Resistance and Capacitance

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Lecture 6

Outline

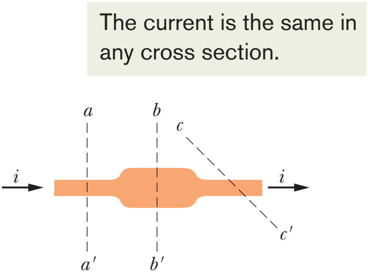
- Electric Charge in Motion
- Resistance and Ohm's Law
- Capacitors and Capacitance
- The Calculation of Capacitance

Electric Current

- We have discussed electrostatics, the physics of stationary charges. From now on, we discuss the physics of **electric currents**, i.e., charge in motion.
- We study steady currents of conduction electrons moving through metallic conductors.
    - The conduction electrons in an isolated length of copper wire are in random motion along either direction so there is no net transport of charge and thus no current through the wire.
    - However, if you connect the ends of the wire to a battery, you slightly bias the flow in one direction, with the result that there now is a net transport of charge and thus an electric current through the wire.

- If charge dq passes through a hypothetical plane in time dt, then the current i through that plane is defined as

$$i = dq/dt.$$



- Under steady-state conditions, the current is the same for all planes that pass completely through the conductor, no matter what their location or orientation, due to the fact that charge is conserved.

- We can find the charge that passes through the plane in a time interval extending from t_1 to t_2 by integration:

$$q = \int dq = \int_{t_1}^{t_2} i dt,$$

  in which the current i may vary with time.
- The SI unit for current is the coulomb per second (C/s), or the ampere (A), which is an SI base unit.

The Direction of Currents

- Current is a scalar. Yet, we often represent a current with an arrow to indicate that charge is moving.
- Convention: A current arrow is drawn in the direction in which positive charge carriers would move, even if the actual charge carriers are negative and move in the opposite direction.

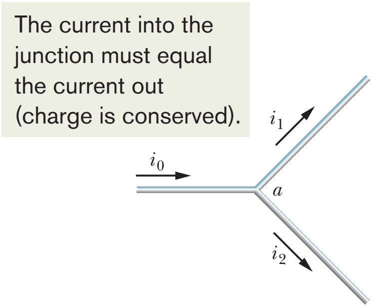


Figure 1: **Junction rule**: When a current i_0 splits at a junction into two branches, the currents in the branches obey $i_0 = i_1 + i_2$.

Current Density

- Current i (a scalar quantity) is related to current density $\vec{J}$ (a vector quantity) by

$$i = \int \vec{J} \cdot d\vec{A}$$

where $d\vec{A}$ is a vector perpendicular to a surface element of area dA and the integral is taken over any surface cutting across the conductor.

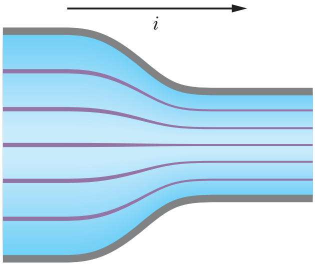


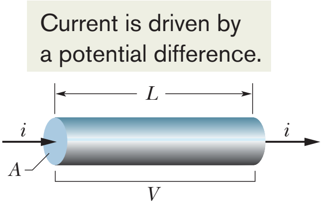
Figure 2: Streamlines representing current density in the flow of charge through a constricted conductor.

Drift Velocity

- When a conductor does not have a current through it, its conduction electrons move randomly, with no net motion in any direction. How do you estimate the average random-motion speed?
- When the conductor does have a current through it, these electrons actually still move randomly, but now they tend to drift with a **drift velocity** $\vec{v}_d$ in the direction opposite that of the applied electric field that causes the current.
- The drift speed is tiny compared with the speeds in the random motion. How tiny?

- The total charge of carriers, each with charge e, in a wire (cross-sectional area A) is

$$q = (nAL)e.$$



- The total charge moves through any cross section in $\Delta t = L/v_d$. The current i is the time rate of transfer of charge across a cross section, so we have

$$i = q/\Delta t = nAev_d \quad \text{or} \quad \vec{J} = ne\vec{v}_d.$$

- Now, can you estimate v_d?

Resistance and Resistivity

- The **resistance** R of a conductor is defined as $R = V/i$, where V is the potential difference across the conductor and i is the current.
- The **resistivity** ρ and **conductivity** σ of a material are related by $\rho = 1/\sigma = E/J$, where E is the magnitude of the applied electric field and J is the magnitude of the current density.
- *Resistance is a property of an object, while resistivity is a property of a material.* (Remember $R = \rho L/A$?)
- The SI unit for resistance is the ohm (Ω). The SI unit for resistivity is the ohm-meter ($\Omega \cdot$m).

Ohm's Law

- Ohm's law is an assertion that the current through a device is *always* directly proportional to the potential difference applied to the device. Why?
- The answer lies in the details of the conduction process at the atomic level. But counter examples exist.

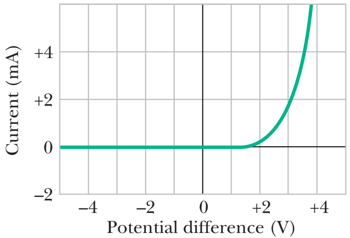
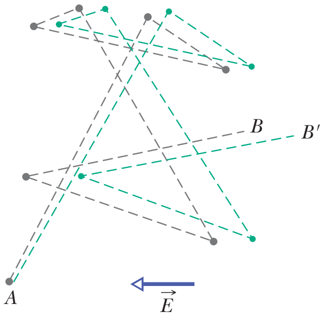


Figure 3: A *pn* junction diode does not obey Ohm's law.

- We consider only conduction in metals, such as copper, in the free-electron model, like the ideal-gas model. The difference is that the electrons collide not with one another but only with atoms of the metal.



- The motion of conduction electrons in an electric field $\vec{E}$ is a combination of the **diffusive motion** due to random collisions (trajectory AB) and the **drift motion** due to $\vec{E}$ (difference between AB and AB').

- An electron of mass m and charge $(-e)$ experiences an acceleration in an electric field $\vec{E}$:

$$\vec{a} = \frac{\vec{F}}{m} = -\frac{e\vec{E}}{m}.$$

- After a typical collision, each electron will "lose its memory" of its previous drift velocity, starting fresh and moving off in a random direction.
- In the average time τ (or **mean free time**) between collisions, the electron will, on average, acquire a drift velocity

$$\vec{v}_d = \vec{a}\tau = -\frac{e\vec{E}}{m}\tau.$$

- Combining this result with $\vec{J} = n(-e)\vec{v}_d$ yields

$$\vec{v}_d = -\frac{\vec{J}}{ne} = -\frac{e\vec{E}\tau}{m},$$

  which we can write as $\vec{E} = \rho\vec{J}$, where

$$\rho = \frac{m}{ne^2\tau}.$$

- For metals under normal circumstances, both n and τ can be considered to be a constant. Thus, metals obey Ohm's law.

- Now, let us estimate the mean free time in typical metals, such as copper.
  - The free electron density in Cu is $n = 8.47 \times 10^{28}$ m^{-3}.
  - Cu has a resistivity $\rho = 1.56$ $\mu\Omega$·cm.
- The mean free time can be estimated by

$$\tau = \frac{m}{\rho n e^2}.$$

- In the classical picture, the mean velocity of an electron at room temperature can be estimated by $(1/2)mv_{\mathrm{th}}^2 = (3/2)k_B T$.
- How far do you expect the electron to travel between two collisions?

Equation of Continuity

- We have implicitly used the (local) conservation of charge here. Namely, if the total charge in some volume changes, the exactly same amount of charge must have passed in or out through the surface. Otherwise, charge accumulation occurs in the volume enclosed by the surface.
- Formally, we have

$$\frac{d}{dt}\int_V \rho(\vec{r},t)dV = \int_V \frac{\partial\rho(\vec{r},t)}{\partial t}dV = -\oint_S \vec{J}(\vec{r},t)\cdot d\vec{A}$$

- To be able to move the derivative under the integral sign this way requires that $\partial\rho/\partial t$ be continuous.
- With the divergence theorem, we can rewrite it in the derivative form (note V is arbitrary)

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\vec{J}$$

- Such an equation of continuity plays an important role in hydrodynamics, hear flow, and diffusion theory, besides electromagnetic theory. It is simply a mathematical expression of a conservation law.

Capacitor and Capacitance

- A **capacitor** consists of two isolated conductors (the plates) with charges $+q$ and $-q$.

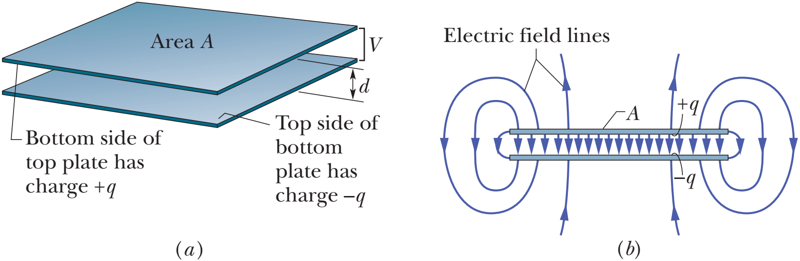
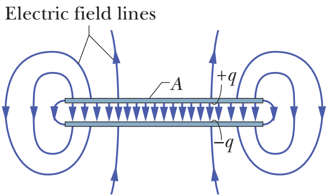


Figure 4: A parallel-plate capacitor, made up of two plates of area A separated by a distance d. Note the edge effect or **fringing**.

- We refer to the charge of a capacitor as being q, the absolute value of these charges on the plates.



- Because the plates are conductors, they are equipotential surfaces; but there is a potential difference between the two plates. For historical reasons, we represent the absolute value of this potential difference with V (not ΔV).
- In practice, we choose a path *from the negative plate to the positive plate* such that $V > 0$.

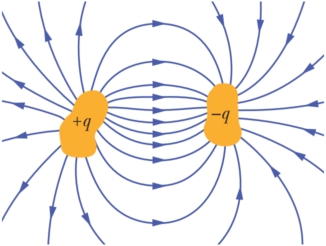
- The charge q and the potential difference V for a capacitor are proportional to each other, i.e.,

$$q = CV,$$

where the proportionality constant C is called the **capacitance** of the capacitor.

- The capacitance is a measure of how much charge must be put on the plates to produce a certain potential difference between them: The greater the capacitance, the more charge is required.

- Capacitance depends only on the geometry of the plates and not on their charge or potential difference. We will see examples below.



- The SI unit of capacitance is the farad (F):

$$1 \text{ farad} = 1 \text{ coulomb per volt.}$$

- The farad is a very large unit. The microfarad ($1\ \mu\text{F} = 10^{-6}$ F) and the picofarad ($1\ \text{pF} = 10^{-12}$ F) are more convenient units in practice.

Calculating the Capacitance

- Our goal here is to calculate the capacitance of a capacitor once we know its geometry. Our plan, which is generalizable, is as follows:
  1. Assume a charge q on the plates;
  2. calculate the electric field $\vec{E}$ between the plates in terms of this charge, using Gauss' law;
  3. knowing $\vec{E}$, calculate the potential difference V between the plates from $V = -\int_{-}^{+} \vec{E} \cdot d\vec{s} = \int_{-}^{+} E ds$ (note the sign);
  4. calculate C from $q = CV$.
- In practice, we neglecting the fringing effect.

Capacitance of a Parallel-Plate Capacitor

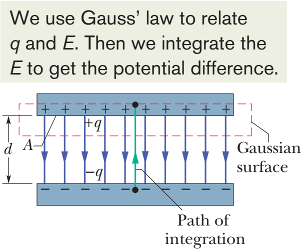


Figure 5: Gaussian surface of a charged parallel-plate capacitor. The capacitance depends only on geometrical factors: the plate area A and the plate separation d.

- We neglect the fringing of the electric field at the edges of the plates, taking $\vec{E}$ to be constant throughout the region between the plates.
- According to Gauss' law, we find $q = \epsilon_0 EA$.
- We also have $V = Ed$, so

$$C = \frac{q}{V} = \frac{\epsilon_0 A}{d}.$$

Capacitance of a Cylindrical Capacitor

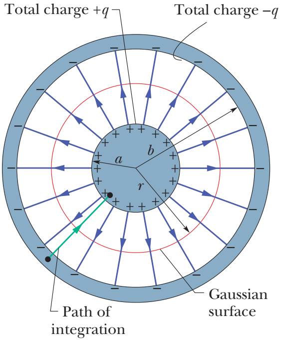


Figure 6: Gaussian surface of a charged cylindrical capacitor of length L formed by two coaxial cylinders.

- Assume that $L \gg b$ so that we can neglect fringing. Each plate contains a charge of magnitude q.
- We choose Gaussian surface as a cylinder of length L and radius r, which is coaxial with the cylinders, closed by end caps. We have

$$q = \epsilon_0 EA = \epsilon_0 E(2\pi rL).$$

- Solving for E and integrating inward, we find

$$V = -\int_{-}^{+} \vec{E} \cdot d\vec{s} = -\frac{q}{2\pi\epsilon_0 L} \int_b^a \frac{dr}{r} = \frac{q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right).$$

- We then have

$$C = \frac{q}{V} = 2\pi\epsilon_0 \frac{L}{\ln(b/a)}.$$

- The capacitance of a cylindrical capacitor, like that of a parallel-plate capacitor, depends only on geometrical factors, in this case the length L and the two radii a and b.

- Apparently, the capacitance is proportional to the length of the cylindrical capacitor, just like the resistance of a cylindrical conductor ($R = \rho L/A$).
- However, the capacitor is connected to a circuit is through its inner and outer conductors, as oppose to the resistor through its two ends.

Quiz 6-1

Summary

- Concepts: electric current i, current density $\vec{J}$, drift speed v_d, resistance R, resistivity ρ, conductivity σ, capacitance C
- Key formulas:

$$i = \frac{dq}{dt}$$

$$\vec{J} = ne\vec{v}_d$$

$$R = \frac{V}{i}$$

$$i = \int \vec{J} \cdot d\vec{A}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$$

$$\vec{E} = \rho \vec{J}$$

- Understand how to derive the resistivity of a metal

$$\rho = \frac{1}{\sigma} = \frac{m}{ne^2\tau}.$$

This also applies to a semiconductor (see Appendix 6A).

- You should spend time on understanding the general procedure to calculate capacitance, which is of the form $C = \epsilon_0 L_{\text{eff}}$ (where L_{eff} depends on the geometry of the capacitor).

$$C = \frac{\epsilon_0 A}{d} \qquad\qquad C = 2\pi\epsilon_0 \frac{L}{\ln(b/a)}$$

Reading

Halliday, Resnick & Krane:

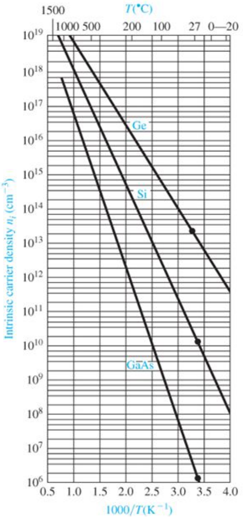
- Chapter 29: The Electrical Properties of Materials
- Chapter 30: Capacitance

Appendix 6A: Semiconductors

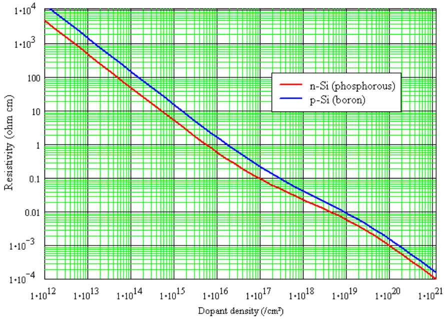
- Semiconducting devices are at the heart of the microelectronic revolution.
- In a semiconductor, n is small but increases very rapidly with temperature as the increased thermal agitation makes more charge carriers available.
- The resistivity of a conductor

$$\rho = \frac{m}{ne^2\tau},$$

also applies to semiconductors.



- Compared to metallic copper, therefore, semiconducting silicon has a much higher resistivity and a temperature coefficient of resistivity that is both large and negative.
- By adding selective impurities to a semiconductor, we can control the density of free charge carriers and thereby can control some of its electrical properties.



Appendix 6B: Superconductors

- In 1911, Dutch physicist Kamerlingh Onnes discovered that the resistivity of mercury absolutely disappears at temperatures below about 4 K.

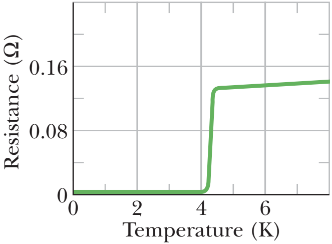
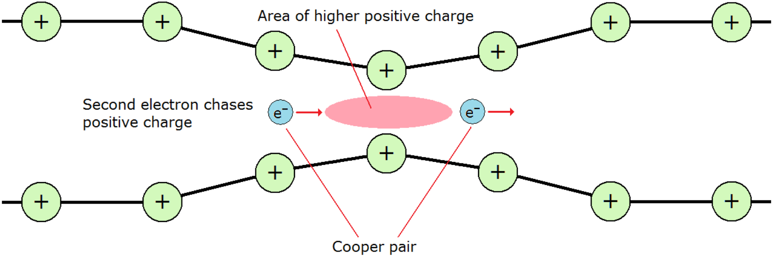


Figure 7: The resistance of mercury drops to zero at a temperature of about 4 K.

- In superconductors, electrons that make up the current move in coordinated pairs (**Cooper pairs**). One of the electrons in a pair may electrically distort the atomic structure of the superconducting material as it moves through, creating nearby a short-lived concentration of positive charge. The other electron in the pair may then be attracted toward this positive charge.



- According to Bardeen, Cooper, and Schrieffer (BCS), such coordination between electrons would prevent them from colliding with the ions of the material and thus would eliminate electrical resistance.
- This phenomenon of superconductivity is of vast potential importance in technology because it means that charge can flow through a superconducting conductor without losing its energy to thermal energy.

Appendix 6C: John Bardeen (1908-91)

- At the Bell Labs, Bardeen, Brattain, and Shockley conducted research on the electron-conducting properties of semiconductors. On Dec. 23, 1947, they unveiled the transistor, which ushered in the electronic revolution.
- The BCS theory (from the initials of Bardeen, Cooper, and Schrieffer) provided a theoretical explanation of the disappearance of electrical resistance in superconductors at temperatures close to absolute zero.

Figure 8: John Bardeen, American physicist who won the Nobel Prize for Physics in both 1956 and 1972.