Light as Electromagnetic Waves

Xin Lu/Gentaro Watanabe

Lecture 15

Outline

- Unifying Light and Electromagnetic Waves
- The Propagation of Light Rays
    - Transmission
    - Reflection
    - Refraction
- Understanding the Laws of Reflection and Refraction
    - Fermat's Principle
    - Huygens' Principle
    - The Electromagnetic Approach

Electromagnetic Waves

- In the previous lecture, we have derived the electromagnetic wave equation in vacuum:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \quad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2},$$

where c is the speed of light, which is fixed nowadays as exactly

$$c = 2.997\ 924\ 58 \times 10^8\ \mathrm{m/s}.$$

(In 1676, the Danish astronomer Ole Roemer became the first person to measure the speed of light.)

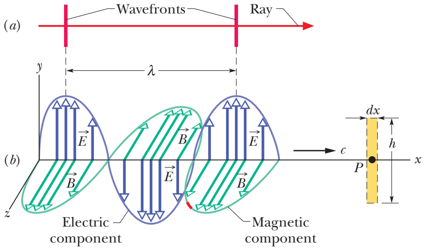
- In Maxwell's time, the discovery was astounding: Light is an electromagnetic wave.
- As Maxwell commented in 1852:

  {"This velocity is very nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws."}

- Note that we have not specified any reference frame. In other words, the speed of light is the same in any inertial frame of reference (see Appendix 15A).

Plane Wave

- Any solution to the wave equation can be written as the superposition of plane waves.



- A plane wave propagating along, e.g., $+x$ direction is

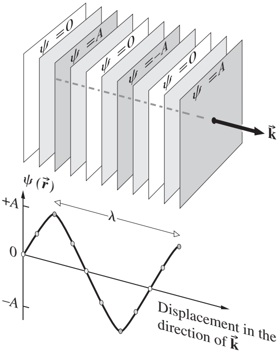
$$\vec{E}(x,t) = \vec{E}_m \cos(kx - \omega t + \phi),$$

$$\vec{B}(x,t) = \vec{B}_m \cos(kx - \omega t + \phi).$$

Wavefronts

- Quite generally, we can consider a plane wave propagating along any direction in three dimensions:

$$\psi(\vec{r}) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \phi).$$



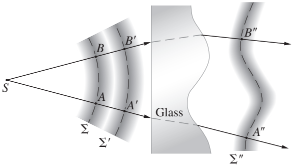
- At any instant a **wavefront** is a surface of constant phase. For the plane wave such a surface is

$$\vec{k}\cdot\vec{r} = \text{constant}.$$

- In reality, wavefronts usually have extremely complicated configurations.

The Propagation of Light Rays

- **Rays** are directed lines perpendicular to the wavefronts that indicate the direction of travel of the wavefronts.
- When the wavelength of light is much smaller than the length scales of your observation, light behaves in a much simpler way:



Electromagnetic waves reduce to light rays.

- The study of the properties of light waves under the approximation that it travels in a straight line is called **geometrical optics**. Our present concern is with the basic phenomena of *transmission, reflection, and refraction*.

Transmission of Light in Matter

- In dielectric materials, the electric field is altered by a factor ϵ_r, the **relative permittivity** (also called the dielectric constant κ).
- In magnetic materials (therefore, not in glass or plastic), the magnetic field is altered by a factor μ_r, the **relative permeability**.
- Therefore, a light wave propagating through any substantive medium travels at a speed

$$v = \frac{1}{\sqrt{\epsilon_r \mu_r}} \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{c}{n},$$

where the **index of refraction** $n = \sqrt{\epsilon_r \mu_r}$.

- In particular, $n = 1.00029$ for air.

Medium	Index	Medium	Index
Vacuum	Exactly 1	Typical crown glass	1.52
Air (STP)[b]	1.00029	Sodium chloride	1.54
Water (20°C)	1.33	Polystyrene	1.55
Acetone	1.36	Carbon disulfide	1.63
Ethyl alcohol	1.36	Heavy flint glass	1.65
Sugar solution (30%)	1.38	Sapphire	1.77
Fused quartz	1.46	Heaviest flint glass	1.89
Sugar solution (80%)	1.49	Diamond	2.42

Table 1: Indices for a wavelength of 589 nm (yellow sodium light). STP means standard temperature (0°C) and pressure (1 atm).

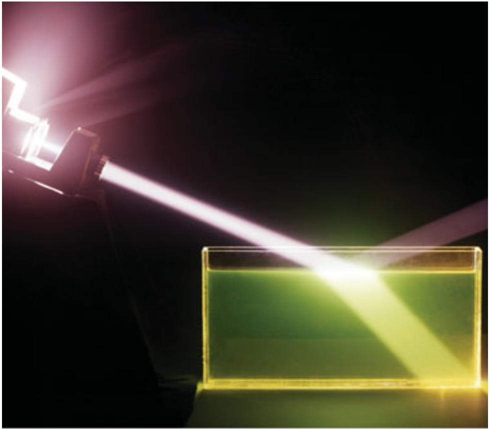
- The dispersion relation becomes

$$\omega = vk = ck/n,$$

hence $k = nk_0$, where k_0 is the wave number in vacuum.

Reflection and Refraction

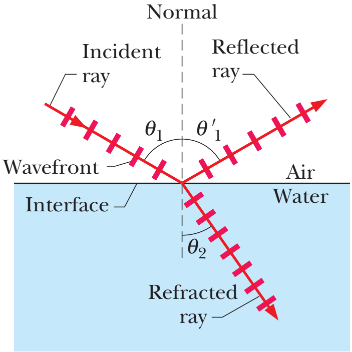
- Light waves travel in approximately straight lines.



{A narrow beam of light, angled downward from the left and traveling through air, encounters a flat water surface.}

- The **incident beam** of light is **reflected** by the surface, as if the beam had bounced from the surface.
- **Law of reflection**: A reflected ray lies in the plane of incidence and has an angle of reflection equal to the angle of incidence (both relative to the normal):

$$\theta_1' = \theta_1.$$



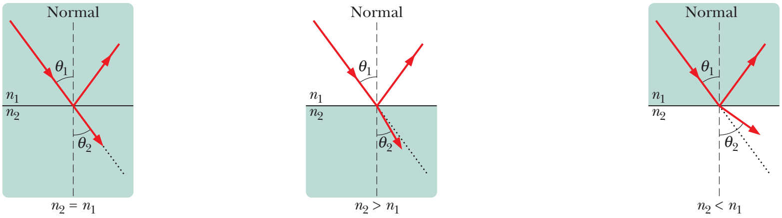
- Light is reflected like an elastic ball bounced from floor.

- The travel of light through a surface (or interface) that separates two media is called **refraction**, and the light is said to be refracted. Unless an incident beam of light is perpendicular to the surface, the beam is said to be *bent* by the refraction.
- **Law of refraction**: A refracted ray lies in the plane of incidence and has an angle of refraction θ_2 that is related to the angle of incidence θ_1 by

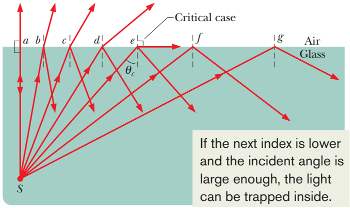
$$n_2 \sin\theta_2 = n_1 \sin\theta_1.$$

The equation is called **Snell's law**.

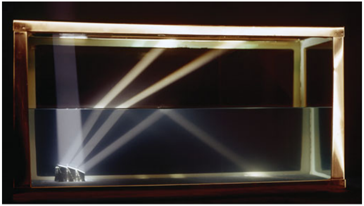
- If $n_2 = n_1$, then $\theta_2 = \theta_1$. The light beam continues in the undeflected direction.
- If $n_2 > n_1$, then $\theta_2 < \theta_1$. Refraction bends the light beam away from the undeflected direction and toward the normal.
- If $n_2 < n_1$, then $\theta_2 > \theta_1$. Refraction bends the light beam away from the undeflected direction and away from the normal.



Total Internal Reflection



(a)



(b)

- With progressively larger angles of incidence θ at the interface, the angle of refraction also increases. At a critical θ_c, the refracted ray points directly along the interface, i.e.,

$$n_{\mathrm{glass}} \sin\theta_c = n_{\mathrm{air}} \sin(\pi/2).$$

For larger θ, all the light is reflected.

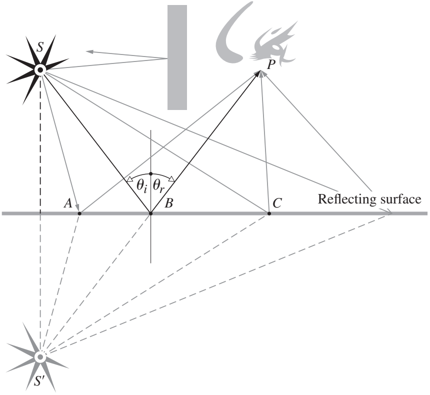
- Total internal reflection has found many applications in telecommunication and medical technology. For example, optical fibers work because of total internal reflection; light follows the zigzag path through the fibers.

- There are two thin bundles of optical fibers in an endoscope, one for light to illuminate the inside of the patient and another for a camera to send the image back to the physician.

Fermat's Principle

- The laws of reflection and refraction, and indeed the manner in which light propagates in general, can be viewed from an entirely different and intriguing perspective afforded us by **Fermat's principle**.
- In 1657 Fermat propounded his celebrated **principle of least time**: *the actual path between two points taken by a beam of light is the one that is traversed in the least time*.
- For a homogeneous medium the principle reduces to the law of the rectilinearity of a ray of light (in accordance with the axiom of geometry that a straight line is the shortest distance between two points).

- For the case of a ray incident on the interface between different media, the laws of the reflection and refraction of light can be obtained from Fermat's principle.



For reflection, the straight-line path $S'BP$, which corresponds to

$$\theta_i = \theta_r,$$

is the shortest possible one.

- For refraction, the transit time from S to P is

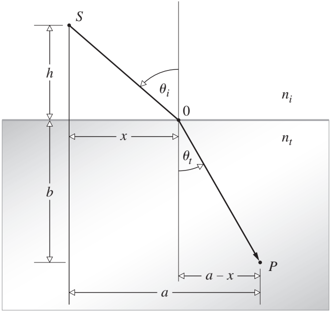
$$t = \frac{\overline{SO}}{v_i} + \frac{\overline{OP}}{v_t} = \frac{\sqrt{h^2 + x^2}}{c/n_i} + \frac{\sqrt{b^2 + (a-x)^2}}{c/n_t}.$$

- To minimize $t(x)$ with respect to variations in x, we set $dt/dx = 0$, i.e.,

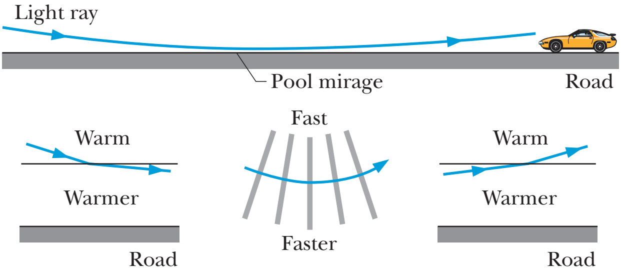
$$\frac{dt}{dx} = \frac{n_i}{c}\frac{x}{\overline{SO}} - \frac{n_t}{c}\frac{a-x}{\overline{OP}} = 0.$$

- Therefore,

$$n_i \sin\theta_i = n_t \sin\theta_t.$$



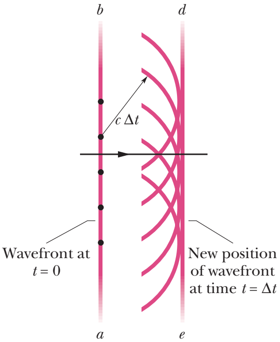
Understanding Mirage with Fermat's Principle

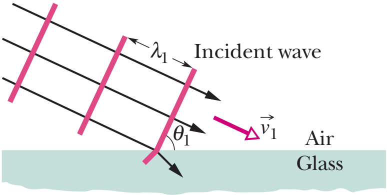


- At very low angles the rays appear to be coming from beneath the road as if reflected in a puddle.
- The effect is easy to view on long modern highways. The only requirement is that you look at the road at near glancing incidence, because the rays bend very gradually.

Huygens' Principle

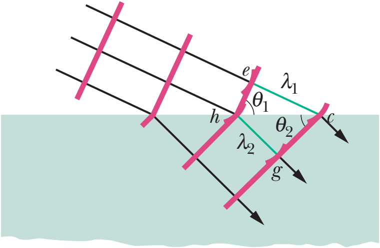
- Christian Huygens' wave theory (1678) is based on a geometrical construction that allows us to tell where a given wavefront will be at any time in the future if we know its present position.
- **Huygens' principle** is: *All points on a wavefront serve as point sources of spherical secondary wavelets. After a time t, the new position of the wavefront will be that of a surface tangent to these secondary wavelets.*





- We arbitrarily choose the wavefronts in the incident light beam to be separated by λ_1, the wavelength in medium 1. Let θ_1 be the angle of incidence.
- Assume that the speed of light in air be v_1 and that in glass be v_2.

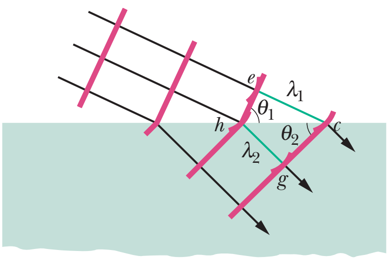
- As the wave moves into the glass, a Huygens wavelet at point e will expand to pass through point c, at a distance of λ_1 from point e.

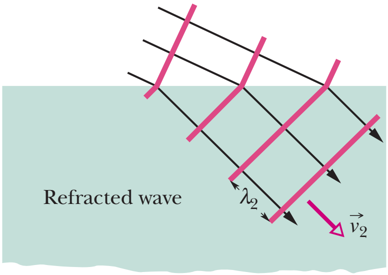


- In this same time interval, a Huygens wavelet at point h will expand to pass through point g, at the reduced speed v_2 and with wavelength λ_2. So,

$$\lambda_1/\lambda_2 = v_1/v_2.$$

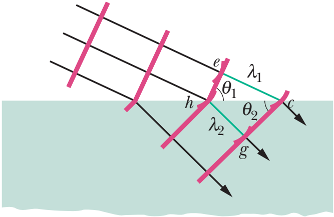
- By Huygens' principle, the refracted wavefront must be tangent to an arc of radius λ_2 centered on h, say at point g; also be tangent to an arc of radius λ_1 centered on e, say at c. Then the refracted wavefront must be oriented as shown. The angle of refraction is actually θ_2.





- Based on geometry, we may write

$$\sin\theta_1 = \frac{\lambda_1}{hc}, \qquad \sin\theta_2 = \frac{\lambda_2}{hc}.$$



- Therefore, with $n = c/v$, we have

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1},$$

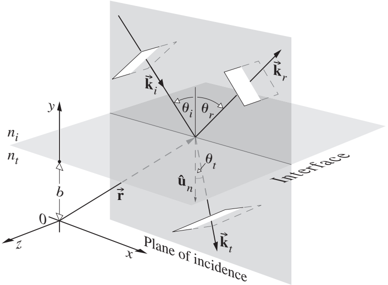
or

$$n_2 \sin\theta_2 = n_1 \sin\theta_1.$$

The Electromagnetic Approach

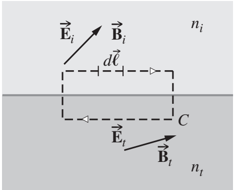
- Suppose that the incident, reflected, and transmitted waves can be written as

$$\vec{E}_i = \vec{E}_{0i}\cos(\vec{k}_i \cdot \vec{r} - \omega_i t)$$
$$\vec{E}_r = \vec{E}_{0r}\cos(\vec{k}_r \cdot \vec{r} - \omega_r t + \phi_r)$$
$$\vec{E}_t = \vec{E}_{0t}\cos(\vec{k}_t \cdot \vec{r} - \omega_t t + \phi_t)$$



- So we have $\vec{E} = \vec{E}_i + \vec{E}_r$ above the interface and $\vec{E} = \vec{E}_t$ below. For simplicity, we consider the case that $\vec{E}_{0i}$, $\vec{E}_{0r}$, $\vec{E}_{0t}$ are constant in time (i.e., linearly polarized).

- The laws of electromagnetic theory lead to certain requirements that must be met by the fields, and they are referred to as the **boundary conditions**.
- For example, we draw a narrow closed path C that runs parallel to the interface inside both media. According to Faraday's induction law



$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}.$$

- The loop can be made so narrow such that there is no flux through C. Define $\hat{u}_n$ to be the unit vector normal to the interface (see figure in the previous page).

- The boundary condition leads to

$$\hat{u}_n \times (\vec{E}_i + \vec{E}_r) - \hat{u}_n \times \vec{E}_t = 0,$$

which is satisfied for all values of time and at any point on the interface. That is,

$$\hat{u}_n \times \vec{E}_{0i} \cos(\vec{k}_i \cdot \vec{r} - \omega_i t) + \hat{u}_n \times \vec{E}_{0r} \cos(\vec{k}_r \cdot \vec{r} - \omega_r t + \phi_r)$$
$$= \hat{u}_n \times \vec{E}_{0t} \cos(\vec{k}_t \cdot \vec{r} - \omega_t t + \phi_t).$$

- This can only be satisfied if $\omega_i = \omega_r = \omega_t$, which means the *charged particles within the media are undergoing forced oscillations at the frequency of the incident wave*.

- Furthermore, for any $\vec{r}$ terminating on the interface,

$$(\vec{k}_i \cdot \vec{r})|_{y=b} = (\vec{k}_r \cdot \vec{r} + \phi_r)|_{y=b} = (\vec{k}_t \cdot \vec{r} + \phi_t)|_{y=b}.$$

- Thus, we find

$$[(\vec{k}_i - \vec{k}_r) \cdot \vec{r}]_{y=b} = \phi_r,$$

  or

$$(\vec{k}_i - \vec{k}_r) \cdot (\vec{r}_1 - \vec{r}_2) = 0,$$

  for any pair of $\vec{r}_1$ and $\vec{r}_2$ terminating on the interface.
- But we also have $\hat{u}_n \cdot (\vec{r}_1 - \vec{r}_2) = 0$, so $(\vec{k}_i - \vec{k}_r)$ is parallel to $\hat{u}_n$, or $k_i \sin\theta_i = k_r \sin\theta_r$.

- Since the incident and reflected waves are in the same medium, $k_i = k_r$, so, finally, $\theta_i = \theta_r$ (the law of reflection).
- Similarly, $(\vec{k}_i - \vec{k}_t)$ is also parallel to $\hat{u}_n$, i.e.,

$$\vec{k}_i \times \hat{u}_n = \vec{k}_t \times \hat{u}_n,$$

or (notice $\vec{k} = nk_0\hat{k}$)

$$n_i(\hat{k}_i \times \hat{u}_n) = n_t(\hat{k}_t \times \hat{u}_n),$$

which is nothing but the law of refraction.

- Note that the law of reflection and the law of refraction only rely on the phase relationship that exists among the phases of $\vec{E}_i$, $\vec{E}_r$, and $\vec{E}_t$ at the boundary.
- There is still an interdependence shared by the amplitudes $\vec{E}_{0i}$, $\vec{E}_{0r}$, and $\vec{E}_{0t}$. The additional constraint can be used to calculate the amplitude of the reflected wave and the transmitted wave (the Fresnel equations). This will lead to the phenomenon of *polarization by reflection* (to be discussed in the lecture on polarization).

Summary

- It is important to understand how light, as electromagnetic waves, propagates, especially at an interface.
  - Laws of reflection and refraction

$$n_1(\hat{k}_1 \times \hat{u}_n) = n_2(\hat{k}_2 \times \hat{u}_n),$$

- Understand how Fermat's principle of least time or Huygens' principle leads to the laws of reflection and refraction.

Reading

Halliday, Resnick & Krane:

- Chapter 39: Light Waves

Appendix 15A: Light in Another Inertial Frame

- Suppose that frame F' is moving with speed v in the $\hat{x}$ direction relative to F, with its axes parallel to those of F. To simplify matters let the space and time origins of F' and F coincide.
- Let a moving object M pass the origin at $t = 0$. The Lorentz transformation relating the space-time coordinates of M in F' and F is

$$x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z,$$

$$t' = \gamma\left(t - \beta\frac{x}{c}\right).$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$ as usual.

- The transformation of the field components is

$$E'_x = E_x,\ \ E'_y = \gamma(E_y - \beta c B_z),\ \ E'_z = \gamma(E_z + \beta c B_y),$$

$$B'_x = B_x,\ \ B'_y = \gamma\left(B_y + \beta\frac{E_z}{c}\right),\ \ B'_z = \gamma\left(B_z - \beta\frac{E_y}{c}\right).$$

- Or, one can write

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \quad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp}).$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}, \quad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - (\vec{v}/c^2) \times \vec{E}_{\perp}).$$

- They result from the same Lorentz transformation that apply to x and t on the previous slide.

- The key is to find invariants that are not changed under Lorentz transformation of the fields. For example,

$$
\begin{aligned}
\vec{E}' \cdot \vec{B}' =& E'_x B'_x + E'_y B'_y + E'_z B'_z \\
=& E_x B_x \\
& + \gamma^2 (E_y B_y + \frac{\beta}{c} E_y E_z - c\beta B_y B_z - \beta^2 E_z B_z) \\
& + \gamma^2 (E_z B_z - \frac{\beta}{c} E_y E_z + c\beta B_y B_z - \beta^2 E_y B_y) \\
=& E_x B_x + \gamma^2 (1-\beta^2)(E_y B_y + E_z B_z) = \vec{E} \cdot \vec{B}.
\end{aligned}
$$

- Similarly, one can show $E'^2 - c^2 B'^2 = E^2 - c^2 B^2$.

- The invariance of the two quantities is a general property of any electromagnetic field.
- For a plane wave, $\vec{B}$ is always perpendicular to $\vec{E}$ and $B = E/c$. Each of the two invariants, $\vec{E} \cdot \vec{B}$ and $E^2 - c^2B^2$, is therefore zero.
- Any Lorentz tranformation of the wave will leave E and B perpendicular and equal in magnitude. *A light wave looks like a light wave in any inertial frame of reference.*
- We arrive again at the postulates of relativity. What would be observed if one could catch up with a light wave?