Chapter 8

Fuzzy Logic

8.1. Fuzzy sets

8.2. Fuzzy relations

8.3. Functions of fuzzy sets. Zadeh Extension Principle

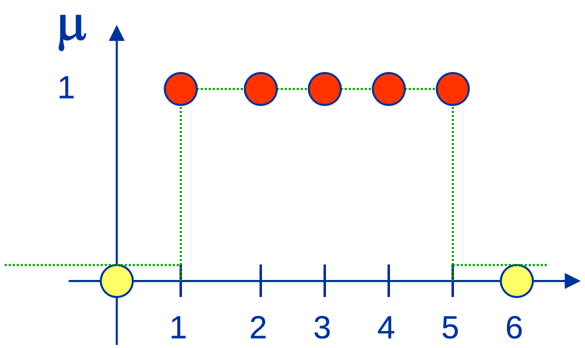
8.4. Inference *modus ponens* and approximate reasoning

8.1. Fuzzy sets

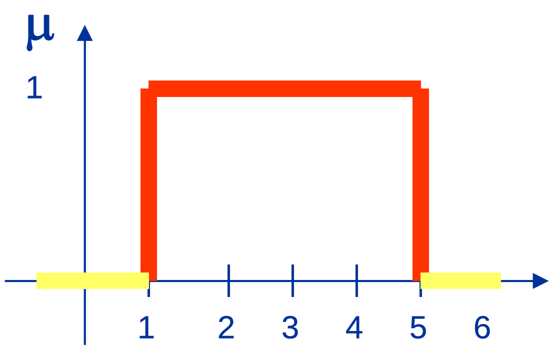
Classic set , crisp set

$A = \{1,2,3,4,5\}$ $\quad \mathbb{R} \triangleq Universe$ $\qquad A = [1,5]$

Discrete



continuous

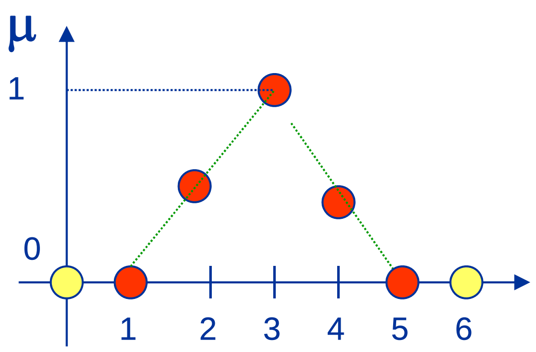


$\mu(x) \triangleq$ characteristic function of the set $= \begin{cases} 1, & if \ x \in A \\ 0, & if \ x \notin A \end{cases}$

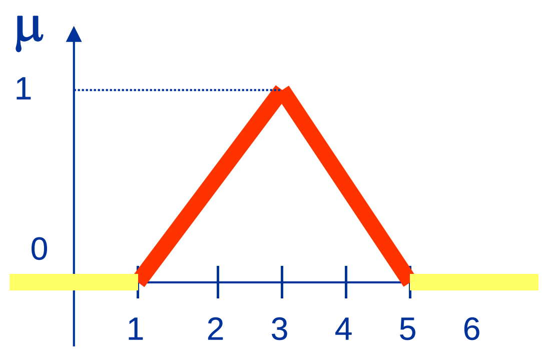
To be or not to be …

Fuzzy set

Discrete



Continuous



$$\underset{\sim}{A} = \left\{ \frac{0}{1} + \frac{0,5}{2} + \frac{1}{3} + \frac{0,5}{4} + \frac{0}{5} \right\}$$

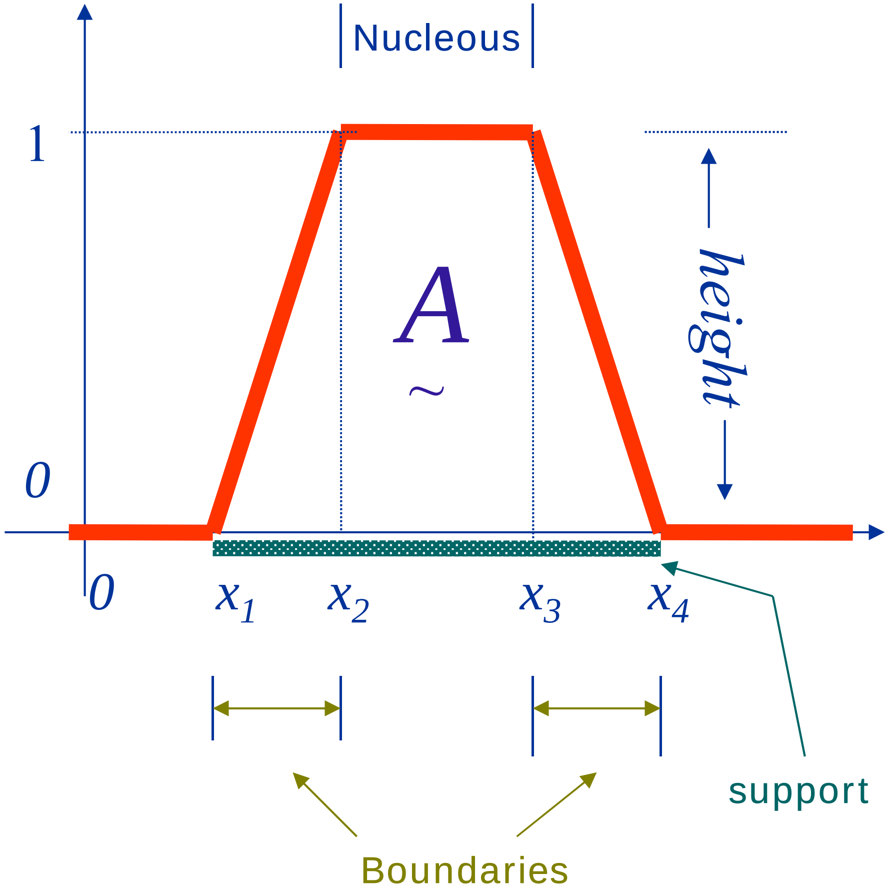
$$\underset{\sim}{A} = \sum_i \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i}$$

enumeration

delimitator

$$\underset{\sim}{A} = \int \frac{\mu_{\underset{\sim}{A}}(x)}{x}$$

$\mu(x) \triangleq$ membership function of the (fuzzy) set $\in [0,1]$

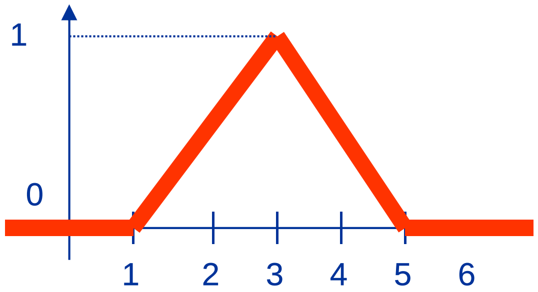


$$\text{nucleous of } \underset{\sim}{A}: \left\{ x \mid \mu_{\underset{\sim}{A}}(x) = 1 \right\}$$

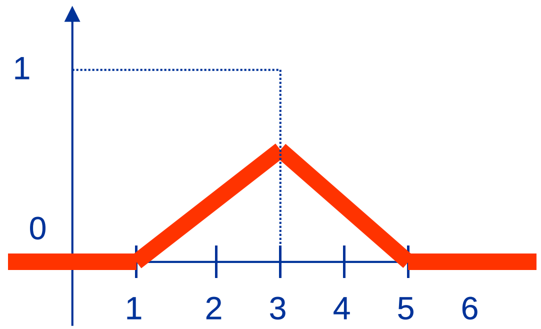
$$\text{support of } \underset{\sim}{A}: \left\{ x \mid \mu_{\underset{\sim}{A}}(x) > 0 \right\}$$

$$\text{boundary of } \underset{\sim}{A}: \left\{ x \mid 0 < \mu_{\underset{\sim}{A}}(x) < 1 \right\}$$

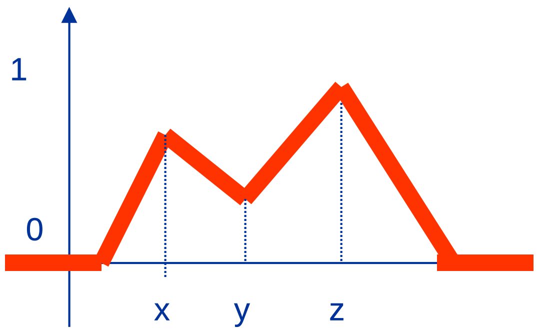
$$\text{height of } \underset{\sim}{A}: \max \mu_{\underset{\sim}{A}}(x)$$



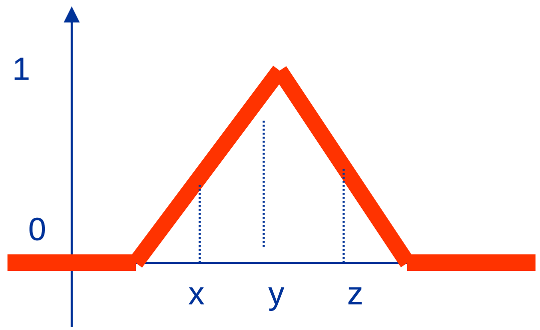
Normal $\sup \mu_{\underset{\sim}{A}}(x) = 1$



Abnormal $\sup \mu_{\underset{\sim}{A}}(x) < 1$



Non convex

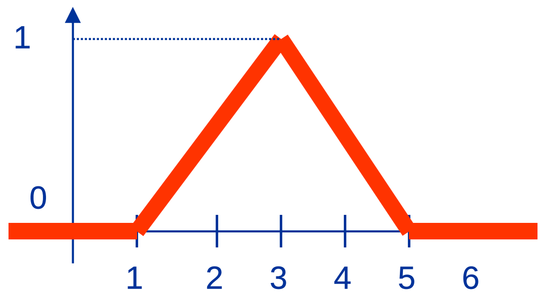


Convex

$$\mu_{\underset{\sim}{A}}(y) \le \min\left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{A}}(z)\right]$$

$$\mu_{\underset{\sim}{A}}(y) \ge \min\left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{A}}(z)\right]$$

Fuzzy number: fuzzy set, convex, normal, with one single element in the nucleous.



$\underset{\sim}{3}$

Operations over fuzzy sets

Let $\underset{\sim}{A}, \ \underset{\sim}{B}, \ \underset{\sim}{C}$ be fuzzy sets in the same Universe X

For one element $x \in X$

Union

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x) = m\acute{a}x\left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right]$$

Intersection

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x) = m\acute{i}n\left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right]$$

Complement

$$\mu_{\bar{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x)$$

Laws of De Morgan

$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$$

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$$

All properties of crisp sets are valid for fuzzy sets, with the exception :

$$A \cup \overline{A} = X$$

$$A \cap \overline{A} = \varnothing$$

$$\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} \neq X$$

$$\underset{\sim}{A} \cap \overline{\underset{\sim}{A}} \neq \varnothing$$

Associativity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C}$$

$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}$$

Distributivity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$$

$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C})$$

Idempotence

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A} \qquad \underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

Identity

$$\underset{\sim}{A} \cup \varnothing = \underset{\sim}{A} \qquad \underset{\sim}{A} \cap \varnothing = \varnothing$$

$$\underset{\sim}{A} \cap X = \underset{\sim}{A} \qquad \underset{\sim}{A} \cup X = X$$

Transitivity

$$\underset{\sim}{A} \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{C} \Rightarrow \underset{\sim}{A} \subseteq \underset{\sim}{C}$$

Involution (double negation)

$$\overline{\overline{\underset{\sim}{A}}} = \underset{\sim}{A}$$

Example

Consider $\underset{\sim}{A}=\left\{\frac{1}{0}+\frac{0,8}{6}+\frac{0,3}{8}\right\}$ $\underset{\sim}{B}=\left\{\frac{0,4}{2}+\frac{0,5}{4}+\frac{1}{8}\right\}$

defined in the Universe of discourse U={0, 2, 4, 6, 8, 10}.

Calculate $\overline{\underset{\sim}{A}}\cap(\underset{\sim}{B}\cup\underset{\sim}{A})$

$$\underset{\sim}{A}=\left\{\frac{1}{0}+\frac{0}{2}+\frac{0}{4}+\frac{0,8}{6}+\frac{0,3}{8}+\frac{0}{10}\right\}\quad \overline{\underset{\sim}{A}}=\left\{\frac{0}{0}+\frac{1}{2}+\frac{1}{4}+\frac{0,2}{6}+\frac{0,7}{8}+\frac{1}{10}\right\}$$

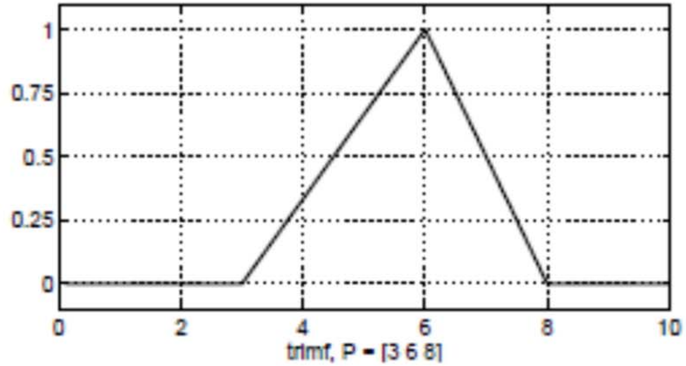
$$\underset{\sim}{B}=\left\{\frac{0}{0}+\frac{0,4}{2}+\frac{0,5}{4}+\frac{0}{6}+\frac{1}{8}+\frac{0}{10}\right\}\quad \overline{\underset{\sim}{B}}=\left\{\frac{1}{0}+\frac{0,6}{2}+\frac{0,5}{4}+\frac{1}{6}+\frac{0}{8}+\frac{1}{10}\right\}$$

Note that A and B, and their complements, must be defined for all elements of the Universe.

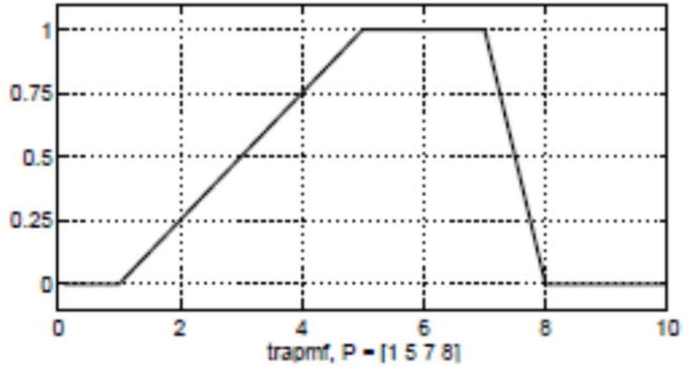
$$\underset{\sim}{B}\cup\underset{\sim}{A}=\left\{\frac{máx(0;1)}{0}+\frac{máx(0,4;0)}{2}+\frac{máx(0,5;0)}{4}+\frac{máx(0;0,8)}{6}+\frac{máx(1;0,3)}{8}+\frac{máx(0,0)}{10}\right\}=\left\{\frac{1}{0}+\frac{0,4}{2}+\frac{0,5}{4}+\frac{0,8}{6}+\frac{1}{8}+\frac{0}{10}\right\}$$

$$\overline{\underset{\sim}{A}}\cap(\underset{\sim}{B}\cup\underset{\sim}{A})=\left\{\frac{mín(0;1)}{0}+\frac{mín(1;0,4)}{2}+\frac{mín(1;0,5)}{4}+\frac{mín(0,2;0,8)}{6}+\frac{mín(0,7;1)}{8}+\frac{mín(1,0)}{10}\right\}=\left\{\frac{0}{0}+\frac{0,4}{2}+\frac{0,5}{4}+\frac{0,2}{6}+\frac{0,7}{8}+\frac{0}{10}\right\}$$

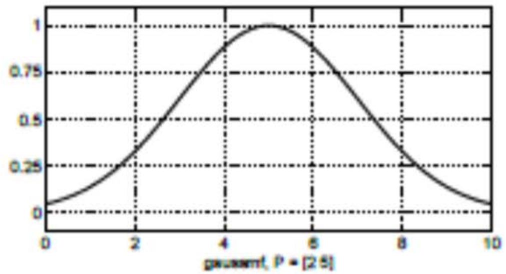
Membership functions implemented in the Fuzzy Logic Toolbox



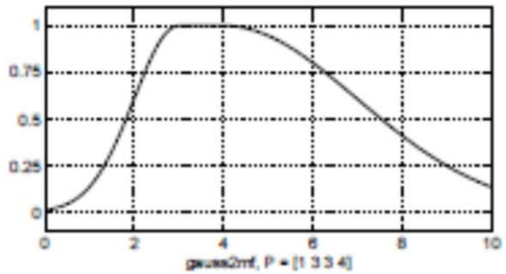
trimf



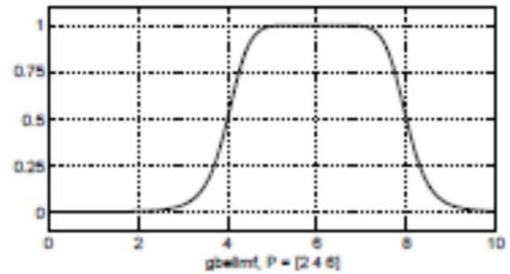
trapmf



gaussmf



gauss2mf

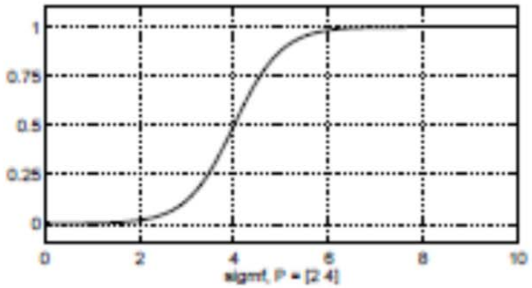


gbellmf

GAUSSMF(X, [SIGMA, C]) =
EXP(-(X - C).^2/(2*SIGMA^2));

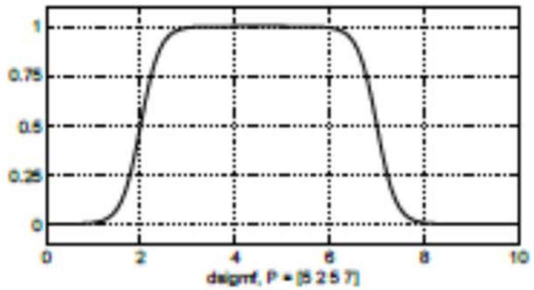
[sig1, c1, sig2, c2]

GBELLMF(X, [A, B, C]) =
1./((1+ABS((X-C)/A))^(2*B))



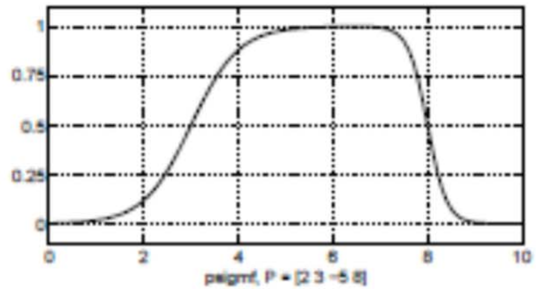
sigmf

f(x;a,c) = 1/(1 + exp(-a(x-c)))



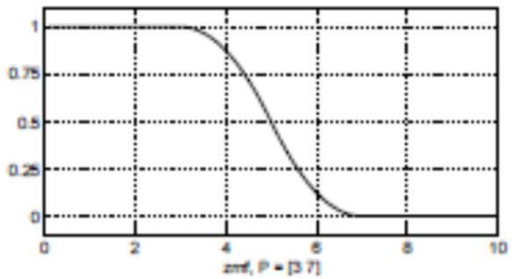
dsigmf

f1(x; a1, c1) - f2(x; a2, c2)



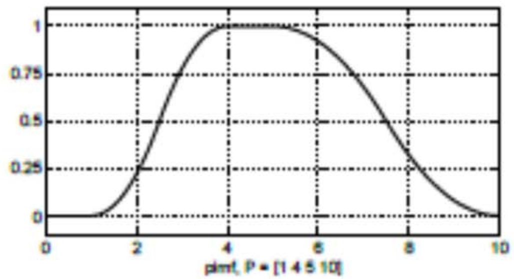
psigmf

PSIGMF(X, PARAMS) =
SIGMF(X, PARAMS(1:2)).*
SIGMF(X, PARAMS(3:4))



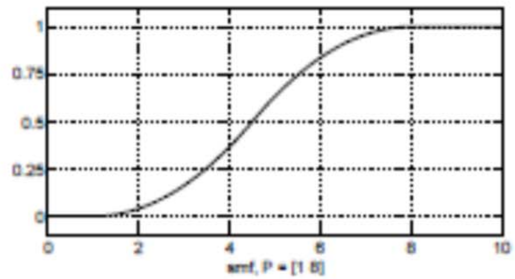
zmf

Z



pimf

Π



smf

S

8.2. Fuzzy relations

Crisp (classic) relations

Relation: mapping between sets (functions of sets)

Cartesian product of two sets

$$X = \{0,1\} \qquad Y = \{a,b\}$$

$$X \times Y = \{(0,a),(0,b),(1,a),(1,b)\}$$

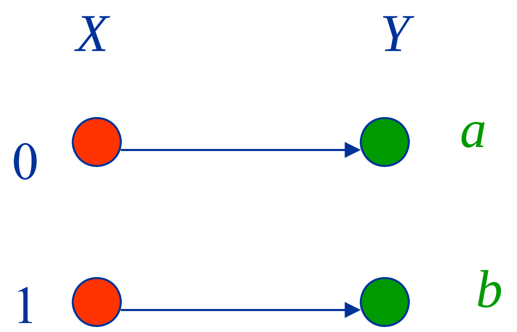
$$Y \times X = \{(a,0),(a,1),(b,0),(b,1)\}$$

$$X \times X = X^2 = \{(0,0),(0,1),(1,0),(1,1)\}$$

Binary relation R in the universes X and Y: any subset of the Cartesian product XxY, made by ordered pairs (x,y) where the 1st belongs to X and the 2nd to Y.

Characteristic function of the binary relation (crisp): a measure of the intensity of the relation:

$$\chi_R(x,y) = \begin{cases} 1, (x,y) \in \text{Relation} \\ 0, (x,y) \notin \text{Relation} \end{cases}$$



$$R = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

R: relational matrix, if X and Y are finite Universes

Example

$$X = \{António, José, Manuel, Tiago\} = \{A, J, M, T\}$$

$$Y = \{Natália, Isabel, Luísa, Catarina\} = \{N, I, L, C\}$$

Cartesian product

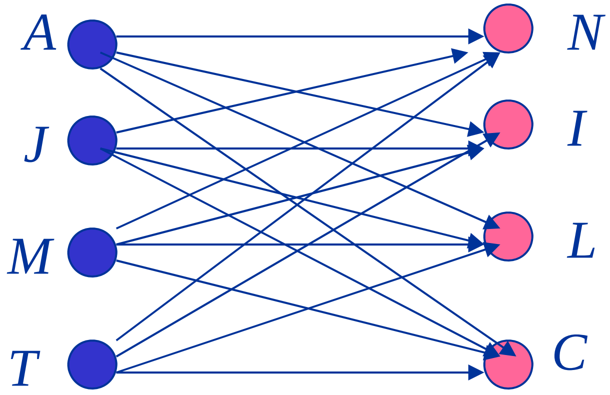
$\{A,N\},\{A,I\},\{A,L\},\{A,C\},$

$\{J,N\},\{J,I\},\{J,L\},\{J,C\},$

$\{M,N\},\{M,I\},\{M,L\},\{M,C\},$

$\{T,N\},\{T,I\},\{T,L\},\{T,C\},$

Sagittal diagram



Relational matrix

1

$$\begin{array}{c|cccc} & N & I & L & C \\ \hline A & 1 & 1 & 1 & 1 \\ J & 1 & 1 & 1 & 1 \\ M & 1 & 1 & 1 & 1 \\ T & 1 & 1 & 1 & 1 \end{array}$$

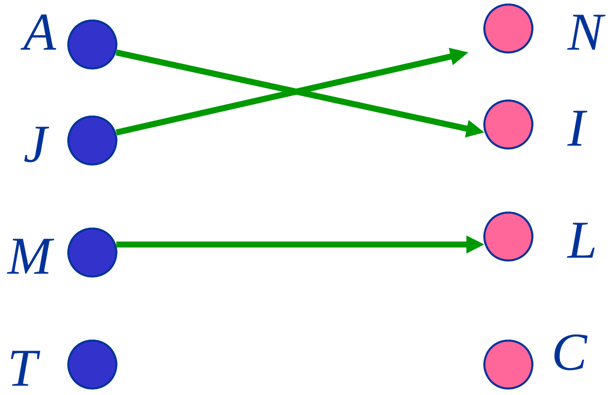
Universal or complete relation: everyone is related to everyone

Relation R: married to

Elements

$$\{A, I\}, \{J, N\}, \{M, L\}$$

Sagittal diagram



Relational matrix

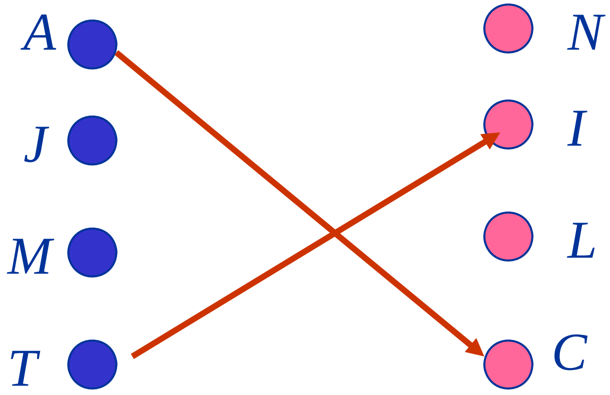
$$\begin{array}{c|cccc|} & N & I & L & C \\ \hline A & 0 & 1 & 0 & 0 \\ J & 1 & 0 & 0 & 0 \\ M & 0 & 0 & 1 & 0 \\ T & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

Relation S: brother to

Elements

$$\{A, C\}, \{T, I\}$$

Sagittal diagram



Relational matrix

$$\begin{array}{c|cccc|} & N & I & L & C \\ \hline A & 0 & 0 & 0 & 1 \\ J & 0 & 0 & 0 & 0 \\ M & 0 & 0 & 0 & 0 \\ T & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

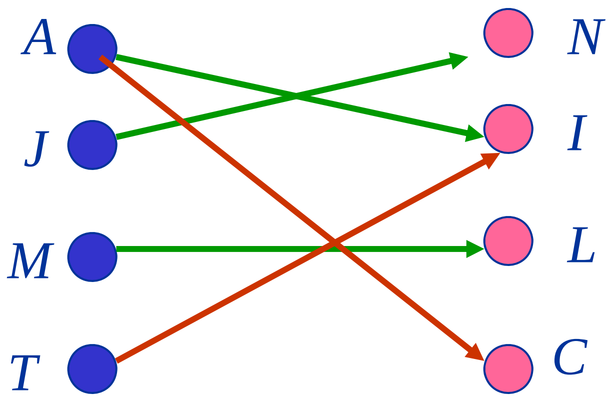
Relation R∪S: married to *or* brother to

Elements

$\{A, I\}, \{J, N\}, \{M, L\},$

$\{A, C\}, \{T, I\}$

Sagittal diagram



Relational matrix

$$\begin{array}{c|cccc} & N & I & L & C \\ \hline A & 0 & 1 & 0 & 1 \\ J & 1 & 0 & 0 & 0 \\ M & 0 & 0 & 1 & 0 \\ T & 0 & 1 & 0 & 0 \end{array}$$

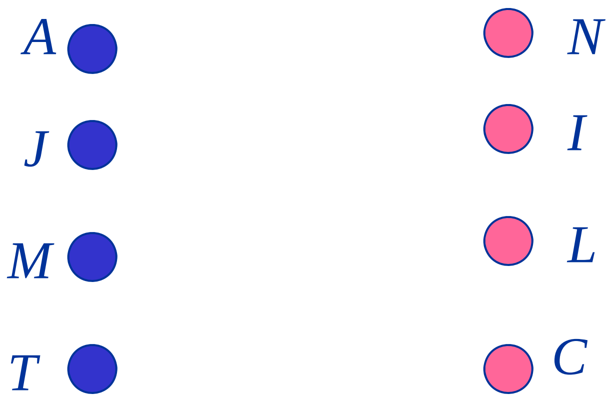
Characteristic function

$$\chi_{R \cup S} = máx\left[\chi_R(x, y), \chi_S(x, y)\right]$$

Relation R∩S: married to *and* brother to

Elements

Sagittal diagram



Relational matrix

$$
\begin{array}{c|cccc|}
 & N & I & L & C \\
\hline
A & 0 & 0 & 0 & 0 \\
J & 0 & 0 & 0 & 0 \\
M & 0 & 0 & 0 & 0 \\
T & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Characteristic function

$$\chi_{R\cap S} = mín\left[\chi_R(x,y), \chi_S(x,y)\right]$$

Relation of infinite cardinality

$X = [0,2] \in \Re$ $\qquad$ $Y = [1,4] \in \Re$

Relation R:

$$x < y$$

It cannot be represented neither by a relational matrix nor by a Sagittal diagram.

Operations over relations

Consider the Universes X and Y , $X \times Y$ their Cartesian product

R and S : binary relations in $X \times Y$

O=[∅] matrix of the null relation

E=[**1**] matrix of the complete relations

One can define the relations:

Union: R ∪S
$$\chi_{R \cup S} = m\acute{a}x\left[\chi_R(x,y), \chi_S(x,y)\right]$$

Intersection: R∩S
$$\chi_{R \cap S} = m\acute{\imath}n\left[\chi_R(x,y), \chi_S(x,y)\right]$$

Complement $\overline{R}$

$$\chi_{\overline{R}} = 1 - \chi_R(x, y)$$

Inclusion $R \subseteq S$

$$\chi_R(x, y) \leq \chi_S(x, y)$$

Empty — Being in $\varnothing$ gives the relational matrix O

Identity — Being in XxY gives the relational matrix **1**

Properties of the relations

commutativity

associativity

distributivity

involution (double negation)

idempotence

Laws of De Morgan

Composition of relations (crisp)

Let: Universes X, Y, Z

Relations

R: (X,Y) relates elements of X with elements of Y

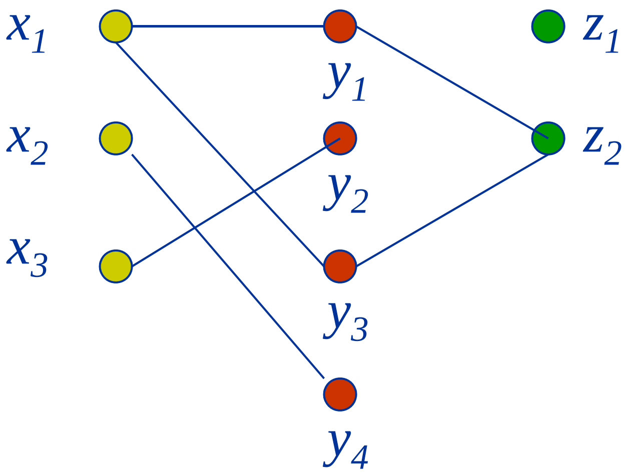
S: (Y,Z) relates elements of Y with elements of Z

T: (X,Z) relates the same elements of X with the same elements of Z

Exists T ? How to find it ?

By composition: T=R o S

Example:



$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

Composition by maximum-minimum, *max-min*

$$\chi_T(x_1, z_2) = max\{min[\chi_R(x_1, y_1), \chi_S(y_1, z_2)], min[\chi_R(x_1, y_2), \chi_S(y_2, z_2)],$$
$$min[\chi_R(x_1, y_3), \chi_S(y_3, z_2)], min[\chi_R(x_1, y_4), \chi_S(y_4, z_2)]\}$$

Composition by maximum-product, *max-prod*

$$\chi_T(x_1, z_2) = máx\{\chi_R(x_1, y_1) \cdot \chi_S(y_1, z_2), \chi_R(x_1, y_2) \cdot \chi_S(y_2, z_2),$$
$$\chi_R(x_1, y_3) \cdot \chi_S(y_3, z_2), \chi_R(x_1, y_4) \cdot \chi_S(y_4, z_2)\}$$

Composition maximum-mínimum *max-min*

$$\chi_T(x_1,z_2) = max\{min[\chi_R(x_1,y_1),\chi_S(y_1,z_2)], min[\chi_R(x_1,y_2),\chi_S(y_2,z_2)],$$
$$min[\chi_R(x_1,y_3),\chi_S(y_3,z_2)], min[\chi_R(x_1,y_4),\chi_S(y_4,z_2)]\}$$

$$\chi_T(x,z) = \vee_{y\in Y}(\chi_R(x,y)\wedge\chi_S(y,z))$$

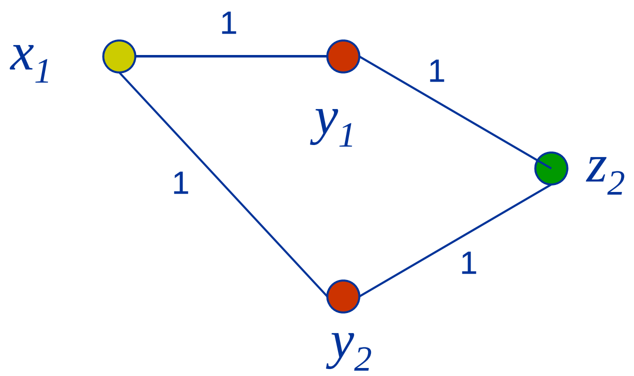
Composition maximum-product, *max-prod*

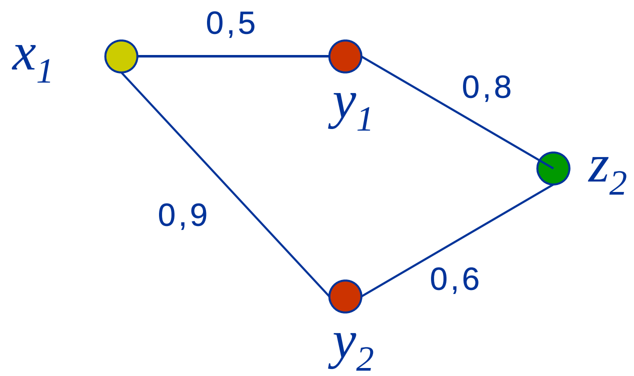
$$\chi_T(x_1,z_2) = max\{\chi_R(x_1,y_1)\cdot\chi_S(y_1,z_2), \chi_R(x_1,y_2)\cdot\chi_S(y_2,z_2),$$
$$\chi_R(x_1,y_3)\cdot\chi_S(y_3,z_2), \chi_R(x_1,y_4)\cdot\chi_S(y_4,z_2)\}$$

$$\chi_T(x,z) = \vee_{y\in Y}(\chi_R(x,y)\bullet\chi_S(y,z))$$

$$\bigvee_{y\in Y}\left(\chi_R(x,y)\bullet\chi_S(y,z)\right)=\bigvee_{y\in Y}\left(\chi_R(x,y)\wedge\chi_S(y,z)\right)\quad ???$$

max-prod = max-min ???





Yes !!

(crisp relations)

No !!

(fuzzy relations)

Fuzzy relations

X, Y Universes do discourse, X x Y Cartesian product

$\underset{\sim}{R}(x, y)$ Fuzzy binary relation: the intensity of the relation is not only 0 or 1, but is in the real interval [0,1]

Characteristic function of the relation $\underset{\sim}{R}(x, y)$

$\mu_{\underset{\sim}{R}}(x, y) \triangleq$ membership value of the ordered pair (x, y) to the relation $\underset{\sim}{R}$

Operations with fuzzy relations

$\underset{\sim}{R}, \underset{\sim}{S}$ fuzzy relations in $X \times Y$, of the Universes X e Y

$\underset{\sim}{R} \cup \underset{\sim}{S}$ $\qquad \mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = max\left[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)\right]$

$\underset{\sim}{R} \cap \underset{\sim}{S}$ $\qquad \mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = min\left[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)\right]$

$\overline{\underset{\sim}{R}}$ $\qquad \mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$

$\underset{\sim}{R} \subseteq \underset{\sim}{S}$ $\qquad \underset{\sim}{R} \subseteq \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$

Properties of the fuzzy relations

commutativity

associativity

distributivity

double negation

idempotence

Laws of De Morgan

but:

$$\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq E$$

$$\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq \mathrm{O}$$

Composition of fuzzy relations

$\underset{\sim}{A}$ fuzzy set defined in the Universe X

$\underset{\sim}{B}$ conjunto fuzzy set defined in the Universe Y

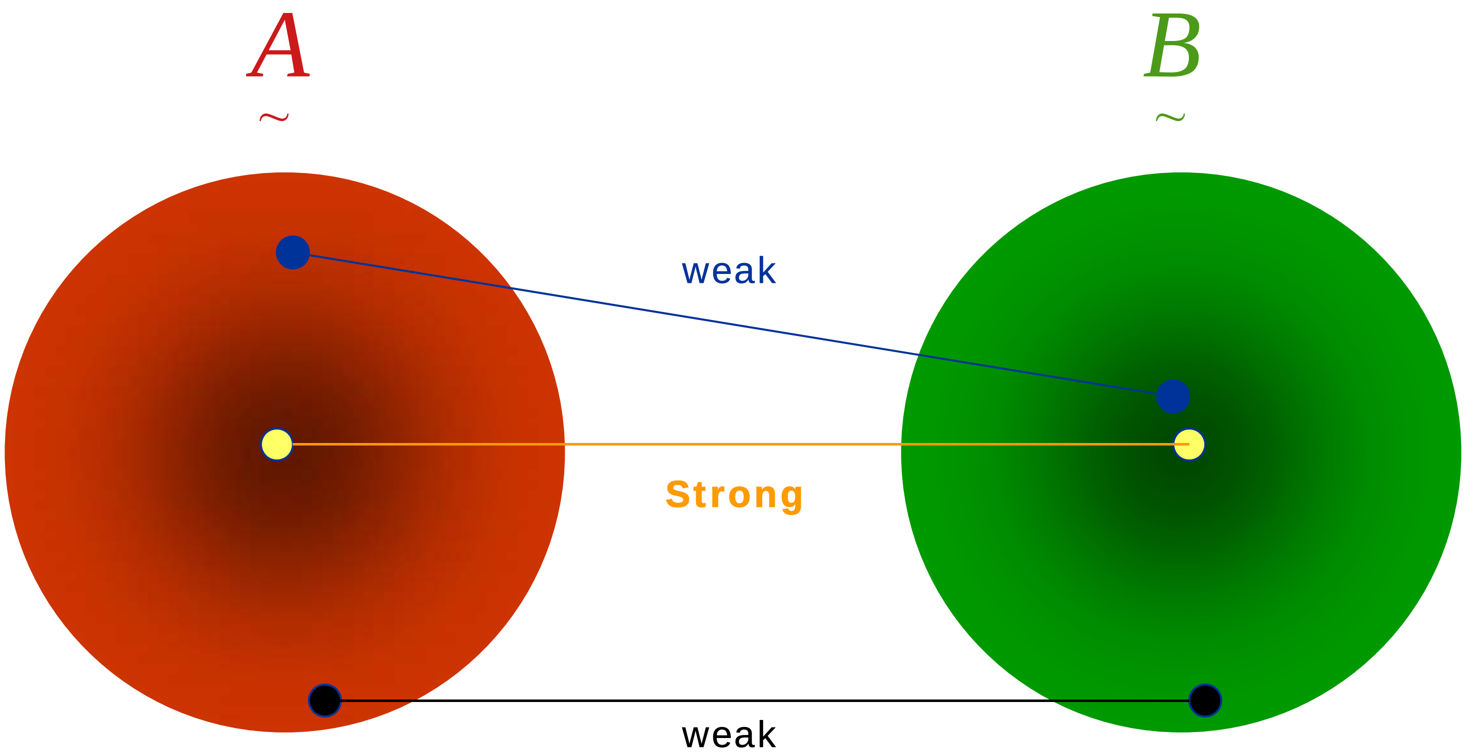
The Cartesian product $\underset{\sim}{A} \times \underset{\sim}{B}$ defines a relation $\underset{\sim}{R}$ in the Cartesian product $X \times Y$

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y$$

Membership function of the fuzzy relation $\underset{\sim}{R}$

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x, y) = min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y))$$

Membership function the fuzzy relation $\underset{\sim}{R}$:



Example

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\} \qquad \underset{\sim}{A} = \left\{\frac{0,2}{x_1} + \frac{1}{x_2}\right\} \qquad \underset{\sim}{B} = \left\{\frac{0,3}{y_1} + \frac{0,9}{y_2}\right\}$$

then

$$\underset{\sim}{R} = \left\{\frac{min(0,2;0,3)}{(x_1, y_1)} + \frac{min(0,2;0,9)}{(x_1, y_2)} + \frac{min(1;0,3)}{(x_2, y_1)} + \frac{min(1;0,9)}{(x_2, y_2)}\right\} =$$

$$= \left\{\frac{0,2}{(x_1, y_1)} + \frac{0,2}{(x_1, y_2)} + \frac{0,3}{(x_2, y_1)} + \frac{0,9}{(x_2, y_2)}\right\} =$$

$$= \underset{\sim}{A} \times \underset{\sim}{B}$$

$$\underset{\sim}{R} = \mu_{\underset{\sim}{A}} \bullet \mu_{\underset{\sim}{B}}^T$$

$$\underset{\sim}{R} = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0,2 & 0,2 \\ x_2 & 0,3 & 0,9 \end{array}$$

$$\underset{\sim}{R} = \begin{bmatrix} 0,2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0,3 & 0,9 \end{bmatrix} = \begin{bmatrix} 0,2 & 0,2 \\ 0,3 & 0,9 \end{bmatrix}$$

min

$\underset{\sim}{R} \triangleq$ fuzzy relation in $X \times Y$

$\underset{\sim}{S} \triangleq$ fuzzy relation in $Y \times Z$

$\underset{\sim}{T} \triangleq$ fuzzy relation in $X \times Z$

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S}$$

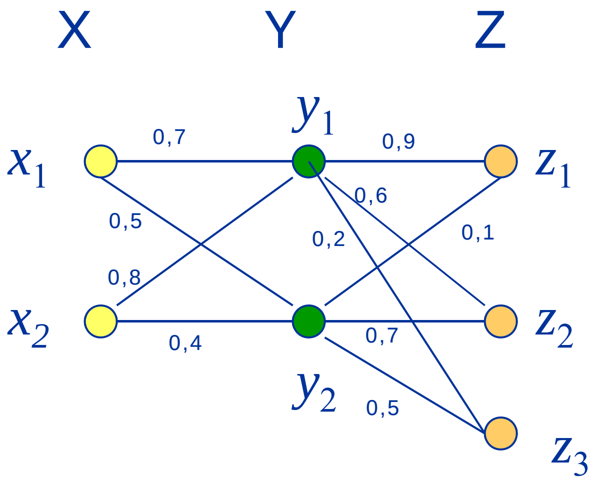
Composition maximum-minimum *max-min*

$$\mu_{\underset{\sim}{T}}(x,z) = \underset{y \in Y}{\vee}(\mu_{\underset{\sim}{R}}(x,y) \wedge \mu_{\underset{\sim}{S}}(y,z)) = max[min(\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{S}}(y,z))]$$

Composition maximum-product *max-prod*

$$\mu_{\underset{\sim}{T}}(x,z) = \underset{y \in Y}{\vee}(\mu_{\underset{\sim}{R}}(x,y) \bullet \mu_{\underset{\sim}{S}}(y,z)) = max[\mu_{\underset{\sim}{R}}(x,y) \bullet \mu_{\underset{\sim}{S}}(y,z))]$$

Example



$$\underset{\sim}{R} \triangleq \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0,7 & 0,5 \\ x_2 & 0,8 & 0,4 \end{array} \qquad \underset{\sim}{S} \triangleq \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0,9 & 0,6 & 0,2 \\ y_2 & 0,1 & 0,7 & 0,5 \end{array}$$

By the composition *max-min*:

$$\mu_T(x_1, z_1) = max\{min[(\mu(x_1, y_1), \mu(y_1, z_1)], min[(\mu(x_1, y_2), \mu(y_2, z_1)]\}$$
$$= max\{min[0,7;0,9], min[0,5;0,1]\} = 0,7$$

By the composition *max-prod*:

$$\mu_T(x_1, z_1) = max[\mu(x_1, y_1) \times \mu(y_1, z_1), (\mu(x_1, y_2) \times \mu(y_2, z_1)]$$
$$= max[0,7 \times 0,9 \; ; \; 0,5 \times 0,1] = 0,63$$

Max-min

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} = \begin{bmatrix} 0,7 & 0,5 \\ 0,8 & 0,4 \end{bmatrix} \circ \begin{bmatrix} 0,9 & 0,6 & 0,2 \\ 0,1 & 0,7 & 0,5 \end{bmatrix} = \begin{bmatrix} 0,7 & 0,6 & 0,5 \\ 0,8 & 0,6 & 0,4 \end{bmatrix}$$

$$= \begin{bmatrix} 0,7 \bullet 0,9 + 0,5 \bullet 0,1 & 0,7 \bullet 0,6 + 0,5 \bullet 0,7 & 0,7 \bullet 0,2 + 0,5 \bullet 0,5 \\ 0,8 \bullet 0,9 + 0,4 \bullet 0,1 & 0,8 \bullet 0,9 + 0,4 \bullet 0,7 & 0,8 \bullet 0,2 + 0,4 \bullet 0,5 \end{bmatrix}$$

$\bullet \triangleq$ minimum operator

$+ \triangleq$ maximum operator

Max-prod

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} = \begin{bmatrix} 0,7 & 0,5 \\ 0,8 & 0,4 \end{bmatrix} \circ \begin{bmatrix} 0,9 & 0,6 & 0,2 \\ 0,1 & 0,7 & 0,5 \end{bmatrix} = \begin{bmatrix} 0,63 & 0,42 & 0,25 \\ 0,72 & 0,48 & 0,20 \end{bmatrix}$$

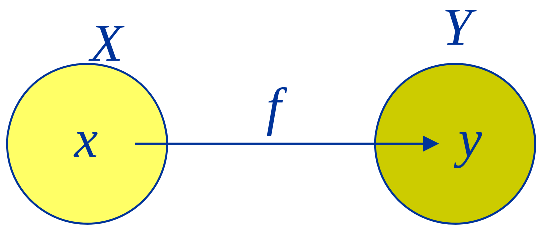
$$= \begin{bmatrix} 0,7 \times 0,9 + 0,5 \times 0,1 & 0,7 \times 0,6 + 0,5 \times 0,7 & 0,7 \times 0,2 + 0,5 \times 0,5 \\ 0,8 \times 0,9 + 0,4 \times 0,1 & 0,8 \times 0,6 + 0,4 \times 0,7 & 0,8 \times 0,2 + 0,4 \times 0,5 \end{bmatrix}$$

$\times \triangleq$ algebraic product operator

$+ \triangleq$ maximum operator

8.3. Function of fuzzy sets. Zadeh Extension Principle

Functions of crisp sets



X, Y two universes

$y = f(x)$, image of x under f, defines the relation R

$$R = \{(x, y) : y = f(x)\} \qquad \chi_R(x, y) = \begin{cases} 1, & \text{se } y = f(x) \\ 0, & \text{se } y \neq f(x) \end{cases}$$

Example

$$X=[-2,-1,0,1,2] \qquad y=4x+2 \qquad Y=[-10,-8,-6,-4,-2,0,2,4,6,8,10]$$

$$R=\begin{array}{c|ccccccccccc} x\backslash y & -10 & -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 \\ \hline -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

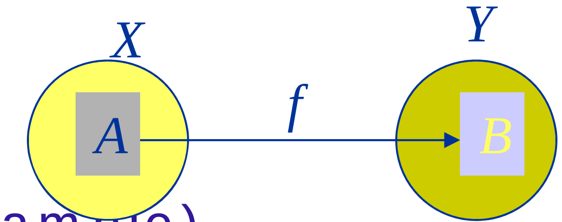
Relational matrix between X and Y

Let A be a set in X and B a set in Y

$B = \{y : \text{ for all } x \in A,\ y = f(x)\}$

$\chi_B(y) = \chi_{f(A)}(y) = \chi_A(x)$, such that $y = f(x)$

Example



$A = \{-1, 0, 1\} \subset X$ (of the previous example)

$B = f(A) = \{-2, 2, 6\}$

$$\chi_A = \left\{ \frac{0}{-2} + \frac{1}{-1} + \frac{1}{0} + \frac{1}{1} + \frac{0}{2} \right\}$$

$$\chi_B = \left\{ \frac{0}{-10} + \frac{0}{-8} + \frac{0}{-6} + \frac{0}{-4} + \frac{1}{-2} + \frac{0}{0} + \frac{1}{2} + \frac{0}{4} + \frac{1}{6} + \frac{0}{8} + \frac{0}{10} \right\}$$

$$\chi_B = \chi_A \circ R = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Example

$$X = \{-2, -1, 0, 1, 2\} \qquad y = x^2 \qquad Y = \{0, 1, 2, 4, 8\}$$

$$R = \begin{array}{c|ccccc} x \setminus y & 0 & 1 & 2 & 4 & 8 \\ \hline -2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$$A = \{-1, 0, 1\} \Rightarrow B = f(A) = \{0, 1\}$$

$$\chi_B(y) = \chi_{f(A)}(y) = \bigvee_{y = f(x)} \chi_A(x)$$

$$\chi_B(y) = \bigvee_{x \in X} (\chi_A(x) \wedge \chi_R(x, y)) = max[min(\chi_A(x), \chi_R(x, y))]$$

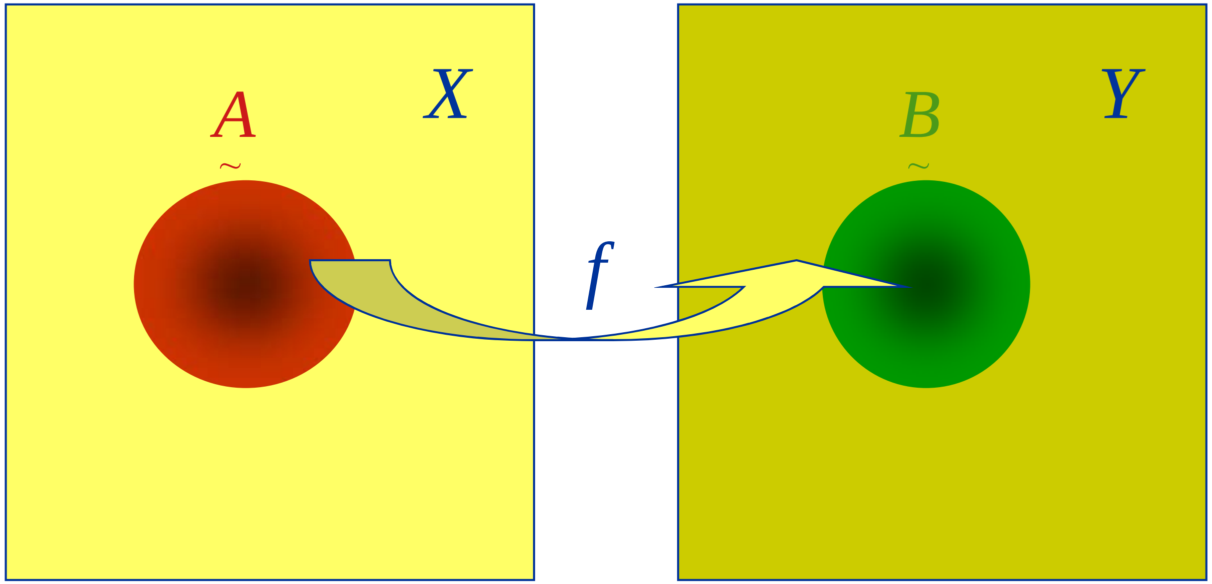
$$A=\left\{\frac{0}{-2}+\frac{1}{-1}+\frac{1}{0}+\frac{1}{1}+\frac{0}{2}\right\} \qquad \chi_A=\begin{bmatrix}0 & 1 & 1 & 1 & 0\end{bmatrix}$$

$$\chi_B=\chi_A \circ R$$

$$\chi_B=max\begin{bmatrix}0 & 1 & 1 & 1 & 0\end{bmatrix}\bullet\begin{bmatrix}0 & 0 & 0 & 1 & 0\\0 & 1 & 0 & 0 & 0\\1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 1 & 0 & 0 & 0\end{bmatrix}$$

$$B=\left\{\frac{1}{0}+\frac{1}{1}+\frac{0}{2}+\frac{0}{4}+\frac{0}{8}\right\}\Rightarrow B=\{0,1\}$$

Functions of fuzzy sets



$$\underset{\sim}{B} = f(\underset{\sim}{A})$$

$$y = f(x)$$

$$\mu_{\underset{\sim}{B}}(y) = \mu_{\underset{\sim}{A}}(x), \ \ (y = f(x)) \text{ if } f \text{ é bijective}$$

$$\mu_{\underset{\sim}{B}}(y) = \bigvee_{f(x)=y} \mu_{\underset{\sim}{A}}(x), \text{ if } f \text{ is not bijective}$$

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \ldots + \frac{\mu_{\underset{\sim}{A}}(x_n)}{x_n} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{\mu_{\underset{\sim}{B}}(x_1)}{y_1} + \frac{\mu_{\underset{\sim}{B}}(x_2)}{y_2} + \ldots + \frac{\mu_{\underset{\sim}{B}}(x_m)}{y_m} \right\} \qquad \mu_{\underset{\sim}{R}}(x,y) = min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y))$$

One may calculate $\underset{\sim}{B}$ by the composition operation

$$\underset{\sim}{B} = \underset{\sim}{A} \circ \underset{\sim}{R}$$

$$\mu_{\underset{\sim}{B}}(y) = \bigvee_{x \in X} (\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{R}}(x,y))$$

$$= \max_x [min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{R}}(x,y)]$$

Zadeh Extension Principle

Consider:

$X_1, X_2, ..., X_n$ and Y universes do discurse

$y = f(x_1, x_2, ..., x_n)$ a mapping in Universe Y

$\underset{\sim}{A}_1, \underset{\sim}{A}_2, ..., \underset{\sim}{A}_n$ fuzzy sets in $X_1, X_2, ..., X_n$

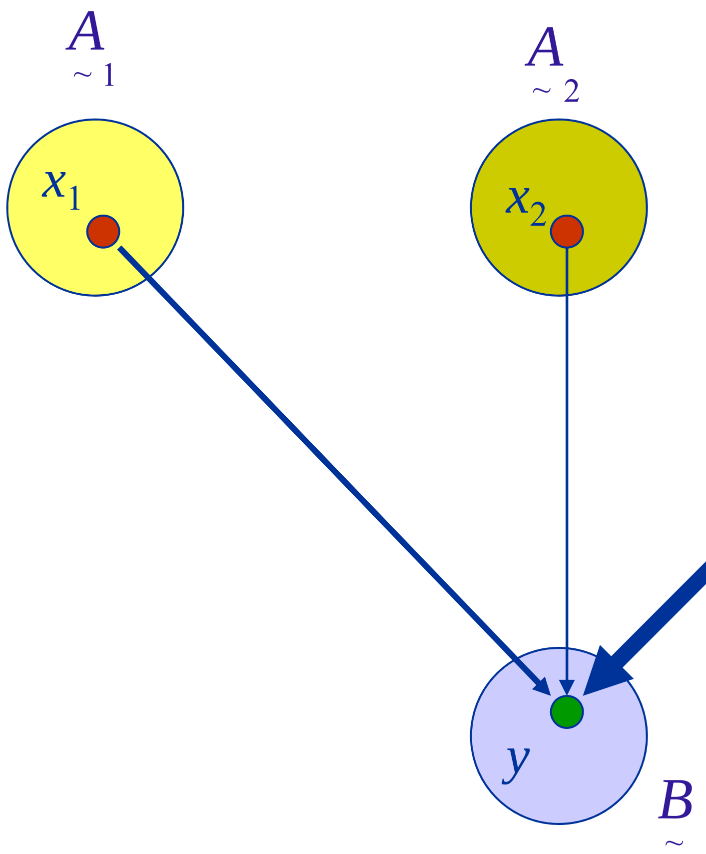
then

$$\underset{\sim}{B} = f(\underset{\sim}{A}_1, \underset{\sim}{A}_2, ..., \underset{\sim}{A}_n)$$

$$\mu_{\underset{\sim}{B}}(y) = \max_{y = f(x_1, x_2, ... x_n)} \left\{ \min[\mu_{\underset{\sim}{A}_1}(x_1), \mu_{\underset{\sim}{A}_2}(x_2), ..., \mu_{\underset{\sim}{A}_n}(x_n)] \right\}$$

... extends to the fuzzy sets the arithmetic and algebraic operations on the crisp sets.

Zadeh Extension Principle

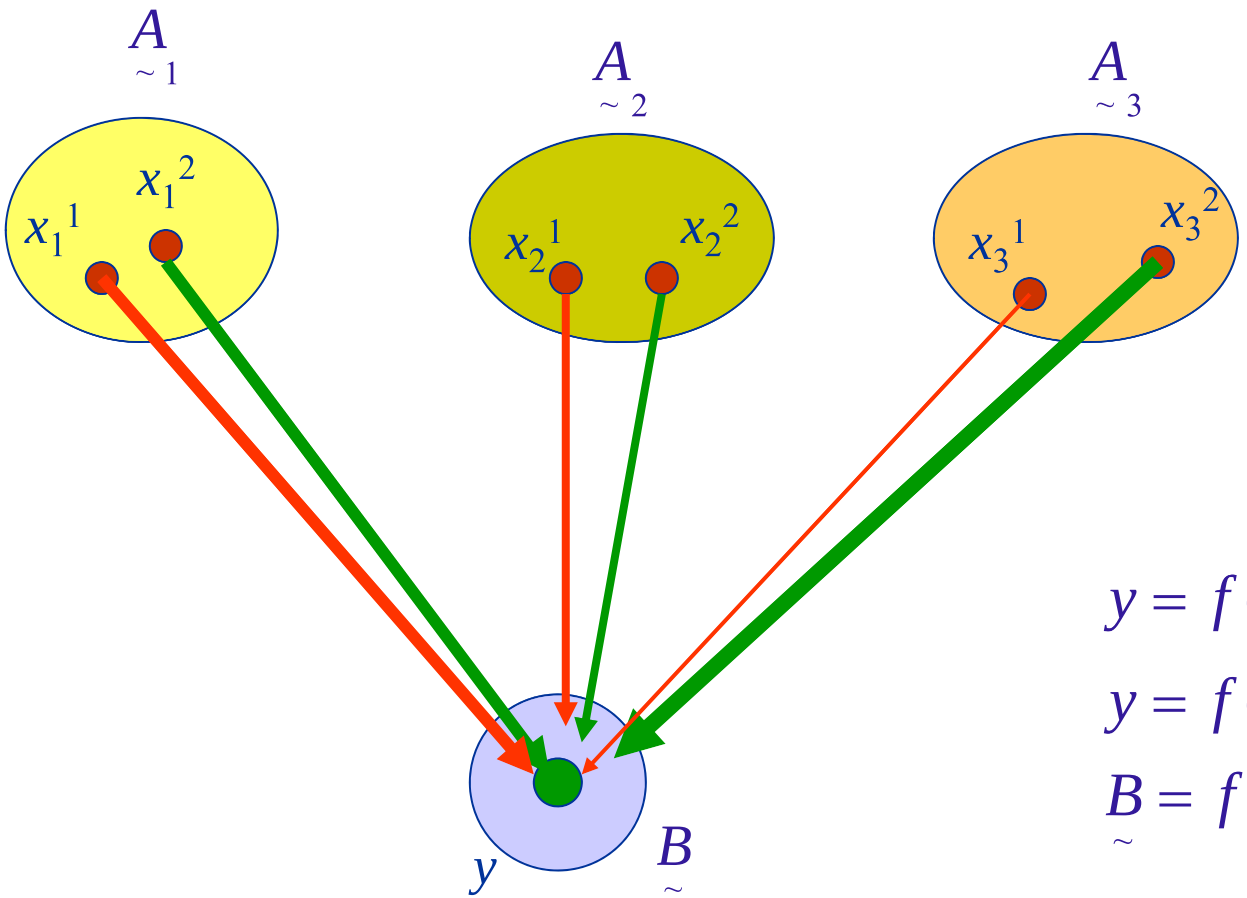


$$y = f(x_1, x_2, x_3)$$

$$\underset{\sim}{B} = f(\underset{\sim 1}{A}, \underset{\sim 2}{A}, \underset{\sim 3}{A})$$

$$\mu_{\underset{\sim}{B}}(y) = min[\mu_{\underset{\sim 1}{A}}(x_1), \mu_{\underset{\sim 2}{A}}(x_2), \mu_{\underset{\sim 3}{A}}(x_3)]$$

Zadeh Extension Principle



$$y = f(x_1^{\ 1}, x_2^{\ 1}, x_3^{\ 1})$$

$$y = f(x_1^{\ 2}, x_2^{\ 2}, x_3^{\ 2})$$

$$\underset{\sim}{B} = f(\underset{\sim 1}{A}, \underset{\sim 2}{A}, \underset{\sim 3}{A})$$

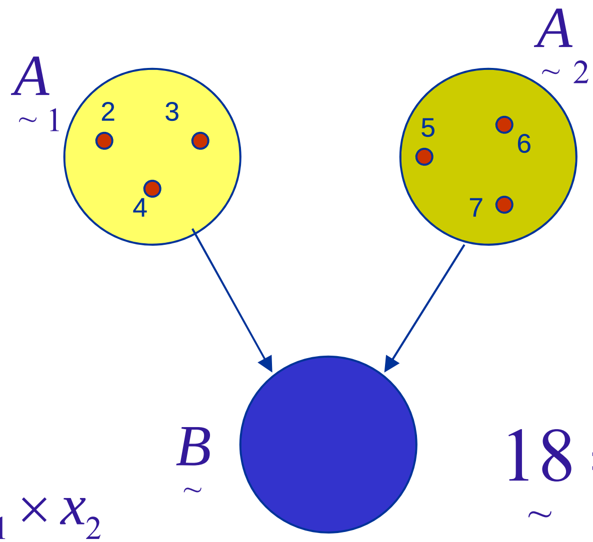
$$\mu_{\underset{\sim}{B}}(y) = max\left\{ min\ [\mu_{\underset{\sim 1}{A}}(x_1^{\ 1}), \mu_{\underset{\sim 2}{A}}(x_2^{\ 1}), \mu_{\underset{\sim 2}{A}}(x_3^{\ 1})], min[\mu_{\underset{\sim 1}{A}}(x_1^{\ 2}), \mu_{\underset{\sim 2}{A}}(x_2^{\ 2}), \mu_{\underset{\sim 2}{A}}(x_3^{\ 2})] \right\}$$

Example

$$\underset{\sim}{A}_1 = \underset{\sim}{3} = \left\{ \frac{0,2}{2} + \frac{1}{3} + \frac{0,3}{4} \right\}$$

$$\underset{\sim}{A}_2 = \underset{\sim}{6} = \left\{ \frac{0,5}{5} + \frac{1}{6} + \frac{0,1}{7} \right\}$$

$$\underset{\sim}{3} \times \underset{\sim}{6} = \underset{\sim}{18}$$



$$\underset{\sim}{18} = ???$$

$$y = f(x_1, x_2) = x_1 \times x_2$$

$$\underset{\sim}{B} = f(\underset{\sim}{A}_1, \underset{\sim}{A}_2) = \underset{\sim}{A}_1 \times \underset{\sim}{A}_2$$

$$\underset{\sim}{B} = \left\{ \frac{}{10} + \frac{}{12} + \frac{}{14} + \frac{}{15} + \frac{}{18} + \frac{}{21} + \frac{}{20} + \frac{}{24} + \frac{}{28} \right\} = \underset{\sim}{18}$$

$$\underset{\sim}{B} = \left\{ \frac{min(0,2;0,5)}{10} + \frac{min(0,2;1)}{12} + \frac{min(0,2;0,1)}{14} + \frac{min(1;0,5)}{15} + \frac{min(1;1)}{18} + \frac{min(1;0,1)}{21} + \frac{min(0,3;0,5)}{20} + \frac{min(0,3;1)}{24} + \frac{min(0,3;0,1)}{28} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0,2}{10} + \frac{0,2}{12} + \frac{0,1}{14} + \frac{0,5}{15} + \frac{1}{18} + \frac{0,1}{21} + \frac{0,3}{20} + \frac{0,3}{24} + \frac{0,1}{28} \right\} = \underset{\sim}{18}$$

8.4. Inference *modus ponens* and approximate reasoning

Classic logic implication:

Universe X, set A in X

Universe Y, set B in Y

$P \triangleq x \in A$ logical proposition P ("x belongs to the set A")

(Truth) $T(P) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$ $\qquad \chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$

$Q \triangleq y \in B$ logical proprosition Q ("y belongs to the set B")

$T(Q) = \begin{cases} 1, y \in B \\ 0, y \notin B \end{cases}$ $\qquad \chi_B(y) = \begin{cases} 1, y \in B \\ 0, y \notin B \end{cases}$

Logical connectives between propositions

–disjunction $(\vee)$, $P \vee Q$ $\quad T(P \vee Q) = max[T(P), T(Q)]$

–conjunction $(\wedge)$, $P \wedge Q$ $\quad T(P \wedge Q) = min[T(P), T(Q)]$

–negation $(-)$, $\overline{P}$ $\quad T(\overline{P}) = 1 - T(P)$

–equivalence $(\leftrightarrow)$, $P \leftrightarrow Q$ $\quad T(P \leftrightarrow Q) = \begin{cases} 1, \text{se } T(P) = T(Q) \\ 0, \text{se } T(P) \neq T(Q) \end{cases}$

-implication $(\rightarrow)$, $P \rightarrow Q$ $\quad P \rightarrow Q = (P \wedge Q) \vee (\overline{Q} \wedge \overline{P}) \vee (\overline{P} \wedge Q) = \overline{P} \vee Q$

P	Q	$P \rightarrow Q$	$\overline{P} \vee Q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

$$T(P \rightarrow Q) = T(\overline{P} \vee Q) = max[T(\overline{P}), T(Q)]$$

... it is true except in the case where the antecedent is true and the consequent is false.

Deductive inference

P proposition defined in a set $A \subset X$

Q proposition defined in a set $B \subset Y$

Tautologies: the main tools for reasoning in traditional logic are proposition that are always true

$$modus\ \ ponens : (A \wedge (A \rightarrow B) \rightarrow B$$

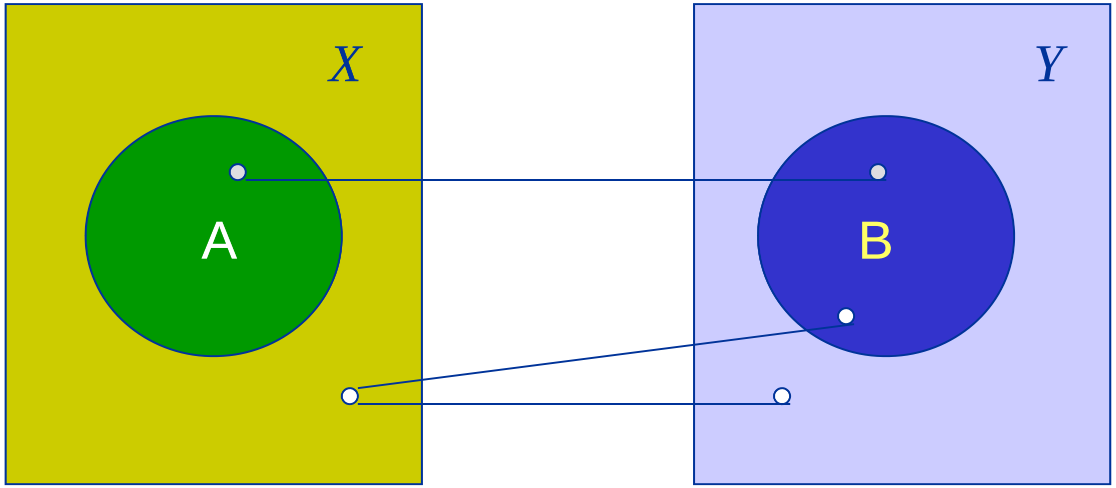
$$modus\ \ tollens : ((A \rightarrow B) \wedge \overline{B} \rightarrow \overline{A}$$

P implies $Q \equiv$

$(or$ "not in A" or "$in\ B$"$) \equiv (\overline{A} \cup B)$ is true

$$x \in A \rightarrow y \in B$$

$$= (x \in A \wedge y \in B) \vee (x \in \overline{A} \wedge y \in \overline{B}) \vee (x \in \overline{A} \wedge y \in B)$$



$x \in A \wedge y \in B$ gives $A \times B$

$x \notin A \wedge y \in B$ gives $\overline{A} \times B$

$x \notin A \wedge y \notin B$ gives $\overline{A} \times \overline{B}$

$$A \times B \cup \overline{A} \times B \cup \overline{A} \times \overline{B}$$
$$= A \times B \cup \overline{A}(B \cup \overline{B})$$
$$= A \times B \cup (\overline{A} \times Y)$$

The relational matrix is composed by the ordered pairs of *AxB* plus the ordered pairs of $\overline{A} \times Y$

$$R = (A \times B) \cup (\overline{A} \times Y)$$

If A **then** B

antecedent consequent

$$R = (A \times B) \cup (\overline{A} \times Y)$$

$$\chi_R(x, y) = max[\chi_A(x) \wedge \chi_B(y), 1 - \chi_A(x) \wedge \chi_Y(y)]$$
$$= max[\chi_A(x) \wedge \chi_B(y), 1 - \chi_A(x) \wedge 1]$$

The ordered pairs that belong to the relation are the Cartesian product AxB plus the ones that do not belong to A and belong to the universe Y.

If another antecedent A' appears, different from A, can we write

If A' **then** B' ??

Which B'

$$B' = A' \circ R = A' \circ [(A \times B) \cup (\overline{A} \times Y)]$$

$$\chi_{B'}(y) = \bigvee_{x \in X} (\chi_{A'}(x) \wedge \chi_R(x, y)) = \max_{x \in X}[\min(\chi_{A'}(x), \chi_R(x, y))]$$

Fuzzy logic implication

A fuzzy proposition $\underset{\sim}{P}$ associated with a fuzzy set $\underset{\sim}{A}$ has truth values

$$T(\underset{\sim}{P}) = \mu_{\underset{\sim}{A}}(x), \qquad 0 \le \mu_{\underset{\sim}{A}}(x) \le 1$$

Universes X e Y $\quad \underset{\sim}{P} \triangleq x \in \underset{\sim}{A}$, $\underset{\sim}{A} \subset X$, $\quad \underset{\sim}{Q} \triangleq y \in \underset{\sim}{B}$, $\underset{\sim}{B} \subset Y$

–disjunction $(\vee)$, $\quad \underset{\sim}{P} \vee \underset{\sim}{Q} \qquad T(\underset{\sim}{P} \vee \underset{\sim}{Q}) = max[T(\underset{\sim}{P}), T(\underset{\sim}{Q})]$

–conjunction $(\wedge)$, $\quad \underset{\sim}{P} \wedge \underset{\sim}{Q} \qquad T(\underset{\sim}{P} \wedge \underset{\sim}{Q}) = min[T(\underset{\sim}{P}), T(\underset{\sim}{Q})]$

–negation $(-)$, $\quad \overline{\underset{\sim}{P}} \qquad T(\overline{\underset{\sim}{P}}) = 1 - T(\underset{\sim}{P})$

–equivalence $(\leftrightarrow)$, $\quad \underset{\sim}{P} \leftrightarrow \underset{\sim}{Q} \qquad T(\underset{\sim}{P} \leftrightarrow \underset{\sim}{Q}) = \begin{cases} 1, \text{if } T(\underset{\sim}{P}) = T(\underset{\sim}{Q}) \\ 0, \text{if } T(\underset{\sim}{P}) \ne T(\underset{\sim}{Q}) \end{cases}$

-normal or Zadeh implication $(\rightarrow)$,

$$\underset{\sim}{P} \rightarrow \underset{\sim}{Q} \qquad \underset{\sim}{P} \rightarrow \underset{\sim}{Q} = (\underset{\sim}{P} \wedge \underset{\sim}{Q}) \vee (\overline{\underset{\sim}{Q}} \wedge \overline{\underset{\sim}{P}}) \vee (\overline{\underset{\sim}{P}} \wedge \underset{\sim}{Q}) = \overline{\underset{\sim}{P}} \vee \underset{\sim}{Q}$$

$$T(\underset{\sim}{P} \rightarrow \underset{\sim}{Q}) = T(\overline{\underset{\sim}{P}} \vee \underset{\sim}{Q}) = max[T(\overline{\underset{\sim}{P}}), T(\underset{\sim}{Q})] = max[(1 - T(\underset{\sim}{P}), T(\underset{\sim}{Q})]$$

Deductive inference *modus ponens*

Let $\underset{\sim}{P} \triangleq x$ belongs to $\underset{\sim}{A}$ $\quad \underset{\sim}{A}$ in Universe X

$\underset{\sim}{Q} \triangleq y$ belongs to $\underset{\sim}{B}$ $\quad \underset{\sim}{B}$ in Universe Y

$\underset{\sim}{P} \rightarrow \underset{\sim}{Q} \equiv$ IF x is $\underset{\sim}{A}$ THEN y is $\underset{\sim}{B}$ can be defined bty the relation $\underset{\sim}{R}$

$\underset{\sim}{R} = (\underset{\sim}{A} \rightarrow \underset{\sim}{B}) = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y)$ (see slide 312, for the crisp case)

$$\mu_{\underset{\sim}{R}}(x, y) = max[\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(y), (1 - \mu_{\underset{\sim}{A}}(x)) \wedge 1] = max[min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)), (1 - \mu_{\underset{\sim}{A}}(x))]$$

... normal or Zadeh implication

If a new antecedent A' appears, which will be B' ?

$$\text{IF } x \text{ is } \underset{\sim}{A}' \text{ THEN } y \text{ is } \underset{\sim}{B}'$$

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}$$

Composition maximum-minimum, *max-min*

$$\mu_{\underset{\sim}{B}'}(y) = \max_{x \in X}[min(\mu_{\underset{\sim}{A}'}(x), \mu_{\underset{\sim}{R}}(x, y)]$$

Composition maximum-product, *max-prod*

$$\mu_{\underset{\sim}{B}'}(y) = \max_{x \in X}[\mu_{\underset{\sim}{A}'}(x).\mu_{\underset{\sim}{R}}(x, y)]$$

IF x is $\underset{\sim}{A}$ THEN y is $\underset{\sim}{B}$

Other forms of implication

Zadeh $$\mu_{\underset{\sim}{R}}(x,y)=max[min(\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(y)),(1-\mu_{\underset{\sim}{A}}(x))]$$

Mamdani $$\mu_{\underset{\sim}{R}}(x,y)=min[\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(y)]$$

Lukasiewicz $$\mu_{\underset{\sim}{R}}(x,y)=min[1,(1-\mu_{\underset{\sim}{A}}(x)+\mu_{\underset{\sim}{B}}(y))]$$

Approximated reasoning

The fuzzy logic, in this case the *modus ponens,* allows to make approximate reasoning. From a fuzzy implication, one extracts the consequent for another antecedent approximated to the previous one.

x is $\underset{\sim}{A}'$

IF x is $\underset{\sim}{A}$ THEN y is $\underset{\sim}{B}$ $\qquad$ $\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y)$

$\Rightarrow y$ is $\underset{\sim}{B}'$ $\qquad$ $\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}$

Bibliography

Fuzzy Logic With Engineering Applications, 3rd Ed. Timothy Ross,Wiley, 2010.

Fuzzy Logic Toolbox Users´s Guide, The Mathworks, 2018.

Fuzzy Logic, H. Zimmermann, Springer Verlag, 1997.

*An Introduction to Fuzzy Sets Analysis and Design, P*edrycz, Witold and Fernando Gomide*,* MIT Press, 1998.

Fuzzy Sets and Fuzzy Logic: Theory and Applications, George J. Klir, Bo Yuan, Prentice Hall 1995.