Closures of Relations

Section 9.4

Definition of Closure

- The *closure* of a relation R with respect to property **P** is the relation obtained by adding the minimum number of ordered pairs to R to obtain property **P**.

Reflexive Closure

- In terms of the digraph representation of *R*:
  - Add loops to all vertices to find the reflexive closure
- In terms of the 0-1 matrix representation:
  - Put 1's on the diagonal to find the reflexive closure
- $r(R)=R\cup\Delta$ where $\Delta=\{(a,a)\mid a\in A\}$

Symmetric Closure

- In terms of the digraph representation of *R*:
  - Add arcs in the opposite direction to find the symmetric closure

  $S(R)= R\cup R^{-1}$
- In terms of the 0-1 matrix representation:
  - Add 1's to the pairs across the diagonals that differ in value

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \quad\Rightarrow\quad \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 1 & 0 \end{matrix}$$

Transitive Closure

- It is very easy to find the reflexive closure and the symmetric closure, but it is difficult to find the transitive closure
- In terms of the digraph representation of *R*:
  - To find the transitive closure, if there is a path from *a* to *b*, add an arc from *a* to *b* (can be complicated)

Transitive Closure

- R={(1,3),(1,4),(2,1),(3,2)} first adding the pairs (1,2),(2,3),(2,4)(3,1) to R obtain R'={(1,3),(1,4),(2,1),(3,2), (1,2),(2,3),(2,4) (3,1) } is not transitive either.
- A path from *a* to *b* in the digraph *G* is a sequence of one or more edges (x_0,x_1), (x_1,x_2), …, (x_{n-1},x_n) in *G* where $x_0=a$ and $x_n=b$. if a=b, the path is called circuit or cycle.

Transitive Closure (Cont.)

- This path is denoted by $x_0, x_1, x_2, \ldots, x_n$ and has length n. the path is called a cycle if it starts and ends at the same vertex.
- Theorem 1: Let R be a relation on a set A, there is a path of length n from a to b if and only if $(a,b) \in R^n$

Proof:

① **Inductive basis**

An edge from *a* to *b* is a path of length 1 which is in $R^1 = R$. Hence the assertion is true for $n = 1$.

② **Inductive step**

There is a path of length n+1 from *a* to *b* if and only if there is an *x* in *A* such that there is a path of length 1 from *a* to *x* and a path of length *n* from *x* to *b*.

From the Induction Hypothesis,

$$(a,x) \in R \qquad (x,b) \in R^n$$

$$(a,b) \in R^n \circ R = R^{n+1}$$

8

Transitive Closure (Cont.)

- $R^* = \cup_1^\infty R^n$, is called the connectivity relation of R,which consists of the (a,b) such that there is path from a to b.
- Theorem 2: the transitive closure of a relation R (denoted by t(R)) equals the connectivity R*
- $R^* = \cup_1^\infty R^i = t(R)$.

Proof

- To prove R* is transitive closure we must prove:
- (1) $R^* \supseteq R$. It is obvious by definition
- (2)R* is transitive. If $(a,b) \in R^*, (b,c) \in R^*$, it implies there is a path from a to b and a path from b to c, hence there is a path from a to c through b.
- (3)R* is minimum. If S is also a transitive relation containing R, then $S \supseteq R^*$. It is obvious that $S^*=S$. since $S \supseteq R$, then $S^* \supseteq R^*$, hence $S \supseteq R^*$.

Lemma 1

- A is a set containing n elements. R is relation on A. if there is a path from a to b, then there is such path with length not exceeding n. if a≠b, there is such path with length not exceeding n-1.
- From this lemma, $t(R)= \cup_1{}^n R^i$

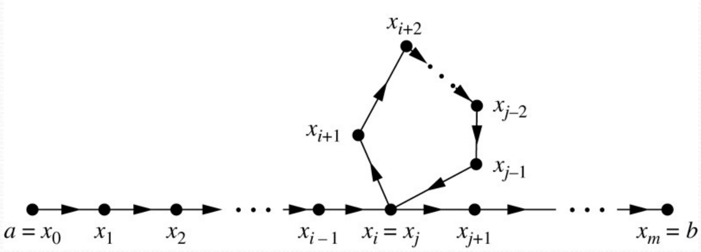


FIGURE 2 Producing a Path with Length Not Exceeding *n*.

Transitive Closure (Cont.)

- Theorem 3 $M_{R^*}= M_R \vee M_R{}^2 \vee M_R{}^3 \vee \ldots \vee M_R{}^n$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad M_R^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = M_R{}^3 = M_{R*}$$

Cont…

Algorithm 1 A procedure for computing the transitive closure

procedure *transitive_closure* ($\mathbf{M}_R$: zero-one $n \times n$ matrix)

$\mathbf{A} := \mathbf{M}_R$

$\mathbf{B} := \mathbf{A}$

for $i := 2$ **to** n

begin

$\mathbf{A} := A \odot \mathbf{M}_R$

$\mathbf{B} := \mathbf{B} \vee \mathbf{A}$

end { **B** is the zero-one matrix for R^* }

Transitive Closure (Cont.)

- Warshall's algorithm an efficient method for computing the transitive closure of a relation.
- Interior vertices of a path: a,$x_1, x_2, \ldots, x_{m-1}$, b. $x_1, x_2, \ldots, x_{m-1}$ are interior vertices
- Matrices: $M_R = W_0, W_1, W_2, \ldots, W_n = M_{R^*}$
- Named after Stephen Warshall in 1960
  - $2n^3$ bit operation
  - Also called Roy-Warshall algorithm, Bernard Roy in 1959
  - Previous algorithm 1 using $2n^3$ (n-1) bit operation

$$n^2(2n-1)(n-1) + (n-1)n^2 = 2n^3(n-1) = O(n^4)$$

FIGURE 3 (9.4,p604)

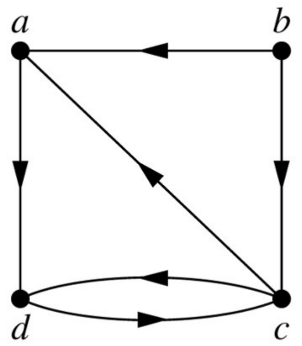


FIGURE 3 The Directed Graph of the Relations *R*.

歐亞書局

P. 551

Warshall's algorithm

- Observation: we can compute W_k directly from W_{k-1}
  - Two cases (Fig. 4)

    (a) There is a path from v_i to v_j with its interior vertices among the first k-1 vertices

    - $w_{ij}^{(k-1)}=1$

    (b) There are paths from v_i to v_k and from v_k to v_j that have interior vertices only among the first k-1 vertices

    - $w_{ik}^{(k-1)}=1$ and $w_{kj}^{(k-1)}=1$

FIGURE 4 (9.4)

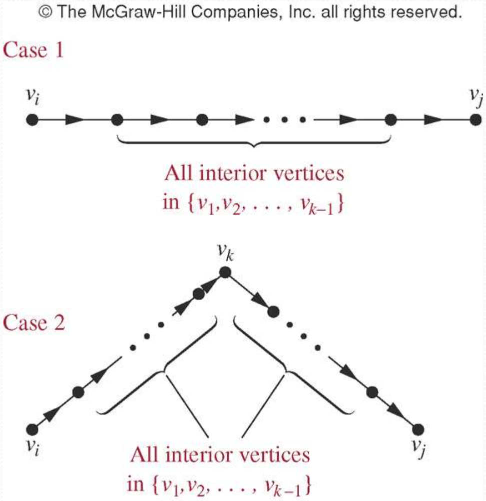


FIGURE 4 p605
Adding v_k to the Set of Allowable Interior Vertices.

歐亞書局 P. 553

Warshall's algorithm

- Lemma 2: Let Wk=$w_{ij}^{(k)}$ be the zero-one matrix that has a 1 in its (i,j)th position iff there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then
  $w_{ij}^{(k)} = w_{ij}^{(k-1)} \vee (w_{ik}^{(k-1)} \wedge w_{kj}^{(k-1)})$,
  whenever I, j, and k are positive integers not exceeding n.

Transitive Closure (Cont.)

- Algorithm 2 warshall algorithm
- Procedure warshall(M_R:n × n zero-one matrix)
- W= M_R
- For k=1 to n
- Begin
-     For I=1 to n
-         Begin
-             For j=1 to n
-             $W_{ij}=w_{ij} \vee (w_{ik} \wedge w_{kj})$
-         End
- End

〖**Example**〗 **Let** $A=\{1,2,3,4,5\}, R=\{(1,1),(1,2),(2,4),(3,5),(4,2)\}, t(R)=?$

Solution:

$$M=\begin{bmatrix}1&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=1}\begin{bmatrix}1&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=2}\begin{bmatrix}1&1&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}$$

$$\xrightarrow{k=3}\begin{bmatrix}1&1&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=4}\begin{bmatrix}1&1&0&1&0\\0&1&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=5}\begin{bmatrix}1&1&0&1&0\\0&1&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}$$

$W_{ij}=w_{ij} \vee (w_{ik} \wedge w_{kj})$
If (w_{ik} == 1)
$W_{ij}=w_{ij} \vee w_{kj}$

20

Homework

Sec.9.4 2, 6, 9(6), 11(6), 20, 28(a), 29

Equivalence Relations

Section 9.5

Section Summary

- Equivalence Relations（等价关系）
- Equivalence Classes（等价类）
- Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

  $$a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$$

  Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s \mid (a,s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative （代表元） of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a+2m, a+2m, \ldots\}$. For example,

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $\quad$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $\quad$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) aRb

(*ii*) $[a] = [b]$

(*iii*) $[a] \cap [b] \neq \emptyset$

Proof: We show that (*i*) implies (*ii*). Assume that aRb. Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

(1) aRb

(2) $[a] = [b]$

(3) $[a] \cap [b] \neq \phi$

- **Show that (2) implies (3)**

$$\left.\begin{array}{l}[a] = [b] \\ \textbf{\textit{R} is reflexive} \Rightarrow \textbf{[\textit{a}] is nonempty}\end{array}\right\} \Rightarrow [a] \cap [b] \neq \phi$$

- **Show that (3) implies (1)**

$$[a] \cap [b] \neq \phi \quad \Rightarrow \exists x \in [a] \cap [b]$$

$$\Rightarrow (a, x) \in R, (b, x) \in R$$

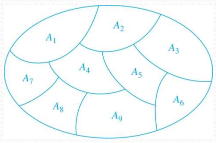
$$\Rightarrow (a, x) \in R, (x, b) \in R$$

$$\Rightarrow (a, b) \in R$$

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S.$



Notation: $pr(A) = \{A_i \mid i \in I\}$

A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set (*continued*)

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

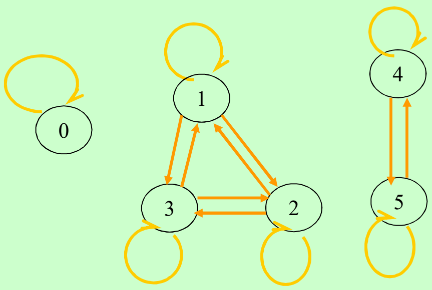
- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

〖Example〗 Find the partition of the set *A* from *R*.

$A = \{0,1,2,3,4,5\}$,

$R = \{(0,0),(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,4),(4,5),(5,4),(5,5)\}$

Solution:



[0]={0}

[1]={1,2,3}

[4]={4,5}

***pr*(*A*)={ [0], [1], [4]}**

34

Question:

Congruence Modulo *m*

$$R=\{(a,b)\mid a\equiv b(\bmod m), a,b\in Z\},\ pr(Z)=?$$

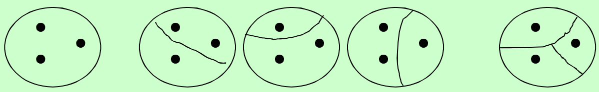
$$pr(Z)=\{[0]_m,[1]_m,\cdots,[m-1]_m\}$$

Question:

$|A|$=3. How many different equivalence relations on the set A are there?

Solution:

an equivalence relation on a set A $\leftrightarrow$ a partition of A



35

The combining of the Equivalence Relations

- Let R and S be equivalence relations on A, how about R∩S, R∪S?
- The answer for R∩S is yes.
- (1) Is it reflexive?:
- Since (a,a) $\in R$ (a,a) $\in S$,then (a,a) $\in$ R∩S for every a $\in A$
- (2) Is it symmetric?
- Since R^{-1}=R S^{-1}=S, then $(R\cap S)^{-1}=R^{-1}\cap S^{-1}=R\cap S$

The combining of the Equivalence Relations

- Is it transitive?
- Since R and S are transitive, then $R^2 \subseteq R$, $S^2 \subseteq S$
- Then $(R \cap S)^2 = R^2 \cap RoS \cap SoR \cap S^2 \subseteq R^2 \cap S^2 = R \cap S$
- So $R \cap S$ is equivalence relation.
- But the answer for $R \cup S$ is No!

【 Theorem 】 If R_1, R_2 are equivalence relations on A, then $R_1 \cup R_2$ is a reflexive and symmetric relation on A.

Proof:

(1) reflexive

$$\forall a \in A \quad \because (a,a) \in R_1, (a,a) \in R_2 \quad \therefore (a,a) \in R_1 \cup R_2$$

(2) symmetric

$$(a,b) \in R_1 \cup R_2 \quad \Rightarrow (a,b) \in R_1 \text{ or } (a,b) \in R_2$$
$$\Rightarrow (b,a) \in R_1 \text{ or } (b,a) \in R_2 \quad \Rightarrow (b,a) \in R_1 \cup R_2$$

Question: transitive?

38

〖**Example**〗 $A=\{a,b,c\}$,

$R_1=\{(a,a),(b,b),(c,c),(a,b),(b,a)\}$

$R_2=\{(a,a),(b,b),(c,c),(b,c),(c,b)\}$

Is $R_1 \cup R_2$ a transitive relation ?

Solution:

$$R_1 \cup R_2=\{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b)\}$$

【 Theorem 】 If R_1, R_2 are equivalence relations on A, then $(R_1 \cup R_2)^*$ is an equivalence relation on A.

Proof:

(1) reflexive

(2) symmetric

(3) transitive

Homework

- Sec. 9.5 3, 10, 16, 36(b), 39, 41

Partial Orderings

Section 9.6

Section Summary

- Partial Orderings（偏序） and Partially-ordered Sets
- Lexicographic Orderings
- Hasse Diagrams
- Lattices（格）
- Topological Sorting

Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering,* or *partial order,* if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation ($|$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

- $(\boldsymbol{Z}^+, |)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset $(S,\preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* （全序） or a *linear order* （线序）. A totally ordered set is also called a *chain*（链）.

Definition 4: $(S,\preccurlyeq)$ is a well-ordered （良序） set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Lexicographic Order

Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

- This definition can be easily extended to a lexicographic ordering on strings (*see text*).

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

〖Example〗 Let $A_1 = A_2 = Z^+$ *and* $R_1 = R_2 = |$. **Then**

(1) (2, 4) < (2, 8) since $x_1 = x_2$ *and* $y_1 R_2 y_2$

(2) (2, 4) is not related under *R* to (2, 6) since $x_1 = x_2$ **but 4 does not divide 6.**

(3) (2, 4) < (4, 5) since $x_1 R_1 x_2$

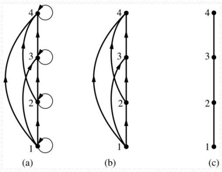
(2, 3, 4, 5) < (2, 3, 8, 2) since $x_1 = x_2$ *and* $y_1 = y_2$ **and 4 divides 8.**

(2, 3, 4, 5) is not related to (3, 6, 8, 10) since 2 does not divide 3.

50

Hasse Diagrams

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

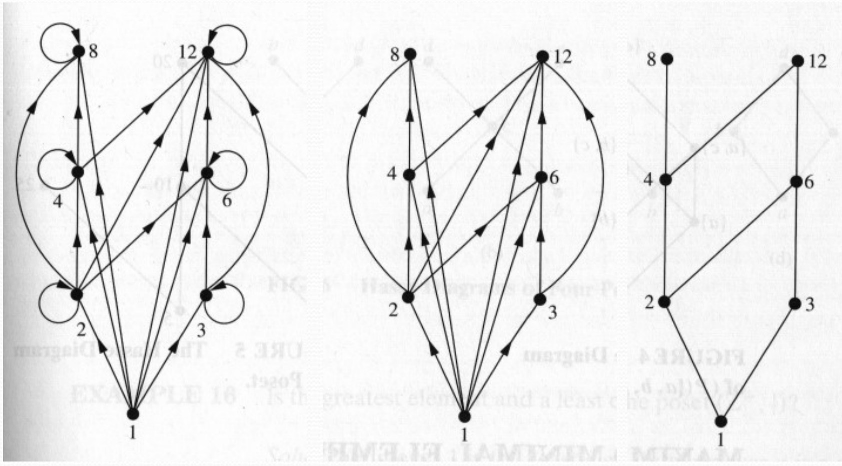


A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).

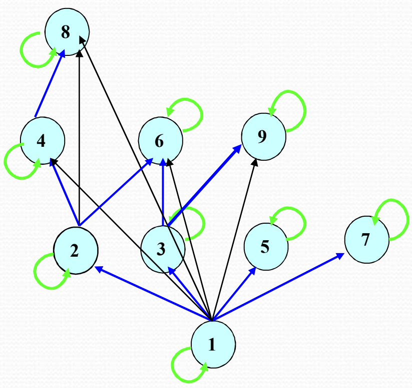
Procedure for Constructing a Hasse Diagram

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

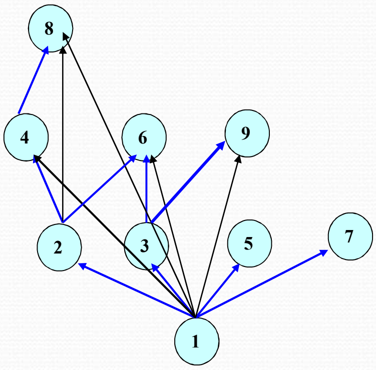
For the poset ({1,2,3,4,6,8,12}, |)



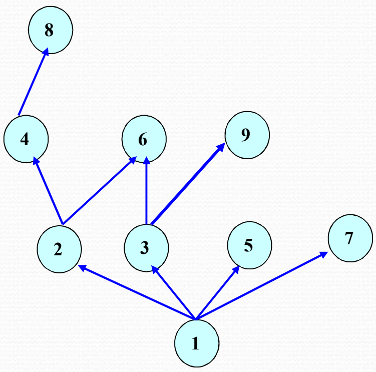
〖**Example**〗 $A = \{1,2,3,4,5,6,7,8,9\}$ $R = \{(a,b) \mid a \mid b, a,b \in A\}$



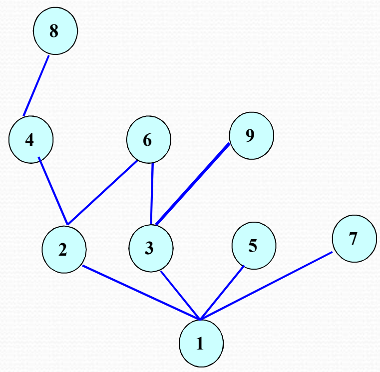
〖**Example**〗 $A = \{1,2,3,4,5,6,7,8,9\}$ $R = \{(a,b) \mid a \mid b, a,b \in A\}$



〖**Example** 〗 $A = \{1,2,3,4,5,6,7,8,9\}$ $R = \{(a,b) \mid a \mid b, a,b \in A\}$

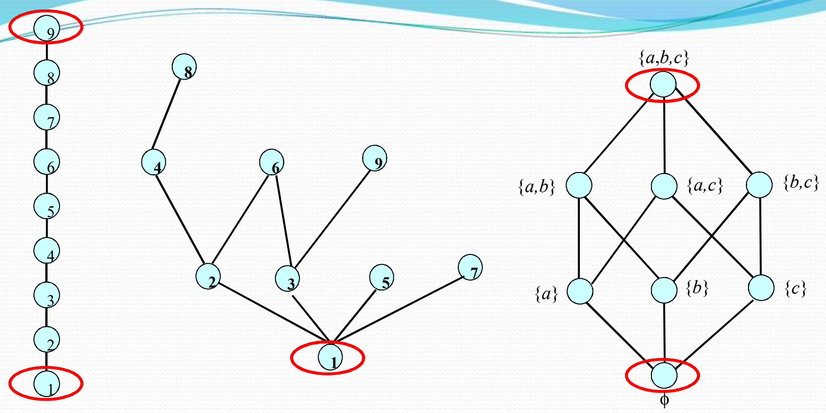


〖**Example**〗 $A = \{1,2,3,4,5,6,7,8,9\}$ $R = \{(a,b) \mid a \mid b, a, b \in A\}$



Hasse Diagram Terminology

- Let $(S, \preccurlyeq)$ be a poset.
- a is *maximal*（极大元） in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \preccurlyeq b$. (**top** of the Hasse diagram)
- a is *minimal* （极小元） in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \preccurlyeq a$. (**bottom** of the Hasse diagram)
- a is the *greatest element* （最大元） of $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in$ S.
- a is the *least element* （最小元） of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.



(1) (2) (3)

	The greatest element	The least element
(1)	9	1
(2)	/	1
(3)	$\{a, b, c\}$	ϕ

59

【 Theorem 】 The greatest and least element of the poset $(A, \leq)$ are unique when they exist.

Proof:

Suppose that a_1 is a greatest element in A. It follows that

$x \leq a_1$ **for every x in A.**

Suppose that a_2 is a greatest element in A. It follows that

$x \leq a_2$ **for every x in A.**

It implies that $a_2 \leq a_1$ *and* $a_1 \leq a_2$

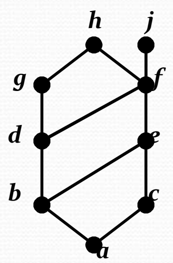
That is $a_1 = a_2$

60

Hasse Diagram Terminology (Cont ..)

- Let A be a subset of $(S, \preccurlyeq)$.
- If $u \in S$ such that $a \preccurlyeq u$ for all $a \in A$, then u is called an *upper bound* of A.
- If $l \in S$ such that $l \preccurlyeq a$ for all $a \in A$, then l is called an *lower bound* of A.
- If x is an upper bound of A and $x \preccurlyeq z$ whenever z is an upper bound of A, then x is called the *least upper bound* of A.
- If y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A, then y is called the *greatest lower bound* of A.

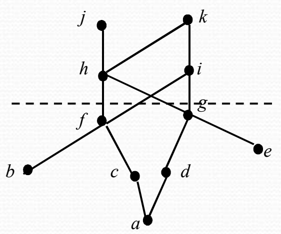
Example



Maximal elements: *h, j*
Minimal elements: *a*
Greatest element: none
Least element: *a*
Upper bound of {*a,b,c*}: {*e, f, j, h*}
Least upper bound of {*a,b,c*}: *e*
Lower bound of {*a,b,c*}: {*a*}
Greatest lower bound of {*a,b,c*}: *a*

【**Example**】 $S = \{a,b,c,d,e,f,g,h,i,j,k\}$

$A = \{a,b,c,d,e,f,g\}, A' = \{h,i,j,k\}$



the upper bounds of set A: ***h, i, k, j***

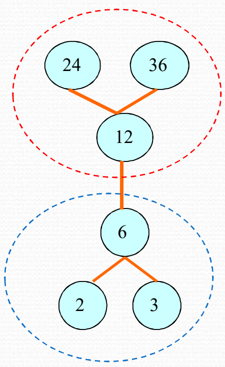
the lower bounds of set A: /

the upper bounds of set A': /

the lower bounds of set A': ***f, g, a, b, c, d, e***

【**Example**】 $A = \{2,3,6,12,24,36\}, B_1 = \{2,3,6\}, B_2 = \{12,24,36\}, R :|$

Determine the maximal elements, minimal elements, greatest element, least element of set *A*, the upper bounds, lower bounds , least upper bound , greatest lower bound of set B_1, B_2 。



***A*:**

maximal elements: 24,36 **minimal elements:2,3**

the greatest element: / **the least element: /**

B_1:

upper bounds: 6,12,24,36 **lower bounds: /**

The least upper bound: 6 **The greatest lower bound:/**

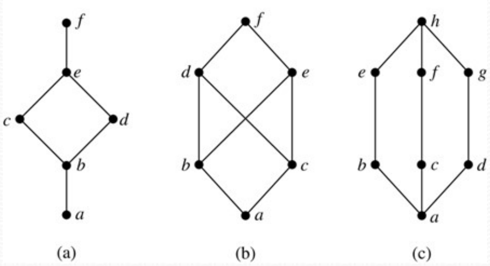
B_2:

upper bounds: / **lower bounds: 12,6,2,3**

The least upper bound: / **The greatest lower bound:12**

Lattices

- A partially ordered set in which every pair of elements has both a least upper bound and greatest lower bound is called a *lattice*.



only b 不是

FIGURE 8 Hasse Diagrams of Three Posets.

(2) $(Z,\leq)$ **?**

lub： the larger of the two elements，

glb： the smaller of the two elements.

Hence, the poset $(Z,\leq)$ **is a lattice.**

Note: Every totally ordered set is a lattice.

(3) $(Z^+,|)$ **?**

$\forall a,b \in Z^+$

lub： the least common multiple，

glb： the greatest common divisor

Hence, the poset $(Z^+,|)$ **is a lattice.**

(4) $(P(s),\subseteq)$ **?**

$\forall s_1,s_2 \in P(s)$ **, lub：** $s_1 \cup s_2$ **glb：** $s_1 \cap s_2$

Hence, the poset $(P(s),\subseteq)$ **is a lattice.**

66

Topological Sorting

- A total ordering $\preccurlyeq$ is said to be compatible with the partial ordering R if a $\preccurlyeq$ b whenever a R b.
  - Topological sorting: constructing a compatible total ordering from a partial ordering
- Lemma 1: Every finite nonempty poset (S, $\preccurlyeq$) has at least one minimal element.

Topological Sorting

- Algorithm 1: topological sorting
  - Procedure topological sort(S, $\preccurlyeq$: finite poset)
    k:=1
    while S$\neq\varnothing$
    begin
      a_k:=a minimal element of S
      S:=S-{a_k}
      k:=k+1
    end
  - (See Fig. 9,Fig. 10, 11)

FIGURE 9 (9.6,p629)

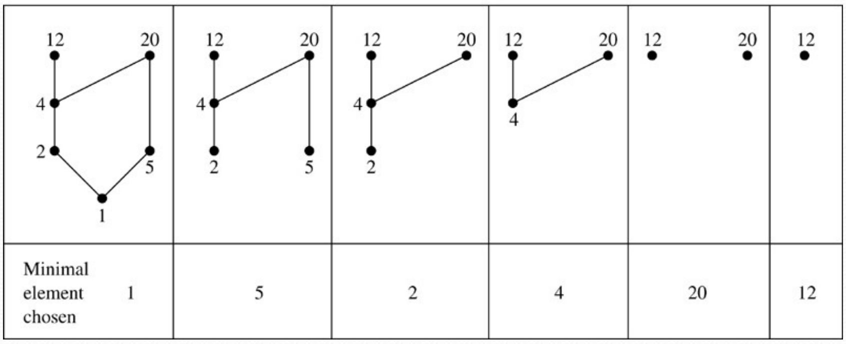


FIGURE 9 A Topological Sort of ({1,2,4,5,12,20}, |).

P. 577

FIGURE 10 (9.6)

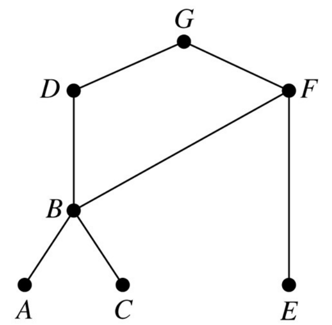


FIGURE 10 The Hasse Diagram for Seven Tasks.

FIGURE 11 (9.6)

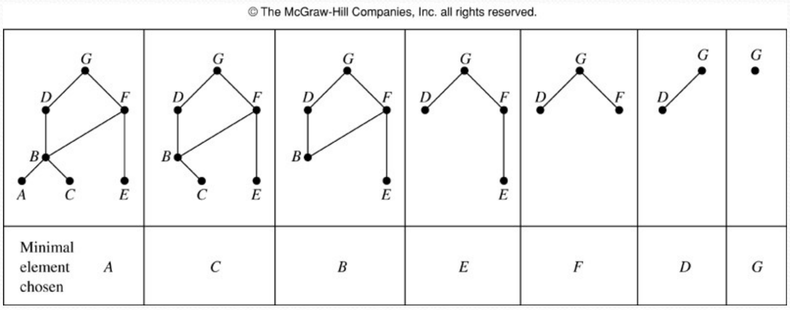


FIGURE 11 A Topological Sort of the Tasks.

Homework

Sec. 9.6 5, 10, 23(a),(c), 32, 44, 66