Alternating-Current Circuits

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Lecture 13

Outline

- Electromagnetic LC Oscillations
- Damped Oscillations in an RLC Circuit
- Forced Oscillations

LC Oscillations

- In RL and RC circuits charge, current, and potential difference grow and decay exponentially. The time scale of the growth or decay is given by a time constant τ, either capacitive or inductive.

$$\tau_C = RC, \quad \tau_L = L/R$$

- Not surprisingly, we obtain from dimension analysis that

$$\tau_{LC} = \sqrt{LC}$$

is also a time constant, characteristic of an LC circuit.

- Therefore, one expects in the case of an LC circuit charge, current, and potential difference vary periodically (sinusoidally). The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations**. Such a circuit is said to oscillate.
- In the *ideal* LC circuit with no resistance, the oscillations continue indefinitely, because of the conservation of energy. There are no energy transfers other than that between the electric field of the capacitor and the magnetic field of the inductor.

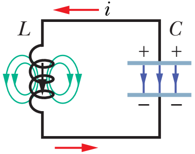
- We can put in a voltmeter to measure the time-varying potential difference (or voltage) v_C that exists across the capacitor C:

$$v_C = q/C,$$

which allows us to find q.

- Current, as measured by an ammeter, is

$$i = dq/dt.$$



The Electrical–Mechanical Analogy

Block–Spring System		*LC* Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
$v = dx/dt$		$i = dq/dt$	

- These correspondences then suggest that
  - q corresponds to x, $1/C$ corresponds to k,
  - i corresponds to $v = dx/dt$, and L corresponds to m.
- The angular frequency of the oscillation for an ideal (resistanceless) LC circuit is then (analogous to $\sqrt{k/m}$)

$$\omega_0 = 1/\tau_{LC} = 1/\sqrt{LC}.$$

LC Oscillations – A Quantitative Analysis

- The total energy U in an oscillating LC circuit is given by

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}.$$

- In the absence of resistance, U remains constant with time.

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0.$$

With $i = dq/dt$ and $di/dt = d^2q/dt^2$, we find

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0.$$

- We have obtained the differential equation that describes the oscillations of a resistanceless LC circuit.
- This contrasts with the mechanical form

$$m\frac{d^2x}{dt^2} + kx = 0,$$

whose general solution is

$$x = A\cos(\omega_0 t + \phi),$$

where $\omega_0 = \sqrt{k/m}$ is an intrinsic property of the block-spring system.

- By analogy, we can write the general solution for the LC circuit as

$$q = Q\cos(\omega_0 t + \phi),$$

where $\omega_0 = 1/\sqrt{LC}$ is an intrinsic property of the circuit.

- Q is the amplitude of the charge variations, and ϕ is the phase constant. These two constants can be determined by, e.g., initial conditions,

$$Q\cos\phi = q\big|_{t=0},$$

$$-\omega_0 Q\sin\phi = \left.\frac{dq}{dt}\right|_{t=0} \equiv i\big|_{t=0}.$$

- Note that the second initial condition can be understood by taking the first derivative of charge q with respect to time t, which gives us the current:

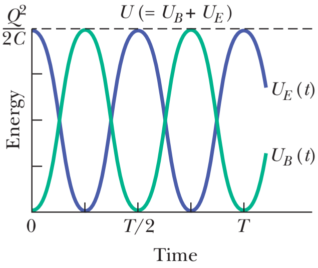
$$i = -I \sin(\omega_0 t + \phi) = I \cos(\omega_0 t + \phi'),$$

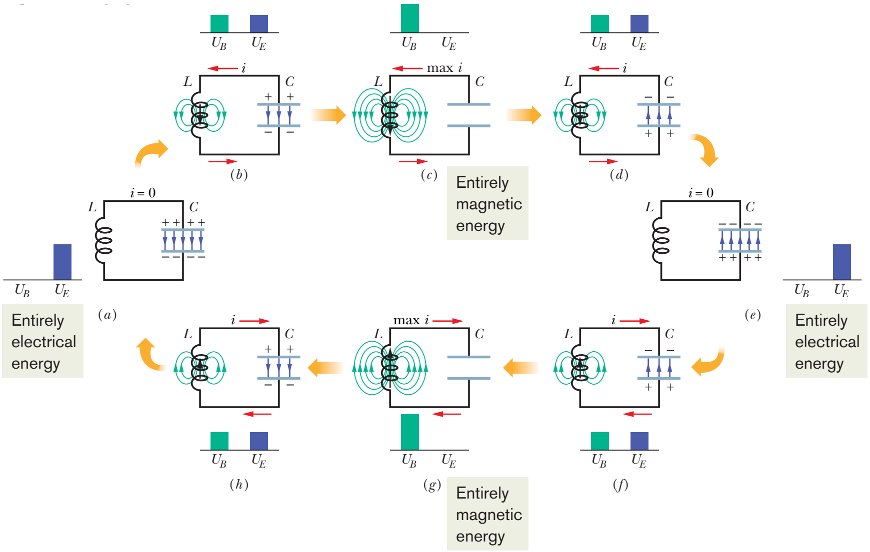
  where $I = \omega_0 Q$ and $\phi' = \phi + \pi/2$.

- In other words, current leads charge (or voltage across the capacitor) in phase by $\pi/2$.

- Therefore, electrical energy and magnetic energy oscillate.
  - The maximum values of U_E and U_B are both $Q^2/2C$.
  - At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
  - When U_E is maximum, U_B is zero, and conversely.

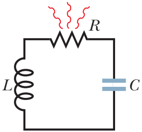
The electrical and magnetic energies vary but the total is constant.



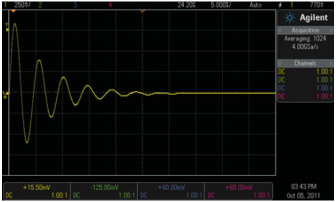


Entirely electrical energy
Entirely magnetic energy
Entirely electrical energy
Entirely magnetic energy
U_B U_E
L C
$i = 0$
max i
i
(a) (b) (c) (d) (e) (f) (g) (h)

- In an *actual* LC circuit, the oscillations will not continue indefinitely because there is always some resistance present that will drain energy from the electric and magnetic fields and dissipate it as *thermal energy* (the circuit may become warmer).



- This is similar to the decay of mechanical oscillations caused by frictional damping in a block-spring system.



The Complex Formalism

- In an RC circuit, the general solution is, up to a constant,

$$q = Qe^{-t/\tau}.$$

- In an LC circuit, the general solution is,

$$q = Q\cos(\omega_0 t + \phi) = \Re\left[Qe^{i\omega_0 t}e^{i\phi}\right].$$

- We can write down the more general solution as

$$q = \Re\left[\tilde{Q}e^{i\tilde{\omega}t}\right] = Qe^{-t/\tau}\cos(\omega t + \phi),$$

where we define a complex amplitude $\tilde{Q} = Qe^{i\phi}$ and a complex frequency $\tilde{\omega} = \omega + i/\tau$.

Damped Oscillations in an RLC Circuit

- With a resistance R present, the total electromagnetic energy U of the circuit is no longer constant; instead, it decreases with time as energy is transferred to thermal energy in the resistance:

$$\frac{dU}{dt} = -i^2 R.$$

- Because of this loss of energy, the oscillations of charge, current, and potential difference continuously decrease in amplitude, and the oscillations are said to be **damped**, just as with the damped block-spring oscillator.

- The differential equation for damped oscillations in an RLC circuit is then

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

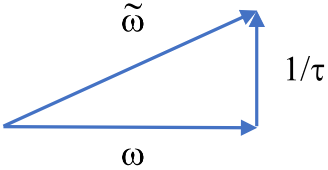
With $i = dq/dt$ and $di/dt = d^2q/dt^2$, we find

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0.$$

- Plugging in $q = \tilde{Q}e^{i\tilde{\omega}t}$, we obtain

$$-L\tilde{\omega}^2 + i\tilde{\omega}R + \frac{1}{C} = 0.$$

- By introducing the complex formalism, we transform the second-order differential equation to a quadratic equation in the complex frequency, which can then be solved by the quadratic formula.



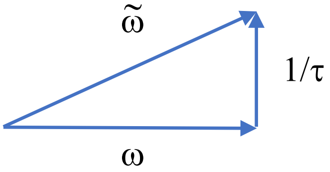
- We can further assume that ω is positive, because we are only interested in the real part of the solution

$$q = Qe^{-t/\tau}\cos(\omega t + \phi).$$

- The solution for $\tilde{\omega} = \omega + i/\tau$ satisfies

$$\omega = \sqrt{\omega_0^2 - (1/\tau)^2} \quad \text{and} \quad 1/\tau = R/(2L),$$

in which $\omega_0 = |\tilde{\omega}| = 1/\sqrt{LC}$.



- What happens when the damping is very strong?

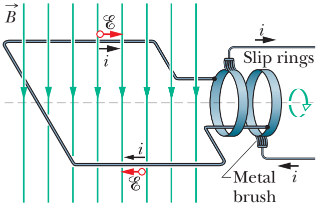
- When $1/\tau < \omega_0$, a real ω can be found and the system still oscillates, but with decreasing amplitude as its energy is converted to heat. The circuit is said to be **underdamped**. Over time the system should come to rest at equilibrium.
- When $1/\tau > \omega_0$, one can only find imaginary ω, which means the frictional force is so great that the system cannot oscillate. The circuit is said to be **overdamped**.
- In between, when $1/\tau = \omega_0$, the circuit is said to be **critically damped**. It is worth noting that *the critical damping gives the fastest return of the system to its equilibrium position*. In engineering design this is often a desirable property.

AC Circuits and Forced Oscillations

- The oscillations in an RLC circuit will not damp out if an external emf device supplies enough energy to make up for the energy dissipated as thermal energy in the resistance R.
- The energy is supplied via oscillating emfs and currents — the current is said to be an **alternating current**, or **ac** for short. These oscillating emfs and currents vary sinusoidally with time, reversing direction 100 times per second and thus having frequency $f = 50$ Hz.

- An ac generator can induce a sinosoidally oscillating emf $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}_m \cos \omega_d t.$$



- When the rotating loop is part of a closed conducting path, this emf drives a sinusoidal current along the path with the same angular frequency ω_d, which then is called the **driving angular frequency**. We can write the current as

$$i = I \cos(\omega_d t + \phi).$$

Natural vs Driving Frequency

- In both undamped LC circuits and underdamped RLC circuits (with sufficiently small enough R), the charge, potential difference, and current oscillate at the circuit's **natural angular frequency** $\omega_0 = 1/\sqrt{LC}$. Such oscillations are said to be **free oscillations**.
- When the external alternating emf is connected, the oscillations of charge, potential difference, and current are said to be **driven oscillations** or **forced oscillations**. These oscillations always occur at the **driving angular frequency**.

Three Simple Circuits

- Assume the potential difference across a circuit element (resistor, capacitor, and inductor) is

$$v(t) = \Re\left(\tilde{V}e^{i\omega_d t}\right),$$

  and the current in the element is

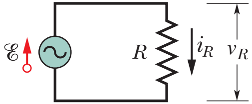
$$i(t) = \Re\left(\tilde{I}e^{i\omega_d t}\right).$$

- We can define **complex impedance** as

$$\tilde{Z} = Ze^{i\phi} = \frac{\tilde{V}}{\tilde{I}}.$$

- In a resistive load,

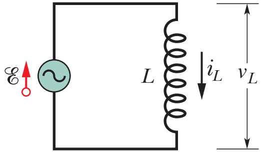
$$\tilde{Z} = R.$$



- In an inductive load,

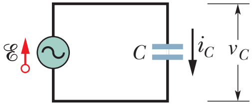
$$\tilde{Z} = i\omega_d L.$$

  - Notice $di(t)/dt = (i\omega_d)i(t)$.



- Similarly, in a capacitive load,

$$\tilde{Z} = \frac{1}{i\omega_d C}.$$



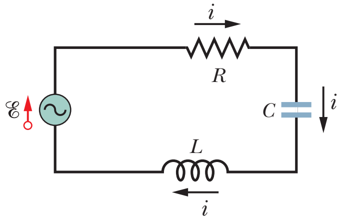
The Series RLC Circuit

- Combining impedances has similarities to the combining of resistors, but the phase relationships make it practically necessary to use the complex impedance method for carrying out the operations.
- Combining series impedances is straightforward:

$$\tilde{Z} = \tilde{Z}_1 + \tilde{Z}_2.$$

- When R, L, and C are in series,

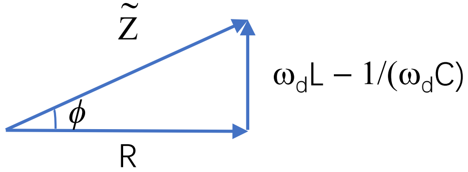
$$\tilde{Z} = R + i\left(\omega_d L - \frac{1}{\omega_d C}\right).$$



- Alternatively, if we write $\tilde{Z} = Ze^{i\phi}$, we find

$$Z\cos\phi = R,$$

$$Z\sin\phi = \omega_d L - \frac{1}{\omega_d C}.$$



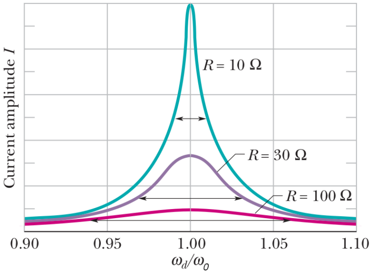
- Therefore, the impedance Z and the phase constant $\tan\phi$ are

$$Z = \sqrt{R^2 + [\omega_d L - 1/(\omega_d C)]^2},$$

$$\tan\phi = \frac{\omega_d L - 1/(\omega_d C)}{R}.$$

Resonance

- When ω_d equals ω_0, the circuit is in **resonance**.
  - The circuit is equally capacitive and inductive ($|Z_C| = |Z_L|$).
  - The current amplitude $I = \mathcal{E}_m/R$ is maximum.
  - Current and emf are in phase ($\phi = 0$).
- The low angular-frequency side of a resonance curve is dominated by the capacitor's impedance, and the high angular-frequency side is dominated by the inductor's impedance.



Summary

- Energy transfers $U = U_E + U_B$, where

$$U_E = \frac{q^2}{2C} \qquad U_B = \frac{Li^2}{2}$$

$$\frac{dU}{dt} = -i^2 R$$

- LC oscillations

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

$$q = Q\cos(\omega_0 t + \phi)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

- Damped oscillations in an RLC circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

$$q = Qe^{-t/\tau}\cos(\omega t + \phi)$$

$$\omega = \sqrt{\omega_0^2 - (1/\tau)^2}$$

$$1/\tau = R/(2L)$$

- Forced oscillations in a series RLC circuit at a driving angular frequency ω_d

$$\mathcal{E} = \mathcal{E}_m \cos \omega_d t$$

$$i = I \cos(\omega_d t + \phi)$$

- Resonance
  - The circuit is equally capacitive and inductive: $1/(\omega_d C) = \omega_d L$, or $\omega_d = \omega_0$.
  - $I = I_{\max} = \mathcal{E}_m / R$
  - $\phi = 0$

Reading

Halliday, Resnick & Krane:

- Chapter 37: Alternating Current Circuits