Research proposal - Quiver varieties and moduli spaces of framed sheaves

Søren Gammelgaard

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Background

There has recently been much interest in Nakajima quiver varieties. They provide interesting examples of hyperkähler manifolds, as well as global or étale-local models of important moduli spaces. Their geometric properties are fairly well understood, for instance any Nakajima quiver variety has symplectic singularities ([1]). My research, both current and proposed, focuses on a specific type of these varieties, namely the Nakajima quiver varieties constructed from McKay quivers associated to finite subgroups Γ of $SL_2(\mathbb{C})$ acting on $\mathbb{C}^2$.

My DPhil thesis – which I'm currently writing – investigates the relationship between such quiver varieties and various moduli spaces associated to the action of Γ on $\mathbb{C}^2$. As part of my thesis, I have been a co-author on two papers in this subject ([4], [5]). The construction of these quiver varieties depends crucially on a stability parameter, and the main line of investigation in these papers concern what happens when this parameter moves into particular rays in its wall-and-chamber structure.

The two papers unite ideas and techniques from different areas of mathematics - they include moduli spaces of modules of associative algebras, representation theory of quivers, and a small bit of symplectic geometry, in addition to the main algebraic-geometric ideas.

In the first, we show that Hilbert schemes of the Kleinian singularities $\mathbb{C}^2/\Gamma$ can be constructed as quiver varieties, while in the second, we investigate 'equivariant Quot schemes' associated to the action of Γ on $\mathbb{C}^2$.

Higher-rank quiver varieties

The definition of these Nakajima quiver varieties also depends on a positive integral *rank* parameter, which we in both these papers set equal to 1. It is a natural question to see what happens when this rank is a larger integer. I have started to investigate this in my DPhil thesis, and I hope to soon make my work on this available as a preprint.

So far, I have considered such 'higher-rank' quiver varieties defined with a stability parameter lying in a specific ray in the wall-and-chamber structure, similar to the one used in our first paper. It turns out that the resulting quiver varieties can be understood as moduli spaces of framed sheaves on a particular stacky compactification of the singularity $\mathbb{C}^2/\Gamma$. This compactification is projective, and luckily tools for working with moduli spaces of framed sheaves on projective stacks have recently been developed by Bruzzo and Sala (see e.g. [3]).

In my two joint papers, we have made extensive use of the technique of *cornering* (see [6, Remark 3.1]), which provides interpretations of Nakajima quiver varieties as fine moduli spaces. This technique remains viable when increasing the rank, and can be used to investigate many faces in the wall-and-chamber structure.

I propose to continue research into such 'higher-rank' Nakajima quiver varieties. Some results are already known: In [12] it is shown that using a generic stability parameter in one chamber gives moduli spaces of framed Γ-equivariant sheaves on $\mathbb{P}^2$; in [11], Nakajima characterises the quiver variety given when choosing a stability parameter from another chamber, finding a description similar to that appearing in my own work. Finally, Nakajima has also given a characterisation of the variety given by choosing the zero parameter in [10].

Work on these varieties could also tie in with a recent conjecture of Bruzzo, Sala, Szabo and Pedrini ([2, Conjecture 4.14]), that moduli spaces of certain framed sheaves on a stacky compactification of the minimal resolution of $\mathbb{C}^2/\Gamma$ are isomorphic to 'higher-rank' Nakajima quiver varieties. (Their conjecture is only stated for the case where Γ is a cyclic group, but I should like to extend it to other cases of Γ as well.)

I hope to place these ideas, including the results from my DPhil thesis, in a larger framework of the Nakajima quiver varieties associated to finite subgroups of $SL_2(\mathbb{C})$. I suspect that many of them can be interpeted as moduli spaces of framed sheaves on other stacky compactifications of partial resolutions of $\mathbb{C}^2/\Gamma$, or as spaces birational to them. Such a framework should extend and generalise that existing for the rank 1-case, which has already been thoroughly investigated (see e.g. [9]).

Applications to K3 surfaces

I would also like to apply my work with quiver varieties to other ares of algebraic geometry. There are, for instance, recent results pointing to a connection of quiver varieties to Hilbert schemes of K3 surfaces and related varieties: In [7], DeHority uses quiver varieties similar to those I've been researching to understand the cohomology of punctual Hilbert schemes on certain K3 surfaces, before extending his results to the cohomology of moduli spaces of rank 1 torsion free sheaves on the K3 surface.

On the other hand, Yamagishi has shown (see [13]) that the Fano variety of lines on certain singular cubic fourfolds has the same type of singularities as the Hilbert square of a surface with all singularities Kleinian. However, the Fano variety of lines on a *smooth* cubic fourfold is famously deformation equivalent to the Hilbert square of a K3 surface.

We constructed (in [4]) the Hilbert square of a Kleinian singularity as a quiver variety with a nongeneric stability parameter. But the NQVs considered by DeHority are constructed with a *generic* parameter. One could guess that this is indicating a deeper connection between Hilbert schemes of K3 surfaces and NQVs. For instance, can the "degeneration" of the Hilbert square of a K3 to the Fano scheme of lines on a singular cubic fourfold be interpreted as parallelling the "degeneration" of a generic stability parameter to a specialised stability parameter (for those quiver varieties that are punctual Hilbert schemes of $\mathbb{C}^2/\Gamma$)?

Singular varieties of dimension ≥ 3

More tentatively, it would be interesting to see whether the techniques we have developed could be extended to cover higher-dimensional singularities of the type $\mathbb{C}^n/\Gamma$.

One can in this case form quiver varieties much as in the 2-dimensional case. There has been some research on these quiver varieties, (e.g. [8]), but much less than in the 2-dimensional case.

The first type of singularity to look at would be the case where $\mathbb{C}^n/\Gamma$ is an isolated cyclic quotient singularity, i.e. where $\Gamma \cong \mathbb{Z}/k$ for some positive integer k, acting by diagonal matrices on $\mathbb{C}^n$, such that the resulting quotient has an isolated singular point.

I conjecture that it is again possible in this case to express the Hilbert schemes $\mathrm{Hilb}^n(\mathbb{C}^n/\Gamma)$ as quiver varieties, constructed with a specific nongeneric stability parameter. I have already made some tentative investigations in this direction, which have had promising results.

References

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