Quiver varieties and moduli spaces attached to Kleinian singularities

joint with A. Craw, Á. Gyenge, B. Szendrői

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We can set

$$\operatorname{Hilb}^n X = \{Z \subset X | \ Z \text{ has dim 0, length } n\}.$$

This is a quasiprojective scheme (Grothendieck)
Compare with $\operatorname{Sym}^n X = (X \times \cdots \times X)/\mathfrak{S}_n$.
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I'm interested in a particular kind of $\operatorname{Hilb}^n(X)$:

Let Γ be a finite subgroup of $SL_2(\mathbb{C})$. It has a tautological action on $\mathbb{C}^2$, and we can form the quotient

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with a map $n\Gamma\text{-}\mathrm{Hilb}\,\mathbb{C}^2 \to \mathrm{Hilb}^n \mathbb{C}^2/\Gamma$, given by $I \mapsto I \cap \mathbb{C}[x,y]^\Gamma$.

Our result

Theorem (Craw-G.-Gyenge-Szendrői 2021)

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- *it has a unique symplectic resolution* *(*$n\Gamma - \mathrm{Hilb}(\mathbb{C}^2) \to \mathrm{Hilb}^n\,\mathbb{C}^2/\Gamma$*)*

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- $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{\mathrm{red}}$ *is irreducible and normal*
- *it has symplectic singularities*
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Main idea: Construct $\mathrm{Hilb}^n\,\mathbb{C}^2/\Gamma$ as a *Nakajima quiver variety.*

Plan:

- What is a NQV (and why should we care?)
- Examples
- Some ideas from our proof
- A glance at a wall-and-chamber structure
- (if time (probably not):) generalisations.

Theorem (Nakajima, Bellamy - Schedler 2020)

Smooth NQVs are hyperKähler.
If singular, they have symplectic singularities.
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So: We're going to show that $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$ is also a Nakajima quiver variety.

What is a Nakajima quiver variety?

Let's take an example: $\mathrm{Hilb}^n \mathbb{C}^2$.
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modulo isomorphism.

We can draw this diagrammatically as

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\mathbb{C} \arrow[r, "1"] & V \arrow[loop above, "\cdot x"] \arrow[loop below, "\cdot y"]
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For a slightly different view, consider the quiver

$$Q = \begin{tikzcd} \rho_\infty \arrow[r, "b"] & \rho_0 \arrow[loop above, "a"] \arrow[loop below, "\bar{a}"] \end{tikzcd}$$

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Now form the path algebra $\mathbb{C}Q$, and let $\Pi_Q = \mathbb{C}Q/(a\bar{a} - \bar{a}a)$. This is the *preprojective algebra.*

A Π_Q-module M is thus a Q-representation satisfying the relations above, and has a dimension vector $(\dim M_\infty, \dim M_0)$.

Then:

Theorem (Nakajima)

There exists a fine moduli space $\mathfrak{M}(1, n)$ of stable Π-modules of dimension $(1, n)$.
We have $\mathfrak{M}(1, n) \xrightarrow{\sim} \mathrm{Hilb}^n(\mathbb{C}^2)$.

We want to use a similar idea to construct $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$.

The McKay correspondence

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List the irreducible representations of Γ as $\{\rho_0, \ldots, \rho_s\}$, with ρ_0 trivial.
Treat these as the vertices in a quiver, where we add an arrow $a\colon \rho_i \to \rho_j$ for every time ρ_j appears as a summand in $\rho_i \otimes L$.
Because L is self-dual, we'll also get an arrow $\overline{a}\colon \rho_j \to \rho_i$.

Examples:

If $\Gamma = \mathbb{Z}/4$:

$$\begin{tikzcd}
& \rho_1 \arrow[dl, shift left] \arrow[dr, shift left] & \\
\rho_0 \arrow[ur, shift left] \arrow[dr, shift left] & & \rho_2 \arrow[ul, shift left] \arrow[dl, shift left] \\
& \rho_3 \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

If $\Gamma = 2D_5$: (a binary dihedral group):

$$\begin{tikzcd}
\rho_1 \arrow[dr, shift left] & & & \rho_4 \arrow[dl, shift left] \\
& \rho_2 \arrow[ul, shift left] \arrow[dl, shift left] \arrow[r, shift left] & \rho_3 \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & \\
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These are *affine Dynkin diagrams!* This is an example of the McKay correspondence.

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Now form the *path algebra* $\mathbb{C}Q$. Choose $\epsilon \colon E \to \{\pm 1\}$,
$\epsilon(a)\epsilon(\bar{a}) = -1$.
Consider the ideal generated by $\mathscr{I} = (\sum \epsilon(a) a\bar{a})$ and set

$$\Pi = \mathbb{C}Q/\mathscr{I}.$$

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Stability

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Stability

Consider modules of Π.
We will choose the *dimension vector* $(1, n\delta)$ for Π-modules – so 1 is the dimension of our modules at the framing vertex, and $\delta_i = \dim \rho_i$. Then choose a *stability parameter* $\theta \in \mathbb{Q}^V$ such that $\theta(1, n\delta) = 0$. A Π-module M is θ-(semi)stable if $\theta(\dim N) > (\geq) 0$ for all $N \subset M$.

Two semistable Π-modules M, M' are S-equivalent if there are filtrations

$$0 = M_1 \subset M_2 \subset \cdots \subset M_{s_1} = M,$$

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$$\bigoplus M_i / M_{i-1} \cong \bigoplus M'_j / M'_{j-1}.$$

Definition (Nakajima quiver variety)

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So: it depends on θ, the quiver Q, and the dimension vector.

Theorem (Folklore, Kuznetsov 2001)

Choose a θ such that $\theta_i > 0$ for all $i \neq \infty$. Then

$$\mathfrak{M}_\theta(1, n\delta) \cong n\Gamma\text{-}\mathrm{Hilb}\,\mathbb{C}^2.$$

Example

Let $\Gamma = \mathbb{Z}/4$, and consider the two $\mathbb{C}[x, y]$-ideals

$$I_1 = \langle x^8, xy^2, x^2y, y^4 \rangle, \quad I_2 = \langle x^6, x^4y, y^3, x^2y^2 \rangle.$$

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Both these are points of $3(\mathbb{Z}/4)\text{--}\operatorname{Hilb}(\mathbb{C}^2)$.

The quiver representation corresponding to $I_1 = \langle x^8, xy^2, x^2y, y^4 \rangle$ is:

$$
\begin{tikzcd}
 & & \mathbb{C}(x \oplus x^5 \oplus y^3) \arrow[dl, shift left, "\cdot y"] \arrow[dr, shift left, "\cdot x"] & \\
\mathbb{C} \arrow[r, shift left, "\cdot 1"] & \mathbb{C}(1 \oplus x^4 \oplus xy) \arrow[l, shift left] \arrow[ur, shift left, "\cdot x"] \arrow[dr, shift left, "\cdot y"] & & C(x^2 \oplus x^6 \oplus y^2) \arrow[ul, shift left, "\cdot y"] \arrow[dl, shift left, "\cdot x"] \\
 & & \mathbb{C}(y \oplus x^3 \oplus x^7) \arrow[ul, shift left, "\cdot x"] \arrow[ur, shift left, "\cdot y"] &
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And for $I_2 = \langle x^6, x^4y, y^3, x^2y^2 \rangle$:

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 & & \mathbb{C}(x \oplus x^5 \oplus x^2y) \arrow[dl, shift left, "\cdot y"] \arrow[dr, shift left, "\cdot x"] & \\
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So how do we get $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$?
We want to focus at the invariant parts - so the ρ_0-vertex.
Set $\theta_0 = (-n, 1, 0, \ldots, 0)$.

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Theorem (Craw-G.-Gyenge-Szendrői)

$$\mathfrak{M}_{\theta_0}(1, n\delta) \cong \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{\mathrm{red}}.$$

As it then is a quiver variety, $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{\mathrm{red}}$ *is irreducible, normal, and has symplectic singularities.*

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I'll sketch some of the proof, but first let's go back to our example.

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So under the map

$$3(\mathbb{Z}/4)\text{--}\,\mathrm{Hilb}(\mathbb{C}^2) \to \mathrm{Hilb}^3(\mathbb{C}^2/(\mathbb{Z}/4)),$$

these should give the same point.

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these should give the same point.
This should be the map $\mathfrak{M}_\theta(1, 3\delta) \to \mathfrak{M}_{\theta_0}(1, 3\delta)$.
Let's see how this works with Π-modules:

Consider again the Π-module corresponding to I_1:

$$
\begin{tikzcd}
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Change the stability parameter to θ_0: we can quotient out by submodules supported away from ρ_0:

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 & & \mathbb{C}(y \oplus x^3) \arrow[ul, shift left, "\cdot x"] \arrow[ur, shift left, "\cdot y"] &
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$$

which corresponds to the ideal $I_m = \langle x^5, y^2, x^2y \rangle$.

Starting with $I_2 = \langle x^6, x^4y, y^3, x^2y^2 \rangle$ will also lead to the same I_m:

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This means that the Π-modules for I_1, I_2 are S-equivalent.
Fact: I_m is *maximal* among Γ-equivariant ideals I such that $I_m \cap \mathbb{C}[x,y]^\Gamma = I_1 \cap \mathbb{C}[x,y]^\Gamma$!

Some idea about the proof:

Problem: the quiver variety $\mathfrak{M}_{\theta_0}(1, n\delta)$ isn't a *fine* moduli space of Π-modules.

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Solution: *Cornering.*

Cornering

Replace Π by $\Pi_0 = (e_\infty + e_0)\Pi(e_\infty + e_0)$, the *cornered preprojective algebra.*

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Analogy: Shut down some metro stations, but allow trains through.
Then there are functors

$$\begin{tikzcd} \Pi\text{-mod} \arrow[r, bend right, "j^*"'] & \Pi_0\text{-mod} \arrow[l, bend right, "j_!"'] \end{tikzcd}$$

and we can similarly build moduli spaces of Π_0-modules.
However, a θ-semistable Π-module becomes a stable Π_0-module, so we get a *fine* moduli space of Π_0-modules.

It's not difficult to show that a Π_0-module gives an ideal of $\mathbb{C}[x,y]^\Gamma$: Π_0 can be constructed from

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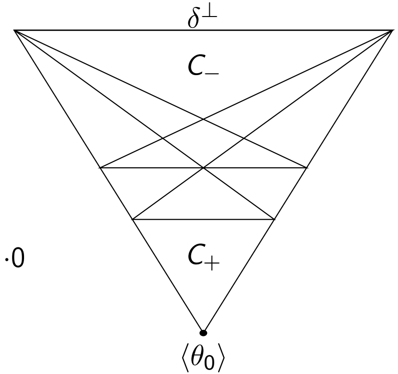
and $\mathbb{C}[x,y]^\Gamma$ also has 3 generators...
e.g. if $\Gamma = \mathbb{Z}/4$, $\mathbb{C}[x,y]^\Gamma = \mathbb{C}[x^4, y^4, xy]$.

Wall-and-chamber structure

Recall that θ varies in a wall-and-chamber structure. There is a complete description of it (Bellamy&Craw 2018)

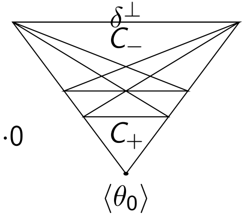
A map of stability parameters



This is a slice of the fundamental chamber in the wall-and-chamber structure for the stability parameter when $\Gamma = \mathbb{Z}/3$, $n = 3$.
You should also imagine the "0" parameter sort of behind these.

You get:

- $\theta \in \delta^{\perp}$: $\mathrm{Sym}^n \widehat{\mathbb{C}^2/\Gamma}$ (Kuznetsov)
- $\theta \in C_-$: $\mathrm{Hilb}^n \widehat{\mathbb{C}^2/\Gamma}$ (Kuznetsov)
- $\theta \in C_+$: $n\Gamma\text{-}\mathrm{Hilb}\,\mathbb{C}^2$ (folklore)
- $\theta = \theta_0$: $\mathrm{Hilb}^n \mathbb{C}^2/\Gamma$ (C.-G.-Gy.-Sz.)
- $\theta = 0$: $\mathrm{Sym}^n \mathbb{C}^2/\Gamma$. (Nakajima (?))



By standard GIT arguments, moving θ around gives morphisms

$$\begin{array}{ccccc} & & & & \mathrm{Hilb}^n(\widehat{\mathbb{C}^2/\Gamma}) \\ & & & & \downarrow \\ & & & & \mathrm{Sym}^n(\widehat{\mathbb{C}^2/\Gamma}) \\ & & & & \downarrow \\ n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2/\Gamma) & \longrightarrow & \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma) & \longrightarrow & \mathrm{Sym}^n(\mathbb{C}^2/\Gamma) \end{array}$$

all birational.

Time (maybe?) to generalise.

Changing the stability parameter

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It turns out that we can give a decent description that (sort of) allows arbitrary dimension vectors v:

Let I be some subset of the vertices in the McKay graph (i.e. a set of irreducible Γ-representations), and let $n_I \in \mathbb{N}^I$. For any $i \in I$, let $R_i = \mathrm{Hom}_\Gamma(\rho_i, \mathbb{C}[x, y])$.

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Then set

$$\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma]) := \left\{ \mathrm{End}\left(\bigoplus_{i \in I} R_i\right) - \text{epimorphisms} \right.$$

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okay, that's fairly complicated, BUT:

- $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{\mathrm{red}}$ is a Nakajima quiver variety!
- In fact it is $\mathfrak{M}_{\theta_I}(1, v)$, where
  1. θ_I is a stability parameter only focusing on those vertices lying in I
  2. v is some dimension vector such that $v_i = n_i$ for any $i \in I$

How do we prove this?

Mainly the same strategies as before.
However, we can avoid ugly case-by-case analysis by carefully manipulating the dimension vectors - this is why we only say "there is *some* v..."
You can find explicit v's that work if you want to though.

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What about $\mathfrak{M}_{\theta_0}(r, n\delta)$?
We'll need to compactify. Embed $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$, $l_\infty := \mathbb{P}^2 \setminus \mathbb{C}^2$.
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Theorem (Varagnolo-Vasserot)

If $\theta \in C_+$, $\mathfrak{M}_\theta(r, n\delta)$ parametrises framed Γ-equivariant sheaves $(\mathscr{F}, \phi_{\mathscr{F}})$ such that

- $\phi_{\mathscr{F}} : \mathscr{F}|_{l_\infty} \cong \mathcal{O}^r_{l_\infty}$,
- $\mathscr{F}$ *is torsion-free of rank* r,
- *satisfy* $h^1(\mathbb{P}^2, \mathscr{F} \otimes \mathscr{I}_{l_\infty}) = n$.

So we could guess that $\mathfrak{M}_{\theta_0}(r, n\delta)$ requires a compactification of $\mathbb{C}^2/\Gamma$.
It turns out that $\mathbb{P}^2/\Gamma$ doesn't work.
So we instead make a *stack*

$$\mathcal{X} = [\mathbb{P}^2 \setminus \{0\}/\Gamma] \cup_{[\mathbb{C}^2 \setminus \{0\}/\Gamma]} \mathbb{C}^2/\Gamma$$

This is a nice stack!

(almost) Theorem, G. 2022

$\mathfrak{M}_{\theta_0}(r, n\delta)$ parametrises framed sheaves $(\mathscr{E}, \phi_{\mathscr{E}})$ on $\mathcal{X}$ where

- $\mathscr{E}$ is torsion-free of rank r,
- $\phi_{\mathscr{E}} \colon \mathscr{E}|_{d_\infty}^r \cong \mathcal{O}_{d_\infty}^r$
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- $h^1(\mathcal{X}, \mathscr{E} \otimes \mathscr{I}_{d_\infty}) = n$.

If $r = 1$, you can (with some work) identify these sheaves with points of $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$.

Some idea of the proof:
There are maps

$$\mathbb{P}^2 \to [\mathbb{P}^2/\Gamma] \to \mathcal{X} \to \mathbb{P}^2/\Gamma.$$

The proof really focuses on pushing sheaves/pulling back sheaves along $q\colon [\mathbb{P}^2/\Gamma] \to \mathcal{X}$.
In this way, many of the properties carry over fairly easily.
On the level of Π-modules: Moving to θ_0-stability, we again quotient out by submodules supported away from ρ_0.
This turns out to correspond to $\varinjlim$ of all sheaves on $[\mathbb{P}^2/\Gamma]$ mapping to the same sheaf on $\mathcal{X}$.

Thank you!