General Physics I

Lecture 12: Applications of Oscillatory Motion

Outline

- The pendulum
- Comparing simple harmonic motion and uniform circular motion
- Damped oscillation and forced oscillation
- Vibration in molecules
- Elastic properties of solids

Simple Pendulum

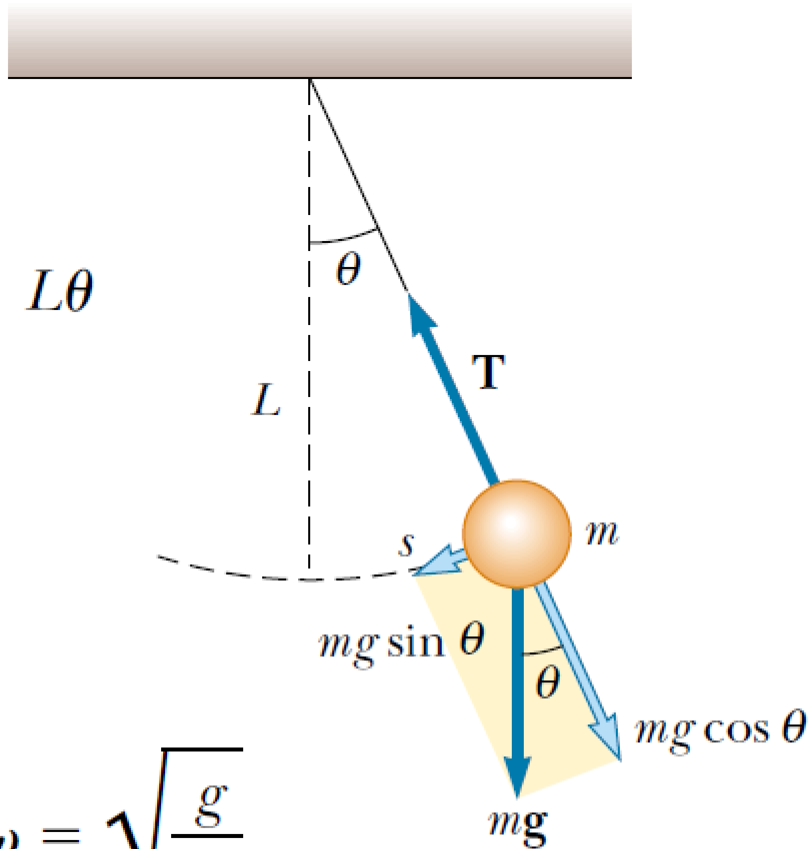
$$\sum F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \qquad s = L\theta$$

$$\sin\theta \approx \theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

$$\theta = \theta_{\max}\cos(\omega t + \phi) \qquad \omega = \sqrt{\frac{g}{L}}$$



Video—MIT单摆周期

Period of the Simple Pendulum

- **The period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity.**

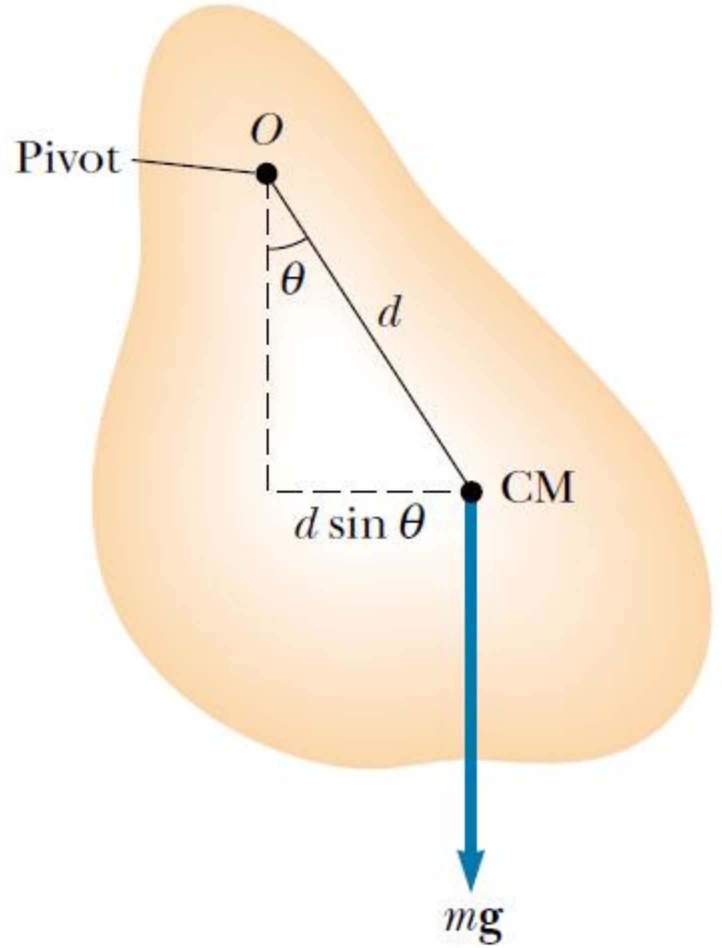
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

Question: Christian Huygens (1629–1695) suggested that an unit of length could be defined as the length of a simple pendulum having a period of exactly 1 s. How long is the length?

$$L = \frac{T^2 g}{4\pi^2} = \frac{(1\ \mathrm{s})^2(9.80\ \mathrm{m/s^2})}{4\pi^2} = \boxed{0.248\ \mathrm{m}}$$

Physical Pendulum

- If a hanging object oscillates **about a fixed axis that does not pass through its center of mass** and the object **cannot be approximated as a point mass**, we cannot treat the system as a simple pendulum. In this case the system is called a **physical pendulum**.



Physical Pendulum

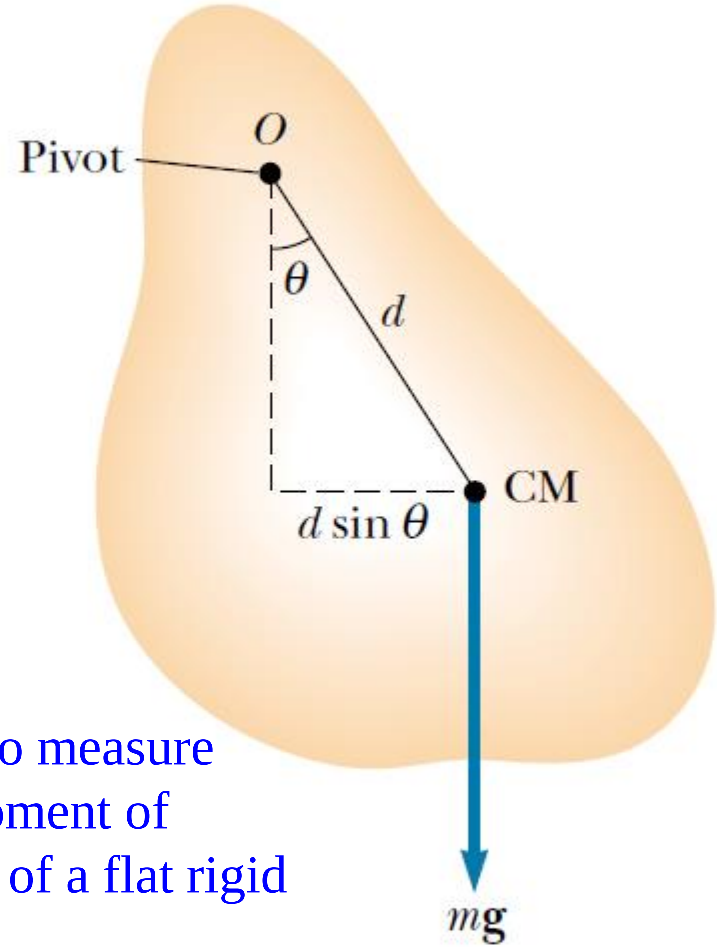
$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

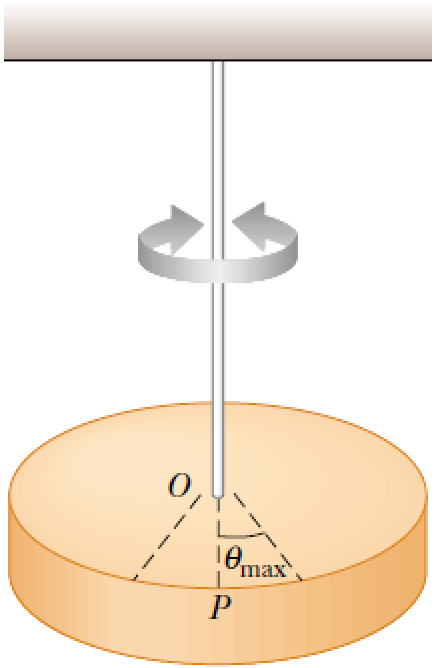
$$\omega = \sqrt{\frac{mgd}{I}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

Used to measure the moment of inertia of a flat rigid body.



Torsional Pendulum



When the body is twisted through some angle, the twisted wire exerts on the body a restoring torque that is proportional to the angular displacement.

$$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

There is no small-angle restriction in this situation as long as the elastic limit of the wire is not exceeded.

Timing & Clocks*…

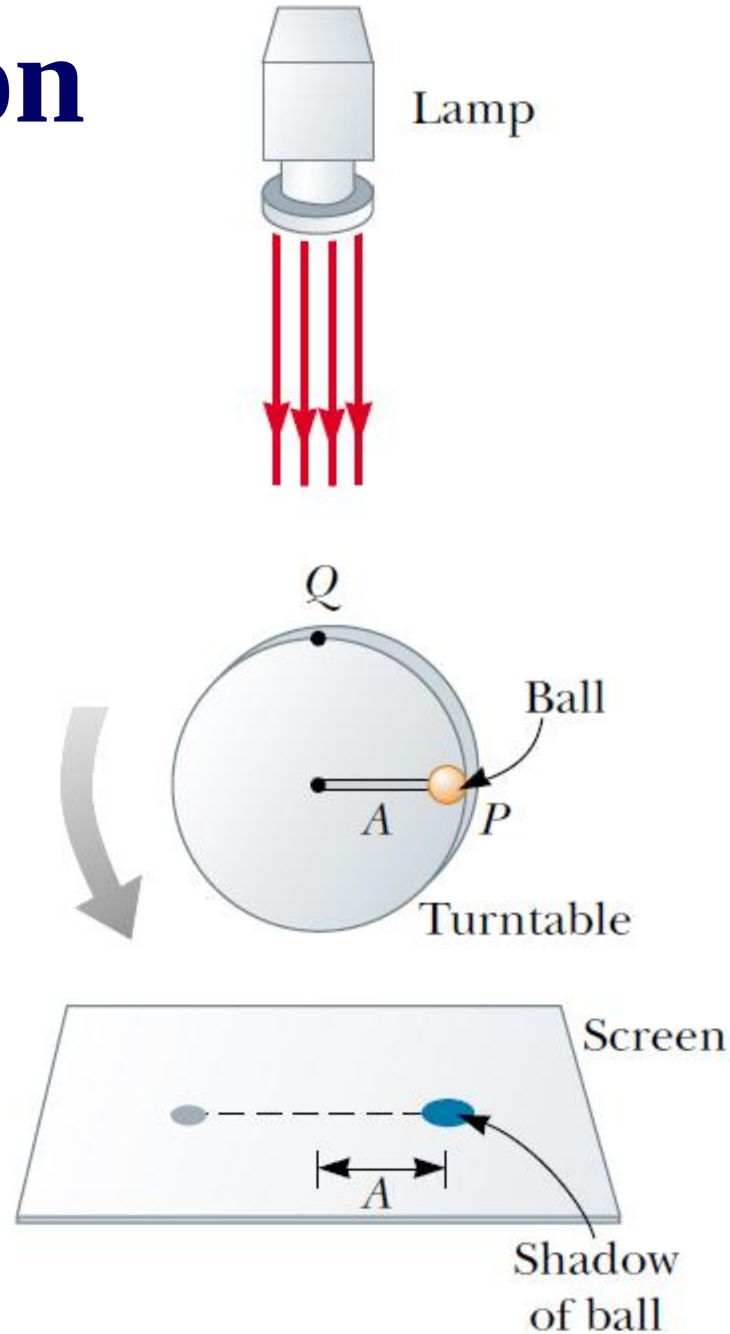
Mechanical Watch:
Hairspring &
Balance Wheel

Quartz clock:
piezoelectric material

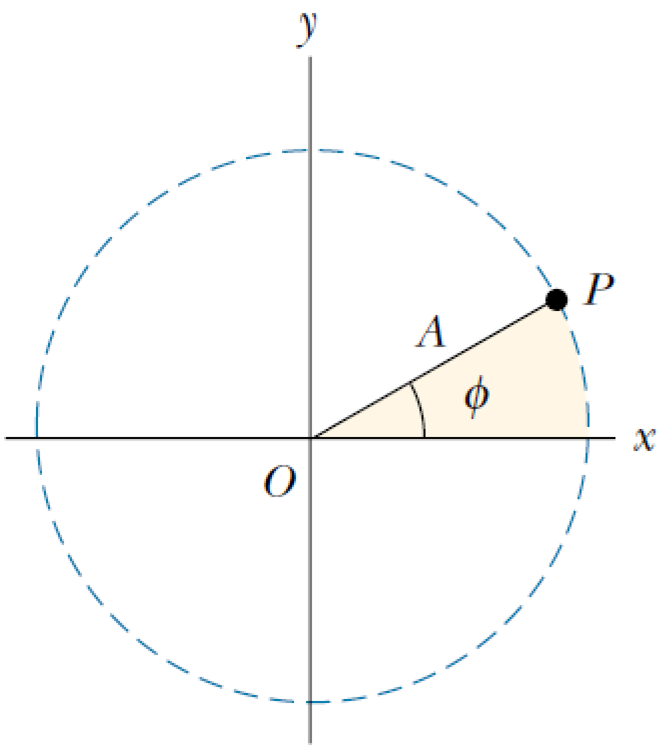
$$f = \frac{1.875104^2}{2\pi} \frac{a}{l^2} \sqrt{\frac{E}{12\rho}},$$

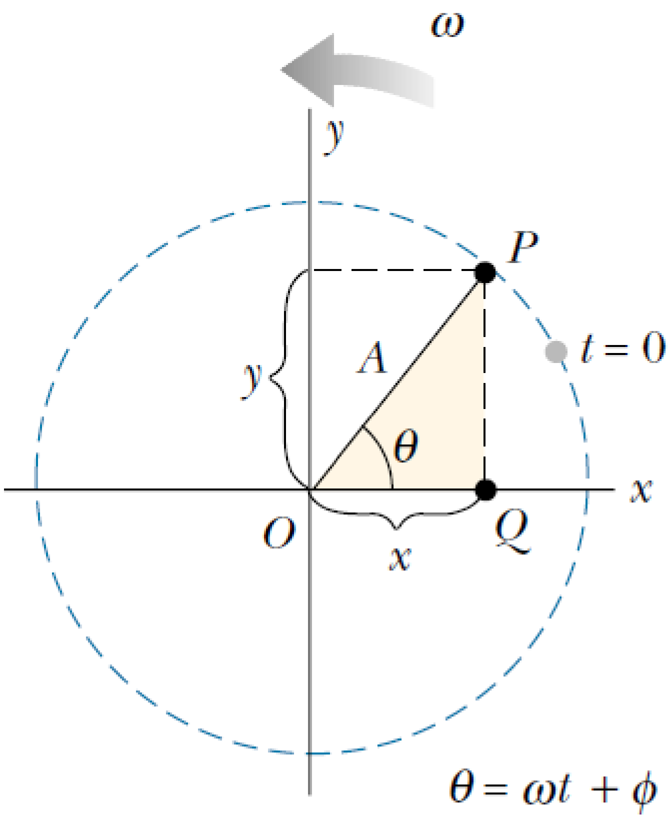
Circular Motion

An experimental setup for demonstrating the connection between simple harmonic motion and uniform circular motion. As the ball rotates on the turntable with constant angular speed, its shadow on the screen moves back and forth in simple harmonic motion.



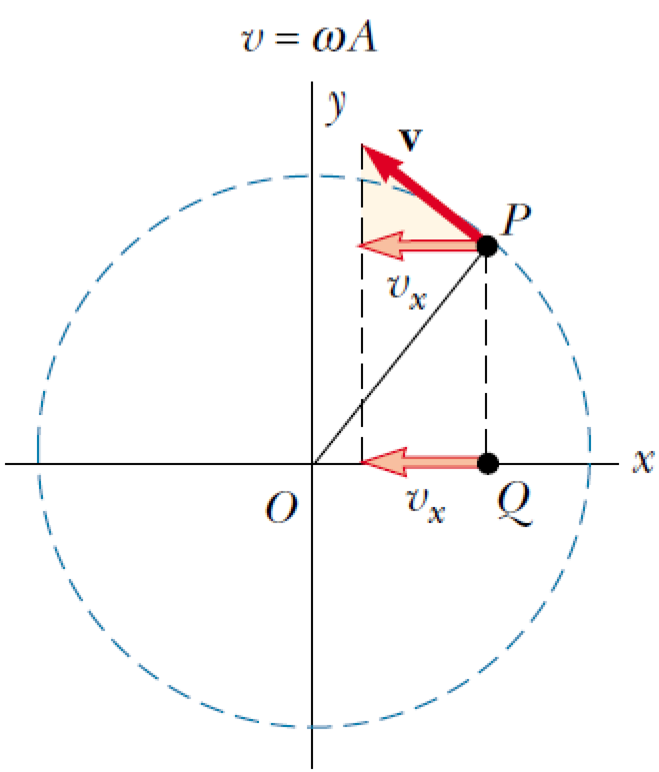
Oscillation vs. Circular Motion

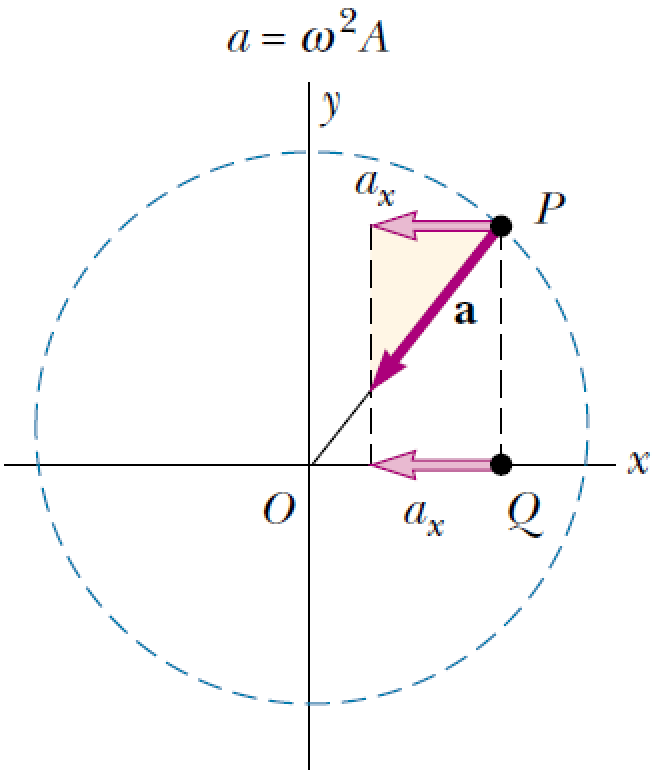




$$x = A\cos(\omega t + \phi)$$

Oscillation vs. Circular Motion





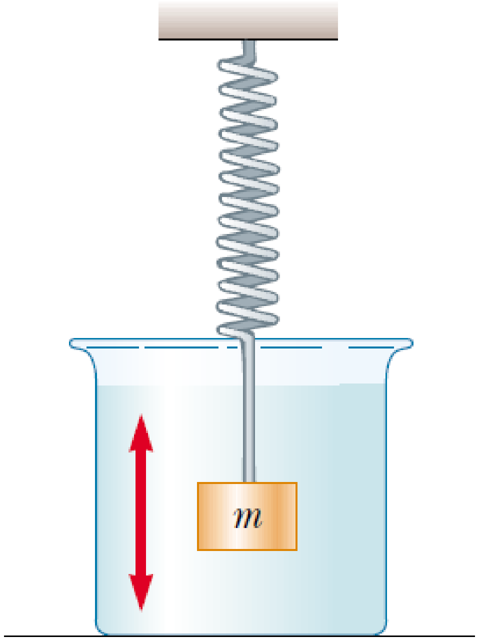
$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \qquad a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi)$$

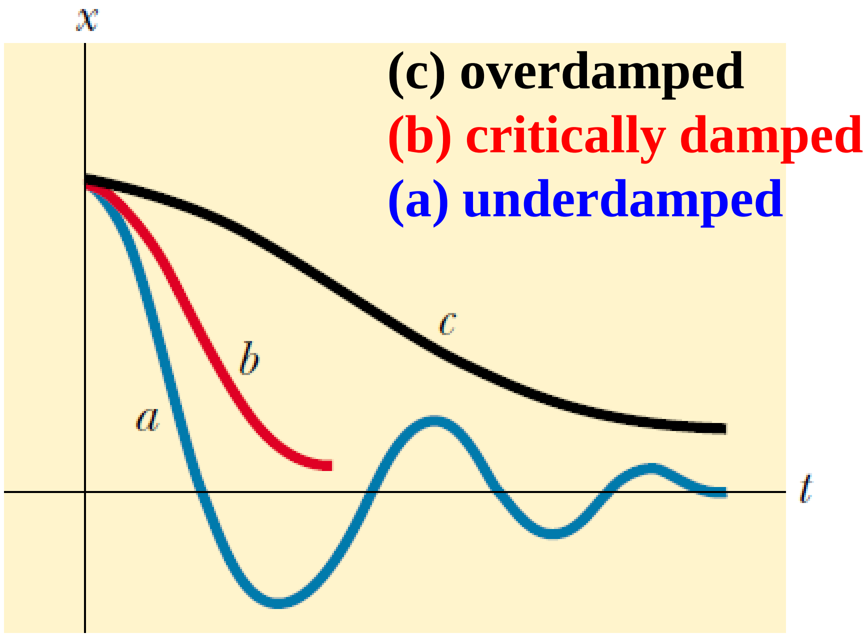
Oscillation vs. Circular Motion

- **Simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.**
- **Uniform circular motion can be considered as a combination of two simple harmonic motions, one along the *x* axis and one along the *y* axis, with the two differing in phase by 90°.**

Damped Oscillator

- When the retarding force is much smaller than the restoring force, the oscillatory character of the motion is preserved but the amplitude decreases in time, with the result that the motion ultimately ceases.



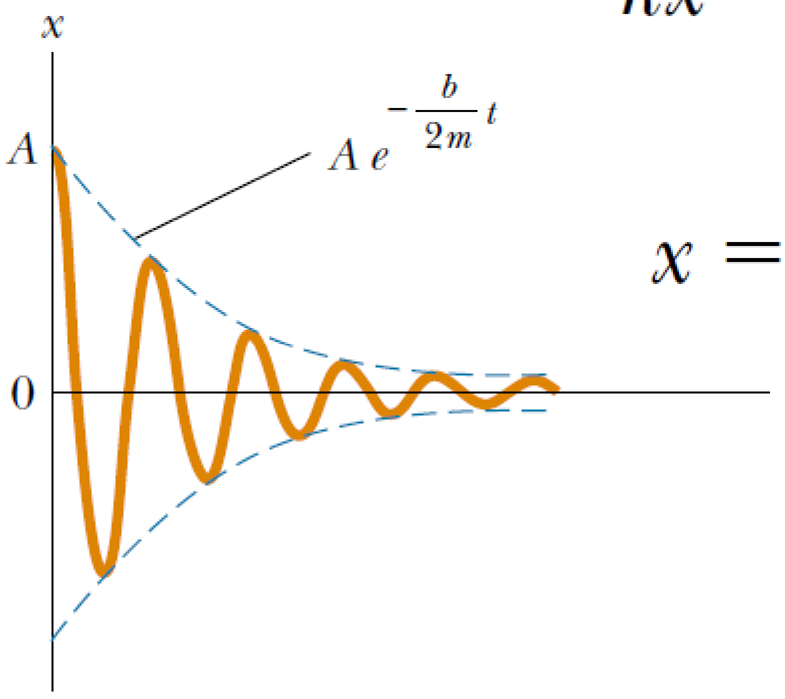


Damped Oscillator

***b*: damping coefficient**

$$\sum F_x = -kx - bv = ma_x$$

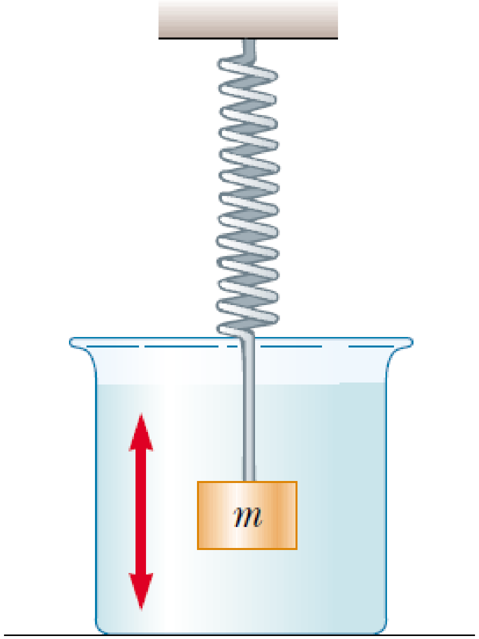
$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

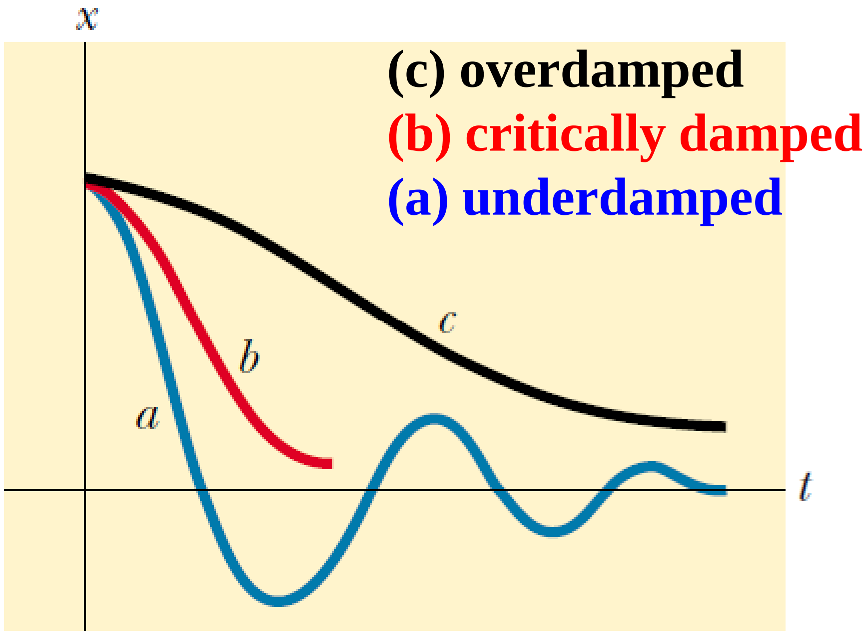


$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Critical Damping





$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

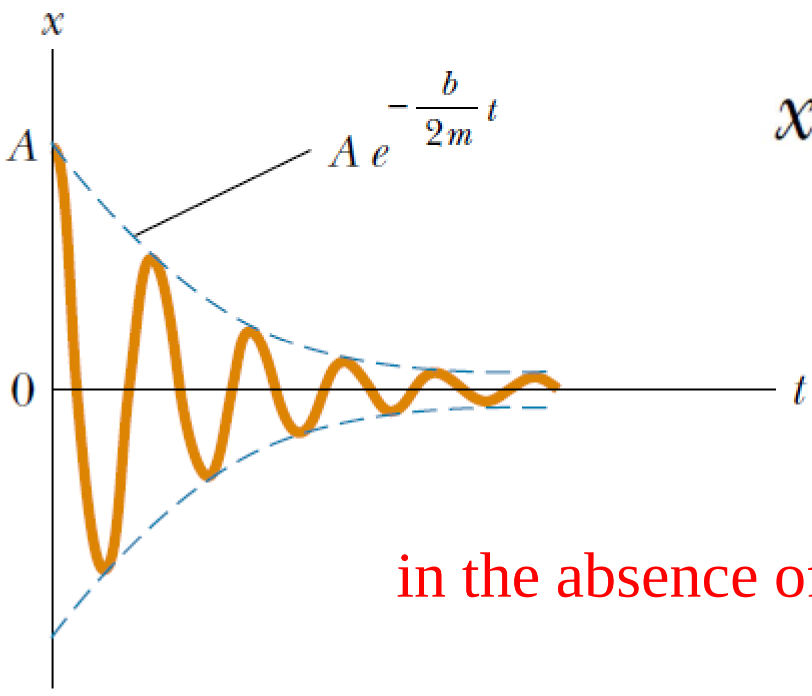
Critical damping:

$$\frac{k}{m} = \left(\frac{b}{2m}\right)^2$$

Forced Oscillation

$$\boxed{F_{\text{ext}} \cos \omega t} - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

We are interested in the underdamped case.



$$x = A'e^{-\frac{b}{2m}t}\cos(\omega' t + \phi')$$

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

in the absence of the driving force

Decomposition of Motion

$$\boxed{F_{\text{ext}} \cos \omega t} - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

In the presence of the driving force

Transient solution

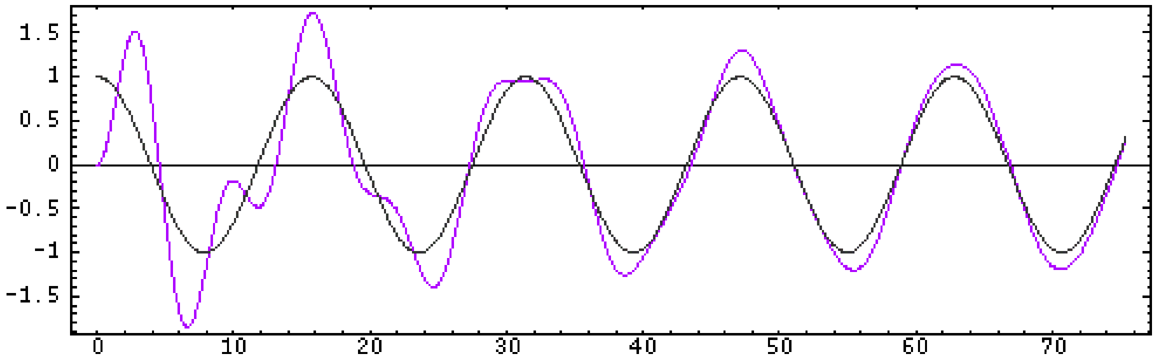
Steady solution

$$x = A'e^{-\frac{b}{2m}t}\cos(\omega' t + \phi') + A\cos(\omega t + \phi)$$

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Slow Drive

$$\boxed{F_{\text{ext}} \cos \omega t} - kx - b\cancel{\frac{dx}{dt}} = m\cancel{\frac{d^2x}{dt^2}}$$

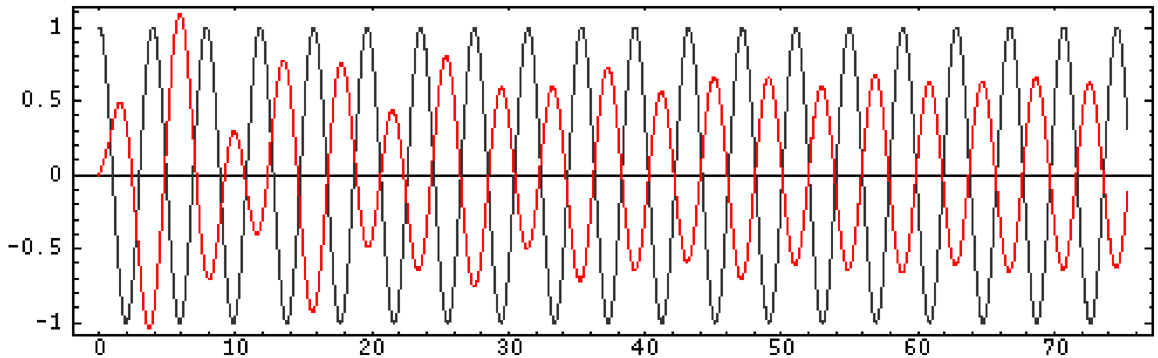


The driving force is slow enough that the oscillator can follow the force after the transient motion decays.

$$\omega < \omega_0 = \sqrt{k/m}$$

Fast Drive

$$\boxed{F_{\text{ext}} \cos \omega t} - \cancel{kx} - b\cancel{\frac{dx}{dt}} = m\frac{d^2 x}{dt^2}$$



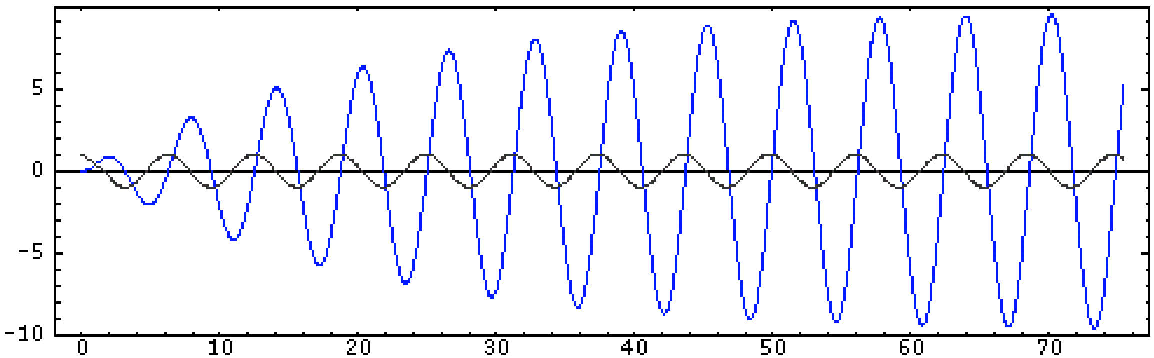
The driving force is fast such that the oscillator cannot follow the force and lags behind (π out of phase). Note that the amplitude is smaller than that for slow drive.

$$\omega > \omega_0 = \sqrt{k/m}$$

At Resonance

$$\boxed{F_{\text{ext}} \cos \omega t} - \cancel{kx} - b\frac{dx}{dt} = m\cancel{\frac{d^2x}{dt^2}}$$

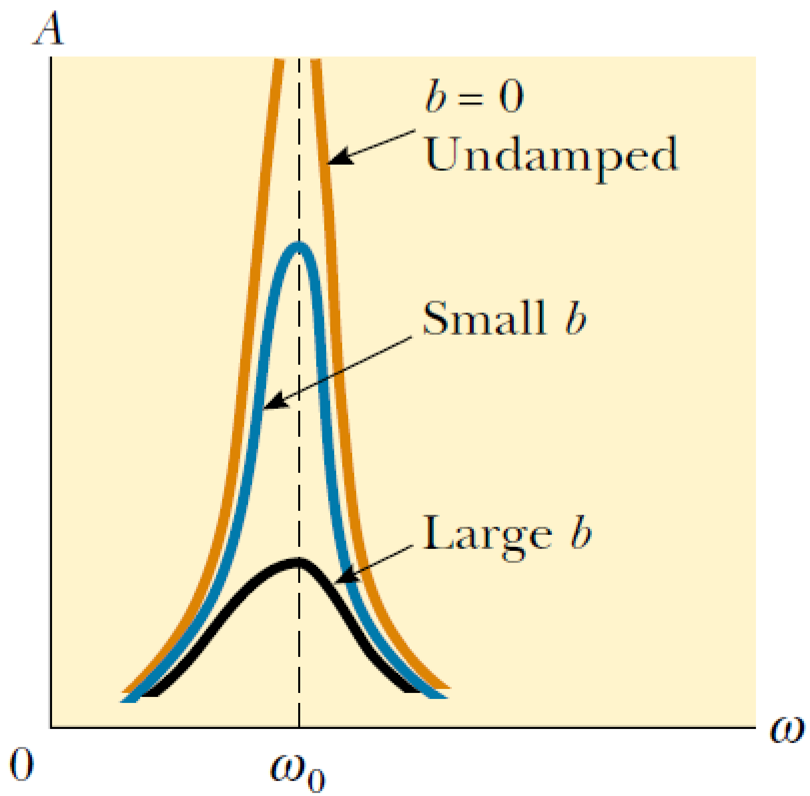
$$\omega = \omega_0 = \sqrt{k/m}$$



The amplitude quickly grows to a maximum. After the transient motion decays and the oscillator settles into steady state motion, the displacement π/2 out of phase with force (displacement lags the force).

Forced Oscillation

$$F_{\text{ext}} \cos \omega t - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$



Steady state:

$$x = A\cos(\omega t + \phi)$$

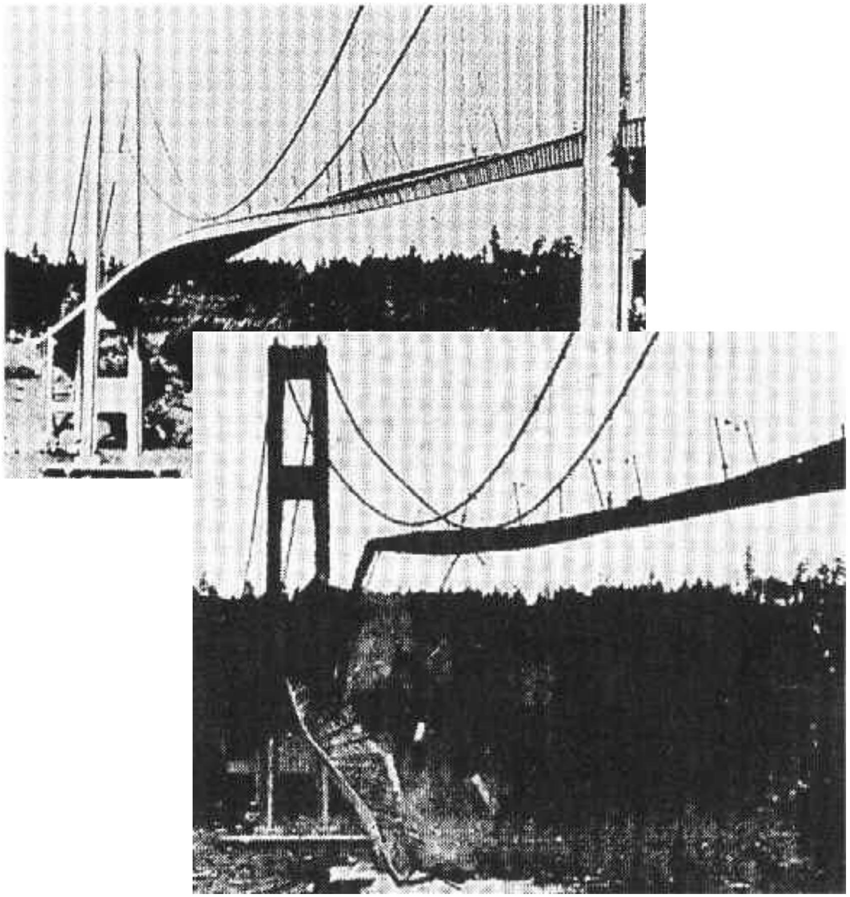
$$A = \frac{F_{\text{ext}}/m}{\sqrt{(\omega^2 - \omega_0{}^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

$$\omega_0 = \sqrt{k/m}$$

Resonance Frequency

- **For small damping, the amplitude becomes very large when the frequency of the driving force is near the natural frequency of oscillation. The dramatic increase in amplitude near the natural frequency ω_0 is called resonance, and for this reason ω_0 is sometimes called the resonance frequency of the system.**
- **At resonance the applied force is in phase with the velocity and that the power transferred to the oscillator is a maximum.**

Resonance Demo



Disastrous consequence!

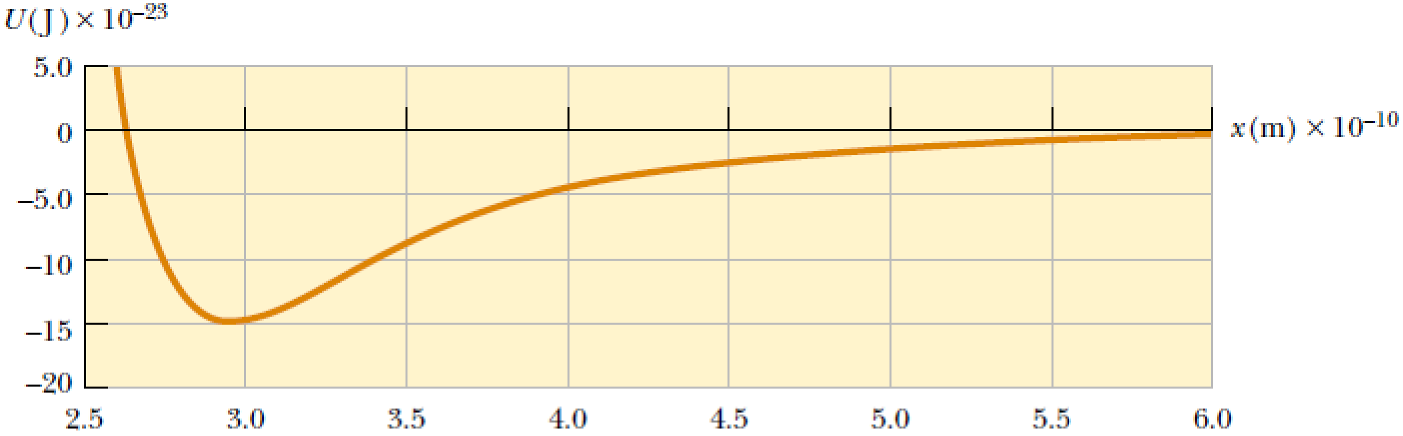
Video—MIT Resonance-2

Video--crashing bridge by wind!

Lennard–Jones Potential

- **The potential energy associated with the force between a pair of neutral atoms or molecules can be modeled by the Lennard–Jones potential energy function:**

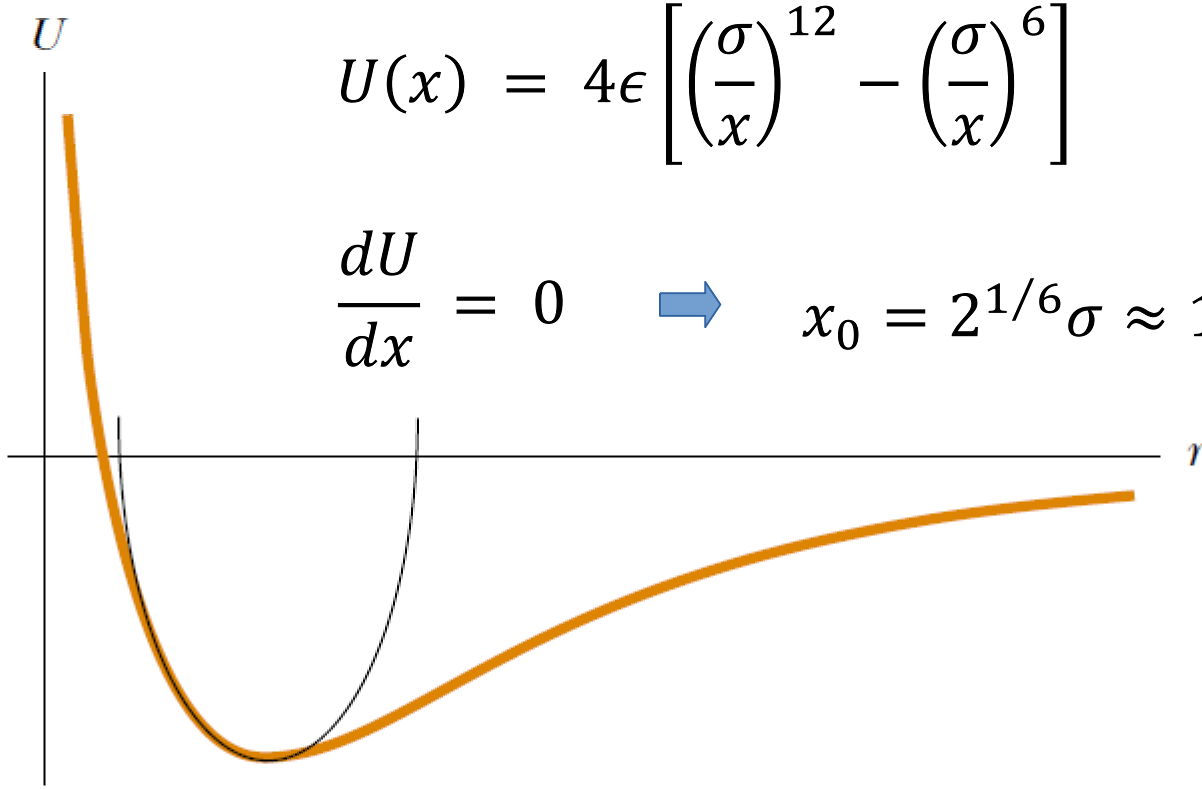
$$U(x) = 4\epsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$



The Equilibrium

$$U(x) = 4\epsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$

$$\frac{dU}{dx} = 0 \quad \Rightarrow \quad x_0 = 2^{1/6}\sigma \approx 1.122\sigma$$



We can approximate the complex atomic/molecular binding forces as tiny springs.

Force Near the Equilibrium

$$U(x) = 4\epsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right] = \epsilon\left[\left(\frac{x_0}{x}\right)^{12} - 2\left(\frac{x_0}{x}\right)^{6}\right]$$

$$F(x) = -\frac{dU(x)}{dx} = \frac{12\epsilon}{x_0}\left[\left(\frac{x_0}{x}\right)^{13} - \left(\frac{x_0}{x}\right)^{7}\right]$$

$$= -\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x - x_0) + O\left((x - x_0)^2\right)$$

$$\approx -\frac{72\epsilon}{x_0^2}(x - x_0)$$

Vibration Frequency

Effective spring constant:

$$k = \frac{72\epsilon}{x_0^2}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{72\epsilon}{\mu x_0^2}}$$

Reduced mass!

Example: Vibration of two water molecules

$$\sigma = 0.32 \times 10^{-9}\, m$$

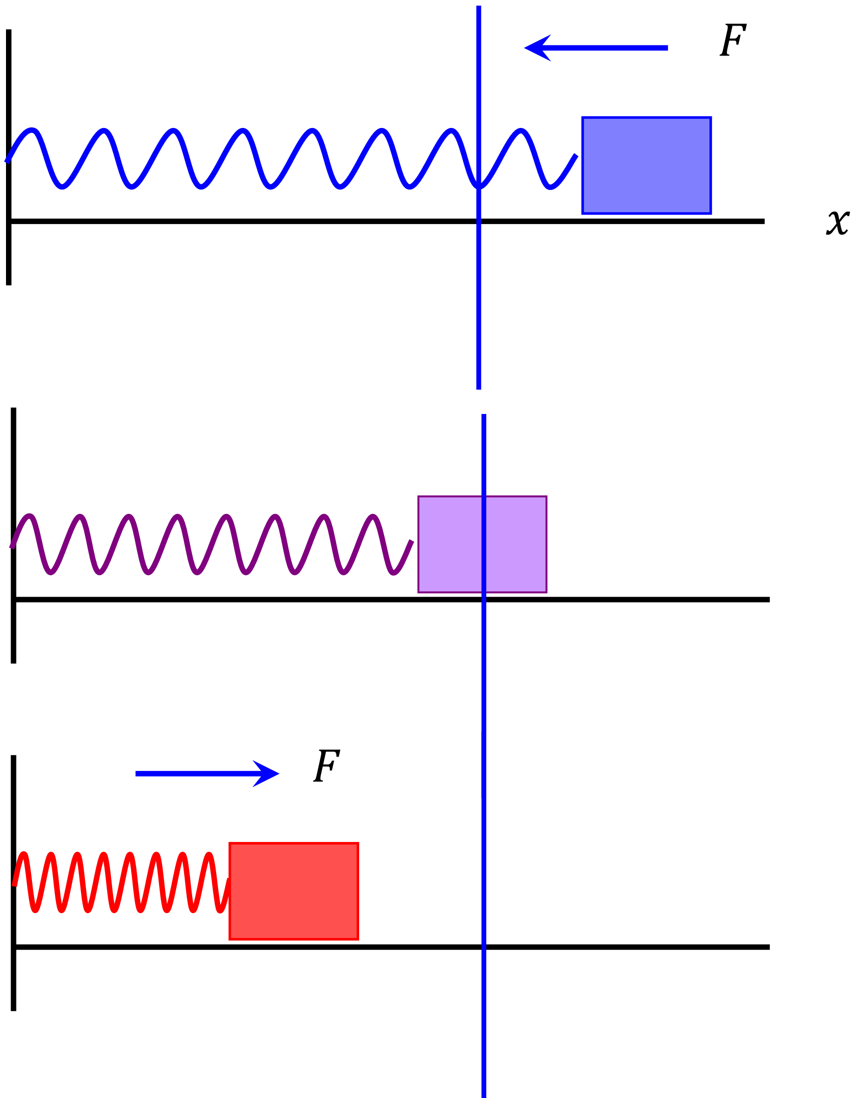
$$\epsilon = 1.08 \times 10^{-21}\, J$$

$$\mu \approx 9m_{proton}$$

Why?

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{72\epsilon}{\mu x_0^2}} \approx 10^{12} Hz$$

Block-Spring System Revisit



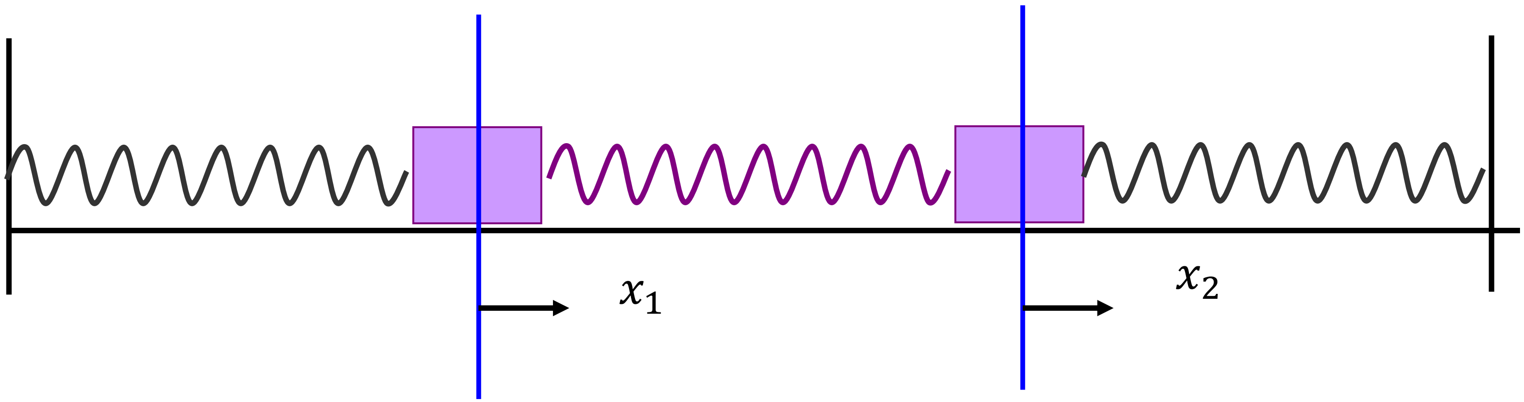
$$U(x) = \frac{1}{2}\mathrm{k}\mathrm{x}^2$$

$$F = -\frac{\mathrm{dU}(x)}{\mathrm{dx}} = -\mathrm{kx} = m\frac{d^2x}{\mathrm{dt}^2}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \omega^2 = \frac{k}{m}$$

$$x = x_0 \cos(\omega t + \varphi)$$

Two Harmonic Oscillators



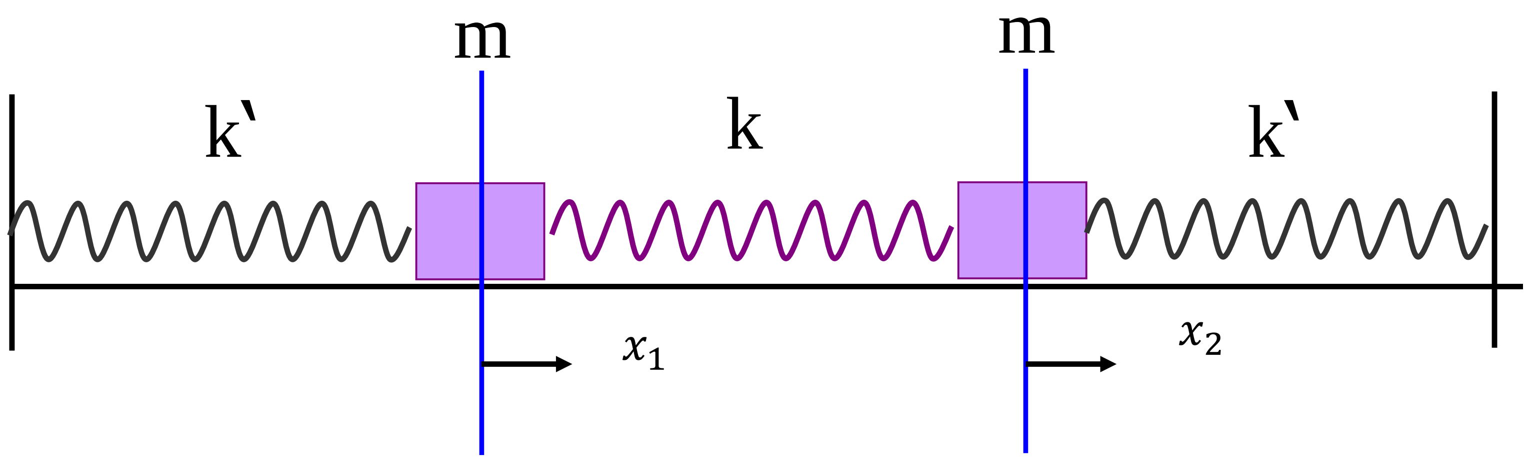
$$m\frac{d^2x_1}{dt^2} = -k'x_1 - k(x_1 - x_2)$$

$$\frac{d^2(x_1 + x_2)}{dt^2} = -\frac{k'}{m}(x_1 + x_2)$$

$$m\frac{d^2x_2}{dt^2} = -k'x_2 - k(x_2 - x_1)$$

$$\frac{d^2(x_1 - x_2)}{dt^2} = -\frac{k' + 2k}{m}(x_1 - x_2)$$

Two Harmonic Oscillators



$$m\frac{d^2x_1}{\mathrm{dt}^2} = -k'x_1 - k(x_1 - x_2)$$

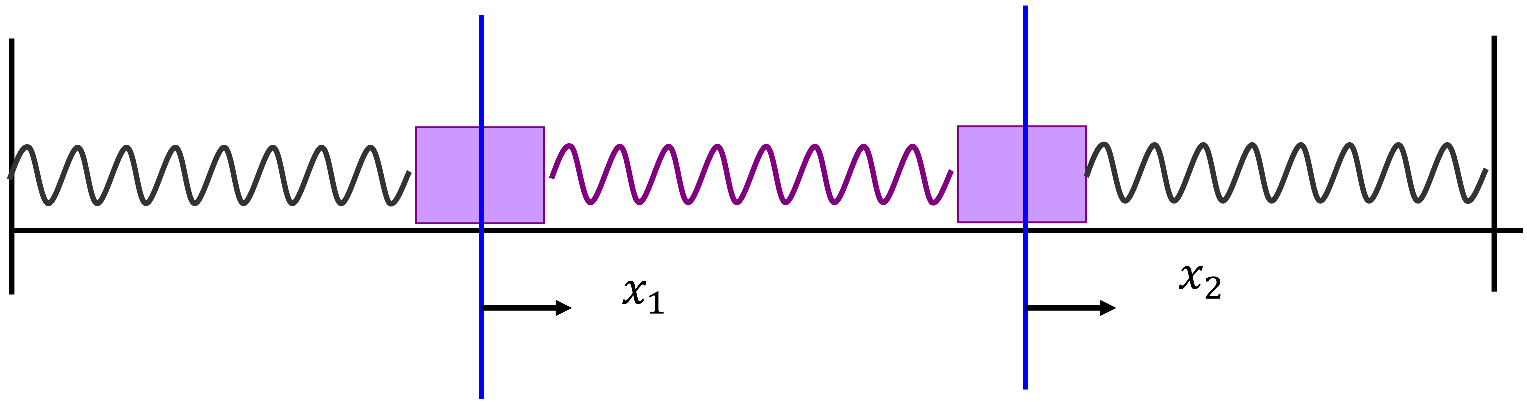
$$m\frac{d^2x_2}{\mathrm{dt}^2} = -k'x_2 - k(x_2 - x_1)$$

$$m\omega^2 x_{10} = (k' + k)x_{10} - \mathrm{k}x_{20}$$

$$m\omega^2 x_{20} = -\mathrm{k}x_{10} + (k' + k)x_{20}$$

Assume $x_i = x_{i0}\cos(\omega t + \varphi)$

Two Harmonic Oscillators



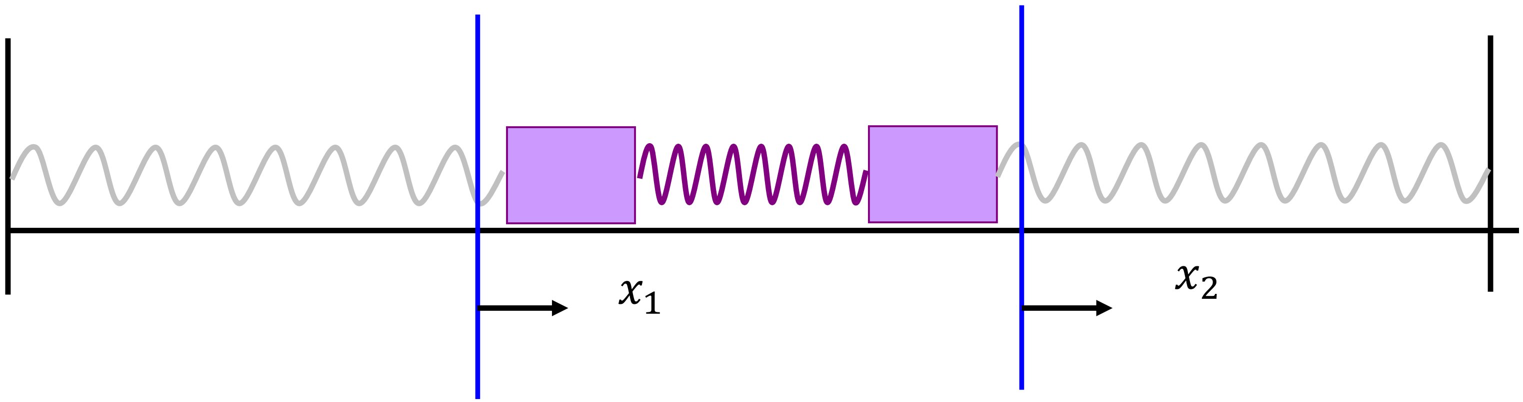
Assume $x_i = x_{i0}\cos(\omega t + \varphi)$

$$m\omega^2 x_{10} = (k' + k)x_{10} - \mathrm{k}x_{20} \qquad m\omega^2 x_{20} = -\mathrm{k}x_{10} + (k' + k)x_{20}$$

Determinant be zero

$$\begin{pmatrix} k' + k - m\omega^2 & -k \\ -k & k' + k - m\omega^2 \end{pmatrix}\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = 0$$

Vibrational Mode



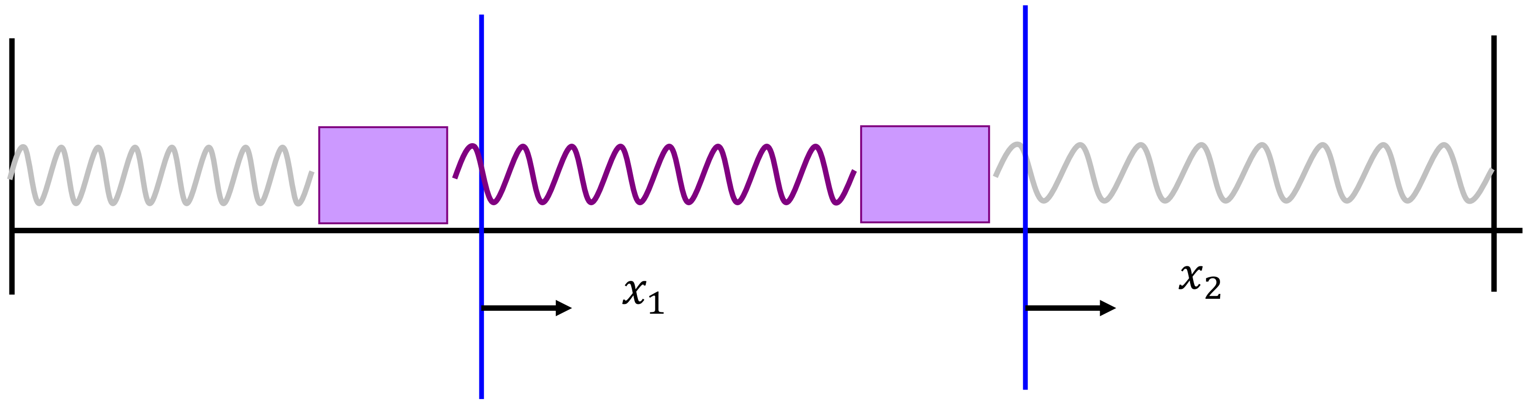
Solution 1:

$$\omega = \sqrt{\frac{k' + 2k}{m}} \xrightarrow[k' \to 0]{} \sqrt{\frac{k}{m/2}}$$

$$k' \to 0 \;\Rightarrow\; x_{10} = -x_{20}$$

Vibration with the reduced mass.

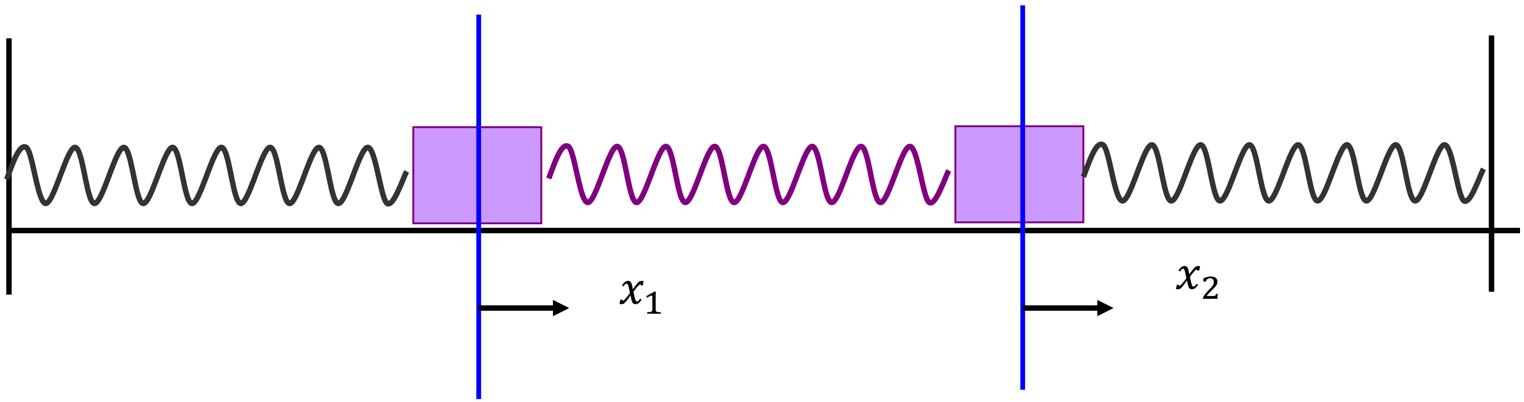
Translational Mode



Solution 1: $$\omega = \sqrt{\frac{k'}{m}} \xrightarrow{k' \to 0} \to 0$$

$$x_{10} = x_{20}$$ **Translation!**

Two Harmonic Oscillators

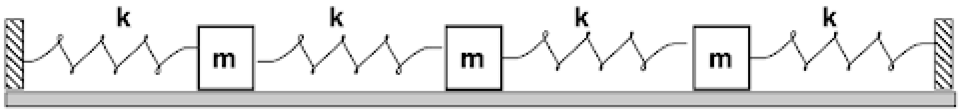


In mathematics language, we solved an eigenvalue problem.

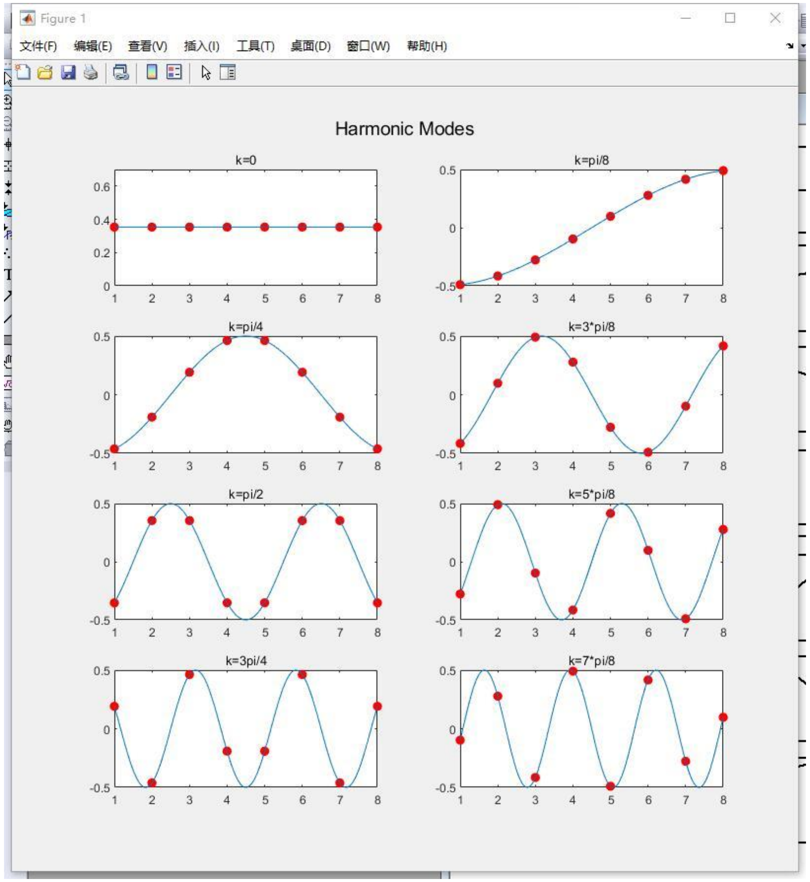
$$\begin{pmatrix} k' + k & -k \\ -k & k' + k \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = m\omega^2 \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

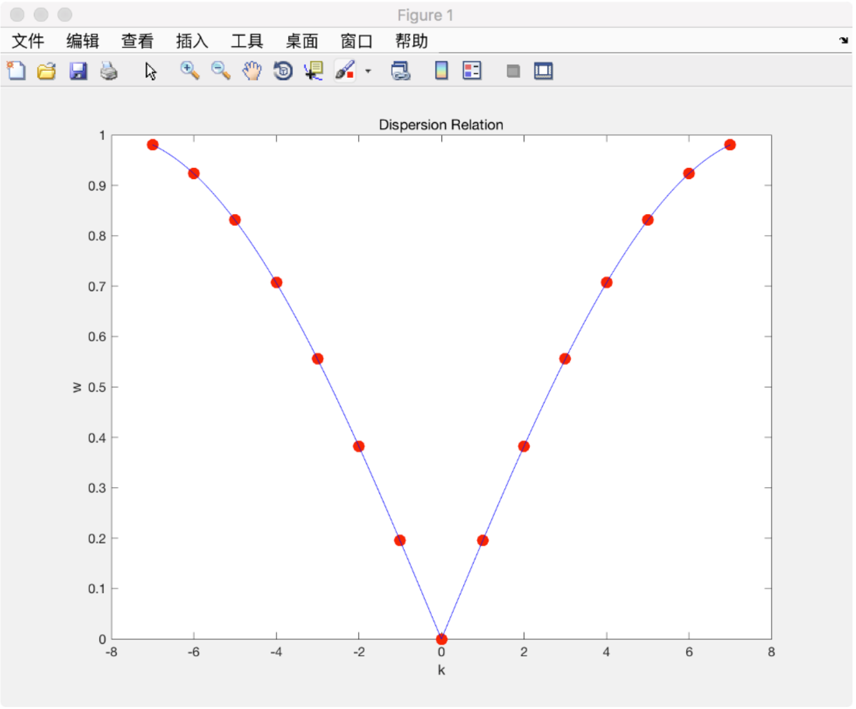
The two eigenvectors are orthogonal to each other. Independent!

Harmonic Modes



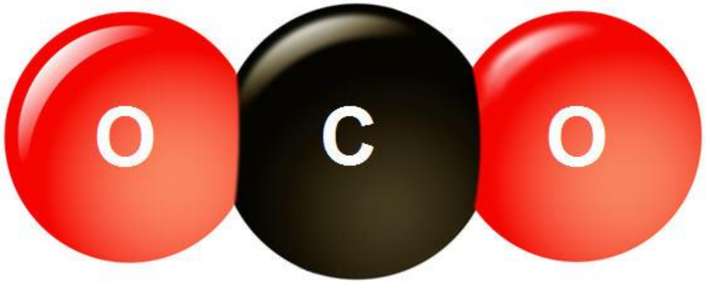
$$\omega = 2\omega_0 \sin(ka/2)$$



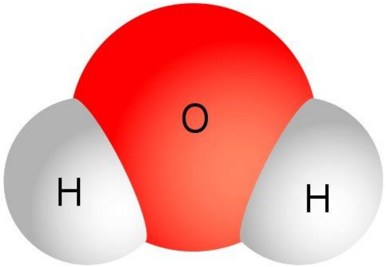


Mode Counting

- **N-atom linear molecule**
  - **Translation: 3**
  - **Rotation: 2**
  - **Vibration: 3N – 5**

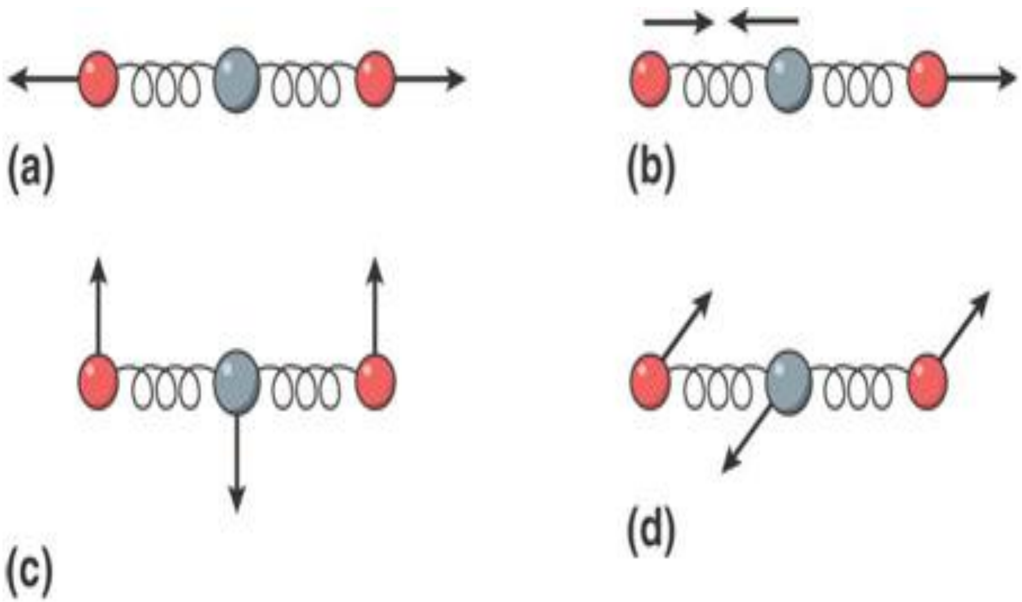


- **N-atom (nonlinear) molecule**
  - **Translation: 3**
  - **Rotation: 3**
  - **Vibration: 3N – 6**



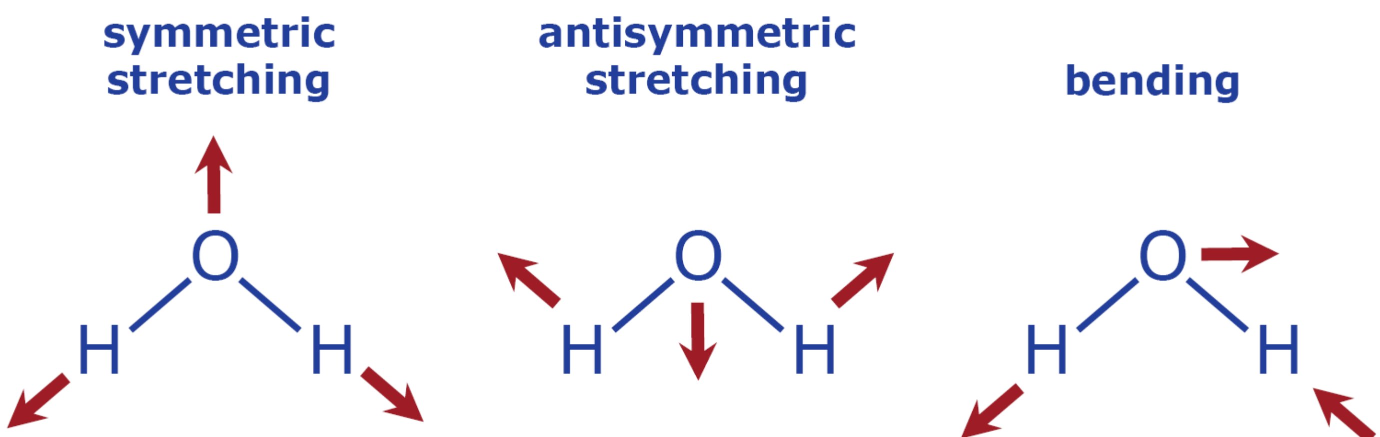
Vibrational Modes of CO_2

- N = 3, linear
  - Translation: 3
  - Rotation: 2
  - Vibration: 3N – 3 – 2 = 4



Vibrational Modes of H_2O

- **N = 3, planer**
  - **Translation: 3**
  - **Rotation: 3**
  - **Vibration: 3N – 3 – 3 = 3**

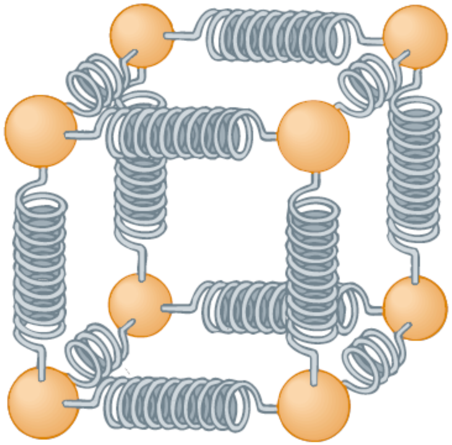


Free molecules: $\tilde{\nu} = 3657\ \mathrm{cm}^{-1}$ $\tilde{\nu} = 1595\ \mathrm{cm}^{-1}$ $\tilde{\nu} = 3756\ \mathrm{cm}^{-1}$

Liquid: $\tilde{\nu} = 3400\ \mathrm{cm}^{-1}$

Solids

- Microscopically, a solid can be regarded as an array of atoms connected by springs (atomic forces).
- Macroscopically, therefore, it is possible to change the shape or the size of a solid
by applying external forces.
As these changes take place,
however, internal forces in the
- object resist the deformation.



Elastic Properties

$$\text{Elastic modulus} \equiv \frac{\text{stress}}{\text{strain}}$$

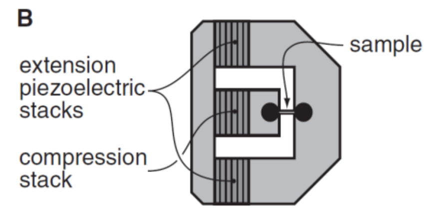
for sufficiently small stresses.

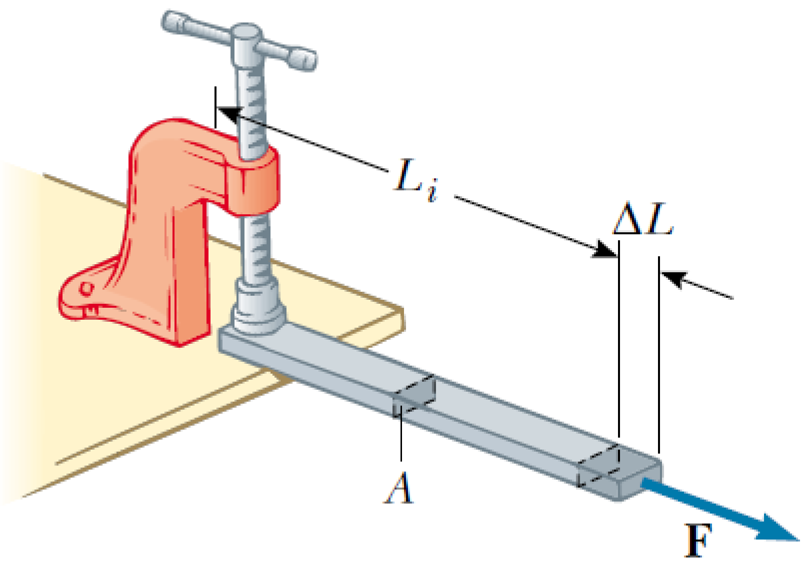
- **Stress: A quantity that is proportional to the force causing a deformation; more specifically, stress is the external force acting on an object per unit cross-sectional area.**
- **Strain: A measure of the degree of deformation.**
- **Elastic modulus: The constant of proportionality depends on the material being deformed and on the nature of the deformation.**

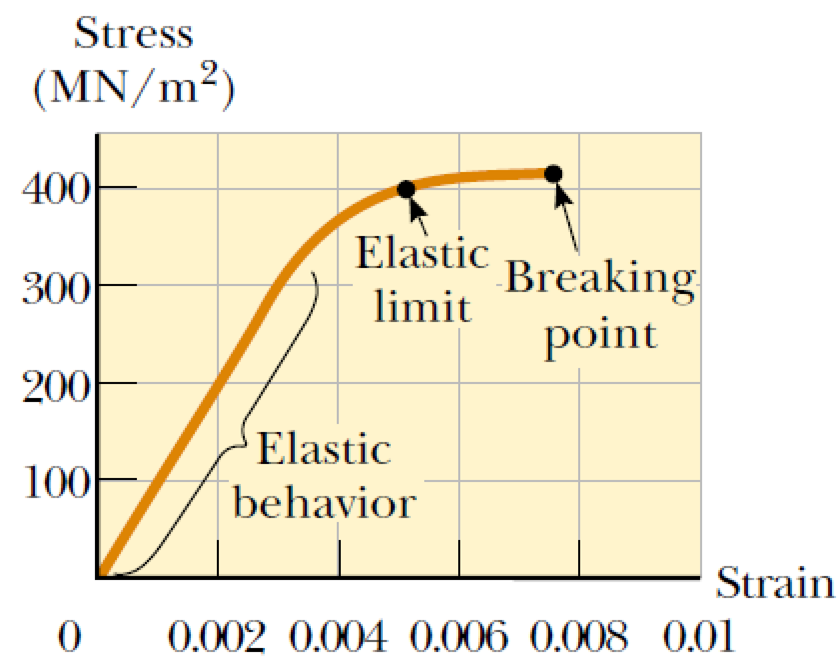
Elasticity in Length

- **Young's Modulus:**

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{\Delta L/L_i}$$



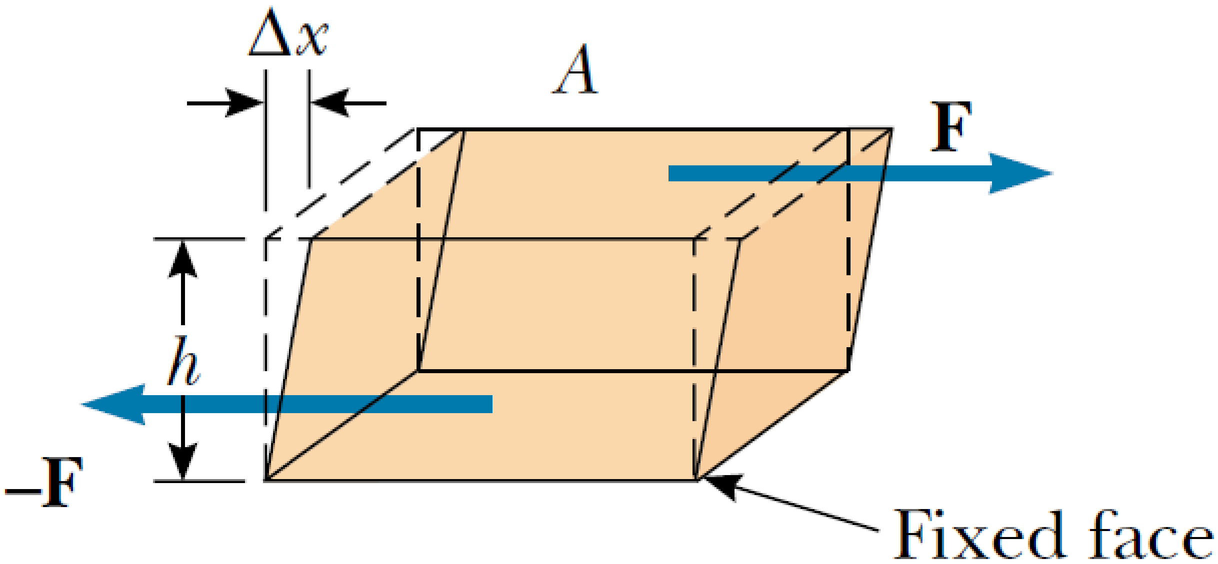




Elasticity of Shape

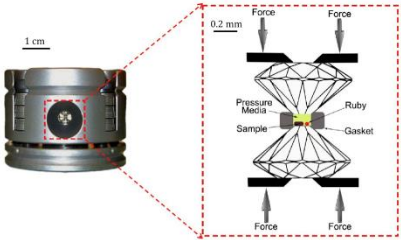
- **Shear Modulus:**

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h}$$



Volume Elasticity

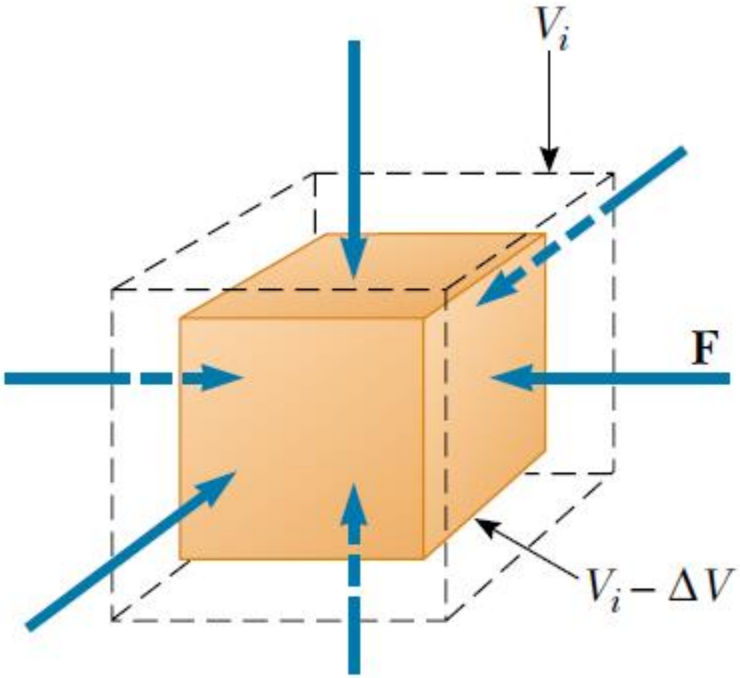
- **Bulk Modulus:**



$$B \equiv \frac{\text{volume stress}}{\text{volume strain}}$$

$$= -\frac{\Delta F/A}{\Delta V/V_i}$$

$$= -\frac{\Delta P}{\Delta V/V_i}$$



Typical Values for Elastic Modulus

Substance	Young's Modulus (N/m^2)	Shear Modulus (N/m^2)	Bulk Modulus (N/m^2)
Tungsten	35×10^{10}	14×10^{10}	20×10^{10}
Steel	20×10^{10}	8.4×10^{10}	6×10^{10}
Copper	11×10^{10}	4.2×10^{10}	14×10^{10}
Brass	9.1×10^{10}	3.5×10^{10}	6.1×10^{10}
Aluminum	7.0×10^{10}	2.5×10^{10}	7.0×10^{10}
Glass	$6.5–7.8 \times 10^{10}$	$2.6–3.2 \times 10^{10}$	$5.0–5.5 \times 10^{10}$
Quartz	5.6×10^{10}	2.6×10^{10}	2.7×10^{10}
Water	—	—	0.21×10^{10}
Mercury	—	—	2.8×10^{10}