General Physics II

Solution #8

2021/12/29

P8-1. Consider a potential energy barrier for electrons,

$$U(x) = \begin{cases} 0 & x < 0 \text{ (region 1)} \\ U_b & 0 < x < L \text{ (region 2)} \\ 0 & x > L \text{ (region 3)} \end{cases}$$

We send a beam of nonrelativistic electrons with mass m from $x = -\infty$ toward the barrier, each with energy $0 < E < U_b$.

(a) Write down the space-dependent trial wave functions in the three regions.

(b) Write down the boundary conditions at $x = 0$ and $x = L$ using the matching of values and slopes.

(c) When the barrier is sufficiently thick and/or high, the wave function at $x = 0$ in region 2 can be simplified to

$$\psi_2(x) \propto e^{-x/\xi}, \quad \xi = \frac{\hbar}{\sqrt{2m(U_b - E)}}.$$

Show that the transmission coefficient (the probability of transmission) is, then, approximated by

$$T = 16\frac{E}{U_b}\left(1 - \frac{E}{U_b}\right) e^{-2L/\xi}.$$

Therefore, the dominant factor in the transmission coefficient is the exponential $e^{-2L/\xi}$.

Solution:

(a)

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \text{ (region 1)} \\ Ce^{-k_b x} + De^{k_b x} & 0 < x < L \text{ (region 2)} \\ Ee^{ikx} & x > L \text{ (region 3)} \end{cases}$$

where $k = \sqrt{2mE}/\hbar$, $k_b = \sqrt{2m(U_b - E)}/\hbar$.

(b) At $x = 0$,

$$A + B = C + D \text{ (matching of values)},$$

$$iAk - iBk = -Ck_b + Dk_b \text{ (matching of slopes)}.$$

At $x = L$,

$$Ce^{-k_b L} + De^{k_b L} = Ee^{ikL} \text{ (matching of values)},$$

$$-Ck_b e^{-k_b L} + Dk_b e^{k_b L} = iEke^{ikL} \text{ (matching of slopes)}.$$

(c) When the barrier is sufficiently thick and/or high, one expects the transmission coefficient is small, so the reflection coefficient is close to one. Therefore, the amplitude of the exponentially increasing term D must be small compared to C. The boundary conditions at $x = 0$ (note that we set $D = 0$) and $x = L$ become:

$$A + B = C \text{ (matching of values)},$$

$$iAk - iBk = -Ck_b \text{ (matching of slopes)},$$

$$Ce^{-k_b L} + De^{k_b L} = Ee^{ikL} \text{ (matching of values)},$$

$$-Ck_b e^{-k_b L} + Dk_b e^{k_b L} = iEke^{ikL} \text{(matching of slopes)}.$$

From the boundary condition in $x = 0$, we can obtain

$$\left|\frac{C}{A}\right|^2 = 4\frac{E}{U_b}.$$

From the boundary condition in $x = L$, we can obtain

$$\left|\frac{E}{C}\right|^2 = 4\frac{U_b - E}{U_b}e^{-2k_b L}.$$

Therefore,

$$T = \left|\frac{E}{A}\right|^2 = 16\frac{E}{U_b}\left(1 - \frac{E}{U_b}\right)e^{-2L/\xi},\ \xi = \frac{\hbar}{\sqrt{2m(U_b - E)}}.$$

P8-2. We have discussed in class that the light emitted by a hydrogen lamp, which contains hydrogen gas, has four discrete wavelengths in the visible range (656 nm, 486 nm, 434 nm, and 410 nm). According to Bohr's theory of the hydrogen atom, the wavelengths of the emitted light are given by

$$\frac{1}{\lambda} = \frac{E_R}{hc}\left(\frac{1}{n^2} - \frac{1}{m^2}\right),$$

where

$$\frac{hc}{E_R} = 912 \text{ Å}.$$

Show that one can use the above information to extract the pair (n, m) for each wavelength.

Solution: If wavelength equals to 410 nm, the equation becomes

$$\frac{91.2}{410} = \frac{1}{n^2} - \frac{1}{m^2} \approx 0.22$$

Therefore, n equals to 1 or 2. If n equals to 1, m is not an integer. Therefore, n equals to 2. The equation becomes

$$\frac{91.2}{410} = \frac{1}{4} - \frac{1}{m^2}, \;\; m \approx 6$$

Similarly, for 434 nm, 486 nm, and 656 nm, n equals to 2, while m equals to 5, 4, and 3.

P8-3. In class we have obtained the ground state energy of the hydrogen atom with the help of Heisenberg's uncertainty principle. Follow the same procedure to estimate the ground state energy of a particle with mass m in a one-dimensional harmonic potential

$$V(x) = \frac{1}{2}kx^2.$$

What is your estimate of the size of the ground-state wave function?

Solution: The uncertainty in momentum is $\Delta p \sim \hbar/\Delta x$ by the uncertainty principle. Therefore the energy of the electron is,

$$E \sim \frac{(\Delta p)^2}{2m} + \frac{1}{2}k\Delta x^2 = \frac{\hbar^2}{2m(\Delta x)^2} + \frac{1}{2}k\Delta x^2.$$

For the minimal energy,

$$\frac{dE}{d(\Delta x)} = 0.$$

Therefore,

$$\Delta x = \left(\frac{\hbar^2}{mk}\right)^{1/4},\ E = \left(\frac{\hbar^2 k}{m}\right)^{1/2}.$$

P8-4. Pauli matrices are defined as

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that the two spin states

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are normalized eigenstates of Z with eigenvalues of $+1$ and -1; that is, $Z\,|0\rangle = |0\rangle$ and $Z\,|1\rangle = -\,|1\rangle$. Therefore, we conclude that $S_z = (\hbar/2)Z$.

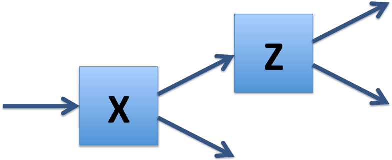
(b) Similarly, we have $S_x = (\hbar/2)X$ and $S_y = (\hbar/2)Y$. The normalized eigenstates of X satisfy $X\left|+\right\rangle = \left|+\right\rangle$ and $X\left|-\right\rangle = -\left|-\right\rangle$. Express $\left|+\right\rangle$ and $\left|-\right\rangle$ in terms of $\left|0\right\rangle$ and $\left|1\right\rangle$.

(c) Suppose a beam of electron with spin $\left|+\right\rangle$ passing through a Stern-Gerlach apparatus measuring S_z. The probabilities of spin being up and down are

$$P(+\hbar/2) = |\langle 0|+\rangle|^2, \quad P(-\hbar/2) = |\langle 1|+\rangle|^2$$

Calculate $P(\pm\hbar/2)$.

(d) The figure show a Stern-Gerlach experiment with an S_x measurement followed by a S_z measurement. If the incident beam has N electrons with spin $S_z = \hbar/2$, how many electrons are there in the two outcoming beams with $S_z = \pm\hbar/2$, respectively?



(e) How many electrons are there in the outcoming beams if the incident beam has N electrons with spin $S_x = \hbar/2$?

Solution:

(a)

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle\,,$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|1\rangle\,.$$

(b) Assume

$$|+\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Therefore,

$$X|+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} = |+\rangle = \begin{pmatrix} b \\ a \end{pmatrix},$$

$$X\left|-\right\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix} = -\left|-\right\rangle = -\begin{pmatrix} c \\ d \end{pmatrix}.$$

So,

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

$$|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

(c)

$$\langle 0|1\rangle = \begin{pmatrix} 1 \; 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Therefore,

$$P(+\hbar/2) = |\langle 0|+\rangle|^2 = \frac{1}{2},$$
$$P(-\hbar/2) = |\langle 1|+\rangle|^2 = \frac{1}{2}.$$

(d) When first measure S_x, the probability of spin being up is

$$P(+\hbar/2) = |\langle +|0\rangle|^2 = \frac{1}{2}.$$

Then second measure S_z, the probabilities of spin being up and down are

$$P(+\hbar/2) = |\langle 0|+\rangle|^2 \cdot |\langle +|0\rangle|^2 = \frac{1}{4},$$
$$P(-\hbar/2) = |\langle 1|+\rangle|^2 \cdot |\langle +|0\rangle|^2 = \frac{1}{4}.$$

Therefore, there are $N/4$ electrons there in the two outcoming beams with $S_z = \pm\hbar/2$, respectively.
(e) When first measure S_x, the probability of spin being up is 1. Then second measure S_z, the probabilities of spin being up and down are

$$P(+\hbar/2) = |\langle 0|+\rangle|^2 = \frac{1}{2},$$

$$P(-\hbar/2) = |\langle 1|+\rangle|^2 = \frac{1}{2}.$$

Therefore, there are $N/2$ electrons there in the two outcoming beams with $S_z = \pm\hbar/2$, respectively.

P8-5. A cubical box of widths $L_x = L_y = L_z = L$ contains eight electrons. What multiple of $h^2/(8mL^2)$ gives the energy of the ground state of this system? Assume that the electrons do not interact with one another, and do not neglect spin.

Solution: In terms of the quantum numbers n_x, n_y, and n_z, the single-particle energy levels are given by

$$E_{n_x,n_y,n_z} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2).$$

The lowest single-particle level corresponds to $n_x = 1$, $n_y = 1$, and $n_z = 1$ and is $E_{1,1,1} = 3(h^2/8mL^2)$. There are two electrons with this energy, one with spin up and one with spin down. The next lowest single-particle level is three-fold degenerate in the three integer quantum numbers. The energy is

$$E_{1,1,2} = E_{1,2,1} = E_{2,1,1} = 6(h^2/8mL^2).$$

Each of these states can be occupied by a spin up and a spin down electron, so six electrons in all can occupy the states. This completes the assignment of the eight electrons to single-particle states.
The ground state energy of the system is

$$E_{\mathrm{gs}} = 2 \cdot 3 \cdot (h^2/8mL^2) + 6 \cdot 6 \cdot (h^2/8mL^2) = 42(h^2/8mL^2).$$

Thus, the multiple of $h^2/8mL^2$ is 42.

P8-6. Consider standing electromagnetic wave in a cubic cavity. We expect that the number of modes N is proportional to the volume V of the cavity.

(a) Show, by dimension analysis, that the number of modes per unit wavelength is then

$$\frac{dN}{d\lambda} = \frac{\alpha V}{\lambda^4},$$

where α is a constant.

(b) Classically, we expect each mode has average energy $k_B T$. Calculate the spectral radiancy $S(\lambda)$, which is power per unit area of the emitter of the radiation per unit wavelength, for a given temperature T.

(c) To explain the experimental result at short wavelengths, Planck (1900) postulated that each mode has average energy

$$\langle E(\lambda)\rangle = \frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1}.$$

Show that the consequent spectral radiancy agrees with the classical result at long wavelengths.

(d) Using Planck's postulation, show that the wavelength $\lambda_{\max}$ at which the $S(\lambda)$ is maximum is (**Wien's displacement law**)

$$\lambda_{\max} T = 2898\ \mu\mathrm{m/K}.$$

(e) Consider the radiation from the Sun's surface with temperature $T = 6000$ K, can you explain why we often see golden sunlight?

Solution: (a) As we expect that the number of modes N is proportional to the volume V of the cavity, dimension analysis tells us the dimensionless number

$$N \propto V/\lambda^3.$$

Taking the derivative with respect to λ, we can express the number of modes per unit wavelength as

$$\frac{dN}{d\lambda} = \frac{\alpha V}{\lambda^4}.$$

(b) In a time interval Δt, the number of modes per unit wavelength passing through a surface of area A is then

$$\frac{dN}{d\lambda} = \alpha \frac{A(cdt)}{\lambda^4},$$

where c is the speed of light. With each mode carrying average energy $k_B T$, the prediction of classical physics for the spectral radiancy, for a given temperature T in kelvins, is

$$S(\lambda) = \frac{1}{A}\frac{1}{\Delta t}\frac{dN}{d\lambda}(k_B T) = \alpha \frac{k_B T}{\lambda^4} c.$$

More quantitative analysis can fix α to be 2π.
(c) Replacing $k_B T$ by

$$\langle E(\lambda) \rangle = \frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1},$$

we obtain Planck's formula

$$S(\lambda) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}.$$

In the long wavelength limit,

$$\frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1} \xrightarrow{\lambda \to \infty} k_B T.$$

So we recover the classical result.

(d) Taking the first derivative of Planck's formula with respect to λ, setting the derivative to zero, and then solving for the wavelength, we obtain

$$\frac{\lambda}{S}\frac{dS}{d\lambda} = -5 + \frac{e^x}{e^x - 1}x = 0,$$

where $x = hc/(\lambda k_B T)$. So, $x \approx 5$.

The wavelength $\lambda_{\max}$ at which the $S(\lambda)$ is maximum (for a given temperature T) is therefore,

$$\lambda_{\max} T = \frac{hc}{xk_B} = 2898\ \mu\mathrm{m/K}.$$

(e) Warm objects emit infrared radiation, which is felt by the skin. For $T = 6000$ K, however, the peak of the spectrum shifts to

$$\lambda_{\max} = 2898/T = 0.483\ \mu\mathrm{m},$$

which is visible yellow. This explains why we see golden sunlight under normal circumstances.

P8-7. Electrons cannot move in definite orbits within atoms, like the planets in our solar system. To see why, let us try to "observe" such an orbiting electron by using a light microscope to measure the electron's presumed orbital position with a precision of, say, 10 pm (a typical atom has a radius of about 100 pm). The wavelength of the light used in the microscope must then be about 10 pm. (a) What would be the photon energy of this light? (b) How much energy would such a photon impart to an electron in a head-on collision? (c) What do these results tell you about the possibility of "viewing" an atomic electron at two or more points along its presumed orbital path? (*Hint*: The outer electrons of atoms are bound to the atom by energies of only a few electron-volts.)

Solution: (a) Using the value $hc = 1240\ \mathrm{nm}\cdot\mathrm{eV}$, we have

$$E = \frac{hc}{\lambda} = \frac{1240\mathrm{nm}\cdot\mathrm{eV}}{10.0\times10^{-3}\mathrm{nm}} = 124\ \mathrm{keV}.$$

(b) The kinetic energy gained by the electron is equal to the energy decrease of the photon:

$$\begin{aligned}\Delta E &= \Delta\left(\frac{hc}{\lambda}\right) = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda+\Delta\lambda}\right) = \left(\frac{hc}{\lambda}\right)\left(\frac{\Delta\lambda}{\lambda+\Delta\lambda}\right)\\ &= \frac{E}{1+\lambda/\Delta\lambda} = \frac{E}{1+\frac{\lambda}{\lambda_C(1-\cos\phi)}}\\ &= \frac{124\mathrm{keV}}{1+\frac{10.0\mathrm{pm}}{(2.43\mathrm{pm})(1-\cos 180^\circ)}} = 40.5\ \mathrm{keV}.\end{aligned}$$

(c) It is impossible to “view” an atomic electron with such a high-energy photon, because with the energy imparted to the electron the photon would have knocked the electron out of its orbit.