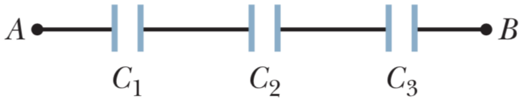
# General Physics II

Solution #2

2020/10/16

**P2-1.** In a three-capacitor, $C_1 = 10.0\ \mu$F, $C_2 = 20.0\ \mu$F, and $C_3 = 25.0\ \mu$F. If no capacitor can withstand a potential difference of more than 100 V without failure, what are (a) the magnitude of the maximum potential difference that can exist between points $A$ and $B$ and (b) the maximum energy that can be stored in the three-capacitor arrangement?

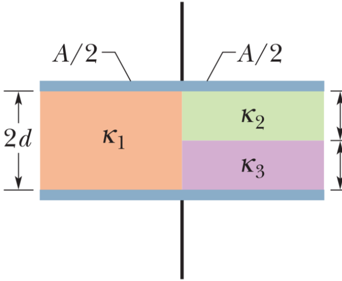


**Solution:**

(a) They each store the same charge, so the maximum voltage is across the smallest capacitor. With 100 V across 10 $\mu$F, then the voltage across the 20 $\mu$F capacitor is 50 V and the voltage across the 25 $\mu$F capacitor is 40 V. Therefore, the voltage across the arrangement is 190 V.

(b) Using $U = \frac{q^2}{2C}$, we sum the energies on the capacitors and obtain $U_{\mathrm{total}} = 0.095$ J.

**P2-2.** A parallel-plate capacitor of plate area $A = 10.5\ \text{cm}^2$ and plate separation $2d = 7.12$ mm. The left half of the gap is filled with material of dielectric constant $\kappa_1 = 21.0$; the top of the




right half is filled with material of dielectric constant $\kappa_2 = 42.0$; the bottom of the right half is filled with material of dielectric constant $\kappa_3 = 58.0$. What is the capacitance?

**Solution:** Let

$$C_1 = \varepsilon_0 (A/2)\kappa_1/2d = \varepsilon_0 A\kappa_1/4d,$$

$$C_2 = \varepsilon_0 (A/2)\kappa_2/d = \varepsilon_0 A\kappa_2/2d,$$

$$C_3 = \varepsilon_0 A\kappa_3/2d.$$

Note that $C_2$ and $C_3$ are effectively connected in series, while $C_1$ is effectively connected in parallel with the $C_2$-$C_3$ combination. Thus,

$$C = C_1 + \frac{C_2 C_3}{C_2 + C_3} = \frac{\varepsilon_0 A\kappa_1}{4d} + \frac{(\varepsilon_0 A/d)(\kappa_2/2)(\kappa_3/2)}{\kappa_2/2 + \kappa_3/2}$$
$$= \frac{\varepsilon_0 A}{4d}\left(\kappa_1 + \frac{2\kappa_2\kappa_3}{\kappa_2 + \kappa_3}\right)$$

With $A = 1.05 \times 10^{-3}$ m$^2$, $d = 3.56 \times 10^{-3}$ m, $\kappa_1 = 21.0$, $\kappa_2 = 42.0$ and $\kappa_3 = 58.0$, we find the capacitance to be

$$C = \frac{(8.85 \times 10^{-12})(1.05 \times 10^{-3})}{4(3.56 \times 10^{-3})} \cdot \left(21.0 + \frac{2 \times 42.0 \times 58.0}{42.0 + 58.0}\right) = 4.55 \times 10^{-11} \text{ F}.$$

**P2-3.** The rain-soaked shoes of a person may explode if ground current from nearby lightning vaporizes the water. The sudden conversion of water to water vapor causes a dramatic expansion that can rip apart shoes. Water has density 1000 kg/$m^3$ and requires 2256 kJ/kg to be vaporized. If horizontal current lasts 2.00 ms and encounters water with resistivity 150 $\Omega \cdot$m, length 12.0 cm, and vertical cross-sectional area $15 \times 10^{-5}$ $m^2$, what average current is required to vaporize the water?

**Solution:** The mass of the water over the length is

$$m = \rho AL = 1000 \cdot 15 \times 10^{-5} \cdot 0.12 = 0.018 \text{ kg}$$

and the energy required to vaporize the water is

$$Q = Lm = 2256 \cdot 0.018 = 4.06 \times 10^{4} \text{ J}.$$

The thermal energy is supplied by joule heating of the resistor:

$$Q = P\Delta t = I^2 R \Delta t$$

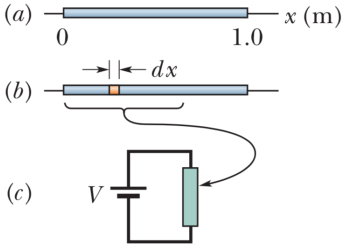
Since the resistance over the length of water is

$$R = \frac{\rho_w L}{A} = \frac{150 \cdot 0.120}{15 \times 10^{-5}} = 1.2 \times 10^5 \ \Omega,$$

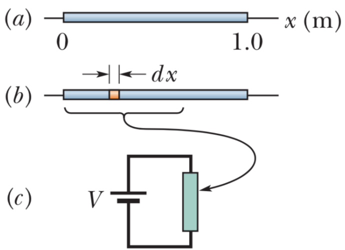
the average current required to vaporize water is

$$I = \sqrt{\frac{Q}{R \Delta t}} = \sqrt{\frac{4.06 \times 10^4}{1.2 \times 10^5 \cdot 2.0 \times 10^{-3}}} = 13.0 \ \mathrm{A}.$$

**P2-4.** There is a rod of resistive material (Figure a). The resistance per unit length of the rod increases in the positive direction of the $x$ axis. At any position $x$ along the rod, the resistance $dR$ of a narrow (differential) section of width $dx$ is given by $dR = 5.00x\ dx$, where $dR$ is in ohms and $x$ is in meters. Figure b shows such a narrow section.



You are to slice off a length of the rod between $x = 0$ and some position $x = L$ and then connect that length to a battery with potential difference $V = 5.0$ V (Figure c). You want the current in the length to transfer energy to thermal energy at the rate of 200 W. At what position $x = L$ should you cut the rod?



**Solution:** From $P = V^2/R$, we have

$$R = 5.0^2/200 = 0.125\ \Omega$$

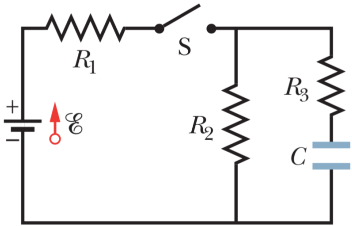
To meet the conditions of the problem statement, we must therefore set

$$\int_0^L 5.00x\ dx = 0.125\ \Omega$$

Thus,

$$\frac{5}{2}L^2 = 0.125 \quad \Rightarrow \quad L = 0.224\ \mathrm{m}.$$

**P2-5.** In a circuit, $\mathcal{E} = 1.2$ kV, $C = 6.5\ \mu$F, $R_1 = R_2 = R_3 = 0.73$ M$\Omega$. With $C$ completely uncharged, switch S is suddenly closed (at $t = 0$). At $t = 0$, what are (a) current $i_1$ in resistor 1, (b) current $i_2$ in resistor 2, and (c) current $i_3$ in resistor 3? At $t = \infty$ (that is, after many time constants), what are (d) $i_1$, (e) $i_2$, and (f) $i_3$? What is the potential difference $V_2$ across resistor 2 at (g) $t = 0$ and (h) $t = \infty$? (i) Sketch $V_2$ versus $t$ between these two extreme times.



**Solution:** Let $i_1$ be the current in $R_1$ and take it to be positive if it is to the right. Let $i_2$ be the current in $R_2$ and take it to be positive if it is downward. Let $i_3$ be the current in $R_3$ and take it to be positive if it is downward. The junction rule produces $i_1 = i_2 + i_3$, the loop rule applied to the left-hand loop produces

$$\varepsilon - i_1 R_1 - i_2 R_2 = 0,$$

and the loop rule applied to the right-hand loop produces

$$i_2 R_2 - i_3 R_3 = 0.$$

Since the resistances are all the same we can simplify the mathematics by replacing $R_1$, $R_2$, and $R_3$ with R.

(a) Solving the three simultaneous equations, we find

$$i_1 = \frac{2\varepsilon}{3R} = \frac{2 \cdot 1.2 \times 10^3}{3 \cdot 0.73 \times 10^6} = 1.1 \times 10^{-3}\mathrm{A}$$

(b) $i_2 = \frac{\varepsilon}{3R} = \frac{1.2\times10^3}{3\cdot0.73\times10^6} = 5.5 \times 10^{-4}$ A,

(c) and $i_3 = i_2 = 5.5 \times 10^{-4}$ A.

At $t = \infty$ the capacitor is fully charged and the current in the capacitor branch is 0. Thus, $i_1 = i_2$, and the loop rule yields $\varepsilon - i_1 R_1 - i_1 R_2 = 0$.

(d) The solution is $i_1 = \frac{\varepsilon}{2R} = \frac{1.2\times10^3}{2\cdot0.73\times10^6} = 8.2 \times 10^{-4}$ A

(e) and $i_2 = i_1 = 8.2 \times 10^{-4}$ A.

(f) As stated before, the current in the capacitor branch is $i_3 = 0$.

We take the upper plate of the capacitor to be positive. This is consistent with current flowing into that plate. The junction equation is $i_1 = i_2 + i_3$, and the loop equations are

$$\varepsilon - i_1 R - i_2 R = 0$$

$$-\frac{q}{C} - i_3 R + i_2 R = 0.$$

We use the first equation to substitute for $i_1$ in the second and obtain

$$\varepsilon - 2i_2 R - i_3 R = 0$$

Thus $i_2 = (\varepsilon - i_3 R)/2R$.

We substitute this expression into the third equation above to obtain

$$-\frac{q}{C} - i_3 R + \frac{\varepsilon}{2} - \frac{i_3 R}{2} = 0.$$

Now we replace $i_3$ with $dq/dt$ to obtain

$$\frac{3R}{2}\frac{dq}{dt} + \frac{q}{C} = \frac{\varepsilon}{2}.$$

This is just like the equation for an $RC$ series circuit, except that the time constant is $\tau = 3RC/2$ and the impressed potential difference is $\varepsilon/2$. The solution is

$$q = \frac{C\varepsilon}{2}(1 - e^{-2t/3RC}).$$

The current in the capacitor branch is

$$i_3(t) = \frac{dq}{dt} = \frac{\varepsilon}{3R} e^{-2t/3RC}.$$

The current in the center branch is

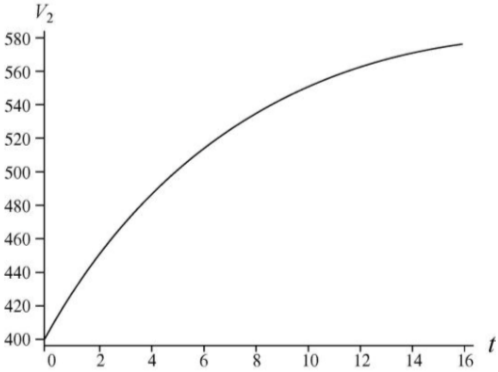
$$i_2(t) = \frac{\varepsilon}{2R} - \frac{i_3}{2} = \frac{\varepsilon}{2R} - \frac{\varepsilon}{6R} e^{-2t/3RC} = \frac{\varepsilon}{6R}(3 - e^{-2t/3RC})$$

and the potential difference across $R_2$ is

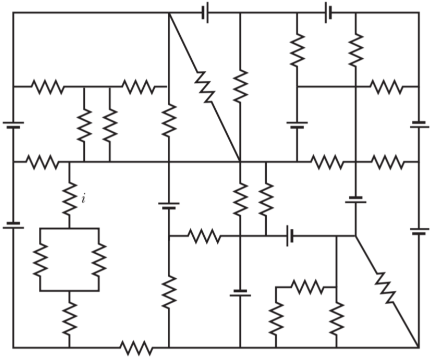
$$V_2(t) = i_2 R = \frac{\varepsilon}{6}(3 - e^{-2t/3RC}).$$

(g) For $t = 0$, $e^{-2t/3RC} = 1$ and
$V_2 = \varepsilon/3 = (1.2 \times 10^3)/3 = 4.0 \times 10^2$ V.
(h) For $t = \infty$, $e^{-2t/3RC} \to 0$ and
$V_2 = \varepsilon/2 = (1.2 \times 20^3)/2 = 6.0 \times 10^2$ V.

(i) A plot of $V_2$ as a function of time is shown in the following graph.



**P2-6.** What are the (a) size and (b) direction (up or down) of current *i*, where all resistances are 4.0 Ω and all batteries are ideal and have an emf of 10 V? (*Hint*: This can be answered using only mental calculation.)



**Solution:** The resistor by the letter $i$ is above three other resistors; together, these four resistors are equivalent to a resistor $R = 10\ \Omega$ (with current $i$). As if we were presented with a maze, we find a path through $R$ that passes through any number of batteries (10, it turns out) but no other resistors, which — as in any good maze — winds "all over the place." Some of the ten batteries are opposing each other (particularly the ones along the outside), so that their net emf is only $\varepsilon = 40$ V.
(a) The current through $R$ is then $i = \varepsilon/R = 4.0$ A.
(b) The direction is upward in the figure.

**P2-7.** A metal sphere of radius 15 cm has a net charge of $3.0 \times 10^{-8}$ C.

(a) What is the electric field at the sphere's surface?
(b) If $V = 0$ at infinity, what is the electric potential at the sphere's surface?
(c) At what distance from the sphere's surface has the electric potential decreased by 500 V?

**Solution:**

(a) The magnitude of the electric field is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{4\pi\varepsilon_0 R^2} = \frac{(3.0 \times 10^{-8})(8.99 \times 10^9)}{(0.15)^2}$$
$$= 1.2 \times 10^4 \ \mathrm{N/C}$$

(b) $V = RE = (0.15)(1.2 \times 10^4) = 1.8 \times 10^3$ V

(c) Let the distance be $x$. Then

$$\Delta V = V(x+R) - V = \frac{q}{4\pi\varepsilon_0}(\frac{1}{R+x} - \frac{1}{R}) = -500 \text{ V}$$

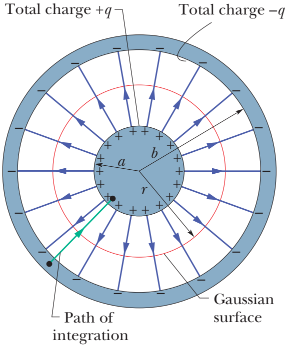
which gives

$$x = \frac{R\Delta V}{-V - \Delta V} = \frac{(0.15)(-500)}{-1800+500} = 5.8 \times 10^{-2} \text{ m}$$

**P2-8.** Consider two concentric spherical shells, of radii $a$ and $b$. Show that the capacitance of the shells is

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

What is the capacitance to a single isolated spherical conductor of radius $R$, then?



**Solution:** As a Gaussian surface we draw a sphere of radius $r$ concentric with the two shells. We have

$$q = \epsilon_0 EA = \epsilon_0 E(4\pi r^2).$$

Solving for $E$, we obtain

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

Hence,

$$V = -\int_-^+ \vec{E} \cdot d\vec{s} = -\frac{q}{4\pi\epsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{b-a}{ab}.$$

Solving for C, we find

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

We can assign a capacitance to a single isolated spherical conductor of radius $R$ by assuming that the “missing plate” is a conducting sphere of infinite radius. To find the capacitance of the conductor, we take the limit $b \to \infty$ and replace $a$ by $R$. Thus we find

$$C = 4\pi\epsilon_0 R.$$

**P2-9.** Show that the curl of a central force $\vec{F}(\vec{r}) = f(r)\hat{r}$ is zero, i.e.,

$$\nabla \times \vec{F}(\vec{r}) = 0.$$

Hence, central forces are conservative.

**Solution:** There are a number of ways of showing that.

(1) If you only know what we taught in class, you can write

$$\vec{F}(\vec{r}) = g(r)\vec{r} = g(r)(x\hat{i} + y\hat{j} + z\hat{k}),$$

where $g(r) = f(r)/r$. The $x$ component of $\nabla \times \vec{F}$ is

$$\begin{aligned}(\nabla \times \vec{F})_x &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ &= \frac{\partial g(r)}{\partial r}\frac{y}{r}z - \frac{\partial g(r)}{\partial r}\frac{z}{r}y \\ &= 0.\end{aligned}$$

So, the curl of a central force is always zero.

(2) One may as well show this by

$$\nabla \times \vec{F} = \nabla \times (g\vec{r}) = (\nabla g) \times \vec{r} + g\nabla \times \vec{r}.$$

Note that $\nabla g$ is radial, hence $(\nabla g) \times \vec{r} = 0$. Obviously, $\nabla \times \vec{r} = 0$.

(3) You may prove this straightforwardly in spherical coordinate system (you can find the formula in an advanced book on E&M). In fact, we can write down the potential energy explicitly as

$$\vec{F} = -\nabla V,$$

where

$$V = -\int f(r)dr.$$

Curl is a well named mathematical term–it denotes the degree of rotation in the vector field (circulation). For this reason, if you go all the way around in a vector field, you'll find that the total integral along that path will depend on the curl of the field in question. If a force had a curl, you could go all the way around and have some net work done, and so it would be nonconservative. A conservative force, on the other hand, cancels itself back out as you go on a closed loop.

The central force has a spherical symmetry and is pointing radially inward/outward. Naturally, its curl is zero.

**P2-10.** Consider a two-dimensional electric field

$$\vec{E}(x, y) = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}.$$

(a) Calculate the curl of the field $\nabla \times \vec{E}$.

(b) Show that the circulation of the field

$$\Gamma = \oint_C \vec{E} \cdot d\vec{s} = 2\pi$$

around a unit circle centered at origin.

Therefore, a vanishing curl does not implies, in general, that the force is conservative. They are equivalent only when the space is simply connected.

**Solution:**

(a) Explicitly, we have

$$E_x = -\frac{y}{x^2 + y^2},$$

$$E_y = \frac{x}{x^2 + y^2}.$$

Taking partial derivatives, we have

$$\frac{\partial E_x}{\partial y} = -\frac{1}{x^2 + y^2} + y\frac{2y}{x^2 + y^2},$$

$$\frac{\partial E_y}{\partial x} = \frac{1}{x^2 + y^2} - x\frac{2x}{x^2 + y^2}.$$

Therefore,

$$\nabla \times \vec{E} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0.$$

(b) The field goes in a circumferential direction with no radial component.

$$
\begin{aligned}
\Gamma &= \oint_C \vec{E} \cdot d\vec{s} \\
&= \int_0^{2\pi} \left( \frac{\hat{\theta}}{r} \right) \cdot (r d\theta \hat{\theta}) \\
&= \int_0^{2\pi} d\theta \\
&= 2\pi
\end{aligned}
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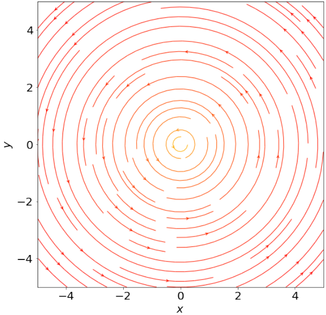


Such a configuration is known as a free vortex. Notice the mathematical singularity at the origin. The space is the punctured plane $(x, y) \neq (0, 0)$. It is not simply connected.