Histogram Specification

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Abstract

This article describes Histogram Specification, whose purpose is to obtain a function $f(p)$ to map the given data into a specified histogram.

This is useful when raw images directly read from sensors are not ideal in brightness or contrast.

1 Problem definition

Given a series of data $P = \{p_1, p_2, ..., p_N\}, p \in \mathbb{N}$, its histogram can be expressed as

$$h_P(p) = n_p, \tag{1}$$

in which, n_p denotes the number of element that values p in sequence P.

Given a desired target histogram $h_Q(q) = n_q$, the objective is to find a monotonic mapping function $f(p) = q$ such that:

$$P' = \{f(q_1), f(q_2\}, ..., f(q_N)\} = \{p'_1, p'_2, ..., p'_N\}, \tag{2}$$

$$h_{P'}(p) = h_Q(p). \tag{3}$$

2 Derivations

The former problem definition is in discrete form. In some extreme but not rare cases, optimal solution may not exist for discrete histogram specification. For example:

$$P = \{1, 1, 1, 1, 1\}, \tag{4}$$

$$Q = \{1, 2, 3, 4, 5\}. \tag{5}$$

In the above scenario, a desired $f(p)$ does not exist.

To simplify the problem, we assume continuous distribution on set P and Q such that $P, Q \in [0,1]^N$. In other word, $h_P(x)$ and $h_Q(x)$ are both continuous.

According to the definition of the problem, several constrains hold as listed below:

$$h_{P'}(x) = h_Q(x), \tag{6}$$

$$\frac{\partial f(x)}{\partial x} \geq 0, \tag{7}$$

in which $x \in [0,1]$.

According to Eq. 6, if the distribution density is identical everywhere, then the cumulative distribution shall also be equal-valued.

$$\int_0^x h_{P'}(z)\,\mathrm{d}z = \int_0^x h_Q(z)\,\mathrm{d}z. \tag{8}$$

The above equation can be transformed into the following form:

$$\int_0^{f(x)} h_P(z)\,\mathrm{d}z = \int_0^x h_Q(z)\,\mathrm{d}z, \tag{9}$$

$$H_P(f(x)) = H_Q(x), \tag{10}$$

where the original functions $H_P(x), H_Q(x)$ are the cumulative intensity functions of $h_P(x), h_Q(x)$. To obtain $f(x)$, the inverse function of H can be applied to both side:

$$H_P^{-1}\left(H_P(f(x))\right) = H_P^{-1}\left(H_Q(x)\right). \tag{11}$$

For cumulative intensity function $H(x) \in [0,1], x \in [0,1]$, the inverse function is simply $H^{-1} = 1 - H$. Hence the equation can be further substituted into:

$$f(x) = 1 - H_P(H_Q(x)), \tag{12}$$

which is the desired optimal solution.

For discrete distributions $P, Q \in \mathbb{R}^N$ with $p_{max} = \max(P)$ and q_{max} respectively, the corresponding mapping function is:

$$f(x) = q_{max} - \frac{q_{max}}{N} H_P\left(\frac{p_{max}}{N} H_Q\left(x\frac{q_{max}}{p_{max}}\right)\right), \tag{13}$$

3 Histogram Equalization

A mapping function $f(x)$ that equalizes the histogram can be computed given that:

$$h_Q(x) = C, \tag{14}$$

where C is a constant value.

Under continuous distribution assumption, we have:

$$H_Q(x) = x, \tag{15}$$
$$f(x) = 1 - H_P(x). \tag{16}$$

Then in the discrete case, the mapping is :

$$f(x) = q_{max} - \frac{q_{max}}{N} H_P(x), \tag{17}$$