Maxwell's Equations and EM Waves

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Lecture 14

Outline

- Maxwell's Equations
- The Traveling Electromagnetic Wave
- Energy Transport

Electrodynamics Before Maxwell

So far, we have encountered the following laws, specifying the divergence and curl of electric and magnetic fields:

- Gauss' law for $\vec{E}$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

- Gauss' law for $\vec{B}$

$$\nabla \cdot \vec{B} = 0$$

- Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Ampere's law

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

A Fatal Inconsistency

- Applying the divergence to Faraday's law, we find

$$\nabla \cdot (\nabla \times \vec{E}) = \nabla \cdot \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \cdot \vec{B}),$$

  which must be zero according to Gauss' law for magnetic fields.

- This is not surprising because the divergence of curl is always zero, i.e., for any vector $\vec{A}$

$$\nabla \cdot (\nabla \times \vec{A}) = 0.$$

- However, when we apply the divergence to Ampere's law, we find

$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot \left(\mu_0 \vec{J}\right) = \mu_0(\nabla \cdot \vec{J}),$$

which is not necessarily zero.

- On the other hand, the local charge conservation leads to the **continuity equation**:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}.$$

So, we must have

$$\nabla \cdot (\nabla \times \vec{B}) = -\mu_0 \frac{\partial \rho}{\partial t} = -\mu_0 \frac{\partial(\epsilon_0 \nabla \cdot \vec{E})}{\partial t} = -\nabla \cdot \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right).$$

How Maxwell's Fixed Ampere's Law

- Maxwell pointed out that the extra divergence can be removed by fixing Ampere's law to be

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_d = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t},$$

where the extra term

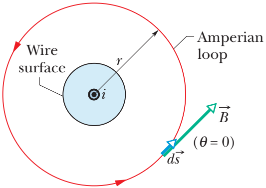
$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

is known as the **displacement current**.

- Maxwell's fix has an aesthetic appeal: Just as a changing magnetic field induces an electric field (Faraday's law), *a changing electric field induces a magnetic field*.

- The integral form of Ampere's law is

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i_{\mathrm{enc}} = \mu_0 \int_S \vec{J} \cdot d\vec{A}.$$

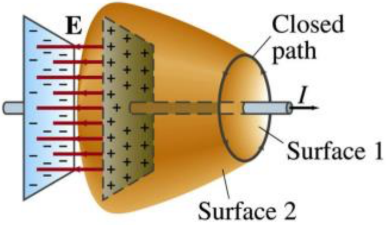


- With Maxwell's correction, we have

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 (i_{\mathrm{enc}} + \int_S \vec{J}_d \cdot d\vec{A}).$$

Current vs Displacement Current

- Now, given an Amperian loop C, we have the freedom to choose an arbitrary surface S that is enclosed by C. The result should be independent of our choice.
- Show that, in the process of charging up the capacitor, the integral form of Ampere-Maxwell's law is valid regardless of the surface chosen.



- The electric field between the capacitor plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{Q}{A},$$

where Q is the charge on the plate and a is its area. Thus,

$$\frac{\partial E}{\partial t} = \frac{J_d}{\epsilon_0} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{I}{\epsilon_0 A}.$$

- If we choose surface 1 to integrate, then $E = 0$ and $I_{\mathrm{enc}} = I$ (contribution from the genuine current).
- If we choose surface 2 to integrate, then $I_{\mathrm{enc}} = 0$, but $\int(\partial\vec{E}/\partial t)\cdot d\vec{A} = I/\epsilon_0$ (contribution from the displacement current).

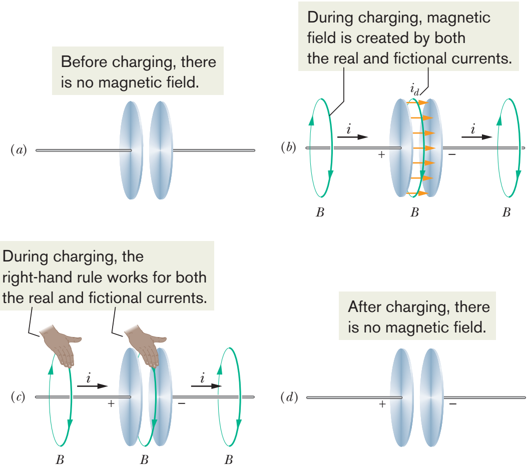


Figure 1: We can view the fictitious displacement current i_d to be simply a continuation of the real current i from one plate, across the capacitor gap, to the other plate.

Maxwell's Equations

- Gauss' law for $\vec{E}$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

- Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Gauss' law for $\vec{B}$

$$\nabla \cdot \vec{B} = 0$$

- Ampere-Maxwell's law

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

- They summarize the entire theoretical content of classical electrodynamics, together with the force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

The More Logical Way

- Charges produce fields

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

- Fields affect charges

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

- Are you satisfied now?

Magnetic Charge

- In a sense, Maxwell's equations beg for magnetic charge to exist.
- Paul Dirac even showed that the existence of magnetic charge would explain why electric charge is quantized.
- However, *no one has ever found any magnetic charge*.
- As far as we know, magnetic charge and magnetic current are zero everywhere; $\vec{B}$ is not on equal footing with $\vec{E}$: there exist stationary source for $\vec{E}$ (electric charge) but none for $\vec{B}$.
- *Nature just is not symmetric, not at all levels.*

Electromagnetic Waves

- Like it or not, the information age in which we live is based almost entirely on the physics of electromagnetic waves. Radio and television, the Internet, and mobile phones rely on the transmission of electromagnetic waves through air, space, or fiber optic cables.
- The challenge for today's engineers is trying to envision what the global interconnection will be in future. The starting point in meeting that challenge is understanding the basic physics of electromagnetic waves.

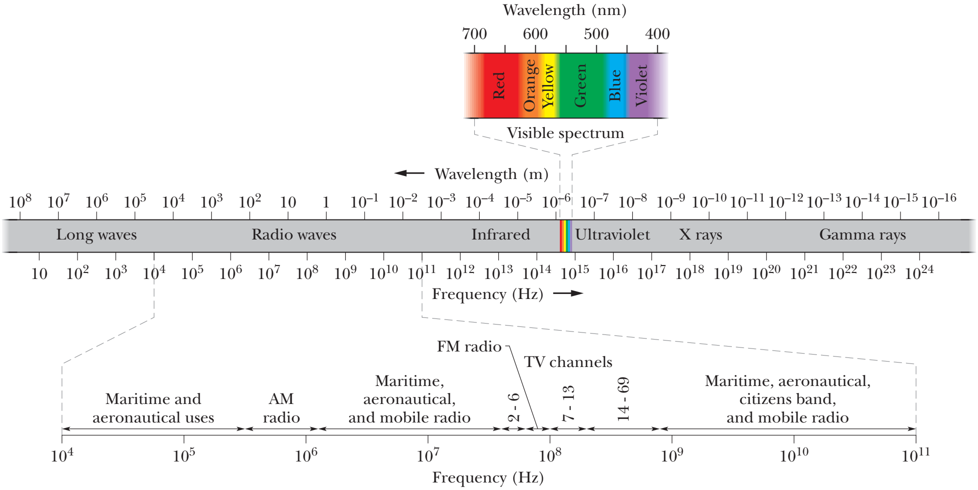


Figure 2: A beam of light is a traveling wave of electric and magnetic fields — an electromagnetic wave — and thus that optics, the study of visible light, is a branch of electromagnetism.

Derivation of the Wave Equation

- In regions of space where there is no charge or current, Maxwell's equation read

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

- They constitute a set of coupled, first-order, partial differential equations for $\vec{E}$ and $\vec{B}$.

- We now show that they can be decoupled by applying the curl to the curls of $\vec{E}$ and $\vec{B}$.

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$
$$= -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$
$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

- Recall that (notice the order)

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B})\vec{C}.$$

- Let $\vec{A} = \vec{B} = \nabla$, we obtain the following identity for an arbitrary vector $\vec{C}$:

$$\nabla \times (\nabla \times \vec{C}) = \nabla(\nabla \cdot \vec{C}) - \nabla^2 \vec{C}.$$

- Since Gauss' laws tell us that $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$ in vacuum, we find that

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}.$$

- Therefore, in vacuum each Cartesian component of $\vec{E}$ and $\vec{B}$ satisfies the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f,$$

where the speed of all electromagnetic waves is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3.00 \times 10^8 \ \mathrm{m/s}.$$

- What is oscillating in an electromagnetic wave is the electric field and magnetic field strength.

How Do Electromagnetic Waves Propagate?

- $\vec{B}$ induces (Faraday) a perpendicular $\vec{E}$, which induces (Maxwell) a perpendicular $\vec{B}$. And so on.
- The resulting variations in the fields travel as the electromagnetic wave.
- We are most interested in plane waves

$$f(x,t) = A\cos(kx - \omega t + \phi),$$

where k is the wave number and ω the angular frequency. They are related to wavelength λ and period T by

$$\lambda = \frac{2\pi}{k}, \quad T = \frac{2\pi}{\omega}.$$

- Then we can write the electric and magnetic fields as sinusoidal functions of position x (along the path of the wave toward $+x$ direction) and time t:

$$\vec{E}(x,t) = \vec{E}_m \cos(kx - \omega t + \phi),$$

$$\vec{B}(x,t) = \vec{B}_m \cos(kx - \omega t + \phi).$$

- Since each component of $\vec{E}$ and $\vec{B}$ satisfies the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f,$$

so we must have $\omega^2 = c^2 k^2$, or $\omega = ck$.

Back to Maxwell's Equations

- Maxwell's equations impose extra constraints on $\vec{E}_m$ & $\vec{B}_m$.
- Note that the divergence and curl can be carried out, e.g., as

$$\nabla \cdot \vec{E}(x,t) = -k\hat{x} \cdot \vec{E}_m \sin(kx - \omega t + \phi),$$

$$\nabla \times \vec{E}(x,t) = -k\hat{x} \times \vec{E}_m \sin(kx - \omega t + \phi).$$

- Similarly, the time derivative can be carried out as

$$\frac{\partial}{\partial t}\vec{E}(x,t) = \omega \vec{E}_m \sin(kx - \omega t + \phi).$$

- From Gauss' laws,

$$\nabla \cdot \vec{E} = 0, \quad \text{and} \quad \nabla \cdot \vec{B} = 0,$$

  we have $(E_m)_x = (B_m)_x = 0$. That is, *electromagnetic waves are transverse*.

- From Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

  we find

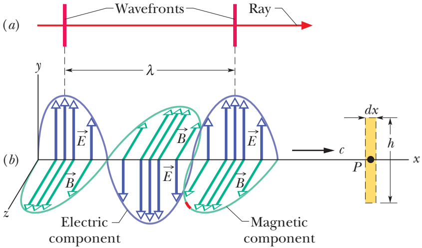
$$-k\hat{x} \times \vec{E}_m = -\omega \vec{B}_m.$$

- Therefore, we have, more generally, in $\hat{k}$

$$\vec{B}_m = \frac{1}{c}(\hat{k} \times \vec{E}_m), \text{ or } \vec{B} = \frac{1}{c}(\hat{k} \times \vec{E}).$$

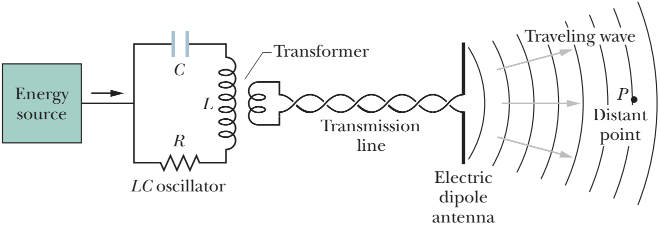
- Alternative, we can write

$$(E_m)_y = c(B_m)_z, \quad (E_m)_z = -c(B_m)_y.$$



- $\vec{E}$ is always perpendicular to $\vec{B}$.
- $\vec{E} \times \vec{B}$ gives the direction in which the wave travels.

Generation of Electromagnetic Waves



- An LC oscillator establishes an angular frequency ω.
- The LC oscillator is coupled by a transformer and a transmission line to an antenna.
- Changing electric and magnetic fields induced by the antenna travel outward at the same angular frequency ω.

Energy Transport

- Since $\vec{E}$ and $\vec{B}$ are perpendicular to each other and $\vec{B} = (\hat{x} \times \vec{E})/c$, the energy densities u_E and u_B for the electric field and magnetic field, respectively, are *equal everywhere* along an electromagnetic wave, i.e.,

$$\frac{\epsilon_0 E^2}{2} = \frac{B^2}{2\mu_0}.$$

- The rate of energy transport per unit area in a plane wave is the product of total energy density and the speed of the electromagnetic wave, i.e.,

$$S = (u_E + u_B)c.$$

- Since both $\vec{E}$ and $\vec{B}$ are sinusoidal function of time,

$$[E^2]_{\text{avg}} = \frac{E_m^2}{2}, \quad [B^2]_{\text{avg}} = \frac{B_m^2}{2}.$$

Similarly, the time average of S, which is known as the **intensity** of the wave, is

$$I = S_{\text{avg}} = \frac{\epsilon_0 E_m^2}{2} c = \frac{B_m^2}{2\mu_0} c.$$

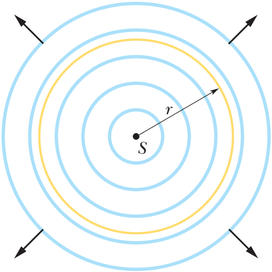
- Dimension analysis tells us that the unit for S or I is the same as that of power per area, whose SI unit is watt per square meter ($\mathrm{W/m^2}$).

Variation of Intensity with Distance

- Intensity I is the time average of the energy transport. It can still vary in space.
- When spherical wavefronts spread from an isotropic point source S with power P_s, the energy of the waves is conserved.
- The intensity I at the sphere must decrease with r as

$$I = \frac{P_s}{4\pi r^2}.$$

The energy emitted by light source S must pass through the sphere of radius r.



Summary

- Maxwell's equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

summarize the entire theoretical content of classical electrodynamics, together with the force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

- In vacuum, electromagnetic waves satisfy

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}.$$

  - Electromagnetic waves are transverse:

$$\hat{k} \cdot \vec{E} = \hat{k} \cdot \vec{B} = 0.$$

  - $\vec{E}$ is always perpendicular to $\vec{B}$:

$$\vec{B} = \frac{1}{c}(\hat{k} \times \vec{E}).$$

  - Electromagnetic waves transport both energy and momentum.

Reading

Halliday, Resnick & Krane:

- Chapter 38: Maxwell's Equations and Electromagnetic Waves

Appendix 14A: The Radiation Pressure

- For a vector notation, we follow John Henry Poynting to define the **Poynting vector** as

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B},$$

which points along the wave-propagating direction $\hat{k}$.

- Since $\vec{B} = (\hat{k} \times \vec{E})/c$, we can write the magnitude of $\vec{S}$ as

$$S = \frac{EB}{\mu_0} = (\epsilon_0 E^2)c = \frac{B^2}{\mu_0}c,$$

which is the same as we defined in the main text.

- The force on the charges in a volume V is

$$F \equiv \frac{dp_{\rm mech}}{dt} = \int_V (\vec{E} + \vec{v} \times \vec{B})\rho dV = \int_V (\rho\vec{E} + \vec{J} \times \vec{B}) dV,$$

  in which ρ and $\vec{J}$ can be eliminated by using Maxwell's equations.
- It is possible (but only after a lot of algebras) to show that, if V is all of space,

$$\frac{\partial}{\partial t}\left(p_{\rm mech} + \frac{1}{c^2}\int_V \vec{S} dV\right) = 0$$

  where $\vec{S}$ is the Poynting vector.

- We can view the integral $\int_V \vec{S} dV / c^2$ as the momentum stored in the electromagnetic fields. In other words, *electromagnetic waves have linear momentum*.
- If the object is free to move and that the radiation is entirely **absorbed** (taken up) by the object. The magnitude Δp of the momentum change of the object is

$$\Delta p = \frac{\Delta U}{c},$$

where ΔU is the energy change in the volume V.

- If the radiation is entirely **reflected** back along its original path, the magnitude of the momentum change of the object is

$$\Delta p = \frac{2\Delta U}{c}.$$

- A flat surface of area A, perpendicular to the path of the radiation, intercepts $\Delta U = AI\Delta t$ in the interval Δt, where I is the intensity of the radiation.
- From Newton's second law, we know that a change in momentum is related to a force by

$$F = \Delta p/\Delta t.$$

- Therefore, the pressure due to the radiation is thus

$$p_r = \frac{F}{A} = \frac{I}{c}$$

in the case of total absorption, but

$$p_r = \frac{2I}{c}$$

in the case of total reflection back along the path.
- Normally, radiation pressures are very small. But a beam of laser light can be focused to a tiny spot to produce much greater radiation pressure.