A small note about Hilbert schemes and Grothendieck rings (INCOMPLETE)

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We investigate the Fano schemes of planes in cubic hypersurfaces. By extending the argument of Galkin and Shinder ([GS14]), we find a relation in the Grothendieck ring involving the Fano scheme of planes, the Fano scheme of lines, and the Hilbert scheme of conic sections. Some applications are discussed.

Notation

Let X be a cubic hypersurface of $\mathbb{P}^n$, $\mathscr{H} := \mathrm{Hilb}^{2t+1}(X)$ the Hilbert Scheme of conic sections on X. Let G be the subscheme of $F_1(X) \times \mathbb{G}(2,n)$ consisting of all pairs $(l \subset P)$, with l a line and P a plane.

Let $\psi\colon \mathscr{H} \dashrightarrow G$ be the rational map defined as follows: A conic section $C \subset X$ lies in a unique plane $P \subset \mathbb{P}^n$. For a general C, this plane will not lie in X, so there will be a residual intersection l. Then l must be a line, and we let $\psi(C) = (l, P)$. Let $U \subset \mathscr{H}$ be the open subset where ψ is well-defined, and Z its complement.

Hence Z is the scheme where each point corresponds to a conic on X lying in a plane contained in X. Similarly, let Z' be the subscheme of G consisting of pairs $(l \subset P)$ with $P \subset X$. Let U' be the complement of Z'. Both Z and Z' project onto $F_2(X)$.

Construction

It follows that we may set up a diagram:

$$
\begin{tikzcd}
U \arrow[rr, "\psi|_U"] \arrow[d, hook] & & U' \arrow[d, hook] & \\
\mathscr{H} \arrow[rr, dotted, "\psi"] & & G \arrow[r, "f"] & F_1(X) \\
Z \arrow[u, hook] \arrow[dr, "\pi_1"] & & Z' \arrow[u, hook] \arrow[dl, "\pi_2"] & \\
& F_2(X) & &
\end{tikzcd}
$$

Here, $\psi|_U$ is an isomorphism. In addition, f is a $\mathbb{P}^{n-2}$-bundle, π_1 a $\mathbb{P}^5$-bundle, and π_2 a $\mathbb{P}^2$-bundle.

Essentially, we have taken the diagram of Galkin and Shinder ([**17**; GS14]), replaced the Hilbert scheme of 0-dimensional length 2-subschemes with $\mathscr{H}$, and replaced other schemes as necessary. Thus we have the following relations in the Grothendieck ring $K_0(Var/\mathbb{C})$:

1. $[U] = [U']$,
2. $[\mathscr{H}] = [U] + [F_2(X)][\mathbb{P}^5]$,
3. $[U'] + [F_2(X)][\mathbb{P}^2] = [G] = [F_1(X)][\mathbb{P}^{n-2}]$

which combine to

$$[\mathscr{H}] = [F_2(X)]\left([\mathbb{P}^5] - [\mathbb{P}^2]\right) + [F_1(X)][\mathbb{P}^{n-2}]. \qquad (*)$$

We then note that $\mathscr{H}$ is smooth in general (Proposition 2.3.4, page 24 of M. F. DeLand's thesis).

We have shown:

Proposition 0.0.1. *The Hilbert Scheme $\mathscr{H}$ of conics on a general cubic hypersurface X is smooth. The relation*

$$[\mathscr{H}] = [F_2(X)]\left([\mathbb{P}^5] - [\mathbb{P}^2]\right) + [F_1(X)][\mathbb{P}^{n-2}]$$

holds in the Grothendieck ring of varieties.

Hodge numbers

There is a "Hodge realization homomorphism" μ_{Hdg} from $K_0(Var/\mathbb{C})$ to the Grothendieck ring $K_0(HS)$ of polarizable pure rational Hodge structures ([GS14,

p. 8]). This means that we can compute the Hodge numbers of $\mathscr{H}$ as long as we know those of $F_1(X)$ and $F_2(X)$. The Hodge numbers of $F_1(X)$ were computed in all dimensions by Galkin and Shinder. Those of $F_2(X)$ are, as far as the author knows, only known when X is five-dimensional. In this case the Hodge diamond of $F_2(X)$ is (see for instance a certain Master's thesis)

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 21 & & 21 & \\
3233 & & 6657 & & 3233 \\
 & 21 & & 21 & \\
 & & 1 & &
\end{array}
$$

So if we apply μ_{Hdg} to $*$, we end up (after an annoyingly lengthy, but straightforward computation) with the following Hodge diamond for $\mathscr{H}$ —the Hilbert scheme of conic sections on a five-dimensional cubic—:

$$
\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&& 1 \\
&&&&&&&&& 0 && 0 \\
&&&&&&&& 0 && 2 && 0 \\
&&&&&&& 0 && 21 && 21 && 0 \\
&&&&&& 0 && 0 && 4 && 0 && 0 \\
&&&&& 0 && 0 && 42 && 42 && 0 && 0 \\
&&&& 0 && 0 && 231 && 447 && 231 && 0 && 0 \\
&&& 0 && 0 && 0 && 84 && 84 && 0 && 0 && 0 \\
&& 0 && 0 && 0 && 3464 && 7106 && 3464 && 0 && 0 && 0 \\
& 0 && 0 && 0 && 0 && 126 && 126 && 0 && 0 && 0 && 0 \\
0 && 0 && 0 && 0 && 3464 && 7107 && 3464 && 0 && 0 && 0 && 0 \\
& 0 && 0 && 0 && 0 && 126 && 126 && 0 && 0 && 0 && 0 \\
&& 0 && 0 && 0 && 3464 && 7106 && 3464 && 0 && 0 && 0 \\
&&& 0 && 0 && 0 && 84 && 84 && 0 && 0 && 0 \\
&&&& 0 && 0 && 231 && 447 && 231 && 0 && 0 \\
&&&&& 0 && 0 && 42 && 42 && 0 && 0 \\
&&&&&& 0 && 0 && 4 && 0 && 0 \\
&&&&&&& 0 && 21 && 21 && 0 \\
&&&&&&&& 0 && 2 && 0 \\
&&&&&&&&& 0 && 0 \\
&&&&&&&&&& 1
\end{array}
$$

Euler characteristics

Even if we are not be able to compute the Hodge numbers of $\mathscr{H}$, it is possible to compute the Euler characteristics $\chi(\Omega^i_X)$ through the Grothendieck ring as in [GS14]

As an example, we find $\chi(\mathscr{O}_{\mathrm{Hilb}^{2t+1}(X)})$ where X is a cubic sixfold.

To do this, we first find the Euler characteristic of $\mathscr{O}_{F_2(X)}$. Then it is a simple computation to use the formula above to find $\chi(\mathscr{O}_{\mathrm{Hilb}^{2t+1}(X)})$

Recall that the Fano scheme $F_2(X)$ is a section of the rank-10 bundle $\mathrm{Sym}^3(\mathscr{U}^\vee)$ on the Grassmannian $\mathbb{G}(2,n)$ of planes in n-dimensional space. Hence there is an exact sequence

$$0 \to \mathscr{T}_{F_2(X)} \to \mathscr{T}_{\mathbb{G}(2,7)}|_{F_2(X)} \to \mathscr{N}_{F_2(X)/\,\mathbb{G}(2,7)} \xrightarrow{\sim} \mathrm{Sym}^3(\mathscr{U}^\vee)|_{F_2(X)} \to 0$$

and it is easy to find that $\chi(\mathscr{O}_{\mathrm{Hilb}^{2t+1}(X)}) = -1131$.

Furthermore, the construction of Galkin and Shinder shows that $\chi(\mathscr{O}_{F_1(X)}) = 1$.

Our formula now shows that

$$\chi(\mathscr{O}_{\mathrm{Hilb}^{2t+1}(X)}) = \chi(\mathscr{O}_{F_2(X)})\chi(\mathscr{O}_{\mathbb{P}^5}) - \chi(\mathscr{O}_{F_2(X)})\chi(\mathscr{O}_{\mathbb{P}}^2) + \chi(\mathscr{O}_{F_1(X)})\chi(\mathscr{O}_{\mathbb{P}^6}) = 1.$$

The number of conic sections on cubics defined over finite fields

Mapping the class of a variety to its Hasse-Weil zeta function is another realization. Thus it should be possible to use the formula to compute the number of conic sections on cubic hypersurfaces defined over finite fields. However, it is necessary to know the number of lines and the number of planes contained in the hypersurface.

Lower bounds for the number of lines were computed by Debarre, Laface and Roulleau in [DLR15]

The number of planes contained in cubic hypersurfaces does not seem to be known in general, and it depends on the defining equation in low dimensions. The number is reasonably easy to find for one specific class of cubic hypersurfaces, namely the Fermat cubics. We give the computation for the cubic Fermat fourfold as an example.

Lemma 0.0.2. *Let $N_{1,q}(F_2)$ be the number of planes contained in the Fermat cubic fourfold defined over the field of $q = p^r$ elements. Assume further that 3 does not divide $p-1$ Then we have*

$$N_{1,q} = 45.$$

Proof. This is a simple computation with [DLR15, section 5.4]. ■

Proposition 0.0.3. *The number of conic sections contained in the Fermat cubic defined over the field of $p^r = q \geq 23$ elements, where 3 does not divide $p-1$ is at least*

$$q^7 - 20q^6 + 235q^5 + 214q^4 + 190q^2 - 20q + 1.$$

Proof. This follows from [DLR15, Theorem 5.2]. ■

Further directions

By generalizing the argument of Galkin and Shinder further, it is possible to give similar formulae in $K_0(Var/\mathbb{C})$ involving $F_k(X)$ and the Hilbert scheme of $k-1$-dimensional quadrics for any k (as long as X has sufficiently large dimension). It is not, however, known whether these Hilbert schemes are smooth.

Bibliography

DebLafRou [DLR15] Olivier Debarre, Antonio Laface, and Xavier Roulleau. "Lines on cubic hypersurfaces over finite fields". In: (2015). DOI: 10.48550/ARXIV.1510.05803. URL: https://arxiv.org/abs/1510.05803.

GalShin [GS14] Sergey Galkin and Evgeny Shinder. *The Fano variety of lines and rationality problem for a cubic hypersurface*. 2014. DOI: 10.48550/ARXIV.1405.5154. URL: https://arxiv.org/abs/1405.5154.