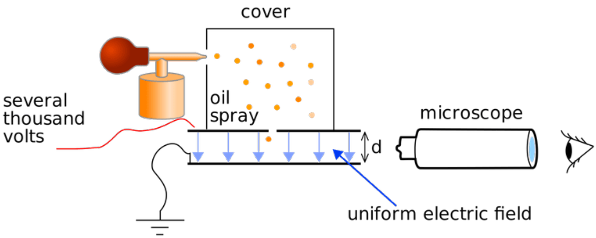
General Physics II

Solution #1

2020/09/29

P1-1. In Milliken's experimental apparatus, oil droplets with different charge q are subject to gravitational force, buoyancy force, and drag force $F_d = 6\pi r \eta v_1$, where r is the droplet radius, η is the viscosity of air and v_1 is the terminal velocity of the droplet. When a uniform electric field E is turned on, the droplet is moving up due to the additional electric force $F_q = qE$ with a terminal speed v_2.



(a) Assume the droplet is spherical. Show that the radius of the droplet is

$$r = \sqrt{\frac{9\eta v_1}{2g(\rho - \rho_{\mathrm{air}})}},$$

where the density of the oil droplet is ρ and the density of air is ρ_{air}. g is the gravitational acceleration.

(b) Calculate the charge of the droplet. Milliken repeated this measurement for a large number of observed droplets and found the charge to be integer multiples of a single number, the fundamental electric charge. Therefore, the experiment confirmed that charge is quantized.

Solution: (a) When the droplet moves down with the terminal speed v_1, the total force on the droplet is zero. That is,

$$\rho V g - \rho_{\mathrm{air}} V g = 6\pi r \eta v_1,$$

where $V = 4\pi r^3/3$ is the droplet volume. Therefore,

$$r = \sqrt{9\eta v_1/2g(\rho - \rho_{\mathrm{air}})}.$$

(b) When the droplet moves up with the terminal speed v_2,

$$qE = \rho V g - \rho_{\mathrm{air}} V g + 6\pi r \eta v_2 = 6\pi r \eta (v_1 - v_2).$$

Therefore,

$$q = \frac{6\pi r \eta}{E}(v_1 - v_2).$$

P1-2. Show that the components of $\vec{E}$ due to a dipole are given, at a distant point P in the xz plane, by

$$E_x = \frac{1}{4\pi\varepsilon_0} \frac{3pxz}{(x^2 + z^2)^{5/2}}$$

and

$$E_z = \frac{1}{4\pi\varepsilon_0} \frac{p(2z^2 - x^2)}{(x^2 + z^2)^{5/2}},$$

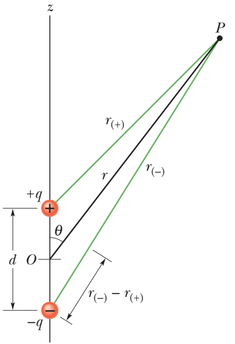
where x and z are coordinates of point P.

Solution: As we have shown in class, the electric potential of the dipole is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{pz}{r^3},$$

where $\cos\theta = z/r$. The electric field components can be obtained as

$$E_x = -\frac{\partial V}{\partial x}; \quad E_z = -\frac{\partial V}{\partial z}.$$



Notice the following identity, for $w = x, y, z$,

$$\frac{\partial f(r)}{\partial w} = f'(r)\frac{\partial r}{\partial w} = f'(r)\frac{w}{r}.$$

Thus, we find in the xz plane ($y = 0$ or $r^2 = x^2 + z^2$)

$$E_x = -\frac{1}{4\pi\varepsilon_0}\frac{(-3pz)}{r^4}\frac{x}{r} = \frac{1}{4\pi\varepsilon_0}\frac{3pxz}{(x^2+z^2)^{5/2}},$$

and

$$E_z = -\frac{1}{4\pi\varepsilon_0}\left[\frac{p}{r^3} + \frac{(-3pz)}{r^4}\frac{z}{r}\right] = \frac{1}{4\pi\varepsilon_0}\frac{p(2z^2-x^2)}{(x^2+z^2)^{5/2}}.$$

P1-3. Suppose N electrons can be placed in either of two configurations. In configuration 1, they are all placed on the circumference of a narrow ring of radius R and are uniformly distributed so that the distance between adjacent electrons is the same everywhere. In configuration 2, $N-1$ electrons are uniformly distributed on the ring and one electron is placed in the center of the ring.

(a) What is the smallest value of N for which the second configuration is less energetic than the first?

(b) For that value of N, consider any one circumference electron—call it e_0. How many other circumference electrons are closer to e_0 than the central electron is?

Solution: We note that for two points on a circle, separated by angle θ (in radians), the directline distance between them is $r = 2Rsin(\theta/2)$. Using this fact, distinguishing between the cases where $N =$ odd and $N =$ even, and counting the pair-wise interactions very carefully, we arrive at the following results for the total potential energies. We use $k = 1/4\pi\varepsilon_0$. For configuration 1 (where all N electrons are on the circle),

$$U_{1,N=even} = \frac{Nke^2}{2R}(\sum_{j=1}^{\frac{N}{2}-1} \frac{1}{sin(j\theta/2)} + \frac{1}{2})$$

$$U_{1,N=odd} = \frac{Nke^2}{2R}(\sum_{j=1}^{\frac{N-1}{2}} \frac{1}{sin(j\theta/2)})$$

where $\theta = \frac{2\pi}{N}$. For configuration 2, we find

$$U_{2,N=even} = \frac{(N-1)ke^2}{2R}(\sum_{j=1}^{\frac{N}{2}-1} \frac{1}{sin(j\theta'/2)} + 2)$$

$$U_{2,N=odd} = \frac{(N-1)ke^2}{2R}(\sum_{j=1}^{\frac{N-3}{2}} \frac{1}{sin(j\theta'/2)}) + \frac{5}{2}$$

where $\theta' = \frac{2\pi}{N-1}$. The results are all of the form $U_{1or2}\ \frac{ke^2}{2R}\times$ a pure number.

In our table below we have the results for those "pure numbers" as they depend on N and on which configuration we are considering.

N	4	5	6	7	8	9	10	11	12	13	14	15
U_1	3.83	6.88	10.96	16.13	22.44	29.92	38.62	48.58	59.81	72.35	86.22	101.5
U_2	4.73	7.83	11.88	16.96	23.13	30.44	39.92	48.62	59.58	71.81	85.35	100.2

The values listed in the U rows are the potential energies divided by $ke^2/2R$. We see that the potential energy for configuration 2 is greater than that for configuration 1 for $N < 12$, but for $N \geq 12$ it is configuration 1 that has the greatest potential energy.
(a) $N = 12$ is the smallest value such that $U_2 < U_1$.
(b) For $N = 12$, configuration 2 consists of 11 electrons distributed at equal distances around the circle, and one electron at the center. A specific electron e_0 on the circle is R distance from the one in the center, and is

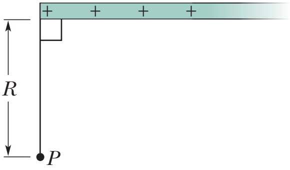
$$r = 2Rsin(\frac{\pi}{11}) \approx 0.56R$$

distance away from its nearest neighbors on the circle (of which there are two — one on each side). Beyond the nearest neighbors, the next nearest electron on the circle is

$$r = 2Rsin(\frac{2\pi}{11}) \approx 1.1R$$

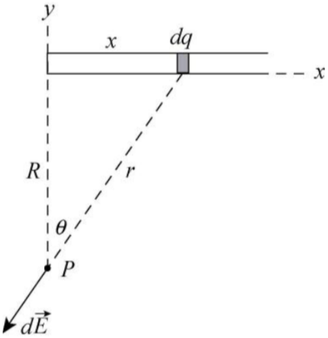
distance away from e_0. Thus, we see that there are only two electrons closer to e_0 than the one in the center.

P1-4. A semi-infinite nonconducting rod has uniform linear charge density λ. Show that the electric field $\vec{E}_p$ at point P makes an angle of 45° with the rod and that this result is independent of the distance R.



Solution: Consider an infinitesimal section of the rod of length dx, a distance x from the left end, as shown in the following diagram. It contains charge $dq = \lambda dx$ and is a distance r from P. The magnitude of the field it produces at P is given by

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{r^2}$$



The x and the y components are

$$dE_x = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{r^2}sin\theta$$

and

$$dE_y = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{r^2}cos\theta$$

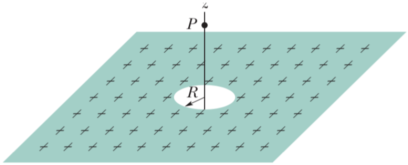
We use θ as the variable of integration and substitute $r = R/cos\theta$, $x = Rtan\theta$ and $dx = (R/cos^2\theta)d\theta$. The limits of integration are 0 and $\pi/2$ rad. Thus,

$$\begin{aligned} E_x &= -\frac{\lambda}{4\pi\varepsilon_0 R}\int_0^{\pi/2} sin\theta d\theta = \frac{\lambda}{4\pi\varepsilon_0 R}cos\theta\Big|_0^{\pi/2} \\ &= -\frac{\lambda}{4\pi\varepsilon_0 R} \end{aligned}$$

$$
\begin{aligned}
E_y &= -\frac{\lambda}{4\pi\varepsilon_0 R}\int_0^{\pi/2} cos\theta d\theta = -\frac{\lambda}{4\pi\varepsilon_0 R} sin\theta\Big|_0^{\pi/2} \\
&= -\frac{\lambda}{4\pi\varepsilon_0 R}
\end{aligned}
$$

We notice that $E_x = E_y$ no matter what the value of R. Thus, $\vec{E}$ makes an angle of 45° with the rod for all values of R.

P1-5. A small circular hole of radius $R = 1.80$ cm has been cut in the middle of an infinite, flat, nonconducting surface that has uniform charge density $\sigma = 4.50\ \mathrm{pC/m^2}$. A z axis, with its origin at the hole's center, is perpendicular to the surface. In unit vector notation, what is the electric field at point P at $z = 2.56$ cm?



Solution: The charge distribution in this problem is equivalent to that of an infinite sheet of charge with surface charge density $\sigma = 4.50 \times 10^{-12}$ C/m^2 plus a small circular pad of radius $R = 1.80$ cm located at the middle of the sheet with charge density $-\sigma$. We denote the electric fields produced by the sheet and the pad with subscripts 1 and 2, respectively.
As we have shown in class, in a uniform charged disk,

$$E_z = \frac{\sigma}{2\varepsilon_0}[1 - \frac{z}{(z^2 + R^2)^{1/2}}]$$

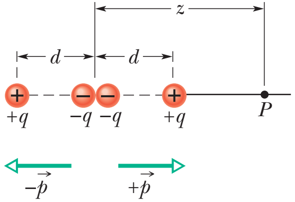
Therefore, the net electric field $\vec{E}$ is then

$$\begin{aligned}\vec{E} &= \vec{E}_1 + \vec{E}_2 \\ &= (\frac{\sigma}{2\varepsilon_0})\hat{k} + \frac{-\sigma}{2\varepsilon_0}(1 - \frac{z}{\sqrt{z^2 + R^2}})\hat{k} \\ &= \frac{\sigma z}{2\varepsilon_0\sqrt{z^2 + R^2}}\hat{k} \\ &= \frac{(4.50 \times 10^{-12})(2.56 \times 10^{-2})}{2(8.85 \times 10^{-12})\sqrt{(2.56 \times 10^{-2})^2 + (1.80 \times 10^{-2})^2}}\hat{k} \\ &= (0.208\ \mathrm{N/C})\hat{k}\end{aligned}$$

P1-6. A type of electric quadrupole consists of two dipoles with dipole moments that are equal in magnitude but opposite in direction. Show that the value of E on the axis of the quadrupole for a point P at a distance z from its center (assume $z \gg d$) is given by

$$E = \frac{3Q}{4\pi\varepsilon_0 z^4},$$

in which $Q = 2qd^2$ is known as the *quadrupole moment* of the charge distribution.



Solution: Consider the point P on the axis, a distance z to the right of the quadrupole center and take a rightward pointing field to be positive. Then the field produced by the right dipole of the pair is given by $qd/2\pi\varepsilon_0(z\breve{}\,d/2)^3$ while the field produced by the left dipole is $-qd/2\pi\varepsilon_0(z+d/2)^3$.
Use the binomial expansions

$$(z-d/2)^{-3} \approx z^{-3} - 3z^{-4}(-d/2)$$

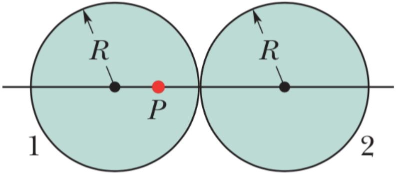
$$(z+d/2)^{-3} \approx z^{-3} - 3z^{-4}(d/2)$$

we obtain

$$
\begin{aligned}
E &= \frac{qd}{2\pi\varepsilon_0(z-d/2)^3} - \frac{qd}{2\pi\varepsilon_0(z+d/2)^3} \\
&\approx \frac{qd}{2\pi\varepsilon_0}[\frac{1}{z^3} + \frac{3d}{2z^4} - \frac{1}{z^3} + \frac{3d}{2z^4}] \\
&= \frac{6qd^2}{4\pi\varepsilon_0 z^4}
\end{aligned}
$$

Since the quadrupole moment is $Q = 2qd^2$, we have $E = \frac{3Q}{4\pi\varepsilon_0 z^4}$.

P1-7. In cross section, two solid spheres with uniformly distributed charge throughout their volumes. Each has radius R. Point P lies on a line connecting the centers of the spheres, at radial distance $R/2.00$ from the center of sphere 1. If the net electric field at point P is zero, what is the ratio q_2/q_1 of the total charges?



Solution: As we have shown in class, if the full charge q enclosed within radius R is uniform, for $r < R$, we have

$$\vec{E} = (\frac{q}{4\pi\varepsilon_0 R^3})\vec{r}$$

Therefore, we have

$$E_1 = \frac{|q_1|}{4\pi\varepsilon_0 R^3} r_1 = \frac{|q_1|}{4\pi\varepsilon_0 R^3}(\frac{R}{2}) = \frac{1}{2}\frac{|q_1|}{4\pi\varepsilon_0 R^2}$$

Also, outside sphere 2 we have

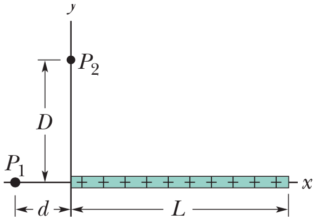
$$E_2 = \frac{|q_2|}{4\pi\varepsilon_0 r^2} = \frac{|q_2|}{4\pi\varepsilon_0 (1.50R)^2}$$

Equating these and solving for the ratio of charges, we arrive at $\frac{q_2}{q_1} = \frac{9}{8} = 1.125$

P1-8. A thin plastic rod of length $L = 10.0$ cm has a nonuniform linear charge density $\lambda = cx$, where $c = 49.9$ pC/m^2.

(a) With $V = 0$ at infinity, find the electric potential at point P_2 on the y axis at $y = D = 3.56$ cm.

(b) Find the electric field component E_y at P_2.



(c) Why cannot the field component E_x at P_2 be found using the result of (a)?

Solution:

(a) Consider an infinitesimal segment of the rod from x to $x + dx$. Its contribution to the potential at point P_2 is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda(x)dx}{\sqrt{x^2+y^2}} = \frac{1}{4\pi\varepsilon_0}\frac{cx}{\sqrt{x^2+y^2}}dx$$

Thus,

$$\begin{aligned}
V &= \int_{rod} dV_P = \frac{c}{4\pi\varepsilon_0}\int_0^L \frac{x}{\sqrt{x^2+y^2}}dx \\
&= \frac{c}{4\pi\varepsilon_0}(\sqrt{L^2+y^2} - y) \\
&= (8.99\times 10^9)(49.9\times 10^{-12}) \\
&\cdot (\sqrt{0.100^2 + 0.0356^2} - 0.0356) \\
&= 3.16\times 10^{-2}\ \mathrm{V}
\end{aligned}$$

(b) The y component of the field there is

$$
\begin{aligned}
E_y &= -\frac{\partial V_P}{\partial y} = -\frac{c}{4\pi\varepsilon_0}\frac{d}{dy}(\sqrt{L^2+y^2}-y) \\
&= \frac{c}{4\pi\varepsilon_0}(1-\frac{y}{\sqrt{L^2+y^2}}) \\
&= (8.99\times 10^9)(49.9\times 10^{-12}) \\
&\cdot (1-\frac{0.0356}{\sqrt{0.100^2+0.0356^2}}) \\
&= 0.298\ \mathrm{N/C}
\end{aligned}
$$

(c) We obtained above the value of the potential at any point P strictly on the y-axis. In order to obtain $E_x(x, y)$ we need to first calculate $V(x, y)$. That is, we must find the potential for an arbitrary point located at (x, y). Then $E_x(x, y)$ can be obtained from $E_x(x, y) = -\partial V(x, y)/\partial x$.

P1-9. An electron is constrained to the central axis of the ring of charge of radius R, with $z \ll R$. Show that the electrostatic force on the electron can cause it to oscillate through the ring center with an angular frequency

$$\omega = \sqrt{\frac{eq}{4\pi\varepsilon_0 m R^3}},$$

where q is the ring's charge and m is the electron's mass.

Solution: The electric field at a point on the axis of a uniformly charged ring, a distance z from the ring center, is given by

$$E = \frac{qz}{4\pi\varepsilon_0(z^2 + R^2)^{3/2}}$$

where q is the charge on the ring and R is the radius of the ring. For q positive, the field points upward at points above the ring and downward at points below the ring. We take the positive direction to be upward. Then, the force acting on an electron on the axis is

$$F = -\frac{eqz}{4\pi\varepsilon_0(z^2 + R^2)^{3/2}}.$$

For small amplitude oscillations $z \ll R$ and z can be neglected in the denominator. Thus,

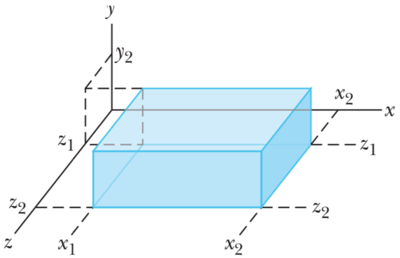
$$F = -\frac{eqz}{4\pi\varepsilon_0 R^3}.$$

The force is a restoring force: it pulls the electron toward the equilibrium point $z = 0$. Furthermore, the magnitude of the force is proportional to z, just as if the electron were attached to a spring with spring constant $k = eq/4\pi\varepsilon_0 R^3$. The electron moves in simple harmonic motion with an angular frequency given by

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{eq}{4\pi\varepsilon_0 m R^3}}$$

where m is the mass of the electron.

P1-10. A box-like Gaussian surface encloses a net charge of $+24.0\varepsilon_0$ C and lies in an electric field given by $\vec{E} = [(10.0 + 2.00x)\hat{i} - 3.00\hat{j} + bz\hat{k}]$ N/C, with x and z in meters and b a constant. The bottom face is in the xz plane; the top face is in the horizontal plane passing through $y_2 = 1.00$ m. For $x_1 = 1.00$ m, $x_2 = 4.00$ m, $z_1 = 1.00$ m, $z_2 = 3.00$ m, what is b?



Solution: The total electric flux through the cube is $\Phi = \oint \vec{E} \cdot d\vec{A}$. The net flux through the two faces parallel to the yz plane is

$$\begin{aligned}
\Phi_{yz} &= \iint [E_x(x = x_2) - E_x(x = x_1)]dydz \\
&= \int_{y_1=0}^{y_2=1} dy \int_{z_1=1}^{z_2=3} dz[10 + 2(4) - 10 - 2(1)] \\
&= 6 \int_{y_1=0}^{y_2=1} dy \int_{z_1=1}^{z_2=3} dz = 12
\end{aligned}$$

Similarly, the net flux through the two faces parallel to the xz plane is

$$\Phi_{xz} = \iint [E_y(y = y_2) - E_y(y = y_1)]dxdz$$

$$= \int_{x_1=1}^{x_2=4} dx \int_{z_1=1}^{z_2=3} dz[-3 - (-3)] = 0$$

and the net flux through the two faces parallel to the xy plane is

$$\Phi_{xy} = \iint [E_z(z = z_2) - E_z(z = z_1)] dx dy$$

$$= \int_{x_1=1}^{x_2=4} dx \int_{y_1=0}^{y_2=1} dy (3b - b) = 6b$$

Applying Gauss' law, we obtain

$$q_{enc} = \varepsilon_0 \Phi = \varepsilon_0 (\Phi_{xy} + \Phi_{xz} + \Phi_{yz})$$

$$= \varepsilon_0 (6.00b + 0 + 12.0) = 24.0\varepsilon_0$$

which implies that $b = 2.00$ N/C·m