The Triangle of Electrostatics

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Lecture 4

Motivation: Calculating the Electric Field

- How many different approaches have you learned to calculate the electric field of a system of charges?
- How are these approaches related?
  - Coulomb's law
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{q\vec{r}}{r^3}$$
  - Gauss' law
$$\epsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{\mathrm{enc}}$$
  - From the electric potential
$$\vec{E} = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

Outline

- Electric Field as a Gradient
- Electric Field with Zero Curl
- Gauss' Law and the Divergence of Electric Field

Gradient

- Ordinary derivative dx/dt of a function $x(t)$, defined in

$$dx = \left(\frac{dx}{dt}\right) dt,$$

  tells us how rapidly $x(t)$ varies when we change t by a tiny amount, dt.
- How fast the function $V(x, y, z)$ varies, however, depends on what direction we move:

$$dV = \left(\frac{\partial V}{\partial x}\right) dx + \left(\frac{\partial V}{\partial y}\right) dy + \left(\frac{\partial V}{\partial z}\right) dz.$$

- We can write dV as a dot product,

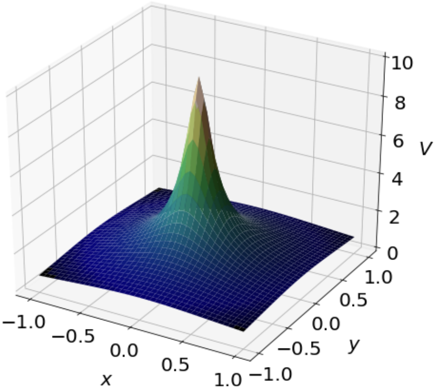
$$dV = \nabla V \cdot d\vec{s},$$

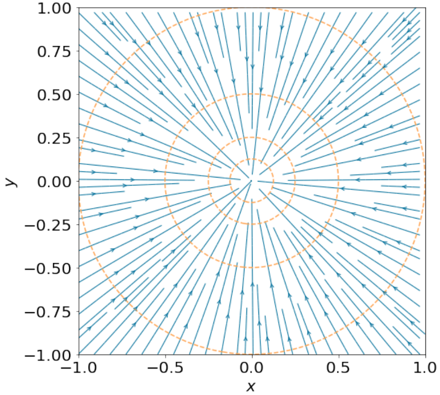
where the **infinitesimal displacement vector** is $d\vec{s} \equiv dx\hat{x} + dy\hat{y} + dz\hat{z}$ and

$$\nabla V \equiv \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

is the **gradient** of V.

- Geometrically, the gradient ∇V points in the direction of maximum increase of the function V (e.g., varying as $1/r$).





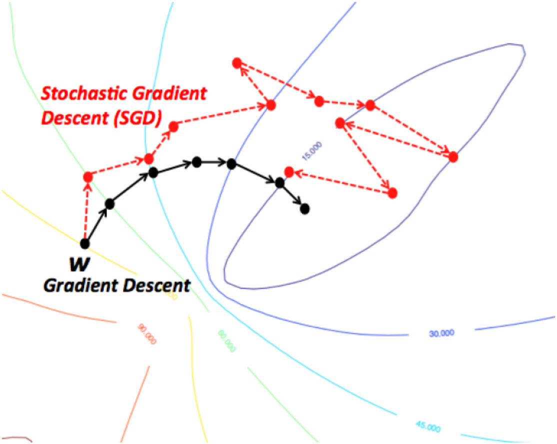


Figure 1: Gradient is useful in many fields, including machine learning.

- We can take one step further to define a **vector operator** that acts upon V as

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z},$$

  which we pronounce as "**del**".

- There are three ways ∇ can act, just as an ordinary vector $\vec{A}$ can multiply,
  - The *gradient*: ∇V
  - The *curl*: $\nabla \times \vec{v}$
  - The *divergence*: $\nabla \cdot \vec{v}$

Electric Field and Electric Potential

- We have learned to find V from $\vec{E}$ via

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}.$$

- On the other hand, we find $\vec{E}$ from V via

$$\vec{E} = -\nabla V.$$

- $\vec{E} = -\nabla V$ is a vector quantity with three components, but V is a scalar. How can one function possibly contain all the information that three independent functions carry?

- For example, we consider $V(r) = 1/r$.

$$E_x = -\frac{\partial}{\partial x}\frac{1}{r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x} = \frac{1}{r^2}\frac{x}{r} = \frac{x}{r^3}$$

  Note that $r = \sqrt{x^2 + y^2 + z^2}$.
- Similarly, we have

$$E_y = \frac{y}{r^3}, \quad E_z = \frac{z}{r^3}$$

- The symmetric expression implies that the three components of $\vec{E}$ are not really as independent as one might think.

- In fact, $\vec{E}$ is a special kind of vector, whose curl is always zero,

$$\nabla \times \vec{E} = 0.$$

- This is not a surprising result in light of the radial nature of the electrostatic field of a point charge. (We will come to the geometrical meaning of curl when we discuss magnetic field.)

- Now, what is the algebraic form of curl?

Curl

- Recall that

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

- From the definition of ∇ we can construct

$$\begin{aligned} \nabla \times \vec{v} =& \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} \\ =& \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \end{aligned}$$

- Now, consider a vector $\vec{v}$, which is a gradient like $\vec{E}$,

$$\vec{v} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}.$$

- From the definition of ∇ we can write, e.g., for the x component (similarly, for the y and z components)

$$\begin{aligned}(\nabla \times \vec{v})_x &= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ &= \frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y} = 0.\end{aligned}$$

So, *the curl of a gradient is always zero*.

The Triangle of Electrostatics

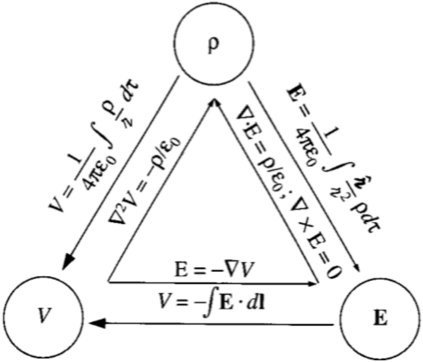
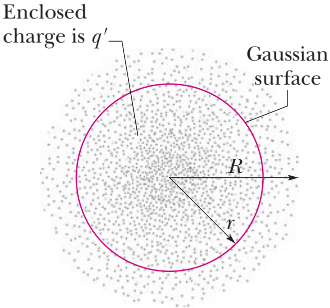


Figure 2: The fundamental quantities and formulas of electrostatics.

- It is generally to your advantage to calculate the potential first, unless the symmetry of the problem admits a solution by Gauss' law.
- How to calculate $\rho(\vec{r})$ (not $q_{\rm enc}$) from $\vec{E}(\vec{r})$ or $V(\vec{r})$?

- Recall, e.g., that for a uniform distribution of charge q of radius R, we have, for $r \leq R$,

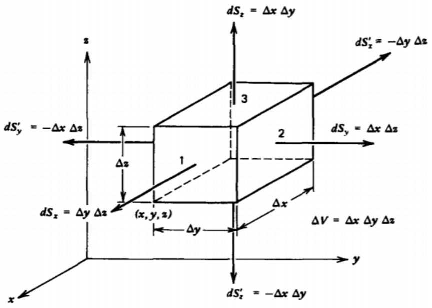
$$\vec{E} = \left(\frac{q}{4\pi\epsilon_0 R^3}\right)\vec{r}.$$



- The integral form of Gauss' law only gives us the total charge inside a Gaussian surface $q_{\mathrm{enc}} = \epsilon_0 \oint \vec{E} \cdot d\vec{A}$.
- However, choosing a sufficiently small surface enclosing a volume ΔV and charge $\rho(\vec{r})\Delta V$, we can show that

$$\frac{\rho(\vec{r})}{\epsilon_0} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{q_{\mathrm{enc}}^{\Delta V}}{\epsilon_0} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint \vec{E}(\vec{r}) \cdot d\vec{A}$$

- To proceed, we choose a Gaussian surface to enclose a small cube centered at $\vec{r}$ with sides Δx, Δy, and Δz, such that $\Delta V = \Delta x \Delta y \Delta z$.
- To evaluate the surface integral we must consider separately the six sides of the cube, each with a multiplication of the component of $\vec{E}$ perpendicular to the surface and the surface area.



- The surface integral over the two surfaces perpendicular to the x axis at $x \pm \Delta x/2$ is

$$\vec{E}\left(x+\frac{\Delta x}{2}, y, z\right)\cdot\hat{x}\Delta y\Delta z+\vec{E}\left(x-\frac{\Delta x}{2}, y, z\right)\cdot(-\hat{x})\Delta y\Delta z$$

$$= E_x\left(x+\frac{\Delta x}{2}, y, z\right)\Delta y\Delta z - E_x\left(x-\frac{\Delta x}{2}, y, z\right)\Delta y\Delta z,$$

which becomes $(\partial E_x/\partial x)\Delta V$ in the small ΔV (hence, small Δx) limit.

- Including contributions from the other four surfaces, we find

$$\frac{\rho(\vec{r})}{\epsilon_0} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \equiv \nabla \cdot \vec{E}.$$

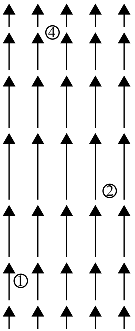
  This is the differential form of the Gauss' law.

- Here, we introduce the **divergence** of a vector $\vec{A}$ as

$$\begin{aligned}\nabla \cdot \vec{A} &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z) \\ &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.\end{aligned}$$

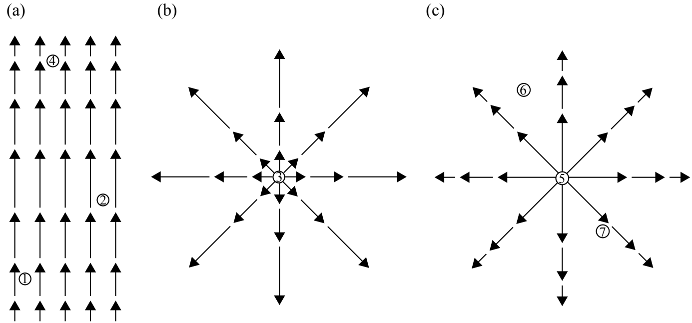
Comments on Divergence

- According to Gauss' law, the only places at which the divergence of the electric field is not zero are those locations at which charge is present. So the divergence is a measure of the tendency of the field to flow away from a (charged) point.
- Nevertheless, the divergence is dependent both on the spreading out and the changing length of field lines. Note that in the right figure $\nabla \cdot \vec{A} = \partial A_z / \partial z$ is not zero in general.



Positive or Negative Divergence?

- Which of the following points have positive (negative) divergence?



Fundamental Theorem for Divergences

- Our discussion on the divergence of $\vec{E}$ illustrates a famous relation between surface integral and volume integral:

$$\oint \text{flow out through the surface} = \int \text{sources/drains within the volume.}$$

- Formally, this is the **fundamental theorem for divergences**:

$$\oint_S \vec{v} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{v}) dV.$$

When to Apply Gauss' Law

- Gauss' law is always true, but not always useful. We can use it in the following situations:
  - Given a symmetric charge distribution, find $\vec{E}$.
  - Given the flux through a closed surface, find the enclosed charge.
  - Given a charge distribution, find the flux through a closed surface surrounding that charge.
  - Given $\vec{E}$ over a surface, find the charge enclosed by the surface.
  - Given $\vec{E}$ in a specified region, find the density of electric charge within that region.

Quiz 4-1

Summary

- In electrostatics, we deal with electric charge density $\rho(r)$, electric field $\vec{E}(r)$, and electric potential $V(r)$. They are related by a series of vector expressions.

$$\nabla \times \vec{E} = 0 \quad \leftrightarrow \quad \vec{E} = -\nabla V$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

- Combining the last two, we obtain **Poisson's Equation**:

$$\nabla^2 V \equiv \nabla \cdot \nabla V = -\frac{\rho}{\epsilon_0}$$

- Experiments observe that the electrostatics force is conservative, or $\oint \vec{E} \cdot d\vec{s} = 0$.

- We are, then, allowed to define electric potential difference:

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}.$$

Or, in differential form

$$\vec{E} = -\nabla V.$$

- Therefore,

$$\nabla \times \vec{E} = 0.$$

This can be derived directly via the fundamental theorem for curls.

- Between $\vec{E}$ and ρ, we have

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho dV$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

- The two forms of Gauss' law are equivalent because of the fundamental theorem for divergences.

Sample Electrostatic Problem

- Describe the electric field and the charge distribution that go with the following potential:

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{e^{-(1/a)(x^2+y^2+z^2)^{1/2}}}{(x^2+y^2+z^2)^{1/2}},$$

where q is a constant charge, and a is a characteristic length.

Reading

Halliday, Resnick & Krane:

- Chapter 25-28.

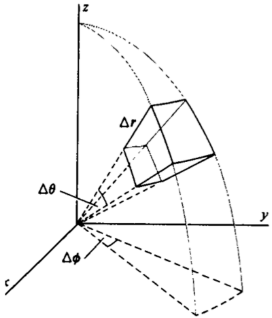
Additional references:

- D. J. Griffiths, Introduction to electrodynamics, 3rd ed., Prentice Hall, 1999.
- D. Fleisch, A student's guide to Maxwell's equations, Cambridge University Press, 2008

Appendix 4A: Div in Spherical Coordinates

- In spherical coordinates, where the components of $\vec{F}$ are F_r, F_θ, and F_ϕ, recall that the general infinitesimal displacement $d\vec{s}$ is (Appendix 2A)

$$d\vec{s} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}.$$



- The infinitesimal volume element of the spherical cuboid is

$$\Delta V = r^2 \sin\theta drd\theta d\phi.$$

- According to the fundamental theorem for divergences,

$$\nabla \cdot \vec{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint \vec{F}(\vec{r}) \cdot d\vec{A}.$$

- One can be convinced that

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.$$

Appendix 4B: Dirac Delta Function

- The divergence of a vector function of the form $\vec{F} = f(r)\hat{r}$ is, therefore,

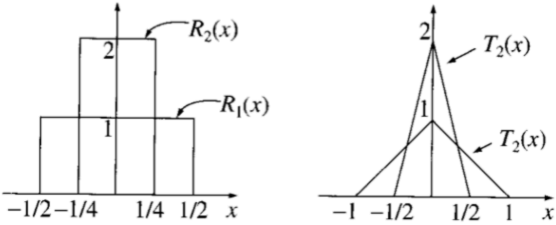
$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 f(r)] = \frac{2}{r}f(r) + f'(r).$$

- In particular, if $f(r) = 1/r^2$, the divergence is precisely zero. This cannot be true because we know that $1/r^2$ is, essentially, the electric field generated by a point charge at the origin. So $\nabla \cdot (\hat{r}/r^2) \neq 0$ at $r = 0$, where $f(r)$ diverges.
- The bizarre situation is because the density of a point charge diverges, but its integral (the total charge) is finite.

- Physicists introduce the Dirac delta function to describe such a mathematical object.

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

- Technically, it is like the limit of a sequence of functions, such as rectangles $R_n(x)$ or isosceles triangles $T_n(x)$.



(a) (b)

- Now we can write

$$\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi\delta^3(\vec{r})$$

where

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

- The divergence of is zero everywhere except at the origin, and yet its integral over any volume containing the origin is a constant of 4π.
- A point particle at point $\vec{r}_0$ with charge q, therefore, has a charge density

$$\rho(\vec{r}) = q\delta^3(\vec{r} - \vec{r}_0)$$