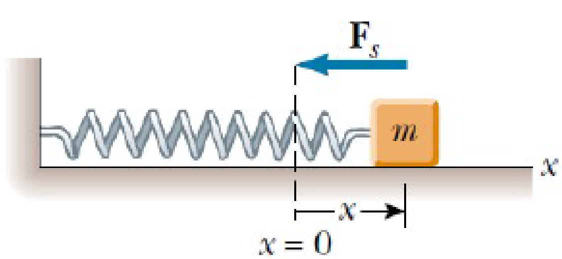
Zhejiang University

Department of Physics

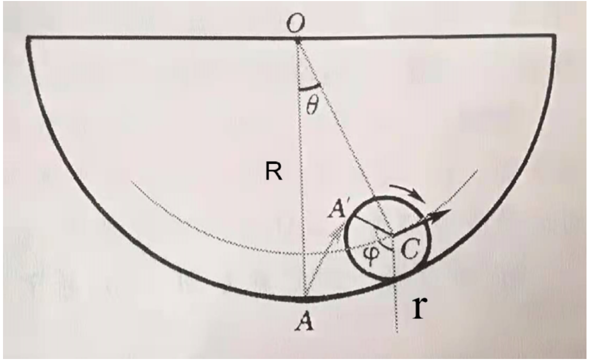
General Physics (H) Problem Set # 6

1. Consider a particle with mass m attached to a massless spring with the force constant k (see the figure below). Now the particle undergoes an oscillating motion around its equilibrium position. The position of the particle measured from the equilibrium position is denoted by x and the momentum of the particle is denoted by p.
(a) Suppose the total energy of the particle is E. Plot the trajectory of the particle on the x-p plane. (Take x for the horizontal axis and p for the vertical axis.) The values at the intersection with x- and p-axes should also be given in terms of m, k, and E.
(b) Suppose this spring is a special spring whose force constant k can be varied. It is known that, if we change the external parameter sufficiently slowly, the area surrounded by the trajectory of the periodic motion in the x-p plane is unchanged ("adiabatic theorem"). Suppose the particle undergoes an oscillatory motion with the energy E_i at the beginning, and then we change the force constant from k_i to k_f very slowly. Derive the energy of the particle E_f at the final state.



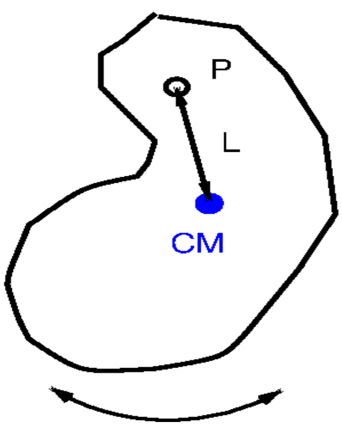
2. Consider a small cylinder with radius r is having a pure rolling motion in a big hollow cylinder with radius R. If the rolling is limited to a small angle θ<<1, the small cylinder will be in a harmonic oscillation:
(a) Calculate the potential energy of the small cylinder in θ;
(b) Calculate the kinetic energy of the small cylinder in d θ/dt;

(c) What is the period of the harmonic oscillation?



3. (a) Let $x_1(t)$ and $x_2(t)$ be solutions of $\ddot{x}^2 + bx = 0$. Show that $x_1(t) + x_2(t)$ is not a solution to this equation.

(b) Consider a physical pendulum formed by a rigid body of mass m with the pivot P around it performs frictionless oscillations and which is at a distance L from the center of mass CM, see the sketch. The moment of inertia of the rigid body is I_p with respect to the pivot. Derive the equation obeyed by small oscillations and find their frequency ω.



4.A damped harmonic oscillator consists of a block (m = 2.00 kg), a spring (k = 10.0 N/m), and a damping force $F = bv$. Initially, it oscillates with an amplitude of 25.0 cm; because of the damping, the amplitude falls to three-fourths of this initial value at the completion of four oscillations. (a) What is the value of b? (b) How much energy has been "lost" during these four oscillations?

5.

Damped oscillator

Consider a one-dimensional damped oscillator, whose equation of motion is given by

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}.$$

In the equation, the first term on the left hand side is the spring force characterized by the spring constant k. The second term is the retarding force that is proportional to the speed and is characterized by the damping coefficient b. The unknown function $x(t)$ is the displacement (as a function of time t) and m is the mass of the oscillator.

(a) Substitute the trial solution $x(t) = e^{pt}$ into the equation and determine the expression of p.

(b) For the over-damped case $b/2m > \sqrt{k/m}$, we are led to two particular solutions with exponential decaying characteristics:

(i) Write down the general solution $x(t)$.

(ii) For large times t, which of the two particular solutions will dominate? Let us introduce a decay time constant τ as $x(t) \sim e^{-t/\tau}$, so that after time τ the displacement $x(t)$ is reduced by a factor $1/e$. Determine τ for large times.

(c) For the under-damped case $b/2m < \sqrt{k/m}$, we get an oscillatory behavior but with an amplitude also decaying in time:

(i) By writing the two arbitrary constants of the general solution in the form $Ae^{i\phi}$ and $Ae^{-i\phi}$, for A and ϕ reals, show that the general solution takes the form $x(t) = 2Ae^{-t/\tau}\cos(\omega_D t + \phi)$. Determine τ and ω_D for this case.

(ii) Given that the oscillator starts from $x = 0$ with velocity v, determine the arbitrary constants. Sketch the solution as a function of time.

(Complex number $i = \sqrt{-1}$. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.)

(d) In practical situations, we would like to minimize the time constant τ by changing the strength of the retarding force (while keeping all other parameters of the oscillator constant). When this is achieved, the system approaches the equilibrium position in the shortest possible time.

(i) Give the form of the damping coefficient b for which this condition is fulfilled.

(ii) Give the corresponding value of τ in the over-damped and under-damped cases, and explain why it is the smallest attainable value for the system.

6.

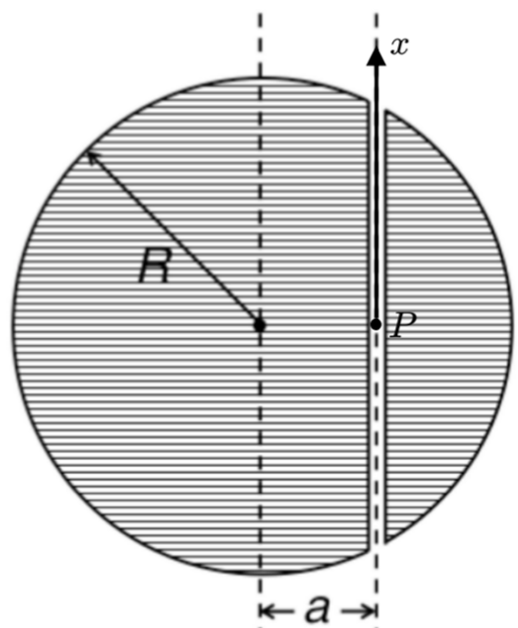
Imagine that we have built a straight tunnel through a planet of mass M and radius R. The vertical distance between the tunnel and the planet's center is a. At time $t = 0$, we drop a particle of mass m into the tunnel from one end of the tunnel. The initial speed of the particle is zero. Assume the tunnel is frictionless and the density of the planet is uniform. Neglect the missing materials from the tunnel and the planet's rotation. The gravitational constant is G.

(a) Calculate the gravity between the particle and the planet when the displacement of the particle from the middle point P of the tunnel is x. The x-axis with P as the origin has been given in the figure.

(Hint: If a particle is in a uniform spherical shell, it feels zero net gravity from the shell. If a particle is out of a uniform spherical shell, it feels nonzero net gravity from the shell as if the mass of the shell is concentrated at the sphere center.)

(b) Express x as a function of time t. How much time does the particle take to reach the other end of the tunnel?

(c) Analyze whether the speed of the particle in the tunnel can reach the first cosmic velocity of the planet if we properly choose a when building the tunnel.



7. Vibrational Modes of Linear Triatomic molecule of CO_2

Consider a linear triatomic molecule of CO_2 [Fig. (a)]. We model the CO_2 molecule by two kinds of balls (with mass m_1 for O atoms and m_2 for C atom) connected by two identical springs with the spring constant k [Fig. (b)], where the force F between the C and O atoms follows the simple relation

$$F = -k\Delta x \, ,$$

where Δx is the deviation from their equilibrium distance a_0.

(a) Calculate three eigenfrequencies for longitudinal vibrational modes along the C-O bond direction (x direction) and illustrate the motion of the atoms in each normal mode.

(b) For transverse vibrational modes, assume that the O atoms are stationary and fixed at $x = \pm a_0$, and the C atom oscillates in the y direction with a small amplitude compared to the equilibrium distance a_0 [Fig. (c)]. Calculate the restoring force acting on C atom as a function of the perpendicular displacement y. Is this vibrational mode a harmonic motion? Explain the reason of your answer.