General Physics I

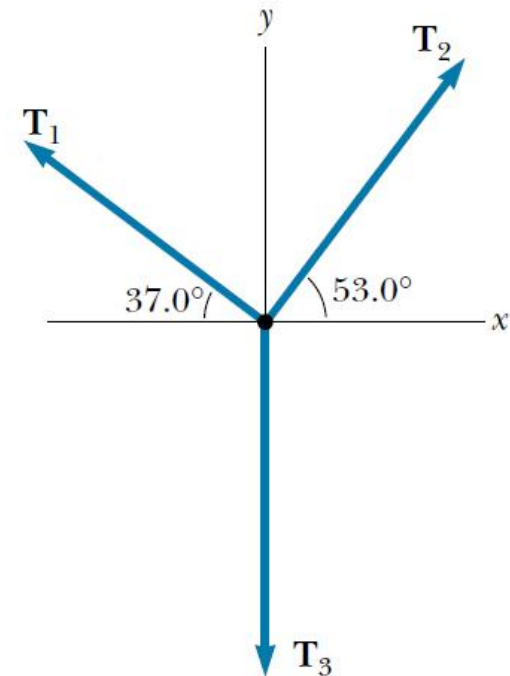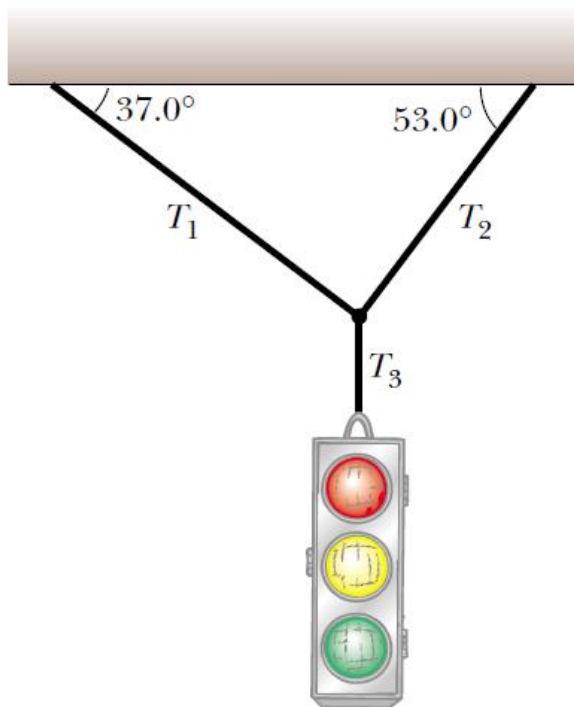# Lecture 11: Simple Harmonic Motion

Outline

- **The conditions for static equilibrium**

- **Motion near the stable equilibrium**

- **Simple harmonic motion**

- **The block-spring system in quantitative treatment**

- **Energy of the simple harmonic oscillator**

For Point-like Objects

• **The necessary condition for equilibrium is that the net force acting on an object be zero.**
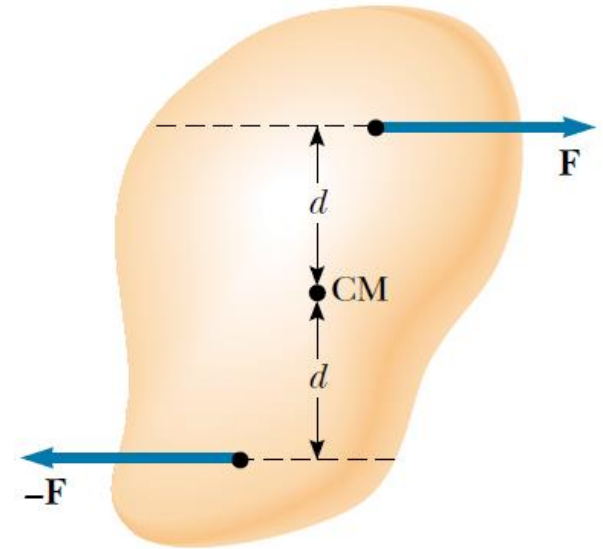
For Extended Objects

●**A second condition is that the net torque about any axis be zero.**

●**The two conditions are the statements of**

- **Translational equilibrium**
- **Rotational equilibrium**

Not in equilibrium!

Static Equilibrium

●An object is in equilibrium means that both the linear acceleration and the angular acceleration are zero.

●If the object is at rest and so has no linear speed or angular speed, the object is in static equilibrium.

Torque

●**The torque t involves the two vectors r and F, and its direction is perpendicular to the plane of r and F. We can establish a mathematical relationship between t, r, and F, using a new mathematical operation called the vector product, or cross product**
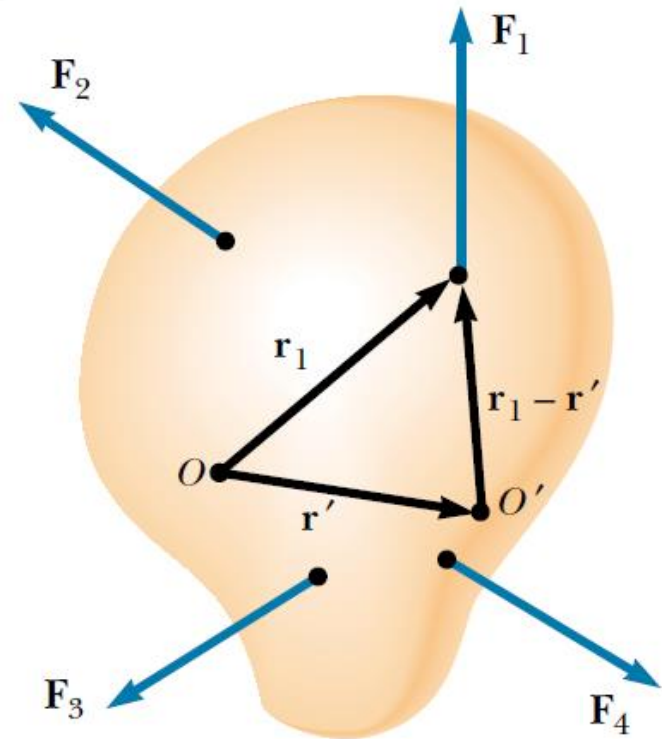
$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}$$

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
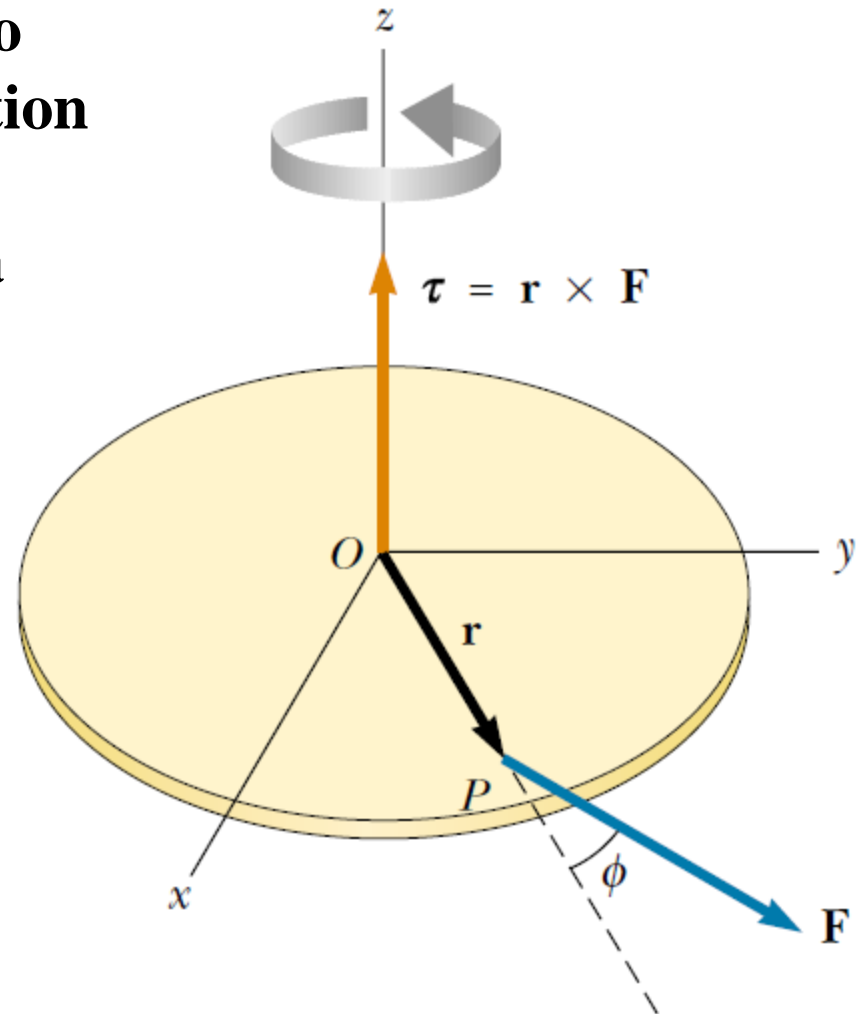
Torque about Any Point?

$$\sum \tau_{O'} = (\mathbf{r}_1 - \mathbf{r}') \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}') \times \mathbf{F}_2 + (\mathbf{r}_3 - \mathbf{r}') \times \mathbf{F}_3 + \cdots$$

$$= \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \cdots - \mathbf{r}' \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots)$$

$$\sum \tau_O = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \cdots$$

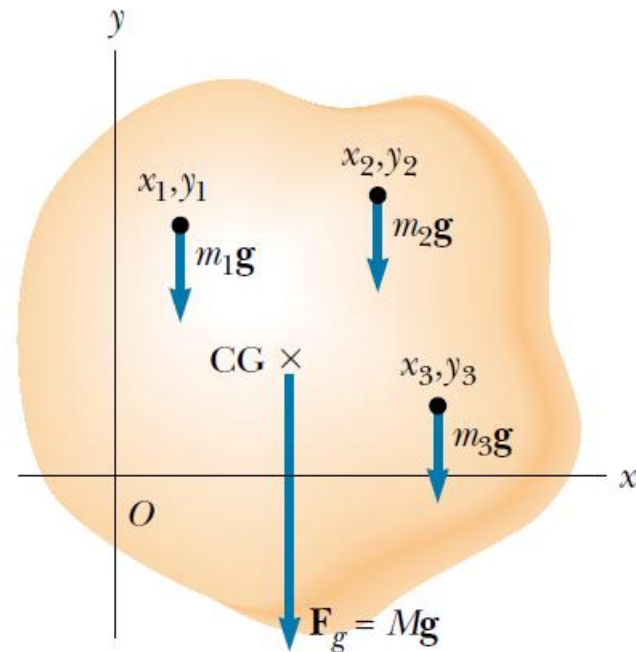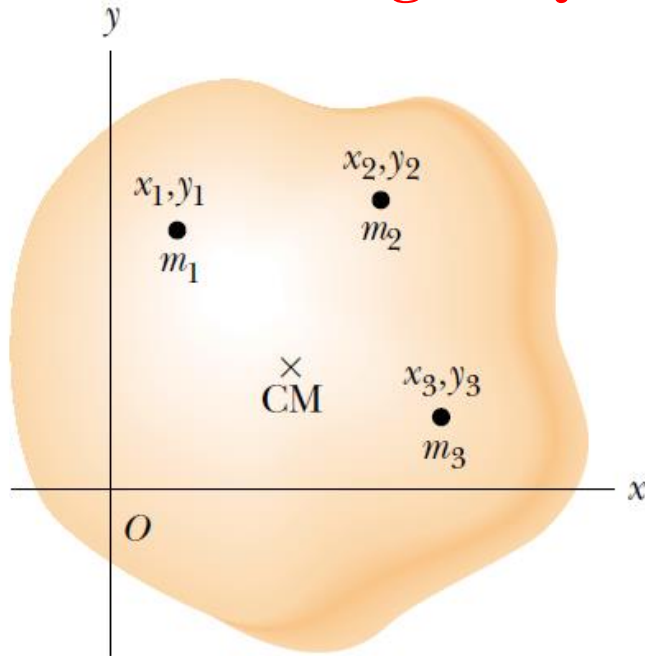$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = 0$$

If an object is in translational equilibrium and the net torque is zero about one point, then the net torque must be zero about any other point.

To compute the torque due to the gravitational force on an object of mass *M*, we need only consider the force *M*g acting at the center of gravity of the object.

The center of gravity is located at the center of mass as long as the object is in a uniform gravitational field.
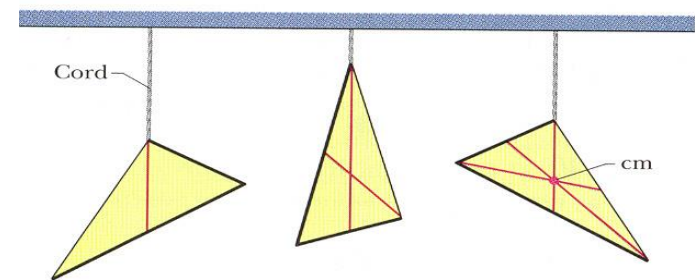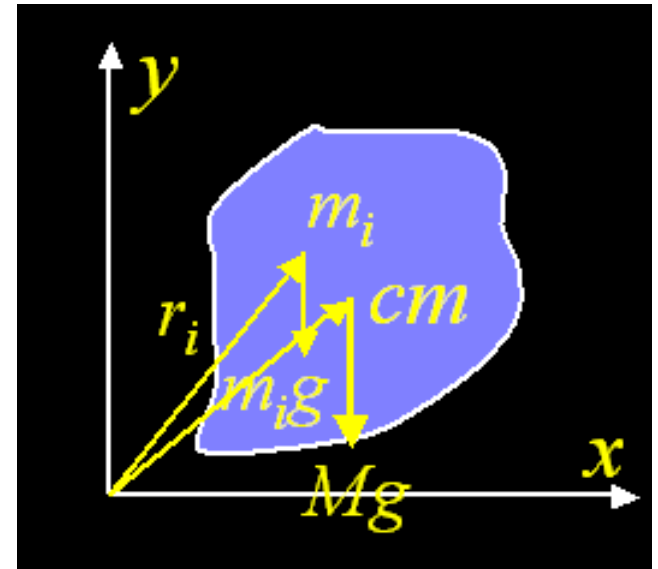
Gravity Torque

The force and torque on the entire body due to gravity is:

$$\vec{F} = \sum m_i \vec{g} = M\vec{g}$$

$$\vec{\tau} = \sum \vec{r}_i \times m_i \vec{g} = \left(\sum m_i \vec{r}_i\right) \times \vec{g}$$
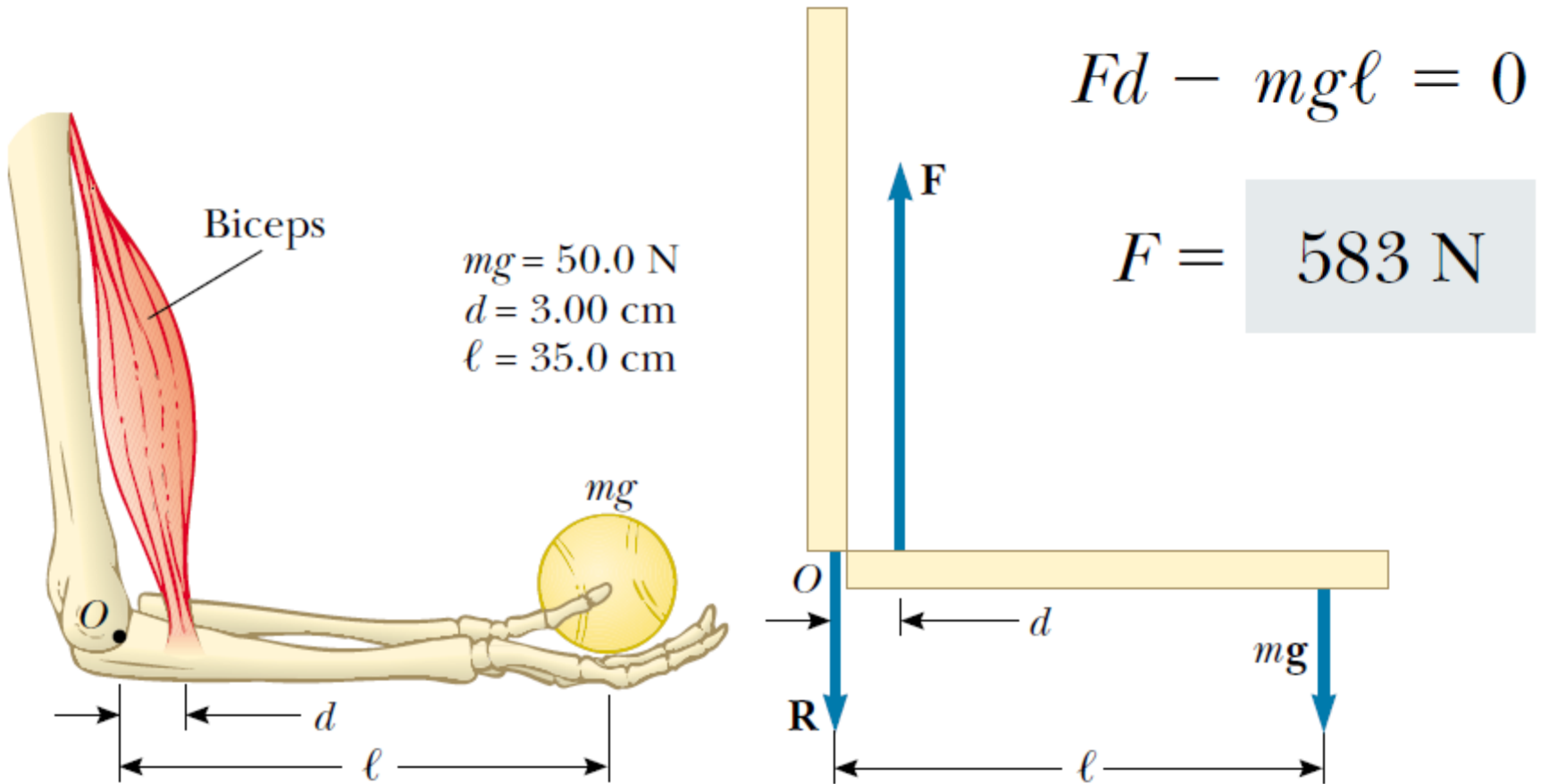
$$= M\vec{r}_{cm} \times \vec{g}$$

$$\boxed{\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{W}}$$

The toque acted on a body due to gravity about any point is equal to the toque of total gravity acted at the CM about this point.
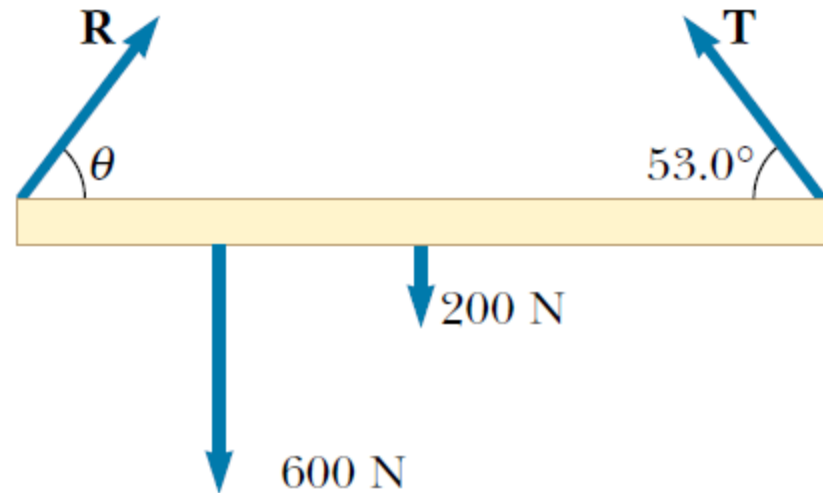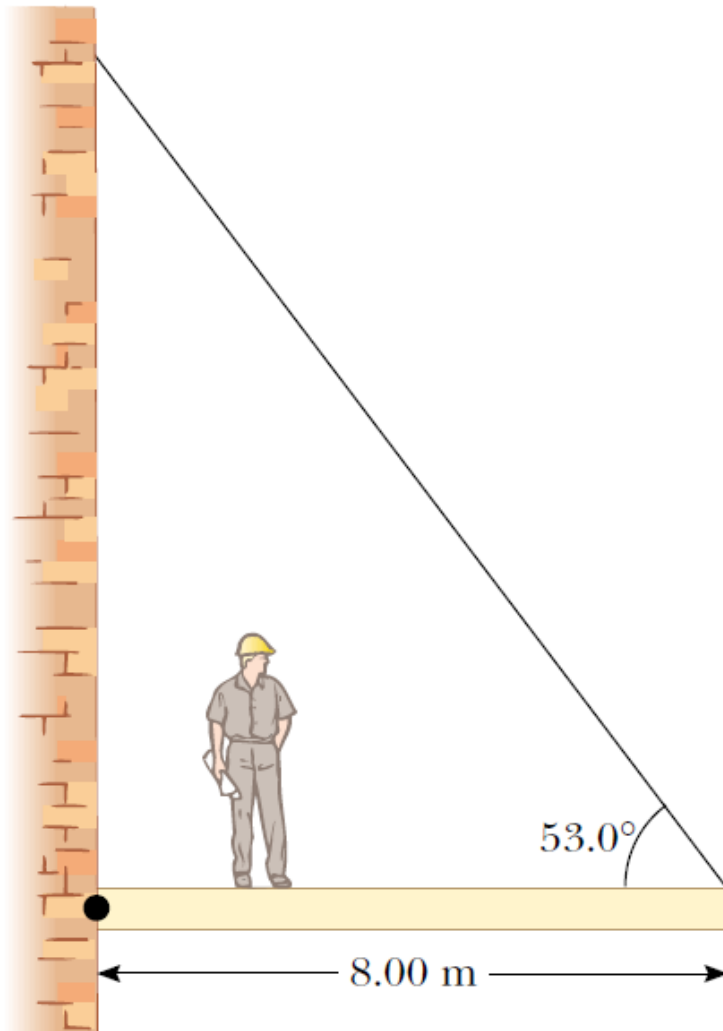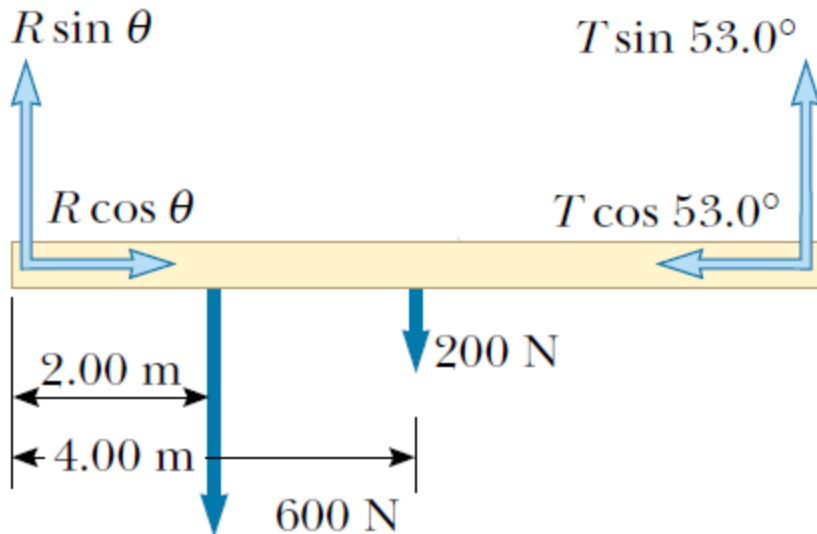
Example: A Weighted Hand

Biceps

$mg = 50.0$ N
$d = 3.00$ cm
$\ell = 35.0$ cm

mg

O

$Fd - mg\ell = 0$

F

$F = \boxed{583 \text{ N}}$

O

d

R

mg

ℓ

Example: Standing on a Beam

Example: Standing on a Beam

$$\sum F_x = R \cos \theta - T \cos 53.0° = 0$$

$$\sum F_y = R \sin \theta + T \sin 53.0°$$

$$- 600 \text{ N} - 200 \text{ N} = 0$$

$$\sum \tau = (T \sin 53.0°)(8.00 \text{ m})$$

$$- (600 \text{ N})(2.00 \text{ m}) - (200 \text{ N})(4.00 \text{ m}) = 0$$
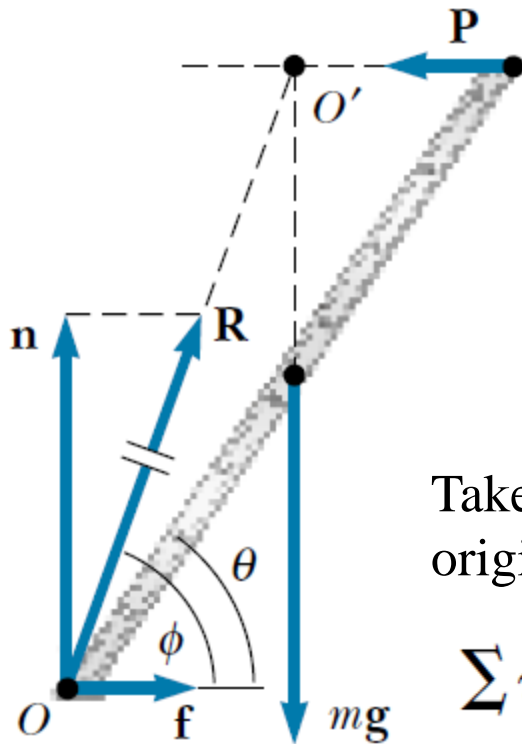
$R \cos \theta = 188 \text{ N}$

$R \sin \theta = 550 \text{ N}$

$T = \boxed{313 \text{ N}}$

Example: The Leaning Ladder

Find the minimum angle at which the ladder does not slip.
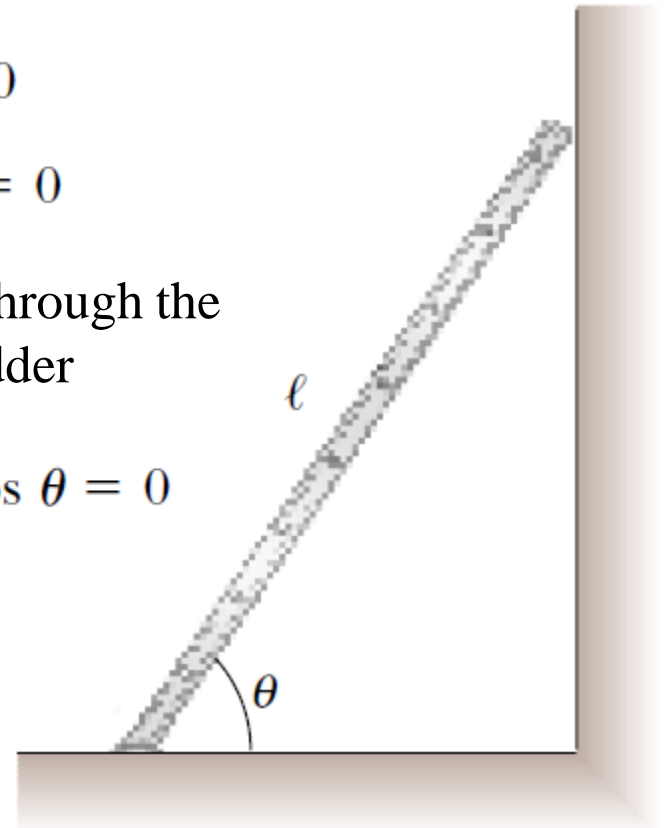
$$\sum F_x = f - P = 0$$

$$\sum F_y = n - mg = 0$$

Take the torques about an axis through the origin O at the bottom of the ladder

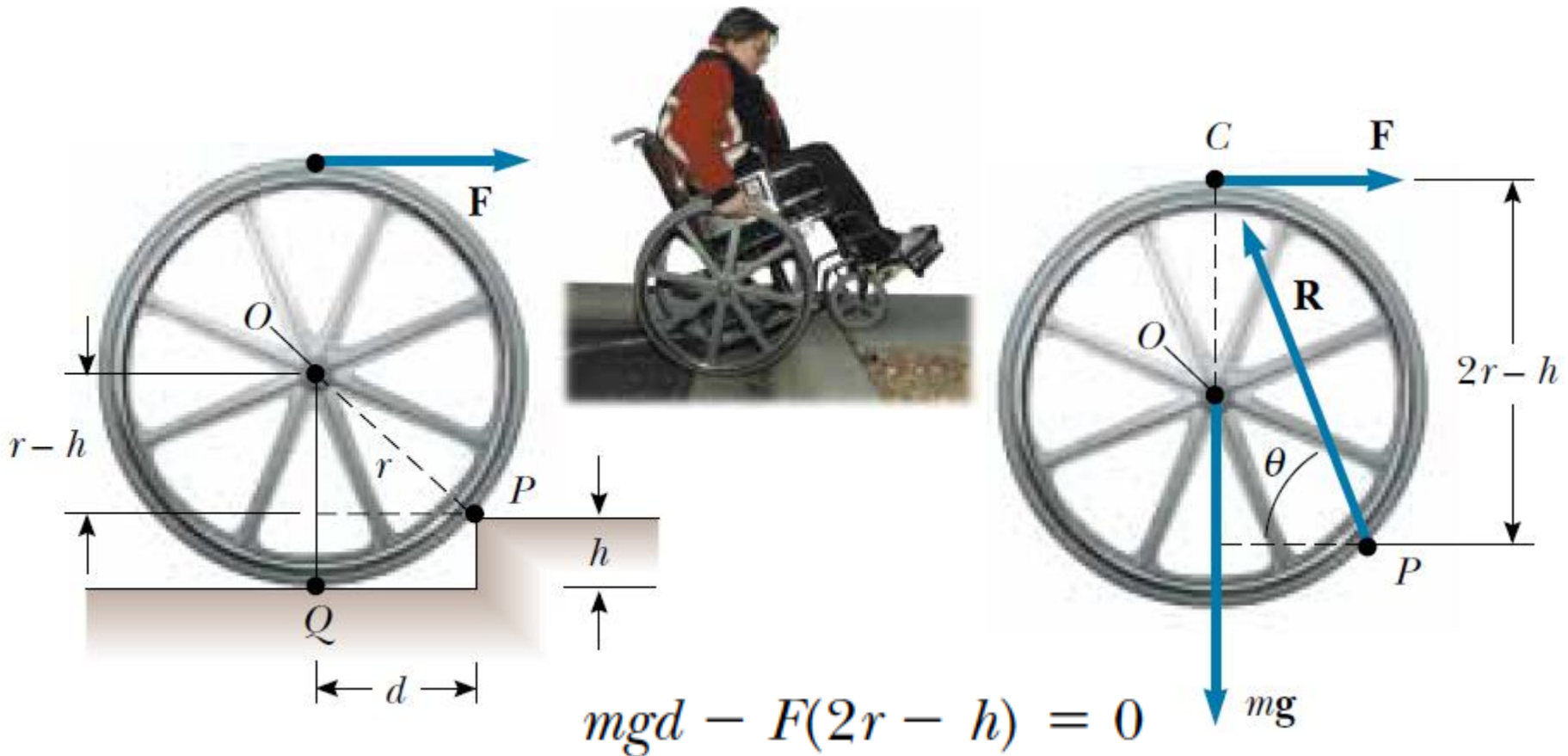$$\sum \tau_O = P\ell \sin \theta - mg \frac{\ell}{2} \cos \theta = 0$$

$$f_s \leq \mu_s n$$

Because the ladder is stationary, the three forces acting on it must all pass through some common point.

More Example: Up the Curb

$$mgd - F(2r - h) = 0$$
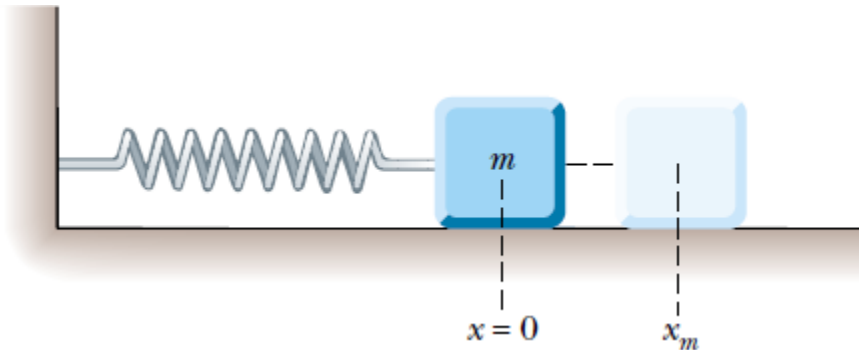
When the wheel is just about to be raised from the street, the reaction force exerted by the ground on the wheel at point Q goes to zero.

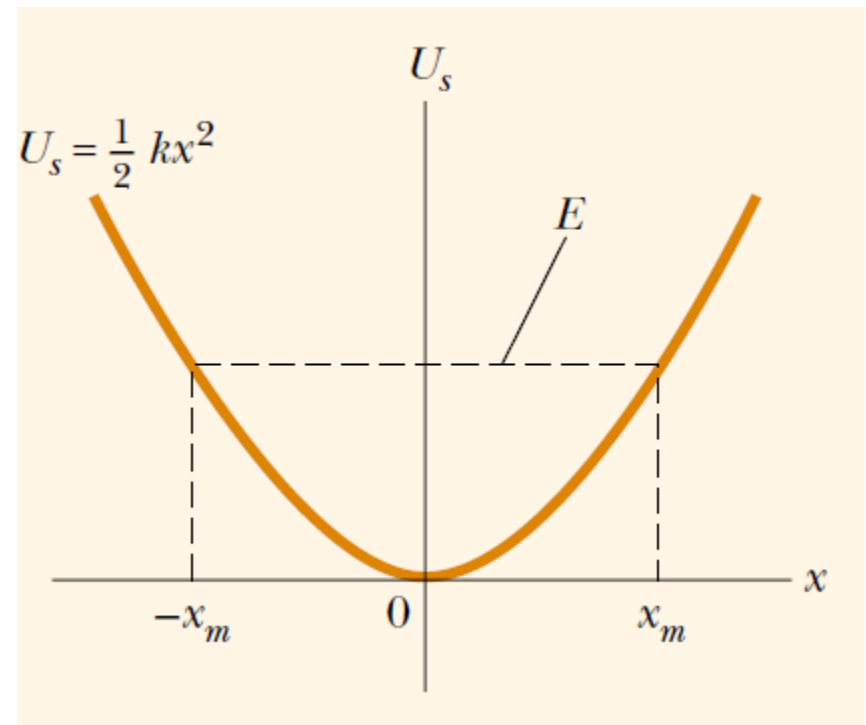# Equilibrium of a System

$$F_s = -\frac{dU_s}{dx} = -kx$$

In general, positions of **stable equilibrium** correspond to points for which $U(x)$ is a **minimum**.
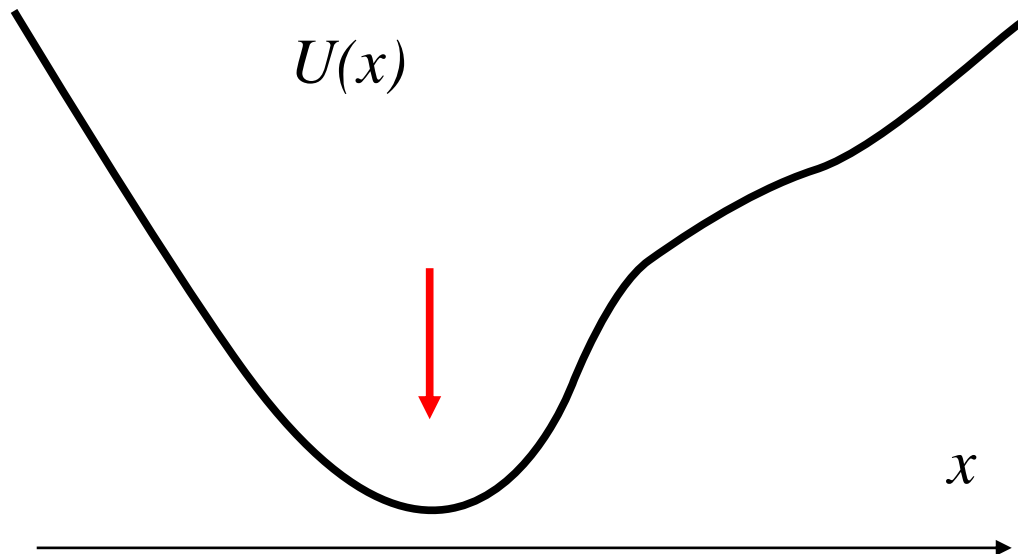
$$\frac{dU}{dx} = 0 \qquad \frac{d^2U}{dx^2} > 0$$

Positions of stable equilibrium correspond to points for which $U(x)$ is a minimum.

Equilibrium ($F = 0$)

$$\frac{dU}{dx} = 0$$

Stable

$$\frac{d^2U}{dx^2} > 0$$

Taylor Expansion

●**We can Taylor expand the potential energy near the equilibrium**

$$U(x) = U(x_0) + \frac{dU}{dx}\bigg|_{x=x_0} (x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2}\bigg|_{x=x_0} (x - x_0)^2 + \cdots$$

●**Quite generically, we expect linear restoring force like in the block-spring system, if we neglect the higher order terms (which are usually small if we stay close enough to the equilibrium).**

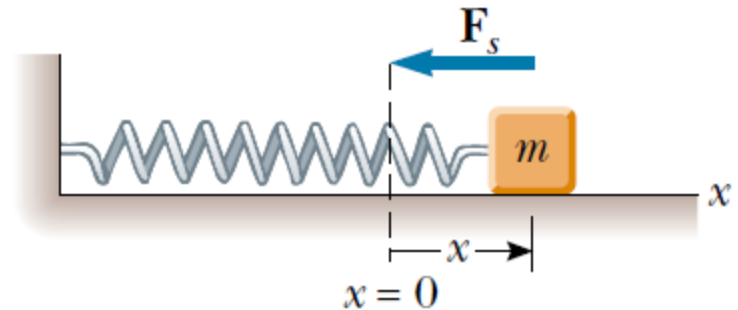●**This universal family of motion is known as the simple harmonic motion.**

Simple Harmonic Motion

●An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed.

$$F_s = -kx = ma$$
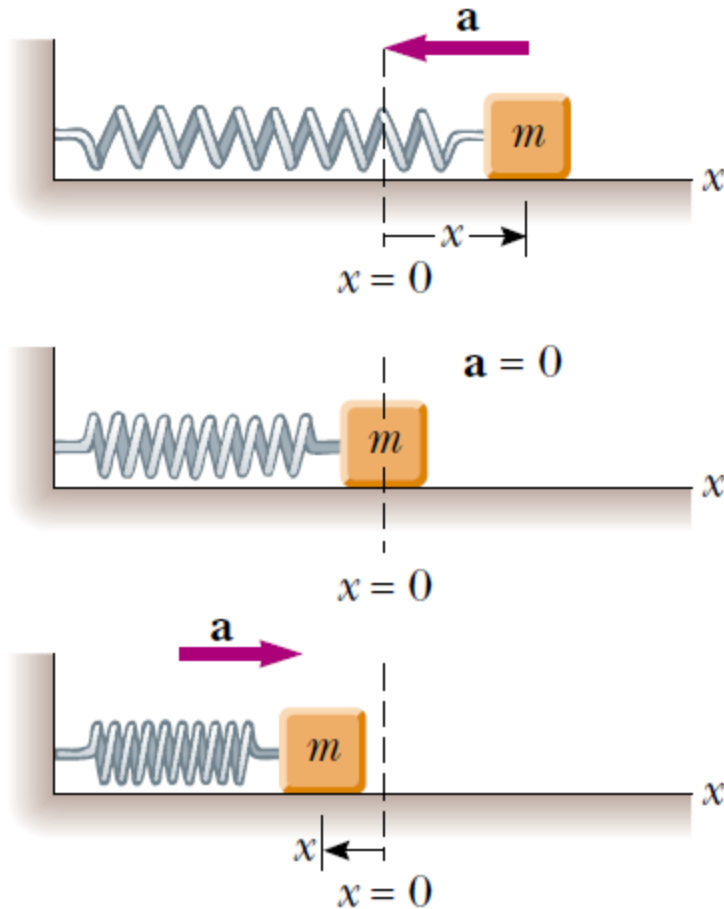
$$a = -\frac{k}{m}x$$

Quantitative Analysis

Define $\quad \omega = \sqrt{k/m}$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

Solution:

$$x = A\cos(\omega t + \phi)$$

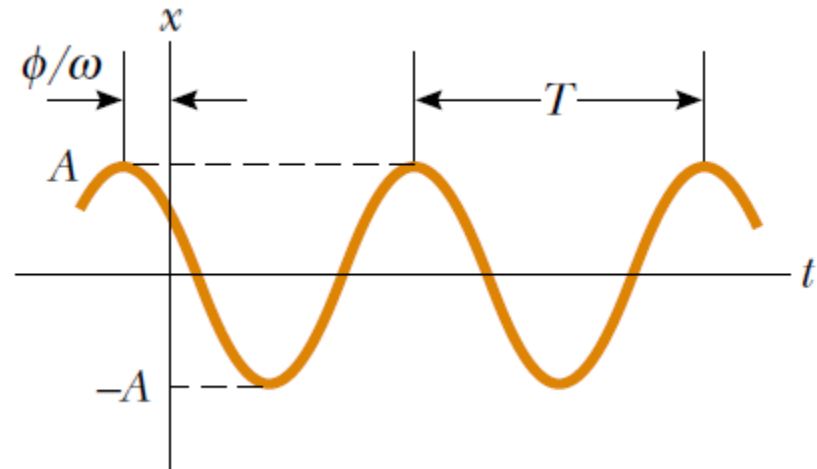Please check that the differential equation is satisfied.

The Meaning of the Solution

$$x = A \cos(\omega t + \phi)$$

A: Amplitude of the oscillation

$T = 2\pi / \omega$: Period of the oscillation

ϕ : Phase constant, or phase angle

Understand how to draw the *x*-*t* curve, and how to read information from the *x*-*t* curve.

Period and Frequency

• The **period** T of the motion is the time it takes for the particle to go through one full cycle.

$$T = \frac{2\pi}{\omega}$$

• The **frequency** represents the number of oscillations that the particle makes per unit time.
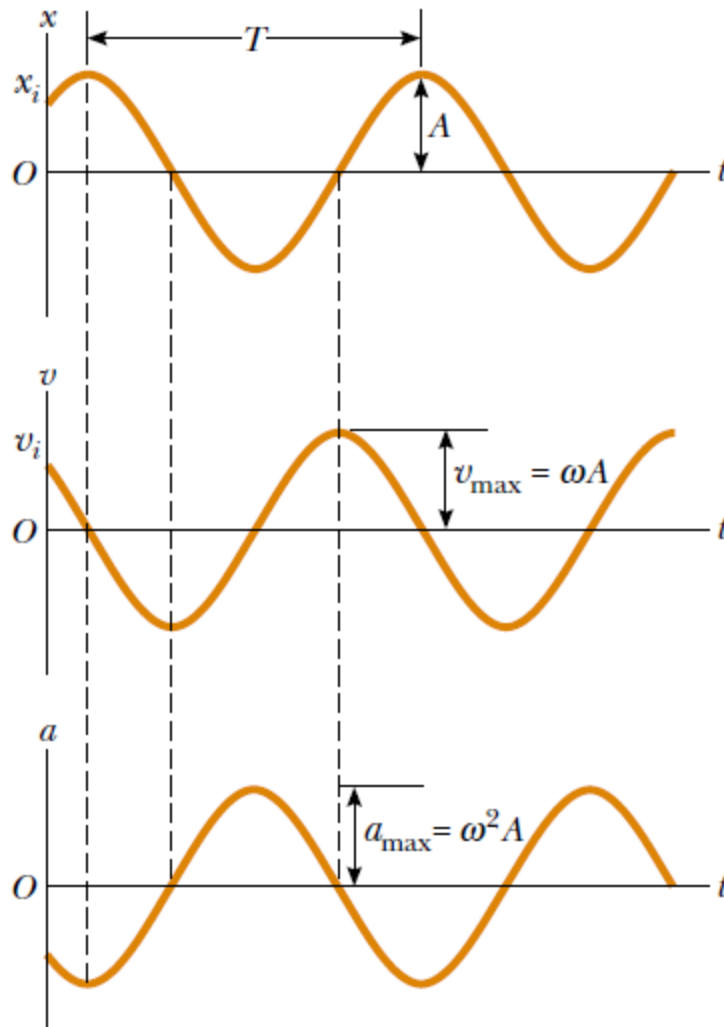
$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

The units of f are cycles per second $= \text{s}^{-1}$, or **hertz** (Hz).

Angular frequency: $\omega = 2\pi f = \dfrac{2\pi}{T}$

$\pi/2$ out of phase with the displacement

π out of phase with the displacement

Important Observations

- **The acceleration of the particle is proportional to the displacement but is in the opposite direction.** This is the necessary and sufficient condition for simple harmonic motion, as opposed to all other kinds of vibration.

- **The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time but are not in phase.**

- **The frequency and the period of the motion are independent of the amplitude.**
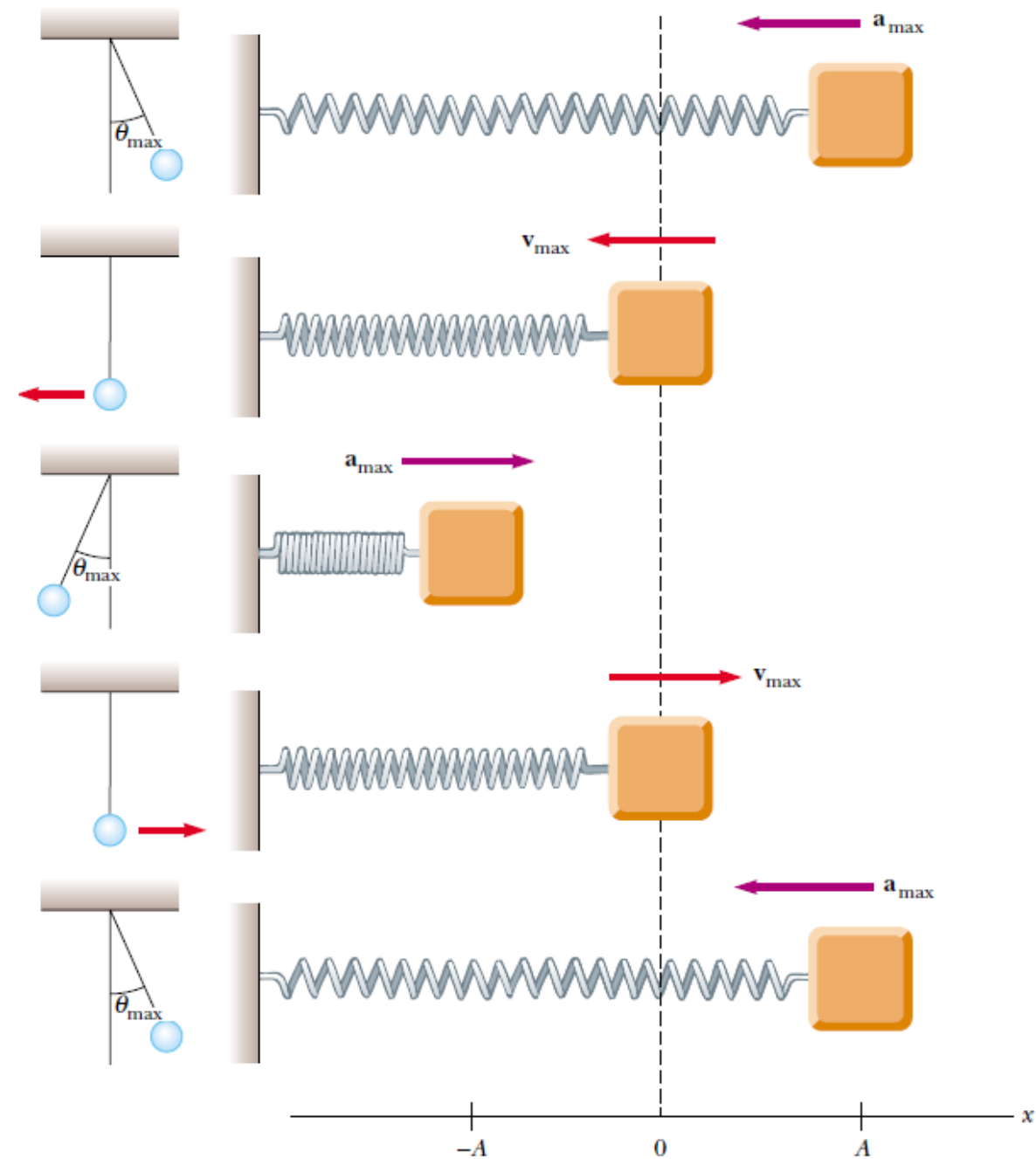
Energy of the Harmonic Oscillator

$$K = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi)$$

$$U = \tfrac{1}{2} kx^2 = \tfrac{1}{2} kA^2 \cos^2(\omega t + \phi)$$

$$E = K + U = \tfrac{1}{2} kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

$$= \tfrac{1}{2} kA^2 \qquad\qquad \omega = \sqrt{k/m}$$

The total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.

t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Energy Transform