Improving Bounds on the Phi-Pebbling Number of Graphs

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Abstract

Graph pebbling was first introduced as a tool for solving a combinatorial number theory conjecture of Erdős and Lemke. Pebbles are moved throughout a graph by removing two from one vertex to place one on an adjacent vertex. We study a pebbling variant called ϕ-pebbling in which each pebble may move once without another being removed. Few bounds have been established on ϕ-pebbling numbers. We establish bounds for graphs of radius one, graphs of diameter two, complete graphs, complete k-partite graphs, paths, hypercubes, fans, Cartesian products, grids, and crowns. The results extend classical pebbling theory, helping to understand Graham's Conjecture as ϕ-pebbling represents moving a pebble through both graphs in a Cartesian product simultaneously.

Graph pebbling

A **graph** $G = (V, E)$ is a set of vertices V connected by edges E. We will study **connected** graphs or those in which there is a sequence of edges between any two vertices.
A **path graph** P_n is one in which two vertices are connected to exactly one other vertex and $n-2$ vertices are connected to exactly two other vertices.

Figure 1. Path graph P_5

What is pebbling?

- Place a non-negative integer number of **pebbles** on a graph.
- Move pebbles according to the following rule: **Take two pebbles away from a vertex, then place one pebble on an adjacent vertex**

Figure 2. A pebbling move, turning 3 pebbles on one vertex into one pebble on each of two adjacent vertices

If, given a **specific configuration of pebbles**, we can **move t pebbles to a vertex**, we say that this vertex is **t-reachable**.

Definition. The **t-pebbling number** of a graph G, denoted $\pi_t(G)$, is the minimum k such that every vertex is reachable by at least t pebbles in any configuration of k pebbles.

Definition of ϕ-pebbling

What is ϕ-pebbling?

- Each pebble is allowed **one free move.**
- Lone pebbles would be unusable in normal pebbling, but in ϕ-pebbling, we can **coalesce lone pebbles onto a mutual neighbor** as free moves, and then proceed normally.

Definition. The **ϕ-pebbling number** of a graph G, denoted $\phi(G)$, is the minimum k such that every vertex is reachable via cut-the-corner moves in any configuration of k pebbles.

Proposition. Asplund et al. [1] prove $\phi(G) \leq \left\lceil \frac{\pi(G)}{2} \right\rceil$.

Motivation for ϕ-pebbling

The **Cartesian product** of two graphs G and H denoted $G \square H$ has vertices

$$V(G) \times V(H) = \{(g,h) : g \in V(G), h \in V(H)\}$$

and an edge between (g_1, h_1) and (g_2, h_2) if

$$g_1 = g_2 \text{ AND } h_1 \text{ is adjacent to } h_2 \quad \text{OR} \quad g_1 \text{ is adjacent to } g_2 \text{ AND } h_1 = h_2.$$

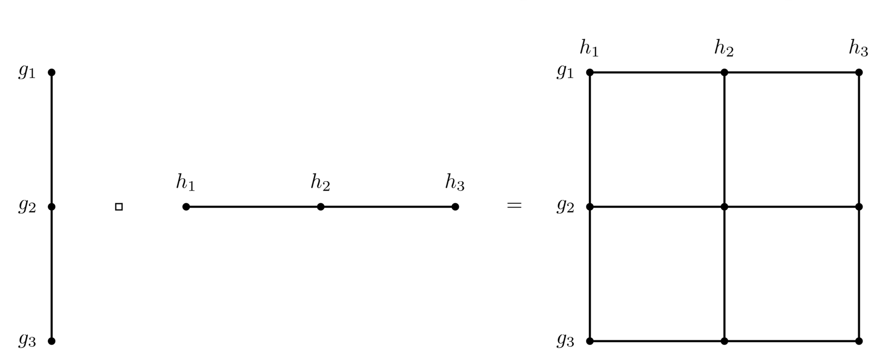

Figure 3. The Cartesian product of P_3 with P_3, which reveals why this product is also known as the "box product"

Graham's Conjecture. Let G and H be graphs with Cartesian product $G \square H$. Then

$$\pi(G \square H) \leq \pi(G)\pi(H).$$

Proposition. Asplund et al. [1] use ϕ-pebbling to prove $\pi(G \square H) \leq 2\pi(G)\pi(H)$.

ϕ-pebbling computations

Definition. A **tree** is a graph in which there is exactly one sequence of non-repeating edges between any two vertices. A **path partition** of a tree is a set of disjoint paths that lies within the tree. We denote a path partition by a non-increasing list $\ell_1, \ldots, \ell_m$ representing the number of vertices of each path in the partition. Path partition $\mathcal{L}$ is **larger** than path partition $\mathcal{L}'$ if it is larger in the first position where they differ [2].

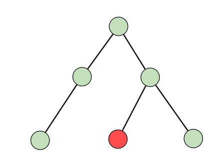

Figure 4. A tree with maximum path partition P_3 and P_1

Proposition: Trees. Let T be a tree with maximum path partition $\ell_1, \ldots, \ell_m$ in non-increasing order. Then

$$\phi(T) = \left(\sum_{i=1}^{m} 2^{\ell_i - 1} \right) - m + 1.$$

Proof sketch.

- Chung [3] proves the pebbling number of a tree is $\pi(T) = \left(\sum_{i=1}^{m} 2^{\ell_i} \right) - m + 1$.
- By the Bunde et al. No-Cycle Lemma [2], we never need to move a pebble in both directions along an edge.
- We can reduce the ϕ-pebbling problem to a standard pebbling number in which each path partition has one fewer edge.

Proposition: Cycles. A **cycle graph** is a connected graph in which each vertex has exactly two connections. Let C_n be a cycle with n vertices. Then

$$\phi(C_n) = \begin{cases} \left\lceil \frac{\pi(C_n)}{2} \right\rceil & n \equiv 1 \text{ (mod 4)} \\ \frac{\pi(C_n)}{2} & n \text{ is even} \\ \left\lfloor \frac{\pi(C_n)}{2} \right\rfloor & n \equiv 3 \text{ (mod 4).} \end{cases}$$

Proof sketch.

- In order to pebble any odd cycle graph C_{2k+1} for some $k \in \mathbb{N}$, place x pebbles on each of two adjacent vertices and attempt to reach the target vertex a distance k away from both vertices. In the example C_5 below, we place pebbles on the bottom two vertices and attempt to reach the top vertex.

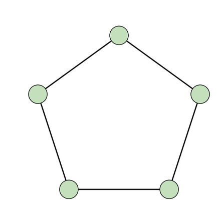

Figure 5. The cycle graph C_5 arranged in a pentagon and connected in a closed loop.

- Then $x + \left\lfloor \frac{x}{2} \right\rfloor$ is the greatest number of pebbles that can reach a vertex that is distance $k-1$ from the target vertex.
- Since $\pi(P_k) = 2^{k-1}$, if x is the largest $x \in \mathbb{N}$ such that $2^{k-1} - 1 = x + \left\lfloor \frac{x}{2} \right\rfloor$, then we cannot reach the root vertex with $2x$ pebbles.
- Thus, $\phi(C_{2k+1}) = 2x + 1$, which is equal to $\left\lceil \frac{\pi(C_n)}{2} \right\rceil$ if k is even and $\left\lfloor \frac{\pi(C_n)}{2} \right\rfloor$ if k is odd.

Proposition: Thorns. For a graph G, the **thorn graph** G^* is obtained by attaching vertices (**thorns**) to each $v \in V(G)$. A thorn adjacent to v is denoted u, and v is its base. For every connected G with thorn graph G^*,

$$\phi(G^*) = \pi_2(G).$$

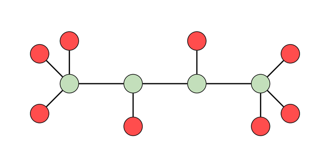

Figure 6. Muthulakshmi et al. [4] shows a thorn graph G^* constructed from the path $G = P_4$. Each green vertex is a base node of G and the red vertices are attached thorns on G^*.

Proof sketch.

- *Upper bound*: Place $\pi_2(G)$ pebbles on G^*. Move thorn pebbles to their bases, leaving $\pi_2(G)$ on G. Since two pebbles can reach any vertex in G, and thorns are one step away, any thorn can also be reached. Thus $\phi(G^*) \leq \pi_2(G)$.
- *Lower bound*: First, place $\pi_2(G) - 1$ pebbles on G such that two cannot reach a target vertex v. Second, move each pebble to a thorn. There is a thorn connected to v that is not reachable via ϕ-pebbling. With the free move, we can place at most $\pi_2(G) - 1$ on G, which is insufficient to reach v with two pebbles or a thorn by one. Thus $\phi(G^*) \geq \pi_2(G)$.

Graphs of diameter two

Definition. The **distance** between two vertices u and v in a graph G is the minimum number of edges between them. The **diameter** of a graph G is the largest distance between any two vertices of G.

Theorem. Let G be a graph of diameter two with n vertices. Then

$$\phi(G) \leq \sqrt{4n+5} - 2.$$

Proof sketch. Using ϕ-pebbling moves, we can reach a vertex v in a graph G with at least one pebble if any of the following holds:

- There is more than one pebble on any single vertex
- A vertex is connected to at least 4 vertices with pebbles
- A vertex with a pebble is connected to at least two other vertices with pebbles
- A vertex connected to v is connected to at least two vertices with pebbles

Given a number of pebbles p, we argue that in a graph of diameter two with fewer than $1 + \frac{3}{2}p + \frac{1}{4}p^2$ vertices each vertex is reachable by at least one pebble. Let v be a target vertex. We need at least p vertices connected to v and p other vertices each with exactly one pebble. In order for G to have diameter two, we need an extra $\frac{1}{4}p^2 - \frac{1}{2}p$ vertices that are not v, neighbors of v, or a vertex with a pebble when p is even. When p is odd, we need at least $\frac{1}{4}p^2 - p + \frac{3}{4}$. By setting $n = 1 + \frac{3}{2}p + \frac{1}{4}p^2$ and solving for p as a function of n, we obtain the desired result.

Example. For any integer p, there is a graph G with $\frac{p^2}{2} + \frac{3p}{2} + 1$ vertices such that $\phi(G) > p$. We build G as follows:

- Create a target vertex v
- Create $N(v)$, with $|N(r)| \geq p$
- Place 1 pebble onto p distinct vertices

We will create this example for $p = 2$

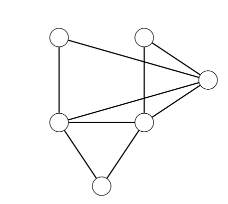

Figure 7. p=2

We see that we have a root r on the bottom, and then $p = 2$ $N(r)$. We connect each $N(r)$, making a complete graph K_3. We then make $p = 2$ $N_2(r)$, and connect each to their corresponding $N(r)$. We connect every $v \in N_2(r)$ through an additional vertex, pictured off to the side. We also connect that vertex to each $N(r)$.

Other results

Cartesian Products: $\phi(G \square H) \leq \min\{\phi(G)(\pi(H) + |H|),\ \phi(H)(\pi(G) + |G|)\}$
Complete k-Partite Graphs: $\phi(K_{a_1, a_2 \ldots a_k}) \leq 2$
Hypercubes: $\phi(Q_n) = \left\lceil \frac{3^n}{2} \right\rceil$
Grids: $\phi(G_{m,n}) = 2^{m+n-3}$
Crowns: $\phi(W_n) \leq 4$[6pt]

Future Work

- Determine an upper bound on $\phi(G)$ that is sharp infinitely often for $\text{diam}(G) = 2$.
- Generalize ϕ-pebbling number of a graph to allow for k free moves per pebble, denoted $\phi_k(G)$. Can we prove that $\phi_{k+1}(G) \leq \left\lceil \frac{\phi_k(G)+1}{2} \right\rceil$?
- Determine whether computing the ϕ-pebbling number of a graph is NP-complete.

Acknowledgments

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