General Physics I

Lecture 7: The Law of Gravity

Outline

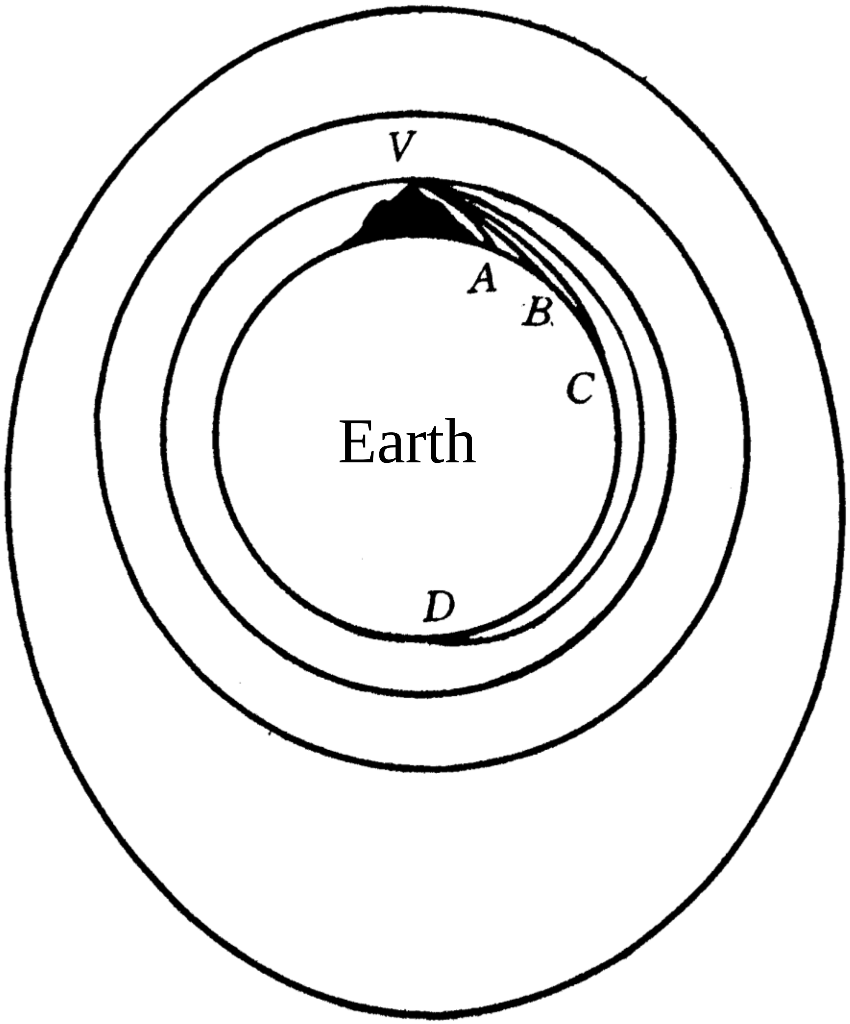
- Newton's law of universal gravitation
- Motion of the planets; Kepler's laws
- Measuring the gravitational constant
- Free-fall acceleration
- Satellite motion
- Rocket propulsion

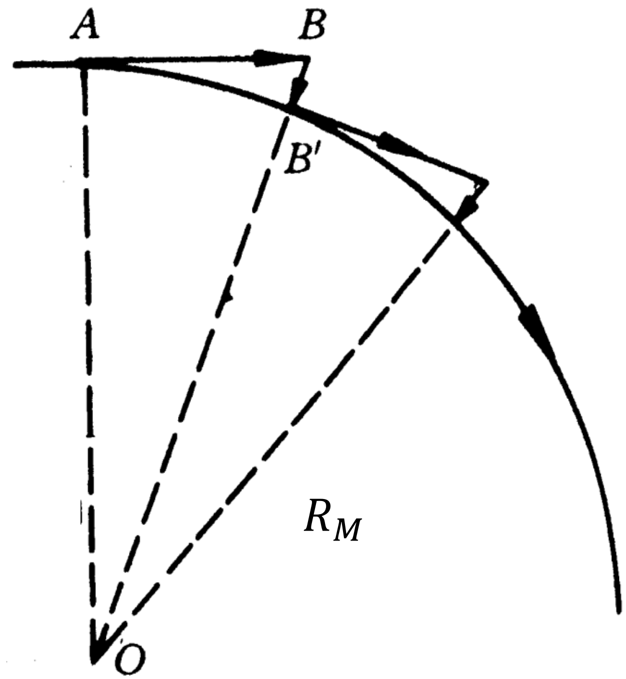
Newton's Law of Universal Gravitation

- **Observations known to Newton:**
  - **Apples fall from trees to the ground.**
  - **Moon circles around the earth.**
  - **Planets moves around the sun.**
- **Newton's unified heavenly and earthly motions.**
  - **Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.**

$$F_g = G\frac{m_1 m_2}{r^2}$$

Newton's Thought Experiment





The Linear Dependence on Mass

- **Near the surface of the earth, we know F = mg. If this force is governed by the same law of gravity, the universal gravition must be proportional to the mass of the object.**

$$F_g \propto m$$

- **Because of Newton's third law, if the force is proportional to the mass of one object, it must be also proportional to the mass of the other object.**

$$F_g \propto m_1 m_2$$

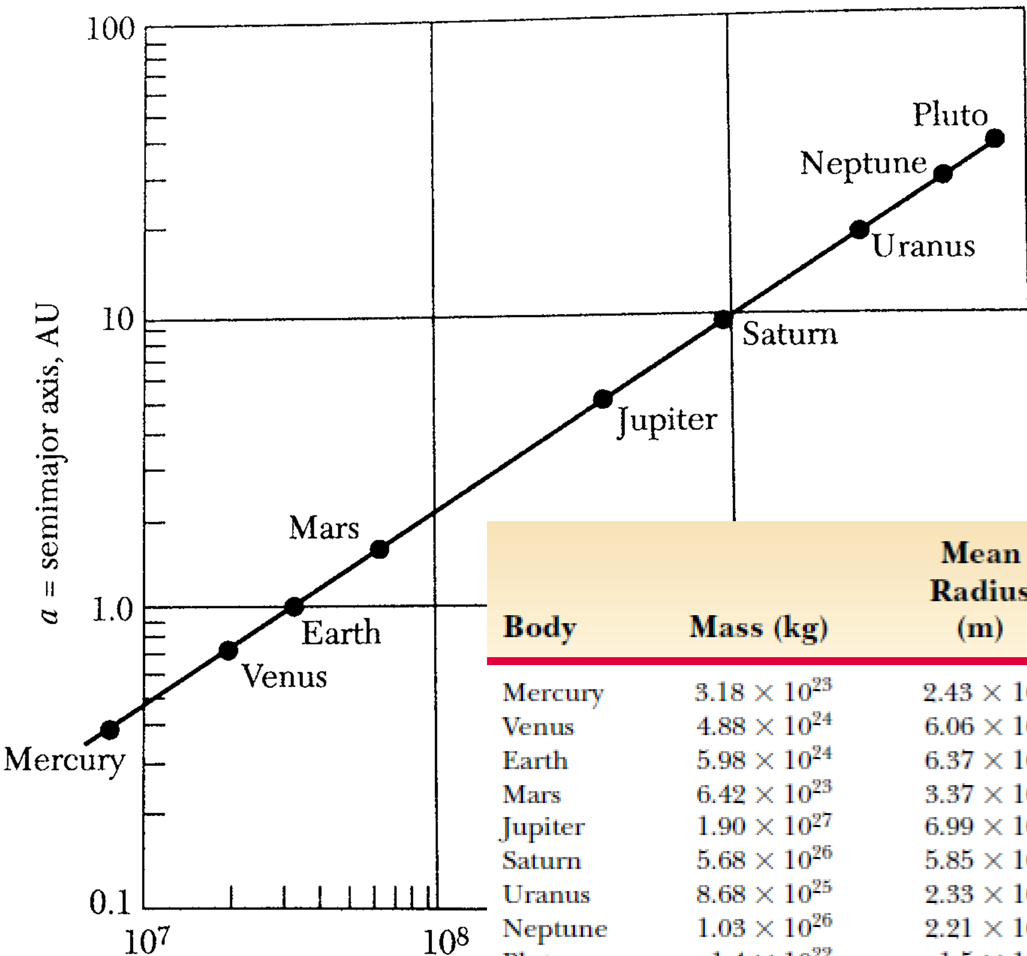
The Inverse-Square Law

- **Question: Based on the knowledge Newton might have, what observation can he used to support his law?**

Motion of planets?

Prove that the square of the orbital period of any planet is proportional to the cube of the radius of the circular orbit – discovered by Johannes Kepler.

Planetary Data



Body	Mass (kg)	Mean Radius (m)	Period of Revolution (s)	Mean Distance from Sun (m)	$\frac{T^2}{r^3}$ (s^2/m^3)
Mercury	3.18×10^{23}	2.43×10^{6}	7.60×10^{6}	5.79×10^{10}	2.97×10^{-19}
Venus	4.88×10^{24}	6.06×10^{6}	1.94×10^{7}	1.08×10^{11}	2.99×10^{-19}
Earth	5.98×10^{24}	6.37×10^{6}	3.156×10^{7}	1.496×10^{11}	2.97×10^{-19}
Mars	6.42×10^{23}	3.37×10^{6}	5.94×10^{7}	2.28×10^{11}	2.98×10^{-19}
Jupiter	1.90×10^{27}	6.99×10^{7}	3.74×10^{8}	7.78×10^{11}	2.97×10^{-19}
Saturn	5.68×10^{26}	5.85×10^{7}	9.35×10^{8}	1.43×10^{12}	2.99×10^{-19}
Uranus	8.68×10^{25}	2.33×10^{7}	2.64×10^{9}	2.87×10^{12}	2.95×10^{-19}
Neptune	1.03×10^{26}	2.21×10^{7}	5.22×10^{9}	4.50×10^{12}	2.99×10^{-19}
Pluto	$\approx 1.4 \times 10^{22}$	$\approx 1.5 \times 10^{6}$	7.82×10^{9}	5.91×10^{12}	2.96×10^{-19}
Moon	7.36×10^{22}	1.74×10^{6}	—	—	—
Sun	1.991×10^{30}	6.96×10^{8}	—	—	—

The Inverse-Square Law

- **Question: Based on the knowledge Newton might have, what observation can he used to support his law?**

Motion of planets

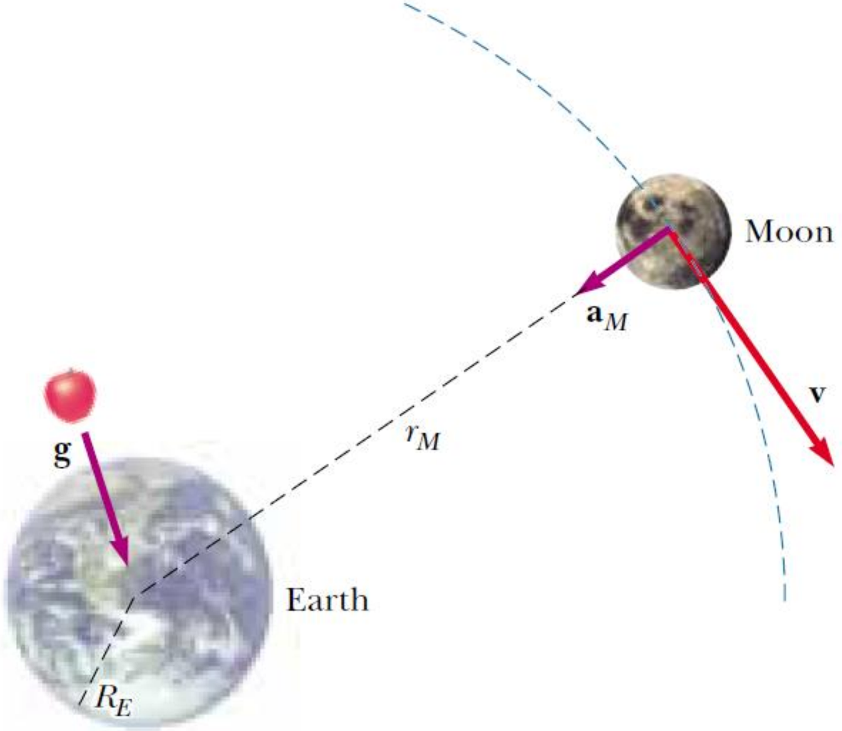
$$m\frac{v^2}{R} = \frac{GMm}{R^2}$$

$$v = \frac{2\pi R}{T} \qquad T^2 = \frac{4\pi^2 R^3}{GM}$$

The square of the orbital period of any planet is proportional to the cube of the radius of the circular orbit – discovered by Johannes Kepler.

The Moon and the Earth

- The earth radius is 6400 km. Eratosthenes (~240 BC) measured the earth radius within 15% error.
- Hipparchus (2nd century BC) measured the earth-moon distance as 400,000 kilometers (6.8% too large).
- So the Moon is roughly 60 Earth radii away.
- The lunar orbit period (a month) is ~27 days.



The Strong Evidence

- **From the lunar distance to earth radius ratio**

- **From the period of the lunar orbit**

Show that the two ways of calculating the moon's acceleration agree.

The Strong Evidence

- **From the lunar distance to earth radius ratio**

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6 \text{ m}}{3.84 \times 10^8 \text{ m}}\right)^2 = 2.75 \times 10^{-4}$$

$$a_M = (2.75 \times 10^{-4})(9.80 \text{ m/s}^2) = 2.70 \times 10^{-3} \text{ m/s}^2$$

- **From the period of the lunar orbit**

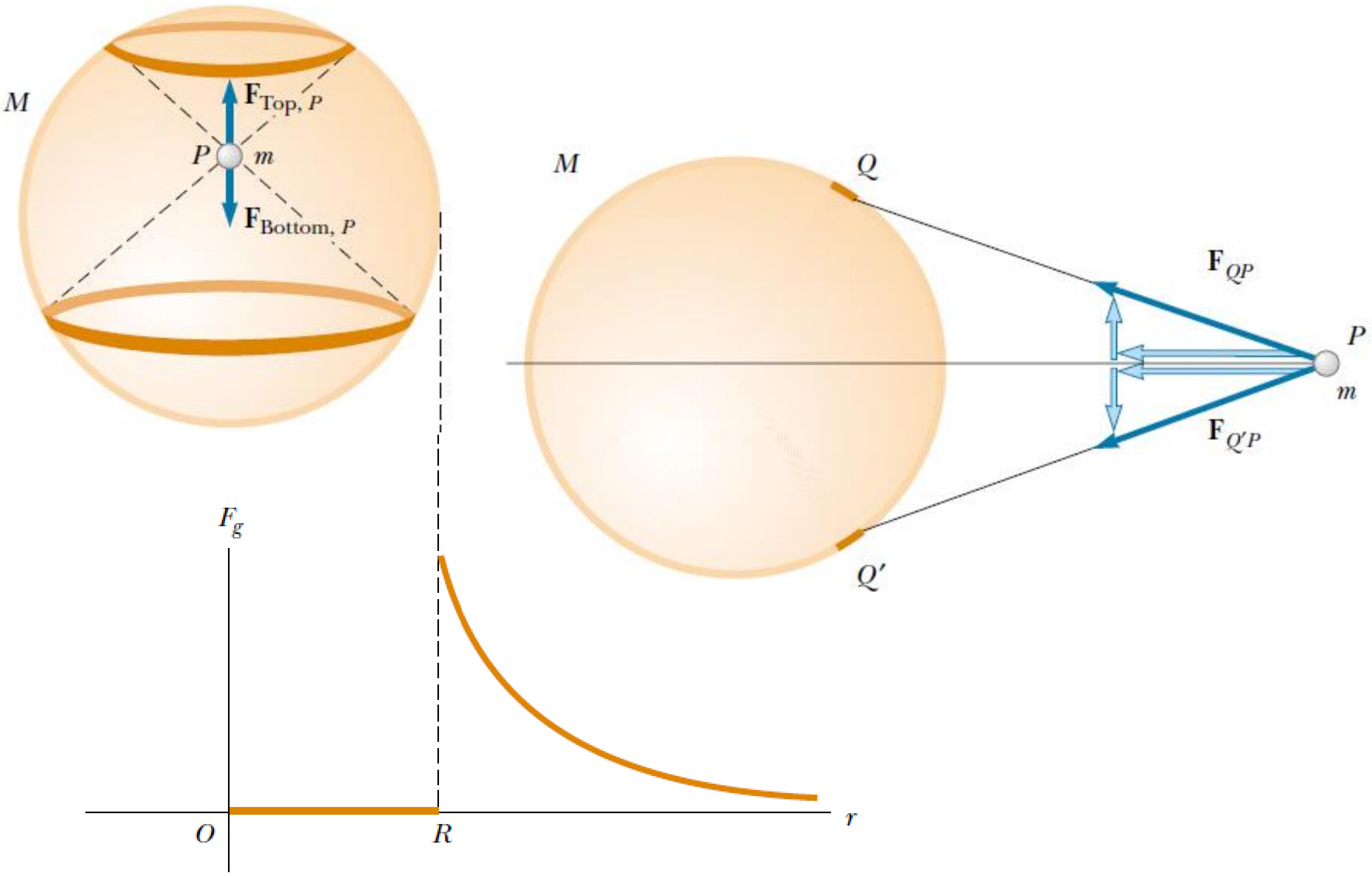
$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2(3.84 \times 10^8 \text{ m})}{(2.36 \times 10^6 \text{ s})^2}$$

$$= 2.72 \times 10^{-3} \text{ m/s}^2 \approx \frac{9.80 \text{ m/s}^2}{60^2}$$

The Assumption

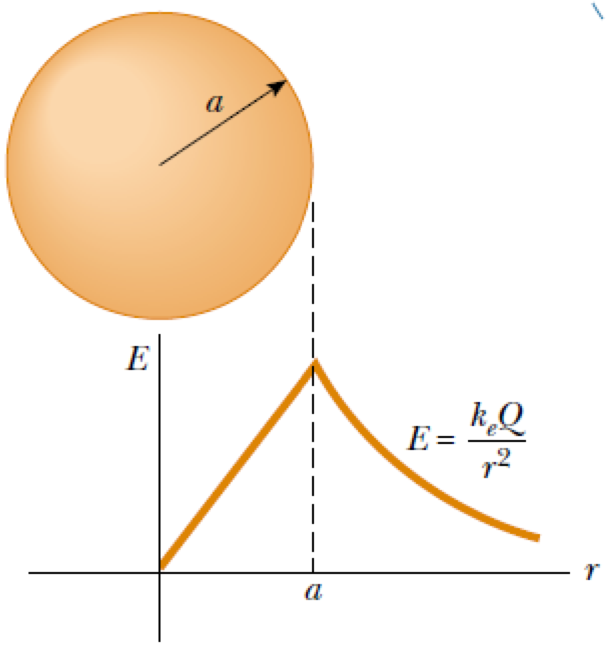
- **To evaluate the acceleration of an object at the Earth's surface, Newton treated the Earth as if its mass were all concentrated at its center.**
- **That is, the gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center.**
- **Newton proved it by calculus in 1687.**

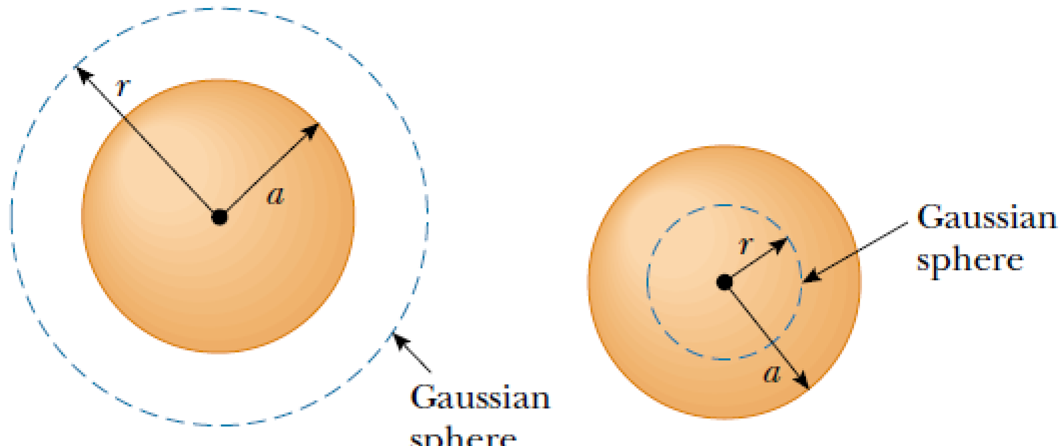
Attraction from a Spherical Mass



E-Field of a Charged Sphere

The electric analogy





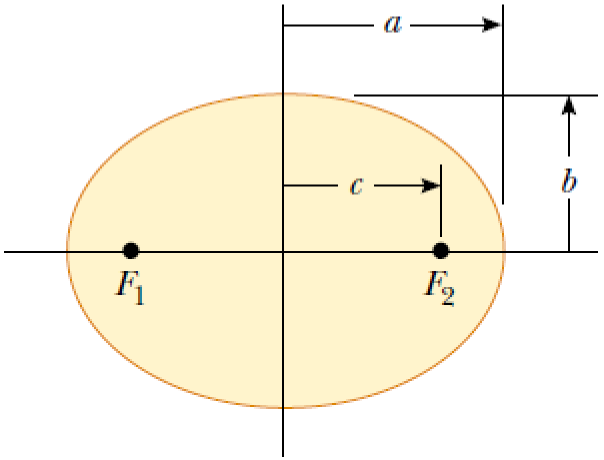
(a) (b)

Gauss's law

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

Kepler's Laws

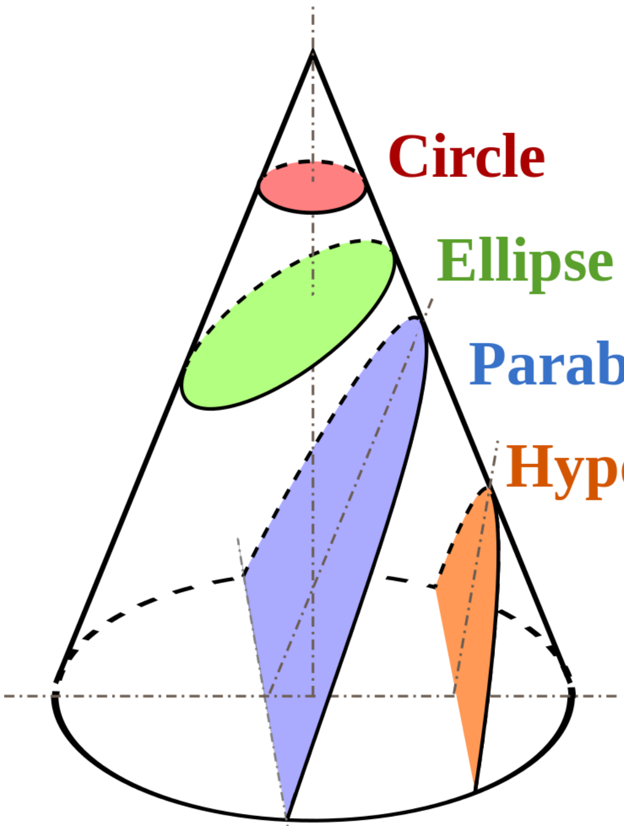
- **All planets move in elliptical orbits with the Sun at one focal point.**
- **(The conservation of angular momentum) The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.**
- **The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.**



http://www.alcyone.com/max/physics/kepler/index.html

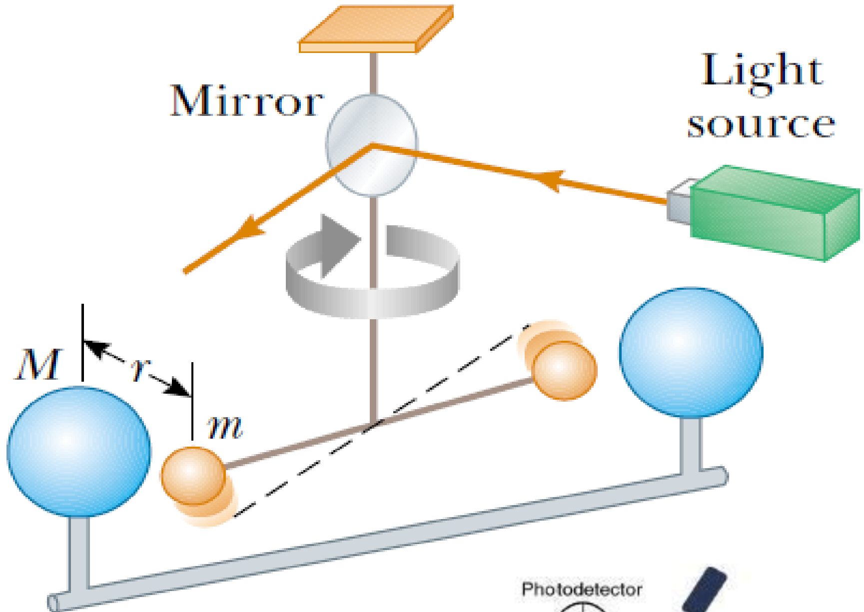
Conic Sections

- According to Newton's analysis, the possible orbits in a gravitational field can take the shape of the figures that are known as conic sections (so called because they may be obtained by slicing sections from a cone, as illustrated in the right figure).

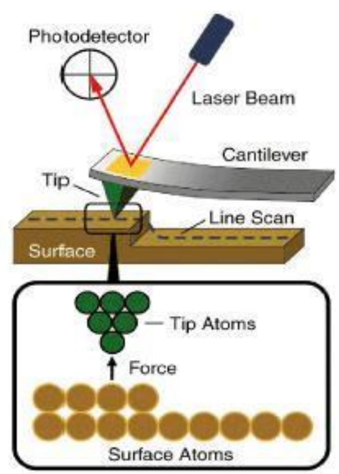


The Gravitational Constant

- In addition to providing a value for G, the results show experimentally that the force is attractive, proportional to the product mM, and inversely proportional to the square of the distance r.



The Cavendish apparatus for measuring G.



The Density of the Earth

- **Can you calculate the density of the earth based on g, G, and the radius of the earth?**

Henry Cavendish (1731–1810) in 1798 measured the constant *G*, which could lead to the amazing result.

The Density of the Earth

- **Can you calculate the density of the earth based on g, G, and the radius of the earth?**

$$mg = G\frac{M_E m}{{R_E}^2} \Rightarrow g = G\frac{M_E}{{R_E}^2}$$

$$\rho_E = \frac{M_E}{V_E} = \frac{M_E}{\frac{4}{3}\pi {R_E}^3} = \boxed{5.50 \times 10^3\,\mathrm{kg/m^3}}$$

Henry Cavendish (1731–1810) in 1798 measured the constant *G*, which could lead to the amazing result.

Common Rock Density

In units of g/cm^3

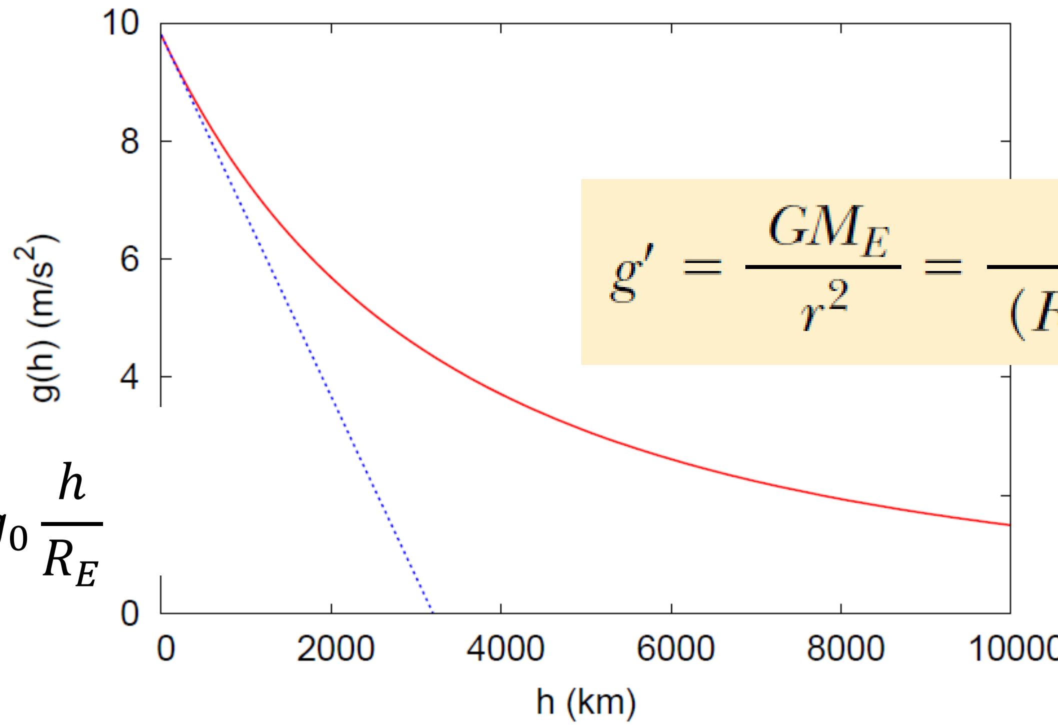
Andesite	2.5 - 2.8	Limestone	2.3 - 2.7
Basalt	2.8 - 3.0	Marble	2.4 - 2.7
Coal	1.1 - 1.4	Mica schist	2.5 - 2.9
Diabase	2.6 - 3.0	Peridotite	3.1 - 3.4
Diorite	2.8 - 3.0	Quartzite	2.6 - 2.8
Dolomite	2.8 - 2.9	Rhyolite	2.4 - 2.6
Gabbro	2.7 - 3.3	Rock salt	2.5 - 2.6
Gneiss	2.6 - 2.9	Sandstone	2.2 - 2.8
Granite	2.6 - 2.7	Shale	2.4 - 2.8
Gypsum	2.3 - 2.8	Slate	2.7 - 2.8

The core of the earth ought to be of higher density!

Variation of Free-Fall Acceleration

- **Now consider an object of mass *m* located a distance *h* above the Earth's surface or a distance *r* from the Earth's center, where $r = R_E + h$.**

$$F_g = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$



$$g' = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2}$$

$$g' \approx g_0 - 2g_0\frac{h}{R_E}$$

Satellite Orbit

Total mechanical energy (neglecting the kinetic energy of the larger body) is conserved:

$$\boxed{E = \frac{1}{2}mv^2 - \frac{GMm}{r}}$$

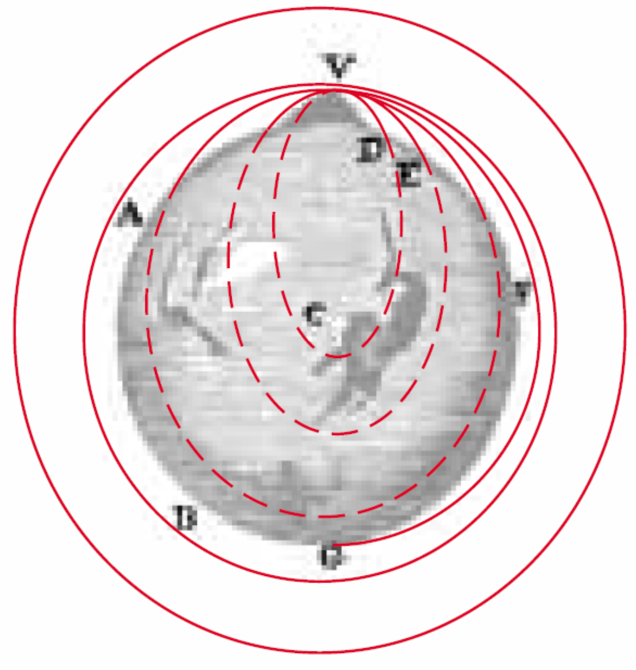
$$\frac{GMm}{r^2} = ma = \frac{mv^2}{r} \Rightarrow \boxed{E = -\frac{GMm}{2r}}$$

Geosynchronous orbit: (T = 1 day = 86 400 s)

$$T = \sqrt{\frac{4\pi^2 R^3}{GM_E}} \Rightarrow r_s = 4.23 \times 10^4\ km$$

Satellite Orbit

- **What is the minimum speed to launch a satellite from the earth?**

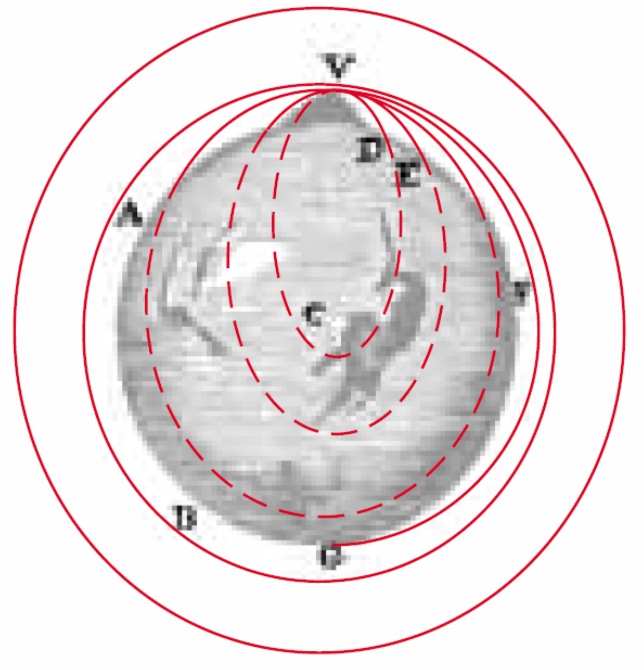


Satellite Orbit

- **What is the minimum speed to launch a satellite from the earth?**

$$\frac{1}{2}mv^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{2R_E}$$

$$v = \sqrt{\frac{GM_E}{R_E}} = 7.8\ km/s$$

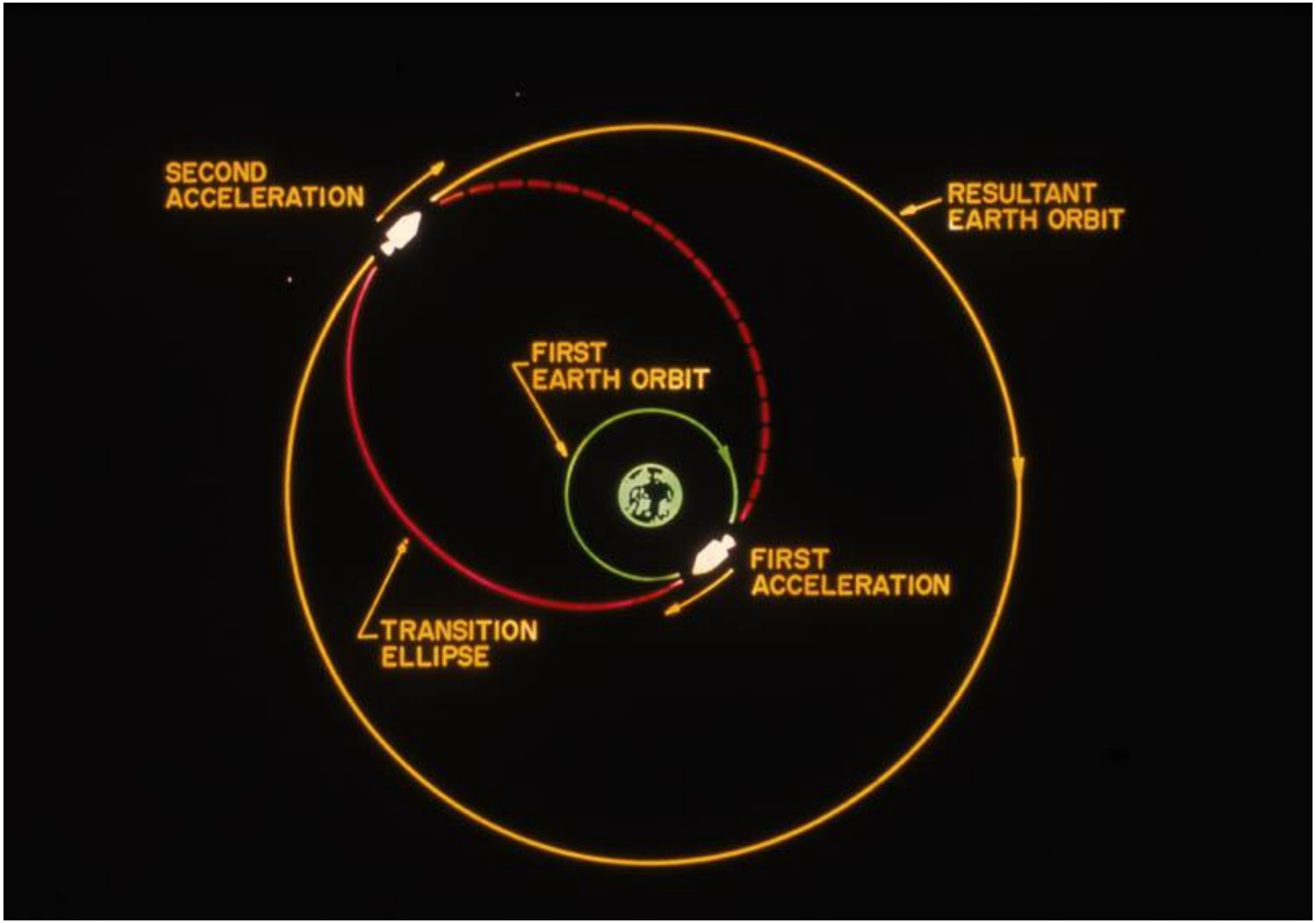


Changing the Orbit of a Satellite

- **The space shuttle releases a 470-kg communications satellite while in an orbit that is 280 km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth. How much energy did the engine have to provide?**

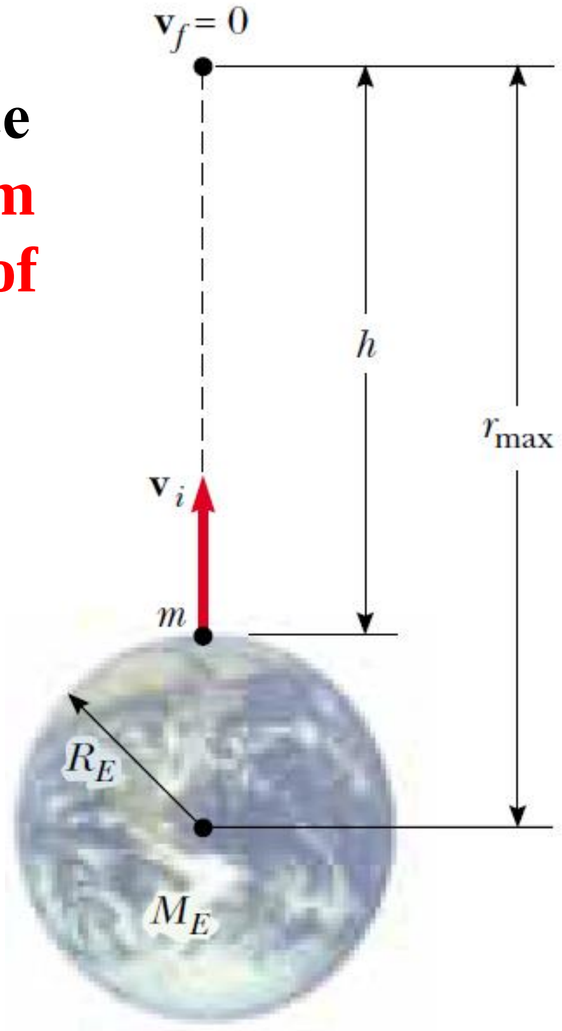
$$E_{\text{engine}} = E_f - E_i = -\frac{GM_E m}{2}\left(\frac{1}{R_f} - \frac{1}{R_i}\right) = \boxed{1.19 \times 10^{10}\ \text{J}}$$

Changing Orbit



Escape Speed (from the Earth)

Suppose an object of mass *m* is projected vertically upward from the Earth's surface with an initial speed. What is the minimum speed for the object to escape the gravity of the earth?



Escape Speed (from the Earth)

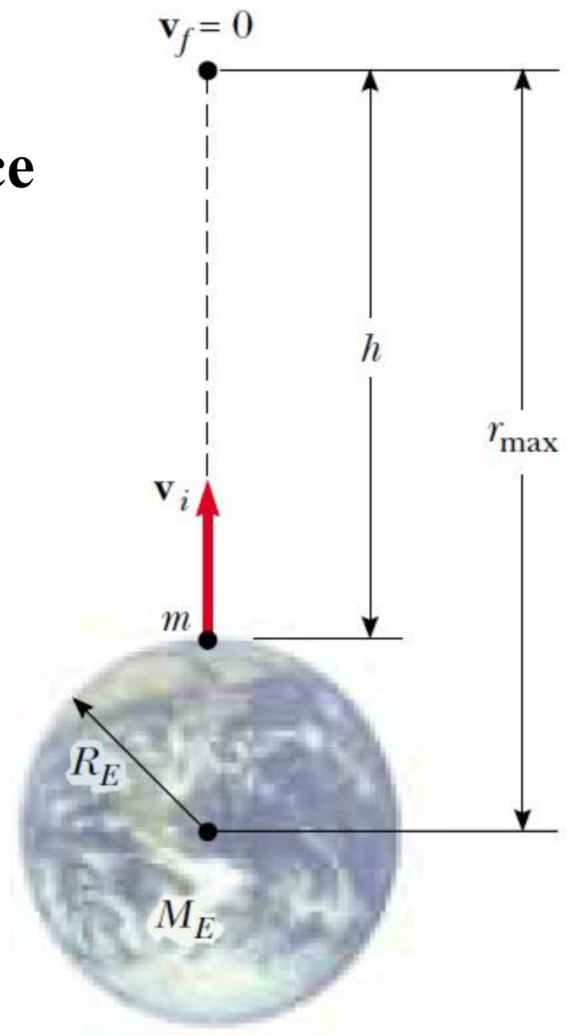
Suppose an object of mass *m* is projected vertically upward from the Earth's surface with an initial speed.

$$\frac{1}{2} m v_i^2 - \frac{G M_E m}{R_E} = -\frac{G M_E m}{r_{\max}}$$

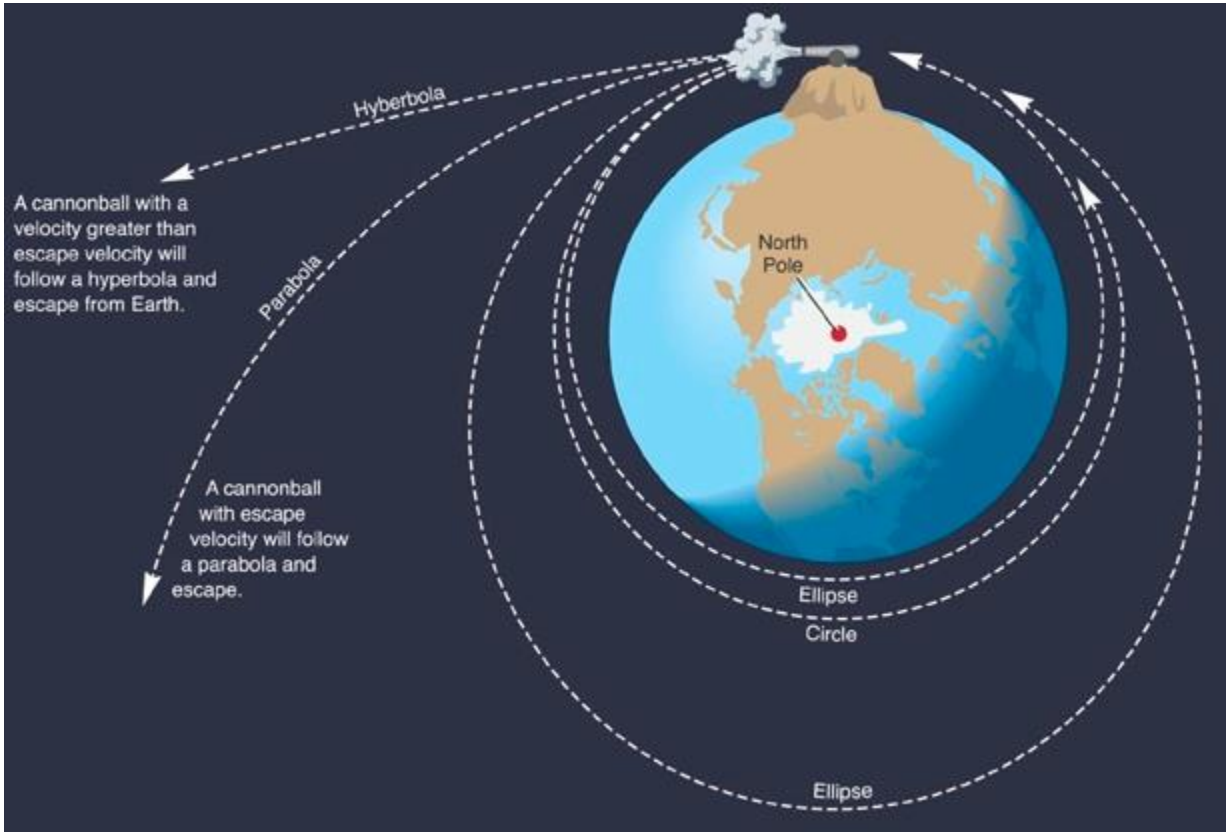
$$v_i^2 = 2 G M_E \left(\frac{1}{R_E} - \frac{1}{r_{\max}} \right)$$

$r_{\max} \rightarrow \infty$

$$\boxed{v_{\mathrm{esc}} = \sqrt{\frac{2 G M_E}{R_E}}} = 11.2\ km/s$$

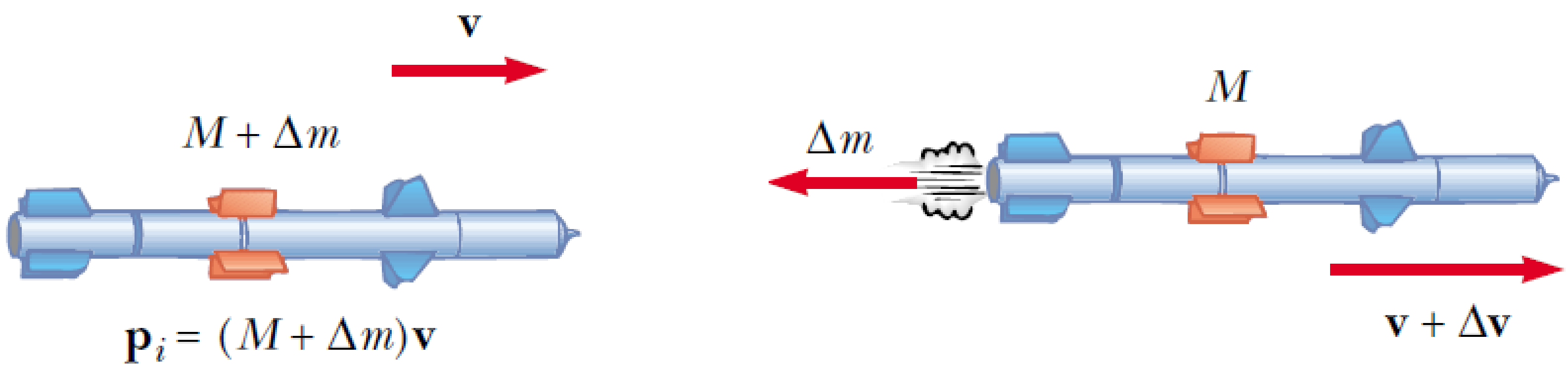


Newton's Mountain



Rocket Propulsion

- **The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel.**



$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

$$M\Delta v = v_e \Delta m$$

Rocket Propulsion

$$M\,\Delta v = v_e\,\Delta m \quad \Rightarrow \quad M\,dv = v_e\,dm = -\,v_e\,dM$$

$$\int_{v_i}^{v_f} dv = -\,v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$

The mass of the rocket without its fuel should be as small as possible and the rocket should carry as much fuel as possible.

The rocket represents a case of the reverse of a perfectly inelastic collision: Momentum is conserved, but the kinetic energy of the system increases (at the expense of chemical potential energy in the fuel).

$$\boxed{M\vec{a} = \vec{F}_{ext} + \vec{v}_{\mathrm{rel}}\frac{\mathrm{d}M}{\mathrm{d}t}} \quad \Rightarrow \quad \boxed{v = v_{\mathrm{rel}} \ln\frac{M_0}{M} - gt}$$

Thrust of a Rocket

- The **thrust** on the rocket is the force exerted on it by the ejected exhaust gases.

$$\text{Thrust} = M\frac{dv}{dt} = \left| v_e \frac{dM}{dt} \right|$$

- The thrust increases as the **exhaust speed** increases and as the rate of change of mass (called the **burn rate**) increases.

Exercise: Rocket Propulsion

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to one-half its mass before ignition?

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

Exercise: Rocket Propulsion

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to one-half its mass before ignition?

$$v_f = v_i + v_e \ln\left(\frac{M_i}{M_f}\right) = \boxed{6.5 \times 10^3 \text{ m/s}}$$

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right| = \boxed{2.5 \times 10^5 \text{ N}}$$

Exercise: The Mass of the Sun

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.156×10^7 s and its distance from the Sun is 1.496×10^{11} m.

Exercise: The Mass of the Sun

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.156×10^{7} s and its distance from the Sun is 1.496×10^{11} m.

$$M_S = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(1.496 \times 10^{11}\ \mathrm{m})^3}{(6.67 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2})(3.156 \times 10^{7}\ \mathrm{s})^2}$$

$$= \boxed{1.99 \times 10^{30}\ \mathrm{kg}}$$

Q: Escape Speed from the Sun

- **What is the minimum speed for a rocket launched from the earth to escape the sun's gravity?**
  - **First, it needs to escape the earth's gravity.**
  - **Second, the residual speed needs to be large enough to escape the sun's gravity.**
  - **Third, one needs to take into account that the earth has a speed in its orbit around the sun.**

Tide, Moon and Sun*