Quiz 2 T5

Let $p>5$ be a prime number. Prove that there are infinite terms a_k in the sequence *1, 11, 111, 1111, ... , 11…1,* (a_k *has k '1's*) such that $p \mid a_k$. （*Hint: Fermat's little theorem*） ***(10%)***

给提示了肯定用得到，关键是想想费马小定理 $a^{p-1} \equiv 1 \pmod p$ 中的a和p在本题中分别是什么

Answer

$10^{p-1} \equiv 1 \pmod p$

$p \mid \frac{10^{p-1}-1}{9}$

可以得到$a_{p-1} = 111 \dots 111(p-1个1) = \frac{10^{p-1}-1}{9}$，即$p \mid a_{p-1}$。

有

$$10^{n(p-1)} \equiv 1^n \equiv 1 \pmod p$$

依此类推，有$p \mid a_{n(p-1)}$对任意正整数n成立，故有无限项，得证。

- 注意要把无限项这一点说出来，有的同学只写出一项就不写下去了。
- 还有些同学用了抽屉原理+同余数两项相减的办法，也很有趣，说理完整的也给分。

Quiz 2 T6

- Given any positive integer m, prove that a multiple of m can be found in the Fibonacci sequence. ***(10%)***

Answer

令斐波那契数列$\{a_n\}$对m取模，得到新数列$\{b_n\}$，其取值范围为0~m-1的整数。二元组:

$$(b_p, b_{p+1})$$

共有m*m种取值。

取值有限，而数列无限。由鸽巢原理，可知至少存在一组p、q使得p>q且二元组:

$$(b_p, b_{p+1}) = (b_q, b_{q+1})$$

向前倒推，根据取模的性质有:

$$(b_{p-1}, b_p) = \left((b_{p+1} - b_p) mod\ m, b_p\right) = \left((b_{q+1} - b_q) mod\ m, b_q\right) = (b_q, b_{q+1})$$

重复以上倒推过程，可以对任意i得到:

$$(b_{p-i}, b_{p-i+1}) = (b_{q-i}, b_{q-i+1})$$

令i=q-1，得到:

$$(b_{p-q+1}, b_{p-q+2}) = (b_1, b_2) = (1,1)$$

可知$b_{p-q} = b_{p-q+2} - b_{p-q+1} = 1 - 1 = 0$，即$a_{p-q}$是m的倍数，得证。

1. 2
4

To use the wireless network in the airport you must pay the daily fee unless you are a subscriber to the service. Express your answer in terms of w: "You can use the wireless network in the airport," d: "You pay the daily fee," and s: "You are a subscriber to the service."

$$w \rightarrow (d \vee s)$$

1. 5
12hkn

Let I (x) be the statement "x has an Internet connection" and C(x, y) be the statement "x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

h) Exactly one student in your class has an Internet connection.

k) Someone in your class has an Internet connection but has not chatted with anyone else in your class.

n) There are at least two students in your class who have not chatted with the same person in your class.

h) $\exists x \forall y (x = y \leftrightarrow I(y))$

k) $\exists x (I(x) \wedge \forall y (x \neq y \rightarrow \neg C(x, y)))$

n) $\exists x \exists y (x \neq y \wedge \forall z \, \neg(C(x, z) \wedge C(y, z)))$

Quiz 1 T5

a) Show the full conjunctive normal form of $(p \leftrightarrow \neg r) \rightarrow (q \leftrightarrow r)$. ***(6%)***

b) Show the full disjunctive normal form of $(p \leftrightarrow \neg r) \rightarrow (q \leftrightarrow r)$. ***(6%)***

Answer

a) $M1 \wedge M6$ (order : p q r)

即: $(p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r)$

b) $m0 \vee m2 \vee m3 \vee m4 \vee m5 \vee m7$ (order : p q r)

即: $(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$

- 注意conjunctive和disjunctive的求法以及区别，不要混淆

Quiz 1 T8

There are 2 piles of stones that begins with 115 stones and 125 stones respectively.

Suppose that two people play a game taking turns removing one, two, or three stones at a time from one of the piles. The person who removes the last stone wins the game.

Show that the first player can win the game no matter what the second player does. ***(8%)***

Answer

- 在125那堆拿掉2个，剩下123个；或在115那堆拿掉2个，剩下113个。
- 这时两堆石头的个数差123-115 = 8。
- 然后不管对手怎么拿，接着让两堆石头的个数差保持为4的倍数即可。

Sec 2.5 T4

- Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
- c) the real numbers with decimal representations consisting of all 1s
- d) the real numbers with decimal representations of all 1s or 9s
    - 注意、理解c)和d)的差别
    - 在c)的表达中注意不要忽略无限小数

c) This set is countable but a little tricky. We can arrange the numbers in a 2-dimensional table as follows:

$.\overline{1}$	$.1$	$.11$	$.111$	$.1111$	$.11111$	$.111111$	$\ldots$
$1.\overline{1}$	1	1.1	1.11	1.111	1.1111	1.11111	$\ldots$
$11.\overline{1}$	11	11.1	11.11	11.111	11.1111	11.11111	$\ldots$
$111.\overline{1}$	111	111.1	111.11	111.111	111.1111	111.11111	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table). Therefore by Exercise 27, the entire set is countable. For an explicit correspondence with the positive integers, we can zigzag along the positive-sloping diagonals as in Figure 3: $1 \leftrightarrow .\overline{1}$, $2 \leftrightarrow 1.\overline{1}$, $3 \leftrightarrow .1$, $4 \leftrightarrow .11$, $5 \leftrightarrow 1$, and so on.

d) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5. All we need to do is choose $d_i = 1$ when $d_{ii} = 9$ and choose $d_i = 9$ when $d_{ii} = 1$ or d_{ii} is blank (if the decimal expansion is finite).

Quiz 1 T6

Let S be a subset of rational numbers, satisfying the following conditions: ***(12%)***

1. If $a \in S$, $b \in S$, then $a+b \in S$, $ab \in S$;
2. For any rational number r, there is one and only one of the three relationship holds:

$$r \in S,\ -r \in S,\ r = 0 .$$

Prove that $S = Q^+$ (Q^+ is the set of positive rational numbers).

- 注意证明的严谨性
- 注意充要性和逻辑性。有些同学只是构造出一个S集合满足条件，如果这样证明的话，必须说明这是唯一可行的S集合，否则若有其他符合条件的S，那么结论就不成立了。
- 从简单的数代入，尝试找一些特殊的点做切入口，例如0、1。

Answer

1. 证明 0 ∉ S
2. 证明 1∈S （1 = 1*1 = （-1）*(-1)）
3. 证明所有正整数 ∈S
4. 任何一个非零有理数r，可以写成 r = p / q，p为正整数，q为非零整数
5. (1/q)*(1/q) ∈S，p ∈S， |q|∈S
6. p * |q| * (1/q)*(1/q) ∈S
7. 所以 |r| ∈S
8. 所有 –|r| ∉ S
9. $S = Q^+$

其他写法言之有理、逻辑完整也给满分

Quiz 1 T7

Use induction to prove that for all nonnegative integer *n*,

a) $f_{5n} \equiv 0 \ (mod\ 5)$, **(6 %)**

b) $f_n{}^2 + f_{n+1}{}^2 = f_{2n+1}$, **(9 %)**

- where f_i denotes the *ith* Fibonacci number.

Answer

a)

①当n=1时，$f_5=5$，成立;

②若n=k时成立，即$f_{5k} \equiv 0 \ (mod\ 5)$;

则n=k+1时，由递推关系得到$f_{5k+5} = 5f_{5k+1} + 3f_{5k} \equiv 0 \ (mod\ 5)$，得证。

b) 用到了两次归纳法，需要耐心

① 当n=1时，$f_1^2 + f_2^2 = 2 = f_3$，成立;

② 若n=k时成立，即$f_k^2 + f_{k+1}^2 = f_{2k+1}$;

则n=k+1时，$f_{k+1}^2 + f_{k+2}^2 = f_{k+1}^2 + (f_k + f_{k+1})^2 = f_{2k+1} + f_{k+1}^2 + 2f_kf_{k+1}$

故只需证$f_{n+1}^2 + 2f_nf_{n+1} = f_{2n+2}$即可。

③ 当n=1时，$f_2^2 + 2f_1f_2 = 3 = f_4$，成立;

④ 若n=k时，$f_{k+1}^2 + 2f_kf_{k+1} = f_{2k+2}$与$f_k^2 + f_{k+1}^2 = f_{2k+1}$都成立

则由①②，知$f_{k+1}^2 + f_{k+2}^2 = f_{2k+3}$成立;

则n=k+1时，$f_{k+2}^2 + 2f_{k+1}f_{k+2} = f_{k+2}^2 + 2f_{k+1}(f_k + f_{k+1}) = f_{k+2}^2 + 2f_kf_{k+1} + 2f_{k+1}^2 = f_{2k+2} + f_{k+1}^2 + f_{k+2}^2 = f_{2k+2} + f_{2k+3} = f_{2k+4}$,

证得$f_{k+2}^2 + 2f_{k+1}f_{k+2} = f_{2k+4}$成立。

由①②③④知，若n=k时，$f_k^2 + f_{k+1}^2 = f_{2k+1}$和$f_{k+1}^2 + 2f_kf_{k+1} = f_{2k+2}$都成立,

则n=k+1时，不仅有$f_{k+1}^2 + f_{k+2}^2 = f_{2k+3}$成立，更能进一步得到$f_{k+2}^2 + 2f_{k+1}f_{k+2} = f_{2k+4}$也成立,

从而归纳得到对任意的n，都有$f_n^2 + f_{n+1}^2 = f_{2n+1}$和$f_{n+1}^2 + 2f_nf_{n+1} = f_{2n+2}$成立，得证。

Sec 6.2 T44

- An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 P.M., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.
- Is the statement in true if 24 is replaced by:
- a) 2 b) 23 c) 25 d) 30 ?

- 四种情况皆为True。
- 尤其在c) d)两小题中，注意课件上最基本的使用鸽巢原理的证明方式只是给出一个充分条件而非必要条件。

- Let ai be the number of matches completed by hour i. Then 1 ⩽ a1 < a2 < ⋯ < a75 ⩽ 125. Also 25 ⩽ a1 + 24 < a2 + 24 < ⋯ < a75 + 24 ⩽ 149. There are 150 numbers a1, ⋯ , a75, a1 + 24, ⋯ , a75 + 24. By the pigeonhole principle, at least two are equal. Because all the ai s are distinct and all the (ai + 24)s are distinct, it follows that ai = aj + 24 for some i > j. Thus, in the period from the (j+1)st to the ith hour, there are exactly 24 matches.

a) The solution of Exercise 41, with 24 replaced by 2 and 149 replaced by 127, tells us that the statement is true.

b) The solution of Exercise 41, with 24 replaced by 23 and 149 replaced by 148, tells us that the statement is true.

c) We begin in a manner similar to the solution of Exercise 41. Look at $a_1, a_2, \ldots, a_{75}, a_1+25, \ldots, a_{75}+25$, where a_i is the total number of matches played up through and including hour i. Then $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$, and $26 \leq a_1+25 < a_2+25 < \cdots < a_{75}+25 \leq 150$. Now either these 150 numbers are precisely all the number from 1 to 150, or else by the pigeonhole principle we get, as in Exercise 41, $a_i = a_j + 25$ for some i and j and we are done. In the former case, however, since each of the numbers $a_i + 25$ is greater than or equal to 26, the numbers $1, 2, \ldots, 25$ must all appear among the a_i's. But since the a_i's are increasing, the only way this can happen is if $a_1 = 1$, $a_2 = 2$, $\ldots$, $a_{25} = 25$. Thus there were exactly 25 matches in the first 25 hours.

d) We need a different approach for this part, an approach, incidentally, that works for many numbers besides 30 in this setting. Let $a_1, a_2, \ldots, a_{75}$ be as before, and note that $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$. By the pigeonhole principle two of the numbers among $a_1, a_2, \ldots, a_{31}$ are congruent modulo 30. If they differ by 30, then we have our solution. Otherwise they differ by 60 or more, so $a_{31} \geq 61$. Similarly, among a_{31} through a_{61}, either we find a solution, or two numbers must differ by 60 or more; therefore we can assume that $a_{61} \geq 121$. But this means that $a_{66} \geq 126$, a contradiction.

Sec 6.3

46. How many ways are there for a horse race with four horses to finish if ties are possible? [*Note:* Any number of the four horses may tie.]

46. We can solve this problem by breaking it down into cases depending on the number of ties. There are five cases. (1) If there are no ties, then there are clearly $P(4,4) = 24$ possible ways for the horses to finish. (2) Assume that there are two horses that tie, but the others have distinct finishes. There are $C(4,2) = 6$ ways to choose the horses to be tied; then there are $P(3,3) = 6$ ways to determine the order of finish for the three groups (the pair and the two single horses). Thus there are $6 \cdot 6 = 36$ ways for this to happen. (3) There might be two groups of two horses that are tied. There are $C(4,2) = 6$ ways to choose the winners (and the other two horses are the losers). (4) There might be a group of three horses all tied. There are $C(4,3) = 4$ ways to choose which these horses will be, and then two ways for the race to end (the tied horses win or they lose), so there are $4 \cdot 2 = 8$ possibilities. (5) There is only one way for all the horses to tie. Putting this all together, the answer is $24 + 36 + 6 + 8 + 1 = 75$.

一点附加知识：斯特林stirling数

- The **signless Stirling number of the first kind** c(n, k), where k and n are integers with $1 \leq k \leq n$, equals the number of ways to arrange n people around k circular tables with at least one person seated at each table, where two seatings of m people around a circular table are considered the same if everyone has the same left neighbor and the same right neighbor.(**Textbook, Page 465.** **)**

  求将n个数划分成m个圆排列的方案数。

- Let $S(n, j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. The numbers $S(n, j)$ are called **Stirling numbers of the second kind**. (**Textbook, Page 453**.)

  求将n个数划分成m个集合的方案数，集合无标号。

无符号第一类斯特林数

递推式

无符号第一类Stirling数的递推式可以从其定义来推导:

考虑其定义如果要将n+1元素构成m个圆排列，考虑第n+1个元素:

(1) 如果n个元素构成了m-1个圆排列，那么第n+1个元素独自构成一个圆排列。方案数:

$$s_u(n, m-1)$$

(2) 如果n个元素构成了m个圆排列，将第n+1个元素插入到任意元素的左边。方案数:

$$n \cdot s_u(n, m)$$

综合两种情况得:

$$s_u(n+1, m) = s_u(n, m-1) + n \cdot s_u(n, m)$$

1	n=0
0 1	n=1
0 1 1	n=2
0 2 3 1	n=3
0 6 11 6 1	n=4
0 24 50 35 10 1	n=5
0 120 274 225 85 15 1	n=6
0 720 1764 1624 735 175 21 1	n=7

第二类斯特林数

递推式

第二类Stirling数的推导和第一类Stirling数类似，可以从定义出发考虑第n+1个元素的情况，假设要把n+1个元素分成m个集合则分析如下：

（1）如果n个元素构成了m-1个集合，那么第n+1个元素单独构成一个集合。方案数 $S(n, m-1)$ 。

（2）如果n个元素已经构成了m个集合，将第n+1个元素插入到任意一个集合。方案数 m*S(n,m) 。

综合两种情况得：

$$S(n+1, m) = S(n, m-1) + m \cdot S(n, m)$$

“pascal” 三角形

n=0	1
n=1	0 1
n=2	0 1 1
n=3	0 1 3 1
n=4	0 1 7 6 1
n=5	0 1 15 25 10 1
n=6	0 1 31 90 65 15 1
n=7	0 1 63 301 350 140 21 1
n=8	0 1 127 966 1701 1050 266 28 1
n=9	0 1 255 3025 7770 6951 2646 462 36 1

Sec6.5 54.

How many ways are there to put 5 different employees into 4 identical offices?

Answer: $\sum_{j=1}^{4} S(5,j) = 1 + 15 + 25 + 10 = 51$

8.6 11 In how many ways can 7 different jobs be assigned to 4 different employees so that each employee is assigned at least one job and the most difficult job is assigned to the best employee?

$3^6 - C(3,1)2^6 + C(3,2)1^6$ =27*27-3*64+3= 540

$4^6 - C(4,1)3^6 + C(4,2)2^6 - C(4,3)1^6$ =64*64-4*27*27+6*64-4=1560

540+1560 = 2100

Quiz 2 T2

a) How many ways are there to distribute 4 balls into 6 boxes, both the balls and boxes are labeled?

exp(6,4)=1296

注意：勿写成exp(4,6)

b) How many ways are there to distribute 4 balls into 6 boxes, if each box must have at most one ball in it, and the balls are unlabeled but the boxes are labeled?

C(6,4)=15

c) How many ways are there to distribute 6 balls into 6 boxes, both the balls and boxes are labeled, and no box is empty?

P(6,6)=6!=720

d) How many ways are there to distribute 6 balls into 6 boxes, both the balls and boxes are labeled, and exactly 3 boxes are not empty?

C(6,3)*(exp(3,6)-3*exp(2,6)+3)=10800

注意：写成这种减法的格式比加法分类讨论更简洁，但最后的+3千万不要漏掉

e) How many ways are there to distribute 6 balls into 6 boxes, if the balls are unlabeled, but the boxes are labeled, and exactly 3 boxes are not empty?

C(6,3)*C(3+3-1,3-1)=200

f) There are 4 kinds of balls colored with Red, Green, Blue and White. The number of each kind is unlimited, and the balls with the same type are unlabeled. How many ways are there to distribute 6 balls into 6 labeled boxes, if there are one or more red balls among them and no box is empty?

exp(4,6)-exp(3,6)=3367

g) Given 4 kinds of balls, the number of each kind is unlimited and the balls with same type are unlabeled. How many ways are there to distribute 6 balls into 6 labeled boxes, if there are exactly 3 kinds of balls and no box is empty?

C(4,3)*(exp(3,6)-exp(2,6)*3+3)=2160

h) There are 3 kinds of balls, and each kind has 3 balls. The balls with the same type are unlabeled. How many ways are there to take out 6 balls?

10

Quiz 2 T4

Find the solution to the following iteration relation: ***(10%)***

$a_n = 4a_{n-1} - 3a_{n-2} + 2^n + 1$ with $a_0 = 1$ and $a_1 = 3$.

这种题目是考试必出题型，没写出来的同学一定要自己亲手算过！

Answer

x*x-4x+3=0

x=1 or x=3

得到a_n = A*3^n+B*2^n+C*n+D

代入n=0、1、2、3的值，解得

A=13/4

B=-4

C=-1/2

D=7/4