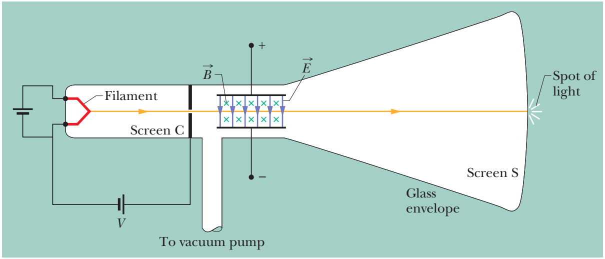
General Physics II

Solution #3

2021/10/27

P3-1. In Thomson's experimental apparatus, charged particles are emitted by a hot filament and form a narrow beam after they pass through a slit. They then pass through a region of crossed $\vec{E}$ and $\vec{B}$ fields, headed toward a fluorescent screen S, where they produce a spot of light.



(a) Set $E = 0$ and $B = 0$ and note the position of the spot on screen S due to the undeflected beam. Turn on $\vec{E}$ and measure the resulting beam deflection. Show that the deflection of the particle at the far end of the plates is

$$y = \frac{|q|EL^2}{2mv^2},$$

where v is the particle's speed, m its mass, q its charge, and L is the length of the plates.

(b) Maintaining $\vec{E}$, now turn on $\vec{B}$ and adjust its value until the beam returns to the undeflected position. Show that the crossed fields allow us to measure the speed of the charged particles passing through them, and therefore,

$$\frac{|q|}{m} = \frac{2yE}{B^2L^2},$$

in which all quantities on the right can be measured.

Solution:

(a) Between the plates, v_x is constant while v_y is accelerated by $\vec{E}$. The acceleration is $\vec{a} = \frac{|q|\vec{E}}{m}$, and the time which the particles spend in the plates is $t = \frac{L}{v}$. Therefore, the deflection is

$$y = \frac{1}{2}at^2 = \frac{1}{2} \cdot \frac{|q|E}{m} \cdot (\frac{L}{v})^2 = \frac{|q|EL^2}{2mv^2}.$$

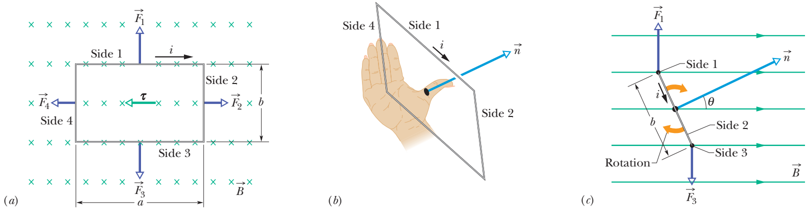
(b) If the beam returns to the undeflected position, the particles are in equilibrium between the plates. Therefore, $|q|\vec{E} = |q|\vec{v} \times \vec{B}$, and $v = \frac{E}{B}$. With (a), we obtain

$$\frac{|q|}{m} = \frac{2yE}{B^2 L^2}.$$

P3-2. For a rectangular loop, of length a and width b and carrying a current i, a uniform perpendicular magnetic field $\vec{B}$ produces a torque, tending to rotate the loop about its central axis. The torque can be expressed to be

$$\vec{\tau} = i\vec{L}_2 \times (\vec{L}_3 \times \vec{B}),$$

where $\vec{L}_2$ and $\vec{L}_3$ are two perpendicular length vectors.



(a) Using the vector triple product identities

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}),$$
$$(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{A}(\vec{B} \cdot \vec{C}) + \vec{B}(\vec{A} \cdot \vec{C}),$$

show that

$$\vec{\tau} = i\vec{A} \times \vec{B},$$

where $\vec{A} = ab\hat{n}$ is the area vector for the loop.

(b) Discuss why the expression $\vec{\tau} = i\vec{A} \times \vec{B}$ holds for all flat coils, *no matter what their shapes are*, provided $\vec{B}$ is uniform.

Solution:

(a) The net torque on the loop is $\vec{\tau} = i\vec{L}_2 \times (\vec{L}_3 \times \vec{B})$. Use $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$, therefore,

$$\begin{aligned} \vec{\tau} &= i\vec{L}_2 \times (\vec{L}_3 \times \vec{B}) \\ &= i\vec{L}_3(\vec{L}_2 \cdot \vec{B}) - i\vec{B}(\vec{L}_2 \cdot \vec{L}_3) \end{aligned}$$

Note that $\vec{L}_2 \cdot \vec{L}_3 = 0$, we obtain

$$\vec{\tau} = i\vec{L}_3(\vec{L}_2 \cdot \vec{B})$$

Use $(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{A}(\vec{B} \cdot \vec{C}) + \vec{B}(\vec{A} \cdot \vec{C})$, therefore,

$$
\begin{aligned}
i\vec{A} \times \vec{B} &= iab\hat{n} \times \vec{B} \\
&= i(\vec{L}_2 \times \vec{L}_3) \times \vec{B} \\
&= -i\vec{L}_2(\vec{L}_3 \cdot \vec{B}) + i\vec{L}_3(\vec{L}_2 \cdot \vec{B})
\end{aligned}
$$

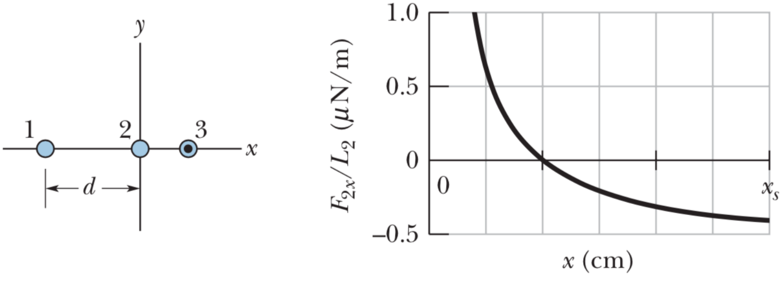
Note that $\vec{L}_3 \cdot \vec{B} = 0$, we obtain

$$
i\vec{A} \times \vec{B} = i\vec{L}_3(\vec{L}_2 \cdot \vec{B}) = \vec{\tau}
$$

(b) From (a), $i\vec{A} \times \vec{B} = i(\vec{L}_2 \times \vec{L}_3) \times \vec{B}$. The magnitude of $\vec{L}_2 \times \vec{L}_3$ equals to the area of the loop, no matter what their shapes are, while $\vec{B}$ is perpendicular to the loop. Therefore, $\vec{\tau} = i\vec{A} \times \vec{B}$ holds for all flat coils, no matter what their shapes are.

P3-3. In cross section, three current-carrying wires that are long, straight, and parallel to one another. Wires 1 and 2 are fixed in place on an x axis, with separation d. Wire 1 has a current of 0.750 A, but the direction of the current is not given. Wire 3, with a current of 0.250 A out of the page, can be moved along the x axis to the right of wire 2. As wire 3 is moved, the magnitude of the net magnetic force $\vec{F}_2$ on wire 2 due to the currents in wires 1 and 3 changes. The x component of that force is F_{2x} and the value per unit length of wire 2 is F_{2x}/L_2.

Figure b gives F_{2x}/L_2 versus the position x of wire 3. The plot has an asymptote $F_{2x}/L_2 = -0.627\ \mu\text{N/m}$ as $x \to \infty$. The horizontal scale is set by $x_s = 12.0$ cm. What are the (a) size and (b) direction (into or out of the page) of the current in wire 2?



(a) (b)

Solution:

(a) The fact that the curve in Figure b passes through zero implies that the currents in wires 1 and 3 exert forces in opposite directions on wire 2. Thus, current i_1 points out of the page. When wire 3 is a great distance from wire 2, the only field that affects wire 2 is that caused by the current in wire 1; in this case the force is negative according to Figure b. This means wire 2 is attracted to wire 1, which implies that wire 2's current is in the same direction as wire 1's current: out of the page. With wire 3 infinitely far away, the force per unit length is given (in magnitude) as 6.27×10^{-7} N/m. We set this equal to $F_{12} = \mu_0 i_1 i_2 / 2\pi d$.

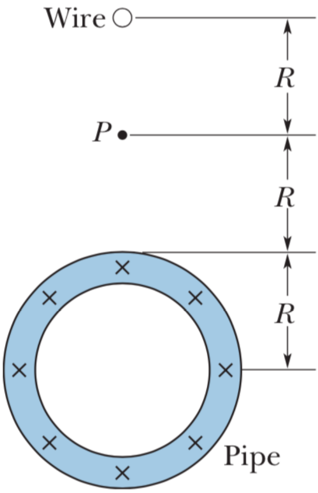
When wire 3 is at $x = 0.04$ m the curve passes through the zero point previously mentioned, so the force between 2 and 3 must equal F_{12} there. This allows us to solve for the distance between wire 1 and wire 2:

$$d = 0.04 \cdot 0.750/0.250 = 0.12 \text{ m}.$$

Then we solve 6.27×10^{-7} N/m $= \mu_0 i_1 i_2 / 2\pi d$ and obtain $i_2 = 0.50$ A.

(b) The direction of i_2 is out of the page.

P3-4. A long circular pipe with outside radius $R = 2.6$ cm carries a (uniformly distributed) current $i = 8.00$ mA into the page. A wire runs parallel to the pipe at a distance of $3.00R$ from center to center. Find the (a) magnitude and (b) direction (into or out of the page) of the current in the wire such that the net magnetic field at point P has the same magnitude as the net magnetic field at the center of the pipe but is in the opposite direction.



Solution:

(a) The field at the center of the pipe (point C) is due to the wire alone, with a magnitude of

$$B_C = \frac{\mu_0 i_{\text{wire}}}{2\pi(3R)} = \frac{\mu_0 i_{\text{wire}}}{6\pi R}$$

For the wire we have $B_{P,\text{wire}} > B_{C,\text{wire}}$. Thus, for $B_P = B_C = B_{C,\text{wire}}$, i_{wire} must be into the page:

$$B_P = B_{P,\text{wire}} - B_{P,\text{pipe}} = \frac{\mu_0 i_{\text{wire}}}{2\pi R} - \frac{\mu_0 i}{2\pi(2R)}$$

Setting $B_C = -B_P$ we obtain $i_{\text{wire}} = 3i/8 = 3 \cdot 8.00 \times 10^{-3}/8 = 3.00 \times 10^{-3}$ A.

(b) The direction is into the page.

P3-5. An electron is shot into one end of a solenoid. As it enters the uniform magnetic field within the solenoid, its speed is 800 m/s and its velocity vector makes an angle of 30° with the central axis of the solenoid. The solenoid carries 4.0 A and has 8000 turns along its length. How many revolutions does the electron make along its helical path within the solenoid by the time it emerges from the solenoid's opposite end? (In a real solenoid, where the field is not uniform at the two ends, the number of revolutions would be slightly less than the answer here.)

Solution: As the problem states near the end, some idealizations are being made here to keep the calculation straightforward (but are slightly unrealistic). For circular motion (with speed, $v_\perp$, which represents the magnitude of the component of the velocity perpendicular to the magnetic field, the period is

$$T = 2\pi r/v_\perp = 2\pi m/eB$$

Now, the time to travel the length of the solenoid is $t = L/v_\parallel$ where $v_\parallel$ is the component of the velocity in the direction of the field (along the coil axis) and is equal to $v\cos\theta$ where $\theta = 30^\circ$. Using $B = \mu_0 in$ with $n = N/L$, we find the number of revolutions made is $t/T = 1.6 \times 10^6$.

P3-6. A student makes a short electromagnet by winding 300 turns of wire around a wooden cylinder of diameter $d = 5.0$ cm. The coil is connected to a battery producing a current of 4.0 A in the wire. (a) What is the magnitude of the magnetic dipole moment of this device? (b) At what axial distance $z \gg d$ will the magnetic field have the magnitude 5.0 μT (approximately one-tenth that of Earth's magnetic field)?

Solution: The cross-sectional area is a circle, so $A = \pi R^2$, where R is the radius. The magnetic field on the axis of a magnetic dipole, a distance z away, is given by $B = \frac{\mu_0}{2\pi}\frac{\mu}{z^3}$.
(a) Substituting the values given, we find the magnitude of the dipole moment to be

$$\mu = Ni\pi R^2 = 300 \cdot 4.0 \cdot \pi \cdot 0.025^2 = 2.4 \ \mathrm{A} \cdot \mathrm{m}^2$$

(b) Solving for z, we obtain

$$z = \left(\frac{\mu_0}{2\pi}\frac{\mu}{B}\right)^{1/3} = \left(\frac{(4\pi \times 10^{-7})(2.36)}{2\pi(5.0 \times 10^{-6})}\right)^{1/3} = 46 \ \mathrm{cm}$$

P3-7. The free electron density in copper is $n = 8.5 \times 10^{28}$ m^{-3}. Copper has a resistivity $\rho = 1.7 \times 10^{-8}$ Ω·m at room temperature. In the classical theory of metals, estimate the mean free path l of electrons in copper, in units of meters?

Solution: $\rho = \frac{m}{ne^2\tau}$, so $\tau = \frac{ne^2\rho}{m}$. Therefore,

$$\begin{aligned}\tau &= \frac{(8.5\times10^{28})\times(1.6\times10^{-19})^2\times(1.7\times10^{-8})}{9.1\times10^{-31}}\\ &= 2.5\times10^{-14}\ \mathrm{s}\end{aligned}$$

From $\frac{1}{2}mv^2 = \frac{3}{2}k_BT$, where T is room temperature,

$$\begin{aligned}v = \sqrt{\frac{3k_BT}{m}} &= \sqrt{\frac{3\times(1.38\times10^{-23})\times300}{9.1\times10^{-31}}}\\ &= 1.2\times10^{5}\ \mathrm{m/s}\end{aligned}$$

Therefore, $l = v\tau \approx 3\times10^{-9}$m.

P3-8. In a Hall effect measurement a copper sample of size $L_x \times L_y \times L_z = 10.0 \times 1.0 \times 0.1$ cm^3 is used. An external magnetic field $B_z = 0.1$ T, as well as a longitudinal voltage $V_x = 10$ V, is applied. Estimate the ratio of the number of carriers that set up the Hall (electric) field to the total number of carriers in the metal. The dielectric constant for copper is assumed to be $\kappa = 1$. What is the ratio if the sample is a semiconductor with $\kappa = 12$? Assume the carrier density and the resistivity of the semiconductor are $n = 10^{22}$ m^{-3} and $\rho = 10^{-2}$ $\Omega{\cdot}$m, respectively.

Solution: In a copper (semiconductor) strip, electrons drift with speed v_d opposite to the current

$$J_x = nev_d.$$

The resistivity at a particular point is defined as the ratio of the electric field to the density of the current it creates at that point:

$$\rho = \frac{E_x}{J_x} = \frac{V_x}{L_x nev_d}.$$

Therefore,

$$v_d = \frac{V_x}{\rho L_x ne}$$

In equilibrium, the electric and magnetic forces are in balance:

$$eE_y = ev_d B_z$$

Therefore,

$$E_y = \frac{B_z V_x}{\rho L_x ne}$$

A parallel-plate capacitor with a dielectric slab (only half part):

$$E_y = \frac{q}{2\kappa\varepsilon_0 L_x L_y} = \frac{B_z V_x}{\rho L_x ne}$$

where q is the number of carriers that set up the Hall (electric) field.

The carrier density of this kind of electrons is

$$n_{\mathrm{Hall}} = \frac{q}{eL_x L_y L_z} = \frac{2\kappa\varepsilon_0 B_z V_x}{e^2 L_z L_x \rho n}$$

The ratio of the number of carriers that set up the Hall field to the total number of carriers in the metal (semiconductor):

$$\frac{n_{\mathrm{Hall}}}{n} = \frac{2\kappa\varepsilon_0 B_z V_x}{e^2 L_z L_x \rho n^2}$$

For copper, $\kappa = 1$, $n = 8.5 \times 10^{28}$ m^{-3}, and $\rho = 1.7 \times 10^{-8}$ $\Omega{\cdot}$m,

$$\frac{n_{\mathrm{Hall}}}{n} \sim 10^{-20}.$$

For the semiconductor, $\kappa = 12$, $n = 10^{22}$ m^{-3}, and $\rho = 10^{-2}$ $\Omega{\cdot}$m

$$\frac{n_{\mathrm{Hall}}}{n} \sim 10^{-11}.$$