Lab Nr. 8, Numerical Calculus

Review of Polynomial Interpolation

I. Summary

1. What **type** of interpolation? It depends on what information is given:

- **only** values of the function are given, e.g. $f(-1), f(0), f(1) \to$ Lagrange
- values of the function are known for **all** nodes and, in addition, for some nodes, **all the derivatives up to some order (not the same for all nodes)** are also given, e.g. $f(-1), f(0), f'(0), f''(0), f(1), f'(1)$ $\to$ Hermite
- **random** values of the **function and/or of the derivatives** are given, e.g. $f(-1), f'(0), f''(0),$ $f'(1), f'''(2) \to$ Birkhoff

2. What **computational formula** is more convenient?

- Lagrange $\to$ Newton's divided (/forward/backward) differences or Aitken's algorithm
- Hermite $\to$ Newton's divided differences (**caution!!**, for multiple nodes, divided differences are computed with *derivatives*)
- Birkhoff $\to$ fundamental (basis) polynomials or directly (determine the coefficients of the polynomial from the interpolation conditions)

3. The **degree** of the interpolation polynomial

- the set of indices I_k consists of the **orders** of the derivatives that are given for each node x_k (the derivative of order 0 is the function itself), e.g.

$$\begin{aligned}
x_0 &= -1 \text{ and we know } f(-1), f'(-1) \to \boldsymbol{I_0 = \{0, 1\}} \\
x_1 &= 0 \text{ and we know } f'(0), f''(0) \to \boldsymbol{I_1 = \{1, 2\}} \\
x_2 &= 1 \text{ and we know } f(1), f'''(1), f^{(iv)}(1) \to \boldsymbol{I_2 = \{0, 3, 4\}} \\
x_3 &= 2 \text{ and we know } f''(2), f^{(v)}(2) \to \boldsymbol{I_3 = \{2, 5\}}
\end{aligned}$$

- the degree of the interpolation polynomial with nodes $x_0, \ldots, x_m$ is

$$\boldsymbol{n = |I_0| + \ldots + |I_m| - 1},$$

where $|A| = \mathrm{card}(A)$ is the number of elements of the set A.

4. Birkhoff interpolation

- The **notation** for Birkhoff fundamental polynomials: $\boldsymbol{b_{ij}}$, where $\boldsymbol{i}$ is the index of the node, $\boldsymbol{j}$ is the order of the derivative that is known for that node, e.g. if we are given

$$f'(-1),\ f'(0),\ f''(0),\ f(1),\ f''(1),$$

that means

- $f'(-1)$: for node $x_0 = -1$, the first derivative is known $\to \boldsymbol{b_{01}}$
- $f'(0),\ f''(0)$: for node $x_1 = 0$, the first and second derivatives are given $\to \boldsymbol{b_{11},\ b_{12}}$

• $f(1),\ f''(1)$: for node $x_2 = 1$, the zero and second derivatives are given $\to \boldsymbol{b_{20}},\ \boldsymbol{b_{22}}$.
So the fundamental polynomials are

$$\boldsymbol{b_{01},\ b_{11},\ b_{12},\ b_{20},\ b_{22}}$$

and the pairs of indices used are

$$\boldsymbol{(0,1),(1,1),(1,2),(2,0),(2,2)}.\ (*)$$

- The interpolation polynomial is then the **linear combination** of the basis polynomials found above, each having as coefficient the value of the function/derivative that it corresponds to:

$$(Bf)(x) \;=\; b_{01}(x)f'(-1) + b_{11}(x)f'(0) + b_{12}(x)f''(0) + b_{20}(x)f(1) + b_{22}(x)f''(1).$$

- Each basis polynomial b_{ij} has (at most) **the same degree** as the overall interpolation polynomial Bf (discussed above) and its coefficients are determined from the conditions

$$b_{ij}^{(j)}(x_i) \;=\; 1,$$

while at **all other combinations of indices used in the writing of the polynomial, it is** $\mathbf{0}$. For the example above (check (*)):

$$\left\{\begin{array}{rcl} \boldsymbol{(0,1)} \to & b_{01}'(x_0) & = \mathbf{1} \\ (1,1) \to & b_{01}'(x_1) & = 0 \\ (1,2) \to & b_{01}''(x_1) & = 0 \\ (2,0) \to & b_{01}(x_2) & = 0 \\ (2,2) \to & b_{01}''(x_2) & = 0 \end{array}\right., \left\{\begin{array}{rcl} (0,1) \to & b_{11}'(x_0) & = 0 \\ \boldsymbol{(1,1)} \to & b_{11}'(x_1) & = \mathbf{1} \\ (1,2) \to & b_{11}''(x_1) & = 0 \\ (2,0) \to & b_{11}(x_2) & = 0 \\ (2,2) \to & b_{11}''(x_2) & = 0 \end{array}\right., \left\{\begin{array}{rcl} (0,1) \to & b_{12}'(x_0) & = 0 \\ (1,1) \to & b_{12}'(x_1) & = 0 \\ \boldsymbol{(1,2)} \to & b_{12}''(x_1) & = \mathbf{1} \\ (2,0) \to & b_{12}(x_2) & = 0 \\ (2,2) \to & b_{12}''(x_2) & = 0 \end{array}\right.,$$

and so on.

- If finding the Birkhoff polynomial **directly**, just impose the interpolation conditions on Bf (after determining its degree **correctly**!). There is just one system in this case, but it is, in general, **more difficult** to solve than the systems for b_{ij} and the remainder is more difficult to find.

5. Check at the end that the polynomial found satisfies the interpolation conditions. If it exists, the interpolating polynomial is unique.

II. Peano's Theorem for the Remainder

- We have an approximating formula

$$\begin{aligned} f(x) &\approx (P_nf)(x), \ \text{or} \\ f(x) &= (P_nf)(x) + (R_n(f))(x), \ \text{for } x \in [a,b], \end{aligned}$$

where $[a,b]$ is the smallest interval containing all the interpolation nodes. The formula has degree of precision (or exactness) $d = n$, if it is exact for **all** polynomials of degree up to n, i.e.

$$\begin{aligned} f(x) &= (P_nf)(x), \ \text{for } f(x) = e_k(x) = x^k,\ k = 0,1,\ldots,n, \\ e_{n+1}(x) &\neq (P_ne_{n+1})(x), \ \text{or, equivalently, } R_n(e_{n+1}) \not\equiv 0. \end{aligned}$$

- The remainder has the form

$$(R_n(f))(x) \quad = \quad \int_a^b K_n(x,t) f^{(n+1)}(t)dt$$

where

$$\begin{aligned} K_n(x,t) &= R_n\left(\frac{(x-t)_+^n}{n!}\right) = \frac{1}{n!}R_n\Big((x-t)_+^n\Big) = \frac{1}{n!}\Big((x-t)_+^n - P_n((x-t)_+^n)\Big), \\ (x-t)_+^n &= ((x-t)_+)^n = \begin{cases} (x-t)^n, & x \ge t \\ 0, & x < t \end{cases}. \end{aligned}$$

- If K has constant sign on $[a,b]$, then (by the MVT)

$$(R_n(f))(x) \quad = \quad \frac{f^{(n+1)}(\xi)}{(n+1)!} R_n(e_{n+1})(x),\ \xi \in [a,b].$$

- The function $F(x) = (x-t)_+^n$ has the derivative

$$F'(x) = \frac{\partial[(x-t)_+^n]}{\partial x} = n(x-t)_+^{n-1}$$

and the integral (this will only be needed later on, in Chapter 4)

$$\int_a^b F(x)dx \quad = \quad \frac{1}{n+1}(x-t)_+^{n+1}\Big|_{x=a}^{x=b} \quad = \quad \frac{1}{n+1}\left[(b-t)_+^{n+1} - (a-t)_+^{n+1}\right]$$

- Peano's theorem can also be used to derive remainder formulas for Lagrange or Hermite interpolation, but we already have other forms for those (recall that Peano's theorem refers to *any* linear functional).

III. Practice Problems

Find the polynomial of minimum degree that interpolates the given data. Determine the remainder for each approximation.

1. $f(0), f(1/2), f(1)$;

Solution. Only *function* values are given, this is **Lagrange** interpolation.
The divided differences table:

$$\begin{array}{c|ccccc} 0 & \boldsymbol{f(0)} & \longrightarrow & \boldsymbol{2\Big(f(1/2)-f(0)\Big)} & \longrightarrow & \boldsymbol{2\Big(f(1)-2f(1/2)+f(0)\Big)} \\ & & \nearrow & & \nearrow & \\ 1/2 & f(1/2) & \longrightarrow & 2\Big(f(1)-f(1/2)\Big) & & \\ & & \nearrow & & & \\ 1 & f(1) & & & & \end{array}$$

Then

$$\begin{aligned} L_2 f(x) &= f(0) + 2\Big(f(1/2)-f(0)\Big)\cdot x + 2\Big(f(1)-2f(1/2)+f(0)\Big)\cdot x(x-1/2) \\ &= (2x-1)(x-1)f(0) + 4x(1-x)f(1/2) + x(2x-1)f(1). \end{aligned}$$

Check that $L_2 f$ satisfies the interpolation conditions. **Note**: this is why it is better to write it in the form above, as a linear combination of the given function values, rather then writing it in the form $a_2 x^2 + a_1 x + a_0$. This second form will be more convenient when we also want to differentiate or integrate the polynomial.
The remainder:

$$R_2 f(x) \;=\; \frac{u(x)}{3!} f'''(\xi) \;=\; \frac{x(x-1/2)(x-1)}{3!} f'''(\xi),\ \xi \in (0,1).$$

If possible, we may try to find a bound for $|u(x)|$ on $[0,1]$ (but this may require Matlab ...).
In this case (it can be done by hand, but this may take some work ...),

$$\max_{x\in[0,1]} |u(x)| = \left| u\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right) \right| = \frac{\sqrt{3}}{36},$$

so a bound for the error is

$$\begin{aligned} |R_2 f(x)| &\le \frac{\sqrt{3}}{36 \cdot 3!} |f'''(\xi)|,\ \xi \in (0,1), \\ |R_2 f(x)| &\le \frac{\sqrt{3}}{216} \|f'''\|. \end{aligned}$$

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2. $f(-1), f(0), f'(0), f(1)$;

Solution. From the data, we have: $x_0 = -1, x_2 = 1$ *simple* nodes and $x_1 = 0$ *double* node, so **Hermite** interpolation. Since $|I_0| = |I_2| = 1$ and $|I_1| = 2$, the degree of the polynomial is $n = 1 + 2 + 1 - 1 = 3$.
The divided differences table:

$$\begin{array}{r|ccccccc}
-1 & \boldsymbol{f(-1)} & \longrightarrow & \boldsymbol{f(0)-f(-1)} & \longrightarrow & \boldsymbol{f'(0)-f(0)+f(-1)} & \longrightarrow & \boldsymbol{\dfrac{f(1)-2f'(0)-f(-1)}{2}} \\
 & & \nearrow & & \nearrow & & \nearrow & \\
0 & f(0) & \longrightarrow & f'(0) & \longrightarrow & f(1)-f(0)-f'(0) & & \\
 & & \nearrow & & \nearrow & & & \\
0 & f(0) & \longrightarrow & f(1)-f(0) & & & & \\
 & & \nearrow & & & & & \\
1 & f(1) & & & & & &
\end{array}$$

Then

$$\begin{aligned} H_3 f(x) &= f(-1) + \Big(f(0) - f(-1)\Big) \cdot (x+1) + \Big(f'(0) - f(0) + f(-1)\Big) \cdot x(x+1) \\ &+ \frac{f(1) - 2f'(0) - f(-1)}{2} \cdot x^2(x+1) \\ &= \frac{1}{2} x^2 (1-x) f(-1) + (1-x^2) f(0) + x(1-x^2) f'(0) + \frac{1}{2} x^2 (x+1) f(1). \end{aligned}$$

Check that $H_3 f$ satisfies the interpolation conditions.
The remainder:

$$R_3 f(x) \;=\; \frac{u(x)}{4!} f^{(iv)}(\xi) \;=\; \frac{x^2(x^2-1)}{4!} f^{(iv)}(\xi),\ \xi \in (-1,1).$$

Here, it's (relatively) easy to find

$$\max_{x\in[-1,1]} |u(x)| = \left| u\left(\pm\frac{\sqrt{2}}{2}\right)\right| = \frac{1}{4}\,,$$

so

$$\begin{aligned} |R_3 f(x)| &\le \frac{1}{4\cdot 4!}|f^{(iv)}(\xi)|,\ \xi\in(-1,1),\\ |R_3 f(x)| &\le \frac{1}{96}||f^{(iv)}||. \end{aligned}$$

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3. $f(0), f'(0), f'(1)$.

Solution. For the node $x_1 = 1$, *only* the derivative is given, so this is **Birkhoff** interpolation with $I_0 = \{0,1\}$ for $x_0 = 0$ and $I_1 = \{1\}$ for $x_1 = 1$. Thus, the degree of the polynomial is $n = 2+1-1 = 2$.

Direct way

The polynomial and its first derivative are

$$\begin{aligned} B_2 f(x) &= ax^2 + bx + c,\\ (B_2 f)'(x) &= 2ax + b. \end{aligned}$$

From the interpolation data, we have

$$\begin{aligned} B_2 f(0) &= \qquad\quad c = f(0),\\ (B_2 f)'(0) &= \quad b \qquad = f'(0),\\ (B_2 f)'(1) &= 2a + b \quad = f'(1), \end{aligned}$$

with solution

$$\begin{aligned} a &= \frac{1}{2}\Big(f'(1) - f'(0)\Big),\\ b &= f'(0),\\ c &= f(0), \end{aligned}$$

so the polynomial is

$$\begin{aligned} B_2 f(x) &= \frac{1}{2}\Big(f'(1) - f'(0)\Big)x^2 + f'(0)x + f(0)\\ &= f(0) + \frac{1}{2}x(2-x)f'(0) + \frac{1}{2}x^2 f'(1). \end{aligned}$$

Check that $B_2 f$ satisfies the interpolation conditions.

With fundamental polynomials

The polynomial is the linear combination

$$B_2 f(x) = b_{00}(x)f(0) + b_{01}(x)f'(0) + b_{11}(x)f'(1).$$

Polynomial $b_{00}(x) = ax^2 + bx + c$ must satisfy the conditions

$$\left\{\begin{array}{rcl} b_{00}(0) & = & 1 \\ b'_{00}(0) & = & 0 \\ b'_{00}(1) & = & 0 \end{array}\right. \iff \left\{\begin{array}{ccccccc} & & & & c & = & 1 \\ & & b & & & = & 0 \\ 2a & + & b & & & = & 0 \end{array}\right. \iff \left\{\begin{array}{rcl} a & = & 0 \\ b & = & 0 \\ c & = & 1 \end{array}\right.$$

So,

$$b_{00}(x) \; = \; 1.$$

Polynomial $b_{01}(x) = ax^2 + bx + c$ must satisfy the conditions

$$\left\{\begin{array}{rcl} b_{01}(0) & = & 0 \\ b'_{01}(0) & = & 1 \\ b'_{01}(1) & = & 0 \end{array}\right. \iff \left\{\begin{array}{ccccccc} & & & & c & = & 0 \\ & & b & & & = & 1 \\ 2a & + & b & & & = & 0 \end{array}\right. \iff \left\{\begin{array}{rcr} a & = & -\frac{1}{2} \\ b & = & 1 \\ c & = & 0 \end{array}\right.$$

So,

$$b_{01}(x) \; = \; -\frac{1}{2}x^2 + x \; = \; \frac{1}{2}x(2-x).$$

Finally, polynomial $b_{11}(x) = ax^2 + bx + c$ must satisfy the conditions

$$\left\{\begin{array}{rcl} b_{11}(0) & = & 0 \\ b'_{11}(0) & = & 0 \\ b'_{11}(1) & = & 1 \end{array}\right. \iff \left\{\begin{array}{ccccccc} & & & & c & = & 0 \\ & & b & & & = & 0 \\ 2a & + & b & & & = & 1 \end{array}\right. \iff \left\{\begin{array}{rcl} a & = & \frac{1}{2} \\ b & = & 0 \\ c & = & 0 \end{array}\right.$$

So,

$$b_{11}(x) \; = \; \frac{1}{2}x^2.$$

Putting everything together,

$$B_2 f(x) \; = \; f(0) + \frac{1}{2}x(2-x)f'(0) + \frac{1}{2}x^2 f'(1).$$

Check that $B_2 f$ satisfies the interpolation conditions.

The remainder

So, we have the formula

$$\begin{array}{rcl} f(x) & = & B_2 f(x) + R_2 f(x), \text{ i.e.} \\ f(x) & = & f(0) + \frac{1}{2}x(2-x)f'(0) + \frac{1}{2}x^2 f'(1) + R_2 f(x). \end{array}$$

We **know** that the degree of precision is 2, because the polynomial $B_2 f$ *has* degree 2 and the interpolating polynomial (when it exists) is **unique**. Nevertheless, it's a good exercise to write the remainder even for $e_0 = 1, e_1 = x$ and $e_2 = x^2$:

$$\begin{array}{rcl} (R_2 e_0)(x) & = & e_0(x) - \left(e_0(0) + \frac{1}{2}x(2-x)e'_0(0) + \frac{1}{2}x^2 e'_0(1)\right) \; = \; 1 - (1+0+0) \; \equiv \; 0, \\ (R_2 e_1)(x) & = & e_1(x) - \left(e_1(0) + \frac{1}{2}x(2-x)e'_1(0) + \frac{1}{2}x^2 e'_1(1)\right) \; = \; x - \left(0 + \frac{1}{2}x(2-x) + \frac{1}{2}x^2\right) \; \equiv \; 0, \\ (R_2 e_2)(x) & = & e_2(x) - \left(e_2(0) + \frac{1}{2}x(2-x)e'_2(0) + \frac{1}{2}x^2 e'_2(1)\right) \; = \; x^2 - \left(0 + 0 + \frac{1}{2}x^2 \cdot 2\right) \; \equiv \; 0, \\ (R_2 e_3)(x) & = & e_3(x) - \left(e_3(0) + \frac{1}{2}x(2-x)e'_3(0) + \frac{1}{2}x^2 e'_3(1)\right) \; = \; x^3 - \left(0 + 0 + \frac{1}{2}x^2 \cdot 3\right) \; = \; x^3 - \frac{3}{2}x^2. \end{array}$$

Then, by Peano's theorem, the remainder is of the form

$$R_2 f(x) = \int_0^1 K_2(x,t) f'''(t)dt,$$

$$K_2(\boldsymbol{x},t) = R_2\left(\frac{(\boldsymbol{x}-t)_+^2}{2!}\right) = \frac{1}{2!}R_2\Big((\boldsymbol{x}-t)_+^2\Big).$$

We have

$$R_2\Big((\boldsymbol{x}-t)_+^2\Big) = (\boldsymbol{x}-t)_+^2 - \Big[(\mathbf{0}-t)_+^2 + \frac{1}{2}x(2-x)\cdot 2(\mathbf{0}-t)_+ + \frac{1}{2}x^2\cdot 2(\mathbf{1}-t)_+\Big]$$

Since $t \in [0,1]$, it follows that $(0-t)_+ = (-t)_+ = 0$ (and, of course, $(0-t)_+^2 = \left((0-t)_+\right)^2 = 0$, as well). Also, it follows that $(1-t)_+ = 1-t$. Thus, so far we have

$$R_2\Big((x-t)_+^2\Big) = (x-t)_+^2 - x^2(1-t).$$

Now we have to consider two cases:

Case **1)** $\underline{0 \le x \le t \le 1}$. Then $(x-t)_+ = 0$ and

$$R_2\Big((x-t)_+^2\Big) = -x^2(1-t) \le 0.$$

Case **2)** $\underline{0 \le t < x \le 1}$. Now $(x-t)_+ = x-t$ and

$$R_2\Big((x-t)_+^2\Big) = (x-t)^2 - x^2(1-t) = t(x^2-2x+t).$$

Let $g(x) = x^2 - 2x + t$. Its roots are $1 \pm \sqrt{1-t}$, both real. Inside the roots' interval $\left(1-\sqrt{1-t}, 1+\sqrt{1-t}\right)$, $g(x)$ has negative sign. It is (relatively) easy to check that

$$0 \le 1-\sqrt{1-t} \le t \le 1 \le 1+\sqrt{1-t}.$$

So, for $x \in (t,1] \subseteq \left(1-\sqrt{1-t}, 1+\sqrt{1-t}\right)$, $g(x)$ has negative sign and, consequently, so does $R_2\Big((x-t)_+^2\Big)$.

Thus, in either case, $K_2(x,t)$ has constant sign. Then

$$R_2 f(x) = \frac{f'''(\xi)}{3!}(R_2 e_3)(x) = \frac{x^3 - \frac{3}{2}x^2}{3!} f'''(\xi) = \frac{2x^3-3x^2}{12} f'''(\xi),\ \xi \in (0,1).$$

Now, for the function $h(x) = 2x^3 - 3x^2$, the derivative is $h'(x) = 6x^2 - 6x$, which is negative on $[0,1]$. So, for $x \in [0,1]$,

$$\begin{aligned} h(1) &\le h(x) \le h(0), \\ -1 &\le h(x) \le 0, \\ 0 &\le |h(x)| \le 1. \end{aligned}$$

Thus,

$$\begin{aligned} |R_2 f(x)| &\le \frac{1}{12}|f'''(\xi)|,\ \xi \in (0,1), \\ |R_2 f(x)| &\le \frac{1}{12}||f'''||. \end{aligned}$$

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