Continuous Charge and Gauss' Law

Xin Lu/Gentaro Watanabe

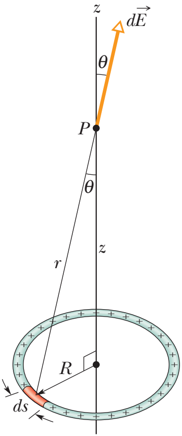
Lecture 2

Outline

- Electric Field due to a Continuous Charge Distribution
- Electric Flux and Gauss' Law
- Application of Gauss' law to Charged Insulators
    - Spherical Symmetry
    - Planar Symmetry
    - Cylindrical Symmetry

The Electric Field Due to a Ring of Charge

- Consider a thin ring of radius R with a uniform, fixed distribution of positive charge along its circumference. We assume a linear charge density λ (the charge per unit length).
- We restrict our interest to an arbitrary point P on the central axis (the axis through the ring's center and perpendicular to the plane of the ring), at distance z from the center point.



Key Ideas

- Mentally divide the ring into differential elements of charge that are so small that we can treat them as though they are particles.
- Each element contributes a field at P in its own direction, so we need to decompose the vector into components.
- Before we separately sum each set of components, check if some set simply all cancels out (due to symmetry).
- We add up the contributions from a continuous distribution of charge by means of integration.

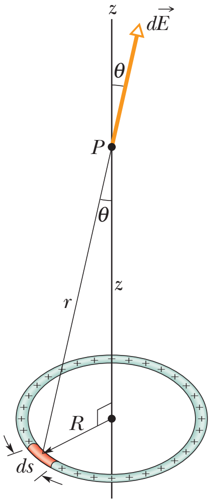
- Let ds be the arc length of the dq element.

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda ds}{z^2 + R^2}\frac{z}{(z^2 + R^2)^{1/2}}$$

- We integrate along the ring, from element to element,

$$E_z = \int dE_z = \frac{qz}{4\pi\epsilon_0(z^2 + R^2)^{3/2}},$$

where $q = \lambda(2\pi R)$ is the total charge of the ring.



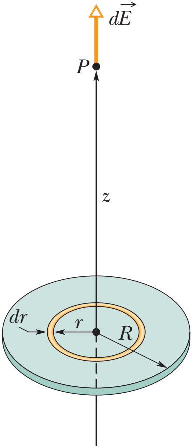
- Never forget to check for limiting cases.
- First, for a point on the central axis with $z \gg R$,

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}.$$

  This is a reasonable result because from a large distance, the ring *looks like* a point charge.
- Second, for a point at the center of the ring, i.e., for z $= 0$, we find $E_z = 0$. This is also a reasonable result because the mirror symmetry across the $z = 0$ plane dictates that $E_z = 0$.

The Electric Field Due to a Charged Disk

- We consider the electric field of a disk of radius R with a uniform, fixed distribution of positive charge on its top surface. The surface charge density (charge per unit area) is σ.
- We restrict our attention to the electric field at an arbitrary point P on the central axis, at distance $z > 0$ from the center of the disk.



- We can recycle the result of the field of a thin ring.
- For a thin ring on the disk at an arbitrary radius r with charge $dq = \sigma dA = \sigma(2\pi r dr)$, its contribution to the electric field at point P is

$$dE_z = \frac{\sigma(2\pi r dr)z}{4\pi\epsilon_0(z^2 + r^2)^{3/2}}.$$

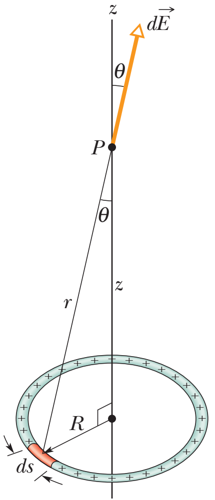
  Integrating the contributions from the center of the disk at $r = 0$ out to the rim at $r = R$, we obtain

$$E_z = \int dE_z = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{(z^2 + R^2)^{1/2}}\right].$$

Quiz 2-1

Motivation of Gauss' Law

- Earlier, our technique to find $\vec{E}$ at points near extended charged objects is the following.
  - First, we split the charge distribution up into charge elements dq, find the field $d\vec{E}$ due to an element, and decompose the vector into components.
  - Then we determine whether the components from all the elements will end up canceling or adding.
  - Finally we sum the adding components by integrating over all the elements.



- The use of symmetry often simplify the solving of such labor-intensive problems. **Gauss' law** is a beautiful relationship between charge and electric field that allows us, in certain symmetric situations, to find the electric field of an extended charged object with a few lines of algebra.

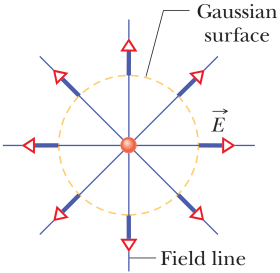


Figure 1: Charge Q.

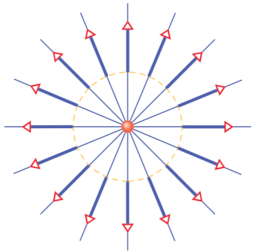


Figure 2: Charge $2Q$.

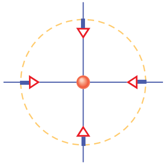


Figure 3: How much charge?

- For a point charge q at the origin, Coulomb's law implies

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

- We can rewrite this equation as

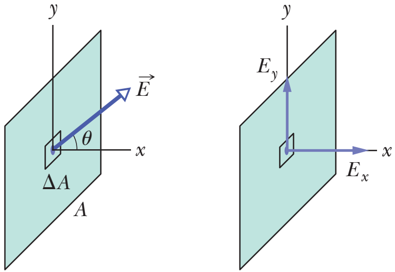
$$(4\pi r^2)E(r) = \frac{q}{\epsilon_0}.$$

- Notice that $4\pi r^2$ is the area of a spherical surface of radius r. What is the physics meaning of $(4\pi r^2)E(r)$? Can we generalize this equation to generic cases? Will it be useful?

Electric Flux

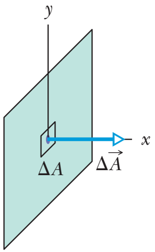
- We introduce **electric flux** to determine how much electric field pierces a surface.
- We begin with a small square patch with area ΔA on a flat surface in a uniform electric field $\vec{E}$.
- The amount of electric field piercing the patch is defined to be the electric flux $\Delta\Phi$ through it:

$$\Delta\Phi = (E\cos\theta)\Delta A.$$



- Alternatively, we define an area vector $\Delta\vec{A}$, which is perpendicular to the patch and which has a magnitude equal to the area ΔA of the patch. Thus,

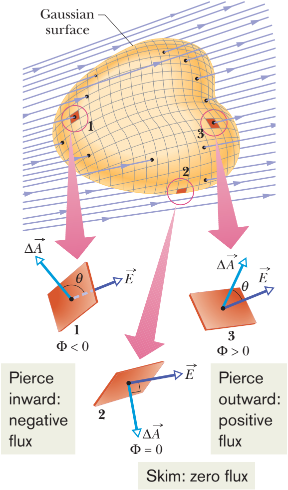
$$\Delta\Phi = \vec{E} \cdot \Delta\vec{A}$$



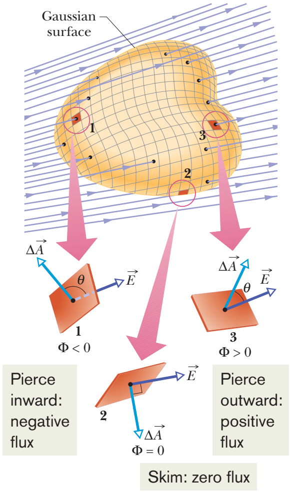
- To find the total flux Φ through a surface, we integrate the dot product

$$\Phi = \int \vec{E} \cdot d\vec{A}.$$

- To relate flux and charge, we need a closed surface, or a **Gaussian surface**, which can sit in a nonuniform electric field.
- As before, we first consider the flux through small square patches. We always draw the area vector $\Delta\vec{A}$ pointing outward from the surface (away from the interior).



- Now, we are interested in not only the piercing components of the field but also on whether the piercing is inward or outward.
- An inward piercing field is negative flux. An outward piercing field is positive flux. A skimming field is zero flux.



- The *net flux* through the closed Gaussian surface is

$$\Phi = \oint \vec{E} \cdot d\vec{A}.$$

  The loop on the integral sign indicates that we must integrate over the entire closed surface, to get the net flux through the surface.
- Electric flux is a scalar, whose SI unit is $\mathrm{N{\cdot}m^2/C}$.

Gauss' Law

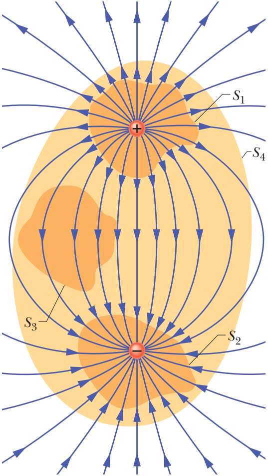
- Gauss' law relates the net flux Φ of an electric field through a closed surface (a Gaussian surface) to the net charge q_{enc} that is enclosed by that surface. It tells us that, in a vacuum or (what is the same for most practical purposes) in air, $\epsilon_0 \Phi = q_{\mathrm{enc}}$, or

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\mathrm{enc}}.$$

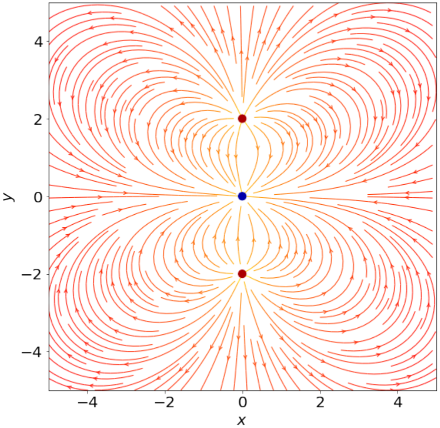
- If q_{enc} is positive, the net flux is outward; if q_{enc} is negative, the net flux is inward.
- Later, we will modify Gauss' law in a material such as mica, oil, or glass.

- Note that $\vec{E}$ is the electric field resulting from all charges, both those inside and those outside the Gaussian surface.
- This statement may seem to be inconsistent, but keep this in mind: The electric field due to a charge outside the Gaussian surface contributes zero net flux through the surface, because as many field lines due to that charge enter the surface as leave it.

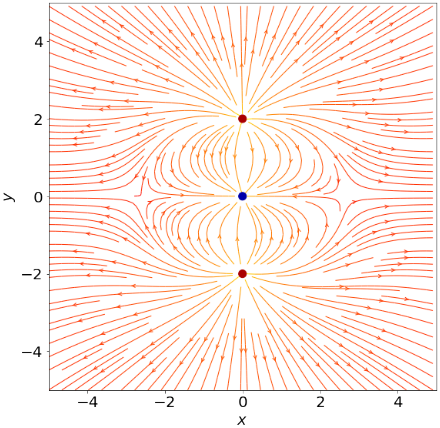
- Consider two particles with charges equal in magnitude but opposite in sign.
  - Surface S_1: The electric field is outward for all points on this surface.
  - Surface S_2: The electric field is inward for all points on this surface.
  - Surface S_3: All the field lines pass entirely through the surface.
  - Surface S_4: There are as many field lines leaving surface S_4 as entering it.



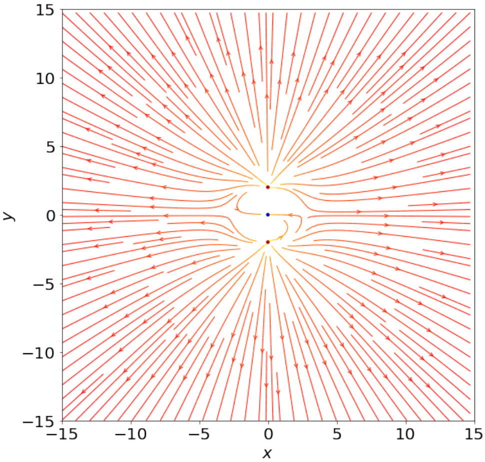
Revisit Quiz 1-1



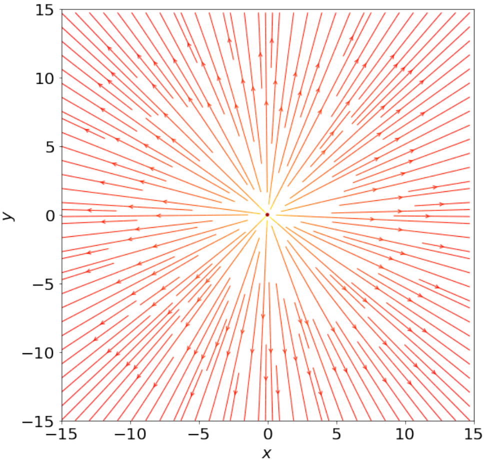
$-2Q$ at $(0, 0)$



$-Q$ at $(0, 0)$



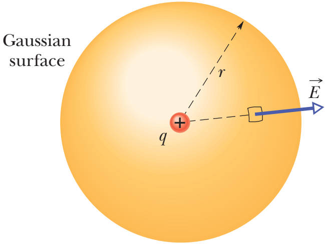
Net charge Q



Point charge Q at $(0, 0)$

Gauss' Law and Coulomb's Law

- As an example, we apply Gauss' law is in finding the electric field of a particle with positive charge q.



- That field has spherical symmetry, so we enclose the particle in a Gaussian sphere that is centered on the particle. From symmetry, we know that $\vec{E}$ is radial, so

$$q = \epsilon_0 \oint \vec{E} \cdot d\vec{A} = \epsilon_0 E \oint dA.$$

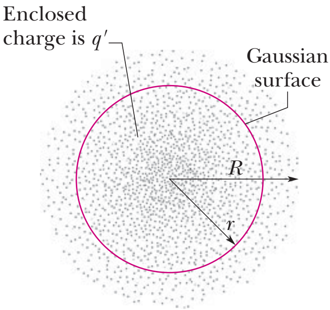
- The remaining integral is just the area ($4\pi r^2$) of the Gaussian surface. Therefore, we have

$$E = \frac{q}{\epsilon_0}\frac{1}{4\pi r^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2},$$

  which is exactly Coulomb's law.
- For historical reason, we sometimes put $k = 1/(4\pi\epsilon_0)$ together. We note, however, that 4π can be regarded as a geometric factor, irrelevant to ϵ_0.
- We live in a three-dimensional space; this also gives us a strong reason to believe that Coulomb's law (as well as the law of universal gravitation) is an inverse-square law, not something like $A/r^{2.00001}$.

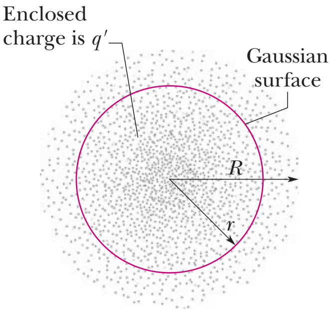
Applying Gauss' Law: Spherical Symmetry

- For a spherically symmetric, uniform distribution of charge q of radius R, the effect of the charge distribution can be summed up *shell by shell*.
- For a Gaussian surface with $r < R$, the inside charge sets up a field on the Gaussian surface as though it is concentrated at the center, while the outside charge does not set up a field.



- If *the full charge q enclosed within radius R is uniform*, the enclosed charge is $q' = q(r/R)^3$, and Gauss' law reads

$$(4\pi r^2)E = \frac{q'}{\epsilon_0}.$$



- Therefore, we have, for $r \leq R$,

$$\vec{E} = \left(\frac{q}{4\pi\epsilon_0 R^3}\right)\vec{r}.$$

Applying Gauss' Law: Planar Symmetry

- For a Gaussian surface as a closed cylinder with end caps of area A, $\epsilon_0(EA + EA) = \sigma A$. Therefore, we find $E = \sigma/(2\epsilon_0)$.

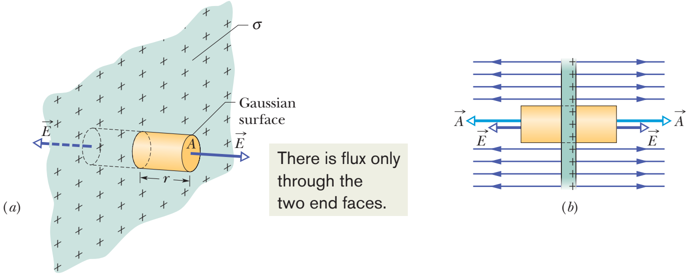


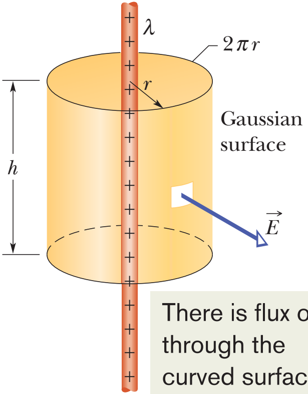
Figure 4: Thin, infinite, nonconducting sheet with a uniform surface charge density σ.

- This result holds for any point at a finite distance from the sheet, which we found earlier by letting the radius of the charge disk going to infinity.
- An infinite sheet of charge is often a good approximation to a finite nonconducting sheet, as long as we are dealing with points close to the sheet and not too near its edges.
- Later, we will compare this result with the electric field outside the surfaces of conductors.

Gauss' Law, Again: Cylindrical Symmetry

- The charge distribution and the field have cylindrical symmetry, so the electric field at any point must be radially inward/outward.
- To find the field at radius r, we enclose a section of the rod with a concentric Gaussian cylinder of radius r and height h, so

$$\Phi = E(2\pi rh).$$



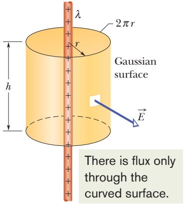
There is flux only through the curved surface.

- According to Gauss' law, we have

$$\epsilon_0 \Phi = \lambda h,$$

where λ is the linear charge density. This yields

$$E = \frac{\lambda h}{\epsilon_0 (2\pi r h)} = \frac{\lambda}{2\pi \epsilon_0 r}.$$



- The direction of $\vec{E}$ is radially outward from the rod if the charge is positive, and radially inward if it is negative.

- This also approximates the field of a finite line of charge at points that are not too near the ends (compared with the distance from the rod).

Summary

- Concepts: continuous charge distribution, electric flux
- Understand how to calculate the electric field due to a continuous charge distribution
- Gauss' law

$$\epsilon_0 \Phi = q_{\text{enc}}, \text{ where } \Phi = \oint \vec{E} \cdot d\vec{A}$$

- Applications of Gauss' law
  - Inside a uniform sphere of charge: $E = qr/(4\pi\epsilon_0 R^3)$
  - An infinite nonconducting sheet: $E = \sigma/(2\epsilon_0)$
  - An infinite line of charge: $E = \lambda/(2\pi\epsilon_0 r)$

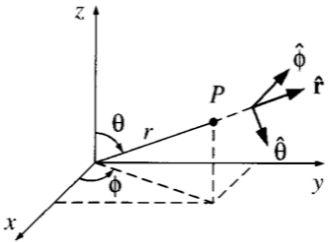
Reading

Halliday, Resnick & Krane:

- Chapter 25. Electric Charge and Coulomb's Law
- Chapter 27. Gauss' Law

Appendix 2A: Spherical Coordinates

- The spherical coordinates (r, θ, ϕ) of a point P are defined in the figure. r is the distance from the origin, the polar angle θ is the angle from the z axis, and the azimuthal angle ϕ is the angle around the x axis.

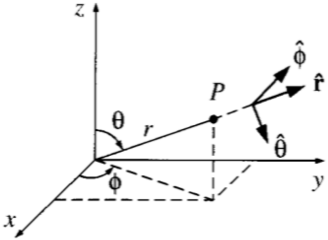


- Note that, in Cartesian coordinates,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

- Any vector can then be expressed in terms of three orthogonal unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ as

$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi},$$



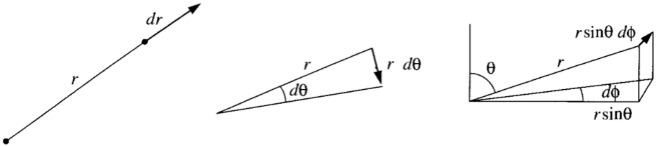
where

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$
$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

- The general infinitesimal displacement $d\vec{s}$ is

$$d\vec{s} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi},$$

as oppose to $d\vec{s} = dx\hat{x} + dy\hat{y} + dz\hat{z}$ in Cartesian coordinates.



- The infinitesimal volume element is, then,

$$dV = r^2 \sin\theta dr d\theta d\phi.$$

Appendix 2B: Surface Integral

- We often encounter the following two types of surface integrals (they are almost the same).
  - The integral of a scalar function, such as mass or charge density.

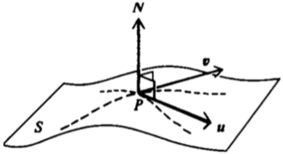
$$\iint_S \sigma(x,y,z)dA$$

  - The integral of the normal component of a vector function, such as electric field.

$$\iint_S \vec{F}(x,y,z)\cdot d\vec{A} = \iint_S \vec{F}(x,y,z)\cdot \hat{n}\ dA$$

The Unit Normal Vector $\hat{n}$

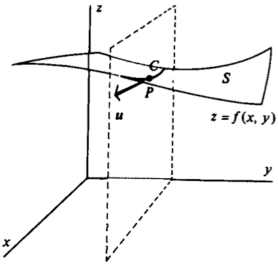
- It is not difficult to define $\hat{n}$ for a flat or a spherical surface. Can you see?
- For an arbitrary surface, however, we need to construct two noncollinear vector $\vec{u}$ and $\vec{v}$ tangent to the surface at some point P. The unit normal vector is defined as



$$\hat{n} = \frac{\vec{N}}{N} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$$

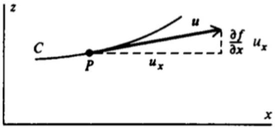
- Consider some surface given by the equation $z = f(x, y)$. We can construct $\vec{u}$ in the xz-plane with an x-component of arbitrary length u_x.

$$\vec{u} = u_x \hat{x} + \frac{\partial f}{\partial x} u_x \hat{z}$$



- We can construct v similarly in the yz-plane. Therefore,

$$\vec{u} \times \vec{v} = \left[-\frac{\partial f}{\partial x} \hat{x} - \frac{\partial f}{\partial y} \hat{y} + \hat{z} \right] u_x v_y$$



- We can calculate $\hat{n}$ as

$$\hat{n} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \frac{-\frac{\partial f}{\partial x}\hat{x} - \frac{\partial f}{\partial y}\hat{y} + \hat{z}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

- In particular, we have

$$\hat{n} \cdot \hat{z} = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

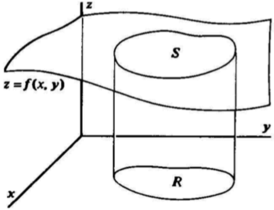
Evaluating Surface Integrals

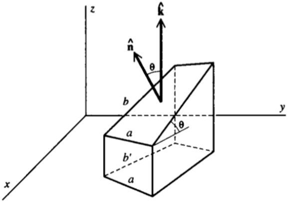
- To evaluate the integral

$$\iint_S \sigma(x, y, z) dA$$

over a portion of the surface $z = f(x, y)$, we cut the surface into rectangles and consider their projection on the xy-plane. The projected area dR of an area dA is

$$dR = \hat{n} \cdot \hat{z} \ dA$$





- Therefore, we can eliminate the apparent z-dependence of the integral and write

$$\iint_S \sigma(x,y,z)dA = \iint_R \sigma(x,y,f(x,y))\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}dxdy$$

- Similarly,

$$\begin{aligned}\iint_S \vec{F}\cdot d\vec{A} &= \iint_R \left(\vec{F}\cdot\hat{n}\right)\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}dxdy \\ &= \iint_R \left[-\frac{\partial f}{\partial x}F_x-\frac{\partial f}{\partial y}F_y+F_z\right]dxdy\end{aligned}$$