General Physics II

Solution #5

2021/11/24

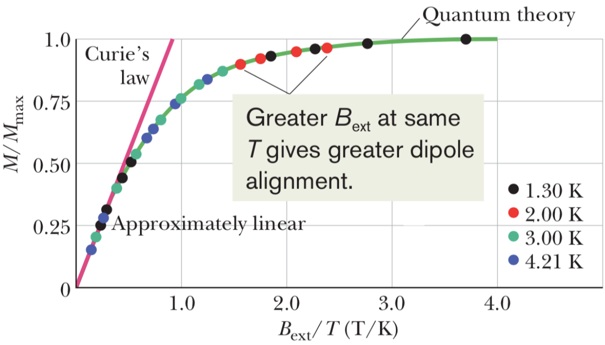
P5-1. Consider a solid containing N atoms per unit volume, each atom having a magnetic dipole moment $\vec{\mu}$. Suppose the direction of $\vec{\mu}$ can be only parallel or antiparallel to an externally applied magnetic field $\vec{B}$ (this will be the case if $\vec{\mu}$ is due to the spin of a single electron). According to statistical mechanics, the probability of an atom being in a state with energy U is proportional to $e^{-U/kT}$, where T is the temperature and k is Boltzmann's constant. Thus, because energy U is $-\vec{\mu} \cdot \vec{B}$, the fraction of atoms whose dipole moment is parallel to $\vec{B}$ is proportional to $e^{\mu B/kT}$ and the fraction of atoms whose dipole moment is antiparallel to $\vec{B}$ is proportional to $e^{-\mu B/kT}$.

(a) Show that the magnitude of the magnetization of this solid is $M = N\mu \tanh(\mu B/kT)$. Here tanh is the hyperbolic tangent function: $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$.

(b) Show that the result given in (a) reduces to $M = N\mu^2 B/kT$ for $\mu B \ll kT$.

(c) Show that the result of (a) reduces to $M = N\mu$ for $\mu B \gg kT$.

(d) Show that both (b) and (c) agree qualitatively with the following figure.



Solution: The orientation energy of a dipole in a magnetic field is given by $U = -\vec{\mu} \cdot \vec{B}$. So if a dipole is parallel with $\vec{B}$, then $U = -\mu B$; however, $U = +\mu B$ if the alignment is anti-parallel. We use the notation $P(\mu) = e^{\mu B/kT}$ for the probability of a dipole that is parallel to $\vec{B}$, and $P(-\mu) = e^{-\mu B/kT}$ for the probability of a dipole that is anti-parallel to the field. The magnetization may be thought of as a "weighted average" in terms of these probabilities.

(a) With N atoms per unit volume, we find the magnetization to be:

$$M = \frac{N\mu P(\mu) - N\mu P(-\mu)}{P(\mu) + P(-\mu)} = \frac{N\mu(e^{\mu B/kT} - e^{-\mu B/kT})}{e^{\mu B/kT} + e^{-\mu B/kT}}$$
$$= N\mu \tanh\left(\frac{\mu B}{kT}\right).$$

(b) For $\mu B \ll kT$ (that is, $\mu B/kT \ll 1$) we have $e^{\pm\mu B/kT} \approx 1 \pm \mu B/kT$, so

$$\begin{aligned} M &= N\mu \tanh\left(\frac{\mu B}{kT}\right) \\ &\approx \frac{N\mu[(1 + \mu B/kT) - (1 - \mu B/kT)]}{[(1 + \mu B/kT) + (1 - \mu B/kT)]} \\ &= \frac{N\mu^2 B}{kT}. \end{aligned}$$

(c) For $\mu B \gg kT$ we have $\tanh(\mu B/kT) \approx 1$, so $M = N\mu \tanh\left(\frac{\mu B}{kT}\right) \approx N\mu$.

(d) One can easily plot the tanh function using, for instance, a graphical calculator.

P5-2. You place a magnetic compass on a horizontal surface, allow the needle to settle, and then give the compass a gentle wiggle to cause the needle to oscillate about its equilibrium position. The oscillation frequency is 0.312 Hz. Earth's magnetic field at the location of the compass has a horizontal component of 18.0 μT. The needle has a magnetic moment of 0.680 mJ/T. What is the needle's rotational inertia about its (vertical) axis of rotation?

Solution: We write the torque as $\tau = -\mu B_h \sin\theta$ where the minus indicates that the torque opposes the angular displacement θ (which we will assume is small and in radians). The small angle approximation leads to $\tau \approx -\mu B_h \theta$, which is an indicator for simple harmonic motion. We then find the period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{\mu B_h}}$$

where I is the rotational inertial that we asked to solve for. Since the frequency is given as 0.312 Hz, then the period is $T = 1/f = 1/(0.312) = 3.21$ s. Similarly, $B_h = 18.0 \times 10^{-6}$ T and $\mu = 6.80 \times 10^{-4}$ J/T. The above relation then yields $I = 3.19 \times 10^{-9}$ kg·m^2.

P5-3. Prove that the displacement current in a parallel-plate capacitor of capacitance C can be written as $i_d = C(dV/dt)$, where V is the potential difference between the plates.

Solution: Let A be the area of a plate and E be the magnitude of the electric field between the plates. The field between the plates is uniform, so $E = V/d$, where V is the potential difference across the plates and d is the plate separation.
Thus, the displacement current is

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d(EA)}{dt} = \varepsilon_0 A \frac{dE}{dt} = \frac{\varepsilon_0 A}{d} \frac{dV}{dt}.$$

Now, $\varepsilon_0 A/d$ is the capacitance C of a parallel-plate capacitor (not filled with a dielectric), so

$$i_d = C \frac{dV}{dt}.$$

P5-4. A plane electromagnetic wave traveling in the positive direction of an x axis in vacuum has components $E_x = E_y = 0$ and $E_z = (2.0\ \mathrm{V/m})\cos[(\pi \times 10^{15}\mathrm{s}^{-1})(t - x/c)]$. (a) What is the amplitude of the magnetic field component? (b) Parallel to which axis does the magnetic field oscillate? (c) When the electric field component is in the positive direction of the z axis at a certain point P, what is the direction of the magnetic field component there?

Solution:

(a) The amplitude of the magnetic field is

$$B_m = \frac{E_m}{c} = \frac{2.0}{2.998 \times 10^8} = 6.67 \times 10^{-9}\mathrm{T} \approx 6.7 \times 10^{-9}\mathrm{T}.$$

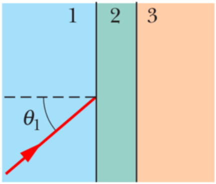
(b) Since the $\vec{E}$-wave oscillates in the z direction and travels in the x direction, we have $B_x = B_z = 0$. So, the oscillation of the magnetic field is parallel to the y axis.

(c) The direction $(+x)$ of the electromagnetic wave propagation is determined by $\vec{E} \times \vec{B}$. If the electric field points in $+z$, then the magnetic field must point in the $-y$ direction.

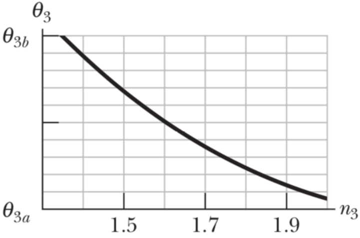
With SI units understood, we may write

$$\begin{aligned}
B_y &= B_m \cos\left[\pi \times 10^{15}\left(t - \frac{x}{c}\right)\right] \\
&= \frac{2.0 \cos[10^{15}\pi(t - x/c)]}{3.0 \times 10^8} \\
&= (6.7 \times 10^{-9}) \cos\left[10^{15}\pi\left(t - \frac{x}{c}\right)\right]
\end{aligned}$$

P5-5. A beam of light in material 1 is incident on a boundary at an angle $\theta_1 = 40^\circ$. Some of the light travels through material 2, and then some of it emerges into material 3. The two boundaries between the three materials are parallel. The final direction of the beam depends, in part, on the index of refraction n_3 of the third material.



(a)



(b)

Figure b gives the angle of refraction θ_3 in that material versus n_3 for a range of possible n_3 values. The vertical axis scale is set by $\theta_{3a} = 30.0^\circ$ and $\theta_{3b} = 50.0^\circ$.
(a) What is the index of refraction of material 1, or is the index impossible to calculate without more information?
(b) What is the index of refraction of material 2, or is the index impossible to calculate without more information?
(c) If θ_1 is changed to 70° and the index of refraction of material 3 is 2.4, what is θ_3?

Solution:

(a) From $n_1 \sin\theta_1 = n_2 \sin\theta_2$ and $n_2 \sin\theta_2 = n_3 \sin\theta_3$, we find $n_1 \sin\theta_1 = n_3 \sin\theta_3$. This has a simple implication: that $\theta_1 = \theta_3$ when $n_1 = n_3$. Since we are given $\theta_1 = 40^\circ$, then we look for a point in Fig.b where $\theta_3 = 40^\circ$. This seems to occur at $n_3 = 1.6$, so we infer that $n_1 = 1.6$.

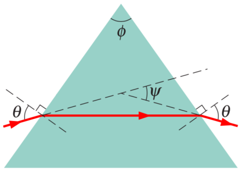
(b) Our first step in our solution to part (a) shows that information concerning n_2 disappears (cancels) in the manipulation. Thus, we cannot tell; we need more information.

(c) From $1.6 \sin 70^\circ = 2.4 \sin\theta_3$ we obtain $\theta_3 = 39^\circ$.

P5-6. A ray is incident on one face of a triangular glass prism in air. The angle of incidence θ is chosen so that the emerging ray also makes the same angle θ with the normal to the other face. Show that the index of refraction n of the glass prism is given by

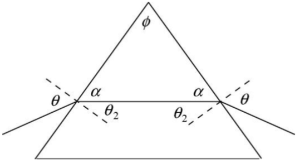
$$n = \frac{\sin \frac{1}{2}(\psi + \phi)}{\sin \frac{1}{2}\phi}$$

where ϕ is the vertex angle of the prism and ψ is the *deviation angle*, the total angle through which the beam is turned in passing through the prism. (Under these conditions the deviation angle ψ has the smallest possible value, which is called the *angle of minimum deviation*.)

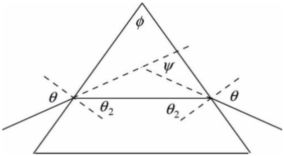


Solution: Consider diagram (a) shown next. The incident angle is θ and the angle of refraction is θ_2. Since $\theta_2 + \alpha = 90^\circ$ and $\phi + 2\alpha = 180^\circ$, we have

$$\theta_2 = 90^\circ - \alpha = 90^\circ - \frac{1}{2}(180^\circ - \phi) = \frac{\phi}{2}.$$



(a)



(b)

s

Next, examine diagram (b) and consider the triangle formed by the two normals and the ray in the interior.

One can show that ψ is given by

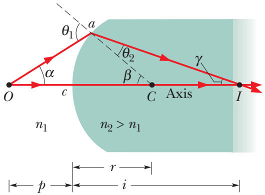
$$\psi = 2(\theta - \theta_2).$$

Upon substituting $\phi/2$ for θ_2, we obtain $\psi = 2(\theta - \phi/2)$ which yields $\theta = (\phi + \psi)/2$. Thus, using the law of refraction, we find the index of refraction of the prism to be

$$n = \frac{\sin\theta}{\sin\theta_2} = \frac{\sin\frac{1}{2}(\phi + \psi)}{\sin\frac{1}{2}\phi}.$$

P5-7. Consider images formed by refraction through spherical surfaces of transparent materials, such as glass. As shown in the figure, light emits from a point object O in a medium with index of refraction n_1. It will refract through a spherical surface of radius r and center of curvature C into a medium of index of refraction n_2. Show that, for light rays making only small angles with the central axis, the object distance p and the image distance i are related by

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{r}.$$



Solution: The incident ray from point object O that falls on point a of a spherical refracting surface is refracted satisfying

$$n_1 \sin \theta_1 = n_2 \sin \theta_2.$$

If α is small, θ_1 and θ_2 will also be small and we can replace the sines of these angles with the angles themselves. Thus, the equation above becomes

$$n_1 \theta_1 \approx n_2 \theta_2.$$

We again use the fact that an exterior angle of a triangle is equal to the sum of the two opposite interior angles. Applying this to triangles COa and ICa yields

$$\theta_1 = \alpha + \beta \text{ and } \beta = \theta_2 + \gamma.$$

Therefore,

$$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta.$$

In radian measure the angles α, β, and γ are

$$\alpha \approx \frac{\bar{a}c}{p};\ \beta = \frac{\bar{a}c}{r};\ \gamma \approx \frac{\bar{a}c}{i}.$$

Therefore,

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{r}.$$

P5-8. When a thin lens with index of refraction n is surrounded by some medium with index of refraction n_{medium}, show that the focal length f is given by

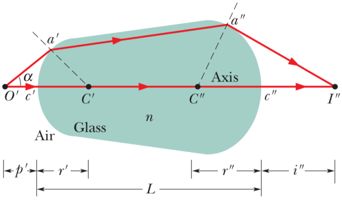
$$\frac{1}{f} = \left(\frac{n}{n_{\mathrm{medium}}} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

where r_1 is the radius of curvature of the lens surface nearer the object and r_2 is that of the other surface.

Solution: Our plan is to consider each lens surface as a separate refracting surface, and to use the image formed by the first surface as the object for the second.

We start with the thick glass "lens" of length L in Fig.a whose left and right refracting surfaces are ground to radii r' and r''. A point object O' is placed near the left surface as shown. A ray leaving O' along the central axis is not deflected on entering or leaving the lens.

A second ray leaving O' at an angle α with the central axis intersects the left surface at point a', is refracted, and intersects the second (right) surface at point a''.

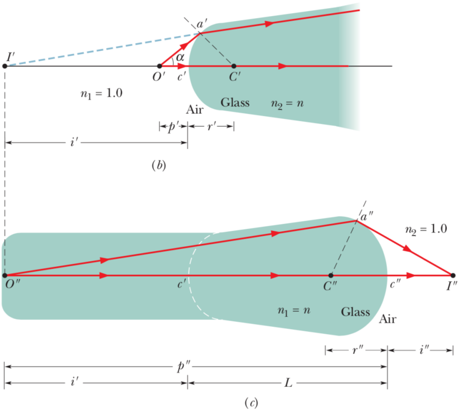


(a)

The ray is again refracted and crosses the axis at I'', which, being the intersection of two rays from O', is the image of point O', formed after refraction at two surfaces.

Figure b shows that the first (left) surface also forms a virtual image of O' at I'. To locate I', we use

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{r}.$$



Putting $n_1 = 1$ for air and $n_2 = n$ for lens glass and bearing in mind that the (virtual) image distance is negative, we obtain

$$\frac{1}{p'} - \frac{n}{i'} = \frac{n-1}{r'}.$$

Figure c shows the second surface again. Unless an observer at point a'' were aware of the existence of the first surface, the observer would think that the light striking that point originated at point I' in Fig.b and that the region to the left of the surface was filled with glass as indicated. Thus, the (virtual) image I' formed by the first surface serves as a real object O'' for the second surface.

The distance of this object from the second surface is

$$p'' = i' + L.$$

To apply

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{r}$$

to the second surface, we must insert $n_1 = n$ and $n_2 = 1$ because the object now is effectively imbedded in glass. Then it becomes

$$\frac{n}{i' + L} + \frac{1}{i''} = \frac{1 - n}{r''}.$$

Let us now assume that the thickness L of the "lens" in Fig.a is so small that we can neglect it in comparison with our other linear quantities (such as p', i'). In all that follows we make this *thin-lens approximation*.

Rearranging the right side lead to

$$\frac{n}{i'} + \frac{1}{i''} = \frac{1-n}{r''}.$$

Therefore,

$$\frac{1}{p'} + \frac{1}{i''} = (n-1)\left(\frac{1}{r'} - \frac{1}{r''}\right).$$

Finally, calling the original object distance simply p and the final image distance simply i leads to

$$\frac{1}{p} + \frac{1}{i} = (n-1)\left(\frac{1}{r'} - \frac{1}{r''}\right),$$

which, with a small change in notation, is

$$\frac{1}{f} = \left(\frac{n}{n_{\mathrm{medium}}} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$