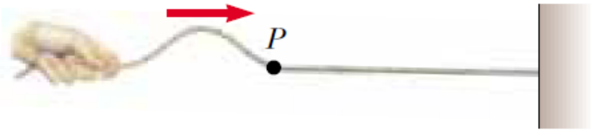
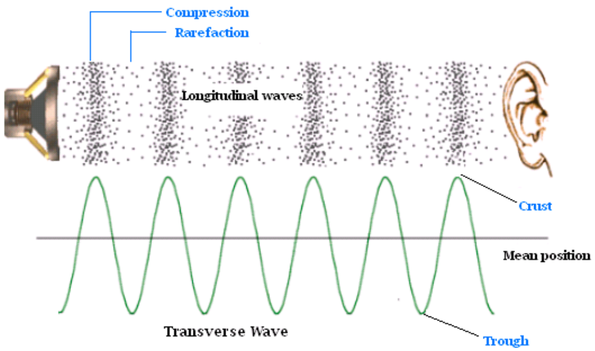
General Physics I

Lecture 14: Sinusoidal Waves

Wave Matters!







The essentially new thing here is that for the first time we consider the motion of something which is not matter, but energy propagated through matter.

- A. Einstein and L. Infeld in *The Evolution of Physics*

Outline

- Sinusoidal waves on strings
  - Linear wave equation
  - Wave forms
  - Rate of energy transfer
- Superposition and interference of sinusoidal waves
  - Beats: Interference in time
  - Standing waves and harmonics
- Non-sinusoidal wave patterns

The Linear Wave Equation

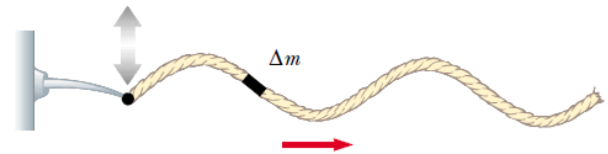
The linear wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

where $$v = a\sqrt{\frac{K}{M}}$$

The linear wave equation applies in general to various types of waves. For sound waves, *y* corresponds to displacement of air molecules from equilibrium or variations in either the pressure or the density of the gas through which the sound waves are propagating. For waves on strings, *y* represents the vertical displacement of the string. In the case of electromagnetic waves, *y* corresponds to electric or magnetic field components.

The Speed of Waves on Strings

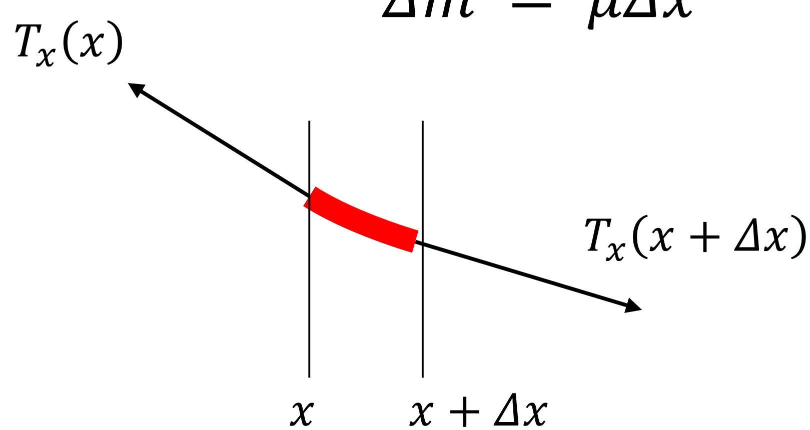


Newton's second law

$$(\mu \Delta x)\frac{\partial^2 y}{\partial t^2} = T_x(x+\Delta x)\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - T_x(x)\left.\frac{\partial y}{\partial x}\right|_{x}$$

$$T_x(x+\Delta x) = T_x(x) \approx T$$

$$\Delta m = \mu \Delta x$$



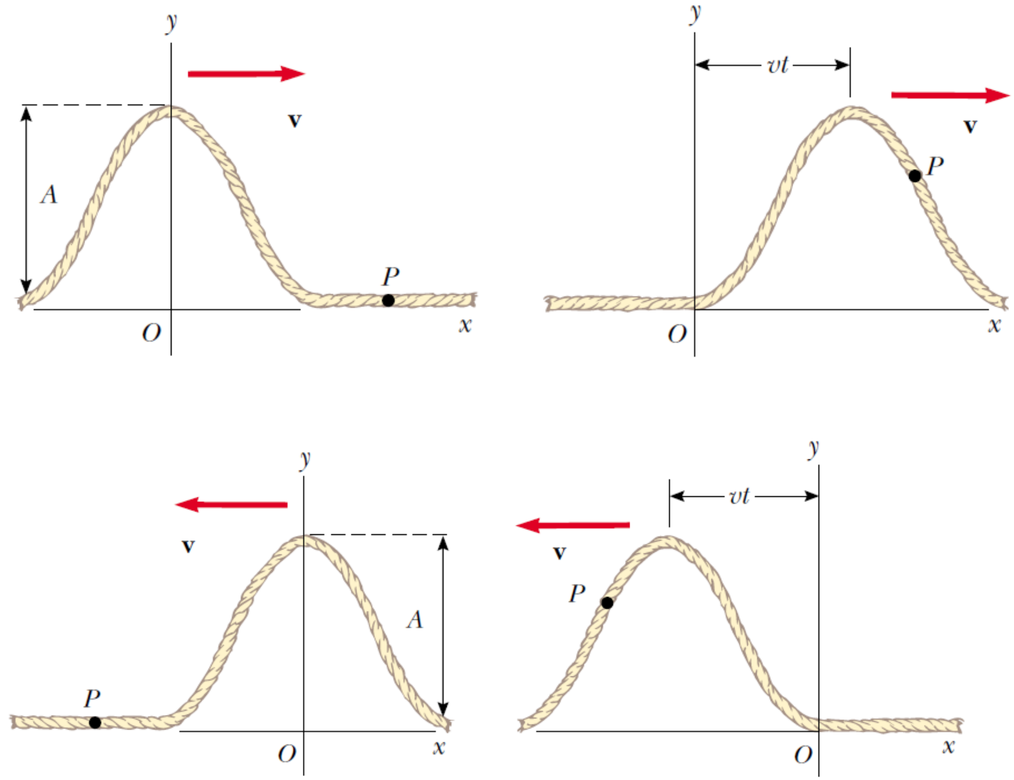
$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$$

$$v = \sqrt{\frac{T}{\mu}}$$

General Solutions

- **The linear wave equation**

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$



- **Wave functions**
  - $y = f(x - vt)$ **and**
  - $y = f(x + vt)$

are obviously solutions to the linear wave equation.

Sinusoidal Waves

- **The linear wave equation**

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

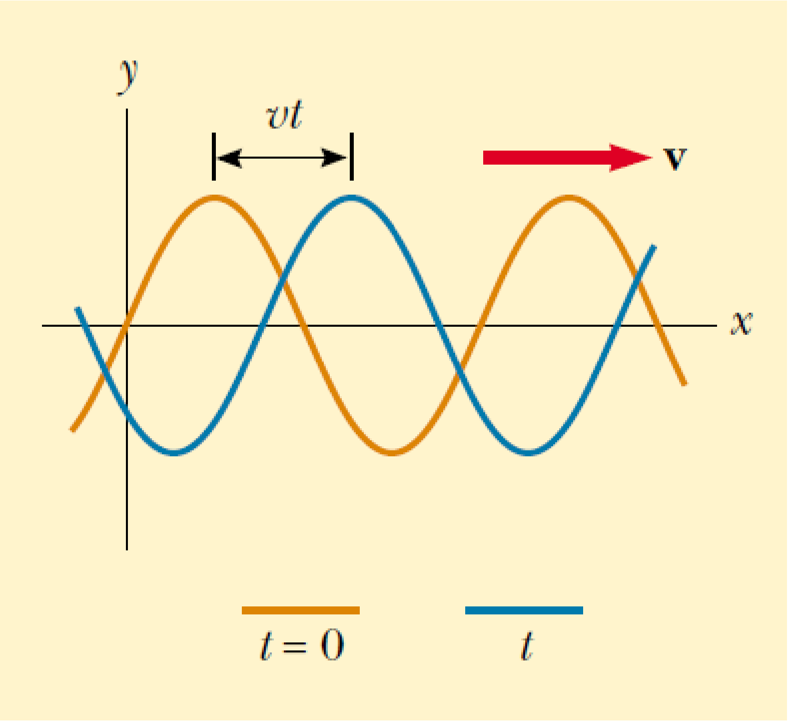
- **The most important family of the solutions are**

$$y = A \sin(kx - \omega t + \phi)$$

angular wave number

angular frequency

phase constant



$$\omega = vk$$

Various Forms

$$y = A\sin(kx - \omega t)$$

$t = 0$

$$y = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

$$\omega \equiv \frac{2\pi}{T}$$

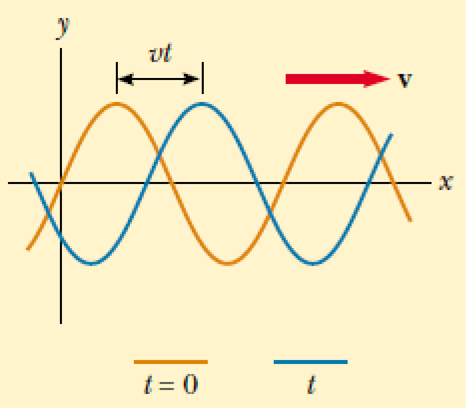
$$k \equiv \frac{2\pi}{\lambda}$$

$$f = \frac{1}{T}$$

$$v = \frac{\omega}{k}$$

$$v = \lambda f$$

$t = 0$



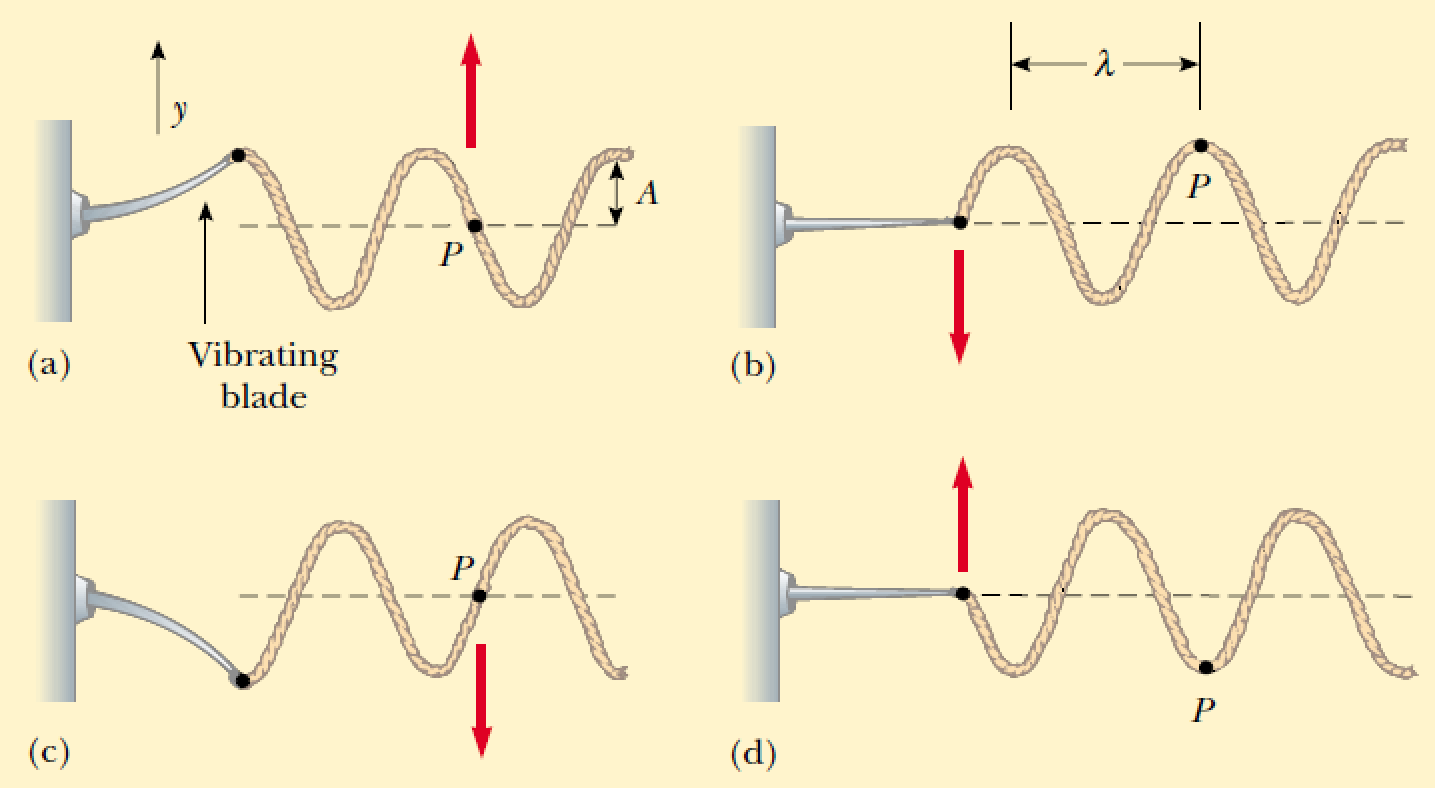
$$v = \frac{\lambda}{T}$$

$$y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

$$y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

Sinusoidal Wave on Strings

- Each particle of the string, such as that at *P*, oscillates vertically with simple harmonic motion.



Sinusoidal Wave on Strings

- **Note that although each segment oscillates in the *y* direction, the wave travels in the *x* direction with a speed *v*. Of course, this is the definition of a transverse wave.**

$$y = A\sin(kx - \omega t)$$

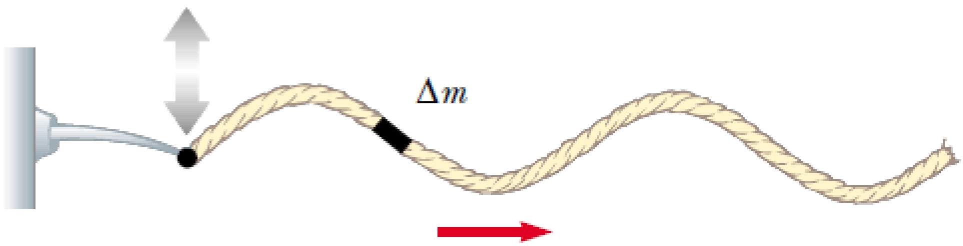
$$v_{y,\,\max} = \omega A$$

$$a_{y,\,\max} = \omega^2 A$$

$$v_y = \left.\frac{dy}{dt}\right]_{x\,=\,\text{constant}} = \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$$

$$a_y = \left.\frac{dv_y}{dt}\right]_{x\,=\,\text{constant}} = \frac{\partial v_y}{\partial t} = -\omega^2 A\sin(kx - \omega t)$$

Rate of Energy Transfer



$$\Delta U = \tfrac{1}{2}(\Delta m)\,\omega^2 y^2$$

$$= \tfrac{1}{2}(\mu \Delta x)\,\omega^2 y^2$$

$$dU = \tfrac{1}{2}\mu\omega^2[A\sin(kx - \omega t)]^2\,dx = \tfrac{1}{2}\mu\omega^2 A^2 \sin^2(kx - \omega t)\,dx$$

We have learned that for simple harmonic oscillation, the total energy $E = K + U$ is a constant, i.e.,

$$dE = \frac{1}{2}\mu\omega^2 A^2 dx$$

The rate of energy transfer: $$P = \frac{dE}{dt} = \frac{1}{2}\mu\omega^2 A^2 v$$

The Principle of Superposition

- The superposition principle states that when two or more waves move in the same linear medium, the net displacement of the medium (that is, the resultant wave) at any point equals the algebraic sum of all the displacements caused by the individual waves.
  - Same frequency, wavelength, amplitude, direction. Different phase.
  - Beats: Different frequency.
  - Standing waves: Same frequency, wavelength, amplitude. Different direction.

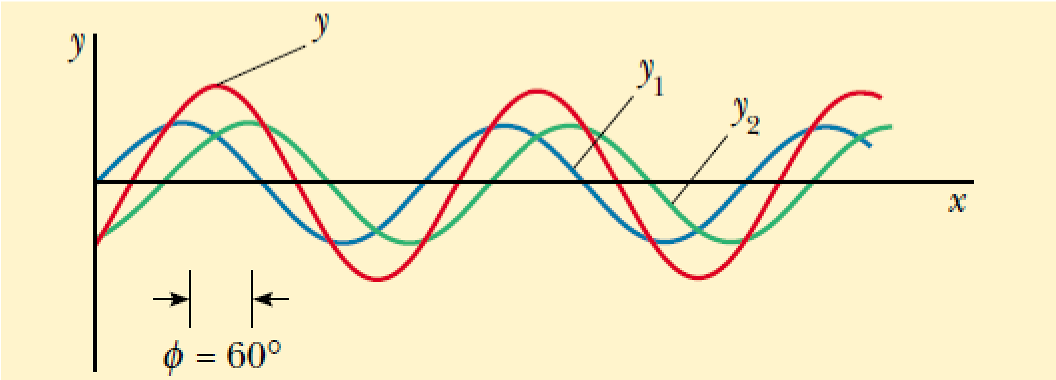
Interference

Same frequency, wavelength, amplitude, direction. Different phase.

$$y_1 = A\sin(kx - \omega t) \qquad y_2 = A\sin(kx - \omega t + \phi)$$

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)]$$

$$= 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$



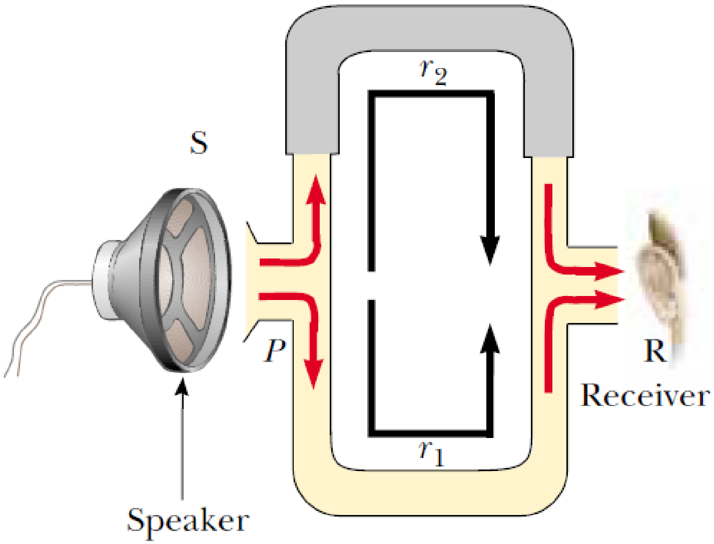
Interference

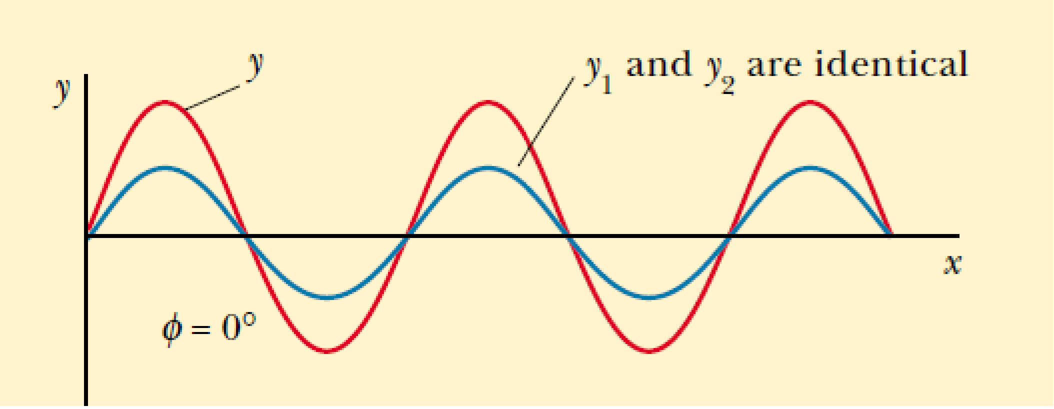
$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

- **When $\cos(\phi/2) = \pm 1$, the waves are said to be everywhere in phase and thus interfere constructively.**
- **When $\cos(\phi/2) = 0$, the resultant wave has zero amplitude everywhere, as a consequence of destructive interference.**

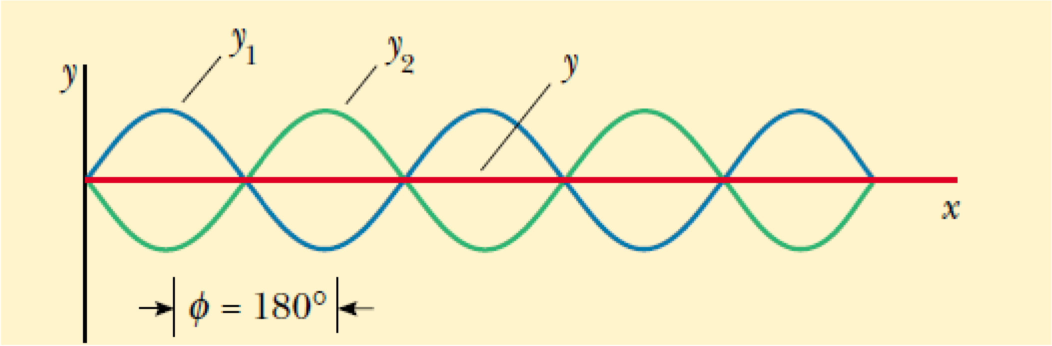
Video： superposition of two coherent wave

Interference of Sound Waves





$$\Delta r = \frac{\phi}{2\pi}\lambda$$



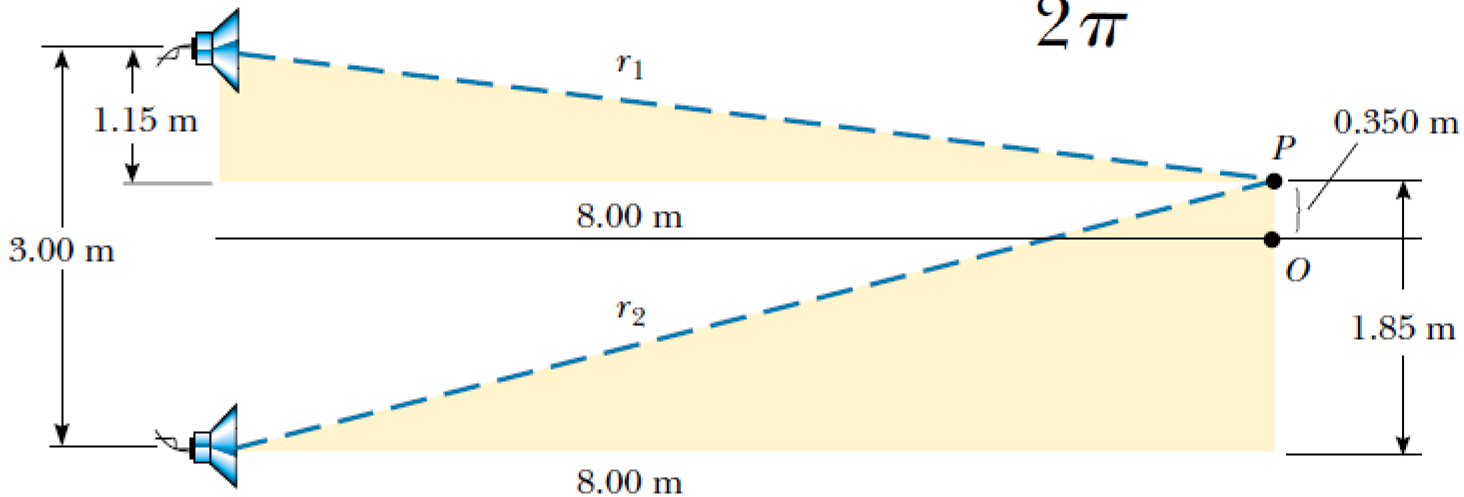
$$\Delta r = |r_1 - r_2|$$

$n\lambda$: in phase

$(n + 1/2)\lambda$: out of phase

Speakers Driven by the Same Source

$$\Delta r = \frac{\phi}{2\pi}\lambda$$



$$\Delta r = (2n)\frac{\lambda}{2} \qquad \text{for constructive interference}$$

$$\Delta r = (2n+1)\frac{\lambda}{2} \qquad \text{for destructive interference}$$

Beating: Temporal Interference

- **Beating is the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies.**

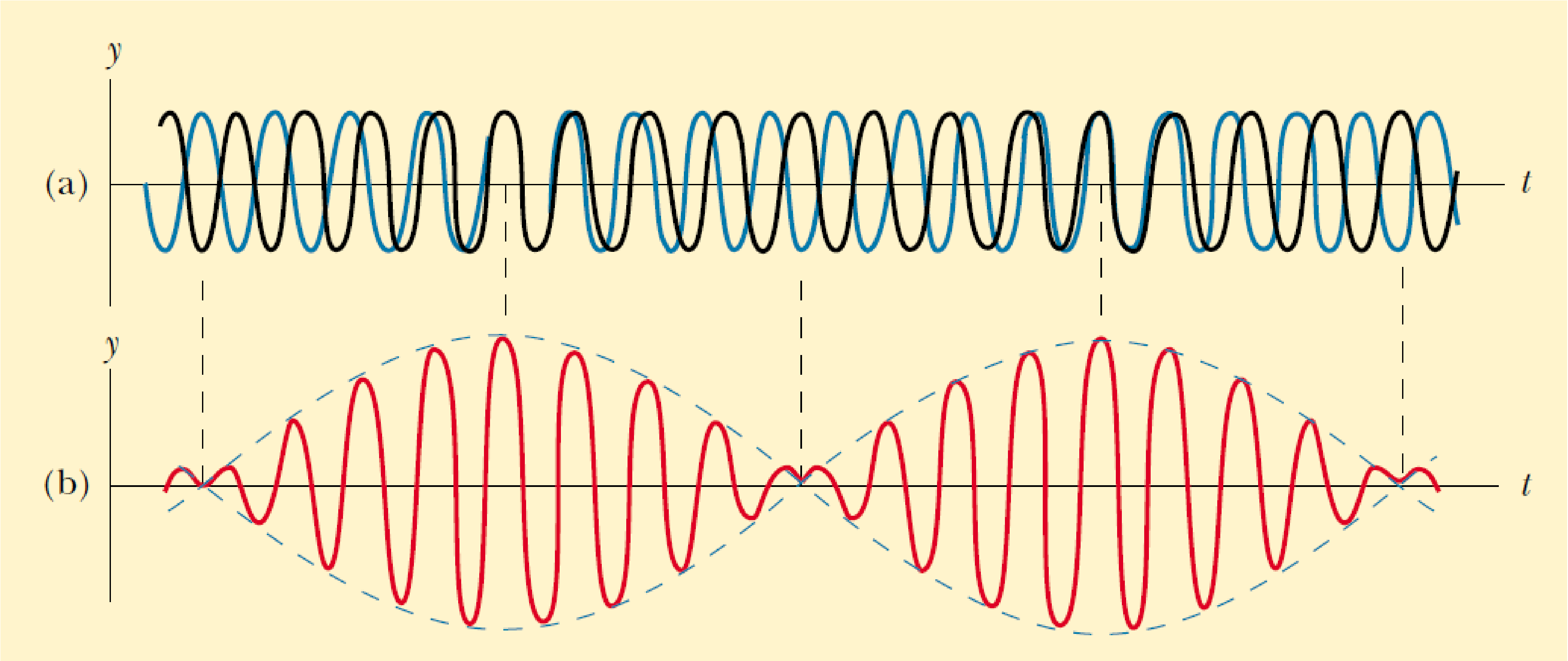
$$y_1 = A\cos\omega_1 t = A\cos 2\pi f_1 t$$

$$y_2 = A\cos\omega_2 t = A\cos 2\pi f_2 t$$

$$y = y_1 + y_2 = A(\cos 2\pi f_1 t + \cos 2\pi f_2 t)$$

$$= \left[2\,A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]\cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t$$

Beating: Temporal Interference



$$y = \left[2\,A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]\cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t$$

Beat Frequency

- The amplitude and therefore the intensity of the resultant sound vary in time.

$$A_{\text{resultant}} = 2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t$$

- The two neighboring maxima in the envelop function are separated by

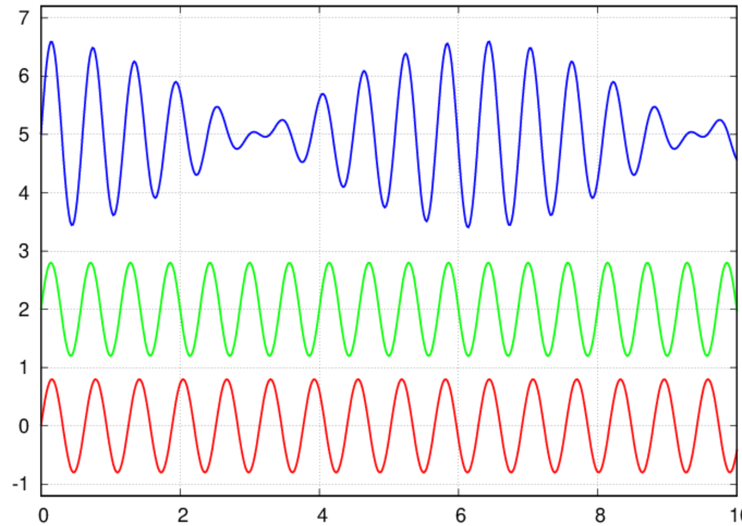
$$2\pi\left(\frac{f_1 - f_2}{2}\right)t = \pi$$

Beat frequency: $f_b = |f_1 - f_2|$

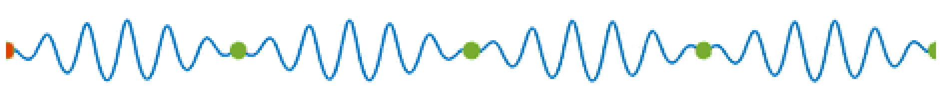
Phase velocity *vs.* Group velocity*

$$y_1(x,t) = A\sin(k_1 x - \omega_1 t)$$
$$y_2(x,t) = A\sin(k_2 x - \omega_2 t)$$

$$k_1 \simeq k_2 \Rightarrow k_2 = k_1 + \Delta k$$
$$\omega_1 \simeq \omega_2 \Rightarrow \omega_2 = \omega_1 + \Delta\omega$$



$$y_1 + y_2 = A\sin(k_1 x - \omega_1 t) + A\sin(k_2 x - \omega_2 t)$$
$$= 2A\sin\left(\left(\frac{k_1+k_2}{2}\right)x - \left(\frac{\omega_1+\omega_2}{2}\right)t\right)\cos\left(\left(\frac{k_1-k_2}{2}\right)x - \left(\frac{\omega_1-\omega_2}{2}\right)t\right)$$
$$\boxed{= 2A\sin\left(k_{\text{avg}}\,x - \omega_{\text{avg}}\,t\right)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)}$$



- Phase velocity:

$$V_P = \omega/k$$

- Group velocity:

$$V_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

- Dispersion Relation:

$$\omega = \omega(k) \neq ck$$

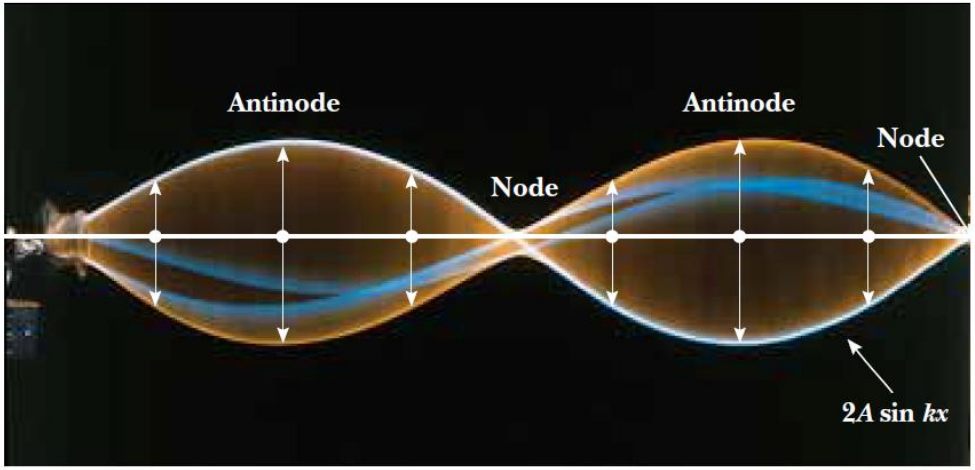
Standing Waves

Same frequency, wavelength, amplitude.
Different direction.

$$y_1 = A\sin(kx - \omega t) \qquad y_2 = A\sin(kx + \omega t)$$

$$y = y_1 + y_2 = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

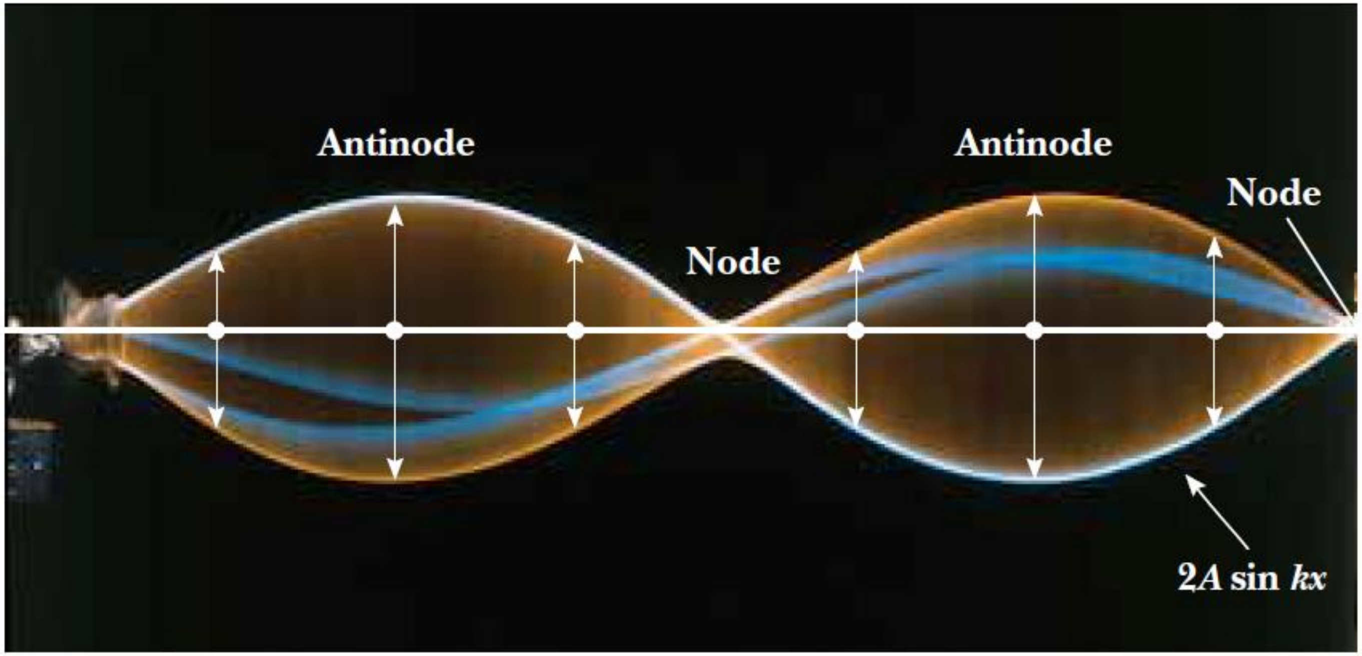
$$= (2A\sin kx)\cos\omega t$$



Standing Waves

- A standing wave is an oscillation pattern with a stationary outline that results from the superposition of two identical waves traveling in opposite directions.
  - No sense of motion in the direction of propagation of either of the original waves.
  - Every particle of the medium oscillates in simple harmonic motion with the same frequency.
  - Need to distinguish between the amplitude of the individual waves and the amplitude of the simple harmonic motion of the particles of the medium.

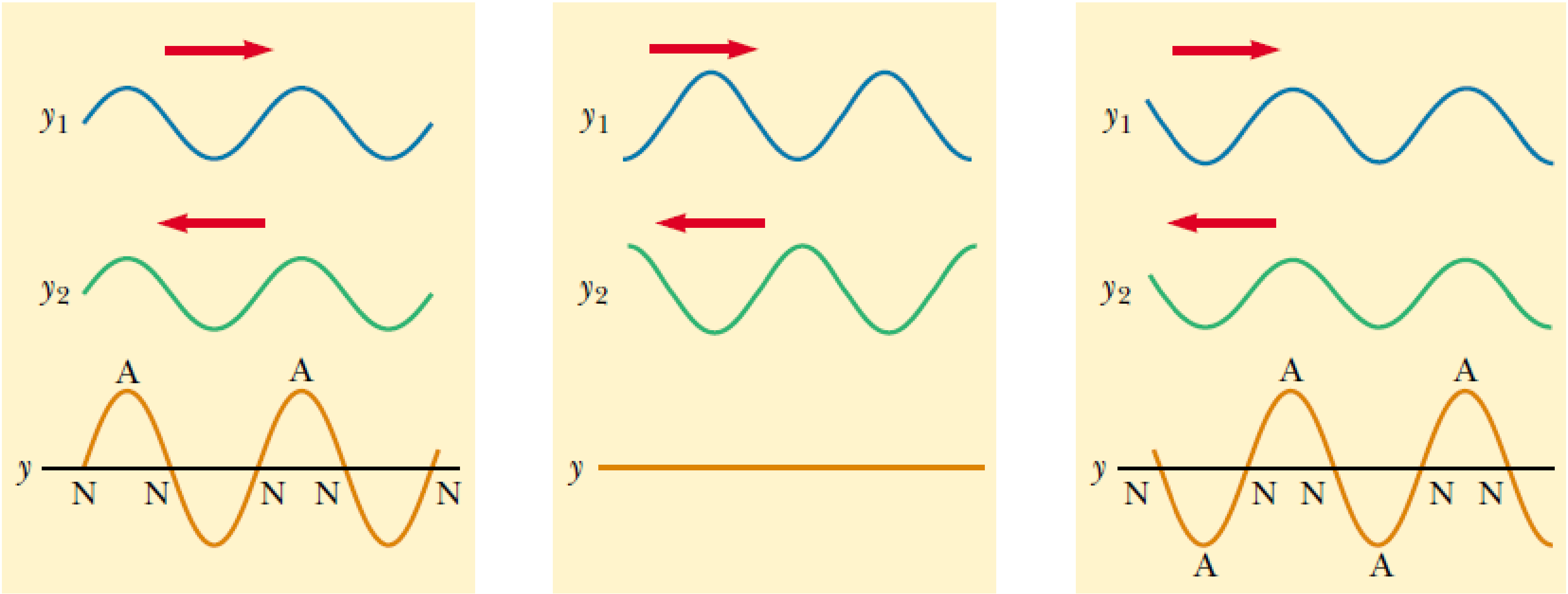
Nodes and Antinodes



Nodes: $kx = n\pi \qquad x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \cdots$

Antinodes: $kx = \left(n + \frac{1}{2}\right)\pi \qquad x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \cdots$

Nodes and Antinodes

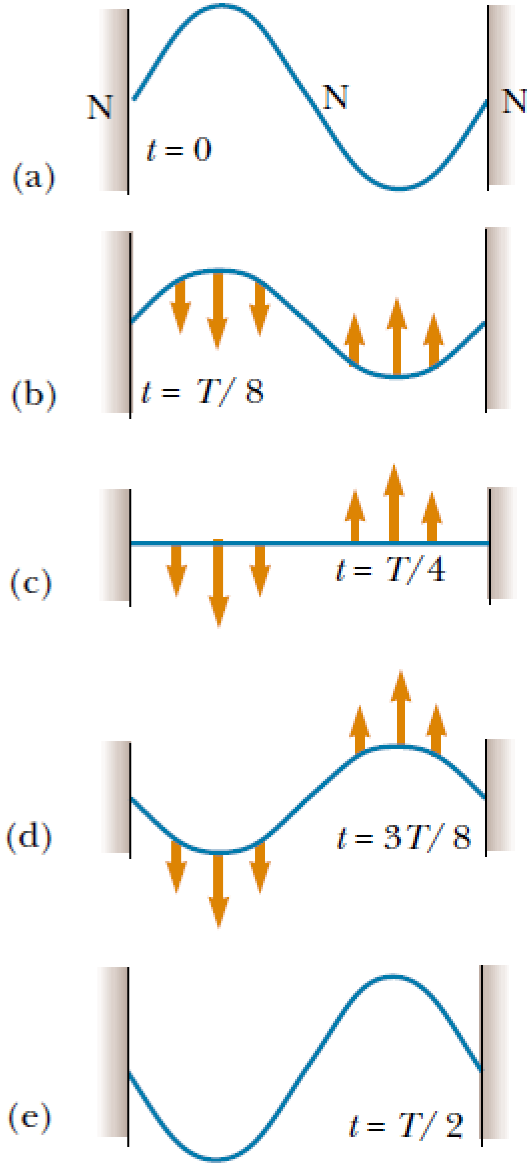


(a) $t = 0$ (b) $t = T/4$ (c) $t = T/2$

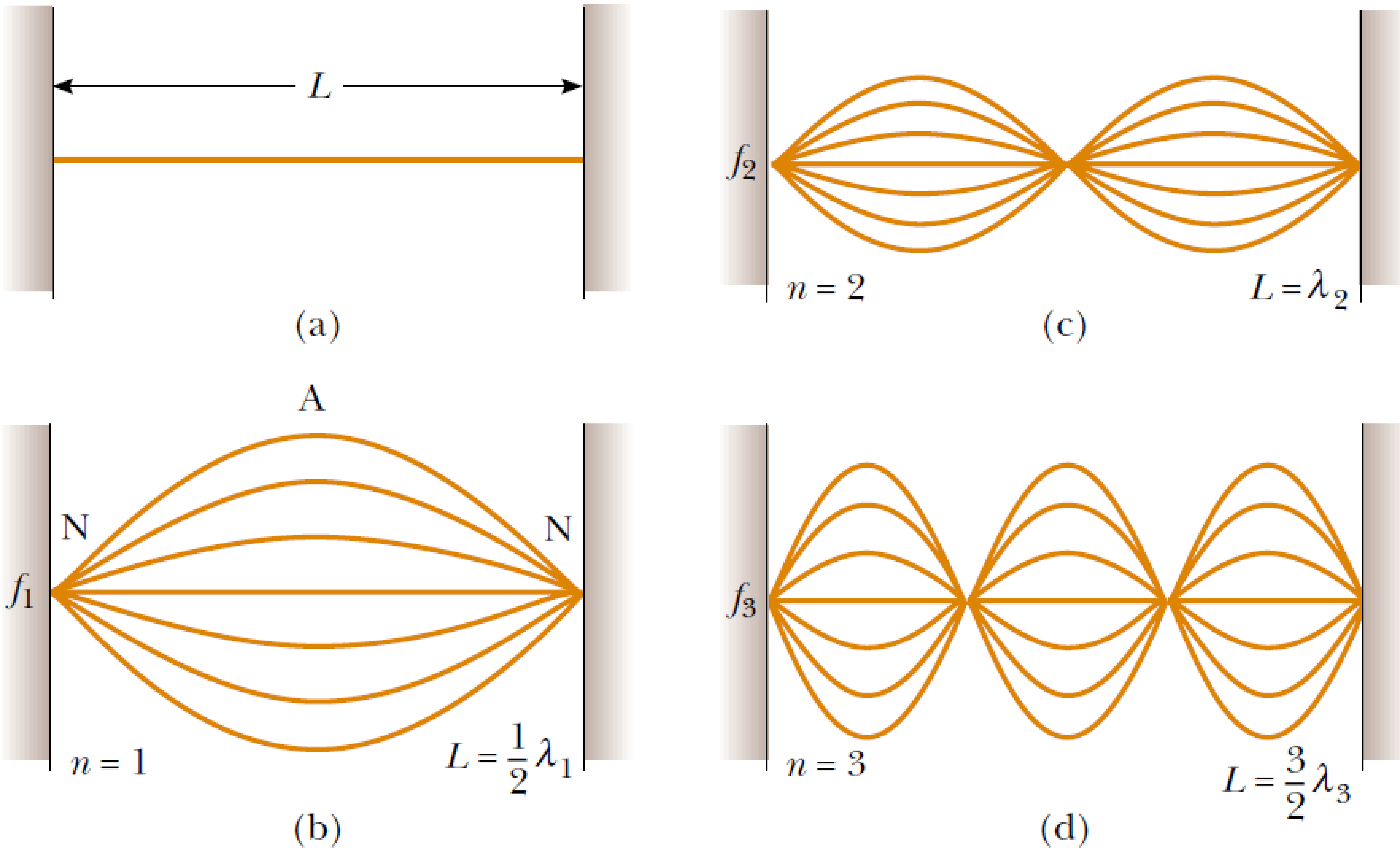
The distance between adjacent antinodes is equal to λ/2.
The distance between adjacent nodes is equal to λ/2.
The distance between a node and an adjacent antinode is λ/4.

Energy in a Standing Wave

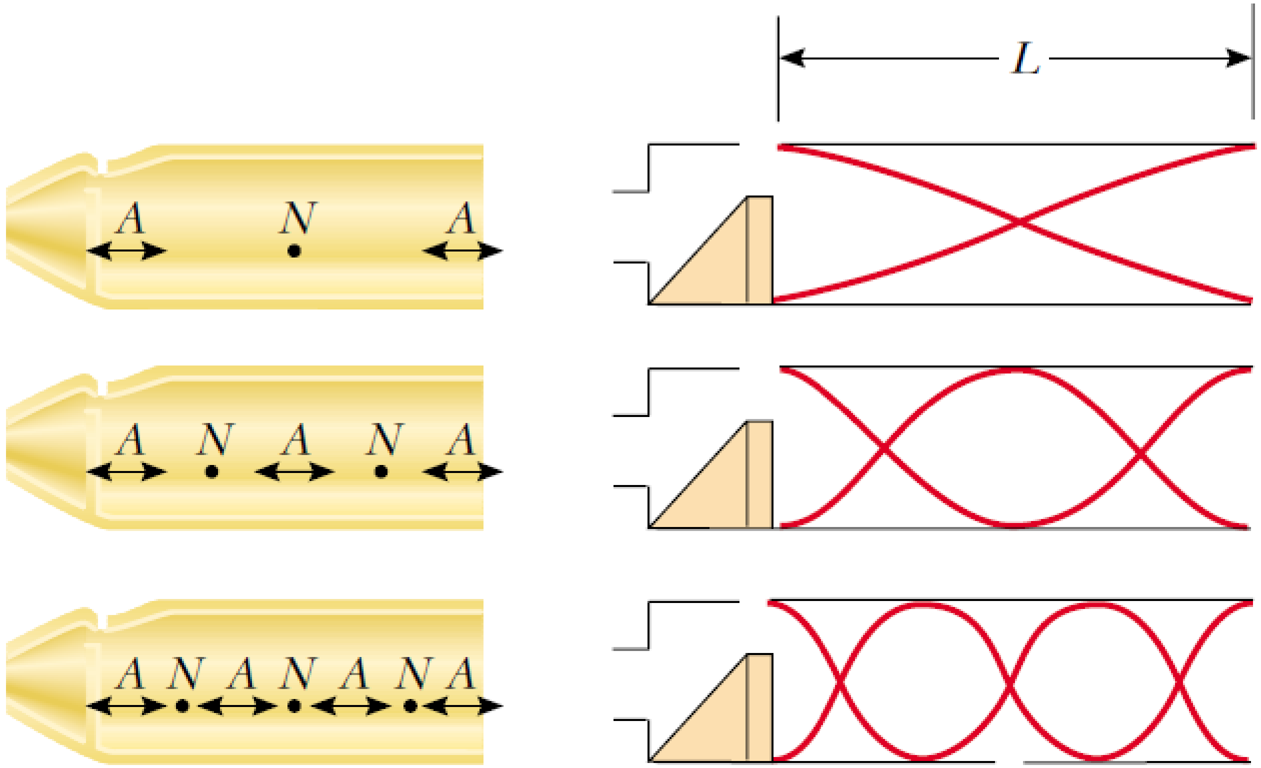
- Except for the nodes, which are always stationary, all points on the string oscillate vertically with the same frequency but with different amplitudes of simple harmonic motion.
- No energy is transmitted along the string across a node, and energy does not propagate in a standing wave.



In a String Fixed at Both Ends



Standing Waves in Air Columns



$$\lambda_1 = 2L$$

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

$$\lambda_2 = L$$

$$f_2 = \frac{v}{L} = 2f_1$$

$$\lambda_3 = \frac{2}{3}L$$

$$f_3 = \frac{3v}{2L} = 3f_1$$

Harmonic Series

- **In general, the wavelength of the various normal modes for a string of length L fixed at both ends are**

$$\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ . \ . \ .$$

- **The frequencies of the normal modes are**

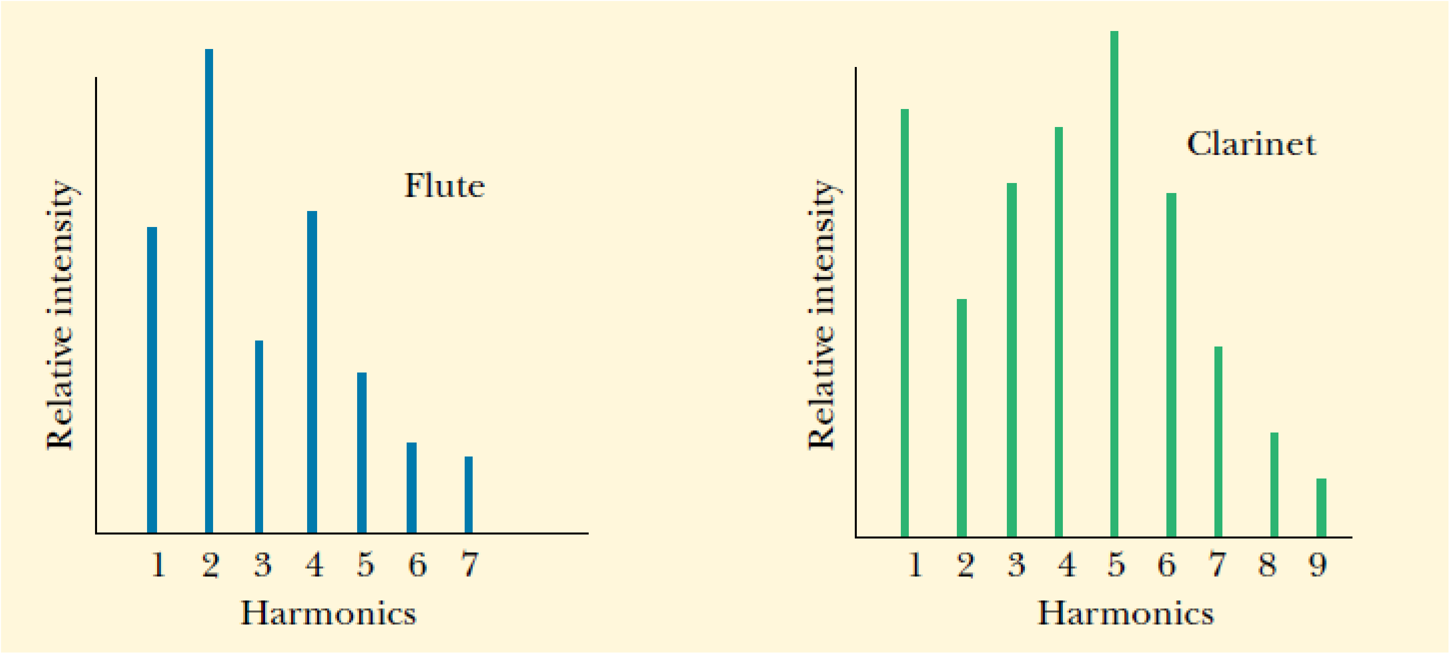
$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \qquad n = 1, 2, 3, \ . \ . \ .$$

- **Frequencies of normal modes that exhibit an integer multiple relationship such as this form a harmonic series, and the normal modes are called harmonics.**

Harmonics in Musical Instruments

- **If we wish to excite just a single harmonic, we need to distort the string in such a way that its distorted shape corresponded to that of the desired harmonic.**
- **If the string is distorted such that its distorted shape is not that of just one harmonic, the resulting vibration includes various harmonics. Such a distortion occurs in musical instruments when the string is plucked (as in a guitar), bowed (as in a cello), or struck (as in a piano). When the string is distorted into a non-sinusoidal shape, only waves that satisfy the boundary conditions can persist on the string. These are the harmonics.**

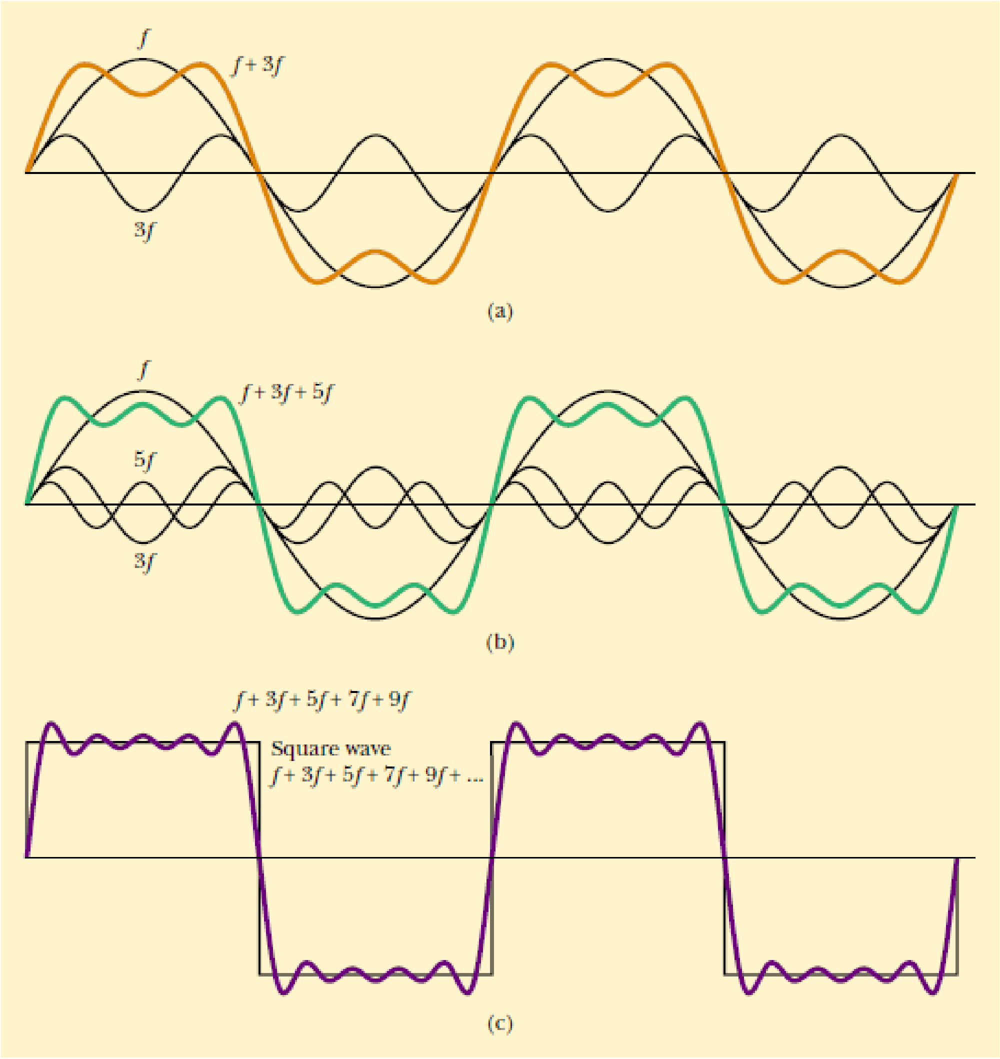
Sound Quality or Timbre



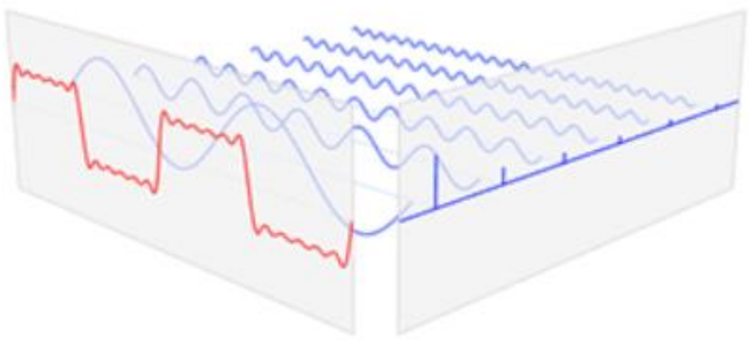
Sounds may be generally characterized by pitch (frequency), loudness (amplitude), and quality.

Video： MIT Frequency spectra of instruments

An "Almost" Square Wave



Fourier analysis of a periodic function



Fourier Analysis

- **Periodic function $f(t)$ with period $T = 2\pi/\omega_1$, i.e., functions such that $f(t + T) = f(t)$ for all t, can be expanded in a Fourier series of the form**

$$f(t) = \sum_{n=0}^{\infty}\left[A_n sin\left(n\frac{2\pi}{T}t\right) + B_n cos\left(n\frac{2\pi}{T}t\right)\right]$$

$$= B_0 + \sum_{n=1}^{\infty} A_n\, sin\left(n\frac{2\pi}{T}t\right) + \sum_{n=1}^{\infty} B_n\, cos\left(n\frac{2\pi}{T}t\right)$$

$$= B_0 + \sum_{n=1}^{\infty} A_n\, sin(n\omega_1 t) + \sum_{n=1}^{\infty} B_n\, cos(n\omega_1 t)$$

Finding Fourier Coefficients

- **Fundamental integrals: for** $n > 0$**,**

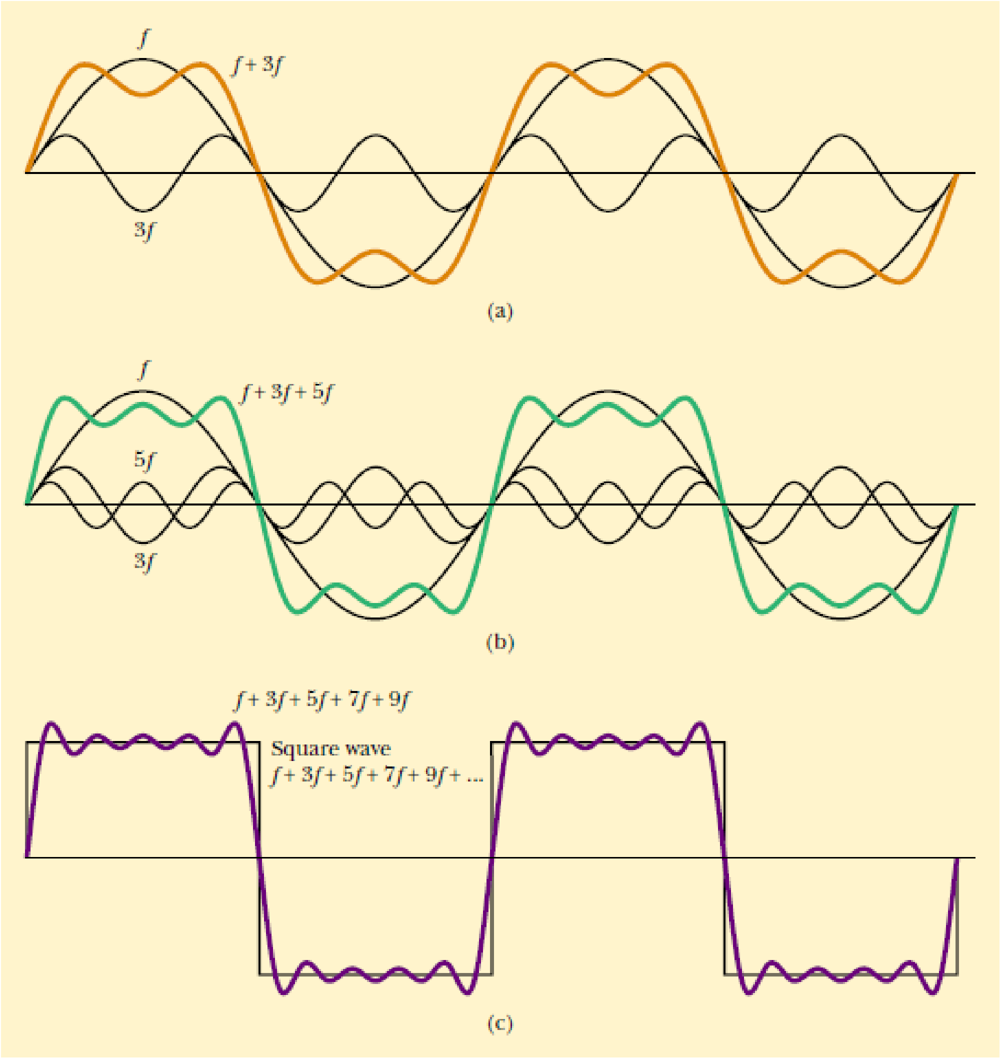
$$\int \sin(n\omega_1 t)dt = 0 \qquad \int \cos(n\omega_1 t)dt = 0$$

$$\Rightarrow \quad B_0 = \frac{2}{T}\int f(t)dt$$

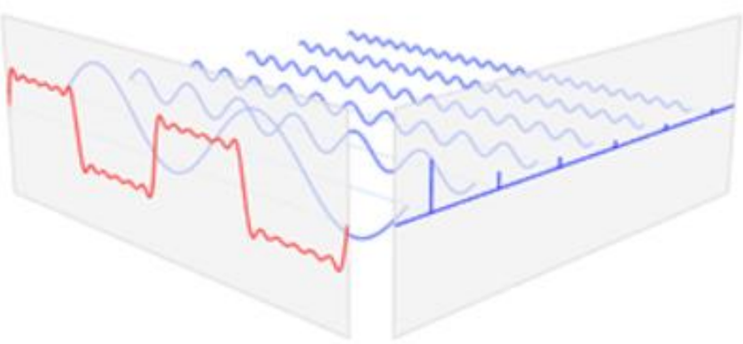
$$A_m = \frac{2}{T}\int f(t)\sin(m\omega_1 t)dt$$

$$B_m = \frac{2}{T}\int f(t)\cos(m\omega_1 t)dt$$

Exercise: Square Wave



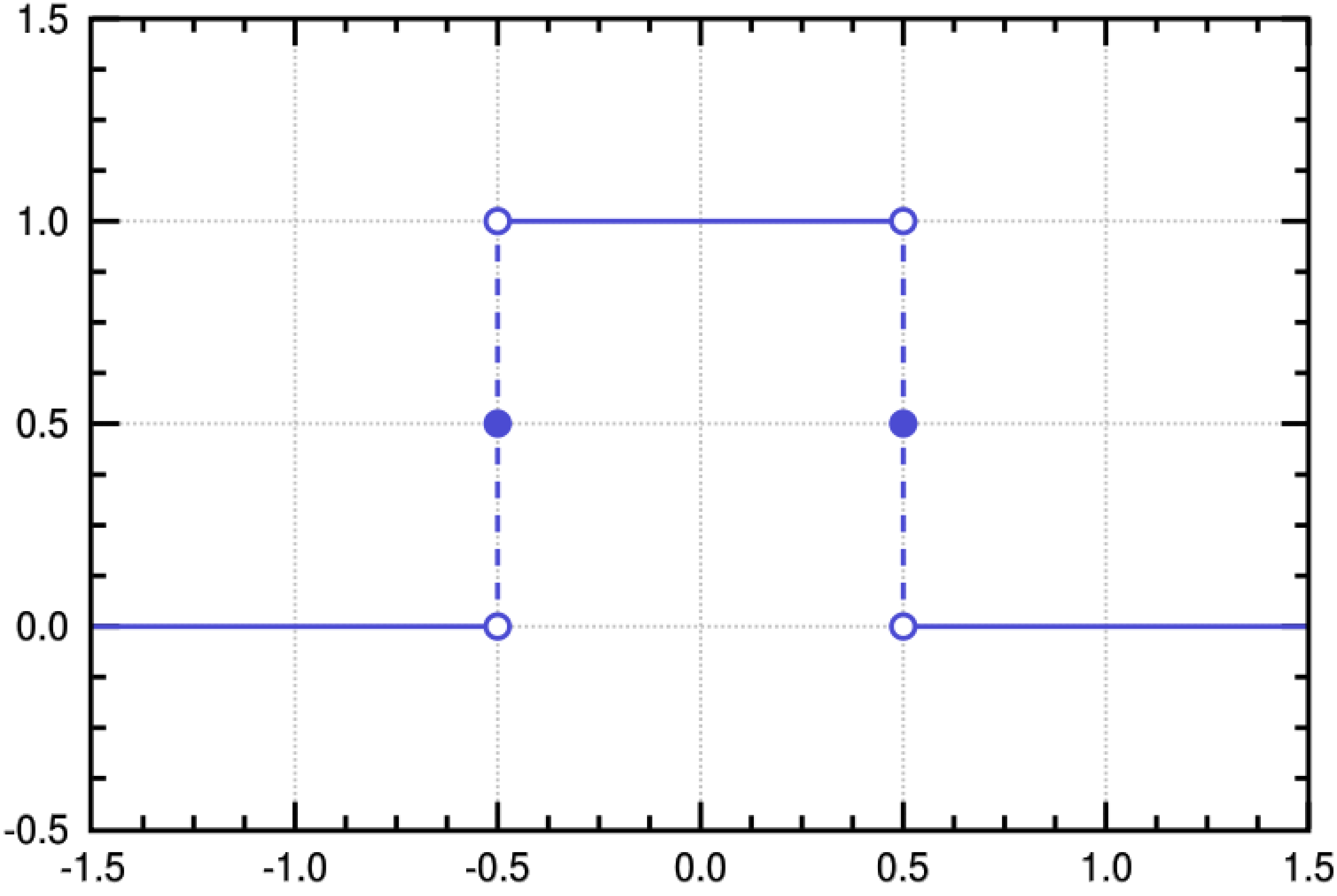
Find out the leading Fourier coefficients of a square wave.



Back to Pulses

- The superposition of sinusoidal wave function can produce not only periodic functions, but also aperiodic pulses.
  - Now continuous frequencies are needed, not just a harmonic series.
  - A finite time width in the pulse corresponds to a finite width in frequency. The broader the signal is, the narrower the frequency spectrum is.

Fourier Analysis of a Pulse



Result: Sinc Function