General Physics I

Lecture 6: Conservation of Momentum

Outline

- **Importance of conservation laws in physics**
  - **Conservation of linear momentum**
- **Applications of the conservation laws**
  - **Collisions**
- **Many-particle systems**
  - **Center of mass**
  - **Collisions in the center-of mass frame**

A Question From You

- We have learn the work-kinetic energy theorem. What happens to it under Galilean transformation?

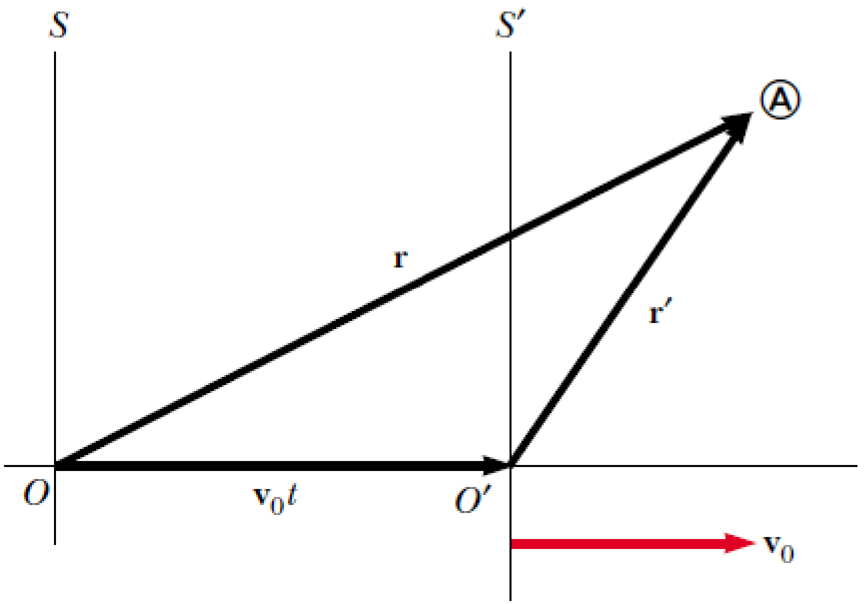
1) Nothing happens, we get the exactly same work-kinetic energy theorem.
2) Since velocity and displacement are reference frame dependent, the work-kinetic energy theorem only works at an inertial frame at rest (w.r.t. fixed stars).
3) Some additional law will emerge to ensure the work-kinetic energy theorem valid in a different inertial frame.

Galilean Transformation

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t$$

$$\frac{d\mathbf{r}'}{dt} = \frac{d\mathbf{r}}{dt} - \mathbf{v}_0$$

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0$$



$$\frac{d\mathbf{v}'}{dt} = \frac{d\mathbf{v}}{dt} - \cancel{\frac{d\mathbf{v}_0}{dt}} \Rightarrow \mathbf{a}' = \mathbf{a}$$

The two inertial observers agree on measurements of acceleration.

Let's Work it out

Assume an object in 1D subject to a total force F for simplicity.

$$W' = Fd' = F(d - v_0 t) = W - v_0 F t$$

$$K' = \frac{1}{2}mv'^2 = \frac{1}{2}m(v - v_0)^2 = K - v_0 m v + \frac{1}{2}mv_0^2$$

According to the work-kinetic energy theorem, $W = K_f - K_i$

$$W' - \left(K'_f - K'_i\right) = W - \left(K_f - K_i\right) - v_0 F t + v_0 m\left(v_f - v_i\right)$$

$$W' = \left(K'_f - K'_i\right) \Leftrightarrow Ft = mv_f - mv_i$$

Impulse-Momentum Theorem

- **The impulse of the force F acting on a particle equals the change in the linear momentum of the particle caused by that force I = Δp.**
  - **Linear momentum**

$$\mathbf{p} \equiv m\mathbf{v}$$

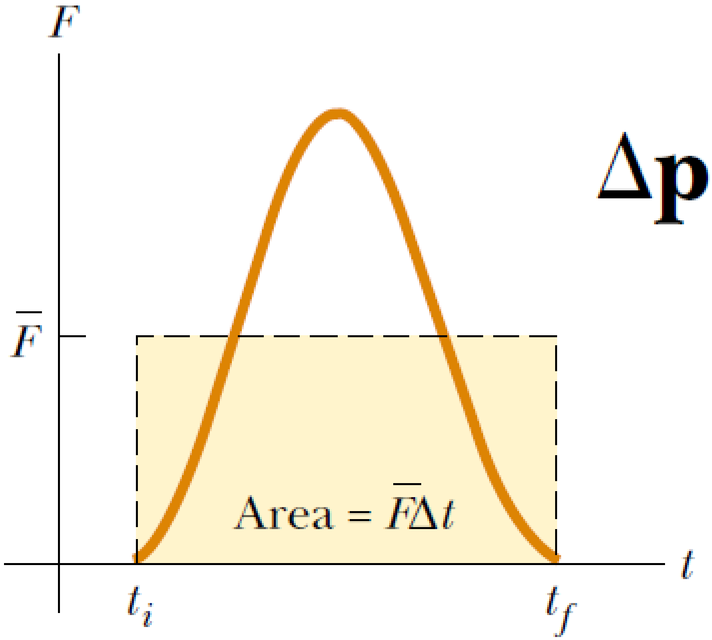
  - **Impulse**

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt$$

Connection to Newton's Laws

- **Assume that a single force F acts on a particle and that this force may vary with time**

$$\mathbf{F} = d\mathbf{p}/dt \quad \Longleftrightarrow \quad d\mathbf{p} = \mathbf{F}\, dt$$

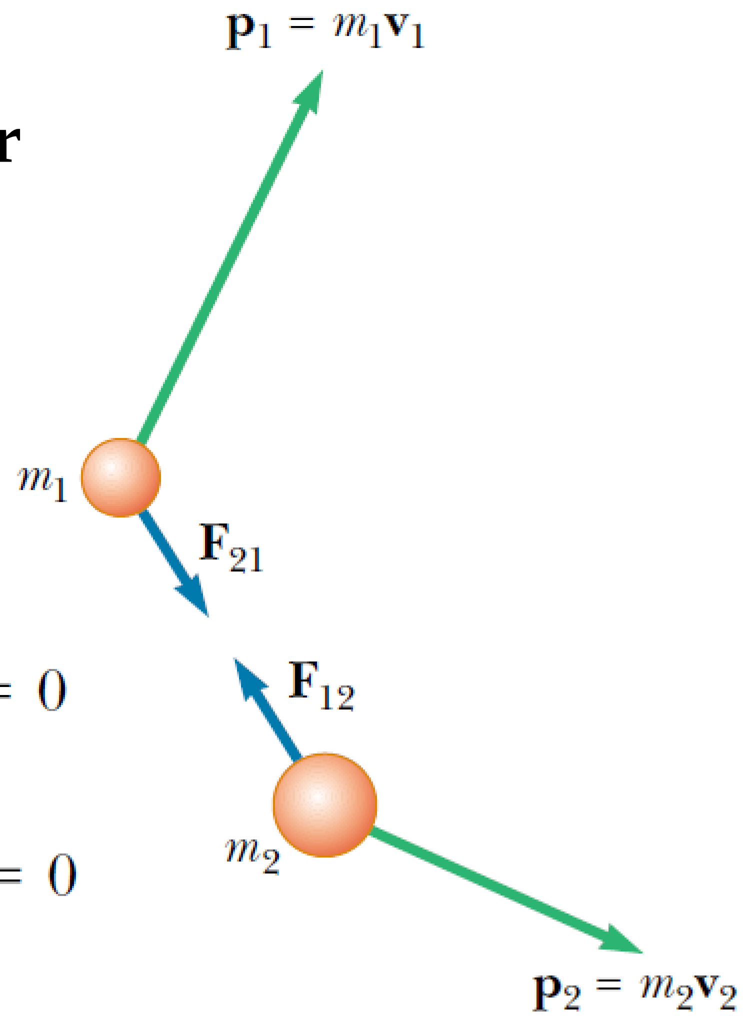


$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}\, dt$$

$$\bar{\mathbf{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f} \mathbf{F}\, dt$$

Two-Particle System

- Consider two particles 1 and 2 that can interact with each other but are isolated from their surroundings.



- Newton's second law

$$\mathbf{F}_{21} = \frac{d\mathbf{p}_1}{dt} \qquad \mathbf{F}_{12} = \frac{d\mathbf{p}_2}{dt}$$

- Newton's third law $\mathbf{F}_{21} + \mathbf{F}_{12} = 0$

$$\Rightarrow \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

Conservation of Linear Momentum

- Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

$$\mathbf{p}_{\text{tot}} = \sum_{\text{system}} \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

- The total momentum of an isolated system at all times equals its initial momentum.

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f}$$

- **The only requirement is that the forces must be internal to the system.**

A Question for You

- **In an isolated system the validity of Galilean invariance and the conservation of kinetic energy (and mass) can lead to the conservation of linear momentum. Can you derive it? Discuss whether your derivation is still valid when some kinetic energy converts into internal energy.**

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 + \Delta\varepsilon_{internal}$$

$$\Rightarrow \quad m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

Inelastic scattering involves internal energy change.

Conservation Laws

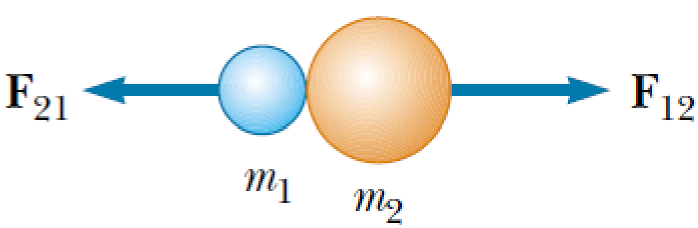
- A conservation law is usually the consequence of some underlying symmetry in the universe.
- Why are conservation laws powerful tools?
  - Can assure something impossible.
  - Applicable even when the force is unknown.
  - An intimate connection with invariance.
  - Convenient in solving for the particle motion.
- First use the relevant conservation laws one by one, then differential equations, computers, etc.

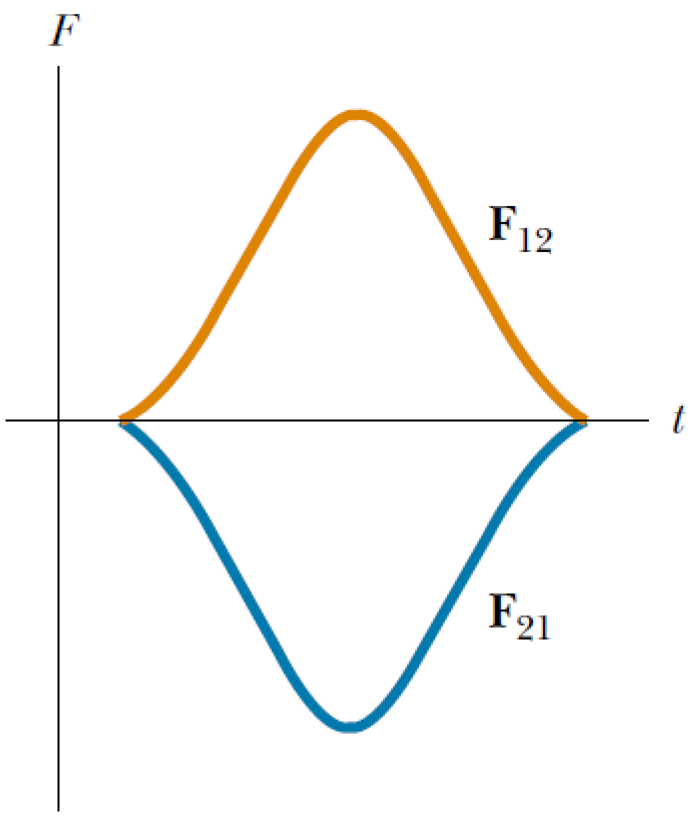
Collision

- The event of two particles' coming together for a short time and thereby producing impulsive forces on each other.

  - These forces are assumed to be much greater than any external forces present.

$$\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$





Elastic and Inelastic Collision

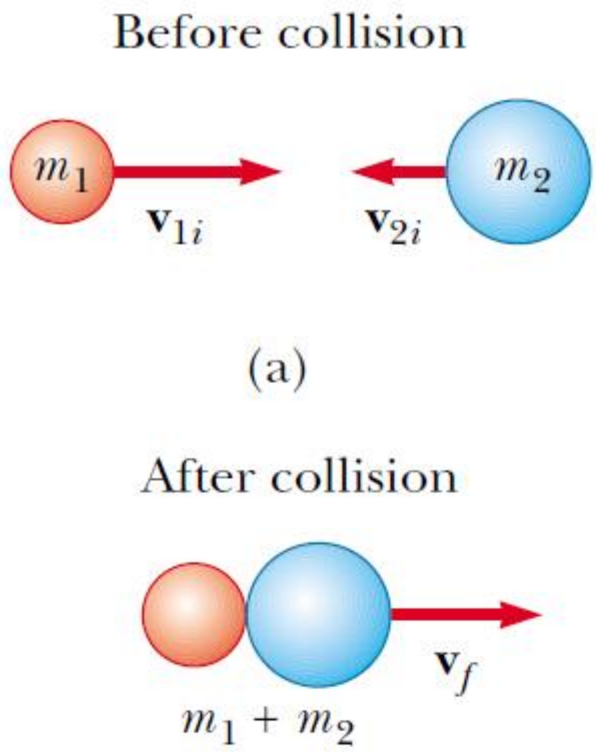
- Momentum is conserved in any collision in which external forces are negligible.
- Kinetic energy may or may not be constant.
  - Elastic collision between two objects is one in which total kinetic energy (as well as total momentum) is the same before and after the collision.
  - Inelastic collision is one in which total kinetic energy is not the same before and after the collision (even though momentum is constant).
  - Kinetic energy is constant only in elastic collisions.

Perfectly Inelastic Collisions

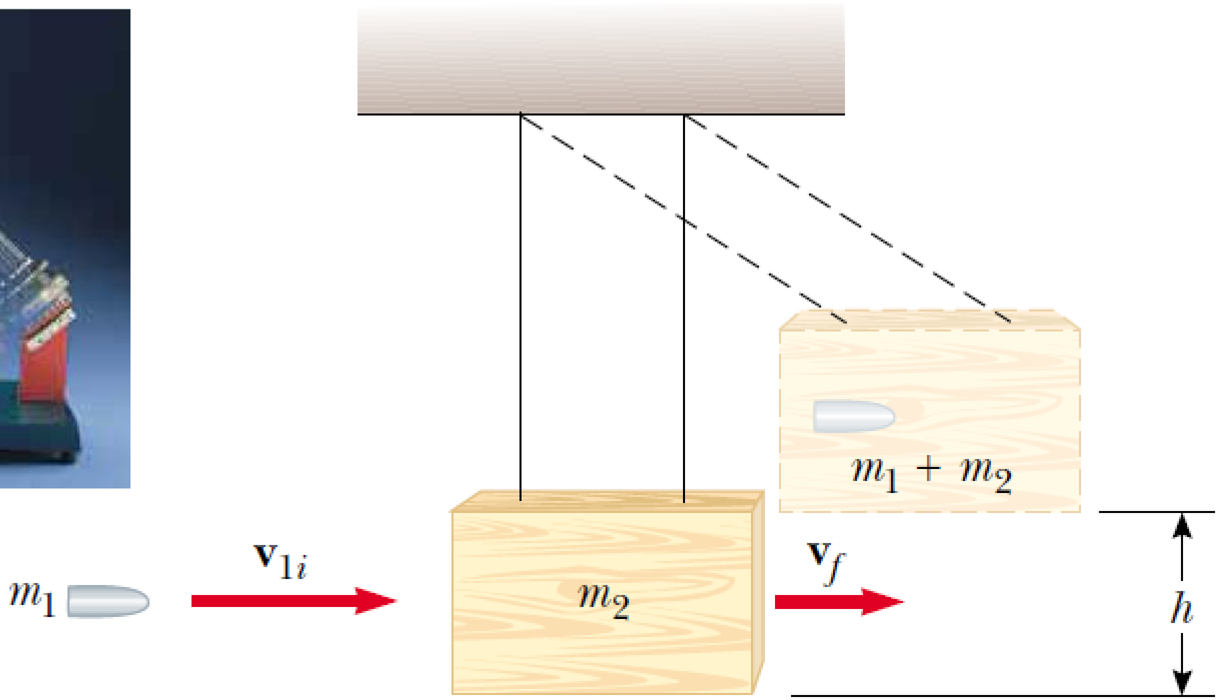
- **When the colliding objects stick together after the collision, the collision is called perfectly inelastic.**

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f$$

$$\boxed{\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2}}$$



Ballistic Pendulum



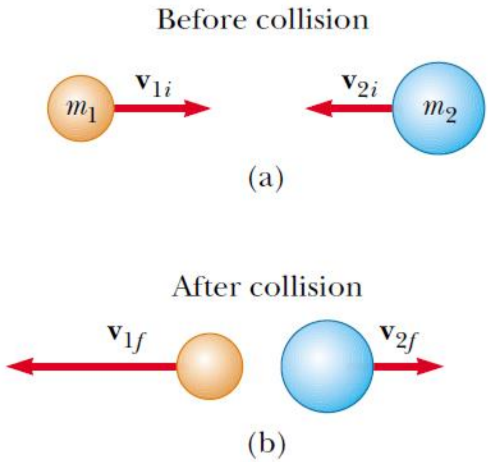
$$v_f = \frac{m_1 v_{1i}}{m_1 + m_2}$$

$$\frac{m_1{}^2 v_{1i}{}^2}{2(m_1 + m_2)} = (m_1 + m_2) g h$$

Elastic Collision in 1D

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$



Show that

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i}$$

Special Cases

- **Particle 2 initially at rest:** $v_{2i} = 0$

(1) $m_1 \ll m_2$

$$v_{1f} \approx -v_{1i}, v_{2f} \approx v_{2i} = 0$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i}$$

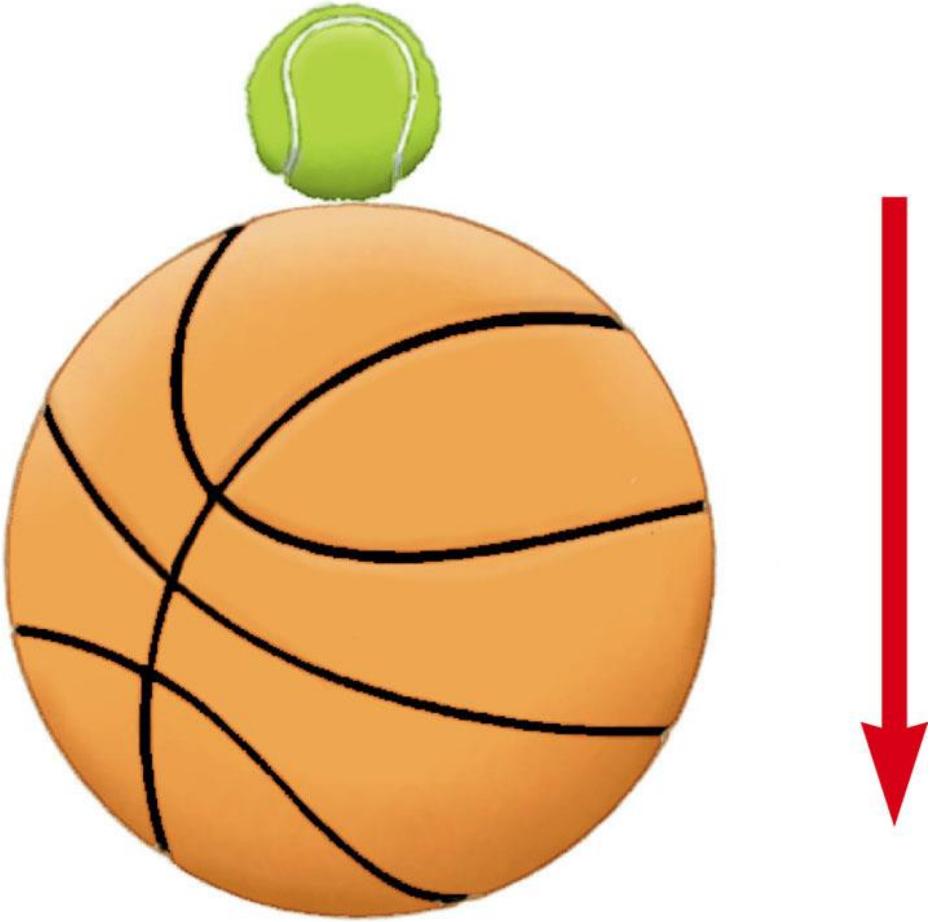
$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i}$$

(2) $m_1 \gg m_2$

$$v_{1f} \approx v_{1i}, v_{2f} \approx 2v_{1i}$$

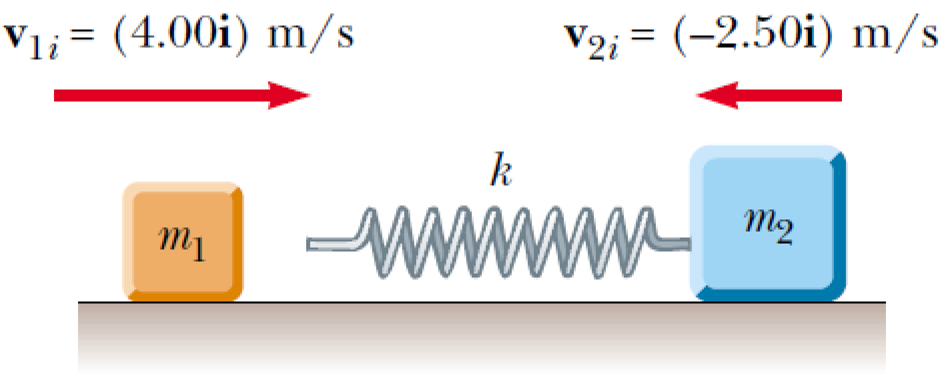
- **Equal masses:** $m_1 = m_2$

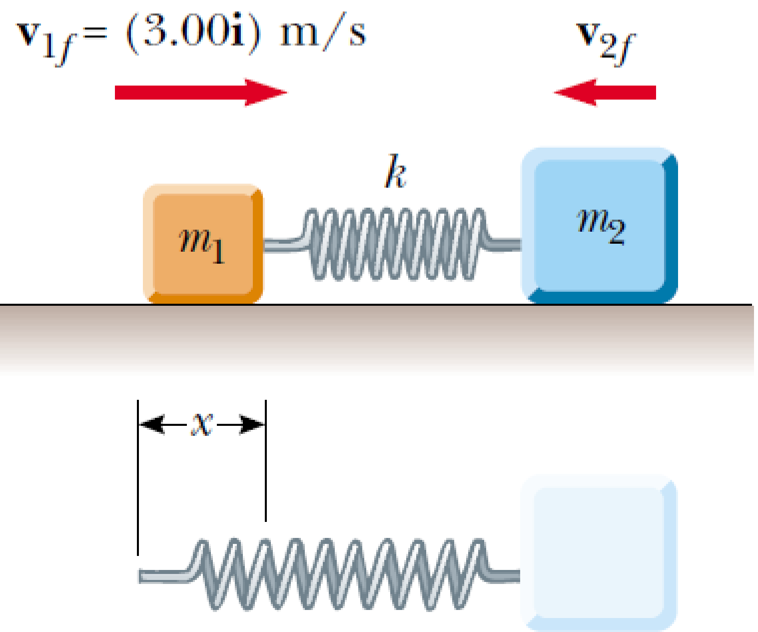
$$v_{1f} = v_{2i}, v_{2f} = v_{1i}$$



Video—MIT impulse

Two-Body Collision with a Spring





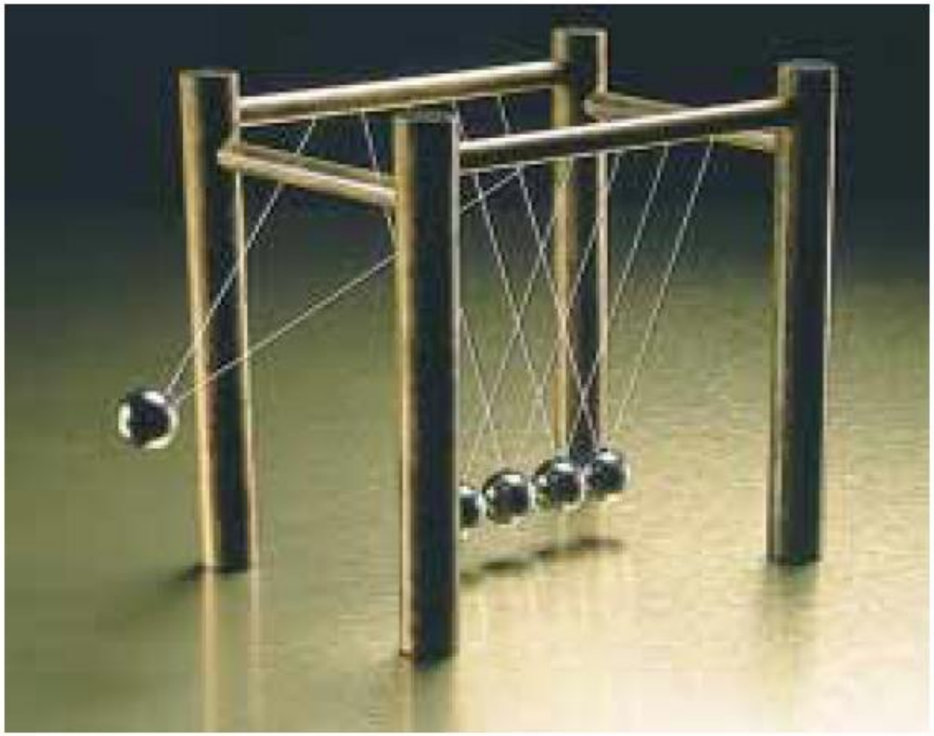
$m_1 = 1.60$ kg $\qquad m_2 = 2.10$ kg

spring constant is 600 N/m

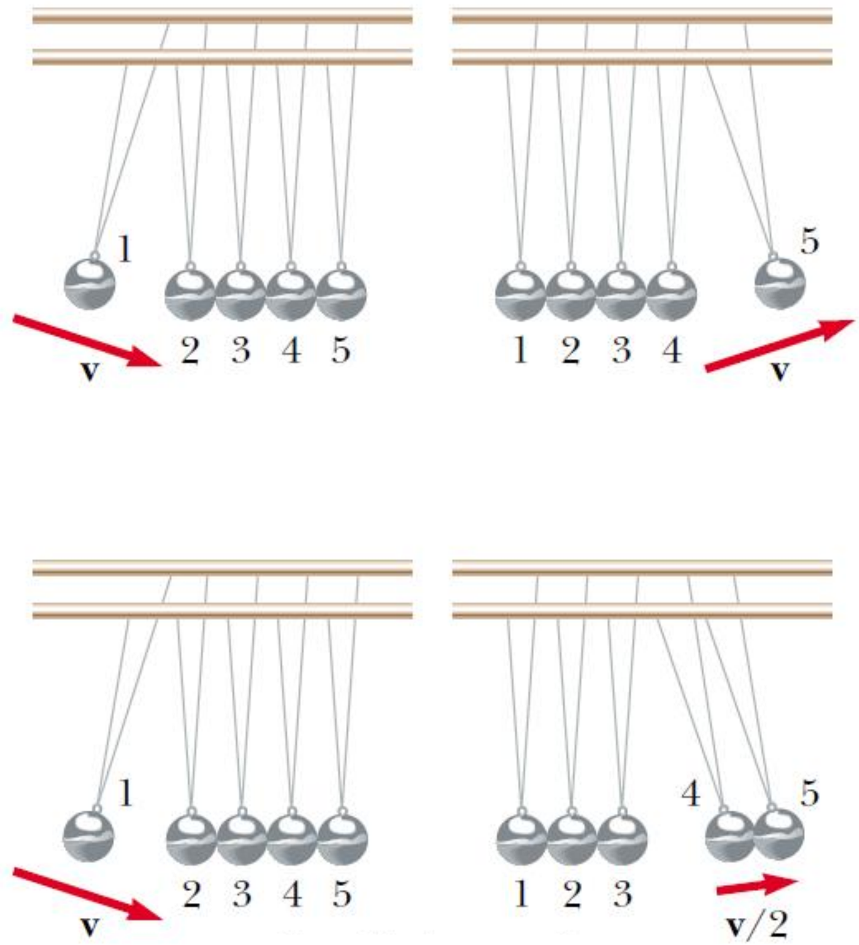
$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 + \frac{1}{2} k x^2$$

Stress Reliever



(a)



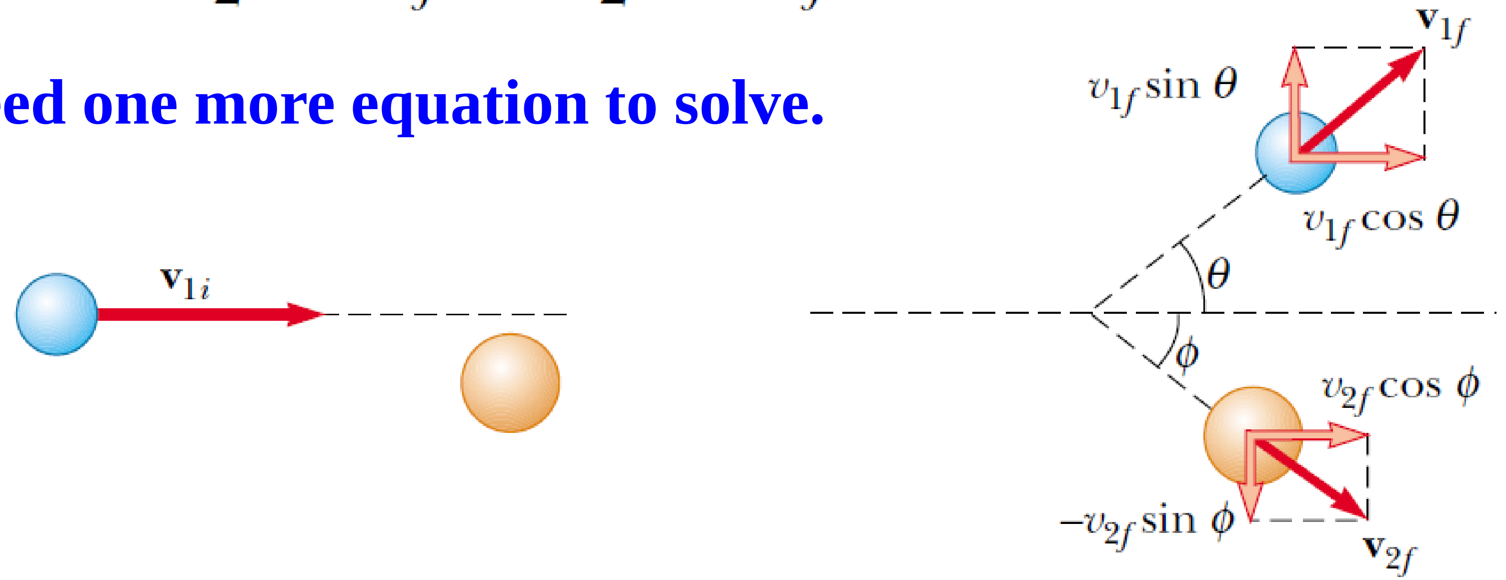
Elastic Collision in 2D

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi$$

$$0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi$$

$$\frac{1}{2} m_1 {v_{1i}}^2 = \frac{1}{2} m_1 {v_{1f}}^2 + \frac{1}{2} m_2 {v_{2f}}^2$$

We need one more equation to solve.



(a) Before the collision (b) After the collision

Example: Proton Collision

Proton 1 collides elastically with proton 2 that is initially at rest. Proton 1 has an initial speed of 3.50×10^5 m/s and makes a glancing collision with proton 2, as was shown in Figure 9.14. After the collision, proton 1 moves at an angle of $37.0°$ to the horizontal axis, and proton 2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.

We are going to show:

Whenever two equal masses collide elastically in a glancing collision and one of them is initially at rest, their final velocities are always at right angles to each other.

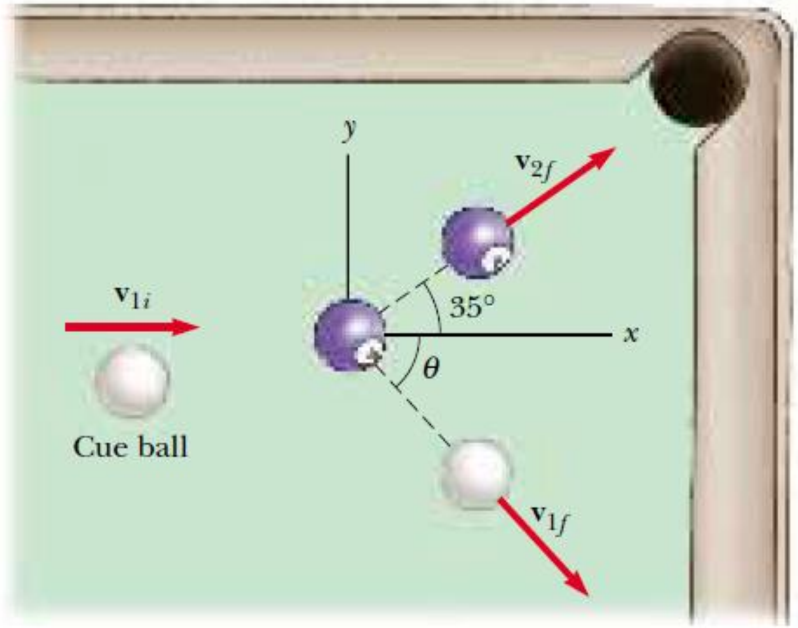
An Equivalent Problem

Conservation of energy

$$v_{1i}^{2} = v_{1f}^{2} + v_{2f}^{2}$$

Conservation of momentum

$$\mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f}$$

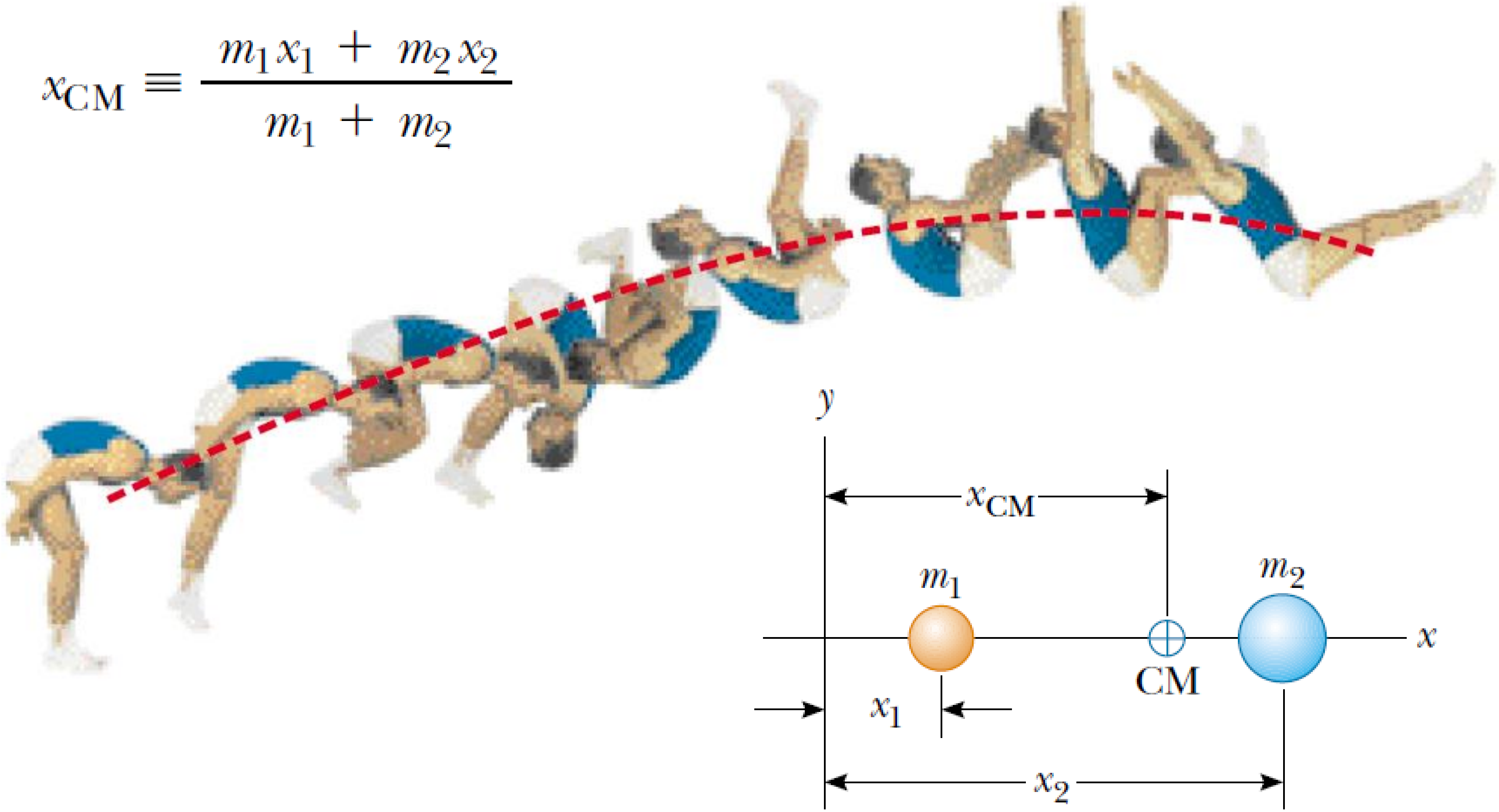


$$v_{1i}^{2} = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f}) = v_{1f}^{2} + v_{2f}^{2} + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

$$\Rightarrow \quad 0 = \mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

The Center of Mass

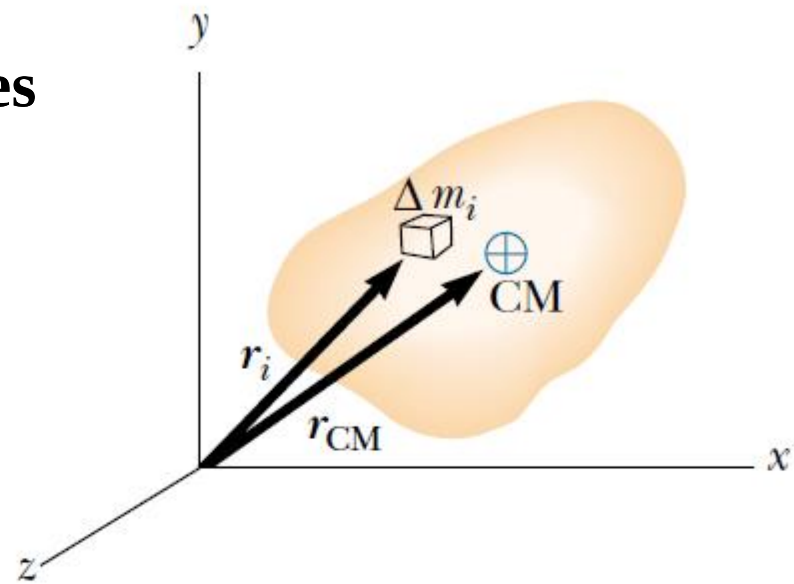
$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



Many-Particle Systems

Consider a system of many particles in three dimensions

$$\mathbf{r}_{\mathrm{CM}} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M}$$



We can think of an extended object as a system containing a large number of particles

$$\mathbf{r}_{\mathrm{CM}} = \frac{1}{M}\int \mathbf{r}\, dm$$

$$x_{\mathrm{CM}} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i\, \Delta m_i}{M} = \frac{1}{M}\int x\, dm$$

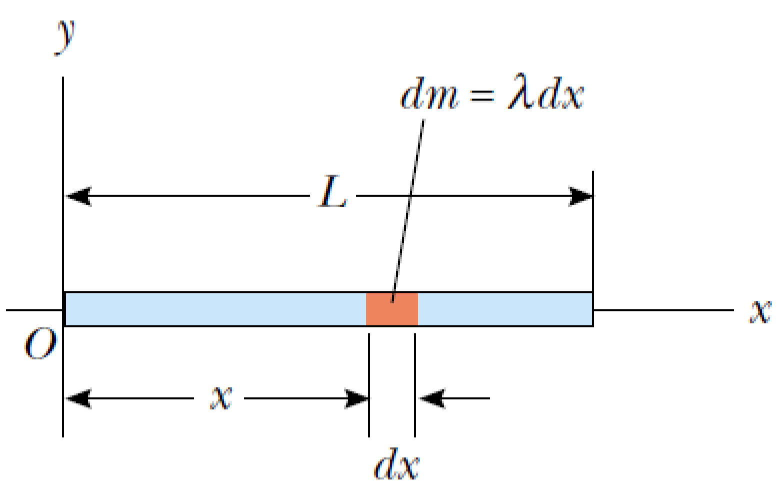
The Center of Mass of a Rod

- **Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.**

$$x_{CM} = \frac{1}{M}\int_0^L x\,\lambda dx$$

$$= \frac{\lambda}{M}\frac{x^2}{2}\bigg|_0^L$$

$$= \frac{\lambda L^2}{2M} = \frac{L}{2}$$



The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.

Nonuniform Rod

- **Suppose a rod is nonuniform such that its mass per unit length varies linearly with x according to**

$$\lambda = \alpha x$$

$$x_{\text{CM}} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx$$

$$= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

$$x_{\text{CM}} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\frac{2}{3}L}$$

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

The CM of a Right Triangle

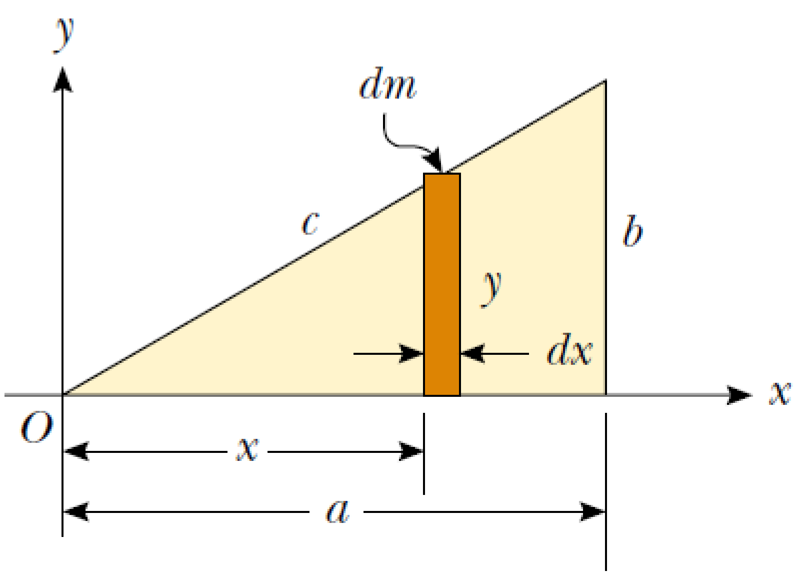
$$x_{\mathrm{CM}} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^a x\left(\frac{2M}{ab}\right)y\, dx = \frac{2}{ab}\int_0^a xy\, dx$$

$$\frac{y}{x} = \frac{b}{a} \qquad \text{or} \qquad y = \frac{b}{a}x$$

$$x_{\mathrm{CM}} = \boxed{\frac{2}{3}a} \qquad y_{\mathrm{CM}} = \boxed{\frac{1}{3}b}$$

$$dm = \frac{\text{total mass of object}}{\text{total area of object}} \times \text{area of strip}$$

$$= \frac{M}{1/2\, ab}(y\, dx) = \left(\frac{2M}{ab}\right)y\, dx$$



Motion of a Many-Particles System

$$\mathbf{r}_{\mathrm{CM}} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M}$$

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{r}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M} \qquad M\mathbf{v}_{\mathrm{CM}} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{\mathrm{tot}}$$

$$\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum_i m_i \mathbf{a}_i$$

$$M\mathbf{a}_{\mathrm{CM}} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i \longrightarrow \boxed{\sum \mathbf{F}_{\mathrm{ext}} = M\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{p}_{\mathrm{tot}}}{dt}}$$

The center of mass of a system of particles of combined mass M moves like an equivalent particle of mass M would move under the influence of the resultant external force on the system.

König theorem-Kinetic energy in CM Frame

$$E_{Ki} = \frac{1}{2}mv_i^2 = \frac{1}{2}m_i\left[(\boldsymbol{v}_i' + \boldsymbol{V}_{cm}).(\boldsymbol{v}_i' + \boldsymbol{V}_{cm})\right] = \frac{1}{2}m_i\left[(\boldsymbol{v}_i^{2\prime} + 2\boldsymbol{v}'_i.\boldsymbol{V}_{cm} + \boldsymbol{V}_{cm}^2)\right]$$

$$E_K = \sum_{i=1}^{n} E_{Ki} = \sum_{i=1}^{n}\left[\frac{1}{2}m_i\boldsymbol{v}_i'^2 + m_i\boldsymbol{v}'_i.\boldsymbol{V}_{cm} + \frac{1}{2}m_i\boldsymbol{V}_{cm}^2\right]$$

$$E_K = \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{v}_i'^2 + \sum_{i=1}^{n}m_i\boldsymbol{v}'_i.\boldsymbol{V}_{cm} + \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{V}_{cm}^2$$

$$\sum_{i=1}^{n}m_i\boldsymbol{r}'_i = 0$$

$$E_K = \frac{1}{2}\boldsymbol{V}_{cm}^2\sum_{i=1}^{n}m_i + \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm}\sum_{i=1}^{n}m_i\boldsymbol{v}'_i$$

$$E_K = \frac{1}{2}\boldsymbol{V}_{cm}^2 M + \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm}\frac{d}{dt}\sum_{i=1}^{n}m_i\boldsymbol{r}'_i$$

$$\boxed{E_K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}\sum_{i=1}^{n}m_i v_i'^2 = E_{Kcm} + E_K'}$$

Total kinetic energy of the system has two parts:

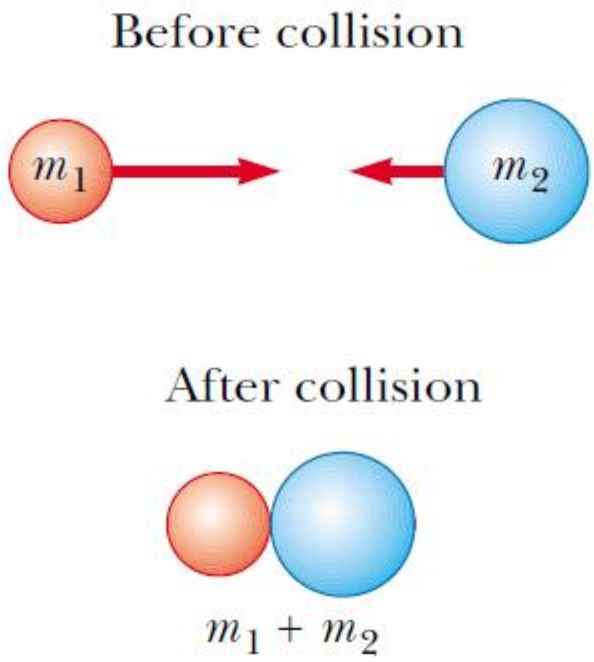
(1) the kinetic energy with all the mass concentrated at the centre of mass

(2) the kinetic energy of motion about the centre of mass.

Collision in the CM Frame

- **Perfectly inelastic scattering**

$$v_{CM} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$



$$u_{1i} = v_{1i} - v_{CM} = \frac{m_2}{m_1 + m_2}(v_{1i} - v_{2i})$$

$$u_{2i} = v_{2i} - v_{CM} = \frac{m_1}{m_1 + m_2}(v_{2i} - v_{1i})$$

$$u_{1f} = u_{2f} = 0$$

$$m_1 u_{1i} + m_2 u_{2i} = m_1 u_{1f} + m_2 u_{2f} = 0$$

In the center-of-mass frame, the total momentum is zero.

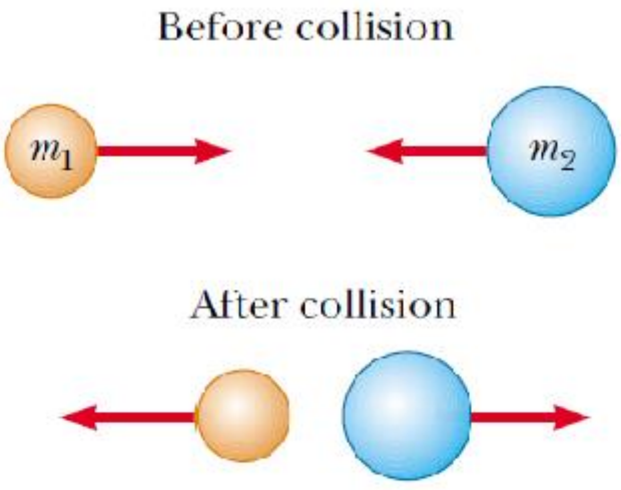
Collision in the CM Frame

- **Elastic scattering**

$$v_{CM} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

$$u_{1i} = v_{1i} - v_{CM} = \frac{m_2}{m_1 + m_2}(v_{1i} - v_{2i})$$

$$u_{2i} = v_{2i} - v_{CM} = \frac{m_1}{m_1 + m_2}(v_{2i} - v_{1i})$$



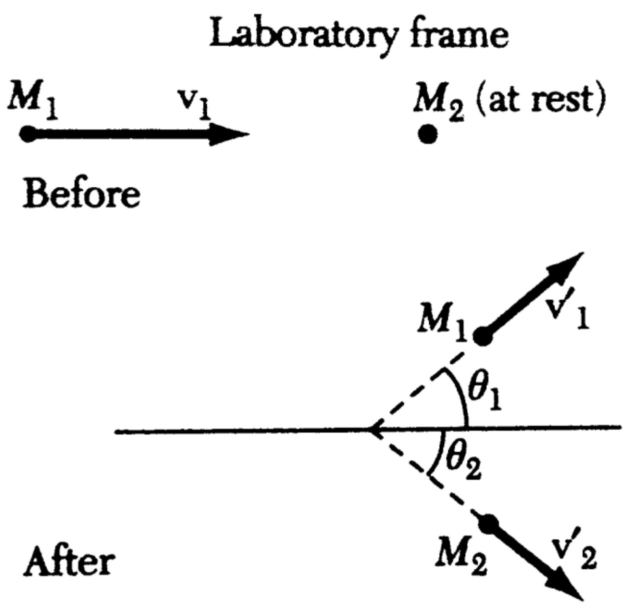
$$u_{1f} = v_{1f} - v_{CM} = \frac{m_2}{m_1 + m_2}(v_{2i} - v_{1i})$$

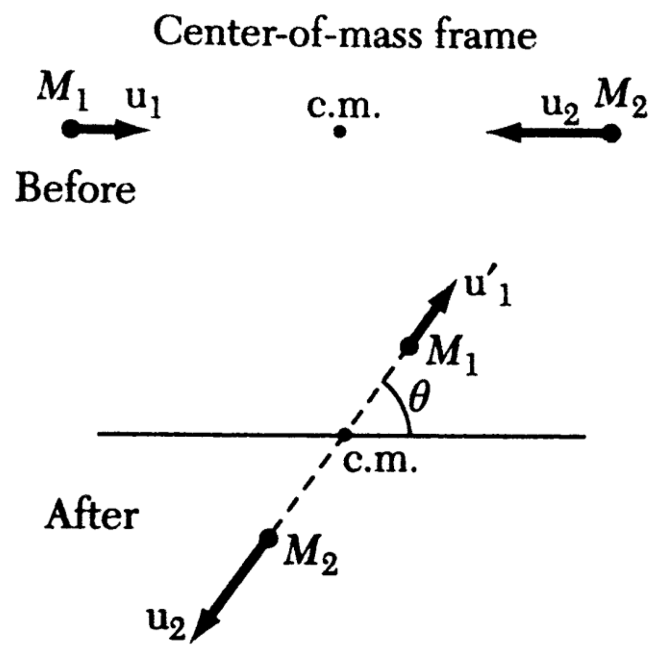
$$u_{2f} = v_{2f} - v_{CM} = \frac{m_1}{m_1 + m_2}(v_{1i} - v_{2i})$$

$$\frac{1}{2}m_1 u_{1i}^2 + \frac{1}{2}m_2 u_{2i}^2 = \frac{1}{2}m_1 u_{1f}^2 + \frac{1}{2}m_2 u_{2f}^2$$

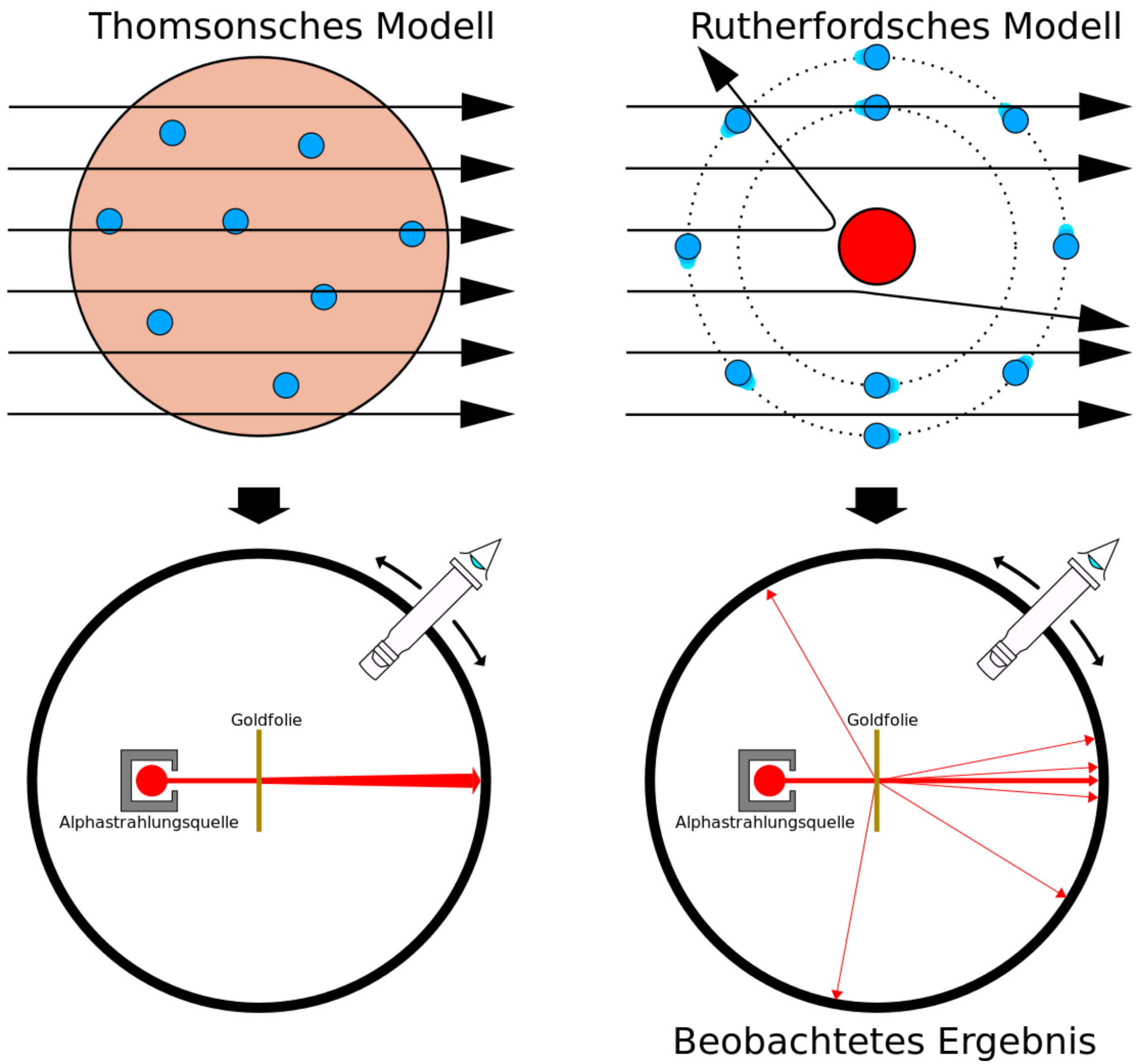
Conservation of kinetic energy

Collision in the CM Frame



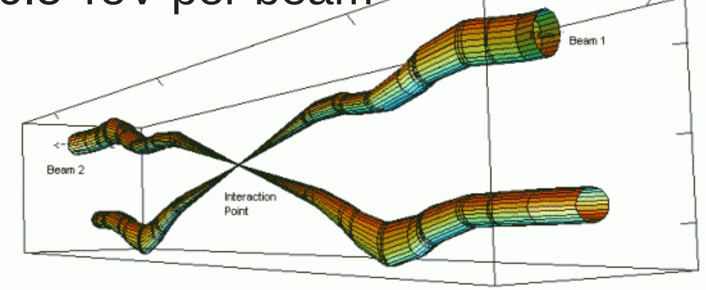


Rutherford Scattering



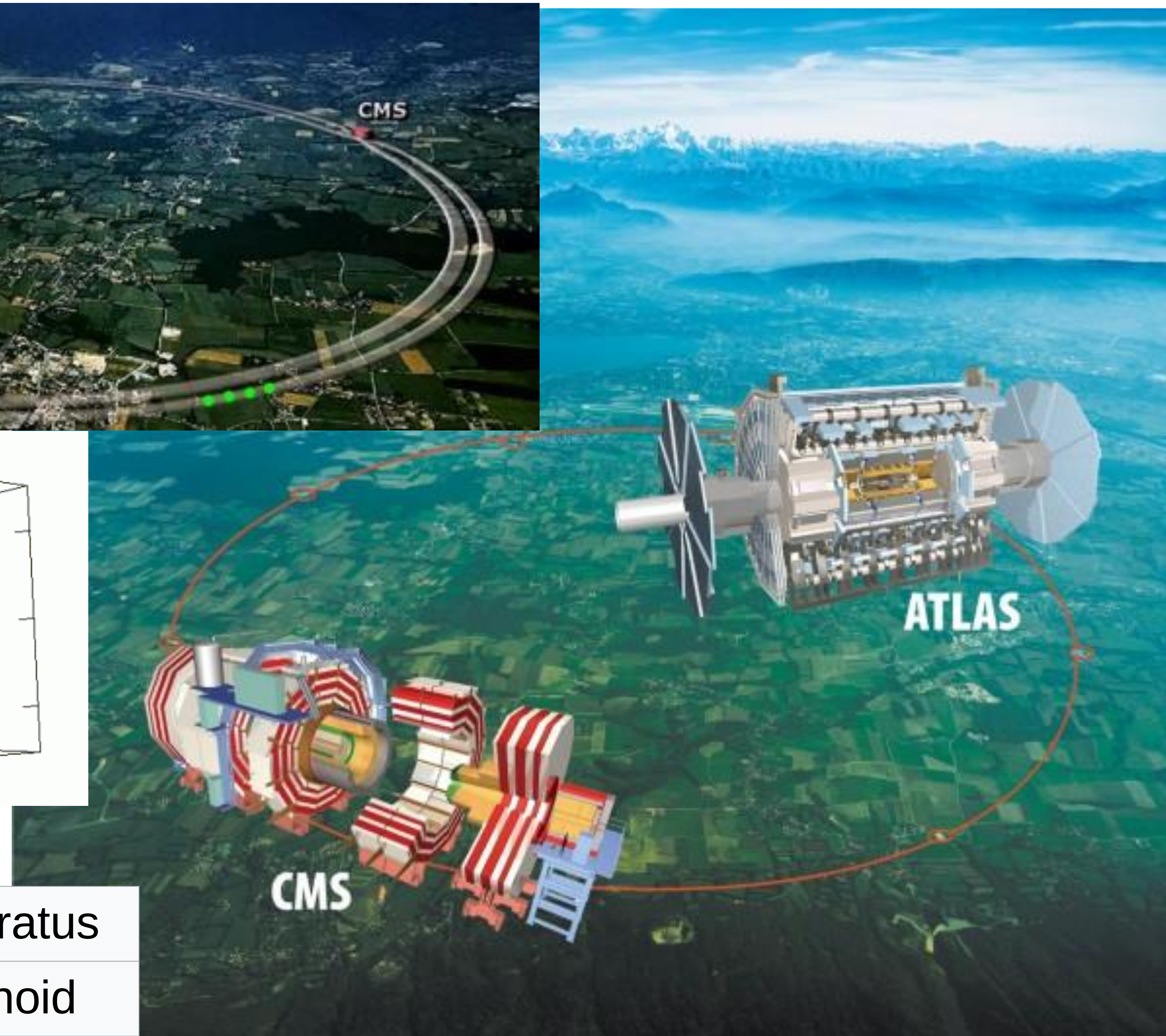
Large Hadron Collider (LHC)

6.5 TeV per beam



Relative beam sizes around IP1 (Atlas) in collision

circumference of 26.7 km



ATLAS	A Toroidal LHC Apparatus
CMS	Compact Muon Solenoid