The Magnetic Field

Xin Lu/Gentaro Watanabe

Lecture 8

Outline

- Charged Particles in a Magnetic Field
  - Circulating Charges
  - Cyclotrons
  - The Hall Effect
- Magnetic Force on a Current-Carrying Wire
- Torque on a Current Loop

The Magnetic Force on a Charged Particle

- The magnetic force on a charged particle is equal to the charge q times the cross product of its velocity $\vec{v}$ and the magnetic field $\vec{B}$ (all measured in the same reference frame):

$$\vec{F}_B = q\vec{v} \times \vec{B}.$$

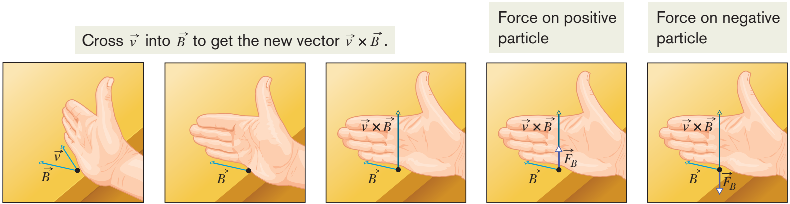


Figure 1: The right-hand rule for a magnetic force.

- The force $\vec{F}_B$ acting on a charged particle moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ is always perpendicular to $\vec{v}$ and $\vec{B}$.
- The SI unit for $\vec{B}$ is the tesla (T), or the newton per coulomb-meter per second.

$$1 \text{ tesla} = 1 \text{ T} = 1 \frac{\text{N}}{\text{C} \cdot (\text{m/s})} = 1 \frac{\text{N}}{\text{A} \cdot \text{m}}.$$

- An earlier (non-SI) unit for $\vec{B}$, still in common use, is the gauss (G), and 1 tesla $= 10^4$ gauss.
- Note that Earth's magnetic field near the planet's surface is about 10^{-4} T ($= 100\ \mu$T or 1 G).

- We can represent magnetic fields with field lines.
  1. the direction of the tangent to a magnetic field line at any point gives the direction of $\vec{B}$ at that point, and
  2. the spacing of the lines represents the magnitude of $\vec{B}$–the magnetic field is stronger where the lines are closer together, and conversely.

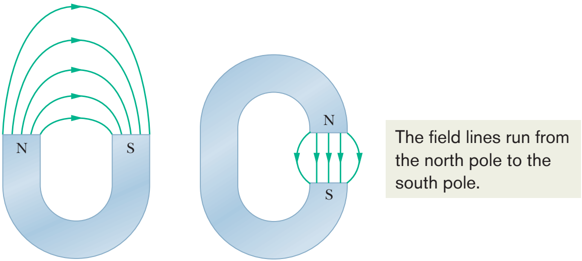


Figure 2: Field lines of a horseshoe magnet and a C-shaped magnet.

- Because a magnet has two poles, it is said to be a **magnetic dipole**. The end of a magnet from which the field lines emerge is called the north pole of the magnet; the other end, where field lines enter the magnet, is called the south pole.

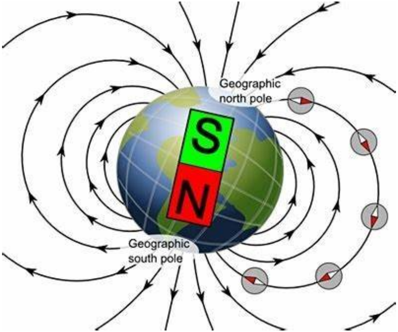


Figure 3: The north pole of Earth is in the Arctic region. Does it make sense to you?

- *Opposite magnetic poles attract each other, and like magnetic poles repel each other.*

The Discovery of Electron (1897)

- J.J. Thomson measured the ratio of mass to charge for the then-unknown charged particle and claimed that it is lighter than hydrogen atom by a factor of more than 1000 (later proved to be 1836.15). He also claimed that these particles are found in all matter.

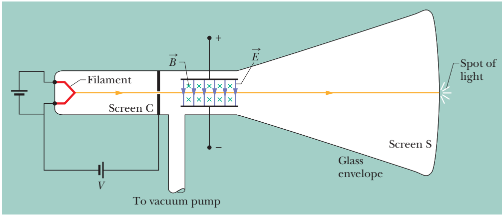
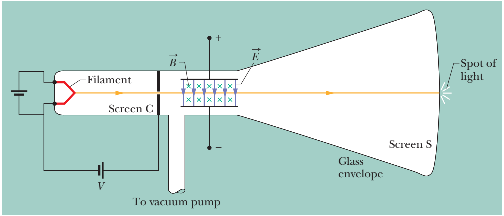
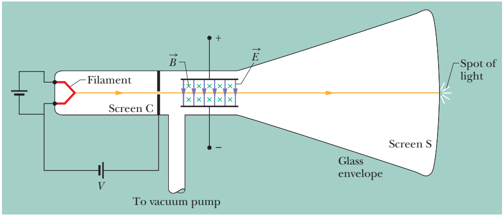


Figure 4: Illustration of Thomson's cathode ray tube.

- Charged particles (now known as electrons) are emitted by a hot filament at the rear of the evacuated tube and are accelerated by an applied potential difference V.



- After they pass through a slit in screen C, they form a narrow beam. They then pass through a region of crossed $\vec{E}$ and $\vec{B}$ fields, headed toward a fluorescent screen S, where they produce a spot of light.



- By controlling the magnitudes and directions of the fields, Thomson could thus control where the spot of light appeared on the screen.
- [Homework] What could Thomson measure to determine the mass-to-charge ratio q/m?

Circulating Charges

- Electrons with speed v move in a uniform magnetic field $\vec{B}$ directed out of the page. As a result, $\vec{B}$ continuously deflects the electrons, causing the electrons to follow a circular path.

$$F = |q|vB = m\frac{v^2}{r}.$$

Therefore, the radius is

$$r = \frac{mv}{|q|B} = \frac{v}{\omega}.$$

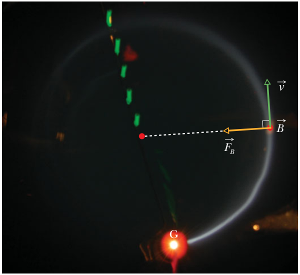


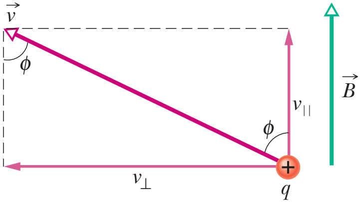
Figure 5: Electrons circulating in a chamber with angular frequency $\omega = |q|B/m$.

- If the velocity of a charged particle has a component parallel to the uniform magnetic field, such that

$$v_{\parallel} = v\cos\phi,$$

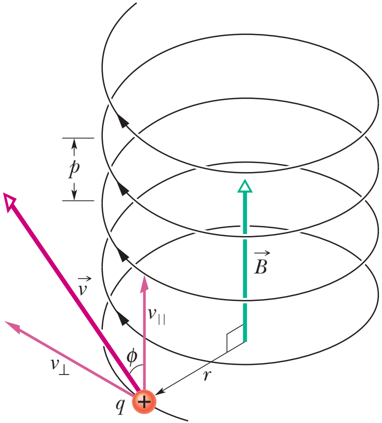
$$v_{\perp} = v\sin\phi.$$

The particle will move in a helical path about the direction of the field vector.



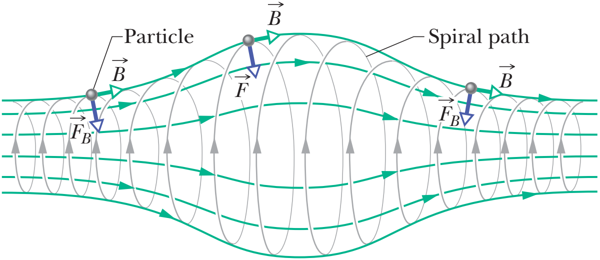
- The perpendicular component $v_\perp$ determines the radius of the helix

$$r = \frac{mv_\perp}{|q|B}.$$



- The parallel component $v_\parallel$ determines the pitch p of the helix – that is, the distance between adjacent turns.

- A charged particle spirals in a nonuniform magnetic field.



- When the field at an end is strong enough, the particle may reflect from that end.

Cyclotrons

- Beams of high-energy electrons and protons are used to probe atoms and nuclei to reveal the fundamental structure of matter.
- Particle accelerators employ a magnetic field to repeatedly bring particles back to an accelerating region, where they gain more and more energy until they finally emerge as a high-energy beam.
- The electric field in the gap alternates in direction. Why and how often?

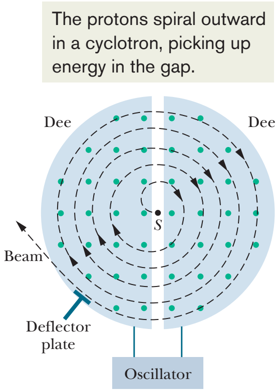


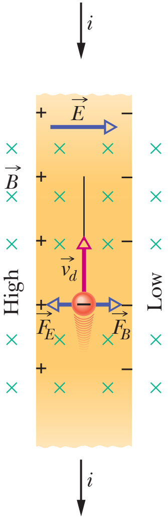
Figure 6: Top view of a cyclotron region.

The Hall Effect (1879)

- A beam of electrons in a vacuum can be deflected by a magnetic field.
- Edwin H. Hall showed that the drifting conduction electrons in a copper strip can also be deflected by a magnetic field.
- This **Hall effect** allows us to find out whether the charge carriers in a conductor are positively or negatively charged. Beyond that, we can measure the number of such carriers per unit volume of the conductor.

- In a copper strip with width d, electrons drift with speed v_d opposite to the current
$i = JA = -nev_d A$,
where A is the cross-sectional area of the strip and n is the carrier density.
- An external field $\vec{B} = B\hat{z}$ deflects drifting electron to one side of the strip, building up a transverse voltage $V = Ed$.
- In equilibrium, the electric and magnetic forces are in balance:

$$eE = ev_d B_z.$$



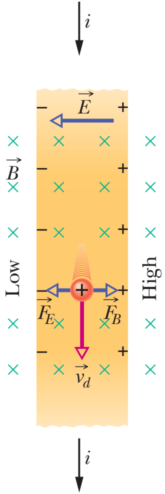
- Thus, we have

$$V = Ed = -\frac{i}{neA}Bd = -\frac{B}{ne}Jd.$$

- Hall resistivity and coefficient are defined as

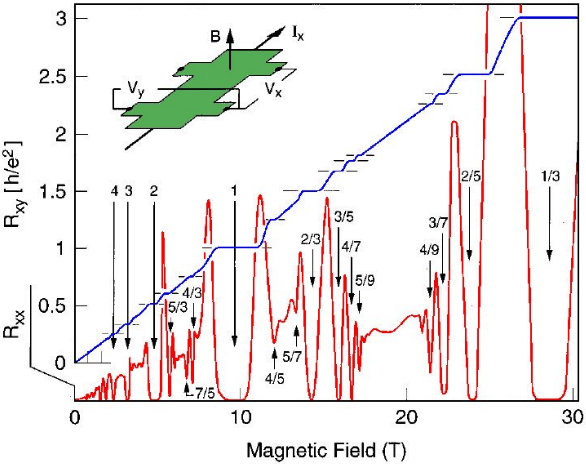
$$\rho_{xy} = \frac{E_y}{J_x} = -\frac{B}{ne}, \quad R_H = \frac{E_y}{B_z J_x} = -\frac{1}{ne}.$$

- The sign of V will change, if the charge carriers in current i are positively charged.



- The Hall effect allows us to measure the number density and the charge type of carriers in a conductor.

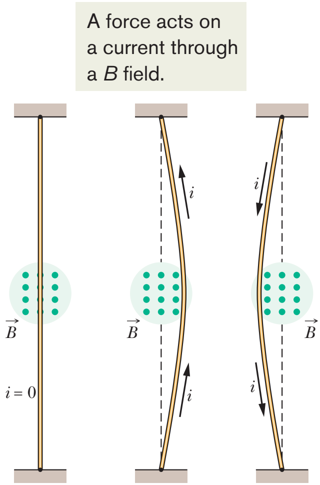
Quantum Hall Effect (1980, 1982)



$$h/e^2 = 25,812.807449(86)\ \Omega$$

Current-Carrying Wire

- In the Hall effect, the magnetic field exerts a sideways force on electrons moving in a wire. This force must then be transmitted to the wire itself, because the conduction electrons cannot escape sideways out of the wire.
- How does the force on the wire change, if we reverse the direction of $\vec{B}$, the direction of i, or the charge type of the carriers?



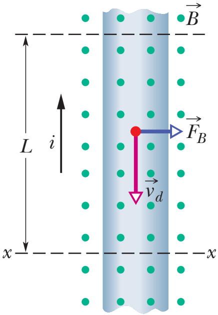
- A conduction electron with drift speed v_d experiences

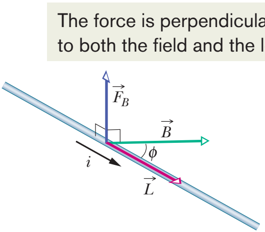
$$\vec{F}_B = e\vec{v}_d \times \vec{B}.$$

- The wire with length L and charge $|q| = i(L/v_d)$ experiences

$$\vec{F} = \frac{iL}{v_d}\frac{e}{|e|}\vec{v}_d \times \vec{B} = i\vec{L} \times \vec{B},$$

where $\vec{L}$ has magnitude L and is directed along the wire segment in the direction of the current.





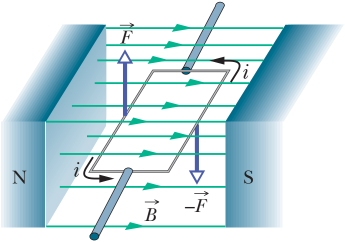
- In practice, the magnetic force on a current-carrying wire can be taken as the defining equation for $\vec{B}$, because it is much easier to measure $\vec{F}$ acting on a wire than that on a single moving charge.
- If a wire is not straight or the field is not uniform, we can imagine the wire broken up into small straight segments and, in the differential limit, we can write

$$d\vec{F} = i d\vec{L} \times \vec{B},$$

and we can find the resultant force on any given arrangement of currents by integrating over that arrangement.

Torque on a Current Loop

- Much of the world's work is done by electric motors.
- The force behind this work is the magnetic force that a magnetic field exerts on a wire that carries a current.



- For a loop of current, magnetic forces produce a torque, tending to rotate it about its central axis. In practice, wires to lead the current into and out of the loop are needed (not shown).

- The net force on the loop is zero since $\vec{F}_1 = -\vec{F}_3$ and $\vec{F}_2 = -\vec{F}_4$.
- To define the orientation of the loop in the magnetic field, we define a normal vector $\vec{n}$ that is perpendicular to the plane of the loop.

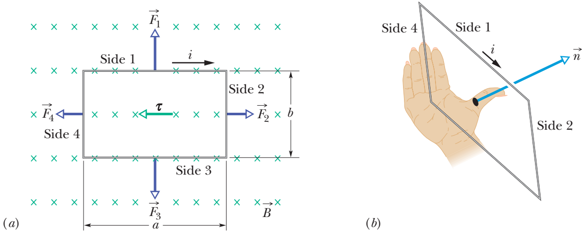
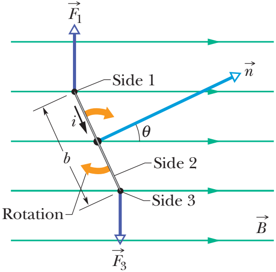


Figure 7: A rectangular loop, of length a and width b and carrying a current i, is located in a uniform magnetic field $\vec{B}$.

- The net torque on the loop

$$\vec{\tau} = -\frac{1}{2}\vec{L}_2 \times \vec{F}_1 + \frac{1}{2}\vec{L}_2 \times \vec{F}_3$$
$$= \vec{L}_2 \times \vec{F}_3,$$



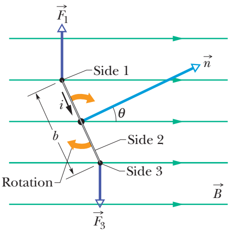
where $\vec{F}_3 = i\vec{L}_3 \times \vec{B}$. Thus,

$$\vec{\tau} = i\vec{L}_2 \times (\vec{L}_3 \times \vec{B}).$$

- The torque τ acts to align the normal vector $\vec{n}$ with the direction of the field. $\vec{L}_2$ and $\vec{L}_3$ are the length vectors along sides 2 and 3, respectively, in the direction of the current.

- Note that $\vec{L}_2 \cdot \vec{L}_3 = 0$ and $\vec{L}_3 \cdot \vec{B} = 0$, we obtain (with Appendix 8A)

$$\begin{aligned}\vec{\tau} &= i\vec{L}_2 \times (\vec{L}_3 \times \vec{B}) \\ &= i(\vec{L}_2 \times \vec{L}_3) \times \vec{B} = i\vec{A} \times \vec{B}.\end{aligned}$$



- Here, we define the area vector for the loop

$$\vec{A} = \vec{L}_2 \times \vec{L}_3 = ab\hat{n}.$$

- The expression $\vec{\tau} = i\vec{A} \times \vec{B}$ holds for all flat coils, *no matter what their shapes are*, provided $\vec{B}$ is uniform.
- [Homework] Validate the expression of the torque and discuss why it works for any flat coil.

- Thus, a current-carrying flat coil placed in $\vec{B}$ will tend to rotate so that $\vec{n}$ has the same direction as $\vec{B}$, just like a bar magnet (a magnetic dipole) placed in the magnetic field.
- To increase the torque, we can replace the single loop of current with a coil of N loops, or turns. The total torque on the coil is then

$$\vec{\tau} = Ni\vec{A} \times \vec{B} = \vec{\mu} \times \vec{B},$$

where $\vec{\mu} = Ni\vec{A}$ is known as the **magnetic dipole moment** of the coil.

Summary

- Magnetic force versus electric force

$$\vec{F}_B = q\vec{v} \times \vec{B}, \quad \vec{F}_E = q\vec{E}$$

$$\vec{F}_B = i\vec{L} \times \vec{B}$$

- Magnetic dipole versus electric dipole

$$\vec{\mu} = Ni\vec{A}, \quad \vec{p} = q\vec{d}$$

$$\vec{\tau}_B = \vec{\mu} \times \vec{B}, \quad \vec{\tau}_E = \vec{p} \times \vec{E}$$

- Charged particle circulating in a magnetic field:

$$\omega = \frac{|q|B}{m}$$

- The Hall effect:

$$R_H = \frac{E}{BJ} = \frac{1}{nq}.$$

Reading

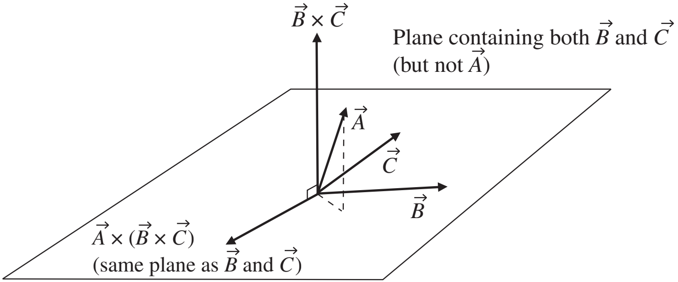
Halliday, Resnick & Krane:

- Chapter 32: The Magnetic Field

Appendix 8A: Vector Triple Product

- **Vector triple product**: $\vec{A} \times (\vec{B} \times \vec{C})$.

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$
$$(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{A}(\vec{B} \cdot \vec{C}) + \vec{B}(\vec{A} \cdot \vec{C})$$



Appendix 8B: Electric Motor

- In a motor, the current in the coil is reversed as $\vec{n}$ begins to line up with $\vec{B}$, so that a torque continues to rotate the coil.
- This automatic reversal of the current is done via a commutator that electrically connects the rotating coil with the stationary contacts on the wires that supply the current from some source.

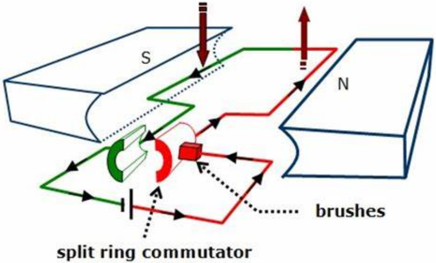


Figure 8: Electric motor with a split-ring commutator.