Section 1.3 (supplement)

Propositional Normal Forms
（命题范式）

College of Computer Science and Technology
Zhejiang University

Propositional Formula

- (1) Each propositional variable is a formula, and the truth values T and F are also formulas.
- (2) If A is formula, so is $(\neg A)$.
- (3) If A and B are formulas, so are $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$.
- (4) A string of symbols is a formula only as it is determined by (finitely many applications of) the above three rules.

College of Computer Science and Technology
Zhejiang University

Example

- **EXAMPLE**

- The following string are formulas:
- $\neg(p \vee q)$, $p \to (q \to r)$, $(p \wedge q) \to r$

- The following string are not formulas:
- $pq \to r$, $\neg p \to q) \to r$

College of Computer Science and Technology
Zhejiang University

Propositional Normal Forms

- Formulas can be transformed into standard forms so that they become more convenient for symbolic manipulations and make identification and comparison of two formulas easier.

College of Computer Science and Technology
Zhejiang University

Normal Forms（范式）

- There are two types of normal forms in propositional calculus:
- the disjunctive normal form(DNF) （析取范式）
- and
- the conjunctive normal form(CNF) （合取范式）.

DEFINITION. A literal(文字) is a variable or its negation

Example p and $\neg p$ are literals, but $p \wedge q$ is not.

Propositional Normal Forms

- DEFINITION. A formula is said to be in disjunctive normal form if it is written as a disjunction, in which all the terms are conjunctions of literals.
- EXAMPLE 1 $(p \wedge q) \vee (p \wedge \neg q)$, $p \vee (q \wedge r)$, $\neg p \vee T$ are in disjunctive normal forms. The disjunction $\neg(p \wedge q) \vee r$ is not in normal form. (Why?)

Clauses（子句）

- In general a formula in disjunctive normal form is
- $(A_{11} \wedge \cdots \wedge A_{1n}) \vee \cdots \vee (A_{k1} \wedge \cdots \wedge A_{kn_k})$
- DEFINITION. Disjunctions (conjunctions) with literals as disjuncts (conjuncts) are called disjunctive (conjunctive) clauses. Disjunctive and conjunctive clauses are simply called clauses.

Clauses (cont…)

Example: $p \wedge q \wedge \neg r$ is a conjunctive clause

$\neg p \vee q \vee \neg r$ is a disjunctive clause

Conjunctive clause is also called basic product and Disjunctive clause is also called basic addition.

DEFINITION. A conjunction with disjunctive clauses as its conjunctions is said to be in Conjunctive normal form.

Cont..

EXAMPLE 2 $p \wedge (q \vee r)$ and $p \wedge F$ are in conjunctive normal form.

However $p \wedge (r \vee (p \wedge q))$ is not in conjunctive form.

In general a formula in conjunctive normal form is

$(A_{11} \vee \cdots \vee A_{1n}) \wedge \cdots \wedge (A_{k1} \vee \cdots \vee A_{knk})$

Conjunctive normal form

Observe the following formulas:

(1) p

(2) $\neg p \vee q$

(3) $\neg p \wedge q \wedge \neg r$

(4) $\neg p \vee (q \wedge \neg r)$

(5) $\neg p \wedge (q \vee \neg r) \wedge (\neg q \vee r)$

(1),(2),(3),(5) are in conjunctive normal forms.
(1),(2),(3),(4) are in disjunctive normal forms

How to obtain normal forms

(1)Eliminating $\rightarrow$, $\leftrightarrow$,|, $\downarrow$

(2) Eliminating $\neg$, $\vee$, $\wedge$ from the scope of $\neg$, such that any $\neg$ has only an atom as its scope. In other words eliminating the parenthesis following behind $\neg$

This method leads to obtaining the disjunctive or conjunctive normal forms.

Example

Convert the following formula into a conjunctive normal form.

$\neg((p \vee \neg q) \wedge \neg r)$

The conjunctive normal form can be found by the following derivations:

$\neg((p \vee \neg q) \wedge \neg r)$

(1) $\neg(p \vee \neg q) \vee (\neg\neg r)$ De Morgan

(2) $\neg(p \vee \neg q) \vee r$ Double negation

Example

(3) $(\neg p \wedge \neg\neg q) \vee r$ De Morgan

(4) $(\neg p \wedge q) \vee r$ Double negation

(5) $(\neg p \vee r) \wedge (q \vee r)$ Distributivity

EXAMPLE 5 Convert the following formula into a conjunctive normal form.

$(p1 \wedge p2) \vee (p3) \wedge (p4 \vee p5)$

Example

- Solution:
- $(p1 \wedge p2) \vee (p3) \wedge (p4 \vee p5) \Leftrightarrow$
- $(p1 \wedge p2) \vee (p3 \wedge (p4 \vee p5)) \Leftrightarrow$
- $((p1 \wedge p2) \vee p3) \wedge ((p1 \wedge p2) \vee (p4 \vee p5))$
- $\Leftrightarrow (p1 \vee p3) \wedge (p1 \vee p4 \vee p5) \wedge (p2 \vee p3) \wedge (p2 \vee p4 \vee p5)$

- Examples

2. Convert the following formula into conjunctive and disjunctive normal forms. $\neg(p\vee q)\leftrightarrow(p\wedge q)$

Solution:

Remark: Formulas (*) is in **CNF** and formulas (**) is in DNF.

$$\neg(p\vee q)\leftrightarrow(p\wedge q)$$
$$\equiv(\neg(p\vee q)\rightarrow(p\wedge q))\wedge((p\wedge q)\rightarrow\neg(p\vee q))$$
$$\equiv((p\vee q)\vee(p\wedge q))\wedge(\neg(p\wedge q)\vee\neg(p\vee q))$$
$$\equiv((p\vee q\vee p)\wedge(p\vee q\vee q))\wedge((\neg p\vee\neg q)\vee(\neg p\wedge\neg q))$$
$$\equiv(p\vee q)\wedge(\neg p\vee\neg q\vee\neg p)\wedge(\neg p\vee\neg q\vee\neg q)$$
$$\equiv(p\vee q)\wedge(\neg p\vee\neg q)*$$
$$\equiv((p\vee q)\wedge\neg p)\vee((p\vee q)\wedge\neg q)$$
$$\equiv(p\wedge\neg p)\vee(q\wedge\neg p)\vee(p\wedge\neg q)\vee(q\wedge\neg q)$$
$$\equiv(q\wedge\neg p)\vee(p\wedge\neg q)**$$

Disjunctive normal forms from truth tables

- So far we have shown how to find the truth table of a logical formula. The reverse is also possible. One can convert any given truth table into a formula.
- The formula obtained in this way is already in disjunctive normal form.
- In fact, the conceptually easiest method to find the normal form of a formula is by using truth tables. Unfortunately, truth tables grow exponentially with the number of variables,

Disjunctive normal forms from truth tables

- which makes this method unattractive for formulas with many variables.
- Generally, a truth table gives truth values of some formula for all possible assignments. The table below gives an example of truth table for a certain formula f. The truth values of f depends on the three variables p, q, r.
- This makes f a truth function, or a Boolean function with arguments

College of Computer Science and Technology
Zhejiang University

(cont…)

- p q r f
- T T T T
- T T F F
- T F T T
- T F F F
- F T T F
- F T F F
- F F T T
- F F F F

College of Computer Science and Technology
Zhejiang University

minterm

- To convert a Boolean function into a formula, one makes use of minterms.
- DEFINITION. A minterm is a conjunction of literals in which each variable is represented exactly once.
- EXAMPLE 7 If a Boolean function has the variables p, q, r then p ^¬q ^ r is a minterm but p ^ ¬q and p ^ p ^ ¬q are not.

College of Computer Science and Technology
Zhejiang University

Representation of minterm

- There are 2^n different minterm for n propositional variables. For example there 4 different minterm for p, q, they are p^q, p^¬q, ¬p ^q, ¬p^¬q. For the sake of simplification, we use m_j denote the minterms. Where j is a integer, its binary representation corresponds the evaluation of variables that make m_j be equal to T.

College of Computer Science and Technology
Zhejiang University

Example

- For three propositional variables,
- $m_0 = \neg p \wedge \neg q \wedge \neg r$ $m_1 = \neg p \wedge \neg q \wedge r$
- $m_2 = \neg p \wedge q \wedge \neg r$ $m_3 = \neg p \wedge q \wedge r$
- $m_4 = p \wedge \neg q \wedge \neg r$ $m_5 = p \wedge \neg q \wedge r$
- $m_6 = p \wedge q \wedge \neg r$ $m_7 = p \wedge q \wedge r$
- If a propositional form is denoted by: $f= m_j \vee m_k \vee \ldots \vee m_l$ we simply denote $f=\sum m(j,k,..,l)$

College of Computer Science and Technology
Zhejiang University

Properties of minterms

- (1) Each minterm is true for exactly one assignment. For example, $p \wedge \neg q \wedge r$ is true if p is T, q is F and r is T. Any deviation from this assignment would make this particular minterm false.
- (2) The conjunction of two different minterm is always false. I.e. $m_i \wedge m_k = F$
- (3) The disjunction of all mintern is T
- $m_0 \vee m_1 \vee \ldots \vee m_{2^n-1} = T$

College of Computer Science and Technology
Zhejiang University

Properties of minterms

- A disjunction of minterms is true only if at least one of its constituents minterms is true. For example,
- $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$
- is only true if at least one of $p \wedge q \wedge r$, $p \wedge \neg q \wedge r$ or $\neg p \wedge \neg q \wedge r$ is true.

Minterms and truth tables

- If a function, such as f, is given by truth table, we know exactly for which assignments it is true. Consequently, we can select the minterms that make the function true and form the disjunction of these minterms.
- The function f, for instance, is true for three assignments:

Disjunctive normal forms from truth tables

- 1. p, q, r are all true.
- 2. p,¬q, r are all true.
- 3. ¬p,¬q, r are all true.
- The disjunction of the corresponding minterms is tautologically equivalent to f, which means that we have the following formula for f:
- $f \Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

Full disjunctive form

- Actually, we have a special type of normal form, the full disjunctive form.
- DEFINITION. If a Boolean function is expressed as a disjunction of minterms, it is said to be in full disjunctive form.

College of Computer Science and Technology
Zhejiang University

How to find Full disjunctive form

- Two methods:
- (1) From truth table;
- (2) From disjunctive form, changing its clauses to minterms.
- Example finding the From disjunctive form of formula
- $f = (p \wedge \neg q) \to (\neg r \wedge \neg p)$

Solution

p	q	r	$p \wedge \neg q$	$\neg r \wedge \neg p$	$(p \wedge \neg q) \to (\neg r \wedge \neg p)$
F	F	F	F	T	T
F	F	T	F	F	T
F	T	F	F	T	T
F	T	T	F	F	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	F	F	T
T	T	T	F	F	T

Cont..

- So $f = \sum m(0,1,2,3,6,7)$
- Solution 2:
- $f = (p \wedge \neg q) \rightarrow (\neg r \wedge \neg p)$
- $= \neg(p \wedge \neg q) \vee (\neg r \wedge \neg p)$
- $= \neg p \vee q \vee (\neg r \wedge \neg p)$
- $= (\neg p \wedge T \wedge T) \vee (q \wedge T \wedge T) \vee (\neg r \wedge \neg p \wedge T)$

College of Computer Science and Technology
Zhejiang University

Cont..

- $= (\neg p \wedge (q \vee \neg q) \wedge (r \vee \neg r))$
- $\vee (q \wedge (p \vee \neg p) \wedge (r \vee \neg r))$
- $\vee (\neg r \wedge \neg p) \wedge (r \vee \neg r)$
- $= (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee$
- $(\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$
- $= m_0 \vee m_1 \vee m_2 \vee m_3 \vee m_6 \vee m_7$

College of Computer Science and Technology
Zhejiang University

Conjunctive Normal Form

- A compound proposition is in *Conjunctive Normal Form* (CNF) if it is a conjunction of disjunctions.
- Every proposition can be put in an equivalent CNF.
- Conjunctive Normal Form (CNF) can be obtained by eliminating implications, moving negation inwards and using the distributive and associative laws.
- Important in resolution theorem proving used in artificial Intelligence (AI).
- A compound proposition can be put in conjunctive normal form through repeated application of the logical equivalences covered earlier.

Conjunctive Normal Form

Example: Put the following into CNF:

$$\neg(p \rightarrow q) \vee (r \rightarrow p)$$

Solution:

1. Eliminate implication signs:

$$\neg(\neg p \vee q) \vee (\neg r \vee p)$$

2. Move negation inwards; eliminate double negation:

$$(p \wedge \neg q) \vee (\neg r \vee p)$$

3. Convert to CNF using associative/distributive laws

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

Conjunctive Normal Form and Disjunctive form

- We can convert disjunctive form to Conjunctive Normal Form directly

Let $f = \sum m(j,k,..,l)$, $g = \sum m(\{0,1,2,....2^{n-1}\}-\{j,k,..,l\})$

Then $f \vee g = T$, $f \wedge g = F$, hence

$f = \neg g = \pi\, M(\{0,1,2,....2^{n-1}\}-\{j,k,..,l\})$

Where M_i = is $\neg\, m_i$ is a disjunction.

Homework

- Find the full disjunctive form from truth table of following formula

$((\neg p \wedge q) \rightarrow (p \leftrightarrow r)) \vee (q \rightarrow \neg r)$