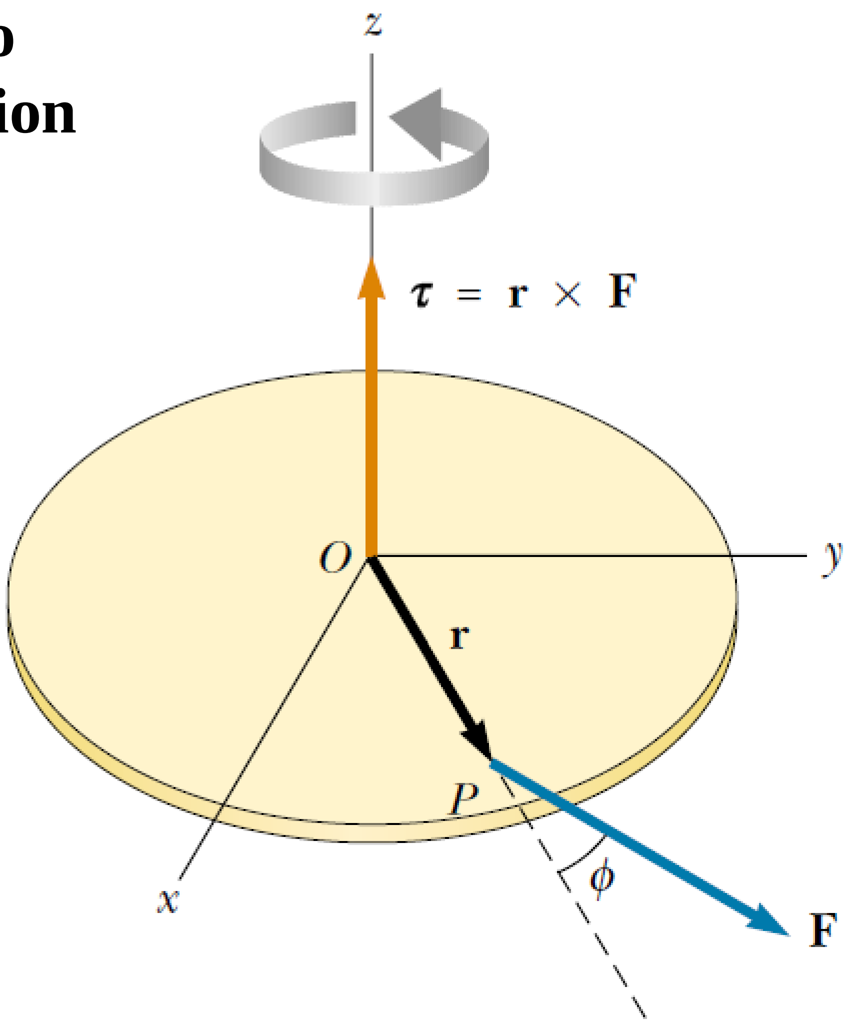
More Notes

- **The result also applies when the forces acting on the mass elements have radial components as well as tangential components.**
- **Although each point on a rigid object rotating about a fixed axis may not experience the same force, linear acceleration, or linear speed, each point experiences the same angular acceleration and angular speed at any instant.**
- **Therefore, at any instant the rotating rigid object as a whole is characterized by specific values for angular acceleration, net torque, and angular speed.**

Vector Representation

- The torque τ involves the two vectors **r** and **F**, and its direction is perpendicular to the plane of **r** and **F**. We can establish a mathematical relationship between τ, **r**, and **F**, using a new mathematical operation called the vector product, or cross product

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}$$



Will discuss more next week.

Rotational Kinetic Energy

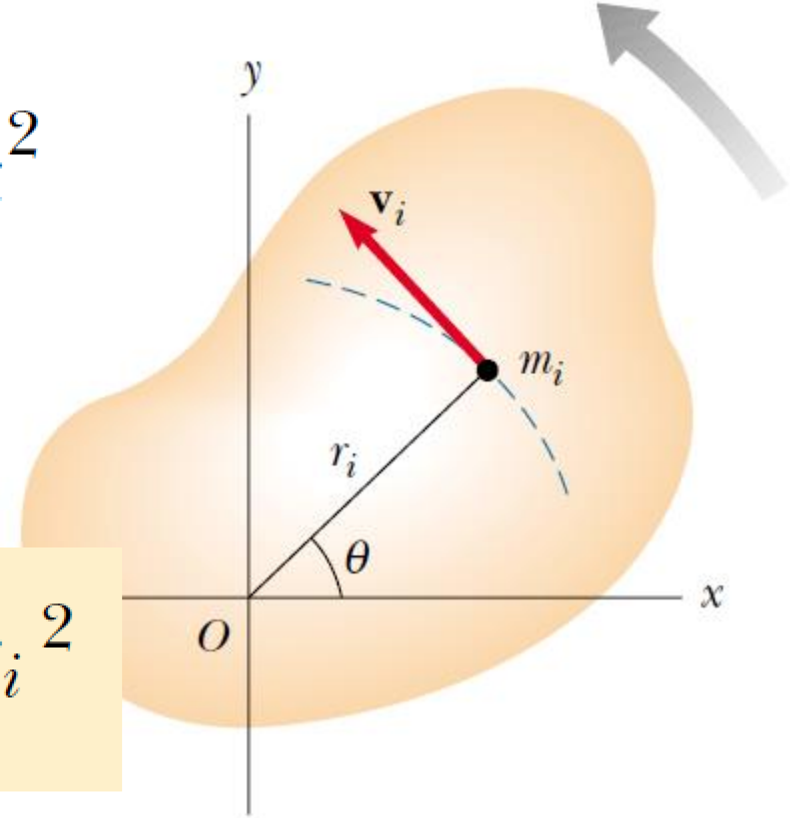
- **The total rotational kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles**

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} \sum_i m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2} I \omega^2$$

$$I \equiv \sum_i m_i r_i^2$$



Analogy Between *I* and *m*

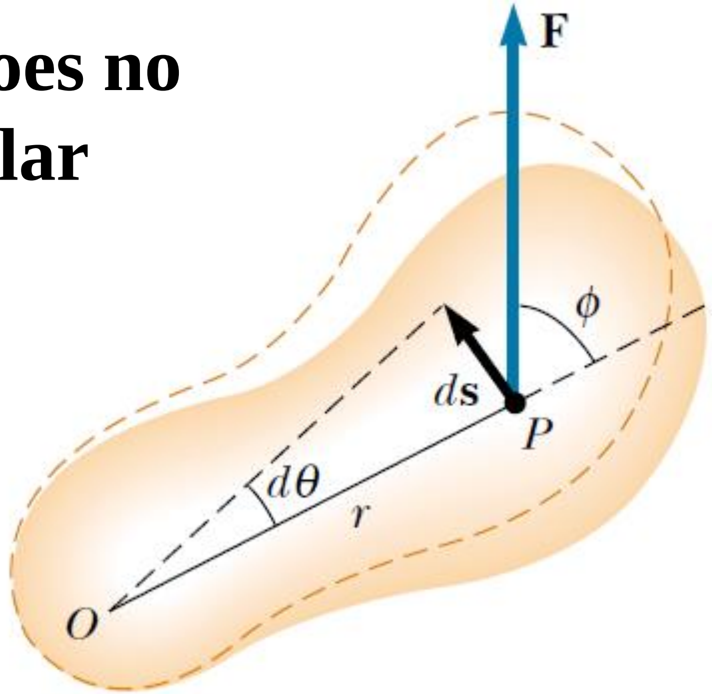
- The moment of inertia is a **measure of the resistance of an object to changes in its rotational motion**, just as mass is a measure of the tendency of an object to resist changes in its linear motion.
- However, mass is an intrinsic property of an object, whereas I **depends on the physical arrangement of that mass**.
- There are situations in which an object's moment of inertia changes even though its mass does not.

Work and Power in Rotation

- **Work done by a single external force F applied at P is**

$$đW = \mathbf{F}\cdot d\mathbf{s} = (F\sin\phi)\,r\,d\theta = \tau\,d\theta$$

- **The radial component of F does no work because it is perpendicular to the displacement.**



- **Instantaneous power**

$$\mathscr{P} = \frac{đW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

Work-Kinetic Energy Theorem

- Using the chain rule from the calculus, we can express the resultant torque as

$$\sum \tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\,\omega$$

$$\Rightarrow \quad \sum \tau \, d\theta = dW = I\omega \, d\omega$$

$$\sum W = \int_{\theta_i}^{\theta_f} \sum \tau \, d\theta = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \tfrac{1}{2}I{\omega_f}^2 - \tfrac{1}{2}I{\omega_i}^2$$

Work–kinetic energy theorem for rotational motion

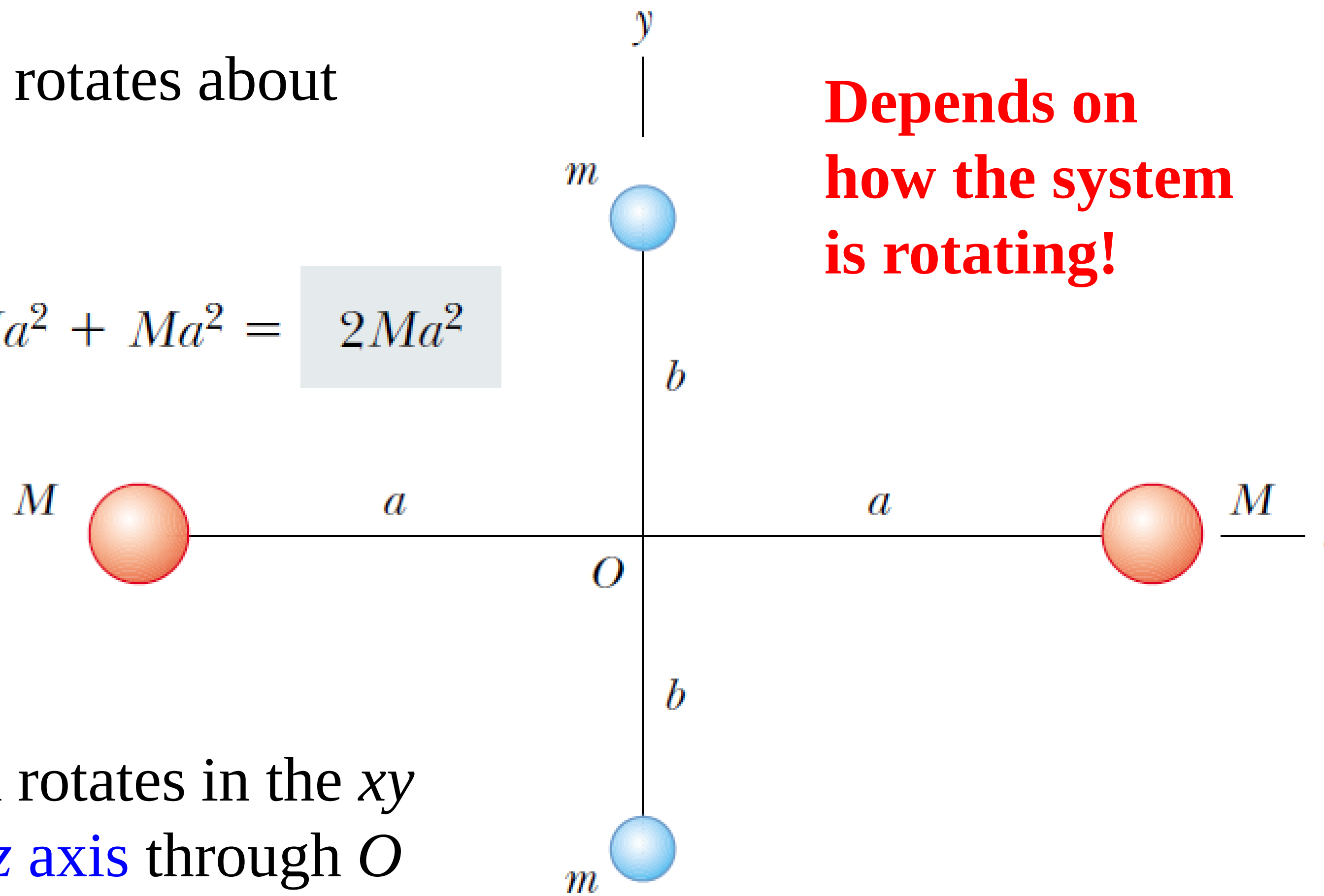
TABLE 10.3 Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Resultant torque $\Sigma\tau = I\alpha$	Resultant force $\Sigma F = ma$
If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f - x_i = v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$	Work $W = \int_{x_i}^{x_f} F_x \, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Resultant torque $\Sigma\tau = dL/dt$	Resultant force $\Sigma F = dp/dt$

Example: The Moment of Inertia

(a) If the system rotates about the y axis

$$I_y = \sum_i m_i r_i^2 = Ma^2 + Ma^2 = \boxed{2Ma^2}$$

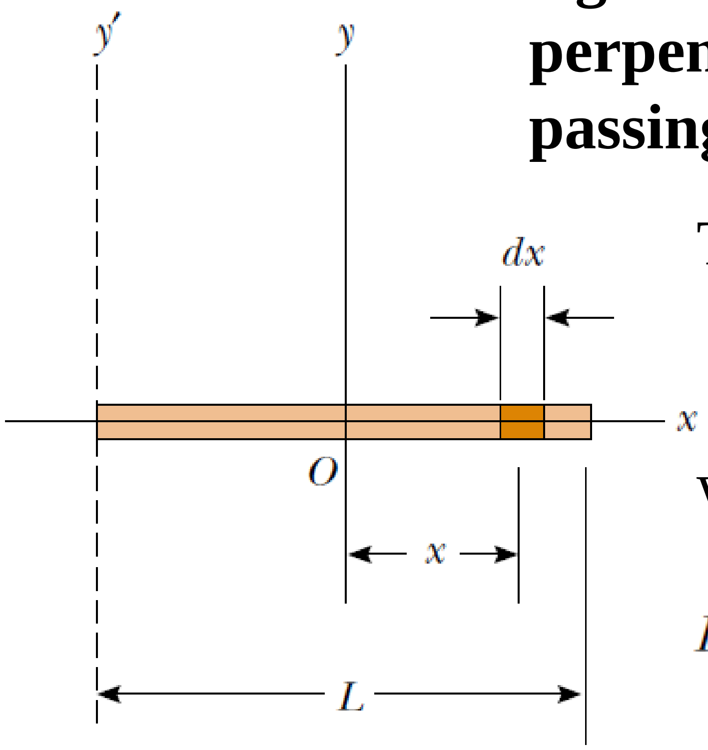


Depends on how the system is rotating!

(b) If the system rotates in the *xy* plane about the z axis through O

$$I_z = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = \boxed{2Ma^2 + 2mb^2}$$

Example: Uniform Rigid Rod



Calculate the moment of inertia of a uniform rigid rod of length *L* and mass *M* about an axis perpendicular to the rod (the *y* axis) and passing through its center of mass.

The shaded length element *dx* has a mass

$$dm = \lambda\, dx = \frac{M}{L}\, dx$$

where λ is the mass per unit length.

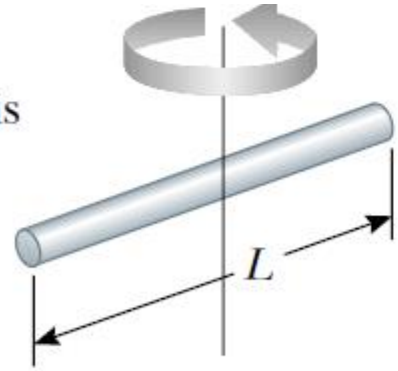
$$I_y = \int r^2\, dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx = \frac{M}{L}\int_{-L/2}^{L/2} x^2\, dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \boxed{\frac{1}{12}ML^2}$$

Calculating Moment of Inertia

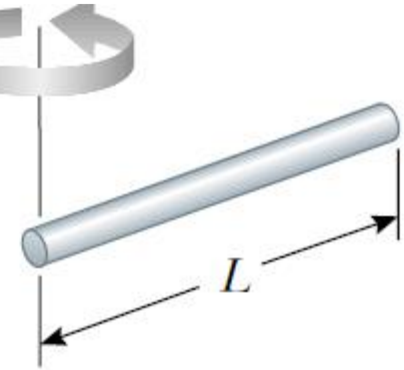
Long thin rod with rotation axis through center

$$I_{CM} = \frac{1}{12} ML^2$$



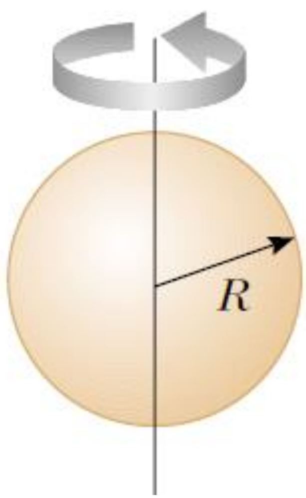
Long thin rod with rotation axis through end

$$I = \frac{1}{3} ML^2$$



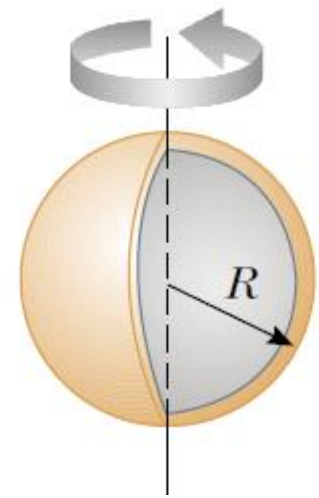
Solid sphere

$$I_{CM} = \frac{2}{5} MR^2$$



Thin spherical shell

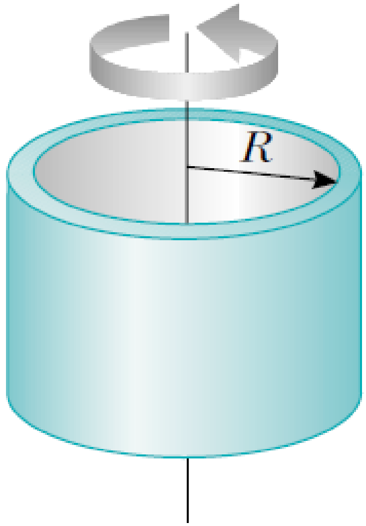
$$I_{CM} = \frac{2}{3} MR^2$$



Calculating Moment of Inertia

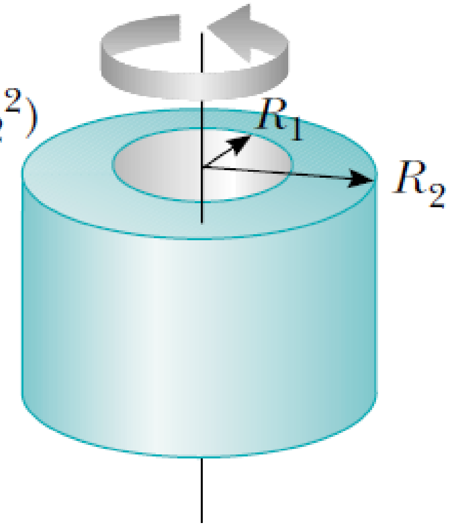
Hoop or cylindrical shell

$I_{CM} = MR^2$



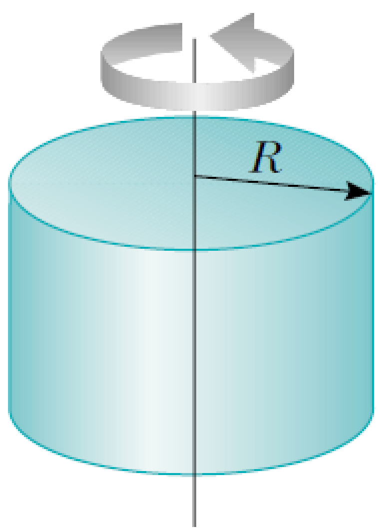
Hollow cylinder

$I_{CM} = \frac{1}{2} M(R_1^2 + R_2^2)$



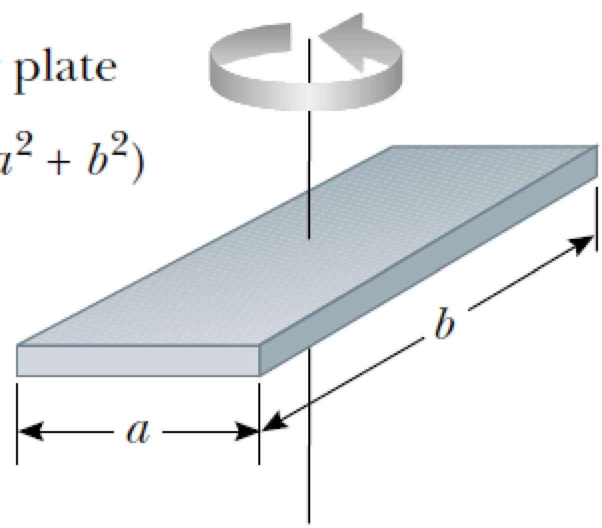
Solid cylinder or disk

$I_{CM} = \frac{1}{2} MR^2$



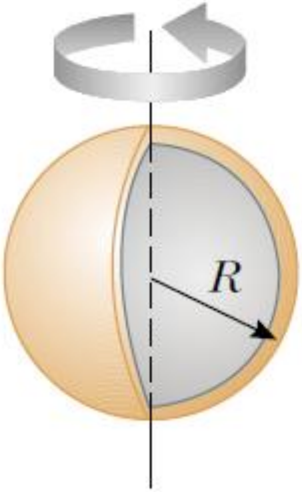
Rectangular plate

$I_{CM} = \frac{1}{12} M(a^2 + b^2)$



Appendix: Spherical Shell

- **First we calculate the moment of inertia of a spherical shell about a diagonal axis.**



$$I_z = \int dm\,(x^2 + y^2)$$

By the rotational symmetry,

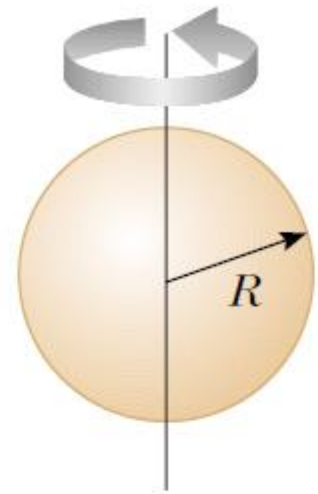
$$I_x = I_y = I_z = \frac{1}{3}\left(I_x + I_y + I_z\right)$$

$$= \frac{2}{3}\int dm\,(x^2 + y^2 + z^2) = \frac{2}{3}\int dm\,R^2 = \frac{2}{3}MR^2$$

Appendix: Solid Sphere

- By dimension analysis, the moment of inertia of a solid sphere (density ρ) is

$$I(R) = cMR^2 \propto c\rho\frac{4\pi}{3}R^5$$



Since a spherical shell can be written as the difference of two solid spheres

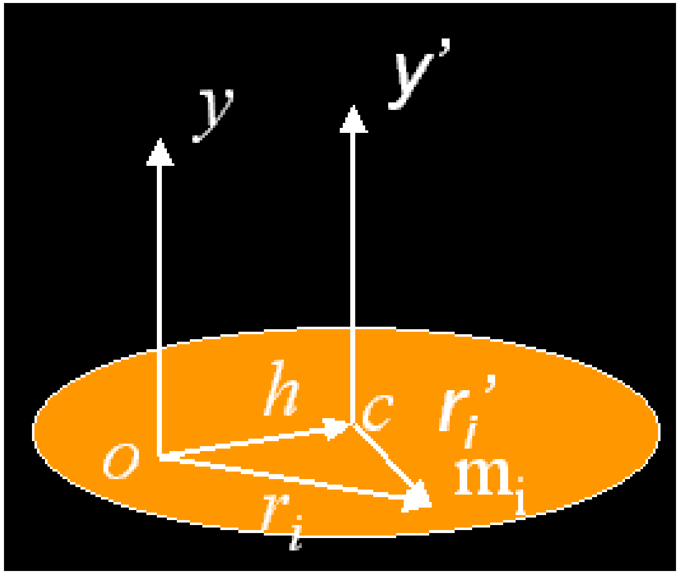
$$I(R+\Delta R) - I(R) = \frac{2}{3}(\rho 4\pi R^2 \Delta R)R^2$$

$$\Delta R \to 0 \qquad dI = \frac{8\pi\rho}{3}R^4 dR$$

$$\Rightarrow \quad I = \frac{8\pi\rho}{15}R^5 = \frac{2}{5}MR^2$$

The parallel-axis theorem

Rotation about an arbitrary axis y



$$I_y = \sum m_i r_i^2 = \sum m_i (\vec{r}_i' + \vec{h})^2$$

$$= \sum m_i r_i'^2 + h^2 \sum m_i + 2 \sum m_i \vec{r}_i' \bullet \vec{h}$$

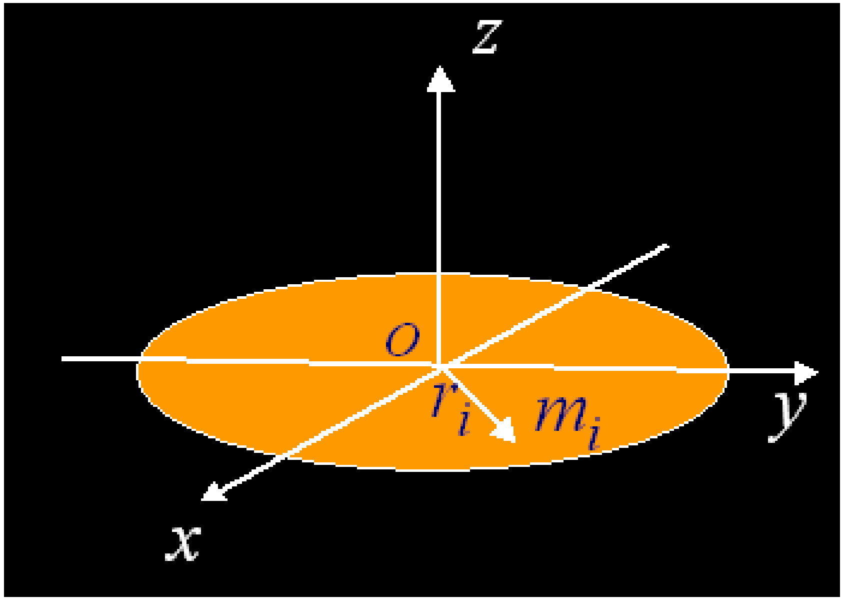
$$= I_{cm} + Mh^2$$

$$I = I_{CM} + mh^2$$

$$\vec{r}_c' = \frac{\sum m_i \vec{r}_i'}{\sum m_i} = 0$$

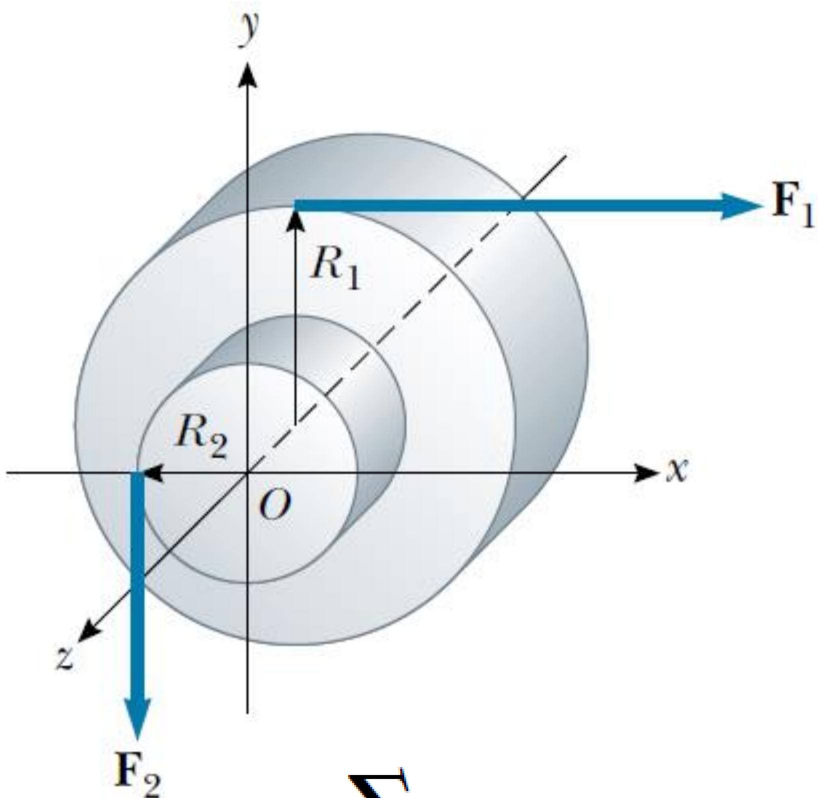
Rotation about an z axis (for Plate)

$$
\begin{aligned}
I_z &= \sum m_i r_i^2 \\
&= \sum m_i (x_i^2 + y_i^2) \\
&= \sum m_i x_i^2 + \sum m_i y_i^2 \\
&= I_y + I_x
\end{aligned}
$$



$$\boxed{I_z = I_y + I_x}$$

Example: Torque on a Cylinder

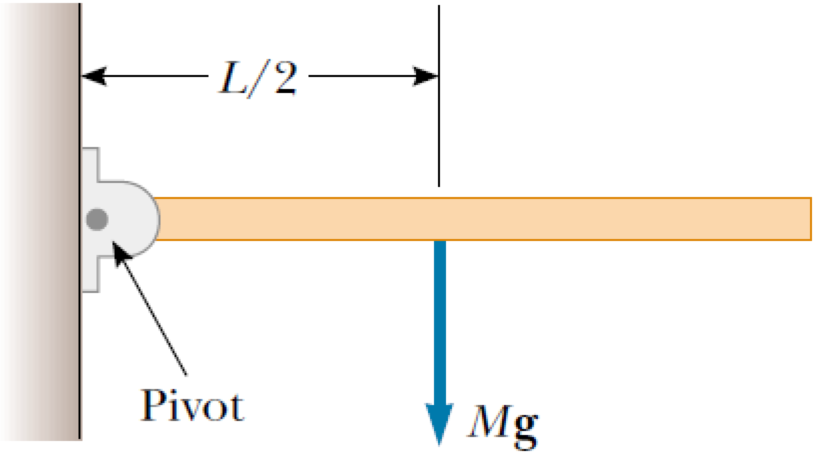


A one-piece cylinder is shaped with a core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the drawing. (a) What is the net torque acting on the cylinder about the *z* axis?

$$\sum \tau = \tau_1 + \tau_2 = \boxed{-R_1F_1 + R_2F_2}$$

Example: Rotating Rod

- **A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What is the initial angular acceleration of the rod and the initial linear acceleration of its right end?**



$$\tau = I\alpha$$

Example: Rotating Rod

- The torque due to this force about an axis through the pivot is

$$\tau = Mg\left(\frac{L}{2}\right)$$

- The moment of inertia for this axis of rotation

$$I = \tfrac{1}{3}ML^2 \longleftarrow$$ **Can you show?**

$$\Rightarrow \quad \alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{1/3\,ML^2} = \boxed{\frac{3g}{2L}}$$

Example: Rotating Rod

- **The linear acceleration of the right end of the rod**

$$a_t = L\alpha = \boxed{\frac{3}{2}g}$$

- **This result—that $a_t > g$ for the free end of the rod—is rather interesting. It means that if we place a coin at the tip of the rod, hold the rod in the horizontal position, and then release the rod, the tip of the rod falls faster than the coin does!**