General Physics I

Lecture 21: Relativistic Energy and Momentum

Outline

- **Relativistic velocity, momentum, and energy**
- **The mass-energy equivalence**
- **Nuclear energy**
- **Twin paradox**

Velocity Transformation

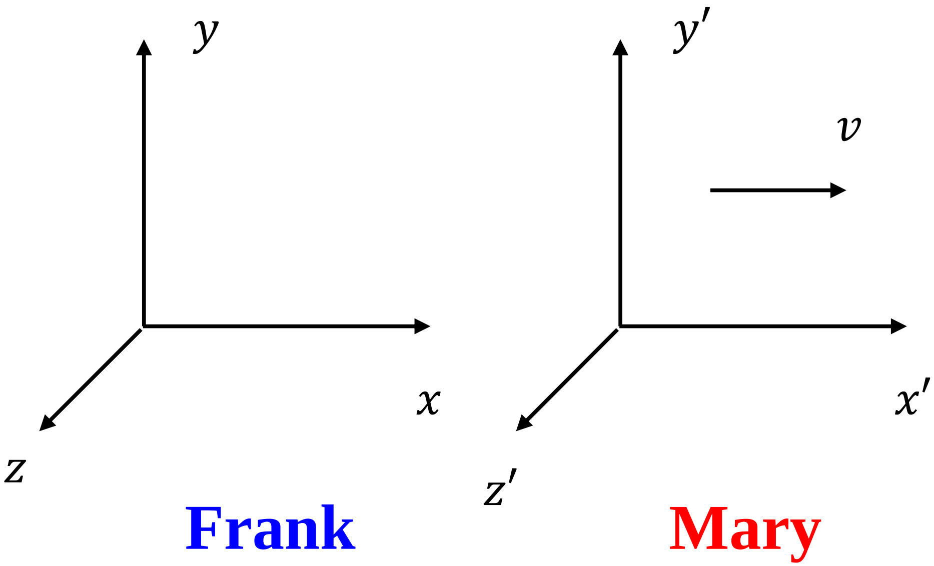
The complete transformation

1) $t' = \gamma(t - vx/c^2)$

2) $x' = \gamma(x - vt)$

3) $y' = y$

4) $z' = z$



$$dx' = \boldsymbol{\gamma}(dx - v\,dt)$$

$$dt' = \boldsymbol{\gamma}\left(dt - \frac{v}{c^2}\,dx\right)$$

Therefore,

$$u'_x = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \dfrac{v}{c^2}\,dx} = \frac{\dfrac{dx}{dt} - v}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}$$

Recognize the velocity addition rule?

Velocity Transformation

The complete transformation

1) $t' = \gamma(t - vx/c^2)$

2) $x' = \gamma(x - vt)$

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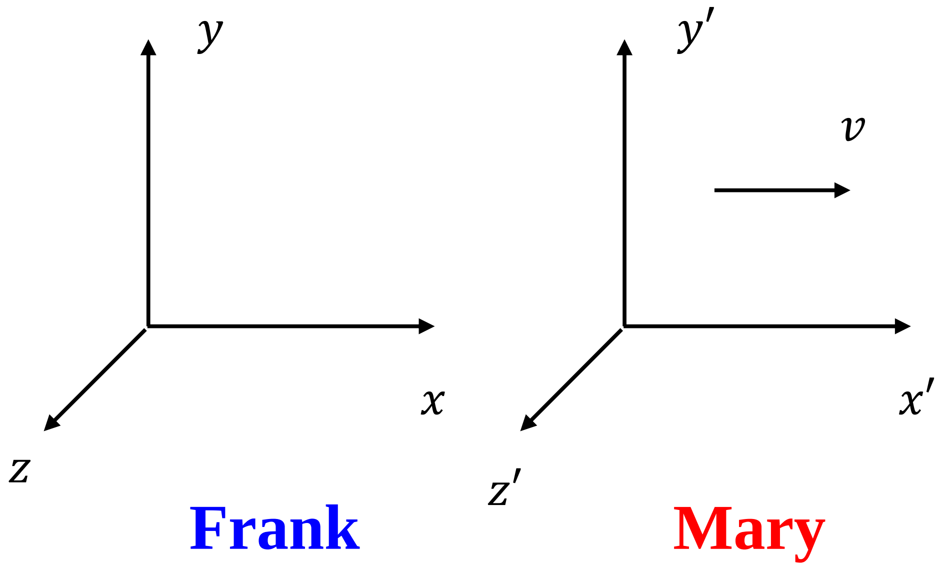


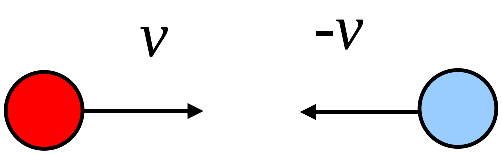
$$u'_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \dfrac{u_x v}{c^2}\right)}$$

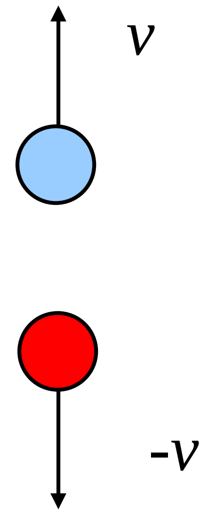
$$u'_z = \frac{u_z}{\gamma\left(1 - \dfrac{u_x v}{c^2}\right)}$$

Linear Momentum

- **Linear momentum *p* must be conserved in all collisions.**

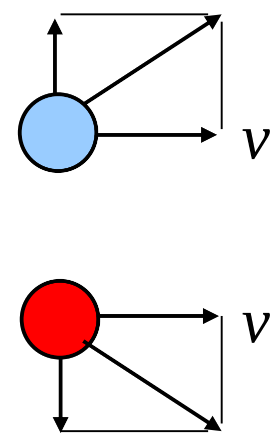


$$\frac{2v}{1\ +\ v^2/c^2}$$



$$p_{xi}\ =\ 0$$

$$p_{xf}\ =\ 0$$



$$p'_{xi}\ =\ \frac{2mv}{1\ +\ v^2/c^2}$$

$$p'_{xf}\ =\ 2mv$$

Linear Momentum

- **Linear momentum p must be conserved in all collisions.**
- **The relativistic value calculated for p must approach the classical value *m*u as u approaches zero.**

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma m\mathbf{u}$$

Please verify the momentum conservation in the previous example of collision.

Newton's second law becomes $\mathbf{F} \equiv \frac{d\mathbf{p}}{dt}$

Relativistic Energy

- We generalize energy from the work-kinetic energy theorem.

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx$$

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m(du/dt)}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}$$

We assume that the particle is accelerated from rest to some final speed *u*.

$$\Rightarrow \quad W = \int_0^t \frac{m(du/dt)\,u\,dt}{\left(1-\frac{u^2}{c^2}\right)^{3/2}} = m\int_0^u \frac{u}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}\,du$$

Relativistic Total Energy

- **Evaluating the integral**

$$W = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2$$

- **We can define**

Rest energy $E_0 = mc^2$

Total energy $E = \gamma mc^2$

Another Invariant

Energy $$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Momentum $$p = \gamma mu = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$E^2 - p^2c^2 = m^2c^4 \qquad c^2t^2 - x^2 = s^2$$ (Remember?)

Likewise, m^2 is an *invariant*, which is independent of the frame of reference. P = (E/c, p_x, p_y, p_z) is often called the *energy-momentum 4-vector*, or the *4-momentum*.

Relativistic Kinetic Energy

- **We can define the relativistic kinetic energy** K

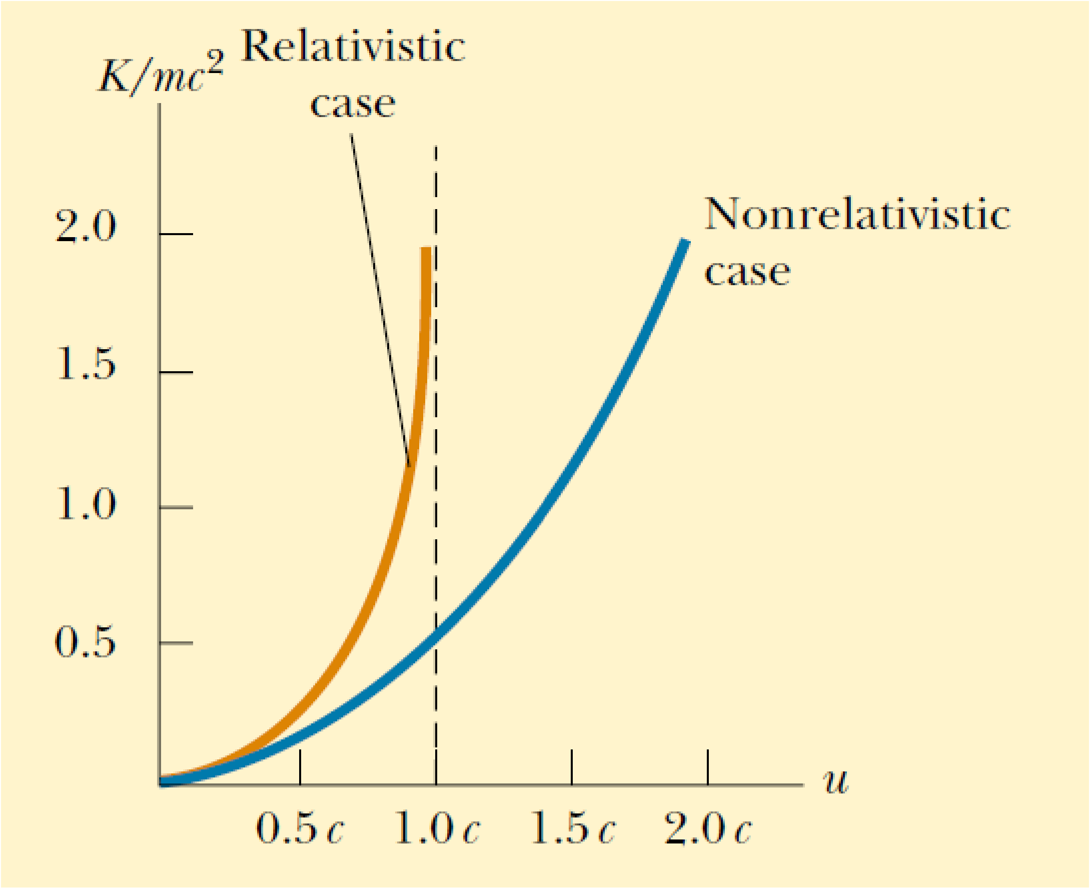
$$K = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2 = \gamma mc^2 - mc^2$$

- **At low** u/c**,**

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \left(1-\frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$$

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) - mc^2 = \frac{1}{2}mu^2$$

A Comparison



Mass-Energy Equivalence

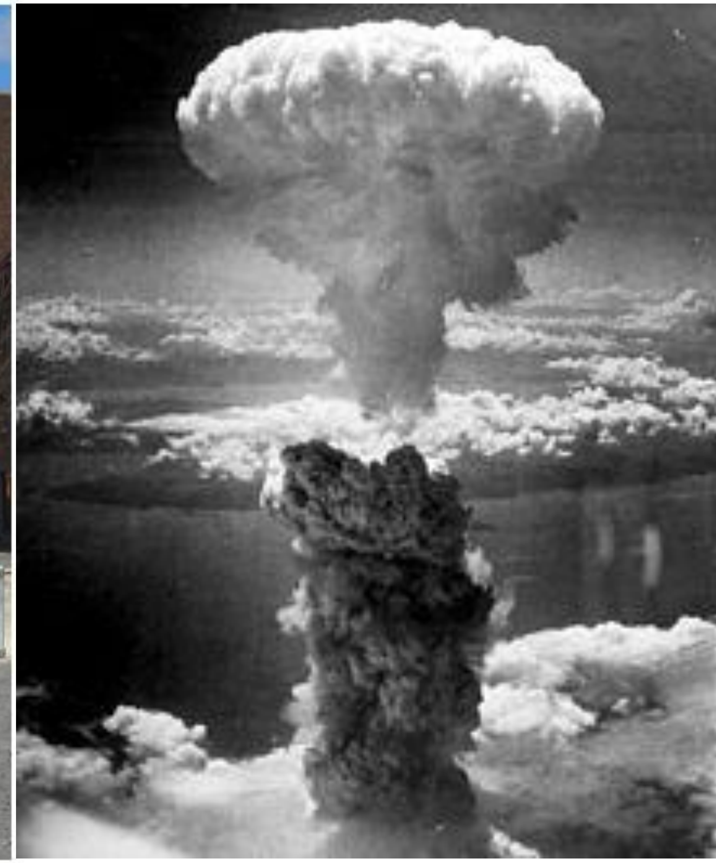
- **A small mass corresponds to an enormous amount of energy. This is a concept fundamental to nuclear and elementary-particle physics.**

$$E = \gamma mc^2 = K + mc^2 = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Even when u = 0, $E_0 = mc^2$

which is enormous, ~ 10^{17} J for m = 1 kg. How to release it?

The Most Famous Equation

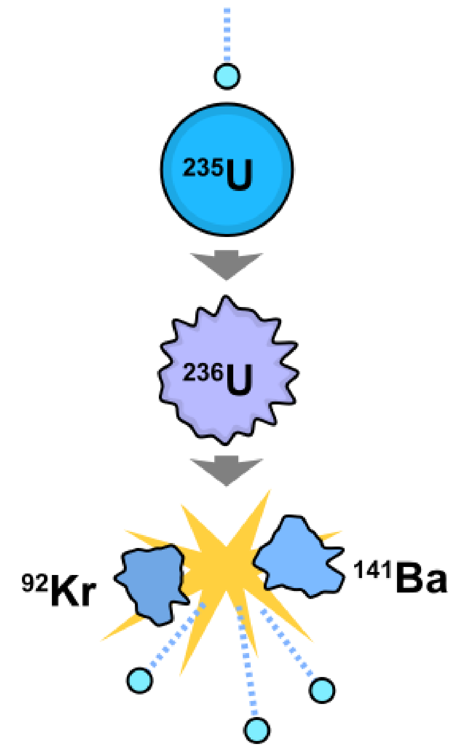


The release of atomic energy has not created a new problem.

– Albert Einstein on *Atomic War or Peace*

Fission of Uranium

- **How much energy is released by the fission of uranium-235 to form barium-142 and krypton-91? The particle masses are**
  - **^{235}U (235.0439 u)**
  - **^{142}Ba (141.9164 u)**
  - **^{91}Kr (90.9234 u)**
  - **^{1}n (1.00866 u)**
  - $^{1}n + {}^{235}U \rightarrow {}^{142}Ba + {}^{91}Kr + 3\,{}^{1}n$



The discovery of nuclear fission by Otto Hahn and Fritz Strassmann opened up a new era in human history. It seems to me that what makes the science behind this discovery so remarkable is that it was achieved by purely chemical means.

– Lise Meitner

Fission of Uranium

1.Determine the change in mass for one U-235 atom:

$$\Delta m = -0.1868\, u$$

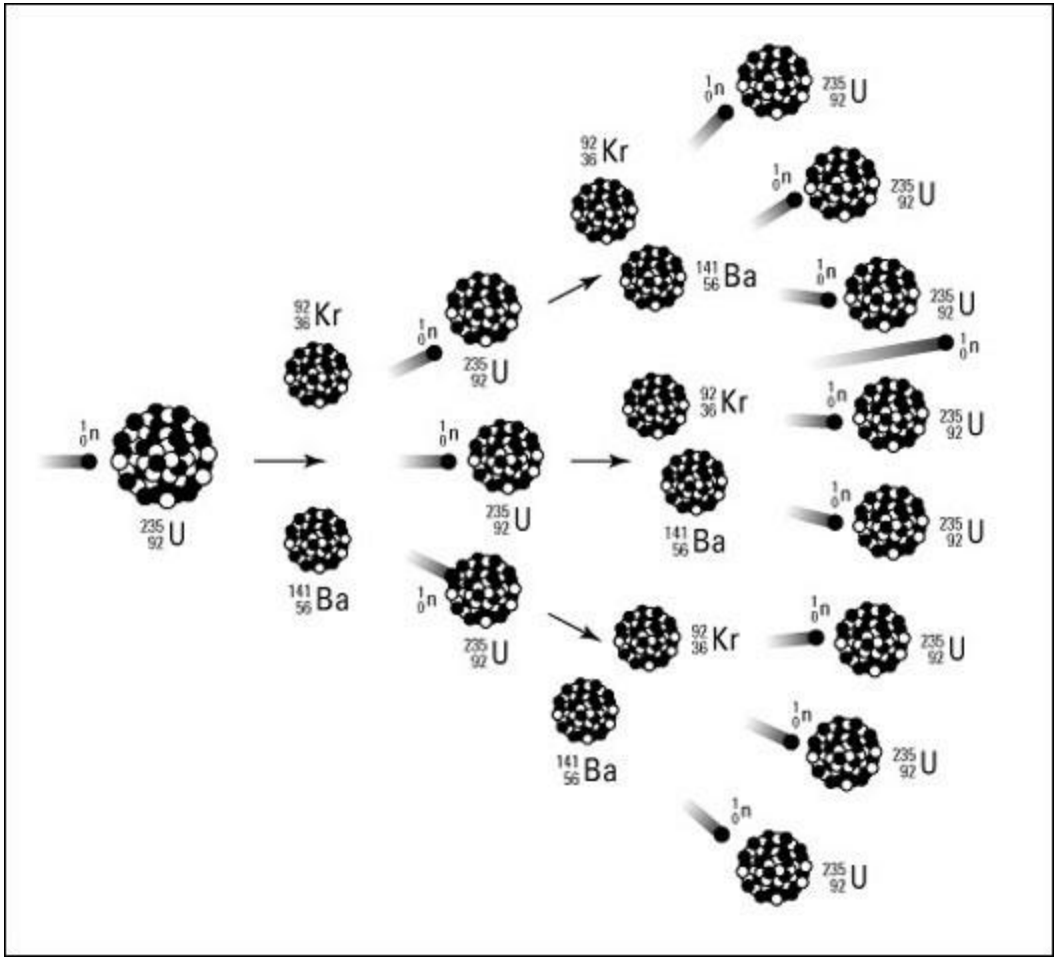
2.This means that for each mole of U-235 atoms (235.0439 g), the corresponding energy by Einstein's equation is

$$E = \Delta mc^2 = 1.681 \times 10^{13}\, J$$
$$= 4.670 \times 10^6\, kWh$$

- (Note that 1 kWh ≈ 0.5 RMB)

The experimental apparatus with which Otto Hahn and Fritz Strassmann discovered nuclear fission in 1938.

Chain Reaction

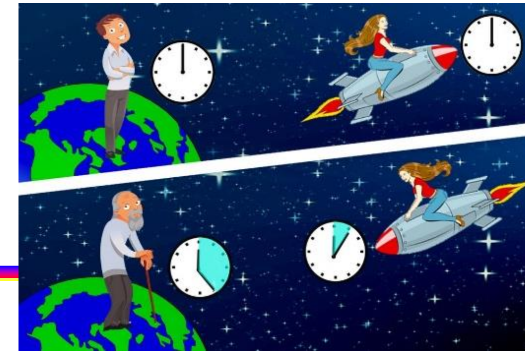


- $^{1}n + {}^{235}U \rightarrow {}^{142}Ba + {}^{91}Kr + 3\;{}^{1}n$

Otto Hahn
Farmhall
1945

"...he had contemplated suicide, when he first recognized the possible military use of fission; now, with the blame of its realization drawn squarely upon his shoulders, suicide again seemed a way to escape his desolation.... Never has social responsibility hit a scientist with such impact.

Twin Paradox



- **Consider Twin Mary (astronaut) and Frank (stockbroker). At age of 30, Mary leaves on a mission to Alpha Centauri B, the nearest planet about the size of Earth outside of our solar system, 4 light years away. Mary will travel at a high speed to the planet and return.**

Mary's biological clock will tick more slowly during her trip, so she will come back younger.

Frank will appear to be moving rapidly with respect to my system. Frank will be younger when I return.

- **This is the twin paradox. Who will be younger?**

Calculation on the Wall Street

- **Assume Mary's spaceship blasts off and quickly reaches a coasting speed of $v = 0.8c$.**
- **Mary will travel 5 years to Alpha Centauri B, and another 5 years back. So Frank will be 30 + 5 + 5 = 40 years old.**
- **Mary's clock is ticking slower and each leg of the trip takes only**

$$5\sqrt{1 - 0.8^2} = 3\,years$$

- **Mary will return at the age of 30 + 3 + 3 = 36.**

Who Is Right?

- Mary will experience acceleration, turning around, and deacceleration during her trip. Her system is, therefore, not an inertial frame.
- There is no paradox at all.
- But can they check their time lapse by sending radio signals during the years?
- Yes. But the frequency will change due to the Doppler effect. Note that this is not time dilation, but how one perceives the frequency of the signals.

Classical Doppler Effect

- **Finally, if both source and observer are in motion, we find the following general relationship for the observed frequency:**

$$f' = \frac{v + v_O}{v - v_S} f$$

The word toward is associated with an increase in observed frequency. The words away from are associated with a decrease in observed frequency.

Analyze Moving Source

- For observer A, the wavelength is shortened to

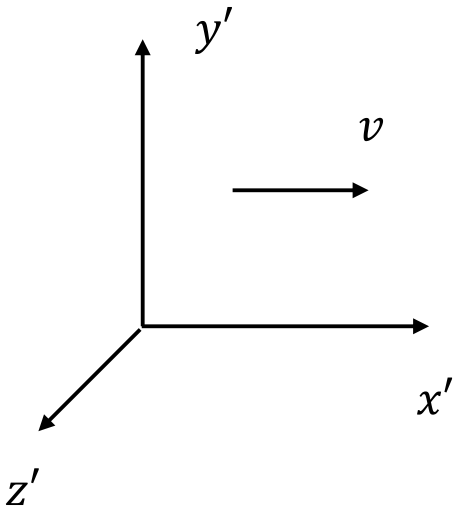
$$\lambda' = \lambda - \Delta\lambda = \lambda - \frac{v_S}{f}$$

- The frequency heard by observer A is

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda - \frac{v_S}{f}} = \left(\frac{1}{1 - \frac{v_S}{v}}\right) f$$

- For observer B, simply use a negative v_S.

Relativistic Doppler Effect

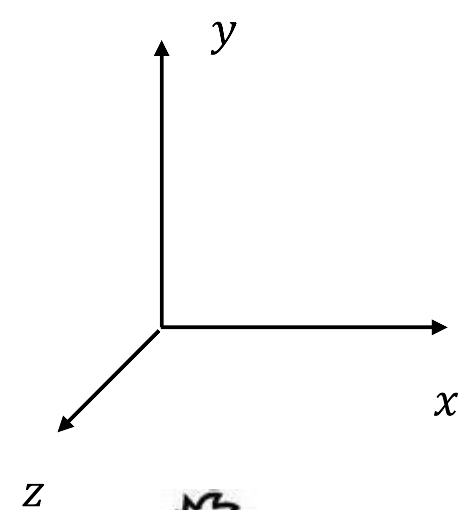


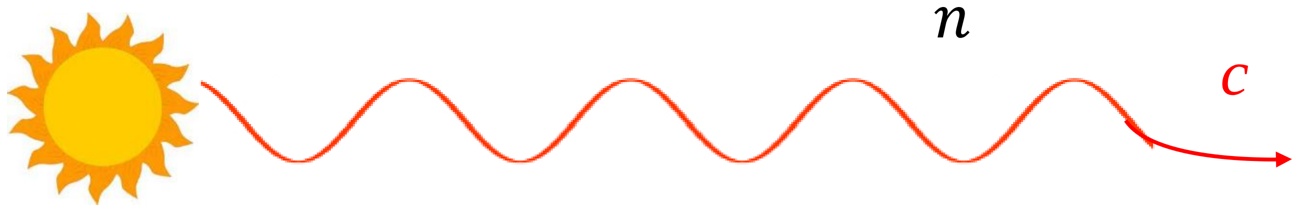
Assume during T measured by Frank, the source emits n waves. The length of the wave train

$$cT - vT$$

The wave length

$$\lambda = \frac{cT - vT}{n}$$





According to the source

$$n = f_0 T' = f_0(sT)$$

$$f = \frac{c}{\lambda} = \frac{s}{1 - v/c} f_0$$

$$= \sqrt{\frac{1 + v/c}{1 - v/c}} f_0$$

Comments

- **When the source and receiver are receding from each other, we can simply replace v by $-v$.**
- **When the source is approaching, the (visible) wavelength is shifted toward shorter wavelengths (blueshift). When the source is receding, the wavelength is shifted toward longer wavelengths (redshift).**
- **The result is also valid when the source is fixed and the receiver approaches it with speed v.**
- **For small v/c, the result conforms with the classical Doppler effect.**

Radio Communication

- **During the space trip to Alpha Centauri B, Mary will send radio signals to Frank at annual intervals. Frank, likewise, will send signals annually.**
- **The frequency of the signals ν_0 = 1 signal/year from the source.**
- **The receiver, moving away at a relative speed $v = 0.8c$, will read the frequency**

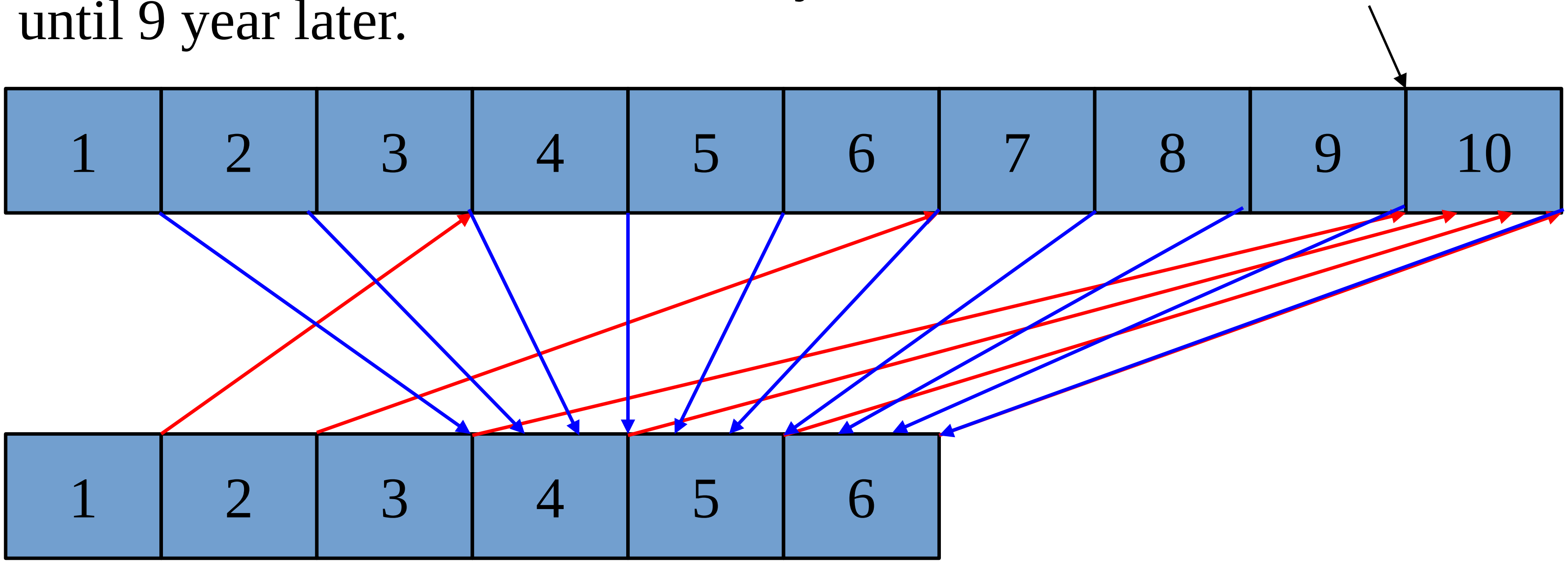
$$\nu = \frac{\sqrt{1 - 0.8}}{\sqrt{1 + 0.8}}\nu_0 = \frac{\nu_0}{3} \qquad \text{(1 signal per 3 years)}$$

- **On the way back, the frequency is $3\nu_0$, or 1 signal every four month.**

Communication Diagram

There is no way Frank knows that Mary has turned around until 9 year later.

The farthest signal takes 4 year to arrive at the earth.



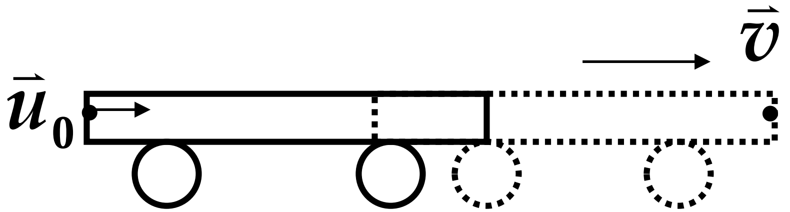
Frank receives 6 messages, so he knows Mary has aged 6 years.

Mary receives 10 messages, so she knows Frank has aged 10 years.

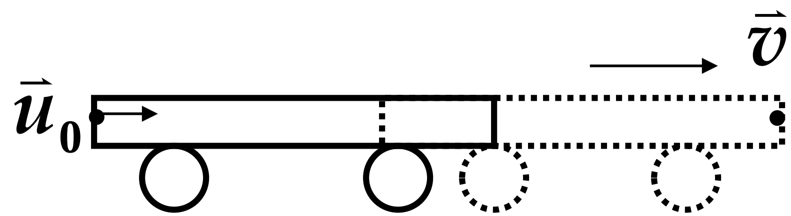
The Happy Ending

- When Mary returns home, Mary will have been paid 10-year salary that will take her only 6 years!
- Frank has invested her 10 years of salary, making her a rich woman at the age of 36.

Example: As shown in the figure, a spaceship of rest length $\boldsymbol{L_0}$ is moving with respect to the Earth at speed $\boldsymbol{v}$, an electron is emitted forward from the back-wall of the spaceship with a speed $\boldsymbol{u_0}$ (respect to the spaceship). Measured from the Earth, what is the distance and the time interval of the electron traveling from the back-wall to the front-wall of the space ship? And the speed of the electron ? (suppose $v = u_0 = \sqrt{3}c/2$)



Solution: **we assumed the Earth as the *S* frame, the spaceship as the *S'* frame**



$$\Delta x' = L_0 \qquad \Delta t' = \frac{L_0}{u_0}$$

$$\Delta x = \frac{\Delta x' + v\Delta t'}{\sqrt{1-\dfrac{v^2}{c^2}}} = 4L_0$$

$$\Delta t = \frac{\Delta t' + \dfrac{v\Delta x'}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{7\sqrt{3}L_0}{3c}$$

Example: Two identical particles of rest mass ***m***, in the laboratory reference frame ,particle A is rest, particle B moves with speed 0.6*c* toward A, they collide and stick together. Find the speed and the rest mass of the resulting particle in the laboratory reference frame.

Solution: in the collision , momentum, energy and mass are conserved

$$p = \frac{m}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} \times 0.6c = 0.75mc$$

$$E = mc^2 + \frac{m}{\sqrt{1-\frac{(0.6c)^2}{c^2}}}c^2 = 2.25mc^2$$

$$m_{\text{rel}} = m + \frac{m}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} = 2.25m$$

$$u = \frac{p}{m_{\text{rel}}} = \frac{0.75mc}{2.25m} = \frac{1}{3}c = 0.33c$$

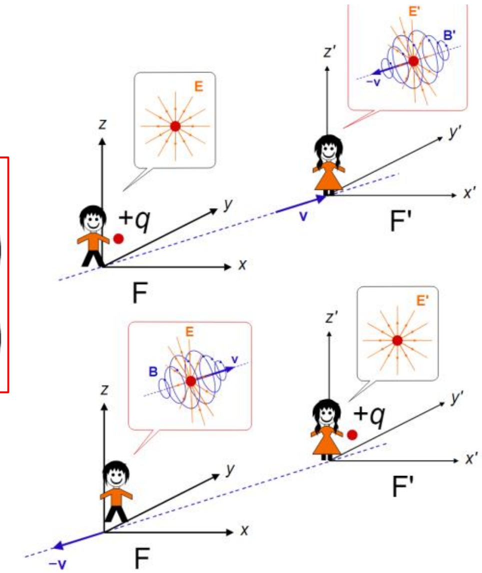
$$m' = m_{\text{rel}}\sqrt{1-\left(\frac{u}{c}\right)^2} = 2.12m$$

Electromagnetic Field of a Moving Charge*

Lorentz Transformation: (φ/c, A_x, A_y, A_z)

$$\varphi' = \gamma\left(\varphi - vA_{\parallel}\right)$$
$$A'_{\parallel} = \gamma\left(A_{\parallel} - \frac{v\varphi}{c^2}\right)$$
$$A'_{\perp} = A_{\perp}$$

$$\boxed{\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma\left(E_y - vB_z\right) & B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ E'_z &= \gamma\left(E_z + vB_y\right) & B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned}}$$



At rest frame

$$\tilde{\varphi} = \frac{e}{4\pi\varepsilon_0\tilde{r}}, \tilde{\vec{A}} = 0$$

where

$$\tilde{r} = c(\tilde{t} - \tilde{t}')$$
$$= \gamma[c(t - t') - \vec{\beta}\cdot(\vec{x} - \vec{x}')]$$
$$= \gamma(r - \vec{\beta}\cdot\vec{r})$$

At Lab frame

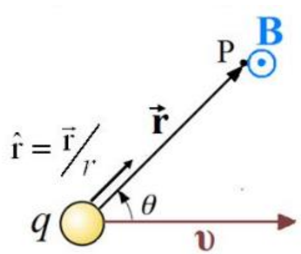
$$\varphi = \gamma\tilde{\varphi} = \gamma\frac{e}{4\pi\varepsilon_0\tilde{r}}$$

$$\vec{A} = \gamma\frac{\vec{v}}{c^2}\tilde{\varphi} = \gamma\frac{e\vec{v}}{4\pi\varepsilon_0 c^2\tilde{r}}$$

$$\boxed{\begin{aligned} \varphi &= \frac{e}{4\pi\varepsilon_0(r - \vec{\beta}\cdot\vec{r})} \\ \vec{A} &= \frac{e\vec{\beta}}{4\pi\varepsilon_0 c(r - \vec{\beta}\cdot\vec{r})} \end{aligned}}$$

Lienard-Wiechert potential

For speed v<<c, Biot–Savart law



$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{q\,\vec{\upsilon}\times\hat{r}}{r^2}}$$ $\quad \hat{r} = \vec{r}/r$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I\,d\boldsymbol{\ell}\times\hat{\mathbf{r}}'}{|\mathbf{r}'|^2}$$

Summary on Relativity

- **The two basic postulates of the special theory of relativity are**
  - **The laws of physics must be the same in all inertial reference frames.**
  - **The speed of light in vacuum has the same value**

$$c = 3.00 \times 10^{8}\ m/s$$

  in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

Summary on Relativity

- **To satisfy the postulates of special relativity, the Galilean transformation equations must be replaced by the Lorentz transformation equations:**

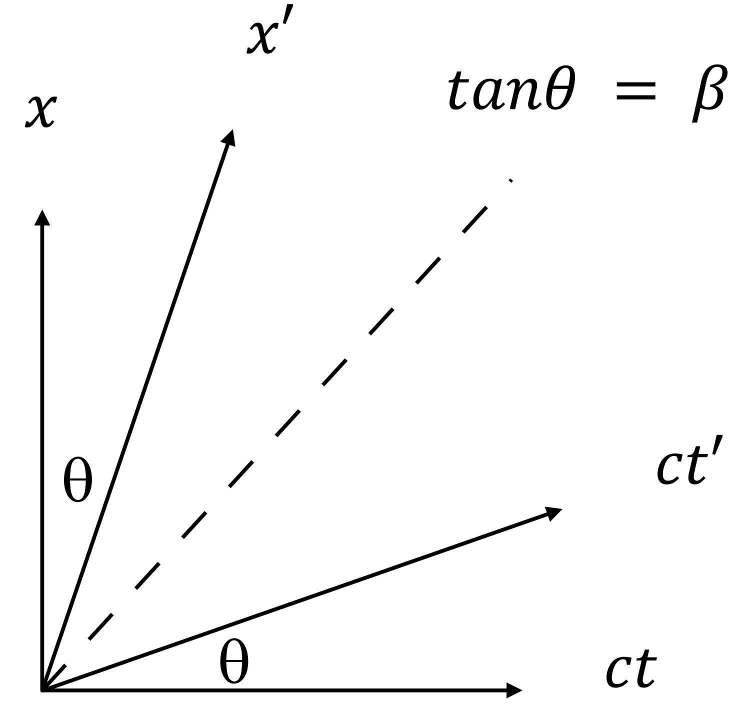
$$t' = \gamma(t - vx/c^2)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$



Summary on Relativity

- **Three consequences of the special theory of relativity are**
  - **Events that are simultaneous for one observer are not simultaneous for another observer who is in motion relative to the first.**
  - **Clocks in motion relative to an observer appear to be slowed down by a factor** $\gamma = (1 - v^2/c^2)^{-1/2}$**. This phenomenon is known as time dilation.**
  - **The length of objects in motion appears to be contracted in the direction of motion by a factor** $1/\gamma = (1 - v^2/c^2)^{1/2}$**. This phenomenon is known as length contraction.**

Summary on Relativity

- **The relativistic form of the velocity transformation equation is**

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad u'_z = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

u_x is the speed of an object as measured in the K frame and

u'_x is its speed measured in the K' frame.

Summary on Relativity

- **The relativistic expression for the linear momentum of a particle moving with a velocity u is**

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma m\mathbf{u}$$

Not relative *v* of *K* & *K'*

- **The relativistic expression for the kinetic energy of a particle is**

$$K = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2 = \gamma mc^2 - mc^2$$

Mass is an invariant