General Physics II

Homework #8

Due 2021/12/29

P8-1. Consider a potential energy barrier for electrons,

$$U(x) = \begin{cases} 0 & x < 0 \text{ (region 1)} \\ U_b & 0 < x < L \text{ (region 2)} \\ 0 & x > L \text{ (region 3)} \end{cases}$$

We send a beam of nonrelativistic electrons with mass m from $x = -\infty$ toward the barrier, each with energy $0 < E < U_b$.

(a) Write down the space-dependent trial wave functions in the three regions.

(b) Write down the boundary conditions at $x = 0$ and $x = L$ using the matching of values and slopes.

(c) When the barrier is sufficiently thick and/or high, the wave function at $x = 0$ in region 2 can be simplified to

$$\psi_2(x) \propto e^{-x/\xi}, \quad \xi = \frac{\hbar}{\sqrt{2m(U_b - E)}}.$$

Show that the transmission coefficient (the probability of transmission) is, then, approximated by

$$T = 16\frac{E}{U_b}\left(1 - \frac{E}{U_b}\right) e^{-2L/\xi}.$$

Therefore, the dominant factor in the transmission coefficient is the exponential $e^{-2L/\xi}$.

P8-2. We have discussed in class that the light emitted by a hydrogen lamp, which contains hydrogen gas, has four discrete wavelengths in the visible range (656 nm, 486 nm, 434 nm, and 410 nm). According to Bohr's theory of the hydrogen atom, the wavelengths of the emitted light are given by

$$\frac{1}{\lambda} = \frac{E_R}{hc}\left(\frac{1}{n^2} - \frac{1}{m^2}\right),$$

where

$$\frac{hc}{E_R} = 912\ \text{Å}.$$

Show that one can use the above information to extract the pair (n, m) for each wavelength.

P8-3. In class we have obtained the ground state energy of the hydrogen atom with the help of Heisenberg's uncertainty principle. Follow the same procedure to estimate the ground state energy of a particle with mass m in a one-dimensional harmonic potential

$$V(x) = \frac{1}{2}kx^2.$$

What is your estimate of the size of the ground-state wave function?

P8-4. Pauli matrices are defined as

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that the two spin states

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are normalized eigenstates of Z with eigenvalues of $+1$ and -1; that is, $Z\,|0\rangle = |0\rangle$ and $Z\,|1\rangle = -\,|1\rangle$. Therefore, we conclude that $S_z = (\hbar/2)Z$.

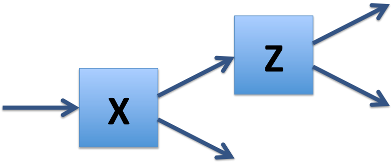
(b) Similarly, we have $S_x = (\hbar/2)X$ and $S_y = (\hbar/2)Y$. The normalized eigenstates of X satisfy $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$. Express $|+\rangle$ and $|-\rangle$ in terms of $|0\rangle$ and $|1\rangle$.

(c) Suppose a beam of electron with spin $|+\rangle$ passing through a Stern-Gerlach apparatus measuring S_z. The probabilities of spin being up and down are

$$P(+\hbar/2) = |\langle 0|+\rangle|^2, \quad P(-\hbar/2) = |\langle 1|+\rangle|^2$$

Calculate $P(\pm\hbar/2)$.

(d) The figure show a Stern-Gerlach experiment with an S_x measurement followed by a S_z measurement. If the incident beam has N electrons with spin $S_z = \hbar/2$, how many electrons are there in the two outcoming beams with $S_z = \pm\hbar/2$, respectively?



(e) How many electrons are there in the outcoming beams if the incident beam has N electrons with spin $S_x = \hbar/2$?

P8-5. A cubical box of widths $L_x = L_y = L_z = L$ contains eight electrons. What multiple of $h^2/(8mL^2)$ gives the energy of the ground state of this system? Assume that the electrons do not interact with one another, and do not neglect spin.

P8-6. Consider standing electromagnetic wave in a cubic cavity. We expect that the number of modes N is proportional to the volume V of the cavity.

(a) Show, by dimension analysis, that the number of modes per unit wavelength is then

$$\frac{dN}{d\lambda} \propto \frac{V}{\lambda^4}.$$

(b) Classically, we expect each mode has average energy $k_B T$. Calculate the spectral radiancy $S(\lambda)$, which is power per unit area of the emitter of the radiation per unit wavelength, for a given temperature T.

(c) To explain the experimental result at short wavelengths, Planck (1900) postulated that each mode has average energy

$$\langle E(\lambda)\rangle = \frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1}.$$

Show that the consequent spectral radiancy agrees with the classical result at long wavelengths.

(d) Using Planck's postulation, show that the wavelength $\lambda_{\max}$ at which the $S(\lambda)$ is maximum is (**Wien's displacement law**)

$$\lambda_{\max} T = 2898\ \mu\mathrm{m/K}.$$

(e) Consider the radiation from the Sun's surface with temperature $T = 6000$ K, can you explain why we often see golden sunlight?

P8-7. Electrons cannot move in definite orbits within atoms, like the planets in our solar system. To see why, let us try to "observe" such an orbiting electron by using a light microscope to measure the electron's presumed orbital position with a precision of, say, 10 pm (a typical atom has a radius of about 100 pm). The wavelength of the light used in the microscope must then be about 10 pm. (a) What would be the photon energy of this light? (b) How much energy would such a photon impart to an electron in a head-on collision? (c) What do these results tell you about the possibility of "viewing" an atomic electron at two or more points along its presumed orbital path? (*Hint*: The outer electrons of atoms are bound to the atom by energies of only a few electron-volts.)