Quiver Varieties and Moduli Spaces of Sheaves on Singular Surfaces

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Abstract

We investigate several types of Nakajima quiver varieties and their connection to the Kleinian singularities $\mathbb{C}^2/\Gamma$.

Such quiver varieties have many good properties: they are for instance irreducible and have symplectic singularities, in particular they are normal.

The first part of this thesis introduces Nakajima quiver varieties, together with the necessary background on projective Deligne-Mumford stacks and framed sheaves. Following that, we prove that the punctual Hilbert schemes (when taken with their reduced scheme structures) $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$ are examples of Nakajima quiver varieties.

We then show that there is a type of 'orbifold Quot scheme' generalising the Hilbert scheme, which can also be identified with a Nakajima quiver variety.

Following this, we investigate another generalisation of the Hilbert scheme: that of a moduli space of framed sheaves on a certain stack compactifying $\mathbb{C}^2/\Gamma$. We show that this moduli space exists as a quasiprojective scheme. We are unable to show that it is isomorphic to a Nakajima quiver variety, but we show that it carries a canonical morphism to a Nakajima quiver variety, and this morphism is a bijection of closed points.

We end by sketching some potential further directions of investigation.

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Chapter 1

Introduction

This thesis investigates a class of algebraic varieties known as *Nakajima quiver varieties*. Introduced by Nakajima, these varieties have several useful geometric properties, and find use in several contexts. The definition of such a quiver variety requires several pieces of data: a quiver, a stability parameter, a deformation parameter, and two dimension vectors. By varying this data, we can construct a large class of varieties, but in this thesis we will investigate certain very special types of them.

To be precise, we will focus on the quivers that are formed from *framed* affine Dynkin diagrams of type ADE. The McKay correspondence [44] establishes connections between such quivers and *Kleinian singularities*, i.e., singularities appearing as quotients of $\mathbb{C}^2$ by finite subgroups of $\mathbf{SL}_2(\mathbb{C})$. In this thesis, we extend the correspondence by constructing several moduli spaces related to Kleinian singularities as Nakajima quiver varieties.

Much of the work that has gone into thesis was done in collaboration with Balázs Szendrői, Ádám Gyenge, and Alastair Craw, and has now been published in the papers [16, 17]. The last chapters are however original.

We will in fact construct three different types of moduli spaces attached to Kleinian singularities. The first is the punctual Hilbert scheme of such singularities. This is in many ways a starting point for the other two types of moduli space, which generalise the punctual Hilbert scheme in different ways.

First, we define a type of *orbifold Quot scheme* associated to the Kleinian singularity. Following that, we generalise the Hilbert scheme in another direction: We construct a projective Deligne-Mumford stack that compactifies the singularity, and on this stack, we define a moduli space of *framed* coherent sheaves satisfying certain cohomological conditions. In special cases, both of these reduce to the Hilbert scheme. We will construct the Hilbert schemes and Quot schemes as Nakajima quiver varieties. We also show that the final moduli space of framed sheaves is very close to being one - we conjecture that it is, but we cannot prove it completely. (See the discussion following Theorem 6.3.1 for some of the difficulties.)

Once and for all, fix a finite subgroup $\Gamma \subset \mathbf{SL}(2, \mathbb{C})$.

Let us write $\{\rho_0, \rho_1, \ldots, \rho_s\}$ for the set of irreducible representations of Γ.

Starting with Γ, we shall use the McKay correspondence to construct a quiver Q_Γ which has $Q_{\Gamma,0} = \{0, 1, \ldots, s\}$ as its vertex set, where i corresponds to the representation ρ_i. From this quiver we can build our different Nakajima quiver varieties. We detail this construction in Section 2.1. Especially, we shall always take the reduced subscheme structure for our quiver varieties.

With a fixed quiver coming from Γ, the definition of a Nakajima quiver variety $\mathfrak{M}_\theta(\mathbf{w}, \mathbf{v})$ depends on a *stability parameter* $\theta \in \mathbb{Q}^{Q_{\Gamma,0}}$, a *framing dimension vector* $\mathbf{w} \in \mathbb{Z}_{\geq 0}^{Q_{\Gamma,0}}$, and a *dimension vector* $\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_{\Gamma,0}}$. In our case, however, we shall always choose the framing vector to having a single nonzero element. We will thus speak of it as the *framing rank*

$$r \coloneqq \mathbf{w}_0,$$

and denote our Nakajima quiver varieties as $\mathfrak{M}_\theta(r, \mathbf{v})$.

1.1 Punctual Hilbert schemes of Kleinian singularities

Consider the action of Γ on $\mathbb{C}^2$. One can associate various Hilbert schemes to the action of Γ on the affine plane $\mathbb{C}^2$ and the Kleinian singularity $\mathbb{C}^2/\Gamma$. For $N \coloneqq |\Gamma|$ and any natural number n, the action of Γ on $\mathbb{C}^2$ induces an action of Γ on the Hilbert scheme $\mathrm{Hilb}^{[nN]}(\mathbb{C}^2)$ of nN points on the affine plane. The scheme $n\Gamma\text{-}\,\mathrm{Hilb}(\mathbb{C}^2)$, parametrising Γ-invariant ideals I in $\mathbb{C}[x, y]$ such that the quotient $\mathbb{C}[x, y]/I$ is isomorphic to the direct sum of n copies of the regular representation of Γ, is a union of components of the fixed point set of the Γ-action on $\mathrm{Hilb}^{[nN]}(\mathbb{C}^2)$. It is thus nonsingular and quasi-projective. One may also consider the Hilbert scheme of n points $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ on the singular surface $\mathbb{C}^2/\Gamma$, parametrising ideals in the invariant ring $\mathbb{C}[x, y]^\Gamma$ that have codimension n. This Hilbert scheme is quasi-projective, and for the rest of this section we endow it with the *reduced* scheme structure.

These two kinds of Hilbert schemes are related by the morphism

$$n\Gamma\text{-}\,\mathrm{Hilb}(\mathbb{C}^2) \longrightarrow \mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \tag{1.1}$$

sending a Γ-invariant ideal I in $\mathbb{C}[x, y]$ to the ideal $I \cap \mathbb{C}[x, y]^\Gamma$; this set-theoretic map is indeed a morphism of schemes by Brion [8, p. 3.4]. By composing with the

Hilbert-Chow morphism of the surface $\mathbb{C}^2/\Gamma$, we see that (1.1) is in fact a morphism of schemes over the affine scheme $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$.

Until recently, not much was known about the schemes $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ for $n > 1$. Gyenge, Némethi and Szendrői [29] computed the generating function of their Euler characteristics for Γ of type A and D (the cyclic and dihedral cases), giving an answer with modular properties. They also conjectured an analogous formula for type E.

In [68], Zheng proved that $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ is always irreducible, and gave a homological characterisation of its smooth points through a detailed analysis of Cohen-Macaulay modules over $\mathbb{C}^2/\Gamma$. Yamagishi [67] studied symplectic resolutions of the Hilbert squares $\mathrm{Hilb}^{[2]}(\mathbb{C}^2/\Gamma)$, and described completely the central fibres of these resolutions, from which he deduced that $\mathrm{Hilb}^{[2]}(\mathbb{C}^2/\Gamma)$ admits a unique symplectic resolution.

For any $n \geq 1$ and with a framing of rank $r = 1$, Kuznetsov [40] determined a pair of cones C^+, C^- in the space of stability parameters for which the corresponding Nakajima quiver variety $\mathfrak{M}_\theta(1, n\delta)$ is isomorphic to the punctual Hilbert scheme $\mathrm{Hilb}^{[n]}(S)$ of the minimal resolution $\widehat{\mathbb{C}^2/\Gamma}$ of $\mathbb{C}^2/\Gamma$ for $\theta \in C^-$, and to the scheme $n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ from (1.1) for $\theta \in C^+$, respectively. Here δ is a fixed dimension vector for the (unframed) McKay quiver. More recently, Bellamy and Craw [5] gave a complete description of the wall-and-chamber structure on the space of stability parameters in this situation, and identified a simplicial cone F containing C^+, C^- that is isomorphic as a fan to the movable cone of $n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ for $n > 1$; in particular, chambers in this simplicial cone correspond one-to-one with projective, symplectic resolutions of $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ (see Figure 3.1 below for an example).

The aim of Chapter 3 is to study the spaces $n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$, $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$, and all possible ways in which the morphism $n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2) \to \mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$from (1.1) can be decomposed, using quiver-theoretic techniques in a uniform way. The main result of Chapter 3 reconstructs the morphism from (1.1) by variation of GIT quotient. Explicitly, we vary a generic stability parameter $\theta \in C^+$ to a parameter θ_0 in a particular extremal ray of the closure of C^+; the induced morphism $\mathfrak{M}_\theta(1, n\delta) \to \mathfrak{M}_{\theta_0}(1, n\delta)$ coincides with the morphism (1.1). As a corollary, we obtain the following result.

Theorem 1.1.1. *Let $\Gamma \subset \mathbf{SL}(2, \mathbb{C})$ be a finite subgroup and let $n \geq 1$. The (reduced) Hilbert scheme $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{red}$ is an irreducible, normal scheme with symplectic, hence rational Gorenstein, singularities. Furthermore, it admits a unique projective, symplectic resolution given by (1.1).*

We thus obtain an interpretation of the quiver variety $\mathfrak{M}_\theta(r, \mathbf{v})$ in the case when $r = 1$ and $\mathbf{v} = n\delta$, and $\theta = \theta_0$, a non-generic parameter lying in a very specific ray in the wall-and-chamber structure on the space of stability parameters.

We reiterate that irreducibility is due originally to Zheng [68]. The existence of a nowhere-vanishing $2n$-form in the type A case, which follows from having symplectic singularities, was shown in the same paper [68, Theorem D], while the existence and uniqueness of the symplectic resolution for $n = 2$ is due to Yamagishi [67].

Because $\mathfrak{M}_{\theta_0}(1, n\delta)$ does not immediately have a *fine* moduli space structure, one of our main tools is to furnish $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ with a quiver-theoretic interpretation as a fine quiver moduli space by the process of 'cornering' [15].

More generally, we provide a fine moduli space description of the quiver varieties $\mathfrak{M}_\theta(1, n\delta)$ for all non-generic stability parameters θ that lie in the closure of the cone C^+, a technique that we develop further in Chapter 4. Our methods give conceptual proofs of the geometric properties of $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ listed in Theorem 1.1.1, and allow us to obtain all possible projective factorisations of the morphism (1.1) by universal properties of the resulting fine moduli spaces. Our proofs for all statements concerning $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ avoid case-by-case analysis with respect to the ADE classification of Γ. We use only one such case-by-case argument for our more general quiver varieties $\mathcal{M}_\theta$ to establish a bound on the dimension vector for quiver representations that are stable with respect to a non-generic stability condition; see Appendix A.

Quiver varieties with degenerate stability conditions identical to ours were considered before in [49]. In recent work, Nakajima [50] uses the main result of our chapter and some results from the representation theory of quantum affine algebras to prove the conjecture of [29].

1.2 Quot schemes of Kleinian singularities

The subsequent chapters investigate what happens when we change the parameters $\theta, \mathbf{v}, r$ in $\mathfrak{M}_\theta(r, \mathbf{v})$ from $\theta_0, n\delta, 1$: first both $\mathbf{v}$ and θ, and second by keeping $\mathbf{v}$ and θ fixed, but increasing r. They thereby generalise Theorem 1.1.1 it in different ways.

After showing in Chapter 3 that (the reduced scheme underlying) the scheme $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$ is irreducible, normal, and that the map

$$\pi_n \colon n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2) \longrightarrow \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{red}$$

is the unique projective symplectic resolution, we generalise. The morphism π_n is induced by variation of GIT quotient for quiver varieties; specifically, if θ is any parameter in the chamber C^+, we will show that for our very specific θ_0, π_n is the morphism from $\mathfrak{M}_\theta(1, n\delta)$ to $\mathfrak{M}_{\theta_0}(1, n\delta)$. We can thus identify, by variation of GIT quotient, all possible ways in which π_n can be factored as a sequence of primitive contractions. These will correspond to subsets of vertices $I \subset Q_{\Gamma,0}$, by associating a face of the wall-and-chamber structure to I, and associating another quiver variety $\mathfrak{M}_{\theta_I}(1, n\delta)$ to I, where θ_I is a generic stability parameter in the facet corresponding to I.

To do this, we consider a family of algebras A_I obtained by cornering, each indexed by a non-empty subset I of the set of irreducible representations of Γ. These already briefly appear in Chapter 3, but we focus on them in Chapter 4. The fine moduli spaces of modules $\mathcal{M}(A_I)(1, n\delta_I)$ for these algebras with very specific dimension vectors are isomorphic to the quiver varieties $\mathfrak{M}_{\theta_I}(1, n\delta)$ mentioned above. For I consisting of the trivial representation only, we will recover the space $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$.

In Chapter 4, we thus have three main goals. First, we aim to understand better the moduli spaces $\mathcal{M}_{A_I}$, constructing them as a kind of Quot scheme that is both natural and geometric. Second, with applications in mind, we consider arbitrary dimension vectors, not just multiples of some set of fixed dimension vectors as above. One interpretation of our results is a new fine moduli space structure, as well as a (noncommutative) geometric interpretation, of a large class of Nakajima quiver varieties for certain non-generic stability parameters. Our third and final goal is to provide proofs that completely bypass any case-by-case analysis of ADE diagrams.

So let, for any non-empty subset $I \subseteq \{0, 1, \ldots, s\}$, $\mathfrak{M}_{\theta_I}(1, v)$ denote the Nakajima quiver variety associated to the affine ADE graph for some dimension vector $(1, v)$, where θ_I is a specific stability condition determined by our choice of I; see (2.5) for the definition.

Theorem 1.2.1. *Let $I \subseteq \{0, \ldots, s\}$ be a non-empty subset and let $n_I \in \mathbb{N}^I$ be a vector. There is an orbifold Quot scheme $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$ such that:*

1. *there is an isomorphism $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma]) \cong \mathcal{M}_{A_I}(1, n_I)$ to a fine moduli space associated to the algebra A_I obtained by cornering;*
2. *when I is a singleton corresponding to a one-dimensional representation of Γ, the orbifold Quot scheme is isomorphic to a classical Quot scheme for $\mathbb{C}^2/\Gamma$;*
3. *the orbifold Quot scheme is non-empty if and only if the quiver variety $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty for some vector $v \in \mathbb{N}^{s+1}$ satisfying $v_i = n_i$ for all $i \in I$, in which*

case there is another vector $\widetilde{v} \in \mathbb{N}^{s+1}$*, also satisfying* $\widetilde{v}_i = n_i$ *for all* $i \in I$*, such that*

$$\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{\mathrm{red}} \cong \mathfrak{M}_{\theta_I}(1, \widetilde{v}),$$

where on the left we take $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$ *with the reduced scheme structure.*

In particular, when it is non-empty, $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{\mathrm{red}}$ *is irreducible, and it has symplectic, hence rational Gorenstein, singularities. It is thus also normal. Moreover, it admits at least one projective symplectic resolution.*

Remark 1.2.2. Fix the dimension vector $\mathbf{v} = n\delta$ again.

1. For $I = \{0\}$, the statement of Theorem 1.2.1 specialises to Theorem 1.1.1 Indeed, the classical Quot scheme from Theorem 1.2.1(2) specialises to the Hilbert scheme in this case by Example 4.1.6, and the isomorphism from Theorem 1.2.1(3) constructs $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{red}$ as a quiver variety by Example 4.4.3 and Theorem 4.4.9. Moreover, the polarising ample bundle on $\mathfrak{M}_{\theta_I}(1, \mathbf{v})$ lies in an extremal ray of the movable cone described in [5], so it admits a unique projective symplectic resolution. In fact, through considering a larger set of dimension vectors satisfying certain combinatorial properties, our arguments here reprove the main result of Chapter 3 without having to resort to any case-by-case analysis of ADE diagrams.

2. For $I = \{1, \dots, s\}$, the stability condition θ_I lies in the relative interior of a wall of the chamber C^+. It follows from [5, Theorem 1.2] that the polarising ample bundle on $\mathfrak{M}_{\theta_I}(1, \mathbf{v})$ lies in the interior of the movable cone, so $\mathfrak{M}_{\theta_I}(1, \mathbf{v})$ admits more than one (in fact, precisely two) projective crepant resolutions.

Thus, while the projective symplectic resolution is unique in Theorem 1.1.1, it is not possible to assert uniqueness in Theorem 1.2.1 above.

While some of our arguments in Chapter 4 follow those of Chapter 3 closely, we introduce Quot schemes in a general noncommutative context, establishing a basic representability result in Proposition 4.1.1 that may be of broader interest. Furthermore, while the morphism

$$\mathfrak{M}_\theta(1, \mathbf{v}) \longrightarrow \mathfrak{M}_{\theta_I}(1, \mathbf{v})$$

induced by varying a generic stability condition θ to a special one θ_I is always surjective for the dimension vector $v = n\delta$ relevant to the case $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$ discussed above, the same is not true in general. Instead, one has to correct $\mathbf{v}$ to obtain a surjective morphism of quiver varieties.

When working with an arbitrary I, it will as in [28, 29] be essential to work with arbitrary dimension vectors, not just multiples of δ.

The structure of Chapter 4 is as follows: In Section 4.1, we define Quot schemes for Kleinian orbifolds and discuss some of their properties. In Section 4.2, we investigate the algebras A_I and the associated fine moduli spaces $\mathcal{M}(A_I)(1, n_I)$, and we identify these moduli spaces with Quot schemes for Kleinian orbifolds. In Section 4.3, we obtain a resolution of singularities for the quiver varieties of interest. Finally, in Section 4.4 we establish the key isomorphism from Theorem 1.2.1(3).

1.3 Moduli spaces of framed modules and projective stacks

The next chapters, Chapters 5 and 6 extend Theorem 1.1.1 in another direction, by investigating the Nakajima quiver varieties $\mathfrak{M}_{\theta_0}(r, n\delta)$ when $r > 1$. Here, θ_0 is again a parameter lying in a special ray in the closure of C^+. It turns out that the resulting quiver varieties parametrise framed torsion-free coherent sheaves on a stacky compactification of the singular surface $\mathbb{C}^2/\Gamma$. We will therefore also investigate the moduli spaces of such sheaves, using theory developed by Bruzzo-Sala [11] and Nironi [53].

1.3.1 Moduli spaces of higher-rank sheaves

In Chapters 3 and 4, we will see several types of quiver varieties $\mathfrak{M}_\theta(1, \mathbf{v})$. We will also see in Section 2.4.1 that for various choices of θ, several different varieties $\mathfrak{M}_\theta(1, n\delta)$ have been identified. Even more, the wall-and-chamber structure on the parameter space for θ was explicitly computed by Bellamy and Craw ([5]).

However, when changing the dimension vector from $(1, n\delta)$ to $(r, \mathbf{v})$ the picture is not as clear.

For Γ a trivial group, it is known that the quiver variety $\mathfrak{M}_\theta(r, n\delta)$ is isomorphic to the moduli space of torsion-free sheaves on $\mathbb{P}^2$ of rank r and second Chern class n, framed along the divisor $l_\infty = V(z)$, the 'line at infinity' [48]. Here, by a sheaf $\mathscr{F}$ of rank r *framed* along l_∞, we mean a pair $(\mathscr{F}, \phi_{\mathscr{F}})$, where $\phi_{\mathscr{F}}$ is an isomorphism $\mathscr{F}|_{l_\infty} \xrightarrow{\sim} \mathscr{O}_{l_\infty}^r$.

Let us then consider a nontrivial Γ. As Γ acts on $\mathbb{C}^2 = \operatorname{Spec}\mathbb{C}[x,y]$, it induces an action on $\mathbb{C}[x,y]$. We extend this to an action on $\mathbb{C}[x,y,z]$, by setting $g \cdot z = z$ for all $g \in \Gamma$. This defines an action of Γ on $\mathbb{P}^2$, fixing the line $l_\infty = \{z = 0\}$ (but not the points of l_∞).

Highly relevant for us is a result of Varagnolo and Vasserot (see Theorem 5.3.1 for the details) stating that, for a generic stability parameter θ lying in a chamber $C^+_{r,n\delta}$, there is a canonical bijection

$$\mathfrak{M}_\theta(r, n\delta)(\mathbb{C}) \xrightarrow{\sim} X_{r,n\delta}.$$

Here $X_{r,n\delta}$ is the set of isomorphism classes of coherent Γ-equivariant sheaves on $\mathbb{P}^2$ together with a framing along the line l_∞, satisfying certain further conditions, see Definition 5.2.3 for the complete definition. In particular, the framing requires the restriction of these sheaves to the divisor l_∞ to be a free sheaf of $\mathcal{O}_{l_\infty}$-modules. For $r = 1$, we can identify $X_{1,n\delta}$ with $n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$.

As in the case where $r = 1$, we wish to specialise θ to a parameter θ_0. We see that for $\theta \in C^+$, changing from $r = 1$ to $r > 1$ requires us to consider sheaves of modules on $\mathbb{P}^2$ instead of $\mathbb{C}^2$. As $\mathfrak{M}_{\theta_0}(1, \mathbf{v})$ turned out to parametrise structures on the singular surface $\mathbb{C}^2/\Gamma$, it is thus natural to expect $\mathfrak{M}_{\theta_0}(r, n\delta)$ to parametrise sheaves of modules on some projective closure of $\mathbb{C}^2/\Gamma$.

The main goal of Chapters 5 and 6 is to provide another example of a quiver variety $\mathfrak{M}_{\theta_0}(r, n\delta)$ by specialising the parameter θ appearing in Varagnolo and Vasserot's result to a θ_0 lying in a certain ray in the parameter space. This quiver variety will turn out to parametrise a similar set $Y_{r,n}$ of isomorphism classes sheaves on a *stack* $\mathcal{X}$, which is a stacky 'compactification' of the scheme quotient $\mathbb{C}^2/\Gamma$. We will construct $\mathcal{X}$ as a quotient of $\mathbb{P}^2$ by Γ, glued together from two patches, where we on one patch take the 'stack-theoretic' quotient, and on another the 'scheme-theoretic' quotient, see Section 5.1 for the full construction.

This 'compactification' is equipped with a morphism of stacks $\mathbb{P}^2 \to \mathcal{X}$, and in particular the image of the divisor l_∞ is a (stacky) divisor $d_\infty = [l_\infty/\Gamma]$. The framed sheaves lying in $Y_{r,n}$ will be framed along d_∞.

The definition is as follows (here $\mathscr{I}_{d_\infty}$ is the ideal sheaf of d_∞ in $\mathcal{X}$):

Definition 1.3.1. Let $Y_{r,n}$ be the set of isomorphism classes of pairs $(\mathscr{F}, \phi_{\mathscr{F}})$, where $\mathscr{F}$ is a torsion-free coherent $\mathcal{O}_{\mathcal{X}}$-module of rank r, with $\dim H^1(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{d_\infty}) = n$, and where $\phi_{\mathcal{F}}$ is an isomorphism (the *framing* isomorphism) $\phi_{\mathcal{F}} \colon \mathscr{F}|_{d_\infty} \xrightarrow{\sim} \mathcal{O}_{d_\infty}^{\oplus r}$.

We shall also define a similar moduli functor.

Definition 1.3.2. Let $\mathcal{Y}_{r,n}$ be the functor associating to any scheme S of finite type the set of isomorphism classes of flat families of sheaves lying in $Y_{r,n}$, parametrised by S.

Our main result is then:

Theorem 1.3.3. *The moduli functor $\mathcal{Y}_{r,n}$ has a fine moduli space $\mathbf{Y}_{r,n}$, the $\mathbb{C}$-valued points of which can be identified with the set $Y_{r,n}$.*

There is a canonical morphism

$$\mathbf{Y}_{r,n} \to \mathfrak{M}_{\theta_0}(r, n\delta)$$

which is bijective on closed points.

We conjecture that this morphism is an isomorphism of the underlying reduced schemes. If this conjecture holds, $\mathbf{Y}_{r,n}$ is a irreducible scheme with symplectic singularities, and is thus normal.

To prove Theorem 1.3.3, we will start by setting up a canonical bijection between $Y_{r,n}$ and $\mathfrak{M}_{\theta_0}(r, n\delta)$. In order to extend this bijection into an morphism of schemes, we will have to introduce *cornered* preprojective algebras, which we do in Section 6.1.

Remark 1.3.4. As for the $r = 1$ case in Theorem 1.1.1, there will be a morphism $\mathfrak{M}_\theta(r, n\delta) \to \mathfrak{M}_{\theta_0}(r, n\delta)$, which is a symplectic resolution. But we're unable to decide whether it is unique, because we do not have an explicit description of the wall-and-chamber structure.

We will detail the construction of $\mathbf{Y}_{r,n}$ in Section 6.2, using theory developed by Bruzzo and Sala in [11].

As part of the proof of Theorem 1.3.3, we will show that there is a functor D from the category of Γ-equivariant coherent shaves on $\mathbb{P}^2$ to the category of coherent sheaves on $\mathcal{X}$. We show in Proposition 5.3.5 that the map $\mathscr{E} \mapsto D(\mathscr{E})$ gives a surjective map of sets $X_{r,n\delta} \to Y_{r,n}$.

An ideal of finite codimension of $\mathbb{C}[x, y]^\Gamma$ can be interpreted as a sheaf on $\mathbb{C}^2/\Gamma$, and can also be extended to an element of $Y_{1,n}$ (see Corollary 6.3.4), in particular, a coherent sheaf of rank 1. We will show, once we have finished Section 6.2, that this *almost* provides another proof of Theorem 1.1.1 - we would need the morphism from Theorem 1.3.3 to be an isomorphism for a complete proof.

In this way, these chapters partially extend the results of Chapter 3 to moduli spaces of sheaves of higher rank.

1.3.2 A related construction

The stack $\mathcal{X}$ may seem a little strange, but similar objects have been constructed before: In [52] Nakajima constructs an orbifold compactification $\widehat{\mathcal{X}}$ of the minimal resolution $\widehat{\mathbb{C}^2/\Gamma}$ of $\mathbb{C}^2/\Gamma$, and shows that there are Nakajima quiver varieties $\mathfrak{M}_{\theta^-}(r, \mathbf{v})$ that can be interpreted as moduli spaces of framed sheaves on $\widehat{\mathcal{X}}$.

Here θ^- is a stability parameter a specific chamber C^- in the space of stability parameters, and the stack $\mathcal{X}'$ is constructed from $\widehat{\mathbb{C}^2/\Gamma}$ in a similar manner to how we shall construct $\mathcal{X}$ from $\mathbb{C}^2/\Gamma$. When $r = 1$, $\mathfrak{M}_{\theta^-}(1, \mathbf{v})$ is the Hilbert scheme of points on $\widehat{\mathbb{C}^2/\Gamma}$, a result due to Kuznetsov [40].

In Appendix C, we extend Nakajima's result by proving (under a mild assumption) that the moduli space of framed sheaves on in some cases carries an intrinsic structure of a fine moduli space of framed sheaves. This was supposed by Nakajima, but not proved. To some extent, this parallels Theorem 1.3.3.

A similar construction appears in a conjecture of Bruzzo, Sala, Pedrini, and Szabo, see [10, Conjecture 4.14].

1.3.3 The structure of the argument for higher-rank sheaves

The necessary background information is in Sections 2.5 and 2.6, collecting some of the stack-theoretic definitions we will make use of. We also provide a short overview on the notion of *projective stacks*, framed sheaves on projective stacks, and moduli spaces of such sheaves.

In Section 5.1, we focus on different ways to take the quotient of $\mathbb{P}^2$ by Γ. Following this, Section 5.2 discusses various sets of coherent sheaves on the quotients introduced in the preceding section, introduces functors to move between them, and investigates which properties of the sheaves are preserved by these functors.

In Section 5.3 and Section 5.4, we prove the first part of Theorem 1.3.3, namely that there is a canonical bijection $\mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C}) \xrightarrow{\sim} Y_{r,n}$. The main idea is to understand the morphism $\mathfrak{M}_{\theta}(r, n\delta) \to \mathfrak{M}_{\theta_0}(r, n\delta)$ induced by variation of GIT, in terms of Γ-equivariant sheaves.

Following that, we use *cornering* again to make $\mathfrak{M}_{\theta_0}(r, n\delta)$ into a fine moduli space (Section 6.1). Then we show that the appropriate moduli space $\mathbf{Y}_{r,n}$ exists. We first show that our stack $\mathcal{X}$ is a *projective* stack. Then we study the framed sheaves in $Y_{r,n}$, showing that they all have the same Hilbert polynomial, and that their Castelnuovo-Mumford regularity is bounded. This enables us to show that they are all $\widehat{\mu}$-stable (Lemma 6.2.8), which finally provides a construction of the space $Y_{r,n}$ as a fine moduli space.

1.4 Further questions

The final chapter contains some sketches of possible directions to continue the investigations of this thesis. One idea would be to investigate the local geometry of our singular Nakajima quiver varieties more, for instance by considering various stratifications of the varieties. To do this, we need some results on the root systems of the Kac-Moody algebras associated to the McKay quiver, and I have sketched a beginning of such work in Section 7.1.

Section 7.2 asks a reasonable conjecture about the smooth locus of $\mathbf{Y}_{r,n}$. Finally, Section 7.3 concerns other possible varieties constructed from quivers, and states some natural questions to continue investigating.

1.5 The appendices

Two proofs – mainly related to Chapter 3 – are relegated to appendices.

The first, Appendix A, is a dimension estimate forming part of the proof of Theorem 1.1.1. It is relegated to this Appendix mainly for its length, and because its techniques are quite different (and more primitive) than the rest of the argument.

The second, Appendix B is the proof of Lemma 2.4.5, a result about the surjectivity of certain morphisms between quiver varieties inducing by varying the stability parameter θ. The result thematically belongs in Chapter 2, but we postpone it, as it requires explicit descriptions of some Nakajima quiver varieties.

The third appendix, Appendix C, is different: It is a fairly straightforward application of theory developed in [10] to extend some results of Nakajima in [52] concerning framed sheaves on a stacky compactification of the minimal resolution of $\mathbb{C}^2/\Gamma$. The results from this Appendix are not used anywhere in the main text.

1.6 Authorship

Chapters 3 and 4 follow closely the papers [16, 17]. I was an equal contributor to the work done on these papers. Parts of this Introduction as well Section 2.1–Section 2.4 are also from these two papers. Sections 2.5 and 2.6) as well as Chapters 5, 6 and 7 and Section 3.6 are entirely my own work.

Appendix A is also adapted from [16], but the other Appendices are original.

1.7 A summary

Let us sum up the interpretations of $\mathfrak{M}_\theta(r, n\delta)$.

In the parameter space for θ, there is a wall-and-chamber structure that determines the geometry of $\mathfrak{M}_\theta(r, n\delta)$. We will not describe the structure in terms of explicit walls and chambers (no such description is known in gnereal), but we can define chambers C^+, C^-, a wall $\delta^\perp$ adjacent to C^-, and describe the faces σ_I of C^+.

Our new results happen on the boundary of C^+, i.e., in the faces σ_I.

1.7.1 Previously known for $r = 1$

It is known that for various choices of θ, $\mathfrak{M}_\theta(1, n\delta)$ is isomorphic to

- $\theta \in C^+$: $n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$ [65, Theorem 2], [64, Theorem 1]
- $\theta \in C^-$: $\operatorname{Hilb}^n(\widehat{\mathbb{C}^2/\Gamma})$ [40, Theorem 43],
- $\theta \in \delta^\perp$, θ generic: $\operatorname{Sym}^n(\widehat{\mathbb{C}^2/\Gamma})$ [40, Remark 42],
- $\theta = 0$: $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$ Well-known, a proof is in [5, Lemma 4.5].

We add to the above list by describing $\mathfrak{M}_\theta(1, n\delta)$ for

- $\theta = \theta_0$: $\operatorname{Hilb}^n(\mathbb{C}^2/\Gamma)_{\text{red}}$ (Theorem 1.1.1),
- $\theta \in \overline{C^+}$: $\operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{\text{red}}$ (Theorem 1.2.1).

We also find interpretations of the varieties $\mathfrak{M}_{\theta_I}(1, \mathbf{v})$ for many different $\mathbf{v}$ in Theorem 1.2.1.

Variation of GIT thus induces the following diagram of birational morphisms:

$$
\begin{tikzcd}
 & & & \operatorname{Hilb}^n(\widehat{\mathbb{C}^2/\Gamma}) \arrow[d] \\
 & & & \operatorname{Sym}^n(\widehat{\mathbb{C}^2/\Gamma}) \arrow[d] \\
n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2) \arrow[r] & \begin{bmatrix} \text{Various types of} \\ \operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{\text{red}} \end{bmatrix} \arrow[r] & \operatorname{Hilb}^n(\mathbb{C}^2/\Gamma)_{\text{red}} \arrow[r] & \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)
\end{tikzcd}
$$

where ‘[Various types of $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{red}$]’ is the commutative diagram given by, for every pair of subsets J, I with $0 \in J \subset I \subset Q_{\Gamma,0}$, and where $n_{Ii} = n_{Ji} = n\delta_i$, the morphism

$$\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{red} \to \mathrm{Quot}_J^{n_J}([\mathbb{C}^2/\Gamma])_{red}.$$

These morphisms all commute, and we have $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{red} = \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{\mathrm{red}}$ for $I = \{0\}$, $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])_{red} = n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ for $I = Q_{\Gamma,0}$.

1.7.2 Previously known for $r > 1$

For $r > 1$, it is known that we obtain the following descriptions of the varieties $\mathfrak{M}_\theta(r, n\delta)$:

- $\theta \in C^+$: The moduli space of framed Γ-equivariant sheaves on $\mathbb{P}^2$ [64]
- $\theta \in C^-$: The moduli space of framed sheaves on $\widehat{\mathcal{X}}$ [52], extended inTheorem C.0.2,
- $\theta \in \delta^\perp$, θ generic: A certain compactification (sometimes called the *Uhlenbeck compactification*) of the moduli space of *locally free* sheaves on $\widehat{\mathcal{X}}$ (the stack from Section 1.3.2), [52, Proposition 1.10].
- $\theta = 0$: Similarly, a compactification of the moduli space of Γ-equivariant locally free sheaves on $\mathbb{P}^2$. [51, Equation 4.1], see also Appendix B.

Here, our new result is a description of $\mathfrak{M}_{\theta_0}(r, n\delta)$, in the sense that it has a canonical morphism to the moduli space $Y_{r,n}$ from Definition 1.3.1, bijective on closed points (taken, again, with the reduced scheme structure).

1.7.3 Different proofs

It might be interesting to note that in this thesis, we provide three different approaches to proving Theorem 1.1.1. The original appears in Chapter 3. Chapter 4 reproves it as a special case of Theorem 1.2.1, and finally it is also a special case of Theorem 1.3.3, assuming that the morphism given by the same Theorem is an isomorphism, as we detail in Corollary 6.3.4.

1.8 Conventions

We work throughout over $\mathbb{C}$. In particular, all schemes are $\mathbb{C}$-schemes and all tensor products are taken over $\mathbb{C}$ unless otherwise indicated. We often use the following partial order on dimension vectors: for any $n \in \mathbb{N}$ and for $u = (u_i), v = (v_i) \in \mathbb{Z}^n$, we define $u \leq v$ if $u_i \leq v_i$ for all $1 \leq i \leq n$.

We also follow the common convention of using 'vector bundle' to mean *locally free sheaf*, and especially 'line bundle' to mean *invertible sheaf*.

Let $\pi\colon X \to Y$ be a projective morphism of schemes over an affine base Y. For a globally generated line bundle L on X, write $|L| \coloneqq \operatorname{Proj}_Y \bigoplus_{k\geq 0} H^0(X, L^{\otimes k})$ for the (relative) linear series of L, and $\varphi_{|L|}\colon X \to |L|$ for the induced morphism over Y.

We will write $\mathsf{Coh}(X)$ for the category of coherent sheaves on a scheme (or stack) X.

If X is equipped with an action of some group G, we write $\mathsf{Coh}_G(X)$ for the category of G-equivariant coherent sheaves on X. $[X/G]$ always denotes the stack-theoretic quotient of X by G, and X/G the scheme-theoretic quotient, if it exists.

We use *orbifold* to mean a smooth Deligne-Mumford stack with generically trivial stabiliser.

If $i\colon X \to Y$ is an embedding of schemes (or stacks), and $\mathscr{F}$ is a $\mathcal{O}_Y$-module, then $\mathscr{F}|_X$ will always mean $i^*\mathscr{F}$, not $i^{-1}\mathscr{F}$.

The *dimension* of a sheaf is the dimension of its support.

Given a sheaf $\mathscr{F}$ on a scheme (or stack) X, we write $h^i(X, \mathscr{F})$ for $\dim H^i(X, \mathscr{F})$.

Chapter 2

Preliminaries

In this chapter we collect the necessary background material on quiver varieties and projective stacks.

We start by describing how a finite group $\Gamma \subset \mathbf{SL}_2(\mathbb{C})$ can be used to define a *McKay quiver*. We then define the Nakajima quiver varieties, in fact we will define them in three equivalent ways. Our Nakajima quiver varieties will depend on a stability parameter, varying in a wall-and-chamber structure. We note some results about this structure, and define the chambers and faces in it that will be of interest later.

In addition to some other algebraic preliminaries – such as the Morita equivalence between two algebras associated to the group Γ – we will introduce a notion of 'concentrated module', which shows up sporadically throughout the thesis, and will be very useful in Chapter 5.

We then give a brief overview of projective Deligne-Mumford stacks. We state the definition of a *framed sheaf* on such a stack, and introduce different notions of stability for such sheaves. We conclude by giving a result due to Bruzzo and Sala [11] about the existence of moduli spaces of stable framed sheaves on projective Deligne-Mumford stacks.

2.1 Quiver varieties from framed McKay quivers

Recall our finite group $\Gamma \subset \mathbf{SL}(2, \mathbb{C})$. Let L denote the given two-dimensional representation of Γ, defined by the inclusion. The *McKay graph* of Γ has vertex set $\{0, 1, \ldots, s\}$ where vertex i corresponds to the representation ρ_i of Γ, and there are $\dim \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes L)$ edges between vertices i and j. Note that, as L is a self-dual Γ-representation, this is symmetric in i and j.

By the McKay correspondence [44], the McKay graph is an extended Dynkin diagram of ADE type.

Let $Q_{\Gamma,0}$ be the vertex set of this graph, and let $Q_{\Gamma,1}$ be the set of all pairs consisting of an edge together with an orientation, we shall think of such a pair as an arrow between two vertices in $Q_{\Gamma,0}$. If $a \in Q_{\Gamma,1}$, we write $\overline{a}$ for the same edge with the opposite orientation. Then $Q_\Gamma \coloneqq (Q_{\Gamma,0}, Q_{\Gamma,1})$ is a quiver, which we will call the *McKay quiver*.

The McKay graph determines an affine Lie algebra $\mathfrak{g}$ (see, e.g., [35, Chapter 1.3]), and we identify its root lattice with $\mathbb{Z}^{Q_{\Gamma,0}}$, such that the minimal imaginary root of $\mathfrak{g}$ is identified with

$$\delta = (\dim \rho_i)_{i \in Q_{\Gamma,0}}.$$

We now fix a positive integer r. Extend Q_Γ by introducing a *framing vertex* ∞, together with r arrows $b_1, \dots b_r$ from ∞ to 0, and r arrows $\overline{b_1}, \dots, \overline{b_r}$ from 0 to ∞. Set $Q_0 = Q_{\Gamma,0} \cup \{\infty\}$, and $Q_1 = Q_{\Gamma,1} \cup \{b_1, \dots, b_r, \overline{b_1}, \dots, \overline{b_r}\}$. Then

$$Q \coloneqq (Q_0, Q_1)$$

forms a quiver, which we will call the *framed McKay quiver.* For each oriented edge $a \in Q_1$ we write $\mathrm{t}(a), \mathrm{h}(a)$ for the tail and head of a respectively. Choose a map $\epsilon \colon Q_1 \to \{-1, 1\}$ such that $\epsilon(a) \neq \epsilon(\overline{a})$ for all $a \in Q_1$.

Choose a vector $\mathbf{v} \in \mathbb{Z}^{Q_{\Gamma,0}}$, with all elements nonnegative. Furthermore, let ρ_∞ be a formal symbol such that $\{\rho_i | i \in Q_0\}$ is a basis for $\mathbb{Z}^{Q_0}$ as a $\mathbb{Z}$-module.

Then we can consider the vector $v = \rho_\infty + \mathbf{v} = (1, \mathbf{v}) \in \mathbb{Z}^{Q_0}$. Let

$$M(v) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{v_{\mathrm{t}(a)}}, \mathbb{C}^{v_{\mathrm{h}(a)}})$$

be the space of all Q-representations of dimension v. The group $G(v) \coloneqq \Pi_{i \in Q_0} \mathrm{GL}(v_i)$ acts naturally on this space by conjugation, with the group $\mathbb{C}^\times$ of diagonal scalar matrices acting trivially. The induced action of the quotient $G_{\mathbf{v}} \coloneqq G(\mathbf{v})/\mathbb{C}^\times$ induces a moment map μ from $M(v)$ to the dual $\mathfrak{g}^\vee$ of the Lie algebra $\mathfrak{g}$ of $G_{\mathbf{v}}$ (see, e.g., [45, §8.1]).

If we identify $\mathfrak{g}$ with $\mathfrak{g}^\vee$ using the trace pairing, we find ([40, p. 676]) that

$$\mu \colon (B) \mapsto \left(\sum_{\substack{\mathrm{t}(a)=i \\ a \in Q_1}} \epsilon(a) B_a B_{\overline{a}} \right)_{i \in Q_0}.$$

The space of characters of $G_{\mathbf{v}}$ can be identified with integer-valued maps in the space

$$\Theta_{r,\mathbf{v}} = \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{Q}) \mid \theta(1, v) = 0\}$$

by the map

$$\chi\colon \theta \mapsto \left(g \mapsto \Pi_{i\in Q_0}(\det g_i)^{\theta_i}\right). \tag{2.1}$$

We call $\Theta_{r,\mathbf{v}}$ the space of stability parameters. As a vector space, it is independent of r, but we indicate r in the notation beacuse we will later define a wall-and-chamber structure on $\Theta_{r,\mathbf{v}}$ that may depend on r.

Definition 2.1.1. ([47, 40]) Given a stability parameter $\theta \in \Theta_{r,\mathbf{v}}$ of $G(v)$, define the *Nakajima quiver variety* to be the GIT quotient

$$\mathfrak{M}_\theta(r,\mathbf{v}) := \left(\mu^{-1}(0)/\!\!/_{\chi(\theta)}G_{\mathbf{v}}\right)_{\mathrm{red}} = \left(\operatorname{Proj}\bigoplus_{k\geq 0}\mathbb{C}[\mu^{-1}(0)]^{k\chi(\theta)}\right)_{\mathrm{red}}$$

Note in particular that we give $\mathfrak{M}_\theta(r,\mathbf{v})$ the reduced scheme structure.

Remarks 2.1.2.

- Note that we reverse the order of the dimension vectors compared to the standard notation appearing in (for example) [40, 64] — we put the framing rank r first.
- Some authors replace the exponent θ_i in (2.1) by $-\theta_i$, and make corresponding sign changes later. This happens, for instance, in [52].
- It is possible to replace $\mu^{-1}(0)$ appearing in Definition 2.1.1 by $\mu^{-1}(\lambda)$ for certain $0 \neq \lambda \in \mathfrak{g}$. This leads, for instance, to moduli spaces of sheaves on *noncommutative* projective planes, see [3].

We will for Chapters 3, 5 and 6 focus on the case $\mathbf{v} = n\delta$ for some integer n.

The structure of these Nakajima quiver varieties is in many ways well understood. We have for instance the following very useful result:

Lemma 2.1.3. *For all $\theta \in \Theta_{\mathbf{v}}$, the scheme $\mathfrak{M}_\theta(r,\mathbf{v})$ is irreducible and normal, with symplectic singularities.*

Proof. See Bellamy and Schedler [6, Theorem 1.2, Proposition 3.21]. ∎

2.2 The preprojective algebra

Let $R(\Gamma)$ denote the representation ring of Γ. Then $\mathbb{Z}^{Q_0} \cong \mathbb{Z} \oplus R(\Gamma)$ considered as $\mathbb{Z}$-modules.

Again, let $v = \rho_\infty + \mathbf{v} \in \mathbb{Z}^{Q_0}$ be nonnegative, and choose some $\theta \in \Theta_{r,\mathbf{v}}$.

Let $\mathbb{C}Q$ be the path algebra of Q. Let $\mathcal{J}$ be the $\mathbb{C}Q$-ideal generated by the expression

$$\sum_{a \in Q_1} \epsilon(a) a\bar{a}. \tag{2.2}$$

We will call $\mathcal{J}$ the (ideal of) *preprojective relations*.

Definition 2.2.1. Set $\Pi := \mathbb{C}Q/\mathcal{J}$. This is the *preprojective algebra* determined by Q.

The preprojective algebra Π does not depend on the choice of the map ϵ [18, Lemma 2.2].

Equivalently, multiplying both sides of this relation by the vertex idempotent of $\mathbb{C}Q$ at vertex i shows that Π can be presented as the quotient of $\mathbb{C}Q$ by the ideal

$$\left(\sum_{\mathrm{h}(a)=i} \epsilon(a) a\bar{a} \mid i \in Q_0 \right). \tag{2.3}$$

A Π-module is then, up to isomorphism, equivalent to the data of a Q-representation on which the preprojective relations $\mathcal{J}$ act by 0, and we will call such a representation a $(Q, \mathcal{J})$-representation.

Given a Π-module M, we write M_i for the vector space which is the part of M supported at the vertex i. By the dimension of a Π-module M, we shall always mean the dimension *vector* $(\dim_0 M, \ldots, \dim_s M) = (\dim M_0, \ldots, \dim M_s)$ of M, and not its dimension as a vector space. For any $\theta \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{N}^{Q_0}, \mathbb{Q})$ and any Π-module M (or Q-representation V), we will write $\theta(M), \theta(V)$ for $\theta(\dim M), \theta(\dim V)$ respectively.

Given a stability parameter θ, recall that a Π-module M is θ-*stable*, respectively θ-*semistable*, if there is no submodule $U \subset M$ such that $\theta(U) \leq 0$, respectively $\theta(U) < 0$. A θ-*polystable* module is a direct sum of θ-stable modules.

Two θ-semistable modules M, N are said to be S_θ-*equivalent*, if there are composition series of θ-semistable modules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{k_1} = M, \qquad 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{k_2} = N$$

such that

1. Every M_i/M_{i-1} and every N_j/N_{j-1} is a θ-stable Π-module, and
2. The polystable modules $\bigoplus_{i=1}^{k_1} M_i/M_{i-1}$ and $\bigoplus_{j=1}^{k_2} N_j/N_{j-1}$ are isomorphic.

A one-dimensional Π-module will be called a *vertex simple* module.

It will be useful to extend S_θ-equivalence to Π-modules of different dimensions.

Definition 2.2.2. Let $I \subset Q_0 \setminus \{\infty\} = Q_{\Gamma,0}$, and suppose that θ is a stability parameter such that $\theta(\rho_i) = 0$ if and only if $i \in I$. Now let M, N be two θ-semistable Π-modules, not necessarily of the same dimension. Let $\widetilde{M}, \widetilde{N}$ be the θ-polystable modules respectively S_θ-equivalent to M and N.

We define M and N to be *R_θ-equivalent* if there are Π-modules M', N' with $M'_j = N'_j = 0$ if $j \notin I$, such that M', N' are (possibly empty) direct sums of vertex simple Π-modules, and that there is an isomorphism

$$\widetilde{M} \oplus M' \simeq \widetilde{N} \oplus N'.$$

It follows that if two modules of the same dimension vector are R_θ-equivalent, they are S_θ-equivalent.

Remark 2.2.3. Usually, the notions of (semi)stability and S-equivalence are used without direct reference to the stability condition. But since we will be considering more than one stability condition, we use notation keeping track of the stability conditions where confusion is possible.

We say that $\theta \in \Theta_{r,\mathbf{v}}$ is *generic* if every θ-semistable Π-module of dimension $(1, \mathbf{v})$ is θ-stable.

The next result provides an alternative interpretation of quiver varieties, which we will be using throughout. (Recall that r is part of our definition of the quiver Q.)

Lemma 2.2.4 ([40, Prop. 8], [36, Proposition 5.3])**.** *There is an isomorphism*

$$\mathfrak{M}_\theta(r, \mathbf{v}) \cong \mathcal{M}od_\Pi((1, \mathbf{v}), \theta)$$

where the right-hand space parametrises S_θ-equivalence classes of θ-semistable Π-*modules of dimension* $(1, \mathbf{v})$*. As such it is a coarse moduli space of θ-semistable* Π-*modules of dimension* $(1, \mathbf{v})$*. If θ is generic, it is a* fine *moduli space.*

This viewpoint was introduced in Kronheimer and Nakajima [39] and studied further by Nakajima [47].

When θ is generic, $\mathfrak{M}_\theta(r, \mathbf{v})$ carries a universal family, i.e., a tautological locally-free sheaf

$$\mathcal{R} \coloneqq \bigoplus_{i \in Q_0} \mathcal{R}_i$$

together with a $\mathbb{C}$-algebra homomorphism $\phi\colon \Pi \to \mathrm{End}(\mathcal{R})$, where $\mathcal{R}_\infty$ is the trivial bundle on $\mathfrak{M}_\theta$ and where $\mathrm{rk}(\mathcal{R}_i) = n \dim(\rho_i)$ for $i \geq 0$.

Finally, we will very often make the following slight abuse of terminology: When writing statements like "a θ-stable Π-module $M \in \mathfrak{M}_\theta(r, \mathbf{v})$," we mean that M is a stable Π-module representing the isomorphism class of Π-modules corresponding to a point $x \in \mathfrak{M}_\theta(r, \mathbf{v})(\mathbb{C})$. We will use similar language for other fine moduli spaces appearing in this thesis.

2.2.1 Other algebras arising from the McKay graph

We briefly describe some other algebras arising from the McKay graph of Γ.

Choosing a basis of $\mathbb{C}^2$, we can write the symmetric algebra of the dual vector space $\mathbb{C}^{2\vee}$ as a polynomial ring $R = \mathbb{C}[x, y]$. The group Γ acts dually on R: for $g \in \Gamma$ and $f \in R$, we have $(g \cdot f)(v) = f(g^{-1}v)$ for all $v \in \mathbb{C}^2$. Let R^Γ denote the Γ-invariant subring of R. Recall that we listed the irreducible representations of Γ as $\{\rho_0, \rho_1, \dots, \rho_s\}$, where ρ_0 is the trivial representation. The decomposition of $R = \mathbb{C}[x, y]$ as an R^Γ-module can be written [27, Proposition 4.1.15]

$$R \cong \bigoplus_{0 \le i \le s} R_i \otimes_{\mathbb{C}} \rho_i, \tag{2.4}$$

where $R_i \coloneqq \mathrm{Hom}_\Gamma(\rho_i, R)$ is an R^Γ-module; note that $R_0 = R^\Gamma$.

In addition to the preprojective algebra Π, we now define two other algebras from the unframed McKay quiver Q_Γ and the framed McKay quiver Q.

Assume $r = 1$, and let A be the quotient of the preprojective algebra Π by the two-sided ideal generated by the class of $\bar{b}$, the unique arrow with head at the ∞-vertex:

$$A \coloneqq \Pi/(\bar{b}).$$

Third, let $\mathbb{C}Q_\Gamma$ denote the path algebra of the (unframed) McKay quiver Q_Γ. The *preprojective algebra of* Q_Γ, denoted Π_Γ, is defined as the quotient of $\mathbb{C}Q_\Gamma$ by an ideal defined similarly to that from (2.2) for the quiver Q_Γ.

Since $\mathbb{C}Q_\Gamma$ is a subalgebra of $\mathbb{C}Q$, we use the same symbol $e_i \in \mathbb{C}Q_\Gamma$ for the idempotent corresponding to the trivial path at vertex i of Q_Γ.

Remark 2.2.5. For each vertex $i \in Q_0$ or $i \in Q_{\Gamma,0}$, we will also denote by e_i the class in A, respectively Π_Γ, of the idempotent at the vertex i. Note that Π_Γ is not a subalgebra of Π. On the other hand, we will see that Π_Γ is isomorphic to a subalgebra of A in Lemma 3.2.2. In particular, the classes e_i, $0 \le i \le s$ are also contained in Π_Γ.

2.2.2 Morita equivalence

The skew group algebra $S \coloneqq R * \Gamma = \mathbb{C}[x, y] * \Gamma$ of the finite subgroup $\Gamma \subset \mathbf{SL}(2, \mathbb{C})$ contains the group algebra $\mathbb{C}\Gamma$ as a subalgebra. For $0 \le i \le s$, choose an idempotent $f_i \in \mathbb{C}\Gamma$ such that $\mathbb{C}\Gamma f_i \cong \rho_i$.

Lemma 2.2.6. *There is an isomorphism $Sf_0 \cong R$ of S-R^Γ-bimodules.*

Proof. As in [18, Lemma 1.1], S has a basis given by all elements of the form $x^j y^k g$, where $j, k \geq 0$ and $g \in \Gamma$. As f_0 is invariant under multiplication by any $g \in \Gamma$, the subspace Sf_0 has a basis given by elements of the form $x^j y^k f_0$ for $j, k \geq 0$. This gives the required isomorphism of vector spaces which is readily seen to be an isomorphism of S-R^Γ-bimodules. ∎

To recall the key result, define $f = f_0 + \cdots + f_s$.

Proposition 2.2.7. *The skew group algebra S and the preprojective algebra Π_Γ are Morita equivalent via an isomorphism $fSf \cong \Pi_\Gamma$; explicitly, the equivalence on left modules is*

$$\begin{array}{rcl} \Phi\colon S\text{-mod} & \longrightarrow & \Pi_\Gamma\text{-mod} \\ M & \mapsto & fM. \end{array}$$

Moreover, there are $\mathbb{C}$-algebra isomorphisms $S \cong \operatorname{End}_{R_0}(R)$ and $\Pi_\Gamma \cong \operatorname{End}_{R_0}\left(\bigoplus_{0\leq i\leq s} R_i\right)$.

Proof. The Morita equivalence is a bi-product of the proof of [56, Proposition 2.13], and may have been first stated explicitly in [18, Theorem 0.1]. The algebra isomorphism $fSf \cong \Pi_\Gamma$ that in particular sends f_i to e_i for $0 \leq i \leq s$ is described in [18, Theorem 3.4]. The description of S as $\operatorname{End}_{R_0}(R)$ follows from [1], spelled out in [42, Theorem 5.12], while the description of Π_Γ follows by applying the isomorphism $fSf \cong \Pi_\Gamma$; see for example Buchweitz [12]. ∎

Corollary 2.2.8. *For any $0 \leq i \leq s$, there is an isomorphism $f_i S f_i \cong e_i \Pi_\Gamma e_i$ of $\mathbb{C}$-algebras, as well as an isomorphism $R_i \cong e_i \Pi_\Gamma e_0$ of $e_i\Pi_\Gamma e_i$-$e_0\Pi_\Gamma e_0$-bimodules.*

Proof. Apply f_i to both sides of fSf while simultaneously applying e_i to both sides of Π_Γ; the algebra isomorphism $fSf \cong \Pi_\Gamma$ then restricts to the algebra isomorphism $f_i S f_i \cong e_i \Pi_\Gamma e_i$. For the second statement, apply e_i and e_0 to the left and right respectively of the isomorphism $\Pi_\Gamma \cong \operatorname{End}_{R_0}\left(\bigoplus_{0\leq i\leq s} R_i\right)$ to obtain $R_i \cong \operatorname{Hom}_{R_0}(R_0, R_i) \cong e_i \Pi_\Gamma e_0$ as required. ∎

We finally look at how the Morita equivalence Φ acts on dimension vectors.

Corollary 2.2.9. *Given an S-module of the form R/J, with dimension vector $\sum_{0\leq i\leq s} v_i \rho_i$, its image under Φ is a Π_Γ-module of the form $\Pi_\Gamma e_0/fJ$ with dimension vector $(v_i) \in \mathbb{Z}^{s+1}$.*

Proof. Applying Lemma 2.2.6 shows that Φ sends the polynomial ring $R \cong Sf_0$ to $fSf_0 = (fSf)f_0 = \Pi_\Gamma e_0$. It follows that for any S-submodule $J \subseteq R$, the quotient R/J is an S-module whose image under Φ is $\Pi_\Gamma e_0/fJ$. Finally, Φ induces a $\mathbb{Z}$-linear isomorphism between the Grothendieck groups of the categories of finite-dimensional left modules over S and Π_Γ which we identify with the representation ring $\mathsf{Rep}(\Gamma)$ and with $\mathbb{Z}^{s+1}$ respectively. ■

2.3 Another construction of Nakajima quiver varieties

There is another construction of the varieties $\mathfrak{M}_\theta(r, \mathbf{v})$, which we will make use of in Chapter 5. We follow the discussion in [64].

Instead of constructing the quiver Q, we could also construct a quiver $Q' = (Q'_0, Q'_1)$, consisting of Q_Γ together with the framing vertex ∞ and just two additional arrows: one $a\colon \infty \to 0$ and another $\overline{a}\colon 0 \to \infty$. Then the data of a representation R' of Q' such that $\dim R'_\infty = r$ is equivalent to that of a representation R of Q such that $\dim r_\infty = 1$.

The McKay quiver Q_Γ is constructed by taking for each vertex an irreducible representation of Γ, and an arrow $\rho_i \to \rho_j$ if $\rho_j \subset L \otimes \rho_i$. This implies that the data of an arbitrary Γ-representation V together with a Γ-equivariant map $B\colon V \to V \otimes L$ is equivalent to that of a Q_Γ-representation.

In particular, choose an r-dimensional vector space W on which Γ acts trivially, and let $V = \bigoplus_{i \in Q_{\Gamma,0}} \mathbb{C}^{\mathbf{v}_i} \otimes \rho_i$. Then the data of an element $x \in \mu^{-1}(0) \subset M(v) = M(\rho_\infty + \mathbf{v})$ can be written as a tuple

$$x = (i_x, j_x, B_x),$$

where

$$i_x\colon W \to \bigwedge^2 L \otimes V, \quad j_x\colon V \to W, \quad B_x\colon V \to L \otimes V$$

are Γ-equivariant maps.

If we let $M(r, \mathbf{v})$ be the space of such tuples for a fixed dimension vector $(r, \mathbf{v})$, the moment map μ can be replaced with the map

$$\mu'\colon M'(v) \to \operatorname{Hom}(V, \wedge^2 L \otimes W), \quad (i_x, j_x, B_x) \mapsto (B_x \wedge B_x - i_x \circ j_x).$$

As before, $M'((r, \mathbf{v}))$ carries a natural action by the group $G_\mathbf{v}$, and the quiver variety $\mathfrak{M}_\theta(r, \mathbf{v})$ can also be defined as the GIT quotient of $\mu'^{-1}(0)$ by this action:

$$\mathfrak{M}_\theta(r, \mathbf{v}) = \mu'^{-1}(0) /\!\!/_{\chi(\theta)} G_\mathbf{v}.$$

Given a Q'-representation M, $M \in \mu^{-1}(0)$ if and only if a similar ideal of preprojective relations acts by 0 on M. To reduce notational clutter, we shall also write this ideal as $\mathcal{J}$, and similarly speak of $(Q', \mathcal{J})$-representations.

Since we shall have to work with Q' -representations of different dimension vectors, it will be convenient to refer explicitly to the Γ-representations V, W. For this reason, we shall write $x = (V_x, W_x, i_x, j_x, B_x)$.

The construction here described was the original definition of quiver varieties ([47]). Our definition using $2r$ framing arrows to set $\dim M_\infty = 1$ for a Π-module M is usually called the *Crawley-Boevey trick*, first explicitly noted in [19].

2.4 Stability parameters

We now go back to the description of $\mathfrak{M}_\theta(r, \mathbf{v})$ as a moduli space for θ-semistable Π-modules.

We will work with several different stability parameters. Let us first, for completeness, give a definition of stability independent of any θ.

Definition 2.4.1. Let M be a Π-module, and let M_0 be the component of M supported at the vertex 0.

A Π-module M is *stable* if there is no strict submodule $N \subsetneq M$ such that $M_0 \subset N$.

Remark 2.4.2. We can translate this into a notion of stability for Q'-representations. Let $M = (W, V, i, j, B) \in \mu^{-1}(0)$ be a Q'-representation. Then M is *stable* if there is no strict subspace $U \subsetneq V$ such that $i(W) \subset U$ and $B(U) \subset L \otimes U$.

If θ is a stability parameter such $\theta(\rho_i) > 0$ for all i, then a Π-module is stable if and only if it is θ-stable.

2.4.1 Wall-and-chamber structure

The set of stability conditions $\Theta_{r,\mathbf{v}}$ admits a preorder $\geq$, where $\theta \geq \theta'$ iff every θ-semistable Π-module is θ'-semistable. This determines a wall-and-chamber structure ([22, 63]), where $\theta, \theta' \in \Theta_{r,\mathbf{v}}$ lie in the relative interior of the same cone if and only if both $\theta \geq \theta'$ and $\theta' \geq \theta$ hold in this preorder, in which case $\mathfrak{M}_\theta(r, \mathbf{v}) \cong \mathfrak{M}_{\theta'}(r, \mathbf{v})$. The interiors of the top-dimensional cones in $\Theta_{r,\mathbf{v}}$ are chambers, while the codimension-one faces of the closure of each chamber are walls. Then $\theta \in \Theta_{r,\mathbf{v}}$ is generic with respect to $\mathbf{v}$, if it lies in some GIT chamber. The vector $(1, \mathbf{v})$ is indivisible, so again, [36, Proposition 5.3] implies that for generic $\theta \in \Theta_{r,\mathbf{v}}$ the quiver variety $\mathfrak{M}_\theta(r, \mathbf{v})$ is the fine moduli space of isomorphism classes of θ-stable Π-modules of dimension vector $(1, \mathbf{v})$.

Let us write

$$C^+_{r,\mathbf{v}} = \{\theta \in \Theta_{r,\mathbf{v}} \mid \theta(\rho_i) > 0 \text{ for all } i.\}$$

This may not in general be a chamber in the wall-and-chamber structure on $\Theta_{r,\mathbf{v}}$, but it is contained in one, and if $\mathbf{v} = (n\delta)$ it is a genuine chamber – see [5, Example 2.1] or [49, Section 2.8] for the case $r = 1$, and Appendix B for $r > 1$.

Every face of the closed cone $C^+_{r,\mathbf{v}}$ is of the form

$$\sigma_I := \left\{\theta \in \overline{C^+_{r,\mathbf{v}}} \mid \theta(\rho_i) > 0 \text{ if and only if } i \in I, \text{ and } \theta(\rho_i) = 0 \text{ otherwise}\right\}$$

for some subset $I \subseteq \{0, \ldots, s\}$. When $\mathbf{v} = n\delta$, these faces are contained in walls of the wall-and-chamber structure on $\Theta_{n\delta}$, though this need not be the case for arbitrary $\mathbf{v} \in \mathbb{N}^{s+1}$. In any case, the parameter

$$\theta_I(\rho_j) := \begin{cases} -\sum_{i \in I} v_i & \text{for } j = \infty \\ 1 & \text{if } j \in I \\ 0 & \text{if } j \in \{0, 1, \ldots, s\} \smallsetminus I \end{cases} \tag{2.5}$$

lies in the relative interior of the face σ_I.

We also set $\delta^\perp = \{\theta \in \Theta_{r,\mathbf{v}} \mid \theta(0, \delta) = 0\}$.

Given a generic $\theta' \in \delta^\perp$, there is a unique chamber $C^-_{r,\mathbf{v}} \subset \Theta_{r,\mathbf{v}}$, such that $\theta' \in \overline{C^-_{r,\mathbf{v}}}$, and for any $\theta \in C^-_{r,\mathbf{v}}$, $\theta(0, \delta) > 0$ [52, p. 707].

There is a Namikawa-Weyl group W acting on $\Theta_{r,\mathbf{v}}$ by reflections (see, e.g., [6, Section 7]). In [5], Bellamy and Craw determine a fundamental domain for the action of W, and show that for generic θ, θ', $\mathfrak{M}_\theta(1, n\delta) \cong \mathfrak{M}_{\theta'}(1, n\delta)$ if and only if there is a $w \in W$ such that $\theta = w(\theta')$. More generally, for arbitrary $\mathbf{v} \in \mathbb{N}^{s+1}$, it is possible to compute the wall-and-chamber structure on $\Theta_{(1,\mathbf{v})}$ by applying recent work of DeHority [20, Proposition 4.8]. However, neither DeHority's not Bellamy and Craw's techniques extend to the cases $r > 1$.

In [47, Theorem 2.8], Nakajima describes certain walls $r^\perp$ in $\Theta_{r,\mathbf{v}}$ such that any $\theta \notin r^\perp$ is generic, i.e., that $\mathfrak{M}_\theta(r, \mathbf{v})$ is smooth. However, it is possible that θ remains generic when moved into some of the walls $r^\perp$.

From now on, for any choice of r and $\mathbf{v}$, we will denote by θ any stability parameter in the cone $C^+_{r,\mathbf{v}}$ except where otherwise indicated. Thus we can use the phrases 'stable module' and 'θ-stable module' We will often leave the framing rank and dimension vector implicit and simply write $C^+ = C^+_{r,\mathbf{v}}$. The next statement, which appeared originally in [51, Section 2], illustrates the importance of $C^+_{r,\mathbf{v}}$ when $r = 1$.

Proposition 2.4.3. *For $\mathbf{v} \in \mathbb{N}^{s+1}$ and $\theta \in C^+_{1,\mathbf{v}}$, there is an isomorphism*

$$\mathrm{Hilb}^{\mathbf{v}}([\mathbb{C}^2/\Gamma]) \cong \mathfrak{M}_\theta(1, \mathbf{v}),$$

where $\mathrm{Hilb}^{\mathbf{v}}([\mathbb{C}^2/\Gamma])$ is the subscheme of $\mathrm{Hilb}(\mathbb{C}^2)$ parametrising Γ-invariant ideals $J \subset \mathbb{C}[x, y]$ such that, as Γ-representations, $\mathbb{C}[x, y]/I \xrightarrow{\sim} \bigoplus \rho_i^{\oplus \mathbf{v}_i}$.

Proof. In the special case $\mathbf{v} = n\delta$, the statement is proved in [40, Theorem 4.6]. However, the argument given there applies for an arbitrary dimension vector $\mathbf{v}$; note in particular that the auxiliary [40, Corollary 3.5] does not need any assumption on the dimension vector (and we have that $\mathrm{Hilb}^v([\mathbb{C}^2/\Gamma])$ is non-empty if and only if $\mathfrak{M}_\theta(1, \mathbf{v})$ is non-empty). ∎

In the case $I = \{0\}$, which is the focus of Chapters 3, 5 and 6, we slightly abuse notation and set

$$\theta_0 = \theta_{\{0\}} = (-n, 1, 0, 0 \ldots, 0),$$

i.e., $\theta_0(\rho_\infty) = -n, \theta_0(\rho_0) = 1, \theta_0(\rho_i) = 0$ for other i, where $n = \mathbf{v}_0$. We fix this parameter for the rest of the thesis.

Whether $C^+_{r,\mathbf{v}}$ is a chamber or not, there are always morphisms $\mathfrak{M}_\theta(r, \mathbf{v}) \to \mathfrak{M}_{\theta_I}(r, \mathbf{v})$ induced by variation of GIT parameter. These morphisms and the interplay between various stability parameters will be very important.

Because the parameters θ_I lie on the boundary of C^+, they have the property that: A stable Π-module M with $\dim_\infty = 1, \dim_i = \mathbf{v}_i$ for all $i \in I$ is θ_I-semistable.

2.4.2 Surjectivity of morphisms induced by GIT

We shall need some results about the morphisms induced by variation of the GIT parameter θ:

Lemma 2.4.4. *Set $r = 1$, and let $\theta \geq \theta'$. The morphism $\pi \colon \mathfrak{M}_\theta(1, v) \to \mathfrak{M}_{\theta'}(1, v)$ obtained by variation of GIT quotient is a projective morphism of varieties over the affine quotient $\mathfrak{M}_0(1, v)$.*

Proof. This is well-known (see for instance [63, Theorem 2.3]). ∎

Lemma 2.4.5. *Let $r, n \in \mathbb{N}$. Then*

1. $\dim \mathfrak{M}_\theta(r, n\delta) = \dim \mathfrak{M}_{\theta_0}(r, n\delta) = 2rn$,

2. *the projective morphism* $\mathfrak{M}_\theta(r, n\delta) \to \mathfrak{M}_{\theta_0}(r, n\delta)$ *induced by varying the parameter* θ *to* θ_0 *is a resolution of singularities,*

3. $C^+_{r,n\delta}$ *is a genuine chamber in* $\Theta_{r,n\delta}$.

We postpone the proof to Appendix B, as we shall need explicit interpretations of the quiver varieties for the proof.

2.4.3 The concentrated module

We now introduce the *concentrated module* of an R_{θ_I}-equivalence class, which we will see uniquely characterises the class.

Let $[M]$ be an S-equivalence class of θ_I-semistable Π-modules of dimension vector $(1, \mathbf{v})$.

Then $[M]$ has a θ_I-polystable member $\widetilde{M}$, which is unique up to isomorphism.

Definition 2.4.6. The *concentrated module (associated to* $[M]$*)* is the unique summand M_{con} of $\widetilde{M}$ such that $\dim_\infty M_{\mathrm{con}} = 1$.

Corollary 2.4.7. *We have* $\dim_i M_{\mathrm{con}} = v_i$ *for* $i \in I$.

Proof. Since every summand N of $\widetilde{M}$ satisfies $\theta_I(N) = 0$, the same must hold for M_{con}. Since $\dim_\infty M_{\mathrm{con}} = 1$, $\theta_I(\dim M_{\mathrm{con}}) = 0$, and $\theta_I(\rho_i) > 0$ precisely for $i \in I$, we must have $\dim_i M_{\mathrm{con}} = v_i$ for $i \in I$. ■

Remark 2.4.8. In a θ_I-polystable module, all summands other than the concentrated module are 1-dimensional vertex simples. To see this, note that removing the vertices ∞ and its incident edges from the framed McKay graph leaves us with an affine Dynkin diagram. Removing the nonzero number of vertices in I and their incident edges thus leaves us with a collection of finite-type Dynkin diagrams.

Then, since a stable summand $N \neq M_{\mathrm{con}}$ must satisfy $\dim_i N = 0$ for $i \in \infty \cup I$ (as $I \neq \emptyset$), we can identify it with a simple module of the preprojective algebra of a quiver of finite type. But such modules are one-dimensional by [61, Lemma 2.2].

We move from S-equivalence to R-equivalence:

Lemma 2.4.9. *Let* M, N *be two* R_{θ_I}*-equivalent* Π*-modules. The concentrated modules of their* S_{θ_I}*-equivalence classes are isomorphic.*

Proof. This immediately follows from Definition 2.2.2 and Remark 2.4.8. ■

It follows that every R_{θ_I}-equivalence class is characterised by a concentrated module, which is unique up to isomorphism. We will often speak of *the* concentrated representation instead of the isomorphism class of such representations.

Corollary 2.4.10. *The concentrated module of an R_{θ_I}-equivalence class is both stable and θ_I-stable.*

Proof. Clear. ■

A reason for introducing the notion of concentrated module is the following result:

Lemma 2.4.11. *Let M be a θ-stable Π-module. Consider M as a θ_I-semistable module, and let M_{con} be the concentrated representation of its R_{θ_I}-equivalence class. Then M_{con} is a quotient of M.*

Proof. Note that M_{con} is, by definition, a subquotient of M. If M_{con} is a quotient of a strict submodule $N \subsetneq M$, we must have $\dim_\infty N = \dim_\infty M = \dim_\infty M_{\mathrm{con}} = 1$. But then $\theta(N) < 0$, contradicting stability of M. ■

2.5 Coherent sheaves on projective Deligne-Mumford stacks

In this short section, we collect some of the background material on stacks, including the important concept of a *projective stack*. We also discuss properties of sheaves on such stacks.

Our basic references for stack theory are [54] and [41].

For projective stacks and moduli spaces of their sheaves, we use [11], [53], and [38].

2.5.1 Deligne-Mumford stacks

We will not require the full machinery of stacks in this thesis – all stacks appearing will be *Deligne-Mumford*. Intuitively, what distinguishes a Deligne-Mumford stack from a scheme is that the points of a Deligne-Mumford stack may have nontrivial finite (and reduced) stabiliser groups.

For the full definition of a Deligne-Mumford stack, see [41, Définition 4.1]. We will not reproduce all of it here, but an important aspect is that a Deligne-Mumford stack

$\mathcal{Y}$ will always have an étale cover (i.e. a surjective étale morphism) $U \to \mathcal{Y}$, where U is a scheme.

This allows us to define many scheme properties for Deligne-Mumford stacks: If P is some property of schemes that is *stable in the smooth topology* (see [54, Lemma 5.1.1] for the terminology), we will say that $\mathcal{Y}$ satisfies P if U does. This holds for instance for $P =$'locally of finite type', $P =$'regular'.

If P is some property of scheme morphisms which is both stable and *local on domain* in the smooth topology, we can use a similar idea (see [54, Definition 8.2.6]) to define when a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ has P.

In the special case when $f\colon \mathcal{X} \to Y$ is a morphism from a Deligne-Mumford stack to a *scheme*, this definition reduces to saying that f has P when the composed morphism $U \to \mathcal{X} \to Y$ has P, for an étale cover $u\colon U \to \mathcal{X}$ with U a scheme.

In the same case, given a sheaf $\mathscr{F}$ on $\mathcal{X}$, we say that $\mathscr{F}$ is flat over Y if the sheaf $u^*\mathscr{F}$ on U is flat over Y (see [Stacks, tag 02JZ] for why this works).

For a straightforward example of a Deligne-Mumford stack, let X be a scheme with a group action $\sigma\colon X \times G \to X$, where G is finite.

Then we can define a group quotient stack $[X/G]$ (see [54, p. 183]), which has the property that giving a morphism $Z \to [X/G]$ is the same as giving a G-principal bundle $E \to Z$ and a G-equivariant morphism $E \to X$. (In fact, we can take this as part of the definition of $[X/G]$.) Especially, there is a quotient morphism $q\colon X \to [X/G]$ corresponding to the trivial G-bundle $X \times G \to X$, giving a commutative diagram

$$\begin{tikzcd} X \times G \arrow[r, "\sigma"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "q"] \\ X \arrow[r, "q"] & {[X/G]} \end{tikzcd} .$$

Such a stack $[X/G]$ is Deligne-Mumford (see, e.g., [21, Theorem 4.21]).

We now record two definitions and a Lemma that will let us compute sheaf cohomology on some Deligne-Mumford stacks.

Definition 2.5.1 ([54, Definition 11.3.2])**.** A separated Deligne-Mumford stack $\mathcal{Y}$ of finite type is *tame* if, for every morphism $x\colon \operatorname{Spec} k \to \mathcal{Y}$ where k is algebraically closed, the automorphism group of x has order invertible in k.

Definition 2.5.2 ([54, Definition 11.1.1])**.** Given a Deligne-Mumford stack $\mathcal{X}$, a *coarse moduli space* for $\mathcal{X}$ is a morphism $t\colon \mathcal{X} \to X$, where X is an algebraic space, such that

- any other morphism from $\mathcal{X}$ to an algebraic space factors through t,

- for every algebraically closed field k, the map of sets $|\mathcal{X}|(k) \to X(k)$ induced by t is bijective, where $|\mathcal{X}|$ is the set of isomorphism classes of morphisms $\operatorname{Spec} k \to \mathcal{X}$.

Lemma 2.5.3 ([54, Proposition 11.3.4]). *Let $\mathcal{X}$ be a tame locally finite Deligne-Mumford stack with finite diagonal, and with a coarse moduli space $f\colon \mathcal{X} \to X$. Then the functor f_* is exact.*

In this thesis, no algebraic spaces other than schemes will appear - all our coarse moduli spaces will luckily turn out to be schemes.

2.5.2 Descent

We will make heavy use of descent of coherent sheaves along group quotients, in the sense that given a scheme X with an action of a group G, there is a canonical equivalence of categories $\mathsf{Coh}([X/G]) \cong \mathsf{Coh}_G(X)$.

This is standard (see, e.g., [54, Exercise 9.H]), but let us show it explicitly: Let X be a sheaf acted on by a group G, and recall (see [45, Definition 1.6]) that a coherent sheaf $\mathscr{F}$ on X is G-equivariant if there is an isomorphism of $\mathcal{O}_{X\times G}$-modules

$$\iota\colon \mathrm{pr}^*\mathscr{F} \cong \sigma^*\mathscr{F},$$

such that there is an equality of $\mathcal{O}_{X\times G\times G}$-module morphisms

$$(\mathrm{id}_X \times m)^*(\iota) = \mathrm{pr}_{12}^*\iota \circ (\sigma \times \mathrm{id}_G)^*\iota.$$

Here $\sigma\colon X \times G \to X$ denotes the group action, $m\colon G \times G \to G$ the multiplication, and $\mathrm{pr}\colon X \times G \to X, \mathrm{pr}_{1,2}\colon X \times G \times G \to X \times G$ the projections.

Since $X \times G = X \times_{[X/G]} X$, this is exactly the condition required for $\mathscr{F}$ to descend along the quotient morphism $t\colon X \to [X/G]$ (see, for instance, [54, Section 4.3]). Similarly, a morphism of equivariant sheaves $f\colon \mathscr{F} \to \mathscr{F}'$ on X satisfies $pr^*f = \sigma^*f$, and so descends along t. Then the equivalence of categories $\mathsf{Coh}([X/G]) \cong \mathsf{Coh}_G(X)$ is given by $t^*(-)$, where $t\colon X \to [X/G]$ is the quotient morphism.

2.5.2.1 Descent on scheme-theoretic quotients

We now record a statement about scheme-theoretic quotients, and show that in some cases these agree with the stack-theoretic quotients. In these cases, we can give an explicit description of how sheaves descend:

Proposition 2.5.4. *Let G be a finite group acting on a scheme X, such that the orbit of every closed point is contained in an open affine subscheme.*

Then:

1. *There is a finite G-invariant morphism $q\colon X \to X/G$ which is a categorical quotient, i.e. with the universal property that any G-invariant morphism $X \to Y$ of schemes factors uniquely through X/G. The structure sheaf of X/G satisfies*
$$\mathcal{O}_{X/G}(U) = (\mathcal{O}_X(U))^G$$
for any open $U \subset X/G$.

2. *If we also assume that the action of G is free, the stack quotient $[X/G]$ is represented by X/G. Furthermore, q is a flat morphism of degree $|G|$, and the functor*
$$q_*(-)^G\colon \mathsf{Coh}_G(X) \to \mathsf{Coh}(X/G)$$
defined by
$$U \mapsto q_*(\mathscr{F})^G(U) = (q_*(\mathscr{F})(U))^G = (\mathscr{F}(q^{-1}(U)))^G$$
is an equivalence. The inverse of $q_*(-)^G$ is given by q^*.

[1]

Proof. This is a special case of [46, Theorem 1, p. 111]. ∎

We will also need to know how sheaf cohomology behaves under descent along group quotients:

Lemma 2.5.5. *Let X be a scheme, acted on by a finite group G. Let $\mathscr{F}$ be a G-equivariant sheaf on X. Write $\mathscr{F}'$ for the sheaf on $[X/G]$ obtained from $\mathscr{F}$ by descent. Then there is a spectral sequence*
$$E_2^{p,q} = H^p(G, H^q(X, \mathscr{F})) \Rightarrow H^{p+q}([X/G], \mathscr{F}').$$

Proof. See, e.g., [24, Theorem A.6].[2] ∎

For us, since we're working over a field of characteristic zero, the functor $M \mapsto M^G$ will always be exact, and we have $H^p([X/G], \mathscr{F}') = H^p(X, \mathscr{F})^G$.

---

[1]Note that there are schemes acted on by finite groups such that there is at least one orbit of a closed point not contained in an affine subscheme: In the line with double origin, with an action of $\mathbb{Z}/2\mathbb{Z}$ interchanging the origins, the two origins do not lie in an affine subscheme. For an example of a complete variety also satisfying this property, see [62].

[2]Their statement is in a slightly different context, but the proof works verbatim here.

Remark 2.5.6. It is very common to use the same notation for $\mathscr{F}$ and $\mathscr{F}'$ - the descent induces, after all, an equivalence of categories $\mathsf{Coh}_G(X) \cong \mathsf{Coh}([X/G])$. We will make the distinction in order to (hopefully) reduce confusion.

2.5.3 Projective stacks

Let $\mathcal{Y}$ be a Deligne-Mumford stack, with a coarse moduli space $f\colon \mathcal{Y} \to Y$.

Definition 2.5.7 ([38, Corollary 5.4, Definition 5.5])**.** The stack $\mathcal{Y}$ is *projective* if the following two conditions are satisfied:

- Y is projective,
- there is a locally free coherent sheaf $\mathscr{V}$ on $\mathcal{Y}$ such that for any point $x\colon \operatorname{Spec} k \to \mathcal{Y}$ with k algebraically closed, the action of the stabiliser group of x on the fibre $x^*\mathscr{V}$ is faithful.

Such a sheaf $\mathscr{V}$ is called a *generating sheaf* for $\mathcal{Y}$. (See [38, p. 10] for an explanation of the terminology.)

Remark 2.5.8. There are other equivalent definitions of what a projective stack is. For instance, we could replace the requirement that there exists a generating sheaf with the requirement that $\mathcal{Y}$ is a global quotient stack (see [38, Corollary 5.4, Definition 5.5]).

For the rest of the chapter, we fix an ample sheaf $\mathcal{O}_Y(1)$ on the coarse moduli space Y, and a generating sheaf $\mathcal{V}$ for $\mathcal{Y}$. (The pair $(\mathcal{O}_Y(1), \mathcal{V})$ is sometimes (e.g. in [11]) called a *polarisation* of the Deligne-Mumford stack $\mathcal{Y}$.)

Definition 2.5.9 ([11, Definition 2.26])**.** Let $\mathscr{F}$ be a coherent sheaf on a projective stack $\mathcal{Y}$. We define the *modified* Hilbert polynomial of $\mathscr{F}$ to be the function of $\nu \in \mathbb{Z}$

$$P_{\mathscr{V}}(\mathscr{F}, \nu) \coloneqq \chi(\mathcal{Y}, \mathscr{F} \otimes f^*\mathcal{O}_Y(\nu) \otimes \mathscr{V}^\vee) = \chi(Y, f_*(\mathscr{F} \otimes \mathscr{V}^\vee) \otimes \mathcal{O}_Y(\nu)).$$

We will usually write *Hilbert polynomial* to mean this modified Hilbert polynomial.

By the last equality (which follows from the projection formula applied to f), we see that $P_{\mathscr{V}}(\mathscr{F}, \nu)$ is indeed a polynomial in ν.

We shall write the Hilbert polynomial as

$$P_{\mathscr{V}}(\mathscr{F}, \nu) = \frac{\alpha_{\mathcal{V},d}\mathscr{F}}{d!}\nu^d + \frac{\alpha_{\mathcal{V},d-1}\mathscr{F}}{(d-1)!}\nu^{d-1} + \cdots + \alpha_{\mathcal{V},0}.$$

(Here $d = \dim \operatorname{Supp} \mathscr{F}$, which by [11, Proposition 2.22] equals $\dim \operatorname{Supp} f_*(\mathscr{F} \otimes \mathscr{V}^\vee)$.
)

Definition 2.5.10 ([53, Definition 4.2])**.** We say that a coherent sheaf $\mathscr{F}$ on $\mathcal{Y}$ is m-regular if $f_*(\mathscr{F} \otimes \mathscr{V}^\vee)$ is m-regular, i.e.,

$$H^i(\mathcal{Y}, \mathscr{F} \otimes \mathscr{V}^\vee \otimes f^*\mathcal{O}_Y(m-i)) = H^i(Y, f_*(\mathscr{F} \otimes \mathscr{V}^\vee)(m-i)) = 0$$

for all $i > 0$.

The *modified* Castelnuovo-Mumford regularity of $\mathscr{F}$ is the smallest integer m such that $\mathscr{F}$ is m-regular.

We will often write *regularity* to mean modified Castelnuovo-Mumford regularity.

Remark 2.5.11. It may seem strange to require the generating sheaf in order to define a Hilbert polynomial and the regularity of $\mathscr{F}$- why not simply consider the Hilbert polynomial $n \mapsto \chi(\mathscr{F} \otimes f^*\mathcal{O}(n))$, where $\mathcal{O}(1)$ is an ample sheaf on Y?

The reason is that such a definition would have undesirable properties. For instance, in the case where $\mathcal{Y} = [\operatorname{Spec}\mathbb{C}/G]$, i.e., the classifying space of G, with G some finite group, the coarse moduli space is simply $\operatorname{Spec}\mathbb{C}$. In this case, defining the Hilbert polynomial in the naïve way above for a sheaf $\mathscr{F}$ on $[\operatorname{Spec}\mathbb{C}/\Gamma]$, we would get a polynomial only depending on the Γ-invariants of $\mathscr{F}$.

For this reason, the existence of the generating sheaf – and especially the properties of the functor $F_{\mathscr{V}}(-)\colon \mathsf{Coh}(\mathcal{Y}) \to \mathsf{Coh}(Y)$ given by $\mathscr{F} \mapsto \mathscr{V}^\vee \otimes \mathscr{F}$ – play a crucial role in the construction of moduli spaces of sheaves on projective stacks. (See [53] for more on the generating sheaf, and [11], especially sections 3 and 4 in the latter.)

2.5.4 Properties of coherent sheaves on a projective Deligne-Mumford stack

Following [11, Definition 2.18], we make the following definition:

Definition 2.5.12. A coherent sheaf $\mathscr{F}$ on a projective stack $\mathcal{X}$ is *torsion-free* if all its nonzero subsheaves are of dimension $\dim \mathcal{X}$.

We also need a notion of *rank* for coherent sheaves on a projective stack $\mathcal{Y}$.

Definition 2.5.13. The *rank* $\operatorname{rk}\mathscr{F}$ of a coherent sheaf $\mathscr{F}$ on $\mathcal{Y}$ is the integer

$$\frac{\alpha_{\mathcal{V},d}\mathscr{F}}{\alpha_d\mathcal{O}_Y} = \frac{\alpha_d(f_*(\mathcal{V}^\vee \otimes \mathscr{F}))}{\alpha_d\mathcal{O}_Y},$$

where $\frac{\alpha_d\mathcal{O}_Y}{d!}$ and $\frac{\alpha_d(f_*(\mathcal{V}^\vee \otimes \mathscr{F}))}{d!}$ are the leading coefficients of the respective ordinary Hilbert polynomials of $\mathcal{O}_Y$ and $f_*(\mathcal{V}^\vee \otimes \mathscr{F})$ on Y.

We shall, however, be using an alternative formulation easier to work with:

Lemma 2.5.14. *Assume that there is an open immersion $V \hookrightarrow \mathcal{Y}$, where V is a scheme. Then a coherent sheaf $\mathscr{F}$ of $\mathcal{O}_{\mathcal{Y}}$-modules has rank $\operatorname{rk}\mathscr{F} = r$ if and only if there is an open substack $\mathcal{U} \subset \mathcal{Y}$ such that $(\mathscr{F} \otimes \mathscr{V}^\vee)|_{\mathcal{U}}$ is locally free of rank r.*

Proof. We have $r = \frac{\alpha_d(f_*(\mathcal{V}^\vee \otimes \mathscr{F}))}{\alpha_d \mathcal{O}_Y}$, which by definition is the rank of the sheaf $f_*(\mathcal{V}^\vee \otimes \mathscr{F})$ on Y. It is then a standard computation that on the locus where the sheaf is locally free, it is locally free of rank r. By our assumption, t restricted to V is an isomorphism. Then there is an open subscheme of V on which $\mathscr{F} \otimes \mathcal{V}^\vee$ is locally free, and we are done. ∎

Of course, if $\mathcal{V}$ is a line bundle, the rank of $\mathscr{F}$ (defined by Definition 2.5.13) equals the rank of $\mathscr{F}|_{\mathcal{U}}$ as a locally free sheaf.

2.6 Moduli spaces of framed sheaves on projective stacks

The theory of moduli spaces of framed sheaves on projective Deligne-Mumford stacks has been developed in [11], we summarise the main points.

Let $\mathcal{Y}$ be a normal irreducible projective Deligne-Mumford stack, with a coarse moduli scheme $\tau\colon \mathcal{Y} \to Y$.

Definition 2.6.1 ([11, Definitions 3.1, 3.3])**.** Let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathcal{Y}$. An $\mathscr{F}$*-framed* sheaf on $\mathcal{Y}$ is a pair $(\mathscr{G}, \phi_{\mathscr{G}})$, where $\mathscr{G}$ is a coherent sheaf on $\mathcal{Y}$, and $\phi_{\mathscr{G}}\colon \mathscr{G} \to \mathscr{F}$ is a sheaf homomorphism.

A morphism of $\mathscr{F}$-framed sheaves $(\mathscr{G}, \phi_{\mathscr{G}}) \to (\mathscr{H}, \phi_{\mathscr{H}})$ consists of a sheaf morphism $f\colon \mathscr{G} \to \mathscr{H}$ such that there is a $c \in \mathbb{C}^\times$ such that $\phi_{\mathscr{G}} \circ f = c\phi_{\mathscr{H}}$.

A morphism $f\colon (\mathscr{G}, \phi_{\mathscr{G}}) \to (\mathscr{H}, \phi_{\mathscr{H}})$ of $\mathscr{F}$-framed sheaves is injective, respectively surjective, respectively an isomorphism, if $f\colon \mathscr{G} \to \mathscr{H}$ is. Clearly, if f is an isomorphism, we can take $c = 1$.

Definition 2.6.2. Let $\mathcal{D} \subset \mathcal{Y}$ be a smooth integral closed substack of codimension 1.

A coherent sheaf on $\mathcal{Y}$ *framed along* $\mathcal{D}$, is a $\mathcal{O}_{\mathcal{D}}^r$-framed sheaf $(\mathscr{G}, \phi_{\mathscr{G}})$, where r is the rank of $\mathscr{G}$, and $\phi_{\mathscr{G}}|_{\mathcal{D}}\colon \mathscr{G}|_{\mathcal{D}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{D}}^r$ is an isomorphism.

Remark 2.6.3. Note that we differ from the definition in [11, p. 5] on several points:

- We make no requirement on $\dim \mathcal{Y}$, and we do not require $\mathcal{Y}$ to be smooth – in fact, we will usually work with a singular $\mathcal{Y}$;

- We do not make any assumptions on the coarse moduli space of $\mathcal{D}$;
- We do not require $\mathscr{G}$ to be locally free in a neighbourhood of $\mathcal{D}$ – in our cases, we will see in Lemmas 5.2.4 and 5.2.11 that this property follows from the other conditions.

Given a $\mathscr{F}$-framed coherent sheaf $(\mathscr{G}, \phi_{\mathscr{G}})$, we set

$$\varepsilon(\phi_{\mathscr{G}}) = \begin{cases} 0 & \text{if } \phi_{\mathscr{G}} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Pick an $\mathscr{F}$-framed coherent sheaf $(\mathscr{G}, \phi_{\mathscr{G}})$ on $\mathcal{Y}$, and set $d = \dim \operatorname{Supp} \mathscr{G}$. Then fix a polynomial

$$\delta(\nu) = \delta_1 \frac{\nu^{d-1}}{(d-1)!} + \delta_2 \frac{\nu^{d-2}}{(d-2)!} + \cdots \delta_d \in \mathbb{Q}[\nu],$$

such that $\delta_1 > 0$.

Then $\mathscr{G}$ is said to be *(semi)stable* (with respect to δ) if:

1. $P_{\mathscr{V}}(\mathscr{G}', n)(\leq) < \frac{\alpha_{\mathscr{V},d}(\mathscr{G}')}{\alpha_{\mathscr{V},d}(\mathscr{G})}(P_{\mathscr{V}}(\mathscr{G}, n) - \varepsilon(\phi_{\mathscr{G}})\delta(\nu))$ for all subsheaves $\mathscr{G}' \subset \ker \phi_{\mathscr{G}}$,
2. $P_{\mathscr{V}_{\mathcal{Y}}}(\mathscr{G}', n) - \delta(\nu)(\leq) < \frac{\alpha_{\mathscr{V},d}(\mathscr{G}')}{\alpha_{\mathscr{V},d}(\mathscr{G})}(P_{\mathscr{V}}(\mathscr{G}, n) - \varepsilon(\phi_{\mathscr{G}})\delta(\nu))$ for all subsheaves $\mathscr{G}' \subset \mathscr{G}$.

Here the notation $(\leq) <$ means: For *stability*, one uses the equations with $<$, and for *semistability*, those with $\leq$.

It will turn out to be easier to work with a slightly modified kind of stability for sheaves, namely $\widehat{\mu}$-stability, which is a type of *slope stability* taking into account the framing sheaf.

Definition 2.6.4 ([11, Definition 3.9]). Let $\mathscr{G}$ be a $\mathscr{F}$-framed sheaf of dimension d on $\mathcal{Y}$, and let $\delta_1 \in \mathbb{Q}$. Then $\mathscr{G}$ is $\widehat{\mu}$-(semi)stable with respect to δ_1 if the following conditions hold:

1. $\ker \phi_{\mathscr{G}}$ is torsion-free,
2. $$\frac{\alpha_{\mathcal{V},d-1}(\mathscr{G}')}{\alpha_{\mathcal{V},d}(\mathscr{G}')}(\leq) < \frac{\alpha_{\mathcal{V},d-1}(\mathscr{G}) - \varepsilon(\phi_{\mathscr{G}})\delta_1}{\alpha_{\mathcal{V},d}(\mathscr{G})}$$
   for all $\mathscr{G}' \subset \ker \phi_{\mathscr{G}}$, and
3. $$\frac{\alpha_{\mathcal{V},d-1}(\mathscr{G}') - \delta_1}{\alpha_{\mathcal{V},d}(\mathscr{G}')}(\leq) < \frac{\alpha_{\mathcal{V},d-1}(\mathscr{G})\delta_1}{\alpha_{\mathcal{V},d}(\mathscr{G})}$$
   for all $\mathscr{G}' \subset \mathscr{G}$ with $\alpha_{\mathscr{G}',d} < \alpha_{\mathscr{G},d}$,

with the same convention as before regarding the notation $(\leq) <$.

The number $\widehat{\mu}(\mathscr{G}, \phi_{\mathscr{G}}) := \frac{\alpha_{\mathcal{V},d}(\mathscr{G})-\delta_1}{\alpha_{\mathcal{V},d-1}(\mathscr{G})}$ is called the *framed hat-slope* of $(\mathscr{G}, \phi_{\mathscr{G}})$.

Let $\delta(\nu) \in \mathbb{Q}[\nu]$ be a polynomial such that $\delta_1/(d-1)!$ is the degree $d-1$-coefficient of δ. We then have a chain of implications for a sheaf $\mathscr{F}$ [11, p. 10]:

$$\widehat{\mu}\text{-stable} \implies \text{stable} \implies \text{semistable} \implies \widehat{\mu}\text{-semistable}, \tag{2.6}$$

where $\widehat{\mu}$-(semi)stability is considered with respect to δ_1, (semi)stability with respect to δ.

Remark 2.6.5. Especially, given a flat family of stable sheaves on $\mathcal{Y}$, being $\widehat{\mu}$-stable is an *open condition*, by the same argument as in [34, Proposition 3.A.1]. (This means: If $\mathscr{G}$ is a sheaf on $\mathcal{Y} \times B$, flat over B, such that for every $b \in B$, $\mathscr{G}_b$ is stable, then the set $\{b \in B \mid \mathscr{G}_b \text{ is } \widehat{\mu}\text{-stable}\}$ is open.)

Now, let

$$\mathcal{M}_{\mathcal{Y}}(\mathscr{V}_{\mathcal{Y}}, \mathcal{O}_Y(1), P_0, \mathscr{F}, \delta(\nu))$$

be the functor associating to any scheme S of finite type the set of isomorphism classes of flat families of $\delta(\nu)$-stable $\mathscr{F}$-framed sheaves with Hilbert polynomial P_0 on $\mathcal{Y}$, parametrised by S.

We then have

Theorem 2.6.6 ([11, Theorem 4.15])**.** *The functor $\mathcal{M}_{\mathcal{Y}}(\mathscr{V}_{\mathcal{Y}}, \mathcal{O}_Y(1), P_0, \mathscr{F}, \delta(\nu))$ is represented by a quasiprojective scheme $\mathbf{M}_{\mathcal{Y}}(\mathscr{V}_{\mathcal{Y}}, \mathcal{O}_Y(1), P_0, \mathscr{F}, \delta(\nu))$.*

Chapter 3

Hilbert schemes of Kleinian singularities

In this chapter, we make the first identification of a moduli space attached to Γ with a Nakajima quiver variety.

Our goal is to prove Theorem 1.1.1, through the identification of $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$ (or at least its underlying reduced structure) with the quiver variety $\mathfrak{M}_{\theta_0}(1, n\delta)$.

Most of this chapter is adapted from the paper [16], of which I was a joint author.

The final section, Section 3.6, was however not part of [16]. In this section, we make a more 'primitive' argument that quickly shows a weakened version of the main result. (This was, in fact, the first result to indicate that Theorem 1.1.1 was provable.) This version has the advantage of being immediately useful to compute with explicit examples of Π-modules. Furthermore, it is a starting point for the generalisation of Theorem 1.1.1 that will happen in Chapters 5 and 6.

For this chapter, we set $r = 1$. The framed McKay quiver Q thus has one arrow $\infty \to 0$ and one arrow $0 \to \infty$.

For a natural number $n \geq 1$ that we fix for the rest of the chapter, consider the dimension vector

$$v \coloneqq (v_i)_{i \in Q_0} \coloneqq \rho_\infty + \sum_{i \geq 0} n \dim(\rho_i)\rho_i = (1, n\delta) \in \mathbb{Z}^{Q_0}.$$

As we are only concerned with the dimension vector $(1, n\delta)$ in this chapter, we will throughout it write $C^+ = C^+_{1,n\delta}$ and

$$\mathfrak{M}_{\theta'} \coloneqq \mathfrak{M}_{\theta'}(1, n\delta)$$

for any $\theta' \in \Theta_{1,n\delta}$.

3.1 Variation of GIT quotient

Variation of GIT quotient for the quiver varieties $\mathfrak{M}_\theta$ was investigated recently by A. Craw with G. Bellamy [5]. The following result records a surjectivity statement that will be useful later on.

Lemma 3.1.1. *Let $\theta, \theta' \in \Theta_v$ satisfy $\theta \geq \theta'$. Then the morphism $\pi\colon \mathfrak{M}_\theta \to \mathfrak{M}_{\theta'}$ obtained by variation of GIT quotient is a surjective, projective and birational morphism of varieties over* $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$.

Proof. If θ is generic and $\theta' = 0$, then the morphism $\mathfrak{M}_\theta \to \mathfrak{M}_0 \cong \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ is a projective symplectic resolution [5, Theorem 4.5] and the result holds. For the general case, combining [6, Lemma 3.22] and [5, Lemma 4.4], we get $\dim \mathfrak{M}_\theta = 2n$. This holds for any $\theta \in \Theta_v$, so $\dim \mathfrak{M}_{\theta'} = 2n$. The morphism $\pi\colon \mathfrak{M}_\theta \to \mathfrak{M}_{\theta'}$ is projective, so the image $Z \coloneqq \pi(\mathfrak{M}_\theta)$ is closed in $\mathfrak{M}_{\theta'}$. Deform θ if necessary to a generic parameter η such that $\eta \geq \theta$. Then the resolution $\mathfrak{M}_\eta \to \mathfrak{M}_0 \cong \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ factors through π by variation of GIT quotient, so $\dim(Z) = 2n$ and hence π is birational onto its image. It follows that Z is an irreducible component of $\mathfrak{M}_{\theta'}$. However, $\mathfrak{M}_{\theta'}$ is irreducible [6, Proposition 3.21]; so π is surjective. ■

Remark 3.1.2. This result improves Lemma 2.4.5, however it depends on results from [5] that require $r = 1$. Our proof of Lemma 2.4.5 does not require $r = 1$.

It is well known that the quiver variety $\mathfrak{M}_\theta$ for $\theta \in C^+$ admits a description as an equivariant Hilbert scheme. Recall from the Introduction that $n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ is the scheme parametrising Γ-invariant ideals $I \subset \mathbb{C}[x, y]$ with quotient isomorphic as a representation of Γ to the direct sum of n copies of the regular representation of Γ.

Theorem 3.1.3 ([64, 65, 40])**.** *Let $\Gamma_n \coloneqq \Gamma^n \rtimes \mathfrak{S}_n \subset \mathbf{Sp}(2n, \mathbb{C})$ denote the wreath product of Γ with the symmetric group $\mathfrak{S}_n$. For $\theta \in C^+$, there is a commutative diagram*

$$\begin{tikzcd} n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2) \arrow[r, "\sim"] \arrow[d] & \mathfrak{M}_\theta \arrow[d, "\pi"] \\ \mathbb{C}^{2n}/\Gamma_n \cong \mathrm{Sym}^n(\mathbb{C}^2/\Gamma) \arrow[r, "\sim"] & \mathfrak{M}_0 \end{tikzcd}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are symplectic resolutions.

We now study partial resolutions of $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ through which the resolution from Theorem 3.1.3 factors. The result of [5, Proposition 6.1] implies that for $n > 1$, the nef cone of $n\Gamma$- $\mathrm{Hilb}(\mathbb{C}^2)$ over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ is isomorphic to the closure $\overline{C^+}$ of the chamber from Section 2.4.1. For $n = 1$, the relation between these two cones is described in [5, Proposition 7.11] (see Remark 3.5.5 for more on the case $n = 1$). In any case, for $n \geq 1$, the partial resolutions of interest can all be obtained as follows: choose a face of $\overline{C^+}$ and any GIT parameter from the relative interior of that face; then perform variation of GIT quotient as the parameter moves to the origin in Θ_v.

Recall our description of the faces of $\overline{C^+}$ and the parameters θ_I from Section 2.4.1

Proposition 3.1.4. *The face poset of the cone $\overline{C^+}$ can be identified with the poset on the set of quiver varieties $\mathfrak{M}_{\theta_I}$ for subsets $I \subseteq \{0, 1, \ldots, s\}$, where edges in the Hasse diagram of the poset are realised by the surjective, projective and birational morphisms $\pi_{I,I'} \colon \mathfrak{M}_{\theta_I} \to \mathfrak{M}_{\theta_{I'}}$.*

Proof. This is standard for variation of GIT quotient apart from surjectivity and birationality of each $\pi_{I,I'}$. This was established in Lemma 3.1.1. ■

Remark 3.1.5. When $I' = \varnothing$ and $I = \{0, \ldots, s\}$, the morphism $\mathfrak{M}_{\theta_I} \to \mathfrak{M}_{\theta_{I'}}$ is the resolution $n\Gamma$-$\mathrm{Hilb}(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ from Theorem 3.1.3. The statement of Proposition 3.1.4 implies that the paths in the Hasse diagram of the face poset of $\overline{C^+}$ from the unique maximal element to the unique minimal element provide all possible ways in which this resolution can be decomposed via primitive morphisms [66].

Example 3.1.6. Consider the case $\Gamma \cong \mu_3$, corresponding to Dynkin type A_2, and $n = 3$. Figure 3.1 shows a transverse slice of the GIT wall-and-chamber structure inside a specific closed cone F in the space Θ_v of stability parameters. According to [5, Theorem 1.2], this decomposition of the cone is isomorphic as a fan to the closure of the movable cone of this particular $n\Gamma$- $\mathrm{Hilb}(\mathbb{C}^2)$, with its natural subdivision into nef cones of birational models. The open subcone C^+ corresponds to the ample cone of $n\Gamma$- $\mathrm{Hilb}(\mathbb{C}^2)$ itself. In Section 3.5 we will focus on the distinguished ray $\langle\theta_0\rangle$ in the boundary of F.

We conclude this section with a lemma that identifies the key geometric fact that makes the chamber C^+ special; our argument depends crucially on this observation. For $\theta \in C^+$ and for any $\theta' \in \Theta_v$, we consider the line bundle $L_{C^+}(\theta') := \bigotimes_{0 \leq i \leq s} \det(\mathcal{R}_i)^{\theta'_i}$ on $\mathfrak{M}_\theta$; the line bundle $L_I := L_{C^+}(\theta_I)$ will play a special role in particular.

Lemma 3.1.7. *Let $\theta \in C^+$. Then*

(i) *for each $\theta' \in \overline{C^+}$, the line bundle $L_{C^+}(\theta')$ on $\mathfrak{M}_\theta$ is globally generated;*

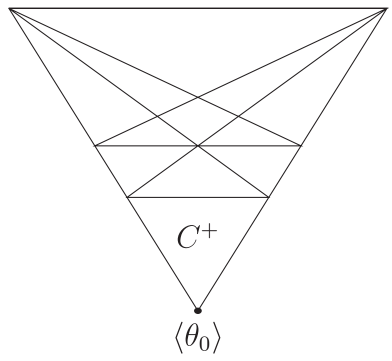


Figure 3.1: Wall-and-chamber structure inside the cone F for $\Gamma \cong \mu_3$ and $n = 3$

(ii) *for any $I \subseteq \{0, \ldots, s\}$, after multiplying θ_I by a positive integer if necessary, the morphism to the linear series of L_I decomposes as the composition of π_I and a closed immersion:*

$$\begin{tikzcd} \mathfrak{M}_\theta \arrow[r, "\varphi_{|L_I|}"] \arrow[d, "\pi_I"'] & {|L_I|}. \\ \mathfrak{M}_{\theta_I} \arrow[ur, hook] & \end{tikzcd} \tag{3.1}$$

Proof. Since $\theta \in C^+$, the tautological bundles $\mathcal{R}_i$ on the quiver variety $\mathfrak{M}_\theta$ are globally generated for $i \in Q_0$ by [15, Corollary 2.4]. Hence $L_{C^+}(\theta')$ is globally generated because $\theta'_i \geq 0$ for all $0 \leq i \leq s$. In particular, since $\theta_I \in \overline{C^+}$, the rational map $\varphi_{|L_I|}$ is defined everywhere. The line bundle L_I induces the morphism $\pi_I \colon \mathfrak{M}_\theta \to \mathfrak{M}_{\theta_I} \subset |L_I|$ by [5, Theorem 1.2], where we take a positive multiple of θ_I if necessary to ensure that the polarising ample bundle on $\mathfrak{M}_{\theta_I}$ is very ample. This proves the result. ■

Remark 3.1.8. We choose a sufficiently high multiple of θ (and the same high multiple of each θ_I) to ensure that the polarising ample line bundle on $\mathfrak{M}_{\theta_I}$ is very ample for every subset $I \subseteq \{0, \ldots, s\}$.

3.2 Deleting an arrow and cornering the algebra

In the framed McKay quiver Q, let $\bar{b} \in Q_1$ be the unique arrow with head at vertex ∞. Recall from Section 2.2.1 that we defined an algebra

$$A = \Pi/(\bar{b}),$$

and that we write the idempotent at vertex i by e_i. Equivalently, if we define a quiver Q^* to have vertex set $Q^*_0 = \{\infty, 0, 1, \ldots, s\}$ and arrow set $Q^*_1 = Q_1 \setminus \{\bar{b}\}$, then A is the quotient of the path algebra $\mathbb{C}Q^*$ by the ideal of relations

$$\left(\sum_{\mathrm{h}(a)=i} \epsilon(a) a\bar{a} \mid 0 \leq i, \mathrm{h}(a), \mathrm{h}(\bar{a}) \leq s \right). \tag{3.2}$$

Since Q^* and Q share the same vertex set, we may consider $v \in \mathbb{Z}^{Q_0}$ as a dimension vector for A-modules and any parameter $\theta \in \Theta_v$ as a stability condition for A-modules of dimension vector v.

Lemma 3.2.1. *For $\theta \in C^+$, the tautological $\mathbb{C}$-algebra homomorphism $\Pi \to \mathrm{End}(\mathcal{R})$ over $\mathfrak{M}_\theta$ factors through A. In particular, the tautological bundle $\mathcal{R}$ on $\mathfrak{M}_\theta$ is a flat family of θ-stable A-modules of dimension vector v.*

Proof. The image of the arrow $\bar{b}$ under the tautological homomorphism $\Pi \to \mathrm{End}(\mathcal{R})$ is a map of vector bundles $\mathcal{R}_0 \to \mathcal{R}_\infty \cong \mathcal{O}_{\mathfrak{M}_\theta}$. Under the isomorphism $\mathfrak{M}_\theta \cong n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$ from Theorem 3.1.3, we may regard the fibre of $\mathcal{R}$ over any closed point as the quotient $\mathbb{C}[x, y]/I$ for some Γ-invariant ideal I of $\mathbb{C}[x, y]$. In this language, the restriction to the 0 vertex is the Γ-invariant part of the quotient $\mathbb{C}[x, y]/I$, and it is well known that the induced map to the one-dimensional vector space at the vertex ∞ vanishes in this case [48, Proposition 2.8]. Thus, as $\mathfrak{M}_\theta$ is non-singular and in particular reduced, the corresponding map $\mathcal{R}_0 \to \mathcal{O}_{\mathfrak{M}_\theta}$ is the zero map, so the tautological $\mathbb{C}$-algebra homomorphism $\Pi \to \mathrm{End}(\mathcal{R})$ factors through A as required. ■

Lemma 3.2.1 allows us to work with the algebra A rather than Π. We can similarly speak of stability parameters for the algebra A rather than the algebra Π, and the notion of a concentrated module (Definition 2.4.6) carries over.

To illustrate why it is convenient to work with A, recall that the preprojective algebra Π_Γ of the unframed McKay quiver Q_Γ is the quotient of $\mathbb{C}Q_\Gamma$ by the ideal generated by the preprojective relations in Q_Γ defined similarly to those in Q as in (2.3). Now, while Π_Γ is not a subalgebra of Π, it is isomorphic to a subalgebra of A as we now show. By a slight abuse of notation, let $b \in A$ denote the class of the arrow b.

Lemma 3.2.2. *The preprojective algebra Π_Γ is isomorphic to the subalgebra $\bigoplus_{0 \leq i,j \leq s} e_j A e_i$ of A. Under this embedding, there is a isomorphism*

$$A \cong \Pi_\Gamma \oplus \Pi_\Gamma b \oplus \mathbb{C}e_\infty$$

of complex vector spaces.

Proof. The quiver Q^* has no arrow with head at vertex ∞, so the subalgebra $\bigoplus_{0\leq i,j\leq s} e_jAe_i$ of A is isomorphic to the quotient algebra $A/(e_\infty)$. The first statement follows since $A/(e_\infty)\cong\Pi_\Gamma$. The decomposition of A as a vector space is immediate from the structure of the quiver Q^*. ■

Let $I\subseteq\{0,1,\dots,s\}$ be non-empty. Define the idempotent $e_I := e_\infty+\sum_{i\in I} e_i$ and consider the subalgebra

$$A_I := e_IAe_I \tag{3.3}$$

of A spanned over $\mathbb{C}$ by the classes of paths in Q^* whose tail and head both lie in the set $\{\infty\}\cup I$. The process of passing from A to A_I is called *cornering*; see [15, Remark 3.1].

We will study moduli spaces of certain finite-dimensional A_I-modules, and for this we must introduce a presentation of the algebra A_I in terms of a quiver with relations.

First, recall that the Γ-module $R := \mathbb{C}[x,y]$ decomposes into isotypical components $R=\bigoplus_{0\leq i\leq s} R_i$, where R_i is the sum of all Γ-submodules of R that are isomorphic to ρ_i. In particular, $R_0=\mathbb{C}[x,y]^\Gamma$ and R_i is a reflexive R_0-module for each $0\leq i\leq s$. Since $\Gamma\subset\mathbf{SL}(2,\mathbb{C})$, the Γ-invariant subring R_0 is well known to admit a presentation of the form

$$R_0\cong\mathbb{C}[z_1,z_2,z_3]/(f), \tag{3.4}$$

leading to the famous description of $\mathbb{C}^2/\Gamma$ as a hypersurface $(f=0)\subset\mathbb{C}^3$. On the other hand, combining Auslander [2] and Reiten–Van den Bergh [56] (compare Buchweitz [12]) gives an isomorphism

$$\Pi_\Gamma\cong\operatorname{End}_{R_0}\Big(\bigoplus_{0\leq i\leq s} R_i\Big) \tag{3.5}$$

of $\mathbb{C}$-algebras. Note that for $0\leq i,j\leq s$, the space $\operatorname{Hom}_{R_0}(R_i,R_j)$ is finitely generated as an R_0-module. One way to see this is to consider the reflexive sheaves $\widetilde{R}_i$ on $\mathbb{C}^2/\Gamma$ determined by the reflexive R_0-modules R_i; then $\operatorname{Hom}_{R_0}(R_i,R_j)$ is the space of sections of the coherent sheaf $\mathcal{H}om(\widetilde{R}_i,\widetilde{R}_j)$ on $\mathbb{C}^2/\Gamma$.

Proposition 3.2.3. *For any non-empty subset $I\subseteq\{0,1,\dots,s\}$, the algebra A_I can be presented as the quotient of the path algebra of a quiver modulo a two-sided ideal of relations.*

Proof. If we regard Π_Γ as a subalgebra of A using Lemma 3.2.2, we see that $e_I' := \sum_{i\in I} e_i$ is the sum of vertex idempotents in Π_Γ, and the isomorphism (3.5) induces an isomorphism

$$e_I'(\Pi_\Gamma)e_I'\cong\operatorname{End}_{R_0}\Big(\bigoplus_{i\in I} R_i\Big). \tag{3.6}$$

Since Π_Γ is isomorphic to $\bigoplus_{0\leq i,j\leq s} e_j A e_i$ by Lemma 3.2.2, it follows that the algebra $e_I'(\Pi_\Gamma)e_I'$ from (3.6) is isomorphic to the subalgebra $e_I' A_I e_I'$ of A_I. For each $i \in I$, we choose three $\mathbb{C}$-algebra generators of $R_0 \subseteq \operatorname{Hom}_{R_0}(R_i, R_i)$ corresponding to the generators in the presentation (3.4), and we extend this to a set of $d_i \geq 3$ generators of $\operatorname{Hom}_{R_0}(R_i, R_i)$ as a $\mathbb{C}$-algebra. Finally, for $i, j \in I$ with $i \neq j$, we choose a finite generating set for $\operatorname{Hom}_{R_0}(R_i, R_j)$ as an R_0-module comprising $d_{i,j} > 0$ generators.

We claim that there exist quivers Q_I and Q_I^* whose path algebras fit into a commutative diagram

$$\begin{tikzcd} \mathbb{C}Q_I \arrow[r] \arrow[d, "\alpha_I"] & \mathbb{C}Q_I^* \arrow[d, "\beta_I"] \\ \operatorname{End}_{R_0}\left(\bigoplus_{i\in I} R_i\right) \arrow[r] & A_I \end{tikzcd} \tag{3.7}$$

of $\mathbb{C}$-algebra homomorphisms where the vertical maps are surjective and the horizontal maps are injective. Given the claim, Lemma 3.2.2 follows because the required quiver is Q_I^* and the ideal of relations is $\ker(\beta_I)$.

To prove the claim, we consider two cases. Suppose first that $0 \in I$. Define the vertex set of Q_I to be I. For the arrow set of Q_I, introduce d_i loops at each vertex $i \in I$ corresponding to our chosen $\mathbb{C}$-algebra generators of $\operatorname{Hom}_{R_0}(R_i, R_i)$, including the three distinguished generators of its subalgebra R_0. Furthermore, for each $i, j \in I$ with $i \neq j$, introduce $d_{i,j}$ arrows from i to j corresponding to our chosen R_0-module generators of $\operatorname{Hom}_{R_0}(R_i, R_j)$. The concatenation of any arrow from i to j with loops at vertex j corresponding to the appropriate elements of $R_0 \subseteq \operatorname{Hom}_{R_0}(R_j, R_j)$ defines paths in Q_I that represent a spanning set for the vector space $e_j A_I e_i = \operatorname{Hom}_{R_0}(R_i, R_j)$. This determines by construction the left-hand epimorphism α_I in (3.7).

Next, define the vertex set of Q_I^* to be $\{\infty\} \cup I$ and define the arrow set by augmenting the arrow set of Q_I with one additional arrow b with tail at ∞ and head at 0. This is well-defined since $0 \in I$, and moreover, $\mathbb{C}Q_I$ is a subalgebra of $\mathbb{C}Q_I^*$ because Q_I^* has no arrows with head at ∞. The lower horizontal map in diagram (3.7) is simply the inclusion of the algebra from (3.6) as the subalgebra $e_I' A_I e_I'$ of A_I. To construct β_I, we need only extend α_I by sending the paths e_∞, b in $\mathbb{C}Q_I^*$ to the classes of e_∞, b in A_I respectively. Surjectivity of β_I follows from the second statement of Lemma 3.2.2. This proves the claim for $0 \in I$.

It remains to consider the case $0 \notin I$. Define $\overline{I} := \{0\} \cup I$ and apply the construction for the case $0 \in \overline{I}$ to obtain the diagram (3.7) for $\overline{I}$. To define the quiver Q_I, we remove from $Q_{\overline{I}}$ the vertex 0 together with all arrows in $Q_{\overline{I}}$ that have head and/or tail at 0. Notice that for each $i, j \in I \subset \overline{I}$, the quiver $Q_{\overline{I}}$ already has $d_{i,j}$ arrows from i

to j corresponding to a set of R_0-module generators of $\mathrm{Hom}_{R_0}(R_i, R_j)$, so the desired $\mathbb{C}$-algebra epimorphism α_I is obtained by restriction from the map $\alpha_{\overline{I}}$ in the diagram (3.7) for $\overline{I}$.

Next, for any $i \in I$, let $\{a'_{i,m} \mid 1 \leq m \leq d_{0,i}\}$ denote the arrows in $Q_{\overline{I}}$ corresponding to our chosen set of generators of $\mathrm{Hom}_{R_0}(R_0, R_i)$. Define the quiver Q^*_I to have vertex set $\{\infty\} \cup I$ and arrow set obtained by augmenting the arrow set of Q_I as follows: for each $i \in I$, introduce arrows $\{a_{i,m} \mid 1 \leq m \leq d_{0,i}\}$ from ∞ to j. Note that $\beta_{\overline{I}}(a'_{i,m}b) \in e_i A_{\overline{I}} e_\infty = e_i A_I e_\infty \subset A_I$. Therefore, if for any path p in $\mathbb{C}Q^*_I$ we define

$$\beta_I(p) = \begin{cases} \alpha_I(p) & \text{for } p \in \mathbb{C}Q_I; \\ e_\infty & \text{for } p = e_\infty; \\ \beta_{\overline{I}}(a'_{j,m}b) & \text{for } p = a_{j,m}, \end{cases}$$

then we determine uniquely a $\mathbb{C}$-algebra homomorphism $\beta_I \colon \mathbb{C}Q^*_I \to A_I$. To show that β_I is surjective, it suffices to check that the image of β_I contains $A_I e_\infty$ because α_I is surjective. For this, consider $\gamma \in A_I e_\infty \subseteq e_I A_{\overline{I}} e_\infty$. Since $\beta_{\overline{I}}$ is surjective, γ can be represented by a linear combination of paths $\gamma_i \in Q^*_{\overline{I}}$, each with tail at ∞ and head at a vertex in I. Every such path γ_i necessarily begins by traversing a path of the form $a'_{i,m}cb$ for some $i \in I$ and $1 \leq m \leq d_{0,i}$, where c is a (possibly empty) composition of loops at vertex 0. Crucially, as $\beta_{\overline{I}}(c) \in \mathrm{Hom}_{R_0}(R_0, R_0) \cong R_0$, the element $\beta_{\overline{I}}(a'_{i,m}c) \in e_i A_{\overline{I}} e_0 = \mathrm{Hom}_{R_0}(R_0, R_i)$ can be written as the image under $\beta_{\overline{I}}$ of a linear combination $\sum_{1 \leq n \leq d_{0,i}} c_n a'_{i,n}$ for some $c_n \in R_0 \subseteq \mathrm{Hom}_{R_0}(R_i, R_i)$. Therefore the start $a'_{i,m}cb$ of the path γ_i satisfies

$$\beta_{\overline{I}}(a'_{j,m}cb) = \beta_{\overline{I}}\Bigg(\sum_{1 \leq n \leq d_{0,j}} c_n a'_{j,n}\Bigg)\beta_{\overline{I}}(b) = \sum_{1 \leq n \leq d_{0,j}} c_n \beta_{\overline{I}}(a'_{j,n}b) = \beta_I\Bigg(\sum_{1 \leq n \leq d_{0,j}} \ell_n a_{j,n}\Bigg),$$

where each ℓ_n is a linear combination of loops in Q^*_I at vertex j satisfying $\alpha_I(\ell_n) = c_n$. Thus, the image in A_I of the beginning of our path γ_i lies in the image of β_I. It follows that each path γ_i arising in the linear combination of γ lies in the image of β_I, because α_I is surjective. Therefore the image of β_I contains $A_I e_\infty$ as required. ∎

Remarks 3.2.4.

1. The quiver Q^*_I that we construct in the proof above has many more arrows than necessary. For example, when $I = \{0, 1, \ldots, s\}$, the algorithm returns a quiver with arrow set containing at least three loops at each vertex, whereas Q^* contains no loops.

2. An alternative proof for Proposition 3.2.3 could be given by exhibiting a finite number of paths in Q^* whose tail and head both lie in the set $\{\infty\} \cup I$, with the property that their classes, up to the preprojective relations, generate the

cornered algebra A_I. While we believe this is indeed possible, the combinatorics of the situation gets rather intricate, especially in the case $0 \notin I$. The proof presented above has the advantage that it avoids case-by-case analysis of Dynkin diagrams.

3.3 Reconstructing quiver varieties via the cornered algebras

In general, the quiver variety $\mathfrak{M}_{\theta_I}$ is the coarse moduli space for S-equivalence classes of θ_I-semistable Π-modules of dimension vector v. However, in the special case $I = \{0, \ldots, s\}$ it may also be regarded as the fine moduli space of isomorphism classes of θ_I-stable A-modules of dimension vector v by Lemma 3.2.1. We now introduce an alternative, fine moduli space construction for each $\mathfrak{M}_{\theta_I}$ using the algebra A_I.

The element

$$v_I := \rho_\infty + \sum_{i \in I} n \dim(\rho_i)\rho_i \in \mathbb{Z} \oplus \mathbb{Z}^I$$

is a dimension vector for A_I-modules, and we consider the stability condition $\eta_I \colon \mathbb{Z} \oplus \mathbb{Z}^I \to \mathbb{Q}$ given by

$$\eta_I(\rho_i) = \begin{cases} -\sum_{i \in I} n \dim(\rho_i) & \text{for } i = \infty \\ 1 & \text{if } i \in I \end{cases}$$

It follows directly from the definition that an A_I-module N of dimension vector v_I is η_I-stable if and only if there exists a surjective A_I-module homomorphism $A_I e_\infty \to N$. The vector v_I is indivisible and η_I is a generic stability condition for A_I-modules.

The quiver moduli space construction of King [36, Proposition 5.3] for finite-dimensional algebras can be adapted to any algebra presented as the quotient of a finite connected quiver by an ideal of relations.

Thus, Proposition 3.2.3 allows us to make the following definition:

Definition 3.3.1. The scheme $\mathcal{M}(A_I)$ is the fine moduli space of η_I-stable A_I-modules of dimension vector v_I.

Let $T_I := \bigoplus_{i \in \{\infty\} \cup I} T_i$ denote the tautological bundle on $\mathcal{M}(A_I)$, where T_∞ is the trivial bundle and T_i has rank $n \dim(\rho_i)$ for $i \in I$. The line bundle

$$\mathcal{L}_I := \bigotimes_{i \in I} \det(T_i)$$

is the polarising ample bundle on $\mathcal{M}(A_I)$ given by the GIT construction.

Lemma 3.3.2. *Let $\theta \in C^+$, and let $I \subseteq \{0, \ldots, s\}$ be any non-empty subset. There is a universal morphism*

$$\tau_I \colon \mathfrak{M}_\theta \to \mathcal{M}(A_I) \tag{3.8}$$

satisfying $\tau_I^*(T_i) \cong \mathcal{R}_i$ for $i \in \{\infty\} \cup I$.

Proof. In light of the universal property of $\mathcal{M}(A_I)$, it suffices to show that the locally-free sheaf

$$\mathcal{R}_I := \bigoplus_{i \in \{\infty\} \cup I} \mathcal{R}_i$$

of rank $1 + \sum_{i \in I} n \dim(\rho_i)$ on the quiver variety $\mathfrak{M}_\theta$ is a flat family of η_I-stable A_I-modules of dimension vector v_I. We saw in Lemma 3.2.1 that we may delete one arrow from Q, giving rise to a $\mathbb{C}$-algebra homomorphism $\phi\colon A \to \operatorname{End}(\mathcal{R})$. Multiplying this on the left and right by the idempotent e_I determines a $\mathbb{C}$-algebra homomorphism $A_I \to \operatorname{End}(\mathcal{R}_I)$ which makes $\mathcal{R}_I$ into a flat family of A_I-modules of dimension vector v_I. To establish stability, write $\bigoplus_{i \in Q_0} \mathcal{R}_{i,y}$ for the fibre of $\mathcal{R}$ over a closed point $y \in \mathfrak{M}_\theta$. The fact that $\bigoplus_{i \in Q_0} \mathcal{R}_{i,y}$ is θ-stable is equivalent to the existence of a surjective A-module homomorphism $Ae_\infty \to \bigoplus_{i \in Q_0} \mathcal{R}_{i,y}$. Applying e_I on the left produces a surjective A_I-module homomorphism $A_I e_\infty \to \bigoplus_{i \in \{\infty\} \cup I} \mathcal{R}_{i,y}$ which in turn is equivalent to η_I-stability of the fibre $\bigoplus_{i \in \{\infty\} \cup I} \mathcal{R}_{i,y}$ of $\mathcal{R}_I$ over $y \in \mathfrak{M}_\theta$. In particular, $\mathcal{R}_I$ is a flat family of η_I-stable A_I-modules of dimension vector v_I. ■

Remarks 3.3.3.

1. An alternative proof of Lemma 3.3.2 uses the fact that the tautological bundles $\mathcal{R}_i$ on $\mathfrak{M}_\theta$ are globally generated for $i \in I$ by [15, Corollary 2.4], in which case one can adapt the proof of [15, Proposition 2.3] to deduce that $\mathcal{R}_I$ is a flat family of η_I-stable A_I-modules of dimension vector v_I. In particular, global generation is the key feature in Lemma 3.3.2, just as in the proof of Lemma 3.1.7. This is not a coincidence; see Theorem 3.3.8.

2. Building on Remark 3.1.8, we now take an even higher multiple of θ if necessary (and the same high multiple of each η_I and each θ_I) to ensure that the polarising ample line bundles on $\mathcal{M}(A_I)$ and on $\mathfrak{M}_{\theta_I}$ are very ample for all relevant $I \subseteq \{0, \ldots, s\}$.

Lemma 3.3.4. *Let $\theta \in C^+$ and assume $I \subseteq \{0, \dots, s\}$ is non-empty. There is a commutative diagram*

$$\begin{tikzcd} & \mathfrak{M}_\theta \arrow[dl, "\pi_I"'] \arrow[dd, "\varphi_{|L_I|}"] \arrow[dr, "\tau_I"] & \\ \mathfrak{M}_{\theta_I} \arrow[dr, hook] & & \mathcal{M}(A_I) \arrow[d, hook, "\varphi_{|\mathcal{L}_I|}"] \\ & {|L_I|} \arrow[r, "\psi"] & {|\mathcal{L}_I|} \end{tikzcd} \tag{3.9}$$

of schemes over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, *where ψ is an isomorphism.*

Proof. The commutative triangle on the left of (3.9) was constructed in Lemma 3.1.7. For the quadrilateral on the right, our choice of η_I ensures that the polarising line bundle $\mathcal{L}_I$ on $\mathcal{M}(A_I)$ is very ample, so the morphism $\varphi_{|\mathcal{L}_I|}$ is well-defined. Since pullback commutes with tensor operations on the T_i, the isomorphisms $\tau_I^*(T_i) \cong \mathcal{R}_i$ for $i \in I$ imply that $L_I = \tau_I^*(\mathcal{L}_I)$. If $\mathcal{O}_{|\mathcal{L}_I|}(1)$ denotes the polarising ample bundle on $|\mathcal{L}_I|$, then

$$(\varphi_{|\mathcal{L}_I|} \circ \tau_I)^*(\mathcal{O}_{|\mathcal{L}_I|}(1)) = \tau_I^*(\mathcal{L}_I) = L_I = \varphi_{|L_I|}^*\Big(\mathcal{O}_{|L_I|}(1)\Big) \tag{3.10}$$

on $\mathfrak{M}_\theta$. The morphism to a complete linear series is unique up to an automorphism of the linear series, so there is an isomorphism $\psi\colon |L_I| \to |\mathcal{L}_I|$ such that $\varphi_{|\mathcal{L}_I|} \circ \tau_I = \psi \circ \varphi_{|L_I|}$ as required.

It remains to show that (3.9) is a diagram of schemes over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$. The Leray spectral sequence for the resolution $\pi\colon \mathfrak{M}_\theta \to \mathfrak{M}_0 \cong \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ gives $H^0(\mathcal{O}_{\mathcal{M}_\theta}) \cong H^0(\mathcal{O}_{\mathcal{M}_0}) \cong (\mathbb{C}[V]^\Gamma)^{\mathfrak{S}_n}$ because $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ has rational singularities. It follows that $\pi = \varphi_{|\mathcal{O}_{\mathfrak{M}_\theta}|}$, i.e. π is the structure morphism of $\mathfrak{M}_\theta$ as a variety over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$. Repeating the argument from (3.10), with the roles of $L_I, \mathcal{L}_I$ and $\mathcal{O}_{|\mathcal{L}_I|}(1)$ played instead by the trivial bundles on $\mathfrak{M}_\theta$, $\mathcal{M}(A_I)$ and $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ respectively, shows that $\mathcal{M}(A_I)$ is a scheme over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$. It follows that (3.9) is a diagram of schemes over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$. ∎

Our goal for the rest of this section is to add a morphism $\iota_I\colon \mathfrak{M}_{\theta_I} \to \mathcal{M}(A_I)$ to diagram (3.9) and to show that ι_I is an isomorphism on the underlying reduced schemes. Consider the functors

$$A\text{-mod} \underset{j_!}{\overset{j^*}{\rightleftarrows}} A_I\text{-mod}$$

defined by $j^*(-) := e_I A \otimes_A (-)$ and $j_!(-) := Ae_I \otimes_{A_I} (-)$. These are two of the six functors in a recollement of the module category A-mod [26]. In particular, j^* is exact, $j^* j_!$ is the identity functor, and for any A_I-module N, the A-module $j_!(N)$ is the maximal extension by $A/(Ae_I A)$-modules; see [15, (3.4)].

Lemma 3.3.5. *Let N be an A_I-module of dimension vector v_I.*

(i) *If there exists a surjective A_I-module homomorphism $A_I e_\infty \to N$, then there exists a surjective A-module homomorphism $Ae_\infty \to j_!(N)$.*

(ii) *The A-module $j_!(N)$ is finite-dimensional and satisfies $\dim_i j_!(N) = \dim_i N$ for all $i \in \{\infty\} \cup I$.*

Proof. The construction of the quiver from Proposition 3.2.3 shows that A is a finitely generated module over the algebra $R_0 \cong e_0 A e_0$. Armed with this observation, the proof of [15, Lemma 3.6] applies verbatim (the notation differs slightly for part (i): our map $A_I e_\infty \to N$ is written $A_C e'_0 \to N$ in *ibid.*). ∎

Lemma 3.3.6. *Let N be an η_I-stable A_I-module of dimension vector v_I. The A-module $j_!(N)$ is θ_I-semistable.*

Proof. Since N is η_I-stable, there is a surjective A_I-module homomorphism $A_I e_\infty \to N$. Lemma 3.3.5 gives a surjective A-module homomorphism $Ae_\infty \to j_!(N)$ and, moreover, the finite dimensional A-module $j_!(N)$ satisfies $\dim_i j_!(N) = \dim_i N$ for $i \in \{\infty\} \cup I$. Recall that $\theta_I(\rho_i) = 0$ for $i \notin \{\infty\} \cup I$, so

$$\theta_I\Big(j_!(N)\Big) = \theta_I\left(\sum_{i\in\{\infty\}\cup I} \dim_i(j_!(N))\rho_i\right) = \eta_I\left(\sum_{i\in\{\infty\}\cup I} \dim_i(N)\rho_i\right) = \eta_I(N) = 0.$$

Now let $M \subset j_!(N)$ be a proper submodule. If $\dim_\infty M = 1$, then surjectivity of the map $Ae_\infty \to j_!(N)$ gives $M = j_!(N)$ which is absurd, so $\dim_\infty M = 0$. But $\theta_I(\rho_i) \geq 0$ for all $i \neq \infty$, so $\theta_I(M) \geq 0$ as required. ∎

Lemma 3.3.7. *Let N be an η_I-stable A_I-module of dimension vector v_I. Then there exists a θ_I-semistable A-module M such that $j^*M \cong N$ and $\dim_i M \leq n \dim(\rho_i)$ for all $i \notin \{\infty\} \cup I$.*

Proof. By Lemma 3.3.6, $j_!(N)$ is θ_I-semistable. If $\dim_i j_!(N) \leq n \dim(\rho_i)$ for $i \notin \{\infty\}\cup I$, then we can simply set $M := j_!(N)$, as $j^* j_!$ is the identity. Otherwise, consider the θ_I-polystable module $\bigoplus_\lambda M_\lambda$ that is S-equivalent to $j_!(N)$. The concentrated module M_{con} is a summand of this polystable module, say $M_{\mathrm{con}} = M_{\lambda_\infty}$.

For each index λ and for all $i \in \{\infty\} \cup I$, we have

$$\dim_i j^* M_\lambda = \dim e_i\Big(e_I A \otimes_A (M_\lambda)\Big) = \dim e_i A \otimes_A M_\lambda = \dim_i M_\lambda.$$

It follows that $\dim_i j^* M_\lambda = 0$ for all $\lambda \neq \lambda_\infty$ and $i \in \{\infty\} \cup I$, and hence $j^* M_\lambda = 0$ for $\lambda \neq \lambda_\infty$.

We claim that $j^*M_{\lambda_\infty} = j^*M_{\mathrm{con}}$ is isomorphic to N. Indeed, the A-module $j_!(N)$ is θ_I-semistable by Lemma 3.3.6, and the θ_I-stable A-modules M_λ are by construction the factors in the composition series of $j_!(N)$ in the category of θ_I-semistable A-modules. It follows from exactness of j^* that the A_I-modules j^*M_λ are the factors in the composition series of $j^*j_!(N) \cong N$ in the category of η_I-semistable A_I-modules. But $j^*M_\lambda = 0$ for $\lambda \neq \lambda_\infty$, so the only nonzero factor of the composition series is j^*M_{con}. It follows that $j^*M_{\mathrm{con}} \cong N$, because the factor j^*M_{con} can only appear once in the composition series.

As a result, the θ_I-stable A-module M_{con} satisfies $j^*M_{\mathrm{con}} \cong N$ and $\dim_i M_{\mathrm{con}} = n\dim(\rho_i)$ for all $i \in I$. Therefore M_{con} is the required A_I-module as long as $\dim_i M_{\mathrm{con}} \leq n\dim(\rho_i)$ for $i \notin \{\infty\} \cup I$. We establish this key inequality in Appendix A. ■

Theorem 3.3.8. *For any non-empty $I \subseteq \{0, \dots, s\}$, there is a commutative diagram of morphisms*

$$\begin{tikzcd} & \mathfrak{M}_\theta \arrow[dl, "\pi_I"'] \arrow[dr, "\tau_I"] & \\ \mathfrak{M}_{\theta_I} \arrow[rr, "\iota_I"] & & \mathcal{M}(A_I), \end{tikzcd} \tag{3.11}$$

where ι_I is an isomorphism of the underlying reduced schemes. In particular, $\mathcal{M}(A_I)$ is irreducible, and its underlying reduced scheme is normal and has symplectic singularities.

Proof. Let $\sigma_I \colon \mathfrak{M}_{\theta_I} \to |\mathcal{L}_I|$ be the composition of the isomorphism ψ of Lemma 3.3.4 with the closed immersion $\mathfrak{M}_{\theta_I} \hookrightarrow |L_I|$ from diagram (3.9). Since σ_I is a closed immersion, it identifies $\mathfrak{M}_{\theta_I}$ with $\mathrm{im}(\sigma_I)$. Surjectivity of π_I and commutativity of diagram (3.9) then imply that $\mathfrak{M}_{\theta_I}$ is isomorphic to the subscheme $\mathrm{im}(\sigma_I \circ \pi_I) = \mathrm{im}(\varphi_{|\mathcal{L}_I|} \circ \tau_I)$ of $|\mathcal{L}_I|$. Since $\mathcal{L}_I$ is the polarising very ample line bundle on the GIT quotient $\mathcal{M}(A_I)$, the closed immersion $\varphi_{|\mathcal{L}_I|}$ induces an isomorphism $\lambda_I \colon \mathrm{im}(\varphi_{|\mathcal{L}_I|}) \to \mathcal{M}(A_I)$. The morphism

$$\iota_I \coloneqq \lambda_I \circ \sigma_I \colon \mathfrak{M}_{\theta_I} \longrightarrow \mathcal{M}(A_I)$$

is therefore a closed immersion. Note that

$$\iota_I \circ \pi_I = \lambda_I \circ \sigma_I \circ \pi_I = \lambda_I \circ \varphi_{|\mathcal{L}_I|} \circ \tau_I = \tau_I,$$

so diagram (3.11) commutes. In order to prove that ι_I is an isomorphism of the underlying reduced schemes, it suffices to show that ι_I is surjective on closed points.

Consider a closed point $[N] \in \mathcal{M}(A_I)$, where N is an η_I-stable A_I-module of dimension vector v_I. Let M be the θ_I-semistable A-module from Lemma 3.3.7. For

$i \notin \{\infty\} \cup I$, define $m_i := n \dim(\rho_i) - \dim_i M \geq 0$ and let $S_i := \mathbb{C}e_i$ denote the vertex simple A-module at vertex $i \in Q_0$. The A-module

$$\overline{M} := M \oplus \bigoplus_{i \in \{0,\dots,s\} \smallsetminus I} S_i^{\oplus m_i}$$

is θ_I-semistable of dimension vector v by construction, and it satisfies $j^*(\overline{M}) = j^*(M) = N$. Write $[\overline{M}] \in \mathfrak{M}_{\theta_I}$ for the corresponding closed point, and let $\widetilde{M}$ be any θ-stable A-module of dimension vector v such that the closed point $[\widetilde{M}] \in \mathfrak{M}_\theta$ satisfies $\pi_I([\widetilde{M}]) = [\overline{M}] \in \mathfrak{M}_{\theta_I}$. Then $j^*(\widetilde{M}) = j^*(\overline{M}) = N$, hence $\tau_I([\widetilde{M}]) = [N]$, and commutativity of diagram (3.11) gives that

$$\iota_I([\overline{M}]) = (\iota_I \circ \pi_I)\big([\widetilde{M}]\big) = \tau_I\big([\widetilde{M}]\big) = [N],$$

so ι_I is indeed surjective. The final statement of Theorem 3.3.8 follows from Lemma 2.1.3 and Lemma 3.3.4. ■

Remarks 3.3.9. 1. If $I \neq \{0,\dots,s\}$, then the stability parameter θ_I lies in the boundary of the GIT chamber C^+, so $\mathfrak{M}_{\theta_I}$ does not admit a universal family of θ_I-semistable Π-modules of dimension vector v. However, the fine moduli space $\mathcal{M}(A_I)$ does carry a universal family T_I of η_I-stable A_I-modules of dimension vector v_I, and hence under the isomorphism of Theorem 3.3.8, the bundle $\iota_I^*(T_I)$ on $\mathfrak{M}_{\theta_I}$ pulls back along π_I to the summand $\bigoplus_{i\in\{\infty\}\cup I} \mathcal{R}_i$ of the tautological bundle on $\mathfrak{M}_\theta$.

2. In the course of the proof of Theorem 3.3.8, we deduce directly that τ_I is surjective on closed points.

3. For $I = \varnothing$, we have $\mathfrak{M}_{\theta_I} \cong \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$. On the other hand, $e_\infty A e_\infty = \mathbb{C}e_\infty$, which does not provide enough information with which to reconstruct $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$.

3.4 Identifying the posets for the coarse and fine moduli problems

We now establish that the morphisms $\iota_I \colon \mathfrak{M}_{\theta_I} \to \mathcal{M}(A_I)$ from Theorem 3.3.8 are compatible with the morphisms $\pi_{I,I'} \colon \mathfrak{M}_{\theta_I} \to \mathfrak{M}_{\theta_{I'}}$ that feature in the poset introduced in Proposition 3.1.4.

Lemma 3.4.1. *For non-empty subsets $I' \subset I \subset \{0, 1, \ldots, s\}$, there is a commutative diagram*

$$\begin{tikzcd}
\mathfrak{M}_{\theta_I} \arrow[r, "\iota_I"] \arrow[d, "\pi_{I,I'}"'] & \mathcal{M}(A_I) \arrow[d, "\tau_{I,I'}"] \\
\mathfrak{M}_{\theta_{I'}} \arrow[r, "\iota_{I'}"] & \mathcal{M}(A_{I'})
\end{tikzcd} \tag{3.12}$$

in which the horizontal arrows are isomorphisms on the underlying reduced schemes and the vertical arrows are surjective, projective, birational morphisms.

Proof. The subbundle $\bigoplus_{i\in\{\infty\}\cup I'} T_i$ of the tautological bundle T_I on $\mathcal{M}(A_I)$ is a flat family of $\eta_{I'}$-stable $A_{I'}$-modules of dimension vector $v_{I'}$, so there is a universal morphism

$$\tau_{I,I'} \colon \mathcal{M}(A_I) \longrightarrow \mathcal{M}(A_{I'})$$

satisfying $\tau_{I,I'}^*(T_i') = T_i$ for $i \in \{\infty\} \cup I'$, where $\bigoplus_{i\in\{\infty\}\cup I'} T_i'$ is the tautological bundle on $\mathcal{M}(A_{I'})$. Now

$$\Big(\tau_{I,I'} \circ \tau_I\Big)^*(T_i') = \tau_I^*(T_i) = \mathcal{R}_i = \tau_{I'}^*(T_i')$$

for all $i \in \{\infty\} \cup I'$, and since this property characterises the morphism $\tau_{I'}$, we have a commutative diagram

$$\begin{tikzcd}
& \mathfrak{M}_\theta \arrow[dl, "\tau_I"'] \arrow[dr, "\tau_I"] & \\
\mathcal{M}(A_I) \arrow[rr, "\tau_{I,I'}"] & & \mathcal{M}(A_{I'}).
\end{tikzcd} \tag{3.13}$$

Proposition 3.1.4 gives a similar commutative diagram expressing the identity $\pi_{I,I'} \circ \pi_I = \pi_{I'}$ for morphisms between quiver varieties, while Theorem 3.3.8 establishes the identities $\iota_I \circ \pi_I = \tau_I$ and $\iota_{I'} \circ \pi_{I'} = \tau_{I'}$. Taken together, these identities show that the maps in all four triangles in the following pyramid diagram commute:

$$\begin{tikzcd}
& \mathfrak{M}_\theta \arrow[dl, "\pi_I"'] \arrow[dr, "\tau_I"] \arrow[ddl, "\pi_{I'}"] \arrow[ddr, "\tau_{I'}"'] & \\
\mathfrak{M}_{\theta_I} \arrow[rr, "\iota_I"] \arrow[d, "\pi_{I,I'}"'] & & \mathcal{M}(A_I) \arrow[d, "\tau_{I,I'}"] \\
\mathfrak{M}_{\theta_{I'}} \arrow[rr, "\iota_{I'}"] & & \mathcal{M}(A_{I'}).
\end{tikzcd} \tag{3.14}$$

To show that the morphisms around the pyramid's square base commute, choose for any closed point $x \in \mathfrak{M}_{\theta_I}$ a lift $y \in \pi_I^{-1}(x) \subset \mathfrak{M}_\theta$. Commutativity of the triangles in the diagram gives

$$\Big(\iota_{I'} \circ \pi_{I,I'}\Big)(x) = \iota_{I'}\big(\pi_{I'}(y)\big) = \tau_{I'}(y) = \tau_{I,I'}\big(\tau_I(y)\big) = \Big(\tau_{I,I'} \circ \iota_I\Big)(x),$$

and since $x \in \mathfrak{M}_\theta$ was arbitrary and π_I is surjective, we have that $\iota_{I'} \circ \pi_{I,I'} = \tau_{I,I'} \circ \iota_I$ as required. ■

We deduce the following.

Proposition 3.4.2. *The face poset of the cone $\overline{C^+}$ can be identified with the poset on the set of fine moduli spaces $\mathcal{M}(A_I)$ for non-empty subsets $I \subseteq \{0, \dots, s\}$ together with $\mathbb{C}^{2n}/\Gamma_n$, where edges in the Hasse diagram of the poset indicating inequalities $\mathfrak{M}(A_I) > \mathfrak{M}(A_{I'})$ and $\mathfrak{M}(A_I) > \mathbb{C}^{2n}/\Gamma_n$ are realised by the universal morphisms $\tau_{I,I'}$ and the structure morphisms $\varphi_{|\mathcal{O}_{\mathcal{M}(A_I)}|}$ respectively.*

3.5 Punctual Hilbert schemes for Kleinian singularities

In this section, we specialise to the case $I = \{0\}$ and study the algebra A_I, before establishing the link between the fine moduli space $\mathcal{M}(A_I)$ and the Hilbert scheme of n points on $\mathbb{C}^2/\Gamma$. It will be convenient to write dimension vectors of A_I-modules as pairs (v_∞, v_0) in this case.

As we saw in Proposition 3.2.3, the algebra A_I can be presented as the path algebra of a quiver modulo an ideal of relations. The relations appear to be fairly complicated, but for $I = \{0\}$ it is possible to give an explicit presentation of A_I; this will turn out to be sufficient for our purposes. To spell this out, recall the construction of the quiver Q_I^* from Proposition 3.2.3 that has vertex set $\{\infty, 0\}$ and arrow set comprising one arrow b from ∞ to 0, and loops $\alpha_1, \alpha_2, \alpha_3$ at vertex 0 corresponding to a set of minimal $\mathbb{C}$-algebra generators of $\mathrm{Hom}_{R_0}(R_0, R_0) = R_0 \cong \mathbb{C}[x, y]^\Gamma$ as shown in Figure 3.2:

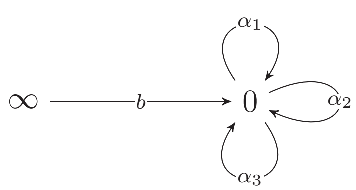


Figure 3.2: The quiver Q_I^* used in the presentation of A_I for $I = \{0\}$.

To state the presentation of A_I in this case, recall the presentation of the algebra $\mathbb{C}[x, y]^\Gamma$ from (3.4).

Lemma 3.5.1. *For $I = \{0\}$, the algebra A_I is isomorphic to the quotient of $\mathbb{C}Q_I^*$ by the two-sided ideal*

$$K = \big(f(\alpha_1, \alpha_2, \alpha_3), \alpha_1\alpha_2 - \alpha_2\alpha_1, \alpha_1\alpha_3 - \alpha_3\alpha_1, \alpha_2\alpha_3 - \alpha_3\alpha_2\big), \tag{3.15}$$

where $f \in \mathbb{C}[z_1, z_2, z_3]$ is the defining equation of the hypersurface $\mathbb{C}^2/\Gamma \subseteq \operatorname{Spec}\mathbb{C}[z_1, z_2, z_3]$.

Proof. In Proposition 3.2.3, we constructed a $\mathbb{C}$-algebra epimorphism $\beta_I \colon \mathbb{C}Q_I^* \to A_I$. The images under β_I of the arrows $\alpha_1, \alpha_2, \alpha_3$ correspond to minimal $\mathbb{C}$-algebra generators of $R_0 \cong e_0 A e_0$, so the ideal K lies in the kernel of β_I. We claim that the induced $\mathbb{C}$-algebra epimorphism

$$\beta_I \colon \mathbb{C}Q_I^*/K \longrightarrow A_I$$

is an isomorphism. Define a $\mathbb{C}$-algebra homomorphism $\gamma_I \colon A_I \to \mathbb{C}Q_I^*/K$ by sending the chosen minimal $\mathbb{C}$-algebra generators of $R_0 \cong e_0 A_I e_0$ to the classes of the arrows $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{C}Q_I^*/K$, and by sending the classes of e_∞ and b in A_I to the classes of the paths e_∞ and b in $\mathbb{C}Q_I^*/K$. This defines a $\mathbb{C}$-algebra homomorphism, because $e_0 A_I e_0$ is a subalgebra of A with quotient $A/(e_\infty)$. Clearly $\gamma_I = \beta_I^{-1}$ as required. ■

Proposition 3.5.2. *For the subset $I = \{0\}$, there is an isomorphism of schemes*

$$\omega_n \colon \operatorname{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \to \mathcal{M}(A_I)$$

over $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$.

Proof. We begin by constructing the morphism of schemes ω_n. Let $\mathcal{T}$ denote the tautological rank n bundle on $\operatorname{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$, and write $\mathcal{O}$ for the trivial bundle. In light of the universal property of $\mathcal{M}(A_I)$, it suffices to show that $\mathcal{O} \oplus \mathcal{T}$ carries a natural structure of a flat family of η_I-stable A_I-modules of dimension vector $v_I = (1, n)$ on $\operatorname{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$. A closed point of $\operatorname{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ corresponds to a codimension n ideal $I \lhd \mathbb{C}[x, y]^\Gamma \cong \mathbb{C}[z_1, z_2, z_3]/(f)$. The quotient vector space $\mathbb{C}[x, y]^\Gamma/I$ is of dimension n, it carries the action of commuting arrows $\alpha_1, \alpha_2, \alpha_3$ satisfying the relation f, and has a distinguished generator $[1] \in \mathbb{C}[x, y]^\Gamma/I$, which can be thought of as the image of a map w from a one-dimensional vector space. Lemma 3.5.1 now shows that we get the data of an A_I-module of dimension vector $(1, n)$. This module is moreover cyclic with generator at vertex ∞, so it is η_I-stable as required. This construction works relatively over the whole of $\operatorname{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$, equipping $\mathcal{O} \oplus \mathcal{T}$ with the structure of a family of η_I-stable A_I-modules as claimed. Moreover, since the bundle $\mathcal{O} \oplus \mathcal{T}$ inducing ω_n has

$\mathcal{O}$ as a summand, and since the trivial bundle on any scheme induces the structure morphism, we see that ω_n commutes with the structure morphisms to $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$.

Reading Lemma 3.5.1 in the opposite direction, an η_I-stable A_I-module of dimension vector $(1,n)$ defines a cyclic $\mathbb{C}[x,y]^\Gamma$-module of dimension n over $\mathbb{C}$. The universal property of $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ then ensures that the flat family $T_\infty \oplus T_0$ of η_I-stable A_I-modules of dimension vector $(1,n)$ over $\mathcal{M}(A_I)$ determines a morphism $\mathcal{M}(A_I) \to \mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$, which is by construction the inverse of the morphism ω_n. ■

We deduce Theorem 1.1.1 announced in the Introduction.

Corollary 3.5.3. *For any $n \geq 1$, the reduced scheme underlying* $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ *is isomorphic to the quiver variety $\mathfrak{M}_{\theta_0}$ for the parameter $\theta_0 = (-n,1,0,\dots,0)$ (compare Figure 3.1). In particular,* $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{\mathrm{red}}$ *is a normal, irreducible scheme over $\mathbb{C}^{2n}/\Gamma_n$ with symplectic singularities that admits a unique projective symplectic resolution, namely the morphism*

$$n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2) \to \mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{\mathrm{red}}$$

that sends an ideal J in $\mathbb{C}[x,y]$ to the ideal $J \cap \mathbb{C}[x,y]^\Gamma$.

Proof. The first statement follows from Theorem 3.3.8 and Proposition 3.5.2, while the geometric properties of $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{red}$ are all inherited from its manifestation as $\mathfrak{M}_{\theta_0}$ via Lemma 2.1.3.

Next we prove the statement about the resolution. In the notation of [5, Theorem 1.2], the extremal ray $\rho_1^\perp \cap \cdots \cap \rho_s^\perp$ of the cone F that contains $\theta_0 = (-n,1,0,\dots,0)$ lies in the closure of precisely one chamber, namely the chamber C^+. Under the isomorphism L_F from *ibid.*, it follows that there is exactly one projective symplectic resolution of $\mathfrak{M}_{\theta_0}$, namely the fine moduli space $\mathfrak{M}_\theta$ for $\theta \in C^+$. By Theorem 3.1.3, this resolution is indeed $\mathfrak{M}_\theta \cong n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$.

To see the last statement of the Corollary, consider the morphism $\tau_I \colon \mathfrak{M}_\theta \to \mathcal{M}(A_I)$ constructed in Lemma 3.3.2. This is obtained by restricting a representation of the framed preprojective algebra Π to the vertices $0,\infty$, noting that the map of vector bundles $\mathcal{R}_0 \to \mathcal{R}_\infty$ is the zero map, and thus we indeed get a representation of A_I. On the other hand, as we discussed before, the isomorphism $\mathfrak{M}_\theta \cong n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ identifies a Π-module with the quotient $\mathbb{C}[x,y]/J$ for a Γ-invariant ideal J of $\mathbb{C}[x,y]$. In this language, the restriction to the 0 vertex is the Γ-invariant part of the quotient $\mathbb{C}[x,y]/I$. The statement follows. ■

Remark 3.5.4. 1. Irreducibility of $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ was first established by Zheng [68] through the study of maximal Cohen–Macaulay modules on Kleinian singularities using a case-by-case analysis following the ADE classification.

2. Uniqueness of the symplectic resolution of $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ was previously known in the special case $n = 2$ by the work of Yamagishi [67, Proposition 2.10].

3. Our approach does not shed light on whether $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ is reduced in its natural scheme structure, coming from its moduli space interpretation.

Remark 3.5.5. For $n = 1$, the statement of Theorem 1.1.1 is well known because $\mathrm{Hilb}^{[1]}(\mathbb{C}^2/\Gamma) \cong \mathbb{C}^2/\Gamma$, while the statement of Theorem 3.3.8 is a framed version of [15, Theorem 1.2] for $\Gamma \subset \mathbf{SL}(2, \mathbb{C})$. Nevertheless, the approach of the current chapter is valid for $n = 1$ and shows in particular that $\mathfrak{M}_{\theta_I} \cong \mathbb{C}^2/\Gamma$ for $I = \{0\}$. In fact, this result follows from [5, Proposition 7.11]. Indeed, *ibid.* constructs a surjective linear map $L_{C^+} \colon \Theta_v \to N^1(S/(\mathbb{C}^2/\Gamma))$ with kernel equal to the subspace spanned by $(-1, 1, 0, \ldots, 0)$, such that $L_{C^+}(C^+)$ is the ample cone of S over $\mathbb{C}^2/\Gamma$. Since $\theta_I = (-1, 1, 0, \ldots, 0)$ for $I = \{0\}$ and $n = 1$, it follows that $\mathfrak{M}_{\theta_I} \cong \mathbb{C}^2/\Gamma$ in that case. In addition, this explicit description of the kernel of L_{C^+} for $n = 1$ shows that the morphisms $\pi_{I,I'}$ and $\tau_{I,I'}$ from Propositions 3.4.2 and 3.1.4 are isomorphisms if and only if $I' \smallsetminus I = \{0\}$.

3.6 An explicit description of the bijection

If we simply wish to describe the bijection of closed points $\mathfrak{M}_{\theta_0}(\mathbb{C}) \xrightarrow{\sim} \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)(\mathbb{C})$, it is possible to make a more explicit argument.

We will construct a map f on closed points:

$$f \colon \mathfrak{M}_{\theta_0}(\mathbb{C}) \to \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)(\mathbb{C})$$

and show that it is a bijection, without using the cornered algebra A_I.

Recall that there is a diagram

$$\begin{tikzcd} n\Gamma\text{-}\mathrm{Hilb}(\mathbb{C}^2) \arrow[r, "\sim"] \arrow[d, "\phi"] & \mathfrak{M}_\theta(1, n\delta) \arrow[d, "\psi"] \\ \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma) & \mathfrak{M}_{\theta_0}(1, n\delta) \end{tikzcd} \tag{3.16}$$

where the top arrow is an isomorphism ([64], [40, Theorem 39]). Furthermore, ϕ and ψ are both surjective, ϕ by [68], and ψ by Lemma 3.1.1. So take some point

$x \in \mathfrak{M}_{\theta_0}$. Choose some $y \in \mathfrak{M}_\theta$ such that $\psi(y) = x$. Then y corresponds to some Γ-invariant ideal $J \subset \mathbb{C}[x, y]$, and we set $I := J \cap \mathbb{C}[x, y]^\Gamma$. We will show that I is independent of the choice of y, and create an inverse to this map. Let x_{con} be the concentrated representation corresponding to the equivalence class x. By the isomorphism in Proposition 3.5.2, x_{con} corresponds to a Γ-invariant $\mathbb{C}[x, y]$-ideal I_{con}.

Consider the poset

$$\mathcal{I} := \{J | J \text{ is a } \Gamma\text{-invariant } \mathbb{C}[x, y]\text{-ideal and } J \cap \mathbb{C}[x, y]^\Gamma = I\},$$

ordered by inclusion. As $\mathbb{C}[x, y]$ is Noetherian, $\mathcal{I}$ has maximal elements. If $K_1 \neq K_2$ were both maximal elements, their sum $K_1 + K_2$ would contain them both and also lie in $\mathcal{I}$, a contradiction. So $\mathcal{I}$ has a unique maximal element $\widetilde{I}$.

We now claim that

Lemma 3.6.1. $I_{\text{con}} = \widetilde{I}$.

Proof. First, note that x_{con} is a quotient of some representation $y \in \mathfrak{M}_\theta$ such that y corresponds to a Γ- invariant ideal J with $J \cap \mathbb{C}[x, y]^\Gamma = I$. This implies that $I_{\text{con}} \supseteq J$. Because x_{con} is θ_0-stable, $\dim_0 x_{\text{con}} = n$, so $I_{\text{con}} \cap \mathbb{C}[x, y]^\Gamma$ is n-dimensional. This means that the superset relation in $I_{\text{con}} \cap \mathbb{C}[x, y]^\Gamma \supseteq J \cap \mathbb{C}[x, y]^\Gamma$ is actually an equality, and it follows that we have $I_{\text{con}} \in \mathcal{I}$.

Suppose that I_{con} is not maximal in $\mathcal{I}$. Then there is some Γ-invariant ideal $J \supsetneq I_{\text{con}}$ such that $J \cap \mathbb{C}[x, y]^\Gamma = I$. If we set M_J to be the stable Π-module corresponding to J, this implies that $\dim M_J < \dim x_{\text{con}}$. Hence M_J is a strict θ_0-semistable submodule of x_{con} with $\dim_0 M_J = 1$. But this contradicts the definition of x_{con}. The equality follows. ∎

Proposition 3.6.2. *There is a canonical bijection on closed points*

$$\text{Hilb}^n(\mathbb{C}^2/\Gamma)(\mathbb{C}) \longleftrightarrow \mathfrak{M}_{\theta_0}(\mathbb{C}).$$

Proof. To define a map $\text{Hilb}^n(\mathbb{C}^2/\Gamma)(\mathbb{C}) \to \mathfrak{M}_{\theta_0}(\mathbb{C})$, let $I \in \text{Hilb}^n(\mathbb{C}^2/\Gamma)$, and let $J \in n\Gamma\text{-}\text{Hilb}(\mathbb{C}^2)$ be a Γ-equivariant ideal such that $J \cap \mathbb{C}[x, y]^\Gamma = I$. Then J corresponds to a θ-stable module $M \in \mathfrak{M}_\theta$. Let $N = \psi(M)$. Then $I \to N$ is our candidate for a map, and we must show that it is well-defined. This is handled by Lemma 3.6.1, which shows that the S-equivalence class of N is determined by its concentrated representation, which is uniquely determined by I.

In the other direction, let $x \in \mathfrak{M}_{\theta_0}$, and let $M \in \mathfrak{M}_\theta$ be a representation such that $\psi(M) = x$.

Then M corresponds to some Γ-invariant ideal $J \in n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$, and we can map M to $I \coloneqq J \cap \mathbb{C}[x,y]^\Gamma$. To see that this map is well-defined, note that x is uniquely determined by the stable Π-module x_{con}. By Lemma 2.2.4, x_{con} corresponds to an ideal I_{con}, such that (as in the proof of Lemma 3.6.1) $I_{\mathrm{con}} \cap \mathbb{C}[x,y]^\Gamma = I$. But since $J \subset I_{\mathrm{con}}$, and also satisfies $J \cap \mathbb{C}[x,y]^\Gamma = I$, any other J would also map to I.

These two maps are clearly mutual inverses. ■

As a corollary, we can see what happens on closed points upon degenerating to the stability condition $\mathbf{0} = (0, 0, \ldots, 0)$. As mentioned in Section 1.7, this is well-known, but it seems worthwhile to give another proof for completeness.

Corollary 3.6.3. *There is a canonical bijection of closed points* $\mathfrak{M}_0(1, n\delta)(\mathbb{C}) \Leftrightarrow \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)(\mathbb{C})$.

Proof. Take $x \in \mathfrak{M}_{\mathbf{0}}$. As $\mathbf{0}$-stability is simplicity, any Π-module $M \in x$ is S-equivalent to the direct sum $\widetilde{M}$ of all its *simple* subquotients. As $\bar{b}$ is represented in M by the zero map (Lemma 3.2.1), one of the summands in $\widetilde{M}$ will be a one-dimensional representation at ∞. Let N be the submodule of M generated by the idempotents at every other vertex. Thus N can be interpreted a module of the preprojective algebra Π_Γ of the unframed quiver. Now assume that there is a simple module P of Π_Γ such that $\dim_0 P > 1$. Consider the endomorphisms $\operatorname{End} P_0$ of the vector space P_0 (the part of the module P supported at the 0-vertex), generated by the commuting images of $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}Q_0^*$ from Lemma 3.5.1. Then $\operatorname{End} P_0$ is an abelian, hence solvable Lie algebra, so by Lie's theorem the α_i have a shared eigenvector $v \in P_0$. Then v generates a strict submodule of P, so P is not simple. Therefore, if P is a simple Π_Γ-module, $\dim_0 R$ is either 0 or 1.

By a straightforward adaptation of the argument in Proposition A.0.1, it follows that simple Π_Γ-modules are either vertex simples, or have dimension δ.

By interpreting the α_i as multiplication by the generators of $\mathbb{C}[x,y]^\Gamma$, again as in Lemma 3.5.1, we find that any Π_Γ-module P with $\dim_0 P = 1$ determines a maximal ideal of $\mathbb{C}[x,y]^\Gamma$. As x is S-equivalent to a polystable representation containing exactly n such modules, this shows that x gives a point of $\operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$.

Now let $y \in \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$. Then y is a set of n points of $\mathbb{C}^2/\Gamma = \operatorname{Hilb}^1(\mathbb{C}^2/\Gamma) = \mathfrak{M}_{\theta_0}(1,\delta) = \mathfrak{M}_0(1,\delta)$. Take the direct sum of the $\mathbf{0}$-semistable Π-modules corresponding to each point of y. This gives a 0-polystable (or semisimple) Π-module of dimension $(n, n\delta)$. But as mentioned above, such a module is S-equivalent to a 0-polystable (or semistable) Π-module in which every summand P with $\dim_\infty P \geq 1$ is a vertex simple module supported at ∞. Removing all but one of these gives a representative of a point in $\mathfrak{M}_0(1, n\delta)$.

$\blacksquare$

Chapter 4

Quot schemes of Kleinian singularities

In this chapter, we provide interpretations of the class of quiver varieties $\mathfrak{M}_{\theta_I}(1, \mathbf{v})$ for arbitrary $\mathbf{v}$. This generalises the main result (Proposition 3.5.2) from the previous chapter, and in fact reprove it with different techniques.

The main ingredient is a deeper analysis of the various moduli spaces of modules of the algebras A_I, which appeared in the previous chapter.

This chapter is a close adaptation of [17], which I was a coauthor of.

For this chapter, we also fix $r = 1$. The underlying quiver Q thus still has a single arrow $b \colon \infty \to 0$, and a single arrow $\bar{b} \colon 0 \to \infty$.

We start by investigating a general class of Quot schemes.

4.1 Orbifold Quot schemes

4.1.1 Quot schemes for modules over associative algebras

Let H denote an arbitrary finitely generated, not necessarily commutative $\mathbb{C}$-algebra, and let M be a finitely generated left H-module. For a scheme X, we denote by H_X the sheaf of $\mathcal{O}_X$-algebras $H \otimes \mathcal{O}_X$, where H is considered as a constant sheaf on X. Similarly, denote by M_X the sheaf of H_X-modules $M \otimes \mathcal{O}_X$.

Consider the functor

$$\mathcal{Q}^n_H(M) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$$

sending a scheme X to the set of isomorphism classes of left H_X-modules Z equipped with a surjective H_X-module homomorphism $M_X \to Z$ such that Z, when considered as an $\mathcal{O}_X$-module, is locally free of rank n.

The following result may be known to experts, though we could not find it in the literature.

Proposition 4.1.1. *The functor $\mathcal{Q}^n_H(M)$ is represented by a scheme* $\mathrm{Quot}^n_H(M)$ *of finite type over* $\mathbb{C}$.

Proof. As H is finitely generated as a $\mathbb{C}$-algebra, we can fix a surjection $H^+ \to H$ with H^+ a free noncommutative $\mathbb{C}$-algebra on a finite number of generators. Then M is also a left H^+-module. Define $J \coloneqq \ker(H^+ \to H)$. Consider a left H^+-module Z that is a quotient of M via a surjective H^+-module morphism $M \to Z$.

Since any element of the two-sided ideal J of H^+ acts trivially on M, and so also on quotients of M, Z is automatically an H-module. Conversely, any quotient of M as an H-module is automatically an H^+-module. It follows that there is a canonical isomorphism of functors $\mathcal{Q}^n_H(M) = \mathcal{Q}^n_{H^+}(M)$.

Thus, we may assume that H is a finitely generated and free $\mathbb{C}$-algebra. Fix a left H-module surjection $\psi\colon H^r \to M$ for some r. Write elements of $\ker\psi$ in the form $\sum_{1\leq k\leq s} w_k e_k$, where the $e_k \in H^r$ are the standard module generators and $w_k \in H$. For any quotient module $q\colon M \to Z$, composition with ψ presents Z as a quotient $\widetilde{q}\colon H^r \to Z$, such that $\ker\psi \subseteq \ker\widetilde{q}$. The equations $\sum_{1\leq k\leq s} w_k\widetilde{q}(e_k) = 0$ give a closed condition on $[\widetilde{q}] \in \mathrm{Quot}^n_H(H^r)$: the vanishing of a collection of vectors in the vector space underlying Z. When trivialising the universal sheaf on some open cover of $Q^n_H(H^r)$, these closed conditions glue together, and we see that this construction realises

$$\mathcal{Q}^n_H(M) \subseteq \mathcal{Q}^n_H(H^r)$$

as a closed subfunctor.

To conclude, it suffices to show that $\mathcal{Q}^n_H(H^r)$ is representable for a free noncommutative $\mathbb{C}$-algebra H. This follows from [4, Theorem 2.5]; while that paper discusses the case when H is freely generated by 3 elements, the proof generalises to finitely many generators. ∎

4.1.2 Quot schemes for Kleinian orbifolds

Let Π_Γ be the preprojective algebra of the unframed McKay quiver and $R = \mathbb{C}[x, y]$ as in Section 2.2.1. Recall that $R_0 = R^\Gamma$, and that for each $0 \leq i \leq s$, the R_0-module $R_i \coloneqq \mathrm{Hom}_\Gamma(\rho_i, R)$ satisfies $R_i \cong e_i\Pi_\Gamma e_0$.

For any subset $I \subseteq \{0, \dots, s\}$, consider the idempotent $e_I \coloneqq \sum_{i\in I} e_i$ in Π_Γ, and define the $\mathbb{C}$-algebra

$$\Pi_{\Gamma I} \coloneqq e_I\Pi_\Gamma e_I \tag{4.1}$$

comprising linear combinations of classes of paths in Q_Γ whose tails and heads lie in the set I. The process of passing from Π_Γ to $\Pi_{\Gamma I}$ is called *cornering*; see [15, Remark 3.1]. Then

$$R_I := \bigoplus_{i\in I} R_i \cong e_I \Pi_\Gamma e_0$$

is naturally a finitely generated left $\Pi_{\Gamma I}$-module.

For a given dimension vector $n_I = (n_i) \in \mathbb{N}^I$, consider the contravariant functor

$$\mathcal{Q}^{n_I}_{\Pi_{\Gamma I}}(R_I) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$$

sending a scheme X to the set of isomorphism classes of left $\Pi_{\Gamma I,X}$-modules Z equipped with a surjective $\Pi_{\Gamma I,X}$-module homomorphism

$$\phi_Z \colon R_I \otimes \mathcal{O}_X \to Z$$

such that each submodule $e_i Z$, for $i \in I$, when considered as an $\mathcal{O}_X$-module, is locally free of rank n_i. Note that we can recover Z from these submodules via

$$Z = \bigoplus_{i\in I} e_i Z.$$

The next result provides the link between this functor and that introduced in Section 4.1.1.

Proposition 4.1.2. *There is a finite decomposition*

$$\mathcal{Q}^{n}_{\Pi_{\Gamma I}}(R_I) = \coprod_{\substack{n_I \\ \sum_{i\in I} n_i = n}} \mathcal{Q}^{n_I}_{\Pi_{\Gamma I}}(R_I) \tag{4.2}$$

into open and closed subfunctors. In particular, each functor $\mathcal{Q}^{n_I}_{\Pi_{\Gamma I}}\Big(R_I\Big)$ is represented by a scheme $\mathrm{Quot}^{n_I}_I([\mathbb{C}^2/\Gamma])$ of finite type over $\mathbb{C}$.

Proof. The dimension vector $n_I = (n_i) \in \mathbb{N}^I$ is locally constant in a flat family of $\Pi_{\Gamma I}$-modules of fixed rank n, with each of the entries being lower semicontinuous by the Schur lemma. So we have a decomposition (4.2) into open and closed subfunctors. As R_I is finitely generated as a left $\Pi_{\Gamma I}$-module, the functor on the left hand side of (4.2) is represented by a scheme of finite type over $\mathbb{C}$ by Proposition 4.1.1. The last claim then follows. ∎

When the index set is a singleton $I = \{i\}$, we often simply write $\mathrm{Quot}^{n_i}_i([\mathbb{C}^2/\Gamma])$ for $\mathrm{Quot}^{n_{\{i\}}}_{\{i\}}([\mathbb{C}^2/\Gamma])$. See Section 4.1.3 for a discussion of this special case.

As is common with Hilbert schemes in other contexts, we consider collections of our Quot schemes for all possible dimension vectors. Define the orbifold Quot scheme for $[\mathbb{C}^2/\Gamma]$ and for the index set I to be

$$\mathrm{Quot}_I([\mathbb{C}^2/\Gamma]) \coloneqq \coprod_{n_I} \mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$$

where $n_I = (n_i)_{i\in I} \in \mathbb{N}^I$ as before. Again, when $I = \{i\}$ is a singleton, we simply write

$$\mathrm{Quot}_i([\mathbb{C}^2/\Gamma]) \coloneqq \coprod_{n_i\in\mathbb{N}} \mathrm{Quot}_i^{n_i}([\mathbb{C}^2/\Gamma]).$$

Denote by $\mathrm{Hilb}([\mathbb{C}^2/\Gamma])$ the Hilbert scheme of Γ-invariant finite colength subschemes of $\mathbb{C}^2$. As explained in more detail in [29, Sect.1.1], this space decomposes as a disjoint union of quasi-projective varieties

$$\mathrm{Hilb}([\mathbb{C}^2/\Gamma]) = \coprod_{v\in\mathbb{N}^{s+1}} \mathrm{Hilb}^v([\mathbb{C}^2/\Gamma]),$$

where

$$\mathrm{Hilb}^v\left([\mathbb{C}^2/\Gamma]\right) = \left\{J \in \mathrm{Hilb}\left([\mathbb{C}^2/\Gamma]\right) \mid H^0(\mathcal{O}_{\mathbb{C}^2}/J) \cong \bigoplus_{i\in\{0,\dots,s\}} \rho_i^{\oplus v_i}\right\}.$$

Lemma 4.1.3. *For $I = \{0,\dots,s\}$, there is an isomorphism*

$$\mathrm{Quot}_I([\mathbb{C}^2/\Gamma]) \cong \mathrm{Hilb}([\mathbb{C}^2/\Gamma])$$

which respects the decompositions on both sides into pieces indexed by $v \in \mathbb{N}^{s+1}$.

Proof. This follows from the Morita equivalence between the algebras S and Π_Γ from Proposition 2.2.7, and its amplification Corollary 2.2.9, together with the fact that a finite colength left S-submodule of $R = \mathbb{C}[x, y]$ is nothing but a finite colength Γ-invariant ideal of $\mathbb{C}[x, y]$. ■

4.1.3 Quot schemes for Kleinian singularities

In this section, we discuss the relation of the construction from Section 4.1.2 to the more classical (commutative) version of the Quot scheme, in the case when $I = \{i\}$ is a singleton. Note that in this case, $R_i \cong e_i\Pi_\Gamma e_0$ is both a left $\Pi_{\Gamma i}$-module and a right $R_0 \cong e_0\Pi_\Gamma e_0$-module.

Let $\mathrm{Quot}_i(\mathbb{C}^2/\Gamma)$ be the scheme parameterising finite codimension R^Γ-submodules of R_i or, equivalently, finite colength subsheaves of the coherent sheaf on the Kleinian singularity $\mathbb{C}^2/\Gamma$ corresponding to R_i. Again, this space decomposes as

$$\mathrm{Quot}_i(\mathbb{C}^2/\Gamma) = \coprod_{n_i \in \mathbb{N}} \mathrm{Quot}_i^{n_i}(\mathbb{C}^2/\Gamma)$$

with $\mathrm{Quot}_i^{n_i}(\mathbb{C}^2/\Gamma) = \left\{M \in \mathrm{Quot}_i(\mathbb{C}^2/\Gamma) \mid \dim R_i/M = n_i\right\}$.

Suppose that $i \in \{0, \ldots, s\}$ satisfies $\dim \rho_i = 1$. Then there is a diagram automorphism of Q_Γ taking vertex i to vertex 0. Pick any such automorphism ι and let $\iota(0) \in \{0, \ldots, s\}$ denote the vertex to which the vertex 0 is mapped under this diagram automorphism. This transformation of the diagram induces an algebra isomorphism

$$\Pi_{\Gamma i} = e_i \Pi_\Gamma e_i \cong e_0 \Pi_\Gamma e_0 = \Pi_{\Gamma 0}.$$

Also, it induces a bijection between left $\Pi_{\Gamma i}$-submodules of $e_i \Pi_\Gamma e_0$ and left $\Pi_{\Gamma 0}$-submodules of $e_0 \Pi_\Gamma e_{\iota(0)}$. Reversing the arrows gives an algebra automorphism $\Pi_{\Gamma 0} \xrightarrow{\sim} \Pi_{\Gamma 0}$ as well as a bijective correspondence between left $\Pi_{\Gamma 0}$-submodules of $e_0 \Pi_\Gamma e_{\iota(0)}$ and right $\Pi_{\Gamma 0}$-submodules of $e_{\iota(0)} \Pi_\Gamma e_0$. Putting these two correspondences together, we get a bijection between the left $\Pi_{\Gamma i}$-submodules of $R_i = e_i \Pi_\Gamma e_0$ and right $\Pi_{\Gamma 0}$-submodules of $R_{\iota(0)} = e_{\iota(0)} \Pi_\Gamma e_0$.

Proposition 4.1.4. *Suppose that $I = \{i\}$ is a singleton such that $\dim \rho_i = 1$. Let $n_i \in \mathbb{N}$ be a dimension vector for the index set $\{i\}$, and denote by $n_{\iota(0)}$ the same dimension vector for the index set $\{\iota(0)\}$ with $\iota(0)$ as above. Then $\mathrm{Quot}_i^{n_i}([\mathbb{C}^2/\Gamma])$ is isomorphic to $\mathrm{Quot}_{\iota(0)}^{n_{\iota(0)}}(\mathbb{C}^2/\Gamma)$.*

Proof. The above correspondence between left $\Pi_{\Gamma i}$-submodules of $e_i \Pi_\Gamma e_0$ and right $\Pi_{\Gamma 0}$-submodules of $e_{\iota(0)} \Pi_\Gamma e_0$ is functorial and preserves the colength. By definition, the scheme $\mathrm{Quot}_{\iota(0)}^{n_{\iota(0)}}(\mathbb{C}^2/\Gamma)$ represents the functor that sends a scheme X to the set of isomorphism classes of right $\Pi_{\Gamma 0, X}$-module quotients $R_{\iota(0), X} \to Z$ which are locally free of rank $n_{\iota(0)}$ over $\mathcal{O}_X$. The claim follows. ∎

Remark 4.1.5. The assumption of Proposition 4.1.4 is satisfied for type A_s and any $0 \leq i \leq s$, type D_s and $i \in \{0, 1, s-1, s\}$, E_6 and $i \in \{0, 1, 6\}$, E_7 and $i \in \{0, 7\}$ and for E_8 and $i = 0$. Here we have used the Bourbaki convention [7] for enumerating the vertices.

Example 4.1.6. For $i = 0$, the assumption of Proposition 4.1.4 is satisfied in all types, with $\iota(0) = 0$ also. Writing $n_0 = n$, we obtain isomorphisms

$$\mathrm{Quot}_0^{n_0}([\mathbb{C}^2/\Gamma]) \cong \mathrm{Quot}_0^{n_0}(\mathbb{C}^2/\Gamma) \cong \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma).$$

4.2 Fine moduli spaces of modules

4.2.1 Cornering and fine moduli spaces

We remind the reader of some constructions from the previous chapter.

Recall that $A = \Pi/(\bar{b})$ is the quotient of the preprojective algebra of the framed McKay quiver of Γ by the two sided ideal $(\bar{b})$, where b^* is the arrow from 0 to ∞.

We defined in (3.3) a subalgebra of A comprising linear combinations of classes of paths in Q whose tails and heads lie in the set $\{\infty\} \cup I$.

Let $n_I \coloneqq (n_i)_{i\in I} = \sum_{i\in I} n_i\rho_i \in \mathbb{N}^I$. Then $(1, n_I) \coloneqq \rho_\infty + n_I \in \mathbb{N}\oplus\mathbb{N}^I$ is a dimension vector for A_I-modules, and we consider the stability condition $\eta_I \colon \mathbb{Z} \oplus \mathbb{Z}^I \to \mathbb{Q}$ given by

$$\eta_I(\rho_i) = \begin{cases} -\sum_{j\in I} n_j & \text{for } i = \infty, \\ 1 & \text{if } i \in I. \end{cases} \tag{4.3}$$

It follows directly from the definition that an A_I-module N of dimension vector $(1, n_I)$ is η_I-stable if and only if there exists a surjective A_I-module homomorphism $A_I e_\infty \to N$. The quiver moduli space construction of King [36, Proposition 5.3] for finite dimensional algebras can be adapted to any algebra presented as the quotient of a finite, connected quiver by an ideal of relations. We established in Proposition 3.2.3 that the algebra A_I has this property. The vector $(1, n_I)$ is indivisible and η_I is a generic stability condition for A_I-modules of dimension vector $(1, n_I)$, so there is a fine moduli space $\mathcal{M}_{A_I}(1, n_I)$ of η_I-stable A_I-modules of dimension vector $(1, n_I)$. As in Section 3.3, let $T_I \coloneqq \bigoplus_{i\in\{\infty\}\cup I} T_i$ denote the tautological vector bundle on $\mathcal{M}_{A_I}(1, n_I)$, where T_∞ is the trivial bundle and T_i has rank n_i for $i \in I$. After multiplying η_I by a positive integer $m \geq 1$ if necessary, we may assume that the polarising ample line bundle $\mathcal{L}_I \coloneqq \bigotimes_{i\in I} \det(T_i)^m$ on $\mathcal{M}_{A_I}(1, n_I)$ given by the GIT construction is very ample.

4.2.2 The Quot scheme as a fine moduli space of modules

We now establish the link between the fine moduli space $\mathcal{M}_{A_I}(1, n_I)$ and the orbifold Quot scheme $\mathrm{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$.

For any non-empty subset $I \subseteq \{0, \dots, s\}$, consider the algebras A_I and $\Pi_{\Gamma I}$ from (3.3) and (4.1) respectively. In Proposition 3.2.3, we used the isomorphism $\Pi_{\Gamma I} \cong \mathrm{End}_{R_0}\left(\bigoplus_{i\in I} R_i\right)$ to introduce quivers Q_I and Q_I^* and a commutative diagram of $\mathbb{C}$-algebra homomorphisms

$$\begin{tikzcd} \mathbb{C}Q_I \arrow[r] \arrow[d, "\alpha_I"] & \mathbb{C}Q_I^* \arrow[d, "\beta_I"] \\ \Pi_{\Gamma I} \arrow[r] & A_I \end{tikzcd}$$

where the vertical maps are surjective and the horizontal maps are injective. In particular, we obtain quiver presentations $\Pi_{\Gamma I} \cong \mathbb{C}Q_I/\ker(\alpha_I)$ and $A_I \cong \mathbb{C}Q_I^*/\ker(\beta_I)$. The vertex set of Q_I^* is $\{\infty\} \cup I$, while the edge set comprises three kinds of arrows: loops at each vertex $i \in I$; arrows between pairs of distinct vertices in I; and arrows from vertex ∞ to vertices in I. Note that Q_I is the complete subquiver of Q_I^* on the vertex set I.

Lemma 4.2.1. *Equip the vector space $\Pi_{\Gamma I} \oplus R_I \oplus \mathbb{C}e_\infty$ with the following multiplication:*

$$(b_1, r_1, c_1e_\infty) \cdot (b_2, r_2, c_2e_\infty) := (b_1b_2, b_1r_2 + r_1c_2, c_1c_2e_\infty).$$

Then there is an isomorphism

$$A_I \cong \Pi_{\Gamma I} \oplus R_I \oplus \mathbb{C}e_\infty$$

of $\mathbb{C}$-algebras, where $R_I = e_I\Pi_\Gamma e_0$ is also a left $\Pi_{\Gamma I}$-module.

Proof. It follows from Lemma 3.2.2 that there is an isomorphism of vector spaces

$$A_I \cong (e_I + e_\infty)(\Pi_\Gamma \oplus \Pi_\Gamma b \oplus \mathbb{C}e_\infty)(e_I + e_\infty) \cong \Pi_{\Gamma I} \oplus e_I\Pi_\Gamma b \oplus \mathbb{C}e_\infty.$$

For the middle summand, consider the map of left $\Pi_{\Gamma I}$-modules

$$R_I = e_I\Pi_\Gamma e_0 \longrightarrow e_I\Pi_\Gamma b \tag{4.4}$$

defined on representing paths in the framed McKay quiver Q by composing with the arrow b on the right. This map is by definition surjective. But it is also injective, as every relation in A involving the arrow b can be factored into the product of b and a relation in Π_Γ. This establishes the isomorphism

$$A_I \cong \Pi_{\Gamma I} \oplus R_I \oplus \mathbb{C}e_\infty$$

of vector spaces. To enhance this to an isomorphism of $\mathbb{C}$-algebras, note that the algebra structure on $A_I \cong \Pi_{\Gamma I} \oplus e_I\Pi_\Gamma b \oplus \mathbb{C}e_\infty$ is given by

$$(b_1, r_1b, c_1e_\infty) \cdot (b_2, r_2b, c_2e_\infty) = \Big(b_1b_2, b_1r_2b + r_1bc_2e_\infty, c_1c_2e_\infty\Big).$$

Since c_2 is a scalar, it commutes with $b = be_\infty$, giving $b_1r_2b + r_1bc_2e_\infty = (b_1r_2 + r_1c_2)b$ in the middle term above. The isomorphism (4.4) allows us to drop the b from the right-hand side, leaving the second statement as required. ∎

Proposition 4.2.2. *For any non-empty subset $I \subseteq \{0, \ldots, s\}$ and $n_I \in \mathbb{N}^I$, there is an isomorphism of schemes*

$$\omega_{n_I} \colon \operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma]) \longrightarrow \mathcal{M}_{A_I}(1, n_I).$$

In particular, $\operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$ *is non-empty if and only if* $\mathcal{M}_{A_I}(1, n_I)$ *is non-empty.*

Proof. Denote by $\mathcal{T}$ the tautological bundle on $\operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$, a family of quotient $\Pi_{\Gamma I}$-modules of R_I of rank $\sum_{i\in I} n_i$. Moreover, let $\mathcal{O}$ be the trivial bundle on $\operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$.

We give $\mathcal{O} \oplus \mathcal{T}$ the structure of a family of A_I-modules using the decomposition from Lemma 4.2.1. Indeed, the $\Pi_{\Gamma I}$-module structure on $\mathcal{T}$ extends to a $\Pi_{\Gamma I}$-module structure on $\mathcal{O} \oplus \mathcal{T}$ by acting by 0 on the first summand. The summand $\mathbb{C}e_\infty$ acts by scaling on $\mathcal{O}$ and trivially on $\mathcal{T}$. Finally, as the bundle $\mathcal{T}$ is a quotient of the trivial family R_I, any element r of the middle summand $R_I \subset A_I$ determines an $\mathcal{O}$-module homomorphism $\phi_s \in \operatorname{Hom}(\mathcal{O}, \mathcal{T})$ via the correspondence $1 \mapsto [r]$. As R_I is a left $\Pi_{\Gamma I}$-module, the $\mathbb{C}$-linear correspondence $s \mapsto \phi_s$ is left $\Pi_{\Gamma I}$-linear. The action of two elements $(b_1, r_1, c_1e_\infty), (b_2, r_2, c_2e_\infty) \in A_I$ on $\mathcal{O} \oplus \mathcal{T}$ therefore composes as

$$\begin{pmatrix} c_1 & 0 \\ \phi_{r_1} & b_1 \end{pmatrix} \circ \begin{pmatrix} c_2 & 0 \\ \phi_{r_2} & b_2 \end{pmatrix} = \begin{pmatrix} c_1c_2 & 0 \\ \phi_{r_1}c_2 + b_1\phi_{r_2} & b_1b_2 \end{pmatrix} = \begin{pmatrix} c_1c_2 & 0 \\ \phi_{r_1c_2+b_1r_2} & b_1b_2 \end{pmatrix}.$$

Lemma 4.2.1 shows that the bundle $\mathcal{O} \oplus \mathcal{T}$ on $\operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$ is a flat family of A_I-modules of dimension vector $(1, n_I)$. Over any closed point, the resulting A_I-module is generated at ∞, and so it is η_I-stable. The universal property of the fine moduli space $\mathcal{M}_{A_I}(1, n_I)$ induces the morphism of schemes ω_{n_I}.

Conversely, applying Lemma 4.2.1 in the opposite direction gives that to any η_I-stable A_I-module of dimension vector $(1, n_I)$, there corresponds a cyclic $\Pi_{\Gamma I}$-module of dimension vector n_I. By applying this to the family given by the tautological bundle T_I on $\mathcal{M}_{A_I}(1, n_I)$, we see that the universal property of the fine moduli space $\operatorname{Quot}_I^{n_I}([\mathbb{C}^2/\Gamma])$ induces the inverse of the morphism ω_{n_I}. ■

4.3 Quiver varieties for the framed McKay quiver

4.3.1 Resolution of singularities

Let $v \in \mathbb{N}^{s+1}$ be a dimension vector and $I \subseteq \{0, \ldots, s\}$. As mentioned in Section 2.4.1, the quiver variety $\mathfrak{M}_{\theta_I}(1, v)$ is singular in general for θ_I as defined in (2.5) above. Moreover, for $\theta \in C_v^+$, the morphism

$$\mathfrak{M}_\theta(1, v) \to \mathfrak{M}_{\theta_I}(1, v)$$

obtained by variation of GIT need not be birational. It is nevertheless possible to find a resolution of $\mathfrak{M}_{\theta_I}(1,v)$ using a quiver variety if one modifies the dimension vector v:

Proposition 4.3.1. *Let $v \in \mathbb{N}^{s+1}$ and let $I \subseteq \{0,\dots,s\}$ be non-empty. Assume further that $\mathfrak{M}_{\theta_I}(1,v)$ is non-empty. There exists a dimension vector $\widetilde{v} \in \mathbb{N}^{s+1}$ satisfying $\widetilde{v}_i \leq v_i$ for $0 \leq i \leq s$ and $\widetilde{v}_i = v_i$ for $i \in I$, such that there is a projective resolution of singularities*

$$\pi_I \colon \mathfrak{M}_{\widetilde{\theta}}(1,\widetilde{v}) \to \mathfrak{M}_{\theta_I}(1,v),$$

where the stability condition $\widetilde{\theta}$ satisfies $\widetilde{\theta} \in C^+_{1,\widetilde{v}}$.

Proof. Following Nakajima [47, Section 6] (c.f. [5, Section 3.5]), there is a finite stratification

$$\mathfrak{M}_{\theta_I}(1,v) = \coprod_{\gamma} \mathfrak{M}_{\theta_I}(1,v)_\gamma$$

by smooth locally-closed subvarieties, where the union is over conjugacy classes of subgroups of the reductive group applied during the GIT construction of $\mathfrak{M}_{\theta_I}(1,v)$, and where the stratum $\mathfrak{M}_{\theta_I}(1,v)_\gamma$ consists of S-equivalence classes of semistable modules with polystable representative having stabiliser in the conjugacy class γ. Since $\mathfrak{M}_{\theta_I}(1,v)$ is irreducible by Lemma 2.1.3, there is a unique dense stratum and we fix a conjugacy class γ such that the dense open stratum in $\mathfrak{M}_{\theta_I}(1,v)$ is $\mathfrak{M}_{\theta_I}(1,v)_\gamma$. Apply [49, Proposition 2.25] to obtain a dimension vector $\widetilde{v} \in \mathbb{N}^{s+1}$ as in the statement of the lemma together with an isomorphism

$$\mathfrak{M}_{\theta_I}(1,v)_\gamma \cong \mathfrak{M}^s_{\theta_I}(1,\widetilde{v}) \times Y, \tag{4.5}$$

where $\mathfrak{M}^s_{\theta_I}(1,\widetilde{v})$ is the fine moduli space of θ_I-stable Π-modules of dimension vector $(1,\widetilde{v})$, and $Y \subset \mathfrak{M}_0(0, v-\widetilde{v})$ is a locally-closed subvariety. Note that $\mathfrak{M}_{\theta_I}(1,v)_\gamma$ is non-empty by assumption, hence so are $\mathfrak{M}^s_{\theta_I}(1,\widetilde{v})$ and Y.

We claim that Y is a closed point. Indeed, $\mathfrak{M}_0(0, v-\widetilde{v})$ parametrises 0-polystable representations, i.e. direct sums of simple representations. The framing is 0-dimensional, and since I is non-empty, there is at least one i such that $v_i = \widetilde{v}_i$. It follows that the representations parameterised by Y are supported on a doubled quiver obtained from an affine ADE diagram with at least one vertex removed. Removing a vertex from an affine ADE diagram leaves a graph in which every connected component is a subgraph of an finite type ADE diagram. Therefore $\mathfrak{M}_0(0, v-\widetilde{v})$ parametrises direct sums of simple representations of preprojective algebras of doubled quivers associated to finite type ADE diagrams. A simple representation of any such quiver is one-dimensional by Remark 2.4.8, so the only polystable representation of dimension vector $(0, v - \widetilde{v})$ is necessarily of the form

$$\bigoplus_{0\leq i\leq s} S_i^{\oplus(v_i-\widetilde{v}_i)}, \tag{4.6}$$

where each S_i is a vertex simple Π-module. Therefore $\mathfrak{M}_0(0, v - \widetilde{v})$ is a closed point, and hence so too is Y because $Y \neq \varnothing$. Thus $\mathfrak{M}_{\theta_I}(1, v)_\gamma \cong \mathfrak{M}^s_{\theta_I}(1, \widetilde{v})$.

For $\widetilde{\theta} \in C^+_{1,\widetilde{v}}$, we claim that the desired morphism $\pi_I \colon \mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v}) \to \mathfrak{M}_{\theta_I}(1, v)$ is induced by sending each $\widetilde{\theta}$-stable Π-module V of dimension vector $(1, \widetilde{v})$ to

$$\pi_I(V) := V \oplus \bigoplus_{0\leq i\leq s} S_i^{\oplus(v_i-\widetilde{v}_i)}.$$

Indeed, if $\widetilde{\mu}$ and μ denote the moment maps for the actions of $G_{\widetilde{v}} := \prod_{0\leq i\leq s} \mathrm{GL}(\widetilde{v}_i)$ and $G_v := \prod_{0\leq i\leq s} \mathrm{GL}(v_i)$ on the representation spaces of Π-modules of dimension vectors $(1, \widetilde{v})$ and $(1, v)$ respectively, then the above assignment induces an inclusion $\widetilde{\mu}^{-1}(0) \hookrightarrow \mu^{-1}(0)$ that is equivariant with respect to the actions of $G_{\widetilde{v}}$ and G_v on $\widetilde{\mu}^{-1}(0)$ and $\mu^{-1}(0)$ respectively. Any submodule of $\pi_I(V)$ is $W \oplus W'$ for submodules $W \subseteq V$ and $W' \subseteq \bigoplus_{0\leq i\leq s} S_i^{\oplus(v_i-\widetilde{v}_i)}$, and we have $\theta_I(W \oplus W') = \theta_I(W) \geq 0$. Thus, the image of the $\widetilde{\theta}$-stable locus of $\widetilde{\mu}^{-1}(0)$ lies in the θ_I-semistable locus of $\mu^{-1}(0)$, and this inclusion induces the morphism π_I as claimed.

It remains to establish the properties of π_I. Adapting the proof from [6, Lemma 2.4] shows that π_I is a projective morphism, while [6, Theorem 1.15] gives that $\mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v})$ is non-singular as $\widetilde{\theta}$ is generic. Our explicit description of π_I shows that it factors via the morphism

$$\mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v}) \to \mathfrak{M}_{\theta_I}(1, \widetilde{v}) \tag{4.7}$$

obtained by variation of GIT quotient. This morphism is birational by [49, Lemma 2.12]. To see that the induced morphism $\mathfrak{M}_{\theta_I}(1, \widetilde{v}) \to \mathfrak{M}_{\theta_I}(1, v)$ is also birational, notice that its restriction to the (open dense) stable locus $\mathfrak{M}^s_{\theta_I}(1, \widetilde{v})$ simply adds the summand (4.6) to each representative Π-module, so it agrees with the isomorphism from (4.5) above. Thus, the image of $\mathfrak{M}^s_{\theta_I}(1, \widetilde{v})$ under this induced morphism is $\mathfrak{M}_{\theta_I}(1, v)_\gamma$ which is dense in $\mathfrak{M}_{\theta_I}(1, v)$ by our choice of γ. This completes the proof. ■

Recall that $\mathfrak{M}_{\theta_I}(1, v)$ is a coarse moduli space of θ_I-polystable modules up to isomorphism [49, Proposition 2.9(2)], and that every θ_I-polystable module has a unique summand that has dimension 1 at the vertex ∞.

Lemma 4.3.2. *Let M be a θ_I-polystable representative for an arbitrary class $[M] \in \mathfrak{M}_{\theta_I}(1, v)$, and let M_{con} be the concentrated module associated to the class. Then*

$$\dim_i M_{\mathrm{con}} \leq \widetilde{v}_i$$

for $0 \le i \le s$ where $\widetilde{v}$ is as in Proposition 4.3.1.

Proof. The morphism π_I from Proposition 4.3.1 is surjective, because quiver varieties are irreducible and π_I is both proper and birational. Therefore $[M] = [\pi_I(V)]$ for some $\widetilde{\theta}$-stable Π-module V of dimension vector $(1, \widetilde{v})$. Each vertex simple S_i arising as a summand of $\pi_I(V)$ has dimension 0 at the vertex ∞, so we must have $M_{\mathrm{con}} \subseteq V$. This implies the lemma. ■

Suppose now that $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty, and apply Proposition 4.3.1 to obtain $\widetilde{v} \le v$ and a resolution of singularities $\pi_I \colon \mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v}) \to \mathfrak{M}_{\theta_I}(1, v)$ for $\theta \in C^+_{\widetilde{v}}$. For any $\theta' \in \Theta_{1,\widetilde{v}}$, there is a line bundle $L_{C^+_{\widetilde{v}}}(\theta') := \bigotimes_{0 \le i \le s} \det(\mathcal{R}_i)^{\theta'_i}$ on $\mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v})$. The line bundle $L_I := L_{C^+_{\widetilde{v}}}(\theta_I)$ plays an important role in what follows.

Lemma 4.3.3. *Let $\theta \in C^+_{\widetilde{v}}$, Then*

1. *for each $\theta' \in \overline{C^+_{\widetilde{v}}}$ the line bundle $L_{C^+_{\widetilde{v}}}(\theta')$ on $\mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v})$ is globally generated;*
2. *for any $I \subseteq \{0, \dots, s\}$, after multiplying θ_I by a positive integer if necessary, the morphism to the linear series of L_I decomposes as the composition of the morphism π_I from Proposition 4.3.1 and a closed immersion:*

$$\begin{tikzcd} \mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v}) \arrow[r, "\varphi_{|L_I|}"] \arrow[d, "\pi_I"'] & {|L_I|}. \\ \mathfrak{M}_{\theta_I}(1, v) \arrow[ur, hook] & \end{tikzcd}$$

Proof. Since $\theta \in C^+_{\widetilde{v}}$, the tautological bundles $\mathcal{R}_i$ on the quiver variety $\mathfrak{M}_{\widetilde{\theta}}(1, \widetilde{v})$ are globally generated for $i \in Q_0$ by [15, Corollary 2.4]. Therefore, $L_{C^+_{\widetilde{v}}}(\theta')$ is globally generated as $\theta'_i \ge 0$ for all $0 \le i \le s$. In particular, since $\theta_I \in \overline{C^+_{\widetilde{v}}}$, the rational map $\varphi_{|L_I|}$ is defined everywhere.

For (2), after taking a positive multiple of θ_I if necessary, we may assume that the polarising ample line bundle L'_I on $\mathfrak{M}_{\theta_I}(1, v)$ is very ample. Our explicit construction of π_I from the proof of Proposition 4.3.1 implies that the G_v-equivariant line bundle on $\mu^{-1}(0)$ determined by the character associated to θ_I restricts under the inclusion $\widetilde{\mu}^{-1}(0) \hookrightarrow \mu^{-1}(0)$ to the $G_{\widetilde{v}}$-equivariant line bundle on $\widetilde{\mu}^{-1}(0)$ determined by the character associated to θ_I. Thus, after descent, we obtain $\pi_I^*(L'_I) = L_I$ as required.

If $\varphi_{|L'_I|} \colon \mathfrak{M}_{\theta_I}(1, v) \to |L'_I|$ is the closed immersion, then

$$(\varphi_{|L'_I|} \circ \pi_I)^*(\mathcal{O}_{|L'_I|}(1)) = \pi_I^*(L'_I) = L_I = \varphi^*_{|L_I|}\Big(\mathcal{O}_{|L_I|}(1)\Big)$$

on $\mathfrak{M}_{\widetilde{\theta}}(1,\widetilde{v})$. The morphism to a complete linear series of a specific line bundle is unique up to an automorphism of the linear series [31, II, Theorem 7.1]. Thus, after a change of basis on $H^0(L'_I)$ if necessary, we have $\varphi_{|L_I|} = \varphi_{|L'_I|} \circ \pi_I$ as required. ■

4.4 Quiver varieties and moduli spaces for cornered algebras

4.4.1 The key commutative diagram

Once and for all, fix a non-empty $I \subseteq \{0, 1, \ldots, s\}$ and a dimension vector $n_I = (n_i)_{i\in I} \in \mathbb{N}^I$.

Proposition 4.4.1. *Let $v = (v_j)_{0\leq j\leq s} \in \mathbb{N}^{s+1}$ be any vector satisfying $v_i = n_i$ for all $i \in I$. Suppose that $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty, and define $\widetilde{v}$ as in Proposition 4.3.1. Then $\widetilde{v}$ satisfies $\widetilde{v}_i = n_i$ for $i \in I$, and there is a commutative diagram*

$$\begin{tikzcd}
 & \mathfrak{M}_{\widetilde{\theta}}(1,\widetilde{v}) \arrow[dl, "\pi_I"'] \arrow[dr, "\tau_I"] & \\
\mathfrak{M}_{\theta_I}(1,v) \arrow[rr, "\iota_I"] & & \mathcal{M}_{A_I}(1,n_I)
\end{tikzcd}$$

of schemes over $\mathbb{C}$ for any $\widetilde{\theta} \in C^+_{1,\widetilde{v}}$, where π_I is surjective and where ι_I is a closed immersion. In particular, the fine moduli space $\mathcal{M}_{A_I}(1, n_I)$ is non-empty.

Proof. The fact that $\widetilde{v}_i = n_i$ for all $i \in I$ is immediate from Proposition 4.3.1. We now construct a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{M}_{\widetilde{\theta}}(1,\widetilde{v}) \arrow[dl, "\pi_I"'] \arrow[dr, "\tau_I"] \arrow[dd, "\varphi_{|L_I|}"] & \\
\mathfrak{M}_{\theta_I}(1,v) \arrow[dr, hook, "\sigma_I"] & & \mathcal{M}_{A_I}(1,n_I) \arrow[dl, hook', "\varphi_I"'] \\
 & {|L_I|} &
\end{tikzcd}$$

of schemes over $\mathbb{C}$, where both morphisms drawn diagonally in the bottom half of the diagram are closed immersions. Commutativity of the triangle on the left is given by Lemma 4.3.3, while surjectivity of π_I is established in the proof of Lemma 4.3.2.

The isomorphism $\mathrm{Hilb}^{\widetilde{v}}([\mathbb{C}^2/\Gamma]) \cong \mathfrak{M}_\theta(1,\widetilde{v})$ from Proposition 2.4.3 allows us to apply the proof of Lemma 3.2.1, using $\widetilde{v}$ in place of $n\delta$ throughout, to conclude that $\mathfrak{M}_{\widetilde{\theta}}(1,\widetilde{v})$ may be regarded as a fine moduli space of A-modules. Applying the proof of Lemma 3.3.2, with $\widetilde{v}_i$ replacing $n\dim\rho_i$ for all $i \in I$, gives the universal morphism τ_I.

For commutativity of the right-hand triangle, recall that the polarising line bundle $\mathcal{L}_I$ on $\mathcal{M}_{A_I}(1, n_I)$ is very ample. As pullback commutes with tensor operations on the T_i, the isomorphisms $\tau_I^*(T_i) \cong \mathcal{R}_i$ for $i \in \{\infty\} \cup I$ from *ibid.* imply that $L_I = \tau_I^*(\mathcal{L}_I)$. If $\varphi_{|\mathcal{L}_I|} \colon \mathcal{M}_{A_I}(1, n_I) \to |\mathcal{L}_I|$ is the closed immersion for $\mathcal{L}_I$, then

$$(\varphi_{|\mathcal{L}_I|} \circ \tau_I)^*(\mathcal{O}_{|\mathcal{L}_I|}(1)) = \tau_I^*(\mathcal{L}_I) = L_I = \varphi_{|L_I|}^*\big(\mathcal{O}_{|L_I|}(1)\big)$$

on $\mathfrak{M}_\theta(1, v)$. Now, just as in the proof of Lemma 4.3.3, we deduce $\varphi_{|L_I|} = \varphi_{|\mathcal{L}_I|} \circ \tau_I$, so the right-hand triangle commutes and hence so does the diagram.

It remains to construct the closed immersion ι_I. Commutativity of the diagram, combined with surjectivity of π_I, shows that $\mathfrak{M}_{\theta_I}(1, v)$ is isomorphic to the closed subscheme $\operatorname{Im}(\sigma_I) = \operatorname{Im}(\sigma_I \circ \pi_I) = \operatorname{Im}(\varphi_I \circ \tau_I)$ of $|L_I|$. The closed immersion φ_I induces an isomorphism $\lambda_I \colon \operatorname{Im}(\varphi_I) \to \mathcal{M}_{A_I}(1, n_I)$ and hence we obtain a closed immersion

$$\iota_I := \lambda_I \circ \sigma_I \colon \mathfrak{M}_{\theta_I}(1, v) \to \mathcal{M}_{A_I}(1, n_I).$$

Since $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty by assumption, it follows that $\mathcal{M}_{A_I}(1, n_I)$ is non-empty. ■

4.4.2 Selecting a suitable dimension vector

We don't yet know whether the morphism τ_I from the key commutative diagram from Proposition 4.4.1 is surjective. We now introduce a collection of dimension vectors that will allow us to establish surjectivity.

For any vector $v \in \mathbb{N}^{s+1}$, it is convenient to write $v_\infty := 1$, so that the dimension vector $(1, v)$ has components v_i for all $i \in Q_0$.

Lemma 4.4.2. *Let $v = (v_j)_{0 \leq j \leq s} \in \mathbb{N}^{s+1}$ satisfy $v_i = n_i$ for all $i \in I$. Let V be a θ_I-stable A-module of dimension vector $(1, v)$. Then $2v_k \leq \sum_{\{e \in Q_1 \mid \mathrm{t}(e) = k\}} v_{\mathrm{h}(e)}$ for all $k \notin \{\infty\} \cup I$.*

Proof. Fix $k \notin \{\infty\} \cup I$ and define

$$V_\oplus := \bigoplus_{\substack{e \in Q_1, \\ \mathrm{t}(e) = k}} V_{\mathrm{h}(e)}.$$

The maps in V determined by arrows with head and tail at vertex k combine to define maps $f \colon V_k \to V_\oplus$ and $g \colon V_\oplus \to V_k$ satisfying $g \circ f = 0$. The proof in Lemma A.0.2 implies that the complex

$$0 \longrightarrow V_k \xrightarrow{\ f\ } V_\oplus \xrightarrow{\ g\ } V_k \longrightarrow 0$$

has nonzero homology only at $V_\oplus$, so $\dim V_\oplus \geq 2\dim V_k$. This proves the claim. ■

For our chosen vector $n_I = (n_i)_{i\in I} \in \mathbb{N}^I$, define

$$\mathcal{V}(n_I) := \left\{(1,v) \in \mathbb{N}^{r+2} \mid v_i = n_i \text{ for } i \in I \text{ and } 2v_k \geq \sum_{\{e\in Q_1 \mid \mathrm{t}(e)=k\}} v_{\mathrm{h}(e)} \text{ for } k \notin \{\infty\}\cup I\right\}$$

(recall that $v_\infty = 1$). Before we prove that this set is non-empty, recall from McKay [44] that for each vertex $0 \leq k \leq s$ in the McKay quiver, we have

$$2\dim(\rho_k) = \sum_{\{e\in Q_1^* \mid \mathrm{t}(e)=k\}} \dim(\rho_{\mathrm{h}(e)}). \tag{4.8}$$

Recall also that $Q_1 = \{\bar{b}\} \cup Q_1^*$, so the indexing set in the sum from (4.8) differs from the set $\{e \in Q_1 \mid \mathrm{t}(e) = 0\}$ only for $k = 0$.

Example 4.4.3. If there exists $n > 0$ such that $n_i = n\dim \rho_i$ for all $i \in I$, then equation (4.8) shows that the vector $(1, v)$ with $v_k = n\dim(\rho_k) = n\delta_k$ for all $0 \leq k \leq s$ lies in $\mathcal{V}(n_I)$.

Lemma 4.4.4. *For any $v \in \mathbb{N}^{s+1}$ satisfying $v_i = n_i$ for all $i \in I$, there exists $(1, v') \in \mathcal{V}(n_I)$ such that $v' - v \geq 0$.*

Proof. Write $K := \{0, 1, .., s\} \smallsetminus I$. Set $v'_\infty = 1$ and $v'_i = n_i$ for $i \in I$. Our goal is to define $v'_k \in \mathbb{N}$ for each $k \in K$ such that $v'_k \geq v_k$ and the inequality

$$2v'_k \geq \sum_{\{e\in Q_1 \mid \mathrm{t}(e)=k\}} v'_{\mathrm{h}(e)} \tag{4.9}$$

holds for all $k \in K$.

For this, define a subset $K' \subseteq K$ as follows. If $0 \in I$, then set $K' = \varnothing$. Otherwise, we have $0 \in K$. In this case, choose any shortest path γ in the McKay graph starting at vertex 0 and ending at some vertex $i \in I$. The set K' of vertices through which γ passes satisfies $K' \subseteq K \smallsetminus \{0\}$. Now, fix $N \gg 0$ and for each $k \in K'$, let d_k denote the distance in the graph from 0 to k. For $k \in K$, define

$$v'_k = \begin{cases} N\dim(\rho_k) - d_k & \text{if } k \in K' \\ N\dim(\rho_k) & \text{otherwise.} \end{cases}$$

Since $N \gg 0$, we have $v'_k \geq v_k$ for all $k \in K$. Also, for all $k \in K$ and $i \in I$, we have $v'_k \gg v'_i$.

It remains to establish the inequality (4.9) for all $k \in K$. To do this, consider three cases, depending on whether $k \in K'$ and whether $k = 0$.

First, suppose $k \in K'$. If there is a vertex $j \in I$ that lies adjacent to k, then (4.9) follows by combining (4.8) with the inequality $v'_k \gg v'_j$. Otherwise, precisely two vertices adjacent to k lie in $\{0\} \cup K'$, while every other vertex adjacent to k lies in K. Then

$$\begin{aligned} 2v'_k = 2(N\dim(\rho_k) - d_k) &= -(d_k - 1) - (d_k + 1) + \sum_{\{e \in Q_1^* \mid \mathrm{t}(e) = k\}} N\dim(\rho_{\mathrm{h}(e)}) \\ &= \sum_{\{e \in Q_1^* \mid \mathrm{t}(e) = k\}} v'_{\mathrm{h}(e)}, \end{aligned}$$

so (4.9) holds because $k \neq 0$.

Next, suppose $k \in K \smallsetminus K'$ with $k \neq 0$. Then each vertex j adjacent to k lies in $\{0, 1, \dots, s\}$ and satisfies $N\dim(\rho_j) \geq v'_j$. Combine these inequalities with (4.8) to obtain

$$2v'_k = 2N\dim(\rho_k) = \sum_{\{e \in Q_1^* \mid \mathrm{t}(e) = k\}} N\dim(\rho_{\mathrm{h}(e)}) \geq \sum_{\{e \in Q_1^* \mid \mathrm{t}(e) = k\}} v'_{\mathrm{h}(e)}.$$

This establishes the inequality (4.9) since $k \neq 0$.

Finally, the only remaining case is when $k = 0 \in K$. If there is a vertex $j \in I$ that lies adjacent to 0, then (4.9) follows by combining (4.8) with the inequality $v'_0 \gg v'_j$ and $v'_\infty = 1$. Otherwise, we have $0 \in K$ and one vertex of K' lies adjacent to 0, giving

$$2v'_0 = 2N = v'_\infty + (-1) + \sum_{\{e \in Q_1^* \mid \mathrm{t}(e) = 0\}} N\dim(\rho_{\mathrm{h}(e)}) = v'_\infty + \sum_{\{e \in Q_1^* \mid \mathrm{t}(e) = 0\}} v'_{\mathrm{h}(e)}.$$

The quiver Q_1 has a unique arrow with tail at 0 and head at ∞, so (4.9) holds for $k = 0$ if $0 \in K$. This completes the proof. ∎

Corollary 4.4.5. *Let $v \in \mathbb{N}^{s+1}$ satisfy $v_i = n_i$ for all $i \in I$. If $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty, then there exists $(1, v') \in \mathcal{V}(n_I)$ such that $v' - v \geq 0$ and $\mathfrak{M}_{\theta_I}(1, v')$ is non-empty.*

Proof. Lemma 4.4.4 determines a vector $(1, v') \in \mathcal{V}(n_I)$ such that $v'_k - v_k \geq 0$ for all $k \in K := \{0, 1, \dots, s\} \smallsetminus I$. Take any θ_I-semistable A-module M of dimension vector $(1, v)$, and let $S_k := \mathbb{C}e_k$ denote the vertex simple A-module at vertex $k \in K$. The A-module

$$\overline{M} := M \oplus \bigoplus_{k \in K} S_k^{\oplus(v'_k - v_k)}.$$

is θ_I-semistable of dimension vector $(1, v')$ by construction. ∎

4.4.3 Quiver varieties as fine moduli spaces

As before, $n_I = (n_i) \in \mathbb{N}^I$ is a dimension vector and $\eta_I \colon \mathbb{Z} \oplus \mathbb{Z}^I \to \mathbb{Q}$ is the stability condition for A_I-modules of dimension vector $(1, n_I)$ defined in (4.3). Let $v = (v_j)_{0 \leq j \leq s} \in \mathbb{N}^{s+1}$ be a dimension vector satisfying $v_i = n_i$ for all $i \in I$, and let $\theta_I \in \Theta_{1,v}$ be the stability condition for A-modules defined in (2.5).

We now use Corollary 4.4.5 to strengthen the statement of Proposition 4.4.1. Before stating the desired result, recall the idempotent $\overline{e}_I = e_\infty + \sum_{i \in I} e_i \in A$ and the algebra $A_I = \overline{e}_I A \overline{e}_I$. Recall now the *recollement* functors

$$\begin{tikzcd} A\text{-mod} \arrow[r, bend left=10, "j_I^*"] & A_I\text{-mod} \arrow[l, bend left=10, "j_{I!}"] \end{tikzcd}$$

defined as in Section 3.3.

In particular, j_I^* is exact, $j_I^* j_{I!}$ is the identity functor, and for any A_I-module N, the A-module $j_{I!}(N)$ provides the maximal extension by $A/(A\overline{e}_I A)$-modules.

Lemma 4.4.6. *Let N be an η_I-stable A_I-module of dimension vector $(1, n_I)$. Then $j_{I!}(N)$ is a θ_I-semistable A-module of finite dimension satisfying $\dim_i j_{I!}(N) = \dim_i N$ for $i \in \{\infty\} \cup I$.*

Proof. The proofs of Lemmas 3.3.5 and 3.3.6 do not rely on the choice of dimension vector, so they apply equally well for the vector $(1, n_I)$. ∎

This gives the following partial converse to Proposition 4.4.1.

Corollary 4.4.7. *Suppose that $\mathcal{M}_{A_I}(1, n_I)$ is non-empty. There exists $v \in \mathbb{N}^{s+1}$ satisfying $v_i = n_i$ for all $i \in I$ such that $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty.*

Proof. Let N denote an η_I-stable A_I-module of dimension vector $(1, n_I)$. Lemma 4.4.6 shows that $j_{I!}(N)$ is a θ_I-semistable A-module of dimension vector $(1, v)$, where $v \in \mathbb{N}^{s+1}$ satisfies $v_i = n_i$ for all $i \in I$. In particular, $\mathfrak{M}_{\theta_I}(1, v) \neq \varnothing$. ∎

Lemma 4.4.8. *Let N be an η_I-stable A_I-module of dimension vector $(1, n_I)$ and let $(1, v) \in \mathcal{V}(n_I)$ be any vector such that $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty. Choose $\widetilde{v}$ as in Proposition 4.3.1. Then there exists a θ_I-semistable A-module M such that $j_I^* M \cong N$ with $\dim_\infty M = 1$, $\dim_i M = n_i$ for all $i \in I$ and $\dim_k M \leq \widetilde{v}_k$ for all $k \in K := \{0, 1, \ldots, s\} \smallsetminus I$.*

Proof. Lemma 4.4.6 shows that $j_{I!}(N)$ is θ_I-semistable. If $\dim_i j_{I!}(N) \leq \widetilde{v}_i$ for $i \notin \{\infty\} \cup I$, then $M \coloneqq j_{I!}(N)$ is the required A-module because $j_I^* j_{I!}$ is the identity. Otherwise, arguing as in Lemma 3.3.7, we find that M_{con} satisfies $j^* M_{\mathrm{con}} = N$.

We established above that $\dim_\infty M_{\mathrm{con}} = 1$ and $\dim_i M_{\mathrm{con}} = n_i$ for all $i \in I$. Thus, to show that M_{con} is the required A-module, it remains to check that

$$\dim_k M_{\mathrm{con}} \leq \widetilde{v}_k \tag{4.10}$$

for $k \in K = \{0, 1, \ldots, s\} \smallsetminus I$. To simplify notation, write $(1, w) \coloneqq \dim M_{\mathrm{con}}$ for some $w \in \mathbb{N}^{s+1}$. We first establish the weaker inequality $w \leq v$ (recall that $\widetilde{v} \leq v$ by Proposition 4.3.1). Combine the inequalities on the components of v coming from the assumption $(1, v) \in \mathcal{V}(n_I)$, together with the inequalities on the components of $(1, w)$ obtained by applying Lemma 4.4.2 to the θ_I-stable A_I-module M_{con}, to obtain the inequality

$$2(v - w)_k \geq \sum_{\{e \in Q_1 \mid \mathrm{t}(e) = k\}} (v - w)_{\mathrm{h}(e)} \tag{4.11}$$

for all $k \in K \coloneqq \{0, \ldots, s\} \smallsetminus I$. The vertex set K, together with all edges of the ADE Dynkin diagram joining two vertices from K, is a disjoint union of simply laced Dynkin diagrams. Let Λ be one of these subdiagrams, and let C be its Cartan matrix. It follows from (4.11) that $C(v - w)|_\Lambda \geq 0$. But the matrix C^{-1} only has nonnegative entries (see, for example, [57, pp. 1.157–1.158]), so $(v - w)|_\Lambda = C^{-1}C(v - w)|_\Lambda \geq 0$. Combining these inequalities for every such subdiagram Λ with the equalities $w_i = v_i$ for $i \in \{\infty\} \cup I$ gives $v \geq w$. This shows that $v_k - w_k \geq 0$ for all $k \in K$. Let $S_k \coloneqq \mathbb{C}e_k$ denote the vertex simple A-module at vertex $k \in K$. The A-module

$$\overline{M} \coloneqq M_{\mathrm{con}} \oplus \bigoplus_{k \in K} S_k^{\oplus(v_k - w_k)}$$

is θ_I-semistable of dimension vector $(1, v)$ by construction. As M_{con} is the unique summand of $\overline{M}$ with dimension 1 at the vertex ∞, we have $w \leq \widetilde{v}$ by Lemma 4.3.2. ■

Theorem 4.4.9. *Let $v = (v_j)_{0 \leq j \leq s} \in \mathbb{N}^{s+1}$ be any vector satisfying $v_i = n_i$ for all $i \in I$. Suppose that $\mathfrak{M}_{\theta_I}(1, v)$ is non-empty. For the vector $(1, v') \in \mathcal{V}(n_I)$ from Corollary 4.4.5, choose a vector $\widetilde{v'}$ as in Proposition 4.3.1. Then there is a commutative diagram*

$$\begin{tikzcd}
 & \mathfrak{M}_{\widetilde{\theta}}\big(1, \widetilde{v'}\big) \arrow[dl, "\pi_I"'] \arrow[dr, "\tau_I"] & \\
\mathfrak{M}_{\theta_I}(1, v') \arrow[rr, "\iota_I"] & & \mathcal{M}_{A_I}(1, n_I)
\end{tikzcd} \tag{4.12}$$

of schemes over $\mathbb{C}$, where ι_I is an isomorphism of the underlying reduced schemes. Thus, $\mathcal{M}_{A_I}(1, n_I)$ is non-empty, irreducible, and its underlying reduced scheme is normal and has symplectic singularities.

Proof. Note first that $\mathfrak{M}_{\theta_I}(1, v')$ is non-empty by Corollary 4.4.5, so we may indeed apply Proposition 4.3.1 to obtain the vector $\widetilde{v'}$. The diagram from Proposition 4.4.1 now takes the form as shown in (4.12). To prove that ι_I is an isomorphism of the underlying reduced schemes, it suffices to show that ι_I is surjective on closed points. The diagram commutes, π_I is surjective and ι_I is a closed immersion, so it suffices to show that τ_I is surjective on closed points. Consider a closed point $[N] \in \mathcal{M}_{A_I}(1, n_I)$, where N is an η_I-stable A_I-module of dimension vector $(1, n_I)$. Let M be the θ_I-semistable A-module from Lemma 4.4.8. Since

$$m_k := v'_k - \dim_k M \geq \widetilde{v'_k} - \dim_k M \geq 0$$

for all $k \in K = \{0, 1, \ldots, s\} \setminus I$ by Proposition 4.3.1 and Lemma 4.4.8, we may once again define

$$\overline{M} := M \oplus \bigoplus_{k \in K} S_k^{\oplus m_k},$$

where S_k is the vertex simple A-module at vertex $k \in K = \{0, \ldots, s\} \cup I$. The A-module $\overline{M}$ is θ_I-semistable of dimension vector $(1, v')$ by construction, and it satisfies $j_I^*(\overline{M}) = j_I^*(M) = N$. Write $[\overline{M}] \in \mathfrak{M}_{\theta_I}(1, v')$ for the corresponding closed point, and let $\widetilde{M}$ be any $\widetilde{\theta}$-stable A-module of dimension vector $\left(1, \widetilde{v'}\right)$ such that the closed point $[\widetilde{M}] \in \mathfrak{M}_{\widetilde{\theta}}\left(1, \widetilde{v'}\right)$ satisfies $\pi_I([\widetilde{M}]) = [\overline{M}] \in \mathfrak{M}_{\theta_I}(1, v')$. Then $j_I^*(\widetilde{M}) = j_I^*(\overline{M}) = N$, hence $\tau_I([\widetilde{M}]) = [N]$. The final statement of the proposition follows from Lemmas 2.1.3 and 2.4.4 and Proposition 4.4.1. ∎

Corollary 4.4.10. *Let $I \subseteq \{0, \ldots, s\}$ be a non-empty subset and let $n_I \in \mathbb{N}^I$. Then*

$$\mathcal{M}_{A_I}(1, n_I) \neq \varnothing \iff \mathfrak{M}_{\theta_I}(1, v) \neq \varnothing \text{ for some } v \in \mathbb{N}^{s+1} \text{ satisfying } v_i = n_i \text{ for } i \in I.$$

Proof. If $\mathcal{M}_{A_I}(1, n_I) \neq \varnothing$, then Corollary 4.4.5 gives a vector $v \in \mathbb{N}^{s+1}$ satisfying $v_i = n_i$ for $i \in I$ such that $\mathfrak{M}_{\theta_I}(1, v) \neq \varnothing$. The converse is immediate from Proposition 4.4.1. ∎

Finally, we prove the results announced in Section 1.2.

Proof of Theorem 1.2.1. The isomorphism in (1), including the case when either space is empty, follows from Proposition 4.2.2. The isomorphism in (2) is proved in Proposition 4.1.4. Statement (3) follows by combining Corollary 4.4.10 and Theorem 4.4.9. The final statement of Theorem 1.2.1 now follows from [6, Theorem 1.5]. ∎

Chapter 5

Sheaves on Quotients of $\mathbb{P}^2$

In this chapter, we introduce several different ways of taking quotients of $\mathbb{P}^2$ by the group Γ, namely the stack quotient $[\mathbb{P}^2/\Gamma]$, the scheme quotient $\mathbb{P}^2/\Gamma$, and an intermediate stack quotient $\mathcal{X}$.

We investigate the behaviour of sheaves on these spaces, and we introduce the set $Y_{r,n}$ of sheaves on $\mathcal{X}$ satisfying certain properties (see Definition 5.2.5). We end this chapter by proving that there is a canonical bijection $Y_{r,n} \xrightarrow{\sim} \mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C})$.

In the next chapter, we will see that $Y_{r,n}$ can be given an intrinsic moduli space structure, and that this bijection extends to a morphism of schemes.

5.1 Quotients of $\mathbb{P}^2$

5.1.1 The stack $\mathcal{X}$

We explain the construction of the stack $\mathcal{X}$ mentioned in Section 1.3.1.

First, we construct the scheme quotient $\mathbb{P}^2/\Gamma$ by considering the action of Γ on the homogeneous coordinate ring $\mathbb{C}[x, y, z]$ of $\mathbb{P}^2$. It is well-known that the invariant ring $\mathbb{C}[x, y]^\Gamma$ is generated as a $\mathbb{C}$-algebra by three homogeneous elements. These three, together with z will then generate the ring $\mathbb{C}[x, y, z]^\Gamma$. Weighting these four elements by their degrees, we find

$$\mathbb{P}^2/\Gamma = \operatorname{Proj} \mathbb{C}[x, y, z]^\Gamma,$$

naturally embedding as a singular surface in a three-dimensional weighted projective space. Let $l_\infty = \{z = 0\} \subset \mathbb{P}^2$, the 'line at infinity'.

Consider the quotient morphism $\mathbb{P}^2 \to \mathbb{P}^2/\Gamma$: The complement of l_∞ is an affine plane, the image of which under this morphism is the Kleinian singularity $\mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[x, y]^\Gamma$. It has a unique singular point, and we write o for the preimage of this point (the 'origin') in $\mathbb{P}^2$.

We may also instead consider the stack quotient $[\mathbb{P}^2/\Gamma]$. There are then surjective morphisms of stacks

$$\mathbb{P}^2 \to [\mathbb{P}^2/\Gamma] \to \mathbb{P}^2/\Gamma.$$

Let now U_f be the open subscheme $\mathbb{P}^2 \smallsetminus (\{o\} \cup l_\infty)$. As Γ acts freely on U_f, Proposition 2.5.4 applies, and the stack $[U_f/\Gamma]$ is represented by a scheme U_f/Γ. As a slight abuse of notation, we will simply write $[U_f/\Gamma] = U_f/\Gamma$ in what follows.

Definition 5.1.1. We define the stack $\mathcal{X}$ as the pushout in the diagram of Deligne-Mumford stacks

$$\begin{tikzcd}
U_f/\Gamma \arrow[r, hook] \arrow[d, hook] & \mathbb{C}^2/\Gamma \arrow[d, dashed] \\
{[\mathbb{P}^2 \smallsetminus \{o\}/\Gamma]} \arrow[r, dashed] & \mathcal{X}
\end{tikzcd}$$

That it is possible to take pushouts of stacks in this manner, follows for instance from [58, Theorem C], which also shows that the two dashed morphisms are open immersions.

Intuitively, $\mathcal{X}$ is a quotient of $\mathbb{P}^2$ by Γ, where we have taken the scheme quotient around 0, and the stack quotient around l_∞.

Set $d_\infty = [l_\infty/\Gamma]$ and $r_\infty = l_\infty/\Gamma$. These will differ as Γ does not act freely on l_∞. Then d_∞ is a closed substack of both $[\mathbb{P}^2/\Gamma]$ and $\mathcal{X}$.

We thus have the following commutative diagram of stack morphisms, where we name the various morphisms as indicated.

$$\begin{tikzcd}
l_\infty \arrow[r, hook, "i"] \arrow[dd, two heads, "\pi|_{l_\infty}"'] & \mathbb{P}^2 \arrow[d, "k"] \arrow[dd, bend left, "\pi"] \\
& {[\mathbb{P}^2/\Gamma]} \arrow[d, two heads, "q"] \\
d_\infty \arrow[r, hook, "j"] \arrow[ur] \arrow[d, two heads] & \mathcal{X} \arrow[d, two heads, "c"] \\
r_\infty \arrow[r, hook] & \mathbb{P}^2/\Gamma
\end{tikzcd} \quad . \tag{5.1}$$

We may note in diagram (5.1) that k is étale. The morphism π is *not* étale, but its restriction to $\mathbb{P}^2 \smallsetminus \{o\}$ (on which we can identify it with k) is.

Furthermore, $c \circ \pi = c \circ q \circ k$ is a finite morphism of schemes, by Proposition 2.5.4.

We shall need some more details on $\mathcal{X}$.

Lemma 5.1.2. *The stack $\mathcal{X}$ is a proper, tame Deligne-Mumford stack of finite type, with finite diagonal.*

Proof. $\mathcal{X}$ is Deligne-Mumford because it is the pushout of the Deligne-Mumford stacks $[(\mathbb{P}^2 \setminus \{o\})/\Gamma]$ and $\mathbb{C}^2/\Gamma$ along the Deligne-Mumford stack U_f.

Explicitly, an étale atlas for $\mathcal{X}$ is given by the map $(\mathbb{P}^2 \setminus \{o\}) \coprod (\mathbb{C}^2/\Gamma) \to \mathcal{X}$ induced by the definition of $\mathcal{X}$.

Then, to show that $\mathcal{X}$ is proper over $\mathbb{C}$, we use [14, Theorem 3.1(2)] to show that $\mathcal{X} \to \mathbb{P}^2/\Gamma$ is proper. Now $\mathbb{P}^2/\Gamma$ is proper, being projective, and since the composition of proper morphisms of stacks is proper ([Stacks, Tag 0CL7]), $\mathcal{X}$ is proper.

To see that $\mathcal{X}$ is of finite type, recall that being of finite type is equivalent to being locally of finite type and quasicompact. It is simple to check on the patches $[\mathbb{P}^2 \setminus \{o\}/\Gamma]$ and $\mathbb{C}^2/\Gamma$ that $\mathcal{X}$ is locally of finite type. To see that it's quasicompact, apply [Stacks, Tag 04YC, (3), (5)] to the composition of morphisms $\mathbb{P}^2 \to [\mathbb{P}^2/\Gamma] \to \mathcal{X}$.

It is tame: Because $\mathcal{X}$ is proper, it is separated. Because $\mathcal{X}$ is Deligne-Mumford, every automorphism group of a geometric point is finite, so its order is invertible in $\mathbb{C}$.

Finally, to determine whether the diagonal morphism $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is finite, it is enough [Stacks, Tag 04XC] to work étale-locally on the target. Now an étale cover of $\mathcal{X} \times \mathcal{X}$ consists of $\mathbb{C}^2 \times \mathbb{C}^2$, $\mathbb{C}^2 \times [\mathbb{P}^2 \setminus \{o\}/\Gamma]$, $[\mathbb{P}^2 \setminus \{o\}/\Gamma] \times \mathbb{C}^2/\Gamma$, and $[\mathbb{P}^2 \setminus \{o\}/\Gamma] \times [\mathbb{P}^2 \setminus \{o\}/\Gamma]$ and it is simple to prove that the base change of $\Delta_{\mathcal{X}}$ to any of these is finite. ■

Similarly, $[\mathbb{P}^2/\Gamma]$ is also a proper tame Deligne-Mumford stack of finite type. with finite diagonal. Since $[\mathbb{P}^2/\Gamma]$ is smooth, the singular locus of $\mathcal{X}$ is precisely the unique singular point of $\mathbb{C}^2/\Gamma \subset \mathcal{X}$.

Let us also note that

$$\mathbb{C}^2/\Gamma \times_{\mathcal{X}} [(\mathbb{P}^2 \setminus \{o\})/\Gamma] = U_f/\Gamma.$$

To see this, we can set up the commutative diagram of fibre products

$$\begin{tikzcd}
\mathbb{C}^2/\Gamma \times_{\mathbb{P}^2/\Gamma} (\mathbb{P}^2 \setminus \{o\})/\Gamma \arrow[r] \arrow[d] & \mathbb{C}^2/\Gamma \arrow[d] \\
(\mathbb{P}^2 \setminus \{o\})/\Gamma \times_{\mathbb{P}^2/\Gamma} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "c"] \\
(\mathbb{P}^2 \setminus \{o\})/\Gamma \arrow[r] & \mathbb{P}^2/\Gamma
\end{tikzcd},$$

where the top-left downwards morphism is induced by the universal property of $(\mathbb{P}^2 \setminus \{o\})/\Gamma \times_{\mathbb{P}^2/\Gamma} \mathcal{X}$.

As

$$(\mathbb{P}^2 \setminus \{o\})/\Gamma \times_{\mathbb{P}^2/\Gamma} \mathcal{X} = [(\mathbb{P}^2 \setminus \{o\})/\Gamma],$$

and

$$\mathbb{C}^2/\Gamma \times_{\mathbb{P}^2/\Gamma} (\mathbb{P}^2 \smallsetminus \{o\})/\Gamma = (\mathbb{C}^2 \smallsetminus \{o\})/\Gamma = U_f/\Gamma,$$

the conclusion follows from the category-theoretical identity $A \times_S B = (A \times_S C) \times_C B$.

5.1.2 Defining a sheaf on $\mathcal{X}$

By a sheaf on $\mathcal{X}$, we mean a sheaf defined on the étale site of $\mathcal{X}$ (see [41, Définition 12.1.(ii)] for details).

Thus a (quasi-)coherent sheaf $\mathscr{F}$ on $\mathcal{X}$ is the data consisting of

- for every étale morphism $u\colon U \to \mathcal{X}$ where U is a scheme, a (Zariski) sheaf $\mathscr{F}_U$ on U,
- for any two étale morphisms $U_i \to \mathcal{X},\ U_j \to \mathcal{X}$, an isomorphism $\psi_{ij}\colon \mathrm{pr}_i^*\mathscr{F}_{U_i} \to \mathrm{pr}_j^*\mathscr{F}_{U_j}$ over the fibre product $U_i \times_{\mathcal{X}} U_j$,
- such that the isomorphisms ψ_{ij} satisfy a *cocycle condition* (see [Stacks, tag 03O7])

Here $\mathrm{pr}_i, \mathrm{pr}_j$ are the projections from $U_i \times_{\mathcal{X}} U_j$ onto the first and second factor.

If $u\colon U \to \mathcal{X}$ is such an étale map, we can take a Zariski cover $\{U_1, \dots, U_k\}$ of U such that for every i, $c \circ u(U_i)$ has empty intersection with either $\{o\}$ or r_∞. Then a Zariski sheaf on U is given by Zariski sheaves on each $U_i, i \in \{1, \dots, k\}$, that glue together on the overlaps $U_i \cap U_j$.

It follows that we may assume that either $c \circ u(U_i) \cap \{o\} = \emptyset$ or $c \circ u(U_i) \cap r_\infty = \emptyset$. In the first case, u factors through $[(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma]$, and in the second, through $\mathbb{C}^2/\Gamma$.

In conclusion, to define an (étale) sheaf $\mathscr{F}$ on $\mathcal{X}$, it is enough to define its restrictions $\mathscr{F}|_{\mathbb{C}^2/\Gamma}$ and $\mathscr{F}|_{[(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma]}$, together with a choice of isomorphism between the restrictions of these sheaves to U_f/Γ. But to define a sheaf on $U_f/\Gamma = [(\mathbb{P}^2 \smallsetminus (\{o\} \cup l_\infty))/\Gamma]$ is equivalent to defining a Γ-equivariant sheaf on $(\mathbb{P}^2 \smallsetminus (\{o\} \cup l_\infty))$.

So we can instead define $\mathscr{F}$ by choosing a sheaf $\mathscr{F}_{(\mathbb{P}^2 \smallsetminus \{o\})}$ on $\mathbb{P}^2 \smallsetminus \{o\}$ and a sheaf $\mathscr{F}_{\mathbb{C}^2/\Gamma}$ on $\mathbb{C}^2/\Gamma$ together with a Γ-equivariant isomorphism of their inverse images on $U_f/\Gamma = \mathbb{C}^2/\Gamma \times_{\mathcal{X}} [(\mathbb{P}^2 \smallsetminus \{o\}/\Gamma)]$ – i.e. an isomorphism of Γ-equivariant sheaves

$$\iota_{\mathscr{F}}\colon (\mathscr{F}_{(\mathbb{P}^2 \smallsetminus \{o\})})|_{U_f} \xrightarrow{\sim} \left((\pi|_{\mathbb{P}^2 \smallsetminus l_\infty})^* \mathscr{F}_{\mathbb{C}^2/\Gamma}\right)|_{U_f}$$

on U_f.

(Since this étale cover of $\mathcal{X}$ consists of only two schemes, the cocycle condition is trivially satisfied.) All our sheaves on $\mathcal{X}$ will be described in this fashion.

Example 5.1.3. We can describe the ideal sheaf $\mathscr{I}_{d_\infty}$ of the closed substack $d_\infty \subset \mathcal{X}$ as follows:

The ideal sheaf $\mathscr{I}_{l_\infty}$ of l_∞ in $\mathbb{P}^2$ is Γ-equivariant, locally generated by a section $z \in \mathcal{O}_{\mathbb{P}^2}$. Over the subscheme U_f, it is isomorphic to the structure sheaf through the morphism locally defined by mapping a section f to $\frac{f}{z}$, since z is invertible.

Thus we can take

$$(\mathscr{I}_{d_\infty})_{(\mathbb{C}^2/\Gamma)} = \mathcal{O}_{\mathbb{C}^2/\Gamma}, \quad (\mathscr{I}_{d_\infty})_{(\mathbb{P}^2 \smallsetminus \{o\})} = \mathscr{I}_{l_\infty}|_{(\mathbb{P}^2 \smallsetminus \{o\})}, \quad \iota_{\mathscr{I}_{d_\infty}} : f \mapsto \frac{f}{z},$$

which completely defines $\mathscr{I}_{d_\infty}$.

5.1.3 Notation

Let's fix some notation for various substacks of $\mathbb{P}^2$ and $\mathcal{X}$. As mentioned, we let o be the "origin" of the affine plane $U_o := \mathbb{P}^2 \smallsetminus l_\infty$, considered as an open subscheme of $\mathbb{P}^2$. Similarly, we set $U_l := \mathbb{P}^2 \smallsetminus \{o\}$.

The image of o under π is the unique singular point of $\mathcal{X}$, call it p.

Similarly, write V_d for the stack $[(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma]$, considered as an open substack of $\mathcal{X}$, and V_p for the singular affine scheme $\mathbb{C}^2/\Gamma = (\mathbb{P}^2 \smallsetminus l_\infty)/\Gamma$, also considered as an open substack of $\mathcal{X}$.

5.1.4 Projectivity of $\mathcal{X}$

We shall need a few more properties of $\mathcal{X}$, namely; we must find its coarse moduli space, and we shall show that $\mathcal{X}$ is a projective stack.

The coarse moduli space of $\mathcal{X}$ is as one should expect:

Lemma 5.1.4. *The scheme $\mathbb{P}^2/\Gamma$ is the coarse moduli space for both $[\mathbb{P}^2/\Gamma]$ and $\mathcal{X}$. Furthermore, the functors*

$$c_*\colon \mathsf{Coh}(\mathcal{X}) \to \mathsf{Coh}(\mathbb{P}^2/\Gamma), \textit{ and } (c \circ q)_*\colon \mathsf{Coh}([\mathbb{P}^2/\Gamma]) \to \mathsf{Coh}(\mathbb{P}^2/\Gamma)$$

are exact.

Proof. By [14, Theorem 3.1]), the coarse moduli space of $[\mathbb{P}^2/\Gamma]$ must be a scheme. As any Γ-invariant morphism of schemes $\mathbb{P}^2 \to Y$ factors through $\mathbb{P}^2/\Gamma$ (Proposition 2.5.4), this shows that any morphism from $[\mathbb{P}^2/\Gamma]$ to a scheme factors through $\mathbb{P}^2/\Gamma$. The second condition of Definition 2.5.2, that c induces a bijection $[\mathbb{P}^2/\Gamma](k) \xrightarrow{\sim} \mathbb{P}^2/\Gamma(k)$ for any algebraically closed field k is clearly satisfied.

Then $\mathbb{P}^2/\Gamma$ is the coarse moduli space of $[\mathbb{P}^2/\Gamma]$.

Since the map $[\mathbb{P}^2/\Gamma] \to \mathbb{P}^2/\Gamma$ factors through $\mathcal{X}$, it follows that $\mathbb{P}^2/\Gamma$ is also the coarse moduli space of $\mathcal{X}$.

The final assertion follows from the tameness of $\mathcal{X}$ and Lemma 2.5.3. ■

Proposition 5.1.5. *The stack $\mathcal{X}$ is projective, and $\mathcal{O}_{\mathcal{X}}(d_\infty)$ is a generating sheaf for $\mathcal{X}$.*

Proof. It is clear that the coarse moduli space of $\mathcal{X}$, namely $\mathbb{P}^2/\Gamma$, is a projective scheme.

We claim that the sheaf $\mathcal{O}_{\mathcal{X}}(d_\infty)$ is a generating sheaf for $\mathcal{X}$. So let $x\colon p = \operatorname{Spec}\mathbb{C} \to \mathcal{X}$ be a point, and assume that it has nontrivial stabiliser. Then $\operatorname{im} x \subset d_\infty$.

Let $\{p_1, \ldots, p_k\} \subset l_\infty$ be the Γ-orbit in $\mathbb{P}^2 \setminus \{o\}$ corresponding to p. There is then, for any p_i, a commutative diagram

$$\begin{tikzcd} p_i \arrow[r, "x_i"] \arrow[d] & \mathbb{P}^2 \setminus \{o\} = U_l \arrow[d, "\pi"] & \\ p \arrow[r, "x"] & {[\mathbb{P}^2 \setminus \{o\}/\Gamma] = V_d} \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

Let Γ_{x_i} be the stabiliser of x_i.

We see that $\pi^*\mathcal{O}_{\mathcal{X}}(d_\infty)|_{V_d} = \mathcal{O}_{\mathbb{P}^2}(l_\infty)|_{U_l}$.

If we think of l_∞ as the projectivisation of $\mathbb{C}^2$, we can then identify the fiber $x_i^*\mathcal{O}_{\mathcal{X}}(d_\infty)$ with the affine line $l_i \subset \mathbb{C}^2$ corresponding to x_i. Now, since Γ acts freely on $\mathbb{C}^2 \setminus \{o\}$, Γ_{x_i} must act freely on $l_i \setminus 0$. Explicitly, if we denote the homogeneous prime ideal of $\mathbb{C}[x, y]$ associated to p_i by $\mathfrak{m}_{p_i}$, the line corresponding to p_i can then be Γ_{x_i}-invariantly identified with the 1-dimensional $\mathbb{C}$-vector space

$$x_i^*\mathcal{O}_(l_\infty) = \left(\frac{1}{z}\mathbb{C}[x, y, z]/(\mathfrak{m}_{p_i})\right)_0.$$

It follows that the action of Γ_{p_i} on this line is faithful.

Finally, let $x\colon \operatorname{Spec} k \to \mathcal{X}$ be another morphism with k an algebraically closed field. Then x either factors through V_p or V_d. In the first case the stabiliser is trivial, and in the second the stabiliser is either trivial, or x factors through one of the morphisms $\operatorname{Spec}\mathbb{C} \to \mathcal{X}$ considered above.

Then the action of the stabiliser groups on the fibres of $\mathcal{O}_{\mathcal{X}}(d_\infty)$ is faithful, and $\mathcal{X}$ is projective.

■

Remark 5.1.6. Because the stabiliser groups Γ_{x_i} have faithful one-dimensional representations, they must be finite cyclic groups. This will be important in Section 6.2.1.

Of course, $[\mathbb{P}^2/\Gamma]$ is also a projective stack, in this case by Remark 2.5.8, simply because it is a global quotient stack.

5.2 Γ-equivariant sheaves on $\mathbb{P}^2$ and sheaves on quotients of $\mathbb{P}^2$ by Γ

In this section, we show how to move between the category of coherent Γ-equivariant sheaves on $\mathbb{P}^2$ and the category of coherent sheaves on $\mathcal{X}$.

5.2.1 Sets of framed sheaves

Associated to the morphism $\pi\colon \mathbb{P}^2 \to \mathcal{X}$, there are functors π^* and π_*. We will, however, need to modify them both. As we will be working with Γ-equivariant sheaves on $\mathbb{P}^2$, we will consider the Γ-invariants of the direct image by π, while taking into account the glued-together nature of $\mathcal{X}$.

More explicitly, we do as follows:

Definition 5.2.1. Let $\mathscr{E}$ be a Γ-equivariant coherent sheaf on $\mathbb{P}^2$. By descent, there is a sheaf $\mathscr{F}$ on V_d such that $(\pi|_{U_l})^*\mathscr{F} = \mathscr{E}|_{U_l}$. As for V_p, restrict $\mathscr{E}$ to $U_o = \mathbb{C}^2$, and form the sheaf $(\pi|_{U_o})_*(\mathscr{E}|_{U_o})^\Gamma$ on V_p.

By Proposition 2.5.4, we have

$$(\pi|_{U_o})_*(\mathscr{E}|_{U_o})^\Gamma|_{U_f/\Gamma} = \mathscr{F}|_{U_f/\Gamma}$$

on $U_f/\Gamma = [U_f/\Gamma]$, so we can glue these sheaves together to a sheaf $D(\mathscr{E})$ on $\mathcal{X}$.

This defines a functor $D\colon \mathsf{Coh}_\Gamma(\mathbb{P}^2) \to \mathsf{Coh}(\mathcal{X})$.

Remark 5.2.2. With the chain of morphisms $\mathbb{P}^2 \to [\mathbb{P}^2/\Gamma] \to \mathcal{X}$, the functor D is of course just the descent along the first morphism followed by the direct image along the second morphism. But the explicit description in the previous paragraph will be useful for our computations.

It is not difficult to see that D is exact: When restricted to U_l, it induces an equivalence of categories, and on the image of U_o we have $D(\mathscr{E})|_{V_p} = \pi_*(\mathscr{E}|_{U_o})^\Gamma$. This is exact since $\pi|_{U_o}$ is a finite morphism and the order of Γ is invertible.

Let us define the sets of interest (see [64, p. 2.3], [48, chapter 2]):

Definition 5.2.3. Let $r \in \mathbb{N}, \mathbf{v} \in \mathbb{N}^{Q_{\Gamma,0}}$.

We write $X_{r,\mathbf{v}}$ for the set of isomorphism classes of $\mathcal{O}_{l_\infty}^r$-framed $\mathcal{O}_{\mathbb{P}^2}$-modules $(\mathscr{E}, \phi_{\mathscr{E}})$ where $\mathscr{E}$ is a Γ-equivariant sheaf of $\mathcal{O}_{\mathbb{P}^2}$-modules, such that $\mathscr{E}$

- is coherent,
- is torsion-free of rank r,
- satisfies $H^1(\mathbb{P}^2, \mathscr{E} \otimes \mathscr{I}_{l_\infty}) \cong \bigoplus_{\rho_i \in R}(\mathbb{C}^{\mathbf{v}_i} \otimes \rho_i)$, where $\mathscr{I}_{l_\infty}$ is the ideal sheaf of $l_\infty \subset \mathbb{P}^2$,

and where $\phi_{\mathscr{E}} \colon \mathscr{E}|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}$ is an isomorphism, the *framing isomorphism.*

It is interesting to note that different sources have slightly different definitions of this set. For instance, Varagnolo and Vasserot (in [64]) do not make the rank an explicit part of the definition. On the other hand, Bruzzo and Sala (in [11]) and Nakajima (in e.g. [51]) add to our definition by requiring local freeness in a neighbourhood of l_∞.

In fact, local freeness in a neighbourhood of the framing follows from the other conditions in Definition 5.2.3:

Lemma 5.2.4. *A framed sheaf $\mathscr{E} \in X_{r,\mathbf{v}}$ is locally free in a neighbourhood of l_∞.*

Proof. Given a reduced noetherian scheme X equipped with a coherent $\mathcal{O}_X$-module $\mathscr{G}$, consider the function

$$\rho_{X,\mathscr{G}}(x) = \dim_{k(x)} \mathscr{G}_x \otimes_{\mathcal{O}_X} k(x),$$

where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. By [31, Exercise II.5.8], $\rho_{X,\mathscr{G}}$ is an upper semi-continuous function, and $\mathscr{G}$ is locally free on an open subscheme $U \subset X$ if and only if $\rho|_U$ is constant.

Let $\mathscr{E} \in X_{r,\mathbf{v}}$. It is straightforward to show that for a point $x \in l_\infty \subset \mathbb{P}^2$, there is an equality

$$\rho_{\mathbb{P}^2,\mathscr{E}}(x) = \rho_{l_\infty,\mathscr{E}|_{l_\infty}}(x).$$

Because $\mathscr{E}$ has rank r, the set $\{x \in \mathbb{P}^2 | \rho_{\mathbb{P}^2,\mathscr{E}}(x) \geq r\}$ is dense, and thus all of $\mathbb{P}^2$.

Then, as $\mathscr{E}|_{l_\infty}$ is free, semicontinuity of $\rho_{\mathbb{P}^2}$ implies that $\mathscr{E}$ is locally free on a neighbourhood of l_∞.

■

For comparison with Definition 5.2.3, let us restate Definition 1.3.1 from the Introduction.

Definition 5.2.5. Let $r, n \in \mathbb{N}$. Write $Y_{r,n}$ for the set of isomorphism classes of $\mathcal{O}^r_{d_\infty}$-framed $\mathcal{O}_{\mathcal{X}}$-modules $(\mathscr{F}, \phi_{\mathscr{F}})$, where $\mathscr{F}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules such that $\mathscr{F}$

- is coherent,

- is torsion-free of rank r,
- satisfies $\dim H^1(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{d_\infty}) = n$, where $\mathscr{I}_{d_\infty}$ is the ideal sheaf of $d_\infty \subset \mathcal{X}$,

and where $\phi_{\mathcal{E}}$ is an isomorphism $\phi_{\mathcal{E}} \colon \mathscr{F}|_{d_\infty} \xrightarrow{\sim} \mathcal{O}_{d_\infty}^{\oplus r}$.

When we write $(\mathscr{E}, \phi_{\mathscr{E}}) \in X_{r,\mathbf{v}}$ or $(\mathscr{F}, \phi_{\mathscr{F}}) \in Y_{r,n}$, we mean that we choose a representative for some isomorphism class. We will often leave the framing isomorphisms $\phi_{\mathscr{E}}, \phi_{\mathscr{F}}$ and framing sheaves $\mathcal{O}_{d_\infty}^{\oplus r}$ implicit, and simply write $\mathscr{F} \in Y_{r,n}$, $\mathscr{E} \in X_{r,\mathbf{v}}$, calling both 'framed sheaves'.

The conditions on these sets of sheaves are clearly parallel. We'll first see what conditions are preserved by $D(-)$, and then what conditions are preserved by π^*. Because $\pi^*\mathscr{F}$ for a framed sheaf $\mathscr{F} \in Y_{r,n}$ may have torsion, we will have to work with the 'torsion-free inverse image' functor π^T, which we introduce in Definition 5.2.8.

5.2.2 Moving from $X_{r,\mathbf{v}}$ to $Y_{r,n}$

Now fix a sheaf $\mathscr{E} \in X_{r,\mathbf{v}}$, where $\mathbf{v}_0 = n$.

Recall the definition of the set $Y_{r,n}$ from Definition 5.2.5. We shall show that $D(\mathscr{E})$ is an element of $Y_{r,n}$, considering each condition separately.

5.2.2.1 Coherence

To see that $D(\mathscr{E})$ is coherent, we can consider its restrictions to each chart.

Since $D(\mathscr{E})|_{V_d}$ is obtained from $\mathscr{E}|_{U_l}$ by descent, it is coherent. Furthermore, $D(\mathscr{E})|_{V_p} = \pi_*(\mathscr{E}|_{U_o})^\Gamma$. Since π is a finite map, it preserves coherence ([31, Exercise II 5.5]). Now $\pi_*(\mathscr{E}|_{U_o})^\Gamma$ equals the intersection of the kernels of the endomorphisms

$$(1-g) \colon \pi_*(\mathscr{E}|_{U_o}) \to \pi_*(\mathscr{E}|_{U_o}),$$

where g runs over all elements of Γ. Kernels of morphisms of coherent sheaves on a locally noetherian scheme are coherent, and so are intersections of coherent sheaves. Then $(\pi_*(\mathscr{E})^\Gamma)|_{U_o} = D(\mathscr{E})_{U_o}$ is coherent.

Thus $D(\mathscr{E})$ is coherent.

5.2.2.2 Framing

To see that $D(\mathscr{E})$ is framed along d_∞, it is sufficient to restrict the sheaf to U_l. Then we are dealing with the descent of a framed sheaf to a group quotient.

Here it is obvious - descent to group quotients commutes with closed immersions, and so $D(\mathscr{E})|_{d_\infty}$ descends from $\mathscr{E}|_{l_\infty}$. Then the framing isomorphism $\phi_{\mathscr{E}} \colon \mathscr{E}|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}$ induces a framing isomorphism $D(\mathscr{E})|_{d_\infty} \xrightarrow{\sim} \mathcal{O}_{d_\infty}^{\oplus r}$.

5.2.2.3 Rank

Let U be an open subscheme of $\mathbb{P}^2$ on which $\mathscr{E}$ is locally free. Then

$$U' = U_f \cap \left(\bigcap_{g \in \Gamma} g(U) \right)$$

is a Γ-invariant open subscheme such that $\mathscr{E}|_{U'}$ is locally free of rank r (see Lemma 2.5.14), and we have $U'/\Gamma \subset U_f/\Gamma \subset \mathcal{X}$.

The morphism $\pi|_{U'} \colon U' \to U'/\Gamma$ is étale and therefore fpqc, and thus we can apply [Stacks, Lemma 05B2], stating that the property of being finite locally free is preserved under descent.

So $D(\mathscr{E})|_{[U'/\Gamma]}$ is locally free of rank r, and so $D(\mathscr{E})$ has rank r.

5.2.2.4 Torsion-freeness

To show that $D(\mathscr{E})$ is torsion-free, it is enough to show that $D(\mathscr{E})|_{U_o}$ is.

Again, since both $U_o = \mathbb{C}^2$ and $V_p = \mathbb{C}^2/\Gamma$ are locally Noetherian schemes, a sheaf on either scheme is torsion-free if and only if it has no subsheaves with support of dimension 1 or lower. Since π is a finite morphism, π_* preserves the dimension of the support of a sheaf. Thus, if $\mathscr{E}$ is a sheaf on U_o, $(\pi_*\mathscr{E})|_{V_p}$ is torsion-free if $\mathscr{E}$ is torsion-free. It is clear that $((\pi_*\mathscr{E})|_{V_p})^\Gamma$ is torsion-free if $\pi_*\mathscr{E}$ is.

Then $D(\mathscr{E})$ is torsion-free.

5.2.2.5 Cohomology

By Lemma 2.5.5, a Γ-equivariant sheaf $\mathscr{E}$ on $\mathbb{P}^2$ satisfies

$$H^i([U/\Gamma], D(\mathscr{E})) = (H^i(U, \mathscr{E}))^\Gamma$$

for any Γ-invariant open subscheme $U \subseteq \mathbb{P}^2$, since $|\Gamma|$ is invertible.

Lemma 5.2.6. *Let $\mathscr{E} \in X_{r,\mathbf{v}}$ for some dimension vector $\mathbf{v}$. Let $n = \mathbf{v}_0$. Then*

$$\dim H^1(\mathcal{X}, D(\mathscr{E}) \otimes \mathscr{I}_{d_\infty}) = n.$$

Proof. By Lemma 5.1.4, both r_* and $(r \circ q)_*$ are exact. The same lemma tells us that

$$H^i([\mathbb{P}^2/\Gamma], \mathscr{A}) = H^i(\mathcal{X}, q_*\mathscr{A}) = H^i(\mathbb{P}^2/\Gamma, (c \circ q)_*\mathscr{A})$$

for any coherent sheaf $\mathscr{A}$ on $[\mathbb{P}^2/\Gamma]$.

On $[\mathbb{P}^2/\Gamma]$, $\mathscr{E}$ descends to a sheaf $\mathscr{F}$. Thus $\mathscr{E} \otimes \mathscr{I}_{l_\infty}$ descends (tautologically) to $\mathscr{F} \otimes \mathscr{I}'_{d_\infty}$, where $\mathscr{I}'_{d_\infty}$ is the ideal sheaf of d_∞ as a substack of $[\mathbb{P}^2/\Gamma]$.

It follows by Lemma 2.5.5 that:

$$H^i(\mathbb{P}^2, \mathscr{F} \otimes \mathscr{I}_{l_\infty})^\Gamma = H^i([\mathbb{P}^2/\Gamma], \mathscr{F} \otimes \mathscr{I}'_{d_\infty}) =$$
$$H^i(\mathcal{X}, q_*(\mathscr{F} \otimes \mathscr{I}'_{d_\infty})) = H^i(\mathbb{P}^2/\Gamma, (c \circ q)_*(\mathscr{F} \otimes \mathscr{I}'_{d_\infty})).$$

It will therefore suffice to show that there is a canonical isomorphism

$$q_*(\mathscr{F} \otimes \mathscr{I}'_{d_\infty}) \xrightarrow{\sim} D(\mathscr{E}) \otimes \mathscr{I}_{d_\infty}.$$

This immediately holds when restricted to V_d.

Since $\mathscr{I}_{d_\infty}|_{[\mathbb{C}^2/\Gamma]} \cong \mathcal{O}_{[\mathbb{C}^2/\Gamma]}$, we have

$$q_*(\mathscr{F} \otimes \mathscr{I}_{d_\infty})|_{V_p} = q_*((\mathscr{F} \otimes \mathscr{I}_{d_\infty})|_{[\mathbb{C}^2/\Gamma]}) \cong q_*(\mathscr{F}|_{[\mathbb{C}^2/\Gamma]}),$$

and we only have to prove that

$$(q_*\mathscr{F})|_{V_p} = (\pi|_{U_o})_*(\mathscr{E}|_{U_o})^\Gamma. \tag{5.2}$$

By definition,

$$(q_*\mathscr{E})(V_p) = \mathscr{E}(V_p \times_{\mathcal{X}} [\mathbb{P}^2/\Gamma]) = \mathscr{E}([\mathbb{C}^2/\Gamma]).$$

Using Lemma 2.5.5, this equals $\mathscr{E}(U_o)^\Gamma = \pi_*(\mathscr{E})^\Gamma(V_p)$. Since V_p and U_o are affine, this is sufficient to show that (5.2) holds.

This concludes the proof. ■

We have now shown

Proposition 5.2.7. *The functor $D(-)$ induces a map of sets from the set $X_{r,n\delta}$ to the set $Y_{r,n}$.*

We will see in Proposition 5.3.5 that this map is surjective.

5.2.3 Inverse images of elements of $Y_{r,n}$

Now let $\mathscr{F} \in Y_{r,n}$. We will not able to show that $\pi^*\mathscr{F}$ lies in $X_{r,n\delta}$, because:

- this sheaf may not be torsion-free, and
- it may not hold that $H^1(\mathbb{P}^2, \pi^*\mathscr{F} \otimes \mathscr{I}_{l_\infty})$, considered as a Γ-representation, has dimension vector $n\delta$.

We will instead show that $\pi^*\mathscr{F}$ modulo torsion lies in $X_{r,\mathbf{v}}$, where $\mathbf{v}$ is some dimension vector satisfying $\mathbf{v}_0 = n$.

Again we consider each criterion for a framed sheaf to belong to $X_{r,\mathbf{v}}$ separately.

5.2.3.1 Torsion

Let us first define the "torsion-free inverse image".

Definition 5.2.8. We set $\pi^T(\mathscr{F}) = (\pi^*\mathscr{F})/\mathrm{tor}$, where tor is the torsion subsheaf of $\pi^*\mathscr{F}$.

It is clear that π^T is functorial: If $\sigma\colon \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on $\mathcal{X}$, then the torsion subsheaf of $\pi^*\mathscr{F}$ is supported on a closed strict subscheme of $\mathbb{P}^2$. It follows that the image of this torsion subsheaf under $\pi^*\sigma$ is also supported on a closed strict subscheme of $\mathbb{P}^2$, i.e., that it is a torsion subsheaf of $\mathscr{G}$.

Remark 5.2.9. Since $\mathscr{F}|_{V_d}$ descends from the sheaf $(\pi^*\mathscr{F})|_{U_l}$, $\mathscr{F}|_{V_d}$ is torsion-free if and only if $\pi^*\mathscr{F}|_{U_l}$ is. It follows that if $\mathscr{F} \in Y_{r,n}$, the torsion subsheaf of $\pi^*\mathscr{F}$ is either zero, or a skyscraper sheaf supported at o.

5.2.3.2 Coherence

It is enough to work locally, and it follows from [31, Prop 5.8] that $(\pi^*\mathscr{F})|_{U_o}$ is coherent. On the other hand, $(\pi^*\mathscr{F})|_{U_l}$ is tautologically coherent.

Then $\pi^*\mathscr{F}$ is coherent, and so is $\pi^T\mathscr{F}$.

5.2.3.3 Framing

Suppose that $\mathscr{F}$ has a rank-r-framing $\phi_{\mathscr{F}}\colon \mathscr{F}|_{d_\infty} \xrightarrow{\sim} \mathcal{O}_{d_\infty}^{\oplus r}$. We can again restrict to V_d. Now $\mathscr{F}|_{V_d}$ is a coherent sheaf on V_d, so by descent $\mathscr{F}$ corresponds to a Γ-equivariant sheaf $\mathscr{E}$ on U_o such that $\pi^*\mathscr{F}|_{V_d} = \mathscr{E}$. Then, in the commutative square

$$\begin{tikzcd} l_\infty \arrow[r, hook, "i"] \arrow[d, "\pi|_{l_\infty}"] & U_l \arrow[d, "\pi"] \\ d_\infty \arrow[r, hook, "j"] & V_d \end{tikzcd}$$

$\phi_{\mathscr{F}}$ induces the isomorphism

$$i^*\pi^*\mathscr{F} = (\pi|_{l_\infty})^* j^*\mathscr{F} \xrightarrow{\sim} (\pi|_{l_\infty})^*\mathcal{O}_{d_\infty}^{\oplus r} = \mathcal{O}_{l_\infty}^{\oplus r},$$

as required. Finally, because any torsion in $\pi^*\mathscr{F}$ is only supported at $\{o\}$, we then have

$$i^*\pi^*\mathscr{F} = i^*\pi^T\mathscr{F} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}.$$

5.2.3.4 Rank

This is immediate: let $V \subset \mathcal{X}$ be an open subscheme such that $\mathscr{F}|_V$ is locally free of rank r. Then $(\pi|_V)^*\mathscr{F}|_V$ is locally free of the same rank, and so $\pi^T\mathscr{F}$ has rank r.

5.2.3.5 Cohomology

Let us show that

Lemma 5.2.10.

$$H^1(\mathbb{P}^2, \pi^*\mathscr{F}) = H^1(\mathbb{P}^2, \pi^T\mathscr{F}), \textit{ and}$$
$$H^1(\mathbb{P}^2, \pi^T\mathscr{F} \otimes \mathscr{I}_{l_\infty}) = H^1(\mathbb{P}^2, \pi^T(\mathscr{F} \otimes \mathscr{I}_{d_\infty}))$$

Proof. If we again let tor be the torsion subsheaf of $\pi^*\mathscr{F}$, the first claim follows from taking the long exact cohomology sequence of

$$0 \to \mathrm{tor} \to \pi^*\mathscr{F} \to \pi^T\mathscr{F} \to 0,$$

since tor is only supported at the origin.

For the second, simply note that the inverse image functor commutes with tensor products, and that $\mathscr{I}_{l_\infty}$ is torsion-free, being a line bundle. ■

It follows from the discussion of the preceding paragraphs that if $\mathscr{F} \in Y_{r,n}$, then $\pi^T\mathscr{F}$ is an element of the set $X_{r,\mathbf{v}}$ for some dimension vector $\mathbf{v}$.

We shall find restrictions on $\mathbf{v}$ in Corollary 5.2.14.

Finally, we have

Lemma 5.2.11. *A framed sheaf $\mathscr{F} \in Y_{r,n}$ is locally free in a neighbourhood of d_∞.*

Proof. Such a sheaf $\mathscr{F}$ is, on any neighbourhood $U \supset d_\infty$ that does not contain p, the descent of an equivariant framed sheaf on $\pi^{-1}(U)$. Now we can simply repeat the proof of Lemma 5.2.4. ■

5.2.4 Correspondences of D and π^T

Now we investigate the compositions of D and π^T.

First, let $\mathscr{E} \in X_{r,\mathbf{v}}$ for some $\mathbf{v}$. Consider the sheaf $\pi^T(D(\mathscr{E}))$. Since $D(\mathscr{E})|_{U_o} = \pi_*(\mathscr{E}|_{V_p})^\Gamma$, there is a homomorphism

$$\iota \colon \pi^*(D(\mathscr{E})) \to \mathscr{E},$$

induced on U_o by the counit of the adjunction $\pi^* \vdash \pi_*$, i.e. $\pi^*\pi_*(\mathscr{E})^\Gamma \to \pi^*\pi_*(\mathscr{E}) \to \mathscr{E}$.

Because $\mathscr{E}$ is torsion-free, ι factors through $\pi^T(D(\mathscr{E}))$.

On U_l, ι is just the identity. Then, because $\ker\iota \subset \pi^*(D(\mathscr{E}))$: must have support contained in $\{o\}$, it is torsion.

We have shown:

Lemma 5.2.12. *Let $\mathscr{E} \in X_{r,n\delta}$. There is a natural injective morphism*

$$\pi^T(D(\mathscr{E})) \hookrightarrow \mathscr{E}.$$

On the other hand, we can show that there is a canonical isomorphism $D(\pi^*\mathscr{F}) \xrightarrow{\sim} \mathscr{F}$. Again, it is enough to show this when restricted to V_p. So let N be a torsion-free $\mathbb{C}[x,y]^\Gamma$-module. Then we need to show that there is an isomorphism of $\mathbb{C}[x,y]^\Gamma$-modules

$$(N \otimes_{\mathbb{C}[x,y]^\Gamma} \mathbb{C}[x,y])^\Gamma \xrightarrow{\sim} N.$$

To see this, note that an element $n' \in (N \otimes_{\mathbb{C}[x,y]^\Gamma} \mathbb{C}[x,y])^\Gamma$ can be written as a sum of elements of the form

$$\frac{1}{|\Gamma|}\left(n \otimes \sum_{g\in\Gamma} g \cdot a\right)$$

for $n \in N$ and $a \in \mathbb{C}[x,y]$, which means that n' lies in the image of the map

$$N \cong N \otimes_{\mathbb{C}[x,y]^\Gamma} C[x,y]^\Gamma \to (N \otimes_{\mathbb{C}[x,y]^\Gamma} \mathbb{C}[x,y])^\Gamma.$$

So the map $N \to (N \otimes_{\mathbb{C}[x,y]^\Gamma} \mathbb{C}[x,y])^\Gamma$ is surjective. On the other hand, since $\mathbb{C}[x,y]^\Gamma$ is a direct summand of $\mathbb{C}[x,y]$ considered as a $\mathbb{C}[x,y]^\Gamma$-module, the map $N \to N \otimes_{\mathbb{C}[x,y]^\Gamma} \mathbb{C}[x,y]$ is injective, and thus

$$N \to (N \otimes_{\mathbb{C}[x,y]^\Gamma} \mathbb{C}[x,y])^\Gamma$$

is an isomorphism. [1]

This proves the first equality of our next Lemma.

Lemma 5.2.13. *Let $\mathscr{F} \in Y_{r,n}$. There are canonical isomorphisms of framed sheaves $\mathscr{F} \xrightarrow{\sim} D(\pi^*\mathscr{F}) \xrightarrow{\sim} D(\pi^T\mathscr{F})$.*

[1] A subring $R \subset S$ such that for any R-module M, the homomorphism $M \to M \otimes_R S$ is injective, is called a *pure* subring. As a generalisation of our case, if G is a linearly reductive affine linear algebraic group acting on a regular Noetherian $\mathbb{C}$-algebra S, then S^G is a pure subring of S, see [32, section 6].

Proof. We need to show the map $D(\pi^*\mathscr{F}) \to D(\pi^T\mathscr{F})$ induced by $\pi^*\mathscr{F} \to \pi^T\mathscr{F}$ is an isomorphism. To do this, let $\mathscr{T} \subset \pi^*\mathscr{F}$ be the torsion subsheaf, which has support contained in $\{o\}$.

Since D is exact, it maps the exact sequence

$$0 \to \mathscr{T} \to \pi^*\mathscr{F} \to \pi^T\mathscr{F} \to 0$$

to

$$0 \to D(\mathscr{T}) \to D(\pi^*\mathscr{F}) = \mathscr{F} \to D(\pi^T\mathscr{F}) \to 0.$$

Now $\operatorname{Supp} D(\mathscr{T}) \subset \{p\}$, which means that $D(\mathscr{T})$ is a torsion subsheaf of $\mathscr{F}$. But $\mathscr{F}$ is torsion-free. Then $D(\mathscr{T}) = 0$, and we are done. ■

Finally, we can prove:

Corollary 5.2.14. *Given $\mathscr{F} \in Y_{r,n}$, we have $\pi^T\mathscr{F} \in X_{r,\mathbf{v}}$ for some $\mathbf{v}$ such that $\mathbf{v}_0 = n$.*

Proof. By Lemma 5.2.13, $\pi^T\mathscr{F} \in X_{r,\mathbf{v}}$ for some vector $\mathbf{v}$. By Lemma 5.2.6, we have $\mathbf{v}_0 = n$. ■

5.3 A construction of Varagnolo and Vasserot, and the map $\mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C}) \to Y_{r,n}$

In this section, we show how Varagnolo and Vasserot's construction in [64] can be extended to give a canonical map of sets $f\colon \mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C}) \to Y_{r,n}$.

First, let $\mathscr{Q}$ be the locally free sheaf of rank 2 on $\mathbb{P}^2$ which is the cokernel in the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathscr{Q} \to 0.$$

5.3.1 The starting bijection

Recall our convention that θ denotes an arbitrary stability parameter in the chamber C^+.

Theorem 5.3.1 ([64, Theorem 1])**.** *There is, for any positive integer r and any nonnegative $\mathbf{v} \in Q_0^{\mathbb{N}}$, a canonical isomorphism*

$$\mathfrak{M}_\theta(r, \mathbf{v}) \xrightarrow{\sim} X_{r,\mathbf{v}}.$$

Proof sketch of Theorem 5.3.1. We sketch the map, and state its properties without proof – see [64, Theorem 1] and its proof for the full details. We will need to use the viewpoint of the quiver variety $\mathfrak{M}_\theta(r, \mathbf{v})$ given in Section 2.3, i.e., we consider the underlying quiver as having only one pair of arrows between the 0-vertex and the framing vertex ∞, but our quiver representations have an r-dimensional vector space at ∞.

So let $N = (V_N, W_N, B_N, i_N, j_N) \in \mu^{-1}(0) \subset M(r, \mathbf{v})$. We construct the Γ-equivariant complex of sheaves

$$C_N^\bullet = \Big[\ \mathscr{O}_{\mathbb{P}^2}(-l_\infty) \otimes V_N \xrightarrow{a_N} \mathscr{O}_{\mathbb{P}^2} \otimes ((V_N \otimes L) \oplus W_N) \xrightarrow{b_N} \mathscr{O}_{\mathbb{P}^2}(l_\infty) \otimes \textstyle\bigwedge^2 L \otimes V_N\ \Big] \tag{5.3}$$

with

$$a_N = \begin{pmatrix} zB - x \cdot e_x - y \cdot e_y \\ zj \end{pmatrix}, \quad b_N = \begin{pmatrix} zB - x \cdot e_x - y \cdot e_y & zi \end{pmatrix}.$$

Here e_x, e_y are the canonical basis vectors of L. We set the terms of $C_N^\bullet$ to be of degree $0, 1$, and 2.

Then $C_N^\bullet$ is a complex of Γ-equivariant $\mathscr{O}_{\mathbb{P}^2}$-modules. We will denote the cohomology objects of the complex $C_N^\bullet$ by $\mathcal{H}^i(C_N^\bullet)$ to indicate that they are sheaves, and to reduce confusion when we compute their sheaf cohomology groups.

Note that the assignment $N \mapsto C_N^\bullet$ is functorial, i.e. a map $N \to N'$ of Π-modules (considered as elements of $M(r, \mathbf{v})$ from Section 2.3) induces a map of complexes $C_N^\bullet \to C_{N'}^\bullet$. As $\mathscr{O}_{\mathbb{P}^2}(i)$ is locally free, this functor is even exact.

For a *stable* N representing a $\mathbb{C}$-point of $\mathfrak{M}_\theta(r, \mathbf{v})$, the complex $C_N^\bullet$ is even a *monad*, that is, the middle $\mathcal{H}^1(C_N^\bullet)$ is the only nonzero cohomology object. This is naturally a Γ-equivariant $\mathscr{O}_{\mathbb{P}^2}$-module, equipped with a framing, and the map $N \mapsto \mathcal{H}^1(C_N^\bullet)$ provides the bijection $\mathfrak{M}_\theta(r, \mathbf{v})(\mathbb{C}) \xrightarrow{\sim} X_{r,\mathbf{v}}$ ([64, Theorem 1]).

We have now made one direction of the bijection from Theorem 5.3.1 explicit, but we can do so for the inverse direction as well:

Let $\mathscr{E} \in X_{r,\mathbf{v}}$, set $V_\mathscr{E} = H^1(\mathbb{P}^2, \mathscr{E} \otimes \mathscr{I}_{l_\infty}) = H^1(\mathbb{P}^2, \mathscr{E}(-l_\infty))$, and let $W_\mathscr{E}$ be an r-dimensional vector space. Associated to any framed torsion-free sheaf on $\mathbb{P}^2$, there is a *Beilinson spectral sequence* (see [48, Section 2.1] for the full details) determining the quiver data.

The E_1-term of the Beilinson spectral sequence associated to $\mathscr{E}$ is, (after tensoring with $\mathcal{O}_{\mathbb{P}^2}(l_\infty)$) then the complex

$$\mathcal{O}_{\mathbb{P}^2}(-l_\infty) \otimes H^1(\mathbb{P}^2, \mathscr{E}(-l_\infty)) \to \mathcal{O}_{\mathbb{P}^2} \otimes H^1(\mathbb{P}^2, \mathscr{E}(-l_\infty) \otimes \mathscr{Q}^\vee) \\ \to \mathcal{O}_{\mathbb{P}^2}(l_\infty) \otimes H^1(\mathbb{P}^2, \mathscr{E}(-l_\infty))$$

which is isomorphic to a monad

$$C^\bullet_{M_{\mathscr{E}}} := \left[\mathcal{O}_{\mathbb{P}^2}(-l_\infty) \otimes V_{\mathscr{E}} \xrightarrow{a_{\mathscr{E}}} \mathcal{O}_{\mathbb{P}^2} \otimes (V_{\mathscr{E}} \otimes L \oplus W_{\mathscr{E}}) \xrightarrow{b_{\mathscr{E}}} \mathcal{O}_{\mathbb{P}^2}(l_\infty) \otimes \Lambda^2 L \otimes V_{\mathscr{E}} \right]$$

of the type from Equation (5.3), and so we can read off linear Γ-equivariant maps $i_{M_{\mathscr{E}}}, j_{M_{\mathscr{E}}}, B_{M_{\mathscr{E}}}$.

This makes $(V_{\mathscr{E}}, W_{\mathscr{E}}, i_{M_{\mathscr{E}}}, j_{M_{\mathscr{E}}}, B_{M_{\mathscr{E}}})$ a stable Π-module (see [48, Section 2.1]), and so, up to isomorphism, it determines a point on $\mathfrak{M}_\theta(r, \mathbf{v})$.

■

Varagnolo and Vasserot prove that Theorem 5.3.1 is a bijection of G_W-sets, where G_W is the product of $\Pi_{i \in Q_0} \mathrm{GL}_{v_i}(\mathbb{C})$ with a torus. But it is not difficult to extend their bijection to an isomorphism of schemes, because we can equip $X_{r,\mathbf{v}}$ with a natural scheme structure:

Consider the moduli space of framed torsion free sheaves $\mathscr{F}$ of rank r on $\mathbb{P}^2$, such that $h^1(\mathbb{P}^2, \mathscr{F} \otimes \mathscr{I}_{l_\infty}) = \sum_{\rho_i \in \mathrm{Rep}\,\Gamma} n\delta_i \mathbf{v}_i$. It is well known that this space has the structure of a quasiprojective scheme, see e.g. [33, Corollary 2.24]. This space carries a Γ-action coming from that of Γ on $\mathbb{P}^2$, defined by $g[\mathscr{F}] = [g^* \mathscr{F}]$ for any $g \in \Gamma$.

There is then a closed locus Z of this moduli space representing the Γ-equivariant framed sheaves $\mathscr{F}$. This Z has different components, depending on the Γ-representation structure on the cohomology groups $H^1(\mathbb{P}^2, \mathscr{F} \otimes \mathscr{I}_{l_\infty})$. One such component will parametrise the framed sheaves lying in $X_{r,\mathbf{v}}$, and it follows that this component is isomorphic to $\mathfrak{M}_\theta(r, \mathbf{v})$.

5.3.1.1 Notation

Let us fix some notation: If $\mathscr{E} \in X_{r,\mathbf{v}}$, we write $(W_{\mathscr{E}}, V_{\mathscr{E}}, B_{\mathscr{E}}, i_{\mathscr{E}}, j_{\mathscr{E}})$ for its associated stable $(Q', \mathcal{J})$-representation (up to isomorphism), and $M_{\mathscr{E}}$ for the resulting Π-module.

In what follows, we will denote the Γ-equivariant $\mathcal{O}_{\mathbb{P}^2}$-module that Theorem 5.3.1 associates to a stable Π-module M by $\widehat{M}$.

5.3.2 Constructing a map $\mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C}) \to Y_{r,n}$

We now construct a map of sets $\mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C}) \to Y_{r,n}$.

Let $x \in \mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C})$. By Lemma 2.4.5, there is a θ-stable Π-module M representing a closed point of $\mathfrak{M}_\theta(r, n\delta)$ that is a lift of x, and this lift corresponds by Theorem 5.3.1 to a framed, torsion-free rank r- Γ-equivariant $\mathcal{O}_{\mathbb{P}^2}$- module $\widehat{M}$.

Proposition 5.3.2. *The framed $\mathcal{O}_{\mathcal{X}}$-module $D(\widehat{M})$ is independent of the choice of M, and is thus only dependent on x.*

In fact, the sheaf $D(\widehat{M})$ is only dependent on the R_{θ_0}-equivalence class of x.

We shall need several intermediate lemmas to prove this. The strategy will be to show that the concentrated representation of an R_{θ_0}-equivalence class gives rise to the same $\mathcal{O}_{\mathcal{X}}$-module as every element of the class.

Lemma 5.3.3. *Suppose that M is a vertex simple Π-module, for some vertex $i \neq 0$, corresponding to the simple Γ-representation ρ_i. Then we have*

$$D(\mathcal{H}^2(C^\bullet_M)) = 0.$$

Proof. We will show that for the étale cover $\{V_p, V_d\}$ of $\mathcal{X}$, we have that $D(\mathcal{H}^2(C^\bullet_M))|_{V_i} = 0$ for $i \in \{p, d\}$.

First, for V_p it is enough to show that $\pi_*((\mathcal{H}^2(C^\bullet_M))|_{U_o})^\Gamma = 0$. As $V_p = \mathbb{C}^2/\Gamma$ is an affine scheme, it suffices to show that this sheaf has no nonzero sections defined on all V_p. This holds if we can show that

$$((\mathcal{H}^2(C^\bullet_M)(U_o))^\Gamma = 0.$$

To do this, note that the end of the complex $C^\bullet_M$ becomes

$$\mathcal{O}_{\mathbb{P}^2} \otimes L \otimes \rho_i \xrightarrow{b_M} \mathcal{O}_{\mathbb{P}^2}(1) \otimes \bigwedge^2 L \otimes \rho_i \longrightarrow 0$$

where, since $B_M = 0$, we have $b_M = (-xe_x - ye_y)$. Now $\bigwedge^2 L$ is the trivial representation, and ρ_i is a nontrivial representation. Over any Γ-invariant open set $U \subset \mathbb{P}^2$ – for instance U_o – a Γ-invariant section of $(\mathcal{O}_{\mathbb{P}^2}(1) \otimes \bigwedge^2 L \otimes \rho_i)$ is of the form

$$\begin{aligned}\sum_{j=1}^{k}((xf_j + yg_j + zh_j) \otimes (e_x \wedge e_y) \otimes v_j)\\ = \sum_{j=1}^{k}(\alpha_j \otimes (e_x \wedge e_y) \otimes v_j),\end{aligned}$$

where f_j, g_j, h_j are sections of $\mathcal{O}_{\mathbb{P}^2}(U)$, and $v_j \in \rho_i$.

Decompose α_j by representation type — i.e. write $\mathcal{O}_{\mathbb{P}^2}(U) = \bigoplus_{\rho_i \in R} V_i$, where V_i is the ρ_i-isotypical component, and set $\alpha_j = \sum \alpha_{j,i}$ with $\alpha_{j,i} \in V_i$. Then we see that $\alpha_{j,0} = 0$, as otherwise $\sum_{j=1}^k \alpha_j \otimes v_j$ cannot be Γ-invariant.

This shows that $h_j = xh_{j1} + yh_{j2}$ for some h_{j1}, h_{j2}. It follows that we have

$$(xf_j + yg_j + zh_j) \otimes (e_x \wedge e_y) \otimes v_j = (x(f_j + zh_{j1}) + y(g_j + zh_{j2})) \otimes (e_x \wedge e_y) \otimes v_j.$$

But this is the image of

$$\Big(-(g_j + zh_{j2}) \otimes e_x + (f_j + zh_{j1}) \otimes e_y \Big) \otimes v_j$$

under b_M. Hence $\operatorname{coker} b_M$ has no Γ-invariant sections.

Second, restrict to V_d: We shall show that $D(\mathcal{H}^2(C^\bullet_M))|_{V_d}$ descends from the zero sheaf, and thus is itself zero. So we shall prove that $\mathcal{H}^2(C^\bullet_M)|_{U_l} = 0$.

Consider the restriction of $\mathcal{H}^2(C^\bullet_M)$ to $U_l = \mathbb{P}^2/\{o\}$. This scheme is the union of two affine subschemes, namely $D_+(x) = \mathbb{P}^2 \setminus V(x)$ and $D_+(y) = \mathbb{P}^2 \setminus V(y)$.

We show that $\mathcal{H}^2(C^\bullet_M)|_{D_+(x)} = 0$, the argument for $D_+(y)$ is completely analogous.

On the distinguished open set $D_+(x)$, any section of $\mathcal{O}_{D_+(x)}(1)$ is a linear combination of elements on the form

$$a = f \otimes (e_x \wedge e_y \otimes v),$$

with $f \in x \cdot \mathbb{C}[y/x, z/x]$, $v \in \rho_i$. We can thus write $f = xg$, with $g \in \mathbb{C}[y/x, z/x]$.

Then a is the image of $g \otimes e_y \otimes v$ under b_M. It follows that $b_M|_{D_+(x)}$ is surjective, and so $H^0(D_+(x), \mathcal{H}^2(C^\bullet_M)) = 0$. But as $\mathcal{H}^2(C^\bullet_M)$ is a coherent sheaf, and $D_+(x)$ is affine, this shows that $\mathcal{H}^2(C^\bullet_M)|_{D_+(x)} = 0$.

This concludes the proof.

■

Lemma 5.3.4. *Let $K = (W_K, V_K, B_K, i_K, j_K)$ be a $(Q', \mathcal{J})$-representation (Section 2.3)such that $W_K = 0, V_{K,0} = 0$.*

Then $D(\mathcal{H}^2(C^\bullet_K)) = 0$.

Proof. First, note that in this case the map

$$b_K \colon \mathcal{O}_{\mathbb{P}^2} \otimes L \otimes V \to \mathcal{O}_{\mathbb{P}^2} \otimes \bigwedge^2 L \otimes V \tag{5.4}$$

is given by $b_k = (B_K - xe_x - ye_y)$. There is by Remark 2.4.8 a filtration

$$0 = K_0 \subset K_1 \subset \cdots \subset K_n = K,$$

where $K_i' = K_i/K_{i-1}$ is a vertex simple (i.e 1-dimensional) Π- module. Thus $V_{K_i'} = K_i' \otimes \rho_i$ is a simple nontrivial Γ-representation, and it follows from Lemma 5.3.3 that $D(\mathcal{H}^2(C^\bullet_{K_i'})) = 0$.

Then the short exact sequences of Π-modules

$$0 \to K_{i-1} \to K_i \to K_i' \to 0$$

give, through Equation (5.3) and applying the exact functor D, sequences

$$D(\mathcal{H}^2(C^\bullet_{K_{i-1}})) \to D(\mathcal{H}^2(C^\bullet_{K_i})) \to D(\mathcal{H}^2(C^\bullet_{K_i'}))$$

exact in the middle. Induction through these establishes the lemma. ∎

Proof of Proposition 5.3.2. Let M_{con} be the concentrated representation of the S_{θ_0}-equivalence class of x. Thus M_{con} is a stable Π-module, and $\dim M_{\mathrm{con}} \leq \dim M = (r, n\delta)$.

Then Lemma 2.4.11 shows that M_{con} is a quotient of M, so there is an exact sequence of Π-modules

$$0 \to K \to M \to M_{\mathrm{con}} \to 0 \tag{5.5}$$

for some Π-module K, which must satisfy $\dim_0 K = \dim_\infty K = 0$.

As M and M_{con} are stable Π-modules, we can apply Theorem 5.3.1 to associate them to Γ-equivariant torsion-free $\mathcal{O}_{\mathbb{P}^2}$-modules $\widehat{M}, \widehat{M}_{con}$ respectively.

Let us use the construction (5.3) on the sequence (5.5). We get an exact sequence of complexes $0 \to C^\bullet_K \to C^\bullet_M \to C^\bullet_{M_{\mathrm{con}}} \to 0$, or if written out:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V_K \arrow[r, "a_K"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2} \otimes ((V_K \otimes L) \oplus W_K) \arrow[r, "b_K"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(1) \otimes \wedge^2 L \otimes V_K \arrow[d] \\
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V_M \arrow[r, "a_M"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2} \otimes ((V_M \otimes L) \oplus W_M) \arrow[r, "b_M"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(1) \otimes \wedge^2 L \otimes V_M \arrow[d] \\
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V_{M_{\mathrm{con}}} \arrow[r, "a_{M_{\mathrm{con}}}"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2} \otimes ((V_{M_{\mathrm{con}}} \otimes L) \oplus W_{M_{\mathrm{con}}}) \arrow[r, "b_{M_{\mathrm{con}}}"] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(1) \otimes \wedge^2 L \otimes V_{M_{\mathrm{con}}} \arrow[d] \\
0 & 0 & 0
\end{tikzcd} . \tag{5.6}$$

Since M and M_{con} are both stable Π-modules,

$$\mathcal{H}^0(C^\bullet_M) = \mathcal{H}^0(C^\bullet_{M_{\mathrm{con}}}) = \mathcal{H}^2(C^\bullet_M) = \mathcal{H}^2(C^\bullet_{M_{\mathrm{con}}}) = 0.$$

It follows that there is an exact sequence of Γ-equivariant $\mathscr{O}_{\mathbb{P}^2}$-modules

$$0 \to \mathcal{H}^1(C^\bullet_K) \to \widehat{M} \to \widehat{M}_{\mathrm{con}} \to \mathcal{H}^2(C^\bullet_K) \to 0$$

where $\mathcal{H}^1(C^\bullet_K) = \ker b_K / \operatorname{im} a_K$, $\mathcal{H}^2(C^\bullet_K) = (\mathscr{O}_{\mathbb{P}^2}(1) \otimes \wedge^2 L \otimes V_K)/\operatorname{im} b_K$.

We now claim that

1. $\mathcal{H}^1(C^\bullet_K) = 0$, and
2. $D(\mathcal{H}^2(C^\bullet_K)) = 0$.

Note that $\dim_\infty K = 0$, so $W_K = 0$, and therefore $C^\bullet_K$ is the complex

$$\mathscr{O}_{\mathbb{P}^2}(-1) \otimes V_K \xrightarrow{a_K} \mathscr{O}_{\mathbb{P}^2} \otimes (V_K \otimes L) \xrightarrow{b_K} \mathscr{O}_{\mathbb{P}^2}(1) \otimes \bigwedge^2 L \otimes V_K \tag{5.7}$$

where a_K, b_K are both given by $zB - x \cdot e_x - y \cdot e_y$ (interpreted appropriately).

But this is a Koszul complex [64, proof of Theorem 1], so $H^1(C^\bullet_K) = 0$. This proves claim (1).

Claim (2) follows from Lemma 5.3.4.

Together, claims (1) and (2) imply that $D(\widehat{M})$ and $D(\widehat{M}_{\mathrm{con}})$ are isomorphic.

Therefore any θ-stable Π-module M which is R_{θ_0}-equivalent to M_{con} will give rise to the same sheaf on $\mathbb{P}^2/\Gamma$.

■

This defines a map of sets $f \colon \mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C}) \to Y_{r,n}$: For a point $x \in \mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C})$, pick any $M \in \mathfrak{M}_\theta(r, n\delta)(\mathbb{C})$ that maps to x. Then f maps x to $D(\widehat{M})$.

5.3.3 Surjectivity of the map $X_{r,n\delta} \to Y_{r,n}$

Proposition 5.3.5. *The map of sets $X_{r,n\delta} \to Y_{r,n}$, given by $\mathscr{F} \mapsto D(\mathscr{F})$, is surjective.*

Proof. Let $\mathscr{E} \in Y_{r,n}$. By Corollary 5.2.14, $\pi^T\mathscr{E}$ is an element of $X_{r,\mathbf{v}}$ for some dimension vector $\mathbf{v}$ such that $\mathbf{v}_0 = n$.

Now, by Theorem 5.3.1, $\pi^T\mathscr{E}$ corresponds to a point of $\mathfrak{M}_\theta(r, \mathbf{v})$, i.e., a stable $(Q, \mathcal{J})$-representation M of dimension $(r, \mathbf{v})$. Then M is R_{θ_0}-equivalent to a θ_0-semistable module M_{con}. By Proposition A.0.1, $\dim M_{\mathrm{con}}$ satisfies

$$\dim_\infty M_{\mathrm{con}} = 1, \quad \dim_1 M_{\mathrm{con}} = n, \quad \dim M_{\mathrm{con}} \leq (1, n\delta).$$

Then, taking the direct sum of M_{con} with the appropriate vertex simple modules will provide a θ_0-semistable Π-module N of dimension $(1, n\delta)$, which lies in the same R_{θ_0}-equivalence class as M_{con}.

Since the map $\mathfrak{M}_\theta(r, n\delta) \to \mathfrak{M}_{\theta_0}(r, n\delta)$ is surjective by Lemma 2.4.5, there is a stable Π-module $M' \in \mathfrak{M}_\theta(r, n\delta)$, in the same R_{θ_0}-equivalence class as N and M_{con}.

It follows from Proposition 5.3.2 that the framed sheaf corresponding to this module is also mapped to $\mathscr{E}$ by D. This concludes the proof. ∎

5.4 An inverse map

In this section, we construct a map $g\colon Y_{r,n} \to \mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C})$, and show that this is an inverse map to the map f constructed in the proof of Proposition 5.3.2.

We now return to our original viewpoint of the quiver varieties, i.e., the underlying quiver has r pairs of arrows between the 0 and ∞-vertices.

Lemma 5.4.1. *There is a canonical map $g\colon Y_{r,n} \to \mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C})$.*

Proof. Let $\mathscr{F} \in Y_{r,n}$. By Lemma 5.2.6 and Corollary 5.2.14, there is a vector $\mathbf{v} \in \mathbb{N}_0^{Q_0}$ satisfying $\mathbf{v}_0 = n$ such that $\pi^T\mathscr{F} \in X_{r,\mathbf{v}}$. This sheaf corresponds by Theorem 5.3.1 to a stable Π-module, which is therefore also θ_0-semistable. Therefore the concentrated module M_{con} of its R_{θ_0}-equivalence class satisfies $M_{\text{con},0} = n$, and by Proposition A.0.1 also satisfies $\dim M_{\text{con}} \leq (1, n\delta)$. But then M_{con} determines a unique S_{θ_0}-equivalence class of Π-modules of dimension $(1, n\delta)$ – i.e. a $\mathbb{C}$-valued point of $\mathfrak{M}_{\theta_0}(r, n\delta)$.

We set this point to be $g(\mathscr{F})$. ∎

It is time to prove:

Proposition 5.4.2. *There is a canonical bijection of sets between $Y_{r,n}$ and $\mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C})$.*

Proof. The only part that remains is to prove that the maps f and g, respectively constructed in Proposition 5.3.2 and Lemma 5.4.1 are mutually inverse.

So let $x \in \mathfrak{M}_{\theta_0}(r, n\delta)$, and let $\mathscr{F} = f(x)$. Then $g(\mathscr{F}) \in \mathfrak{M}_{\theta_0}(r, n\delta)$, but it is not clear whether $g(\mathscr{F})$ equals x. We shall show that any two elements of $X_{r,n\delta}$ that both are mapped to $\mathscr{F}$ by f, correspond to S_{θ_0}-equivalent Π-modules; this implies that $g(\mathscr{F}) = x$.

Let $\{\mathscr{E}_i\}_{i\in J}$ be the set of all coherent framed Γ-equivariant torsion free sheaves on $\mathbb{P}^2$ such that $D(\mathscr{E}_i) = \mathscr{F}$. This set is nonempty, as by Proposition 5.3.5, $\pi^T\mathscr{F}$ is a member.

As $D(\mathscr{E}_i)|_{V_p} = \pi_*(\mathscr{E}_i|_{U_o})^\Gamma$, the counit of the adjunction $\pi_* \vdash \pi^*$ induces for every $\mathscr{E}_i$ a map

$$\pi^*\mathscr{F} = \pi^*(D(\mathscr{E}_i)) \to \mathscr{E}_i.$$

Since $\mathscr{E}_i$ is torsion-free, this map annihilates the torsion subsheaf of $\pi^*\mathscr{F}$, and we obtain for every i a map $\tau_i \colon \pi^T\mathscr{F} \to \mathscr{E}_i$.

Set $\mathscr{E} = \pi^T\mathscr{F}$. Then for any i, the morphism $\tau_i \colon \mathscr{E} \to \mathscr{E}_i$ is injective by Lemma 5.2.12.

Let $\mathscr{K}_i \coloneqq \operatorname{coker} \tau_i$, a torsion sheaf supported at o.

For any $\mathscr{E}_i$, the short exact sequence

$$0 \to \mathscr{E} \to \mathscr{E}_i \to \mathscr{K} \to 0$$

induces a sequence of complexes

$$C^\bullet_{M_\mathscr{E}} \to C^\bullet_{M_{\mathscr{E}_i}} \to C^\bullet_{M_\mathscr{K}}$$

exact in the middle. (Note that the construction of the complex $C^\bullet_{M_\mathscr{K}}$ works just as for $\mathscr{E}$, even though $\mathscr{K}$ is torsion.)

Since $\mathscr{K}$ is supported only at o, we have

$$H^1(\mathbb{P}^2, \mathscr{K}_i(-l_\infty)) = H^1(\mathbb{P}^2, \mathscr{K}_i(-l_\infty) \otimes \mathscr{Q}^\vee) = 0,$$

and thus $C^\bullet_{M_\mathscr{K}} = 0$. It follows that there is for every i a surjection of complexes $C^\bullet_{M_\mathscr{E}} \to C^\bullet_{M_{\mathscr{E}_i}}$. Hence we have surjections of θ_0-semistable modules $M_\mathscr{E} \to M_{\mathscr{E}_i}$, which implies that $M_\mathscr{E}$ and $M_{\mathscr{E}_i}$ are R_{θ_0}-equivalent – and so any two $M_{\mathscr{E}_i}$ must also be R_{θ_0}-equivalent.

Therefore any two sheaves in $X_{r,n\delta} \cong \mathfrak{M}_\theta(r, n\delta)$ mapped to $\mathscr{F}$ by $D(-)$ must correspond to S_{θ_0}-equivalent Π-modules, and so they determine the same point of $\mathfrak{M}_{\theta_0}(r, n\delta)$. But this shows that $g(f(M)) = M$.

On the other hand, choose a sheaf $\mathscr{F} \in Y_{r,n}$ and set $M = g(\mathscr{F})$. By construction, the concentrated Π-module R_{θ_0}-equivalent to M is R_{θ_0}-equivalent to the stable module corresponding to $\pi^T\mathscr{F}$. But Proposition 5.3.2 tells us that, for any two sheaves $\mathscr{E}_1, \mathscr{E}_2$ on $\mathbb{P}^2$ corresponding to elements of one R_{θ_0}-equivalence class, we have $D(\mathscr{E}_1) = D(\mathscr{E}_1)$. Since $D(\pi^T(\mathscr{F})) = \mathscr{F}$, by Lemma 5.2.13, it follows that $f(g(\mathscr{F})) = \mathscr{F}$.

Thus f and g are mutually inverse maps. The proof is complete. ∎

Chapter 6

Construction of the moduli space

In this chapter, we prove that there is a natural moduli space $\mathbf{Y}_{r,n}$ which is a quasiprojective scheme, such that $\mathbf{Y}_{r,n}(\mathbb{C}) = Y_{r,n}$. We also show that the bijection $\mathfrak{M}_{\theta_0}(r, n\delta)(\mathbb{C}) \to Y_{r,n}$ from Proposition 5.4.2 can be extended to a morphism $\mathbf{Y}_{r,n} \to \mathfrak{M}_{\theta_0}(r, n\delta)$.

We conjecture that this morphism is actually an isomorphism (at least of the underlying reduced schemes), but we are unfortunately unable to prove it.

There are two strands to this argument:

First, as in Chapter 3, the Nakajima quiver variety $\mathfrak{M}_{\theta_0}(r, n\delta)$ does not *a priori* carry the structure of a *fine* moduli space. We will therefore have to construct a *cornered* preprojective algebra Π_0, and consider the moduli space $\mathcal{M}(\Pi_0)(1, n)$ of stable Π_0-modules for this algebra (for some appropriate stability condition). This will be a fine moduli space, and we shall as in Chapter 3 show that there is an isomorphism of underlying reduced schemes between $\mathfrak{M}_{\theta_0}(r, n\delta)$ and $\mathcal{M}(\Pi_0)(1, n)$.

The second strand is to show that $Y_{r,n}$ can be given the structure of a moduli space. For this, we will use the results in Section 2.6.

We combine these strands in Section 6.3.

6.1 The cornered preprojective algebra

We return to the original construction of Nakajima quiver varieties (Definition 2.1.1). In other words, the underlying quiver Q consists of an unframed McKay quiver, to which we add a framing vertex ∞, together with r pairs of arrows $(b_1, \overline{b_1}), \ldots, (b_r, \overline{b_r})$ between ∞ and the 0-vertex, where the b_i go from ∞ to 0, and the $\overline{b_i}$ in the opposite direction. Our quiver varieties $\mathfrak{M}_{\theta'}(r, \mathbf{v})$ parametrise S_θ-equivalence classes of modules of the preprojective algebra of Q, with dimension vector $(1, \mathbf{v})$ - especially, the dimension at ∞ will always be 1.

Similarly to Sections 3.2, 3.3 and 4.2, we will now interpret $\mathfrak{M}_{\theta_0}(r, n\delta)$ as a fine moduli space modules of an auxiliary algebra Π_0.

The proofs involved can be copied almost verbatim from Section 3.3, so we only give a brief sketch of the details.

Let $e_0, e_\infty \in \Pi$ be the vertex idempotents for the vertices $0, \infty$ respectively. We set $\Pi_0 = (e_0 + e_\infty)\Pi(e_0 + e_\infty)$, and call this the *cornered preprojective algebra.*

(The algebra Π_0 resembles the algebra A_0 appearing in Chapters 3 and 4. The difference is that in Π_0, we have r pairs of arrows between the 0-vertex and the framing vertex ∞, whereas the A_0-algebra only had a single arrow from $\infty$to 0, and none in the opposite direction.)

Then $v = (1, n) = \rho_\infty + n\rho_0$ is a dimension vector for Π_0-modules.

We also obtain a stability condition $\eta_0 \colon \mathbb{Z}\rho_0 \oplus \mathbb{Z}\rho_\infty \to \mathbb{Q}$ for Π_0-modules of dimension $(1, n)$, given by $\eta(\rho_\infty) = -n, \quad \eta(\rho_0) = 1$.

We will first need some results on finite generation. Note that the argument we apply in Proposition 6.1.1 does not directly work in the setting considered in Chapter 3 and Chapter 4, because the cornering to an arbitrary nonempty set of vertices considered there makes the resulting combinatorics more complicated.

We use the fact that $\mathbb{C}Q$ is a graded algebra, with elements graded by their *length* as paths in Q. Because the preprojective relations are homogeneous with respect to length, we obtain an induced grading by length on Π, $\Pi_0/(e_\infty)$ and their subalgebras.

Proposition 6.1.1. *The algebra Π_0 is finitely generated.*

Proof. Write $b_1, \dots, b_r$ for the elements in Π coming from the arrows in Q going from ∞ to 0, $\overline{b_1}, \dots, \overline{b_r}$ for their opposites. Let e_∞ be the idempotent at ∞.

Recall from (3.4) that the subquotient algebra $\Pi_0/(e_\infty) = e_0(\Pi/(e_\infty))e_0$ of Π is isomorphic to $\mathbb{C}[x, y]^\Gamma$, and so generated by three classes a_1, a_2, a_3. We choose elements $\alpha_1, \alpha_2, \alpha_3 \in e_0\Pi e_0$ such that for each i, α_i is a preimage with minimal length of a_i.

Consider an element $p \in \Pi_0$. We will show that p can be generated by the classes of the paths $\{\alpha_i\}, \{b_i\}, \{\overline{b_i}\}$.

It is clear that if either endpoint of p is the ∞-vertex, p is the composition of some $b_i, \overline{b_i}$ with the class of a path starting and ending at 0.We may assume that p is a class of paths starting and ending at the 0-vertex. Let p' be the image of p in $\Pi_0/(e_\infty)$ - this element can clearly be expressed by the three classes a_i, so we can write

$$p = p' + \sum_{i=1}^{m_p} k_i, \quad k_i \neq 0$$

with $k_i \in (e_\infty) \subset \Pi_0$. Now each k_i can be written as a product of classes of paths based at the 0-vertex, at least one of which must be of the form $(b_i\overline{b_j})$. The other factors of k_i (say $\{p_j\}|_{j\in J}$, for some finite index set J) have nonzero images in $\Pi_0/(e_\infty)$, and must each have shorter length than p.

Now repeat this construction for each p_j in place of p, leading to expressions

$$p_j = p_j' + \sum k_{i,j_\ell},$$

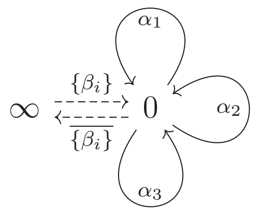
with $k_{i,j_\ell} \in (e_\infty)$, each of which again must have some factors $p_{j_k} \in (e_\infty)$. We repeat the same construction again with the p_{j_k} in place of p, and continue similarly.

Since the length of the paths considered decreases in every step, the process must eventually conclude. When that has happened, we will have written p as generated by the classes $a_1, a_2, a_3, \{b_i\}, \{\overline{b_i}\}$. It follows that Π_0 is finitely generated. ■

Corollary 6.1.2. *There is a fine moduli space* $\mathcal{M}(\Pi_0)(1,n) = \mathcal{M}_{\eta_0}(\Pi_0)(1,n)$*, parametrising* η_0*-stable* Π_0*-modules of dimension* $(1,n)$*.*

Proof. Since Π_0 is finitely-generated by Proposition 6.1.1, this again follows from the result of King [36, Proposition 5.3]. ■

Remark 6.1.3. It is not difficult to see that this proof shows that Π_0 can be written as a quotient of the path algebra of this quiver:



,

where the dashed arrows between ∞ and 0 are each meant to stand in for r arrows. The homomorphism from the path algebra for this quiver to the algebra Π_0 maps $\beta_i \mapsto b_i, \overline{\beta_i} \mapsto \overline{b_i}$, and each loop α_i at the 0 vertex to one of the a_i's.

Now the arguments of Section 3.3 can be adapted: There are functors

$$j^*\colon \Pi{-}\mathrm{mod} \to \Pi_0{-}\mathrm{mod}, \quad j_!\colon \Pi_0{-}\mathrm{mod} \to \Pi{-}\mathrm{mod}.$$

defined by $j^*(M) = (e_\infty + e_0)\Pi \otimes_\Pi M$, and $j_!(N) = \Pi(e_\infty + e_0) \otimes_{\Pi_0} N$. Especially $j^*j_!$ is the identity functor for Π_0-modules.

Theorem 6.1.4. *The functors j^*, $j_!$ induce a morphism*

$$\iota_J \colon \mathfrak{M}_{\theta_0}(r, n\delta) \to \mathcal{M}(\Pi_0)(1, n)$$

which is an isomorphism of the underlying reduced schemes. Especially, $\mathcal{M}(\Pi_0)(1, n)$ is irreducible, and its underlying reduced scheme is normal, with symplectic singularities.

The proof of this claim is an adaptation of the arguments appearing in Section 3.3.

The difference is that, since we know from Proposition 6.1.1 that Π_0 is finitely generated, we do not need to delete the arrows $b_i \colon 0 \to \infty$ – which we cannot do anyway, as Lemma 3.2.1 does not hold when $r > 1$.

Especially, note that a Π_0-module M of dimension vector $(1, n)$ is η_0-semistable if and only if there is a surjective morphism $\Pi_0 e_\infty \to M$.

We start by proving an analogue of Lemma 3.1.7.

Recall from Lemma 2.2.4 that the quiver variety $\mathfrak{M}_\theta(r, n\delta)$ is equipped with a universal sheaf, i.e., a direct sum of locally free sheaves

$$\mathcal{R} = \bigoplus_{i \in Q_0} \mathcal{R}_i.$$

with $\operatorname{rk} \mathcal{R}_i = n\delta_i$, $\operatorname{rk} \mathcal{R}_\infty = 1$, and a homomorphism $\Pi \to \operatorname{End}(\mathcal{R})$.

Lemma 6.1.5. *The line bundle $\mathcal{R}_0$ is globally generated, and after multiplying by a positive integer if necessary, the morphism $\mathfrak{M}_\theta(r, n\delta) \to |\mathcal{R}_0|$ governed by $\mathcal{R}_0$ factorises as*

$$\begin{tikzcd}
\mathfrak{M}_\theta(r, n\delta) \arrow[r, "\varphi_{\mathcal{R}_0}"] \arrow[d, "\pi_0"] & {|\mathcal{R}_0|} \\
\mathfrak{M}_{\theta_0}(r, n\delta) \arrow[ur, hook] &
\end{tikzcd}$$

where π_0 is the morphism induced by variation of GIT parameter.

Proof. The proof is almost a repeat of that of Lemma 4.3.3. The bundle $\mathcal{R}_0$ is globally generated because the tautological bundle on $\mathfrak{M}_{\theta(r,n\delta)}$ is.

Then by construction, the ample line bundle $\mathcal{R}'_0$ on $\mathfrak{M}_{\theta_0}(r, n\delta)$ furnished by the GIT construction is precisely that descending from $\mathcal{O}_{\mu^{-1}(0)}$, linearised by the character χ_{θ_0}. (see [5, p. 6.2]). Replace $\mathcal{R}'_0$ by some very ample tensor power, so that its associated morphism is a closed immersion. However, by construction of the GIT quotient, we obtain $\pi_0^*(\mathcal{R}'_0) = \mathcal{R}_0$. We now argue completely as in Lemma 4.3.3:

If $\varphi_{|\mathcal{R}'_0|} \colon \mathfrak{M}_{\theta_0}(r, n\delta) \to |\mathcal{R}'_0|$ is the closed immersion, then

$$(\varphi_{|\mathcal{R}'_0|} \circ \pi_I)^*(\mathcal{O}_{|\mathcal{R}'_0|}(1)) = \pi_I^*(\mathcal{R}'_0) = \mathcal{R}_0 = \varphi^*_{|\mathcal{R}_0|}\Big(\mathcal{O}_{|\mathcal{R}_0|}(1)\Big)$$

on $\mathfrak{M}_{\theta_0}(r, n\delta)$. By [31, II, Theorem 7.1], the morphism to a complete linear series of a specific line bundle is unique up to automorphism of the linear series Thus, after a change of basis on $H^0(\mathcal{R}_0')$ if necessary, we have $\varphi_{|\mathcal{R}'|} = \varphi_{|\mathcal{R}_0'|} \circ \pi_I$ as required.

■

Proof of Theorem 6.1.4. To prove this, we can repeat the proofs of Theorem 3.3.8 and its prerequisites Lemma 3.3.2-Lemma 3.3.7 *verbatim*,omitting any reference to morphisms over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, and replacing A_I by Π_0, and any reference to Lemma 3.1.7 by Lemma 6.1.5. ■

Remark 6.1.6. Take a framed sheaf $\mathscr{F} \in Y_{r,n}$, and find its corresponding $x \in \mathfrak{M}_{\theta_0}(r, n\delta)$ with Proposition 5.4.2.

Then use Theorem 6.1.4 to find a η_0-stable Π_0-module $M = (M_\infty, M_0)$ corresponding to $\mathscr{F}$. If we trace through the arguments, we find that $M_0 = H^1(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{d_\infty})$, so the three α_i-arrows in Remark 6.1.3 act by endomorphisms of this cohomology group.

Let us also note:

Corollary 6.1.7. *Let $\mathscr{F} \in Y_{r,n}$ be the framed sheaf corresponding to the η_0-stable Π_0-module M.*

Then $\pi^T\mathscr{F}$ corresponds to the stable Π-module $j_!(M)$.

Proof. The Γ-equivariant sheaf $\pi^T\mathscr{F}$ corresponds to a stable Π-module N of dimension $(1, \mathbf{v})$ for some $\mathbf{v}$ such that $\mathbf{v}_0 = n$, by Corollary 5.2.14. By Lemma 5.2.13, $D(\pi^T(\mathscr{F})) \cong \mathscr{F}$, so $M = j^*(N) = j^*(j_!(M))$.

Let $\mathscr{E}$ be the Γ-equivariant sheaf on $\mathbb{P}^2$ corresponding to $j_!(M)$. As in the proof of Proposition 5.4.2, there is an injective morphism $\pi^T\mathscr{F} \hookrightarrow \mathscr{E}$.

By passing from sheaves to Π-modules, we obtain (again, as in the proof of Proposition 5.4.2) a surjective homomorphism $q\colon N \to j_!(M)$. Then, by [15, Lemma 3.8], there is an isomorphism $N \xrightarrow{\sim} j_!(M)$. ■

6.2 A scheme structure for $Y_{r,n}$

We show that $Y_{r,n}$ carries a natural structure as a scheme, and represents the moduli functor $\mathcal{Y}_{r,n}$.

It will turn out that $\mathcal{Y}_{r,n}$ is represented by an open subscheme of a moduli space of the type $\mathbf{M}_{\mathcal{Y}}(\mathscr{V}_{\mathcal{Y}}, \mathcal{O}_Y(1), P_0, \mathcal{O}^r_{d_\infty}, \delta(\nu))$ from Section 2.6.

Recall from Definition 1.3.2 that we have defined $\mathcal{Y}_{r,n}$ to be the functor associating to any scheme S of finite type the set of isomorphism classes families of $\mathcal{O}^r_{d_\infty}$-framed sheaves $(\mathscr{F}, \phi_{\mathscr{F}})$ on $\mathcal{X} \times S$, flat over S such that every $\mathcal{O}^r_{d_\infty}$-framed sheaf $(\mathscr{F}_s, \phi_{\mathscr{F}_s})$ on $\mathcal{X}$ satisfies the criteria of Definition 1.3.1.

Theorem 6.2.1. *There is a quasiprojective scheme $\mathbf{Y}_{r,n}$ representing the functor $\mathcal{Y}_{r,n}$.*

The proof will take up most of the remaining chapter.

We must thus establish that

- there is a $\delta_1 \in \mathbb{Q}$ such that any $\mathscr{F} \in Y_{r,n}$ is $\hat{\mu}$-stable with respect to δ_1.
- the modified Hilbert polynomial $P_{\mathcal{O}_X(d_\infty)}(\mathscr{F}, n)$ is independent of the choice of $\mathscr{F} \in Y_{r,n}$, and
- there is a bound for the modified Castelnuovo-Mumford regularities of the framed sheaves in $Y_{r,n}$.

6.2.1 Equivariant sheaves on $\mathbb{P}^2$

The isomorphism classes of Γ-equivariant line bundles on $\mathbb{P}^2$ form a group under $\otimes$, which we denote $\mathrm{Pic}_\Gamma(\mathbb{P}^2)$. This group is usually called the *equivariant Picard group*, see e.g. [59] or [45].

Forgetting the equivariant structure gives a homomorphism $t\colon \mathrm{Pic}_\Gamma(\mathbb{P}^2) \to \mathrm{Pic}(\mathbb{P}^2)$.

For instance, the line bundle $\mathcal{O}_{\mathbb{P}^2}(l_\infty)$ has a Γ-equivariant structure induced by the action of Γ on $\mathbb{P}^2$, and it maps to $\mathcal{O}_{\mathbb{P}^2}(1)$ under t. Especially, the ideal sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-l_\infty) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{l_\infty} \to 0$$

is equivariant.

By [37, Proposition 4.2], the set of elements of $\mathrm{Pic}_\Gamma(\mathbb{P}^2)$ that are inverse images under $\pi \circ c$ of line bundles on $\mathbb{P}^2/\Gamma$ consists exactly of those line bundles $\mathscr{L}$ such that, for each closed point $x \in \mathbb{P}^2$, the action of the stabiliser group Γ_x on fibres of $\mathscr{L}$ is trivial.

We know from Remark 5.1.6 that the stabiliser groups of any point $x \in \mathbb{P}^2 \smallsetminus \{o\}$ are all finite cyclic. In addition, over o, the stabiliser $\Gamma_o = \Gamma$ acts trivially on the fibre of $\mathcal{O}_{\mathbb{P}^2}(l_\infty)$. Therefore there must be some minimal m such that the tensor power $\mathcal{O}_{\mathbb{P}^2}(l_\infty)^{\otimes m}$ lies in the inverse image of $\mathrm{Pic}(\mathbb{P}^2/\Gamma)$ under $\pi \circ c$. This tensor power must furthermore be (very) ample.

From now on, fix $\mathcal{O}_{\mathbb{P}^2/\Gamma}(1)$ to be a line bundle on $\mathbb{P}^2/\Gamma$ such that

$$\pi^* c^* \mathcal{O}_{\mathbb{P}^2/\Gamma}(1) = \mathcal{O}_{\mathbb{P}^2}(l_\infty)^{\otimes m}.$$

Because $\mathbb{P}^2/\Gamma$ is projective and $c \circ \pi\colon \mathbb{P}^2 \to \mathbb{P}^2/\Gamma$ is finite, by [31, Exercise III.5.7.d], $\mathcal{O}_{\mathbb{P}^2/\Gamma}(1)$ is ample.

6.2.2 Equivariant Euler characteristics

We now introduce an 'equivariant Euler characteristic' χ_G as a computational tool, following Ellingsrud and Lønsted [23].

Recall that the standard Euler characteristic χ for coherent sheaves on a scheme X can be interpreted as a homomorphism $\chi\colon K(X)^f \to \mathbb{Z}$, where $K(X)^f$ is the subgroup of the Grothendieck group $K(X)$ given by the equivalence classes of sheaves $\mathscr{F}$ with finite-dimensional cohomology groups $H^i(X, \mathscr{F})$.

Now, let G be a finite group acting on X, let Irr_G be its set of irreducible representations, and let $R_{\mathbb{C}}G$ be the representation ring of G over $\mathbb{C}$. If $\mathscr{F}$ is a G-equivariant sheaf on X, each cohomology group $H^i(X, \mathscr{F})$ carries an induced G-representation structure, and we denote its image in $R_{\mathbb{C}}G$ by $[H^i(X, \mathscr{F})]$.

Let $K_G(X)^f$ be the finite-dimensional equivariant K-theory of X: Take the abelian group formed by taking the free abelian group on isomorphism classes of G-equivariant coherent $\mathcal{O}_X$-modules with finite-dimensional cohomology groups. Then $K_G(X)^f$ is the quotient of this group by, for every short exact sequence of equivariant $\mathcal{O}_X$-module homomorphisms

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0,$$

the expression $[\mathscr{F}] + [\mathscr{G}] - [\mathscr{E}]$.

There is then a map $\chi_G\colon K_G(X) \to R_{\mathbb{C}}G$, defined by

$$\mathscr{F} \mapsto \sum_{i \geq 0} (-1)^i [H^i(X, \mathscr{F})].$$

(This is sometimes called a *Lefschetz trace*, see e.g. [23, Definition 1.1].)

It is then clear that the ordinary Euler characteristic χ is the composition of χ_G with the homomorphism $R_{\mathbb{C}}G \to \mathbb{Z}$ defined by $\sum_{\rho_j \in \mathrm{Irr}_G} n_j [\rho_j] \mapsto \sum_{\rho_j \in \mathrm{Irr}_G} n_j \dim \rho_j$ - here we have used Maschke's theorem to say that any G-representation splits into a direct sum of irreducible representations.

Furthermore, there are 'isotypic projection' group homomorphisms $p_j\colon R_{\mathbb{C}}G \to \mathbb{Z}$, given by $p_j\colon \sum_{\rho_j \in \mathrm{Irr}_G} n_j [\rho_j] \mapsto n_j$.

Let us return to the case $G = \Gamma, X = \mathbb{P}^2$. We will from now on write $\chi_0 \coloneqq p_0 \circ \chi_\Gamma$.

Let's use this to compute Euler characteristics for sheaves on our stack $\mathcal{X}$.

Lemma 6.2.2. *Let $\mathscr{E} \in \mathsf{Coh}(\mathcal{X})$. Let $\mathscr{F}$ be any Γ-equivariant coherent sheaf on $\mathbb{P}^2$ such that $D(\mathscr{F}) = \mathscr{E}$. Then*

$$\chi(\mathcal{X}, \mathscr{E}) = \chi_0(\mathbb{P}^2, \mathscr{F}).$$

Proof. This follows from the fact that $\mathcal{X}$ and $[\mathbb{P}^2/\Gamma]$ share $\mathbb{P}^2/\Gamma$ as their coarse moduli space, together with Lemma 2.5.5. ∎

6.2.3 Hilbert polynomials of framed sheaves on $\mathcal{X}$

Proposition 6.2.3. *Let $\mathscr{F} \in Y_{r,n}$.*

The function

$$H_\Gamma(\mathscr{F}, \nu) \coloneqq \nu \mapsto \chi_0(\mathbb{P}^2, \pi^T \mathscr{F}(\nu l_\infty))$$

does not depend on the choice of $\mathscr{F}$.

Proof. The sheaf $\pi^T\mathscr{F}$ lies in some $X_{r,\mathbf{v}}$ such that $\mathbf{v}_0 = n$ by Corollary 5.2.14, and so $\dim(H^1(\mathbb{P}^2, \pi^T\mathscr{F}(-l_\infty)))^\Gamma = n$.

It follows from [48, Lemma 2.4] that

$$H^i(\mathbb{P}^2, \pi^T\mathscr{F}(-p \cdot l_\infty)) = 0$$

for $i = 0, 2; p = 1, 2$.

Especially, we find

$$\chi_0(\pi^T\mathscr{F}(-l_\infty)) = n.$$

We can for every $\nu \in \mathbb{Z}$ set up short exact Γ-equivariant sequences

$$0 \to \pi^T\mathscr{F}((\nu-1)l_\infty) \to \pi^T\mathscr{F}(\nu l_\infty) \to \pi^T\mathscr{F}|_{l_\infty}(\nu l_\infty) \to 0,$$

which is short exact because $\pi^T\mathscr{F}$ is locally free in a neighbourhood of l_∞.

Now $\mathscr{F}|_{l_\infty}(\nu l_\infty)) = \mathcal{O}_{l_\infty}(\nu l_\infty)^{\oplus r}$, which shows (since the long exact sequences of cohomology become long exact sequences of Γ-representations) that

$$\chi_0(\mathbb{P}^2, \pi^T\mathscr{F}(\nu l_\infty)) = \chi_0(\mathbb{P}^2, \pi^T\mathscr{F}((\nu-1)l_\infty)) + \chi_0(\mathcal{O}_{l_\infty}(\nu l_\infty)^{\oplus r}).$$

Then, for any ν, $\chi_0(\mathbb{P}^2, \pi^T\mathscr{F}(\nu l_\infty))$ can be iteratively computed from $H_\Gamma(\mathcal{O}_{l_\infty})$ and $\chi_0(\mathbb{P}^2, \pi^T\mathscr{F}(-l_\infty)) = n$. This concludes the proof.

∎

The function $H_\Gamma(\mathscr{F}, \nu)$ is not in general a polynomial. A slight adaptation of the next proof shows that it is however a *quasi-polynomial*, i.e., there is a set of numerical polynomials $p_1, \ldots, p_m \in \mathbb{Q}[\nu]$ such that $H_\Gamma(\mathscr{F}, \nu) = p_i(\nu)$ for $\nu \equiv i \bmod m$. We will not make use of this.

Corollary 6.2.4. *All sheaves $\mathscr{F} \in Y_{r,n}$ have the same modified Hilbert polynomial.*

Proof. Let $\mathscr{F} \in Y_{r,n}$.

Since the dual of $\mathcal{O}(d_\infty)$ is $\mathscr{I}_{d_\infty}$, we have

$$P_{\mathcal{O}(d_\infty)}(\mathscr{F}, \nu) = \chi(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{(d_\infty)} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu)).$$

To compute these Euler characteristics, we shall work on $\mathbb{P}^2$.

First, we claim that

$$\begin{aligned} D(\pi^*\Big(\mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu)\Big)) &= D(\pi^T\Big(\mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu)\Big)) \\ &= \mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu). \end{aligned} \tag{6.1}$$

To see this, it is enough to restrict to some affine open U trivialising $\mathscr{I}_{(d_\infty)} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu)$, such that $p \in U \subset U_p$. Here we have

$$D\left(\pi^*(\mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu))\right)|_U \xrightarrow{\sim} D(\pi^*(\mathscr{F}|_U),$$

$$D\left(\pi^T(\mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu))\right)|_U \xrightarrow{\sim} D(\pi^T(\mathscr{F}|_U),$$

which by Lemma 5.2.13 both are canonically isomorphic to $\mathscr{F}|_U$.

By construction of $\mathcal{O}_{[\mathbb{P}^2/\Gamma]}(1)$, we have

$$\begin{aligned} \pi^T\left(\mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu)\right) &= \pi^T\mathscr{F} \otimes \mathscr{I}_{l_\infty} \otimes \mathcal{O}_{\mathbb{P}^2}(\nu m \cdot l_\infty) \\ &= \pi^T\mathscr{F}(\nu(m-1) \cdot l_\infty). \end{aligned} \tag{6.2}$$

Now we can apply Lemma 6.2.2, which tells us that

$$\chi(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{(d_\infty)} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(\nu)) = \chi_0(\mathbb{P}^2\, \pi^T\mathscr{F}(\nu(m-1) \cdot l_\infty)).$$

So we find that $P_{\mathcal{O}(d_\infty)}(\mathscr{F}, \nu)$ is given by

$$\nu \mapsto H_\Gamma(\pi^T\mathscr{F}, \nu m - 1),$$

which by Proposition 6.2.3 is the same for all $\mathscr{F} \in Y_{r,n}$.

■

6.2.4 Regularity

For a coherent sheaf $\mathscr{G}$ on $\mathbb{P}^2$, set $P(\mathscr{G}, \nu) := \chi(\mathbb{P}^2, \mathscr{G} \otimes \mathcal{O}_{\mathbb{P}^2}(\nu))$ to be the (ordinary) Hilbert polynomial.

We will now show that the set $\{\pi^T\mathscr{F} | \mathscr{F} \in Y_{r,n}\}$ is *bounded*, i.e., the set of Hilbert polynomials

$$\{P(\pi^T\mathscr{F}, \nu) \mid \mathscr{F} \in Y_{r,n}\}$$

is finite. We also show that there is a number ρ which is an upper bound for the (modified) regularity of the sheaves in $Y_{r,n}$. Recall that we in Definition 2.5.10 defined the regularity of $\mathscr{F} \in Y_{r,n}$ to be the regularity of $c_*(\mathscr{F} \otimes \mathcal{O}(d_\infty)^\vee)$.

Remark 6.2.5. The more usual definition, in [53, Definition 4.10] (for DM stacks) originally for schemes in [SGA6, XIII–1.12], would say: $Y_{r,n}$ is *bounded* if there is a scheme T of finite type over $\mathbb{C}$ and a coherent sheaf $\mathscr{G}$ on $\mathcal{X} \times T$ such that every element of $Y_{r,n}$ is contained in the fibres of $\mathscr{G}$ over closed points of T. The equivalence with our definition is proved in [53, Theorem 4.12], assuming that the set of sheaves $Y_{r,n}$ satisfies a cohomological condition.

Lemma 6.2.6. *The set of Hilbert polynomials $\{P(\pi^T\mathscr{F}, \nu)\}$ for $\mathscr{F} \in Y_{r,n}$ is finite.*

Proof. This is because the Hilbert polynomial is completely determined by the rank of $\pi^T\mathscr{F}$ and the Euler characteristics $\chi(\mathbb{P}^2, \pi^T\mathscr{F}(-l_\infty))$ and $\chi(\mathbb{P}^2, \pi^T\mathscr{F}(-2l_\infty))$.

If we let M be the Π-module associated to $\pi^T(\mathscr{F})$ by Theorem 5.3.1, the latter two are isomorphic (as Γ-representations) to $\bigoplus_{i \in Q_{\Gamma,0}} M_i \otimes \rho_i$.

But the dimension vector of M is bounded: Let N be the Π_0-module corresponding to $\mathscr{F}$. Then, by Corollary 6.1.7, $M = j_!(N)$. We can then follow the proof of [15, Lemma 3.6.(ii))]: We have that Π is finitely-generated as a $e_0\Pi e_0$-module. Let $a_1, \ldots, a_r$ be a set of generators for $\Pi(e_0 + e_\infty)$ as a right $(e_0 + e_\infty)\Pi(e_0 + e_\infty)$-module. Then as a vector space, $\Pi(e_\infty + e_0) \otimes_{\Pi_0} N \subset N^r$ is finite-dimensional, so $M = j_!(N) = \Pi(e_\infty + e_0) \otimes_{\Pi_0} N$ is finite-dimensional, and its dimension vector $\dim M$ must satisfy $\sum_{i \in Q_0} \dim_i M \leq r(\dim_0 N + \dim_\infty N)$.

So the set of possible dimension vectors for M is finite, and we are done. ■

Lemma 6.2.7. *There is an integer p, depending only on r and n, such that every framed sheaf $\mathscr{F} \in Y_{r,n}$ is p-regular.*

We need to show that there is an $p \in \mathbb{Z}$ such that for any $\mathscr{F} \in Y_{r,n}$, the groups

$$H^i(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(p - i))$$

vanish for all i.

For any numerical polynomial $P \in \mathbb{Q}[\nu]$, set

$$Y_P \subset Y_{r,n} = \{\pi^T\mathscr{F} \mid \mathscr{F} \in Y_{r,n}, P(\pi^T\mathscr{F}, \nu) = P\}.$$

Then, by [9, Theorem 2.5], each Y_P is bounded. This means that given a polynomial P, each $\pi^T\mathscr{F} \in Y_P$ is p_P-regular for some p_P depending on P. But since the set of nonempty Y_P is finite by Lemma 6.2.6, $\pi^T\mathscr{F}$ is $p' := \max\{p_P\}$-regular for each $\mathscr{F} \in Y_{r,n}$.

From the proof of Corollary 6.2.4, we know that the cohomology groups

$$H^i(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(m' - i))$$

vanish when the groups

$$H^i(\mathbb{P}^2, \pi^T(\mathscr{F} \otimes \mathscr{I}_{d_\infty} \otimes c^*\mathcal{O}_{\mathbb{P}^2/\Gamma}(m'-i))) = H^i(\mathbb{P}^2, \pi^T\mathscr{F} \otimes \mathcal{O}_{\mathbb{P}^2}\Big((m(m'-i)-1)\Big)) \quad (6.3)$$

do.

Now set p to be an integer greater than $p'/(m+2)$.

Then, for $i = 1$ Equation (6.3) becomes

$$H^1(\mathbb{P}^2, \pi^T\mathscr{F} \otimes \mathcal{O}_{\mathbb{P}^2}((mm' - m - 1)).$$

If we choose $m' > p$, then $mm' - m > p'$, so by p'-regularity of $\pi^T\mathscr{F}$, this group vanishes.

A similar computation shows that if $m' > p$,

$$H^2(\mathbb{P}^2, \pi^T\mathscr{F} \otimes \mathcal{O}_{\mathbb{P}^2}((mm' - 2m - 2)) = 0.$$

It follows that all $\mathscr{F} \in Y_{r,n}$ are p-regular.

6.2.5 Stability of framed sheaves on $\mathcal{X}$

We now know that every framed sheaf lying in $Y_{r,n}$ shares the same modified Hilbert polynomial, and thus have the same framed hat-slope. In addition, there is an integer p such that the framed sheaves are all p-regular.

We need one more ingredient before we can construct the moduli space: we need the sheaves in $Y_{r,n}$ to fulfill a $\widehat{\mu}$-stability condition.

Lemma 6.2.8. *There is a $\delta_1 \in \mathbb{Q}$ such that each $\mathscr{F} \in Y_{r,n}$ is $\widehat{\mu}$-stable with respect to δ_1.*

Proof. Let $\mathscr{F} \in Y_{r,n}$.

Because we know that the Hilbert polynomial $P_{\mathcal{O}_X(d_\infty)}(\mathscr{F}, \nu)$ is independent of $\mathscr{F}$, and they are all p-regular, we can apply the *Grothendieck lemma* in its stacky version [53, Lemma 4.13, Remark 4.14]. This tells us that there is a constant C, depending only on the modified Hilbert polynomial and modified Castelnuovo-Mumford regularity of $\mathscr{F}$, such that any purely 2-dimensional subsheaf $\mathscr{F}' \subset \mathscr{F}$ satisfies $\widehat{\mu}(\mathscr{F}') \leq C$. Fix such a C.

Let then $\mathscr{F}' \subset \mathscr{F}$. Because $\mathscr{F}$ is torsion-free, $\mathscr{F}'$ is pure of dimension 2. Now, the modified Hilbert polynomial and the modified Castelnuovo-Mumford regularity of $\mathscr{F}$ do not depend on the choice of F, and thus neither does its framed hat-slope $\widehat{\mu}(\mathscr{F})$. We will thus write $\widehat{\mu} \coloneqq \widehat{\mu}(\mathscr{F})$.

We can rewrite condition 2 in Definition 2.6.4 to

$$\widehat{\mu}(\mathscr{F}') - \widehat{\mu}(\mathscr{F}) \leq \delta_1 \left(\frac{1}{\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}')} - \frac{1}{\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F})} \right). \tag{6.4}$$

We then have $\widehat{\mu}(\mathscr{F}') \leq C$, which implies that Equation (6.4) will hold whenever

$$\delta_1 \geq \frac{(C - \widehat{\mu})(\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}')\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}))}{\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}) - \alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}')}$$

Note that the maximal value of the right-hand side is attained when $\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}') = \alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}) - 1$.

It follows that if we set

$$\delta_1 \geq (C - \widehat{\mu})\Big((\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}) - 1\Big)\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F})),$$

condition 2 of Definition 2.6.4 will always hold.

Arguing similarly for condition 1) of Definition 2.6.4, we find that it will hold if

$$\delta_1 \geq (C - \widehat{\mu})(\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F})).$$

It is thus sufficient to pick

$$\delta_1 \geq \max\Big\{(C - \widehat{\mu})\Big((\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}) - 1\Big)\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F})), \quad (C - \widehat{\mu})(\alpha_{\mathcal{O}_{\mathcal{X}}(d_\infty),2}(\mathscr{F}))\Big\}.$$

■

It follows that there is a $\delta(\nu) \in \mathbb{Q}[\nu]$ such that the sheaves in $Y_{r,n}$ are stable with respect to δ.

Corollary 6.2.9. *There is a quasiprojective scheme $\mathbf{Y}_{r,n}$ which is a fine moduli space for the functor $\mathcal{Y}_{r,n}$.*

Proof. Let P be the Hilbert polynomial common to all elements of $Y_{r,n}$, and let $\delta(\nu) = \delta_1\nu + \delta_0$ be a linear polynomial such that all elements of $Y_{r,n}$ are $\widehat{\mu}$-stable with respect to δ_1. By Corollary 6.2.4 and Lemma 6.2.8,such P, δ exist. Then, by Theorem 2.6.6, there exists a quasiprojective scheme

$$\mathbf{M}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}(d_\infty), \mathcal{O}_{\mathbb{P}^2/\Gamma}(1), P, i_*\mathcal{O}_{l_\infty}^{\oplus r}, \delta(\nu))$$

parametrising $\mathcal{O}_{l_\infty}^{\oplus r}$-framed sheaves $(\mathscr{F}, \phi_{\mathscr{F}})$ that are $\delta(\nu)$-stable, and have modified Hilbert polynomial equal to P.

It is clear that there is a subscheme

$$\mathbf{Y}_{r,n} \subset \mathbf{M}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}(d_\infty), \mathcal{O}_{\mathbb{P}^2/\Gamma}(1), P, i_*\mathcal{O}_{l_\infty}^{\oplus r}, \delta(\nu))$$

representing $\mathcal{Y}_{r,n}$.

Now, given an $\mathcal{O}_{l_\infty}^{\oplus r}$-framed sheaf $(\mathscr{F}, \phi_{\mathscr{F}})$, the map $\phi_{\mathscr{F}} \colon i_*\mathcal{O}_{l_\infty}^{\oplus r} \to \mathscr{F}$ may not necessarily be the adjoint of an *isomorphism* $\mathcal{O}_{l_\infty}^{\oplus r} \xrightarrow{\sim} i^*\mathscr{F} = \mathscr{F}|_{l_\infty}$. Furthermore, $\mathscr{F}$ may not be torsion-free. It is, however, easy to see that if $\phi_{\mathscr{F}}$ is the adjoint of an isomorphism, the isomorphism class of the pair

$$(\mathscr{F}, \phi_{\mathscr{F}})$$

lies in $\mathbf{Y}_{r,n}$.

So to show that $\mathbf{Y}_{\mathbf{r},\mathbf{n}}$ is quasiprojective, we only need to show that the locus $\mathbf{Y}_{r,n} \subset \mathbf{M}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}(d_\infty), \mathcal{O}_{\mathbb{P}^2/\Gamma}(1), P, i_*\mathcal{O}_{l_\infty}^{\oplus r}, \delta(\nu))$ is open.

There are two parts to this:

- Show that torsion-freeness is an open condition;
- Show that the framing morphism being an isomorphism is an open condition.

For torsion-freeness, this is simply [11, Proposition 2.31]. (Had we been working with schemes instead of projective stacks, this would have been [43, Proposition 2.1])

As for the second point, every flat family of framed sheaves $\{(\mathscr{F}, \phi_{\mathscr{F}} \colon i^*\mathscr{F} \to \mathcal{O}_{d_\infty}^{\oplus r})\}$ on $\mathcal{X}$ restricts to a family of morphisms $\{\phi_{\mathscr{F}} \colon i^*\mathscr{F} \to \mathcal{O}_{d_\infty}^{\oplus r}\}$ on d_∞. In such a family, the condition $\operatorname{coker} \phi_{\mathscr{F}} = 0$ is open by semicontinuity, and on the locus where $\phi_{\mathscr{F}}$ is surjective, $\ker \phi_{\mathscr{F}} = 0$ is similarly an open condition.

∎

Corollary 6.2.10. *The functor D from Definition 5.2.1 induces a scheme morphism*

$$X_{r,\mathbf{v}} \to \mathbf{Y}_{r,n}$$

for any $\mathbf{v} \in \mathbb{N}^{Q_{\Gamma,0}}$ such that $\mathbf{v}_0 = n$.

Proof. It is enough to show the universal family $\mathscr{U}$ on $X_{r,\mathbf{v}}$ induces a flat family of sheaves in $Y_{r,n}$.

So $\mathcal{U}$ is a sheaf on $\mathbb{P}^2 \times X_{r,\mathbf{v}}$. Since D works relatively, we have a sheaf $D(\mathcal{U})$ on $\mathcal{X} \times X_{r,\mathbf{v}}$, and we only need to show that it is flat over $X_{r,\mathbf{v}}$. To see this, we show that $D(\mathscr{U})|_{V_d \times X_{r,\mathbf{v}}}$ and $D(\mathcal{U})|_{V_p \times X_{r,\mathbf{v}}}$ are flat over $X_{r,\mathbf{v}}$. The first is clear, while the second reduces to showing the following claim:

If M is a Γ-equivariant $\mathbb{C}[x,y] \otimes R$-module (for some ring R) which is flat as a R-module, then M^Γ is a $\mathbb{C}[x,y]^\Gamma \otimes R$-module flat over R.

To show this, let $0 \to K \to E \to F \to 0$ be a short exact sequence of R-modules. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & M^\Gamma \otimes K & \xrightarrow{j} & M^\Gamma \otimes E & \longrightarrow & M^\Gamma \otimes F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M \otimes K & \longrightarrow & M \otimes E & \longrightarrow & M \otimes F & \longrightarrow & 0 \end{array}.$$

Here the downward homomorphisms are injective because M^Γ is a direct summand of M. This shows that j is injective, so M^Γ is flat.

■

6.3 Proof of the main result

We will now prove the main result of this chapter:

Theorem 6.3.1. *There is a morphism of schemes $\mathbf{Y}_{r,n} \to \mathfrak{M}_{\theta_0}(r, n\delta)$, extending the bijection from Theorem 1.3.3.*

We already know from Corollary 3.5.3 that $\mathfrak{M}_{\theta_0}(1, n\delta)$ is isomorphic to $(\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma))_{\mathrm{red}}$. It is not immediately obvious, but we will see that $\mathbf{Y}_{1,n}$ is isomorphic to $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$ in Corollary 6.3.4.

Proof. We already know from Theorem 6.1.4 that there is an isomorphism of reduced schemes

$$\iota\colon \mathfrak{M}_{\theta_0}(r, n\delta) \xrightarrow{\sim} \mathcal{M}(\Pi_0)(1,n)_{\mathrm{red}},$$

so we will find a morphism $\omega\colon (\mathbf{Y}_{r,n}) \to \mathcal{M}(\Pi_0)(1,n)$ such that $\iota \circ \omega$ is, on the level of closed points, the required bijection.

To construct ω, consider the universal property of the scheme $\mathcal{M}(\Pi_0)(1,n)$. It is enough to show that there is a coherent sheaf on $(\mathbf{Y}_{r,n})$ carrying the structure of a flat family of Π_0-modules of dimension $(1,n)$.

To ease notational clutter, set $\mathbf{Y} \coloneqq (\mathbf{Y}_{r,n})$, and let $\mathscr{U}$ be the universal sheaf of $\mathbf{Y}$, so especially $\mathscr{U}$ is a sheaf on $\mathbf{Y}\times\mathcal{X}$, flat over $\mathbf{Y}$. Write $p_1\colon \mathbf{Y}\times\mathcal{X} \to \mathbf{Y}, \quad p_2\colon \mathbf{Y}\times\mathcal{X} \to \mathcal{X}$ for the projections. We make a slight abuse of notation to set $\mathscr{U}\boxtimes\mathscr{I}_{d_\infty} = \mathscr{U}\otimes p_2^*(\mathscr{I}_{d_\infty})$.

Now let $x\colon \operatorname{Spec}\mathbb{C} \to \mathbf{Y}$ be a point, and form the Cartesian square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "x'"] \arrow[d, "p_1'"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{Y}\times\mathcal{X} \arrow[d, "p_1"] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "x"] & \mathbf{Y} \end{tikzcd} .$$

Let $\mathscr{F} \in Y_{r,n}$ be the fibre of $(p_1)_*\mathscr{U}$ over x.

We then have

$$R^2(p_1')_*x'^*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty}) = H^2(\mathcal{X}, \mathscr{F}\otimes\mathscr{I}_{d_\infty}).$$

But the proof of Corollary 6.2.4 tells us that this group vanishes.

Similarly, $(p_1')_*x'^*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty}) = H^0(\mathcal{X}, \mathscr{F}\otimes\mathscr{I}_{d_\infty}) = 0$.

Because $\mathcal{X}$ is a proper stack by Lemma 5.1.2, p_1 is a proper morphism of DM stacks. We can thus apply a *proper base change* result for tame algebraic stacks, stated by Nironi (but see Hall [30, Theorem A] for a generalisation).

By [53, Theorem 1.7], since $R^2(p_1')_*x'^*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty}) = 0$, it follows that

$$x^*R^1(p_1)_*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty}) = R^1(p_1')_*x'^*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty}) = H^1(\mathcal{X}, \mathscr{F}\otimes\mathscr{I}_{d_\infty}).$$

This is independent of the choice of x, and we have shown that passing from the sheaf $\mathscr{U}$ to the sheaf $R^1(p_1)_*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty})$ on $\mathbf{Y}$ is a relative way of replacing each sheaf $\mathscr{F}\in Y$ by its cohomology group $H^1(\mathcal{X}, \mathscr{F}\otimes\mathscr{I}_{d_\infty})$. [53, Theorem 1.7] also shows, since $(p_1')_*x'^*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty}) = 0$, that $R^1(p_1)_*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty})$ is a locally free sheaf of $\mathcal{O}_{\mathbf{Y}}$-modules of rank n.

Recall from Remark 6.1.6 the construction of Π_0-modules corresponding to the framed sheaves in Y from Lemma 5.4.1 and Theorem 6.1.4. Especially, we have that $\mathscr{F}\in Y$ corresponds to a Π_0-module M such that $M_0 = H^1(\mathcal{X}, \mathscr{F}\otimes\mathscr{I}_{d_\infty})$. This construction now extends to $R^1(p_1)_*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty})$ – the endomorphisms $\{a_i\}$ of M_0 from Proposition 6.1.1 extend to sheaf endomorphisms of $R^1(p_1)_*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty})$, and the r pairs $\{(b_i,\overline{b_i})\}$ of framing maps $b_i\colon \mathbb{C}\to M_0$, $\overline{b_i}\colon M_0\to\mathbb{C}$ likewise extend to r pairs of sheaf morphisms between $\mathcal{O}_{\mathbf{Y}}$ and $R^1(p_1)_*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty})$.

This provides $\mathcal{O}_{\mathbf{Y}}\oplus R^1(p_1)_*(\mathscr{U}\boxtimes\mathscr{I}_{d_\infty})$ with the structure of a flat family of η_0-stable Π_0-modules. By the universal property of $\mathcal{M}(\Pi_0)(1,n)$, this induces the required morphism

$$\omega\colon \mathbf{Y}\to\mathcal{M}(\Pi_0)(1,n).$$

■

Conjecture 6.3.2. *The morphism ω is an isomorphism.*

Of course, I expect this to hold, but it is difficult to find an inverse to ω.

We ought to obtain a sheaf $\mathcal{V}$ on $\mathcal{M}(\Pi_0)(1,n) \times \mathcal{X}$ such that the fibre of $\mathcal{V}$ over any point in $\mathcal{M}(\Pi_0)(1,n)$ lies in $Y_{r,n}$. However, I can't see a way of doing this relatively, i.e., such that $\mathcal{V}$ is flat over $\mathcal{M}(\Pi_0)(1,n)$.

Corollary 6.3.3. *Assuming Conjecture 6.3.2, the scheme $\mathbf{Y}_{r,n}$ is irreducible, with symplectic singularities (and so it is normal).*

Proof. This follows from Lemma 2.1.3. ■

We finally re-prove a statement from 3:

Corollary 6.3.4. *Assuming Conjecture 6.3.2, there is an isomorphism*

$$\mathfrak{M}_{\theta_0}(1,n\delta) \xrightarrow{\sim} \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{red}.$$

Proof. In light of 6.3.1, it is enough to prove that there is an isomorphism $\mathbf{Y}_{1,n} \xrightarrow{\sim} \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$.

This is not too hard: let $\mathscr{F} \in Y_{1,n}$. We shall show that $\mathscr{F}|_{V_d} \in \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$.

Then $\pi^T\mathscr{F}$ is an element of some $X_{1,\mathbf{v}}$ with $\mathbf{v}_0 = n$, especially it is an ideal sheaf on $\mathbb{P}^2$. Now $D(\pi^T\mathscr{F}) \cong \mathscr{F}$ by Lemma 5.2.13, and because D is left exact, $\mathscr{F}$ is an ideal sheaf on $\mathcal{X}$. Now we can tensor the ideal sheaf sequence for $\mathscr{F}$ with $\mathscr{I}_{d_\infty}$, obtaining

$$0 \to \mathscr{F} \otimes \mathscr{I}_{d_\infty} \to \mathscr{I}_{d_\infty} \to \mathcal{O}_{\mathcal{X}}/\mathscr{F} \otimes \mathscr{I}_{d_\infty} \to 0$$

which is a short exact sequence because $\mathscr{I}_{d_\infty}$ is locally free. Because $\mathscr{F}$ is trivial along d_∞, it is the ideal sheaf of a zero-dimensional subscheme not intersecting d_∞, and so we have $\mathcal{O}_{\mathcal{X}}/\mathscr{F} \otimes \mathscr{I}_{d_\infty} \cong \mathcal{O}_{\mathcal{X}}/\mathscr{F}$. Taking the long exact sequence of cohomology, we find that

$$h^1(\mathcal{X}, \mathscr{F} \otimes \mathscr{I}_{d_\infty}) = h^0(\mathcal{O}_{\mathcal{X}}/\mathscr{F}) = n,$$

so $\mathscr{F}$ is indeed a point of $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$.

In the other direction, take a colength n zero-dimensional subscheme $Z \subset \mathbb{C}^2/\Gamma$, and let $\mathscr{I}_Z$ be the ideal sheaf of Z as a substack of $\mathcal{X}$. It is straightforward to check that $\mathscr{I}_Z \in Y_{1,n}$, and this gives the inverse map to the previous construction.

Both these constructions work relatively, and so we get an isomorphism $\mathbf{Y}_{1,n\delta} \xrightarrow{\sim} \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$.

■

Chapter 7

Root systems and conjectures

In this final chapter, we collect some sketches of possible directions in which the results of this thesis could be extended. Most of this chapter will therefore be more sketched than the rest of the thesis.

7.1 Root systems and stratifications

There are different ways of stratifying the quiver varieties that we have constructed, but it is not completely clear if they agree.

I have carried out some investigations into these stratifications, in the case where the quiver variety is $\mathfrak{M}_{\theta_0}(1, n\delta) \cong \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{red}$. First we need to know something more about the *root systems* associated to our quivers.

7.1.1 Root systems

Let A be the adjacency matrix for the framed extended ADE Dynkin diagram corresponding to the quiver Q. Then the matrix $C = 2\mathrm{Id} - A$ is a *generalised Cartan matrix*. To such a matrix, we can associate a complex Lie algebra $\mathfrak{g}$, with root system $R \subset \mathbb{Z}^{Q_0}$. This $\mathfrak{g}$ is a *Kac-Moody algebra* [35].

An important characteristic of such algebras is that the set of roots of a Kac-Moody algebra can be split into sets of *real* and *imaginary* roots. (Occasionally, as in [6], the set of imaginary roots is further subdivided into isotropic and anisotropic roots.) We follow [6] in describing the roots.

On the space $\mathbb{Z}^{Q_0}$, we can (following [19, 6]) define the *Euler form* $(-,-)$ by $(u, v) = uCv$. Finally, given any $v \in \mathbb{N}^{Q_0}$, set $|u|^2 = (u, u)$, and let $p(v) = 1 - \langle v, v\rangle = 1 - \frac{1}{2}(v, v)$.

Let e_i be the unit vector at vertex i. As there are no loops in Q, we find that $p(e_i) = 0$. There is a reflection map $s_i \colon \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ defined by $s_i(v) = v - (v, e_i)e_i$.

We say that a vector $\alpha \in \mathbb{Z}^{Q_0}$ is a *real* root α of the Kac-Moody algebra if it can be obtained from some e_i by a series of reflections. By linearity, it follows that if α is real, we have $p(\alpha) = 0$.

For the rest of this section, we set $r = 1$.

The set F of all nonzero $\alpha \in \mathbb{Z}^{Q_0}$ with connected support such that $(\alpha, e_i) \leq 0$ for all i is called the *fundamental region.*[6, Section 2.2].

A vector α is an *imaginary root* if it can be obtained from $\pm$ an element of F by a series of reflections.

It is known that for any imaginary root α, we have $p(\alpha) \geq 1$ [35, Proposition 5.2]. Thus any root α with $p(\alpha) = 0$ is real.

As mentioned, root δ is called the *minimal imaginary root.* This is justified as $(0, \delta)$ lies in F, satisfies $p(0, \delta) = 1$, and if r is a root with $r < \delta$, one can check that $p(r) = 0$, so r is a real root. As mentioned in Section 2.1, δ is also the minimal imaginary root for the affine Lie algebra associated to the extended *unframed* Dynkin diagram, when interpreted as an element of $\mathbb{Z}^{Q_0 \setminus \{\infty\}}$.

We will write R^+ for the set of positive roots (both real and imaginary), i.e., roots with no negative component.

7.1.2 Stratifications

There are various ways of stratifying the scheme $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$. Some of these are equivalent, but there is one stratification introduced by Zheng [68] whose relation with the others is unclear.

Consider then $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma) = \mathfrak{M}_{\theta_0}(1, n\delta)$, and let for $x \in \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$, x_{con} denote the concentrated module attached to it in Section 3.6. We can define a stratification by, for every $u \leq n\delta$ setting $H_u = \mathfrak{M}^s_{\theta_0}(1, u)$ to be the subscheme consisting of all $x \in \mathfrak{M}_{\theta_0}$ such that $\dim x_{\mathrm{con}} = u$. Then

$$\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma) = \bigsqcup_{u \leq n\delta} H_u,$$

and we would like to describe the structure of H_u better.

This stratification is the same as that appearing in [5], [6] (and in the proof of Proposition 4.3.1), but since any θ_0-stable representation M with $M_0 = 0$ is one-dimensional by Remark 2.4.8, it is simplified here.

Let us try to determine the possible dimension vectors of the concentrated representations (when $H_u \neq \emptyset$) — or more generally, the dimension of θ-stable representations for an arbitrary $\theta \in \Theta_{1,n\delta}$. So choose a stability parameter θ. Let Σ_θ be the set of dimension vectors of θ-stable Π-modules, and set $R^+_\theta = \{\alpha \in R^+ | \theta(\alpha) = 0\}$. Then it is known ([5, Theorem 3.9], [6, Section 2.3]) that we have

$$\Sigma_\theta = \left\{ \alpha \in R^+_\theta \mid p(\alpha) > \sum_{i=0}^{k} p(\beta^{(i)}) \text{ for all possible sums } \alpha = \sum_{i=0}^{k} \beta^{(i)}, k > 0, \beta^{(i)} \in R^+_\theta \right\}.$$

7.1.3 Determining Σ_{θ_0}

Again let θ_0 be the stability parameter with $\mathfrak{M}_{\theta_0} \cong \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$. Then it is quite easy to determine whether a given dimension vector v lies the set Σ_{θ_0}.

First, note that if $i \notin \{0, \infty\}$, e_i lies in Σ_{θ_0}. (Clearly it satisfies $\theta_0(e_i) = 0$.) Furthermore, an easy computation shows that $p(\rho_i) = 0$. Given any $v = (1, n, v_1, v_2, \ldots, v_r) \in \mathbb{Z}^{Q_0}$, it follows that $v \in \Sigma_{\theta_0}$ if and only if every $u \in R^+_\theta$ such that $u < v$ has the property $p(u) < p(v)$.

This makes it easy to determine if $v \in \Sigma_{\theta_0}$ given the set $R^+_{\theta_0,v} := \{u \in R^+_{\theta_0} | u < v\}$. In order to compute $R^+_{\theta_0,v}$, we can use the following technique to determine whether any given dimension vector lies in R^+_θ for any θ in low-rank cases (all type E, also types $A_1 ... A_7$ and $D_4 ... D_8$). Recall [35] that a generalised Cartan matrix A is said to be *hyperbolic* if

- A is indecomposable symmetrizable of indefinite type, and
- Every connected proper subdiagram of the Dynkin diagram associated to A is of finite or affine type.

By inspection (or by looking through the catalogue in [13]), we see that both conditions hold for the Cartan matrices of the framed extended versions of the Dynkin diagrams $A_1, \ldots, A_8; D_4, \ldots, D_8; E_6, E_7, E_8$.

For such matrices, there is a nice description of the root system by Kac:

Proposition 7.1.1 ([35, Proposition 5.10]). *Let A be a generalized Cartan matrix, of finite, affine, or hyperbolic type. Then*

1. *$v = \sum_{i \in Q_0} k_i e_i$ is a real root if and only if $|v|^2 > 0$ and $k_i |e_i|^2 / |v|^2 \in \mathbb{Z}$ for all i.*
2. *$v = \sum_{i \in Q_0} k_i e_i$ is an imaginary root if $v \neq 0$ and $|v|^2 \leq 0$.*

This makes it computationally easy to verify whether any given dimension vector lies in R^+, and thus it is simple to determine R_θ^+.

It is then not much more difficult to find Σ_{θ_0} for the framed extended Dynkin diagrams $A_1, \ldots, A_8; D_4, \ldots, D_8; E_6, E_7, E_8$.

7.1.4 Interpreting $p(v)$

We know that $\dim H_u = 2p(u)$ by [6, Lemma 3.23].

But we may give another interpretation of the number $p(v)$:

Corollary 7.1.2. *Let $I \subset \mathbb{C}[x,y]^\Gamma$, and set v to be the dimension of the concentrated Π-module M_{con} associated to I in Section 3.6. Then we have*

$$\#\{q \in V(I) | q \neq 0\} \leq p(v).$$

Finally, $v = (1, n\delta)$ if and only if I does not vanish at p (the singular point of $\mathbb{C}[x,y]^\Gamma$).

Proof. Let $\bigoplus_{i=1}^k N_i$ be the polystable representation S-equivalent to M_{con} with respect to the degenerate condition 0, and let $u_i = \dim N_i$. Then it follows from [5, section A.1] that

$$\mathfrak{M}_0(v) \xrightarrow{\sim} \mathfrak{M}_0(u_1) \times \cdots \times \mathfrak{M}_0(u_k).$$

and has dimension $\sum_{i=1}^k p(u_i)$. In addition, we know that $\mathfrak{M}_{\theta_0}(v) \to \mathfrak{M}_0(v)$ is surjective by variation of GIT and that $\dim \mathfrak{M}_{\theta_0}(v) = p(v)$ [5, Theorem A.1]. So $p(v) \geq \sum_{i=1}^k p(u_i)$. Now there are (by Corollary 3.6.3) three possibilities for each i.

Firstly, $u_i = (0, \delta)$, in which case N_i corresponds to the ideal of a Γ-orbit in $\operatorname{Spec} \mathbb{C}[x,y]$ that does not contain 0. In this case $p(u_i) = 1$, and $\mathfrak{M}_0(u_i)$ gives a point $p \neq 0$ in $\mathbb{C}^2/\Gamma$.

Secondly, we may have $u_i = e_0$. Then N_i corresponds to the ideal (x, y) of $\mathbb{C}[x,y]$, which gives the point 0 of $\mathbb{C}^2/\Gamma$. In this case $p(u_i) = 0$.

Finally, we may have $u_i = e_j, j \neq 0$. Then N_i has no Γ-invariant part, and does not correspond to anything in $\mathbb{C}^2/\Gamma$. In this case, $p(u_i) = 0$ as well.

Summing up proves the first claim.

For the second, suppose not. Then $M_Z$is S_0-equivalent to a Π_0-module $M'_Z \oplus S_i$, where S_i is a vertex simple module supported away from 0. This implies that S_i is a subquotient of M_Z. As M_Z is θ_0-stable, S_i cannot be a submodule of M_Z, so there are submodules $R \subset Q \subset M_Z$ such that $Q/R = e_i$.

Then $\dim_0 R = \dim_0 Q$. Now R, Q must be generated as Π modules by R_0, Q_0 respectively, as if not, there is a submodule of M_Z supported away from 0. But then we must have $R = Q$, so $Q/R = 0$. This is a contradiction, and so the final claim holds.

Conversely, given an ideal $I \subset \mathbb{C}[x,y]^\Gamma$ of colength n that does not vanish at p, its preimage in $n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$ is unique. Then the S_{θ_0}-equivalence class of Π-modules corresponding to I only has a single element (up to isomorphism). This element must then equal the concentrated module associated to the same S_{θ_0}-equivalence class, which must then have dimension $(1, n\delta)$. ■

There are counterexamples to replacing $\leq$ by $=$ in Corollary 7.1.2.

For a simple one, let $\Gamma = \mathbb{Z}/4\mathbb{Z}$, acting diagonally on $\mathbb{C}^2$, and consider the ideal $I = (x^8, y^4, xy) \in \operatorname{Hilb}^2(\mathbb{C}^2/\Gamma)$. The concentrated Π-module associated to this ideal in Section 3.6 has dimension vector $v = (1, 2, 1, 1, 1)$ (it corresponds to the Γ-equivariant ideal $(x^5, y) \subset \mathbb{C}[x,y]$). We compute that $p(v) = 1$, but I is only supported at the singularity.

7.1.5 Another stratification

In [68, Remark 3.4], Zheng suggests a different stratification of $\operatorname{Hilb}^n \mathbb{C}^2/\Gamma$. Namely, let $i \colon Z \hookrightarrow \mathbb{C}^2/\Gamma$ be a closed subscheme of length n. The first syzygy module S_1 of $\mathscr{I}_Z$ can be considered as an $\mathcal{O}_X$-module. Then S_1 is uniquely determined up to stable isomorphism, and we can stratify $\operatorname{Hilb}^n \mathbb{C}^2/\Gamma$ by these isomorphism classes. (Two $\mathcal{O}_X$-modules $\mathscr{M}, \mathscr{N}$ are *stably isomorphic* if there is some k such that $\mathscr{M} \oplus \mathcal{O}_X^k \simeq \mathscr{N}$.)

Conjecture 7.1.3. *These two stratifications of* $\operatorname{Hilb}^n(\mathbb{C}^2/\Gamma)$*– one by syzygies, the other by* $\dim M_{\mathrm{con}}$ *– agree.*

But it is not yet even clear to me whether either is a refinement of the other.

Zheng shows that the smooth locus of $\operatorname{Hilb}^n(\mathbb{C}^2/\Gamma)$ is precisely those ideals that have finite homological dimension ([68, Theorem 4.1]), which is the dense stratum in his stratification. The smooth locus of $\operatorname{Hilb}^n(\mathbb{C}^2/\Gamma)$ is, by Corollary 7.1.2 also the dense stratum when stratifying by $\dim M_{\mathrm{con}}$. The last claim of Corollary 7.1.2 also shows that this stratum consists precisely of those ideals that do no vanish at p

It is, however, not obvious to me how one might compare the other strata.

7.2 Smoothness

What is the smooth locus of a Nakajima quiver variety?

Proposition 7.2.1. *Suppose that the locus $\mathfrak{M}_{\theta_I}(r, \mathbf{v})^s$ of θ_I-stable Π-modules is nonempty. Then this is precisely the smooth locus of $\mathfrak{M}_{\theta_I}(r, \mathbf{v})$.*

Proof. Note that $(1, \mathbf{v})$ lies in the set Σ_{θ_0} precisely when there is a stable Π-module of dimension $(1, \mathbf{v})$ by [6, Theorem 1.3], which is equivalent to saying that $\mathfrak{M}_{\theta_I}(r, \mathbf{v})^s \neq \emptyset$. Then this is just a special case of [6, Theorem 1.15]. ■

In the case where $(r, \mathbf{v}) = (1, n\delta)$ and $\mathfrak{M}_{\theta_I}(1, n\delta) \xrightarrow{\sim} \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)_{\mathrm{red}}$, we already know that the Π-modules of dimension $(1, n\delta)$ that are not θ_0-stable, correspond to ideals of subschemes Z such that the singular point of $\mathbb{C}^2/\Gamma$ lies in Z.

It is perhaps more interesting to look at the case $(r, n\delta)$ for $r \geq 2$. In this case we make the following

Conjecture 7.2.2. *Suppose that $\mathbf{Y_{r,n}}$ is singular at the point $\mathscr{F}$. Then the singular point $p \in \mathcal{X}$ lies in the complement of the substack of $\mathcal{X}$ on which $\mathscr{F}$ is locally free.*

This is true for $r = 1$ by Corollary 7.1.2, but I do not know how to show this in general.

7.3 Other quiver constructions

7.3.1 Framed flags of sheaves

It may be possible to extend our results to cover *framed flags* of sheaves.

Consider the set of triples $(\mathscr{E}, \mathscr{F}, \phi_{\mathscr{F}})$, consisting of a torsion-free sheaf $\mathscr{F}$ on $\mathbb{P}^2$, with $\phi_{\mathscr{F}} \colon \mathcal{O}_{l_\infty}^r \xrightarrow{\sim} \mathscr{F}|_{l_\infty}$ a framing along l_∞, and a subsheaf $\mathscr{E}$ of $\mathscr{F}$ such that the quotient $\mathscr{F}/\mathscr{E}$ is supported away from l_∞. Such a triple is a *framed flag* of sheaves on $\mathbb{P}^2$.

If we set $r = rk(\mathscr{E}) = rk(\mathscr{F}), n = c_2(\mathscr{E}), l = h^0(\mathbb{P}^2, \mathscr{F}/\mathscr{E})$ we can write $F(r, n, l)$ for the moduli space of isomorphism classes of such triples where these invariants are fixed.

In [25], von Flach and Jardim investigated what they term *enhanced ADHM quivers*, i.e., quivers of the form

$$Q^e = \begin{tikzcd} \infty \arrow[r, bend left, "i"] & 0 \arrow[l, bend left, "j"] \arrow[loop above, "a"] \arrow[loop below, "\overline{a}"] & 1 \arrow[l, "k"] \arrow[loop above, "b"] \arrow[loop below, "\overline{b}"] \end{tikzcd} .$$

To this quiver, they attach an *enhanced preprojective algebra*

$$\Pi^e := \mathbb{C}Q^e/(a\overline{a} - \overline{a}a + ij, ak - kb, b\overline{b} - a\overline{a}, \overline{a}k - k\overline{b}, jk),$$

and investigate moduli spaces of Π^e-modules satisfying an appropriate stability condition θ^e. Especially, they find:

Proposition 7.3.1 ([25, Theorem 18])**.** *There is a quasiprojective scheme $\mathfrak{M}^e_\theta$ which is a fine moduli space of θ^e-stable Π^e-modules of dimension $(r, n+l, l)$.*

There is an isomorphism $\mathfrak{M}^e_\theta \xrightarrow{\sim} F(r, n, l)$.

This extended the work of dos Santos ([60]) that shows that nested punctual Hilbert schemes $\mathrm{Hilb}^{r,n}(\mathbb{C}^2)$ can be similarly constructed.

If we replace the vertices $0, 1$ by unframed McKay quivers, and similarly adapt the relations defining Π^e, it is straightforward to see that we obtain nested *equivariant* Hilbert schemes of points on $\mathbb{C}^2$, and flags of *equivariant* framed sheaves on $\mathbb{P}^2$.

We could then define a stability parameter θ^e_0, which would satisfy $\theta^e_0(e_i) \neq 0$ only for i the framing vertex or the two vertices coming from the 0-vertex in the unframed McKay quiver and investigate the moduli space of θ^e_0-semistable Π^e-modules of appropriate dimension. Ideally, this would lead to a construction of flags of framed sheaves on $\mathcal{X}$, with nested Hilbert schemes on $\mathbb{C}^2/\Gamma$ as a special case. But I have not succeeded in proving that this works. One reason is that it is not clear to me whether the map from the space of 'θ^e-stable' modules to that of 'θ^e_0-stable' modules is surjective, another is that I do not know how to handle the dimension vectors of Π^e-modules when adapting the functor $j_!(-)$ from Section 3.3.

7.3.2 More on Nakajima quiver varieties

Finally, perhaps the most natural question to ask after this thesis would be: Is there an interesting description of the quiver varieties $\mathfrak{M}_{\theta_I}(r, \mathbf{v})$ for arbitrary $r, \mathbf{v}$, where θ_I lies in the boundary of $C^+_{r,\mathbf{v}}$?

Such a description would simultaneously extend the results of Chapters 4 and 6. In Chapter 4 we found an interpretation of $\mathfrak{M}_{\theta_I}(1, \mathbf{v})$ as 'orbifold Quot schemes' attached to the orbifold $[\mathbb{C}^2/\Gamma]$. It therefore seems natural to conjecture that $\mathfrak{M}_{\theta_I}(r, \mathbf{v})$

parametrises some sort of ‘higher-rank quotient modules’, again equipped with a framing, on the stack $[\mathbb{P}^2/\Gamma]$. For $I = 0$ (and $n = \mathbf{v}_0$), this should specialise to the scheme $\mathbf{Y}_{r,n}$ constructed in Chapter 6, whereas for $I = \{0, \dots, s\}$, it should specialise to Varagnolo and Vasserot's description from Theorem 5.3.1.

It is, however, not obvious to me what such a definition of a ‘higher-rank quotient module’ could be. In the rank 1-case, given a choice of vertices $I \subseteq \{0, \dots, s\}$ we had a natural object to consider quotients of, namely the $\Pi_{\Gamma I}$-module $R_I = \bigoplus_{i \in I} R_i = e_I \Pi_\Gamma e_0$.

But for $r > 1$, even just in the case $I = \{0\}$, I do not see any way to interpret the framed sheaves lying in $Y_{r,n}$ as 'quotients' of some ambient object.

A further complication is the fact that for $r = 1$, for any stability parameter θ_I, the arrow $\bar{b}\colon 0 \to \infty$ acts by 0 on a θ_I-semistable module. This is, as in Proposition 4.2.2, what in that case allows us to identify a θ_I-semistable Π_I-module with a $\Pi_{\Gamma I}$-module which is a quotient of R_I. But for $r > 1$, we cannot assume that the arrows $\overline{b_1}, \dots, \overline{b_r}$ act by 0. Thus, even if we could identify a point in $\mathfrak{M}_{\theta_I}(r, \mathbf{v})$ with a quotient of some fixed Π_I-module (and I reiterate that I cannot see a way to do this), we would have to work to find a connection to the geometry of the stacks $[\mathbb{P}^2/\Gamma]$ or $\mathcal{X}$ — or maybe some third related stack.

Assuming that these difficulties can be overcome, I would tentatively hope to create a complete catalogue of the varieties $\mathfrak{M}_{\widetilde{\theta}}(r, \mathbf{v})$ for all $\widetilde{\theta} \in \Theta_{r,\mathbf{v}}$, not just in the boundary of C^+, but such a description does not (as far as I know) even exist for $r = 1$ yet.

Appendix A

Bounding the dimension vectors of θ_I-stable modules

A.0.1 The key statement

We use the term 'diagram' to mean 'framed extended Dynkin diagram', and use the notation A_i, D_i, E_i for the framed (with r pairs of arrows) extended versions of these Dynkin diagrams. An A- or Π-module M of the appropriate type naturally determines a representation V of the corresponding quiver Q^* or Q, respectively, that satisfies the relations from equation (3.2); we will call these simply 'quiver representations' below, and identify an A-module with a Π-module on which the arrow $\bar{b}\colon 0 \to \infty$ acts by 0. The notion of θ_I-stability for M defines a notion of θ_I-stability for V.

For $i \in Q_0 = Q_0^* = \{\infty, 0, 1, \dots, r\}$ we write $v_i \coloneqq \dim_i V$, and for $0 \leq i \leq r$ we write $\delta_i \coloneqq \dim(\rho_i)$, so that the regular representation $\delta = \sum_{0 \leq i \leq r} \delta_i \rho_i$ coincides with the minimal imaginary root of the affine Lie algebra associated to the extended Dynkin diagram.

The goal of this appendix is to prove the following result, which we require in the proof of Lemma 3.3.7.

Proposition A.0.1. *Let $I \subseteq \{0, 1, \dots, s\}$ be a non-empty subset. Assume that V is a θ_I-stable quiver representation with $v_\infty = 1$ and $v_i = n\delta_i$ for $i \in I$ and some fixed natural number n.*

Assume that either:

- *$r = 1$ (and thus $\bar{b}$ acts by 0), or*
- *$0 \in I$.*

Then $v_j \leq n\delta_j$ for $j \notin I \cup \{\infty\}$.

The proof splits into two cases according to whether or not $0 \in I$. We first treat the case $0 \in I$ that is required for Chapters 3 and 6. We then go on to study the case $0 \notin I$ in a lengthy case-by-case analysis beginning in Section A.0.2 - for this we require $r = 1$, so that there is only a single edge between the vertices ∞ and 0 in the diagram.

Our main tool for proving Proposition A.0.1 is the following estimate, the proof of which is inspired by a result of Crawley-Boevey [19, Lemma 7.2]. This inequality is the only consequence of θ_I-stability that we use in the subsequent numerical argument.

Lemma A.0.2. *Let V be a θ_I-stable quiver representation. If $i \notin I \cup \{\infty\}$, then $2v_i \leq \sum_{\{a \in Q_1 \mid \mathrm{h}(a)=i\}} v_{\mathrm{t}(a)}$.*

Proof. Define

$$V_\oplus \coloneqq \bigoplus_{\substack{a \in Q_1, \\ \mathrm{h}(a)=i}} V_{\mathrm{t}(a)}.$$

The maps in V determined by arrows with tail and head at vertex i combine to define maps $f \colon V_i \to V_\oplus$ and $g \colon V_\oplus \to V_i$ satisfying $g \circ f = 0$.

If $\ker(f) \neq 0$, then V admits a nonzero subrepresentation W such that $W_i = \ker(f)$ and $W_j = 0$ for $j \neq i$. But then W is a proper, nonzero subrepresentation of V satisfying $\theta_I(W) = 0$, thereby contradicting θ_I-stability of V. Thus f is injective. Similarly, if $\mathrm{im}(g) \subsetneq V_i$, then V admits a subrepresentation U such that $U_i = \mathrm{im}(g)$ and $U_j = V_j$ for $j \neq i$. Then U is a proper, nonzero subrepresentation of V satisfying $\theta_I(U) = \theta_I(V) = 0$ which again contradicts θ_I-stability of V, so g is surjective. It follows that the complex

$$0 \longrightarrow V_i \xrightarrow{f} V_\oplus \xrightarrow{g} V_i \longrightarrow 0 \tag{A.1}$$

has nonzero homology only at $V_\oplus$, so $\dim V_\oplus \geq 2 \dim V_i$. ■

Proof of Proposition A.0.1 in the case $0 \in I$. Let v' be the restriction of the stable dimension vector v to the underlying extended Dynkin diagram, and let C be the Cartan matrix of the same extended diagram. Define $u = v' - n\delta$. We will be done once we show that $u_i \leq 0$ for all i.

We can rephrase Lemma A.0.2 as saying that $(Cv')_i < 0$ for $i \notin I$. Since $C\delta = 0$, this also implies that $(Cu)_i \leq 0$ for all i. As in Remark 2.4.8, removing $I \cup \{\infty\}$ from the diagram leaves a collection of finite-type Dynkin diagrams. Let Q' be any such subdiagram, let $C_{Q'}$ be its Cartan matrix and let $u_{Q'}$ be the restriction of u to Q'. As

$u_i = 0$ for any $i \in I$, it follows that $(C_{Q'}u_{Q'})_i \leq 0$ for all i. Now $C_{Q'}^{-1}$ has only positive coefficients (see, e.g. [57, p. 1.157]), and so

$$u_{Q'} = C_{Q'}^{-1} C_{Q'} u_{Q'}$$

also satisfies $u_{Q',i} \leq 0$ for all $i \in Q'$. Then $u_i \leq 0$ for all i, giving $v_i' \leq n\delta_i$ for all i as required. ■

Remark A.0.3. The original proof of Proposition A.0.1 in the case $0 \in I$ used a lengthy case-by-case argument, similar to that which follows for the case $0 \notin I$. We are grateful to a referee for [16] for suggesting this more elegant approach (see also [50, p.6]). Unfortunately, we were unable to extend this technique to the case $0 \notin I$. Indeed, if we define $V_{\oplus}$ as the sum of all vector spaces indexed by adjacent vertices in the McKay quiver - i.e. excluding the framing vertex - then the complex (A.1) can have nonzero homology at the second V_i.

A.0.2 Strategy and preparatory results for the case $0 \notin I$

We now lay the foundation for the proof of Proposition A.0.1 in the case $0 \notin I$. *From now on, we will assume that* $r = 1$.

We argue by contradiction, performing a case-by-case analysis on Dynkin diagrams. The basic idea is as follows. First, if the inequality $v_i > n\delta_i$ holds for a vertex i but not its neighbour j, we deduce a basic inequality (A.2) and show that this inequality can be 'pushed along' the branches of the diagram (see Lemma A.0.4). If the diagram branches at a trivalent vertex, then we push the inequality further along at least one branch (see Lemma A.0.5). This leads either to a contradiction or to strong constraints on $\dim V$.

Lemma A.0.4. (i) *Let $i, i-1$ be adjacent vertices of the diagram. If $v_i > n\delta_i$ and $v_{i-1} \leq n\delta_{i-1}$, then*

$$\delta_{i-1} v_i > \delta_i v_{i-1}. \tag{A.2}$$

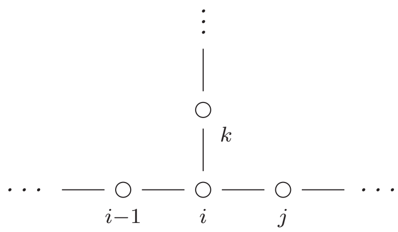
(ii) *Suppose the vertex $i \notin I$ is bivalent, and neither of its neightbours is ∞:*

$$\cdots \;—\underset{i-1}{\circ}—\underset{i}{\circ}—\underset{i+1}{\circ}—\; \cdots \tag{A.3}$$

Then $\delta_{i-1}v_i > \delta_i v_{i-1}$ implies $\delta_i v_{i+1} > \delta_{i+1} v_i$. If in addition $v_i > n\delta_i$, then $v_{i+1} > n\delta_{i+1}$.

Proof. Part (i) is immediate. Since i and ∞ are not neighbours, $2\delta_i = \delta_{i-1} + \delta_{i+1}$ holds. Part (ii) follows by combining this equality with the assumed inequality $\delta_{i-1}v_i > \delta_i v_{i-1}$ and $2v_i \leq v_{i-1} + v_{i+1}$ coming from Lemma A.0.2. The last statement is again immediate. ∎

Lemma A.0.5. *Suppose that the diagram has a trivalent vertex $i \notin I$, not adjacent to the vertex ∞:*



(A.4)

and assume that $\delta_{i-1}v_i > \delta_i v_{i-1}$.

(i) *At least one of the inequalities $\delta_j v_i < \delta_i v_j$ and $\delta_k v_i < \delta_i v_k$ must hold.*

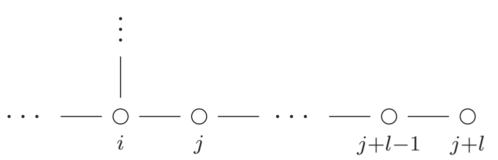
Suppose now that $v_i > n\delta_i$, that $\delta_j v_i < \delta_i v_j$ holds, and furthermore that the branch starting at j does not branch further. Then

(ii) *the branch starting at j does not contain any vertices in I, and*

(iii) *the same branch must terminate at the framing vertex ∞, and in this case $\delta_i v_j = \delta_j v_i + 1$.*

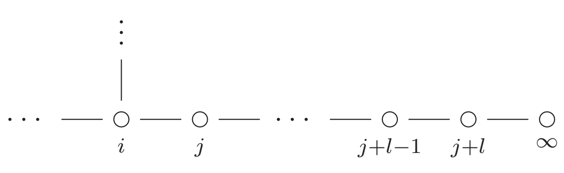
Remark A.0.6. The only framed extended Dynkin diagrams where a trivalent vertex is adjacent to the framing vertex are of type A_i for $i > 1$. We handle the case of such a vertex not being in I in Lemma A.8.

Proof. For (i), combining $2\delta_i = \delta_{i-1}+\delta_j+\delta_k$ with $2v_i \leq v_{i-1}+v_k+v_j$ and $\delta_{i-1}v_i > \delta_i v_{i-1}$ leads to $\delta_j v_i + \delta_k v_i < \delta_i v_j + \delta_i v_k$ which implies the result. For (ii) and (iii), we denote the vertices as



(A.5)

if the branch does not contain the framing vertex, or



(A.6)

if it does. To simplify notation, we take $j - 1 = i$ in the following argument. One of the following must occur.

- *The branch contains another vertex in I.* Suppose that j' is the node with smallest index on the branch such that $j' \neq i$ and $j' \in I$. Lemma A.0.4(ii) gives $\delta_{j'-1}v_{j'} > \delta_{j'}v_{j'-1}$ and $v_{j'} > n\delta_{j'}$, contradicting $j' \in I$.

- *The branch contains no vertices in $I \cup \infty$.* Repeated applications of Lemma A.0.4(ii) show that $\delta_{j+l-1}v_{j+l} > \delta_{j+l}v_{j+l-1}$. However, since $2\delta_{j+l} = \delta_{j+l-1}$, this implies $2v_{j+l} > v_{j+l-1}$, contradicting Lemma A.0.2.

- *The branch contains no vertices of I, and terminates at ∞.* We prove a slightly stronger statement, namely that for any vertex $m \neq \infty$ on the branch, we have $\delta_{m-1}v_m = \delta_m v_{m-1} + 1$. We proceed by induction on the number of edges that lie between ∞ and m. For the base case $m = j + l$, note that $\delta_{j+l-1}v_{j+l} > \delta_{j+l}v_{j+l-1}$ implies $2v_{j+l} > v_{j+l-1}$. However, since $2v_{j+l} \leq v_{j+l-1} + 1$ by Lemma A.0.2, we must have $2v_{j+l} = v_{j+l-1} + 1$. If there is more than one edge between ∞ and m, then the induction hypothesis gives $\delta_m v_{m+1} = \delta_{m+1}v_m + 1$. Combining this with $2v_m \leq v_{m-1} + v_{m+1}$ from Lemma A.0.2 and $2\delta_m = \delta_{m+1} + \delta_{m-1}$ shows that $\delta_{m-1}v_m \leq \delta_m v_{m-1} + 1$. Lemma A.0.4(ii) gives $\delta_{m-1}v_m > \delta_m v_{m-1}$ and the result follows.

This concludes the proof. ■

A.0.3 Proof for the case $0 \notin I$, types A_1 and D_4

Lemma A.0.7. *Proposition A.0.1 holds for A_1 and D_4.*

Proof. For type A_1, we have the diagram

$$\underset{\infty}{\circ} — \underset{0}{\circ} = \underset{1}{\circ} \tag{A.7}$$

where the symbol $=$ indicates that the diagram has two edges. The only remaining case is $J = \{1\}$. A straightforward adaptation of Lemma A.0.2 shows that if $I = \{1\}$, then $2v_0 \leq 2v_1 + 1 = 2n + 1$, giving $v_0 \leq n$.

For type D_4, the diagram is:

$$\begin{tikzcd} & & \underset{1}{\circ} \arrow[d, dash] & \\ \underset{\infty}{\circ} \arrow[r, dash] & \underset{0}{\circ} \arrow[r, dash] & \underset{2}{\circ} \arrow[r, dash] \arrow[d, dash] & \underset{3}{\circ} \\ & & \underset{4}{\circ} & \end{tikzcd} . \tag{A.8}$$

Suppose without loss of generality that $1 \in I$. Then any other vertex i with $v_i > n\delta_i$ will, by Lemma A.0.5 or Lemma A.0.2 give that $v_2 > 2n$. The same lemmas show that

$$4v_2 \leq 2v_0 + 2v_1 + 2v_3 + 2v_4 \leq 2n + 3v_2 + 1 \tag{A.9}$$

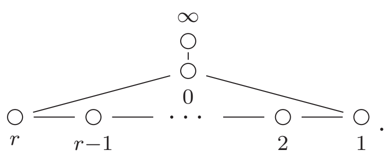
and thus $v_2 \leq 2n+1$. So $v_2 = 2n+1$, but then Lemma A.0.2 gives that $v_1 = v_3 = v_4 = n$. Plugging this into (A.9) gives $6n + 3 = 3v_2 \leq 6n + 1$, a contradiction. ■

A.0.4 Proof when $0 \notin I$, the general case

For the rest, we need to handle each diagram type separately.

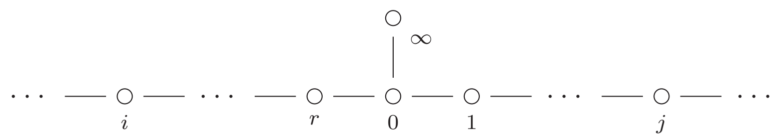
Lemma A.0.8. *Proposition A.0.1 holds for any diagram of type A_i with $i > 1$.*

Proof. We number the vertices as follows:



(A.10)

Assume that some vertex $k' \neq \infty$ has $v_{k'} > n\delta_{k'} = n$. Since $0 \notin I$, we may consider a subdiagram



(A.11)

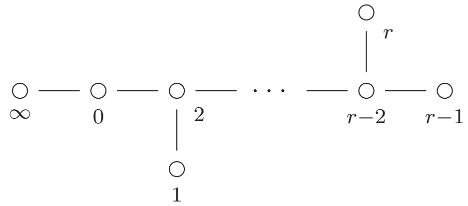
where i, j (possibly equal) are the only vertices in I, with k' some vertex in this subdiagram. We can without loss of generality assume $0 \leq k' < j$. Then there are adjacent vertices $k, k+1$ such that $k' \leq k,\ k+1 \leq j$ with $v_k > n \geq v_{k+1}$. Repeatedly applying Lemma A.0.4 gives

$$v_0 > v_1 > n. \tag{A.12}$$

There must also be adjacent vertices $l, l+1$ between i and 0 such that $v_{l+1} > v_l$. In a similar way, this leads to $v_0 > v_r$. Combining with (A.12), we deduce $2v_0 > v_1 + v_r + 1$, contradicting Lemma A.0.2. ■

Lemma A.0.9. *Proposition A.0.1 holds for diagrams of type D_i, $i > 4$.*

Proof. We number the vertices as follows:



(A.13)

Our proof of Proposition A.0.1 for the case $0 \in I$ leaves only three possible configurations for the nodes in I when $0 \notin I$, up to symmetry of the diagram. We prove Proposition A.0.1 by contradiction in each case.

1. *There is an i such that $2 \leq i \leq r-1$, $v_i > n\delta_i = 2n$, and all $j \in I$ have $i < j$.* Let k be maximal among the vertices such that $v_k > n\delta_k$. If $k \leq r-2$, we have $\delta_{k+1}v_k > \delta_k v_{k+1}$. Otherwise, $k = r-1$. We must have $I = \{r\}$, and by Lemma A.0.5 we get $\delta_{r-3}v_{r-2} > \delta_{r-2}v_{r-3}$. By symmetry, the case $k = r$ also leads to $\delta_{r-3}v_{r-2} > \delta_{r-2}v_{r-3}$.

   Both cases lead (by Lemma A.0.4) to $v_2 > v_3$, that is, $v_2 - 1 \geq v_3$. Then Lemma A.0.5 gives $2v_0 = v_2 + 1$ and $2v_1 \leq v_2$. Combining these with Lemma A.0.2 leads to
   $$4v_2 \leq 2v_3 + 2v_1 + 2v_0 \leq 4v_2 - 1,$$
   which is absurd.

2. *$v_1 > n\delta_1 = n$, and all $j \in I$ have $j > 2$.* This implies $v_2 > 2n$. Let j be the least vertex such that $v_j \leq n\delta_j$. Applying Lemmas A.0.4 and A.0.5 to the vertices $j-1, j$ (or if $j = r$, the vertices $r, r-2$) we again find $v_2 > v_3$. Then the conclusion of case (1) applies.

3. *$v_0 > n\delta_0$, and all $j \in I$ have $j \geq 2$.* If $2 \in I$, we have $v_2 = 2n$, and then $v_1 > n$ leads to $2v_1 > 2n+1$, contradicting Lemma A.0.2. If $2 \notin I$ we can again take j as the least vertex with $v_j \leq n\delta_j$ and argue as in case (2).

Therefore, if D_i has $i > 4$, then Proposition A.0.1 holds. ■

To conclude, we consider the diagrams E_6, E_7 and E_8, which must all be dealt with separately.

Lemma A.0.10. *Proposition A.0.1 holds for type E_8.*

Proof. We number the vertices as follows:

$$\begin{array}{ccccccccc} & & \circ & & & & & & \\ & & |\,{\scriptstyle 2} & & & & & & \\ \underset{1}{\circ} - & \underset{3}{\circ} - & \underset{4}{\circ} - & \underset{5}{\circ} - & \underset{6}{\circ} - & \underset{7}{\circ} - & \underset{8}{\circ} - & \underset{0}{\circ} - & \underset{\infty}{\circ} \end{array} \tag{A.14}$$

This time, we split the possible configurations for I across the diagram in the case $0 \notin I$ into four possibilities. Let k be the minimal vertex with $v_k > n\delta_k$. The possible configurations are:

1. *$k > 4$ and all $j \in I$ have $j < k$, or $k = 0$.* By Lemma A.0.4 and Lemma A.0.5, we find that $v_0 > n\delta_0 = n$. The same lemmas show that $\delta_{k+1}v_k + 1 = \delta_k v_{k+1}$. Let us temporarily use the designation 9 for the vertex marked 0. By Lemma A.0.2, we get
$$2\delta_k v_k \leq \delta_k v_{k+1} + \delta_k v_{k-1} \leq \delta_{k+1}v_k + 1 + \delta_k n\delta_{k-1}$$
implying $\delta_{k-1}(v_k - n\delta_k) \leq 1$. But this contradicts $v_k > n\delta_k$.

2. *$k = 4$ and all $j \in I$ have $j < k$*: We have
$$2v_4 \leq v_2 + v_3 + v_5 \leq n\delta_2 + n\delta_3 + v_5 = 7n + v_5.$$
Since we also have (Lemma A.0.5) $5v_4 + 1 = 6v_5$, this implies that $7v_5 - 2 \leq 35n$. But since $v_5 > 5n$, this is impossible.

3. *$k = 2$ and at least one of the vertices 1 and 3 are in I*: By Lemma A.0.2, we must have $v_4 \geq 6n + 1$. By Lemma A.0.4 applied to the vertex chain $1, 3, 4$, we find $6v_3 < 4v_4$. Then Lemma A.0.5 shows that $6v_5 = 5v_4 + 1$. Now, if $v_3 \leq n\delta_3$, the same lemma and Lemma A.0.2 imply
$$12v_4 \leq 6v_2 + 6v_3 + 6v_5 \leq 8v_4 + 24n + 1$$
leading to $24n + 4 \leq 4v_4 \leq 24n + 1$, a contradiction. So suppose that $1 \in I$, and $v_3 > 4n$. By Lemma A.0.4, we get $6v_3 < 4v_4$. As above, we find
$$12v_4 \leq 6v_2 + 6v_3 + 6v_5 \leq 8v_4 + 6v_3 + 1$$
leading to $4v_4 \leq 6v_3 + 1$. This implies that $4v_4 = 6v_3 + 1$, which has no integer solutions. Hence we have a contradiction.

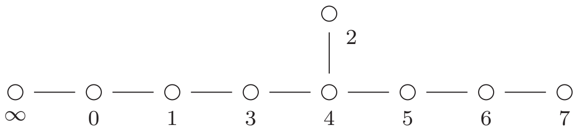
4. *$k = 1$ or $k = 3$, and so I only consists of 2*: Suppose that $v_2 = n\delta_2 = 3n$. Then, by Lemma A.0.5 and Lemma A.0.4, we get $v_4 > 4n$, say $v_4 = 4n + t,\ t > 0$. But then Lemma A.0.5 and Lemma A.0.2 give
$$12v_4 \leq 6v_2 + 6v_3 + 6v_5 \leq 18n + 4v_4 + 5v_4 + 1$$
leading to $18n + 3t = 3v_4 \leq 18n + 1$, a contradiction.

Therefore Proposition A.0.1 holds for E_8. ■

In [16], we did not include the proof of Proposition A.0.1 for types E_7 and E_6. I include those cases here.

Lemma A.0.11. *Proposition A.0.1 holds for type E_7.*

Proof. The vertices are numbered as follows:



(A.15)

For this diagram, we split what remains to consider after Lemma A.0.5 in five cases.

1. *$v_i > n\delta_i$ for $i = 0, 1,$ or 3, and $j \in I$ for some j along the path from i to and including* 4: The argument from the first case of Lemma A.0.10 applies here as well.

2. $2 \in I$: We have $v_2 = \delta_2 n = 2n$. Let k be the maximal vertex with $v_k > n\delta_k$. If $k = 0, 1,$ or 3, case 1 applies. If $k = 6, 7$ or 8, combining Lemma A.0.5 and Proposition A.0.2 lead to $v_4 > 4n$, say $v_4 = 4n + t$ with $t > 0$, and $4v_5 \leq 3v_4$. Lemma A.0.5 shows that $4v_3 = 3v_4 + 1$.

   But $8v_4 \leq 4v_2 + 4v_5 + 4v_3 \leq 8n + 6v_4 + 1$, giving $8n + 2t = 2v_4 \leq 8n + 1$, a contradiction.

3. *$5 \in I$, and none of $0, \ldots, 4$ are in I*: We have $v_5 = \delta_5 n = 3n$. As above, let k be the maximal vertex with $v_k > n\delta_k$. If $k = 0, 1,$ or 3, case 1 applies. If $k = 2$ or 4, we get $v_4 > 4n$ – say $v_4 = 4n + t,\ t > 0$. Then Lemma A.0.5 gives $4v_3 = 3v_4 + 1$, and by Proposition A.0.2

   $$8v_4 \leq 4v_3 + 4v_2 + 4v_5 \leq 5v_4 + 12n + 1,$$

   leading to $12n + 3t = 3v_4 \leq 12n + 1$, a contradiction.

4. *$6 \in I$, and none of $0, \ldots, 5$ are in I*: We have $v_6 = \delta_6 n = 2n$. Choose k as above. If $k < 5$, the argument of case 3 works. So suppose that $k = 5$, so $v_5 > \delta_5 n = 3n$. Then Lemma A.0.4 gives $3v_4 > 4v_5$. As above, Lemma A.0.5 combines with Proposition A.0.2 to show

   $$8v_4 \leq 4v_3 + 4v_2 + 4v_5 \leq 5v_4 + 4v_5 + 1$$

   but since $3v_4 > 4v_5$, this gives $3v_4 = 4v_5 + 1$. But then there are no integer solutions of $4v_3 = 3v_4 + 1$, a contradiction.

5. *The only vertex in I is 7.*: We have $v_7 = n$. Choose k as above. If $k < 6$, the argument of case 4 works. So suppose that $k = 6$, so $v_6 > 2n$. By Lemma A.0.4, we can then say $2v_5 > 3v_6$ and $3v_4 > 4v_5$.

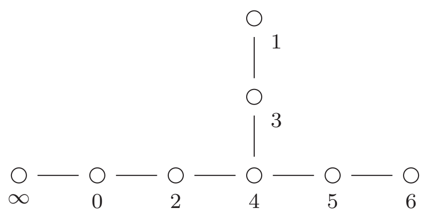
   By Lemma A.0.5, we obtain $4v_3 = 3v_4 + 1$ and $4v_2 \leq 2v_4$. Then Proposition A.0.2 gives
   $$8v_4 \leq 4v_3 + 4v_5 + 4v_2 \leq 5v_4 + 1 + 4v_5.$$
   Since we know that $3v_4 > 4v_5$, this implies $3v_4 = 4v_5 + 1$.

   But then we also have
   $$6v_5 \leq 3v_6 + 3v_4 = 3v_6 + 4v_5 + 1,$$
   so $2v_5 = 3v_6 + 1$. Similarly we get $v_6 = 2n + 1$. Backtracking, we see that this implies $v_4 = 4n + 3$. But then the equation $4v_3 = 3v_4 + 1$ has no integer solutions, a contradiction.

Thus we have covered every case, and Proposition A.0.1 holds for type E_7 as well. ■

Lemma A.0.12. *Proposition A.0.1 holds for type E_6.*

Proof. The vertices are numbered as follows:



(A.16)

By symmetry, we reduce the cases remaining from Lemma A.0.5 to four.

1. $3 \in I$ *and some i with $i \neq 1, 3$ has $v_i > n\delta_i$*: Let k be the maximal vertex with $v_k > n\delta_k$. If $v_i > n\delta_i$ for i equal to either 5 or 6, Lemma A.0.5 and Proposition A.0.2 lead to $2v_4 > 3v_5$ and $v_4 > n\delta_4 = 3n$, i.e, $v_4 \geq 3n + 1$. Thus $2v_4 > 3v_3$ and we can apply Lemma A.0.5 to conclude that $3v_2 = 2v_4 + 1$. Combining with Proposition A.0.2 gives
   $$6v_4 \leq 3v_3 + 3v_2 + 3v_5 < 6n + 4v_4 + 1,$$
   contradicting $v_4 \geq 3n + 1$.

   If $k = 2$, we have $3v_2 > 2v_4$, implying by Lemma A.0.5 that $v_2 = 2v_1 + 1 \geq 2n + 3$ and $2v_4 = 3v_2 + 1 \geq 6n + 8$, contradicting maximality of k. If $k = 0$, we have $2v_0 > 2v_2 + 1$, contradicting Proposition A.0.2.

2. *$3 \notin I, 1 \in I$ and some $i \neq 1$ has $v_i > n\delta_i$*: We have $v_1 = n$. Suppose that $v_3 > 2n$, say $v_3 = 2n + t,\ t > 0$. Then we can apply Lemma A.0.4 to get

$$2v_4 > 3v_3 = 6n + 3t. \tag{$*$}$$

Now Lemma A.0.5 gives us

$$3v_2 = 2v_4 + 1 \tag{$**$}$$

and $2v_4 \geq 3v_5$. Combining with Proposition A.0.2 leads to

$$6v_4 \leq 3v_3 + 3v_2 + 3v_5 \leq 6n + 3t + 4v_4 + 1,$$

which, when combined with ($*$) implies $2v_4 = 3v_3 + 1$, giving $6v_3 = 4v_4 - 2 \leq 3v_4 + 3n$. But then $v_4 \leq 3n + 2$, implying by ($*$) that $t = 1$ and $v_4 = 3n + 2$. But then ($**$) has no integer solutions.

Suppose then that $v_3 \leq 2n$. Then the proof of case 1 applies.

3. *$4 \in I,\ v_i > n\delta_i$ for $i = 0$ or $i = 2$*: The proof of Case (1) in Lemma A.0.10 applies.

4. *$2 \in I,\ v_0 > n\delta_0 = n$*: This implies that $2v_0 > 2n+1$, contradicting Proposition A.0.2.

Hence we have covered all cases, and Proposition A.0.1 holds for type E_6 as well. ■

Proof of Proposition A.0.1 in the case $0 \notin I$. Our case-by-case analysis is given in Lemma A.0.7 to Lemma A.0.12. ■

Appendix B

Surjectivity of morphisms induced by variation of GIT

We prove Lemma 2.4.5.

For this Appendix, let us write

$$\mathfrak{M}_\theta = \mathfrak{M}_\theta(r, n\delta), \quad \mathfrak{M}_{\theta_0} = \mathfrak{M}_{\theta_0}(r, n\delta), \text{ and } \mathfrak{M}_0 = \mathfrak{M}_0(r, n\delta).$$

First, we need the following description of the variety $\mathfrak{M}_0(r, \mathbf{v})$ due to Nakajima: Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of an integer L, and set

$$\mathrm{Sym}^L_\lambda(\mathbb{C}^2) = \left\{ \sum_{i=1}^r \lambda_i [x_i] \mid x_i \neq 0 \text{ and } x_i \neq x_j \text{ for } i \neq j \right\}.$$

Lemma B.0.1 ([51, Equation 4.1])**.**

$$\mathfrak{M}_0(r, \mathbf{v}) = \bigsqcup_{\substack{\lambda, \mathbf{v}' \\ m\delta + \mathbf{v}' \leq \mathbf{v}}} \mathfrak{M}^s_0(r, \mathbf{v}') \times (\mathrm{Sym}^{(m|\Gamma|)}_\lambda \mathbb{C}^2)^\Gamma. \tag{B.1}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i)$ is a partition of m, and the space $\mathfrak{M}^s_0(r, \mathbf{v}')$ parametrises Γ-equivariant locally free *sheaves, $(\mathscr{E}, \phi_{\mathscr{E}})$ framed along l_∞, of rank r, satisfying*

$$H^1(\mathbb{P}^2, \mathscr{E} \otimes \mathscr{I}_{l_\infty}) \cong \bigoplus \rho_i^{\mathbf{v}'_i}$$

as Γ-representations. Furthermore, the map $X_{r,\mathbf{v}} \cong \mathfrak{M}_\theta(r, \mathbf{v}) \to \mathfrak{M}_0(r, \mathbf{v})$ can be identified with

$$(\mathscr{E}, \phi_{\mathscr{E}}) \mapsto ((\mathscr{E}^{\vee\vee}, \phi_{\mathscr{E}}), \mathrm{Supp}(\mathscr{E}^{\vee\vee}/\mathscr{E})).$$

Here we use Lemma 5.2.4 to say that $\mathscr{E}$ is locally free in a neighbourhood U of l_∞. Then the morphism $\mathscr{E} \to \mathscr{E}^{\vee\vee}$ is an isomorphism over U, so $\phi_{\mathscr{E}}$ is also a framing morphism for $\mathscr{E}^{\vee\vee}$, and the support of $\mathscr{E}^{\vee\vee}/\mathscr{E}$ must be contained in the affine scheme $\mathbb{P}^2 \setminus l_\infty = \mathbb{C}^2$.

Let $t\colon \mathfrak{M}_\theta \to \mathfrak{M}_{\theta_0}$ be the projective morphism induced by variation of GIT. We will prove that t is a resolution of singularities.

Proof of Lemma 2.4.5. Our strategy is to show that the subscheme $\mathfrak{M}^s_{\theta_0}$ consisting of θ_0-stable modules is nonempty, and therefore dense in $\mathfrak{M}_\theta$. By general results on Nakajima quiver varieties (see, e.g., [47, p. 2.6]), $\mathfrak{M}_\theta$ is smooth, and if $\mathfrak{M}^s_{\theta_0} \neq \emptyset$,

$$\dim \mathfrak{M}^s_{\theta_0} = \dim \mathfrak{M}_\theta = 2rn.$$

Since a θ_0-stable module is θ-stable, $\mathfrak{M}^s_{\theta_0} \neq \emptyset$ will imply that t is birational, and since t is projective, also that it is surjective. Since $\mathfrak{M}_\theta$ is smooth, t will then be a resolution of singularities.

So it is enough to find an element of $\mathfrak{M}^s_{\theta_0}$.

It follows from Lemma B.0.1 and Remark 2.4.8, that there is a decomposition

$$\mathfrak{M}_0 = \bigsqcup_{\substack{m,m' \\ m\delta+m'\delta=n\delta}} \mathfrak{M}^s_0(r, m'\delta) \times \operatorname{Sym}^m(\mathbb{C}^2/\Gamma). \tag{B.2}$$

Let us then choose

$$S \subset (\operatorname{Sym}^{(n|\Gamma|)}_{(1,1,\ldots,1)} \mathbb{C}^2)^\Gamma \subset \operatorname{Sym}^n(\mathbb{C}^2/\Gamma),$$

consisting of n distinct Γ-orbits of $|\Gamma|$ points in $\mathbb{C}^2$ (thus no orbit including the origin), and let $\mathscr{I}_S$ be the ideal sheaf of these $n|\Gamma|$ points, considered as a subscheme of $\mathbb{P}^2$.

Then the sheaf $\mathscr{E} = \mathscr{I}_S \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus r-1}$ is an element of $X_{r,n\delta} \cong \mathfrak{M}_\theta$, mapping to $x = ((\mathcal{O}_{\mathbb{P}^2}^{\oplus r}, \mathrm{id}_{\mathcal{O}_{l_\infty}}), S)$ in $\mathfrak{M}_0$. Especially $\mathscr{E}$ corresponds, through Theorem 5.3.1, to a θ-stable Π-module $M_{\mathscr{E}}$.

Now note that an element $(p_1 + p_2 + \cdots + p_n) \in \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$ corresponds to a direct sum of n simple Π_Γ-modules. Each of these simple modules is either a vertex simple supported at 0 (for a p_i equal to the singular point $p \in \mathbb{C}^2/\Gamma$), or has dimension δ (for a p_i different from p).

The Π-module corresponding to $\mathcal{O}_{\mathbb{P}^2}^{r-1}$, as an element of $X_{r,0}$, is the vertex simple module supported at ∞.

Thus the 0-polystable (i.e., semisimple) Π-module corresponding to $x \in \mathfrak{M}_0$ can thus be written as a direct sum

$$M_x \coloneqq N_\infty \oplus N_1 \oplus \cdots \oplus N_n,$$

where N_∞ is a vertex simple module, supported only at ∞, and each N_i is simple, of dimension vector $(0, \delta)$. Especially, there is no vertex simple summand supported at another vertex than ∞.

This means that the module $t(M_{\mathscr{E}})$ is θ_0-stable. For if it was not, its θ_0-polystable equivalent would, by Corollary 2.4.7, be the direct sum of a concentrated module and vertex simple modules supported away from the 0-vertex. Then the image M_x of $M_{\mathscr{E}}$ in $\mathfrak{M}_0$ would also have vertex simple summands away from this vertex, which contradicts its construction.

Thus $\mathfrak{M}^s_{\theta_0}$ is nonempty. This concludes the proof of (2) of Lemma 2.4.5

As for (3): we will show: Let $r > 1$, and let $I \subset Q_{\Gamma,0}$. Assume that $\mathfrak{M}_{\theta_I}(1, n\delta)$ is singular. Then $\mathfrak{M}_{\theta_I}(r, n\delta)$ is also singular, i.e., θ_I is not a generic stability parameter in $\Theta_{r,\mathbf{v}}$.

We know that the morphism $\pi_I \colon n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2) \xrightarrow{\sim} \mathfrak{M}_\theta(1, n\delta) \to \mathfrak{M}_{\theta_I}(1, n\delta)$ is surjective (for instance by Lemma 2.4.5).

So choose some $M_I \in \mathfrak{M}_\theta(1, n\delta)$, which is not θ_I-stable, meaning that there is a Π-submodule $N \subset M_I$ such that N is supported on I.

Now set $\mathscr{I} \in n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$ to be the ideal sheaf corresponding to M_I. Then $\mathscr{I} \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus r-1}$ is a point of $X_{r,n\delta}$, corresponding to a θ-stable module M. By the definition of M, precisely one of the arrows $b_i \colon \infty \to 0$ in Q_1 will act as a nonzero map in M (and all the other arrows between ∞ and 0 are the zero map). It follows that N will also be a submodule of M. Thus M is not θ_I-stable.

Finally, we need to show that the general element of $\mathfrak{M}_{\theta_I}(r, n\delta)$ corresponds to a θ_I-stable Π-module.

So choose $\mathscr{I}' \in n\Gamma\text{-}\operatorname{Hilb}(\mathbb{C}^2)$ such that the θ-stable Π-module corresponding to $\mathscr{I}'$ is also θ_I-stable. Then the Π-module corresponding to $\mathscr{I}' \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus r-1}$ will also be θ_I-stable. ■

Appendix C

Compactified minimal resolutions

In this Appendix, we slightly expand Nakajima's interpretation of the varieties $\mathfrak{M}_{\theta^-}(r, \mathbf{v})$ from [52]. Here θ^- is a stability parameter lying in a chamber $C^-_{r,\mathbf{v}}$ from Section 2.4.1. There may be several such chambers, we fix one for the rest of this Appendix.

Let $\widehat{\mathcal{X}}$ be the stack formed by gluing together the minimal resolution $\widehat{\mathbb{C}^2/\Gamma}$ of $\mathbb{C}^2/\Gamma$ with the stack quotient $[(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma]$ along $(\mathbb{C}^2 \smallsetminus \{o\})/\Gamma$. More formally, we could define it as the pushout in the following diagram of stacks:

$$\begin{tikzcd} \mathbb{C}^2 \smallsetminus \{o\}/\Gamma \arrow[r] \arrow[d] & \widehat{\mathbb{C}^2/\Gamma} \arrow[d, dotted] \\ {[\mathbb{P}^2 \smallsetminus \{o\}/\Gamma]} \arrow[r, dotted] & \widehat{\mathcal{X}} \end{tikzcd}$$

Then $\widehat{\mathcal{X}}$ is a stack containing the ALE space $\widehat{\mathbb{C}^2/\Gamma}$ (see [52, section 3], where $\widehat{\mathcal{X}}$ is constructed in a more differential-geometric way).

The stack $\widehat{\mathcal{X}}$ comes equipped with tautological bundles $(\mathcal{R}_i)_{i \in Q_{\Gamma,0}}$ [52, p. 705]. Let $\mathcal{R} = \bigoplus_{i \in Q_{\Gamma,0}} \mathcal{R}_{i^\vee} \otimes \rho_i$, where $i^\vee$ is the vertex in Q_Γ such that $\rho_{i^\vee}$ is the dual representation of ρ_i.

As before, set $d_\infty \coloneqq \widehat{\mathcal{X}} \smallsetminus \widehat{\mathbb{C}^2/\Gamma} = [l_\infty/\Gamma]$.

Lemma C.0.1. *The stack $\widehat{\mathcal{X}}$ is a 2-dimensional projective orbifold.*

Its coarse moduli space is the projective scheme $\widetilde{\mathbb{P}^2/\Gamma}$ defined by gluing together $\widehat{\mathbb{C}^2/\Gamma}$ and $(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma$ along $(\mathbb{C}^2 \smallsetminus \{o\})/\Gamma$, and $\mathcal{O}_{\widehat{\mathcal{X}}}(d_\infty)$ is a generating sheaf for $\widehat{\mathcal{X}}$.

Proof. First, $\widehat{\mathcal{X}}$ is smooth because both $[(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma]$ and $\widehat{\mathbb{C}^2/\Gamma}$ are. The generic stabiliser is trivial because the scheme $\widehat{\mathbb{C}^2/\Gamma}$ is an open substack of $\widehat{\mathcal{X}}$.

Finding the coarse moduli space of a Deligne-Mumford stack can be done locally [14, p. 2]. Clearly the coarse moduli space of $[(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma]$ is $(\mathbb{P}^2 \smallsetminus \{o\})/\Gamma$, and we glue this to $\widetilde{\mathbb{C}^2/\Gamma}$ (which is its own coarse moduli space), to obtain $\widetilde{\mathbb{P}^2/\Gamma}$.

To see that $\mathcal{O}_{\widehat{\mathcal{X}}}(d_\infty)$ is a generating sheaf, we can argue precisely as in Proposition 5.1.5. ■

We will write $c\colon \widehat{\mathcal{X}} \to \widetilde{\mathbb{P}^2/\Gamma}$ for the map to the coarse moduli scheme.

We then have:

Theorem C.0.2 ([52, Theorem 2.2])**.** *Let $\mathbf{v} \in \mathbb{N}^{Q_{\Gamma,0}}$, and choose a positive integer r. Let $\theta^- \in C^-_{r,\mathbf{v}}$.*

There is then a canonical bijection between $\mathfrak{M}_{\theta^-}(r, \mathbf{v})(\mathbb{C})$ and the set $\widehat{Y}_{r,\mathbf{v}}$ of torsion free sheaves $\mathscr{F}$ on $\widehat{\mathcal{X}}$ of rank r, framed along d_∞, such that

- *The framing $\phi_{\mathscr{F}}\colon \mathscr{F}|_{d_\infty} \xrightarrow{\sim} \mathcal{O}_{d_\infty}^{\oplus r}$ is an isomorphism,*
- *there is an isomorphism of Γ-representations*

$$H^1(\widehat{\mathcal{X}}, \mathscr{F} \otimes \mathcal{R}^\vee(-d_\infty)) \xrightarrow{\sim} \bigoplus_{0 \le i \le s} \mathcal{R}_i^{\oplus v_i}.$$

(Note that the cohomology groups inherit a Γ-representation structure from the Γ-action on $\mathcal{R}$.)

Nakajima's paper [52] phrases this result in greater generality, and states the second condition in terms of the first Chern class of $\mathscr{F}$. Our statement in terms of cohomology groups is equivalent, see [52, p. 120].

Nakajima conjectures [52, p. 709] that the set $\widehat{Y}_{r,\mathbf{v}}$ can be given an intrinsic complex-analytic structure. The goal of this Appendix is to show that it can be given an intrinsic scheme structure as the moduli space of framed sheaves on a smooth projective stack, as long as we make a small stack-theoretic assumption (Assumption C.0.3) that probably holds true.

We will now show that the space $\widehat{Y}_{r,\mathbf{v}}$ can be given the structure of a fine moduli space, just as $Y_{r,n}$ could, and that at least in one direction, we can write out a morphism between moduli spaces extending Nakajima's bijection.

Our arguments are in many respects a repeat of those already seen for $Y_{r,n}$.

As in Section 6.2.1, let m be the smallest positive integer such that the stabiliser group of every point $x\colon \operatorname{Spec} k \to \widehat{\mathcal{X}}$ acts trivially on the fibre of $\mathcal{O}_{\widehat{\mathcal{X}}}(md_\infty)$ over x, where k is algebraically closed. Because Γ is finite, m exists. Then, let $\mathcal{O}_{\widetilde{\mathbb{P}^2/\Gamma}}(1)$ be a line bundle on the coarse moduli space $\widetilde{\mathbb{P}^2/\Gamma}$ such that

$$c^*\mathcal{O}_{\widetilde{\mathbb{P}^2/\Gamma}}(1) = \mathcal{O}_{\widehat{\mathcal{X}}}(m \cdot d_\infty),$$

which exists by [55, Proposition 6.1].

Assumption C.0.3. *For the rest of this Appendix, we assume that $\mathcal{O}_{\widetilde{\mathbb{P}^2/\Gamma}}(1)$ is ample. I have not found a way to prove this.*

Lemma C.0.4. *Under Assumption C.0.3, every framed sheaf in $\widehat{Y}_{r,\mathbf{v}}$ has the same modified Hilbert polynomial.*

Proof. This can be done very similarly as in Proposition 6.2.3.

Let $\mathscr{F} \in \widehat{Y}_{r,\mathbf{v}}$.

We know from [52, Section 5(ii)] that

$$H^q(\widehat{\mathcal{X}}, \mathscr{F}(-d_\infty)) = H^q(\widehat{\mathcal{X}}, \mathscr{F}(-2 \cdot d_\infty)),$$

and that for $q = 0, 2$ these groups vanish.

Then, by taking Euler characteristics in the short exact sequences

$$0 \to \mathscr{F}((-\nu - 1)d_\infty) \to \mathscr{F}(-\nu d_\infty) \to \mathscr{F}(-\nu d_\infty)|_{d_\infty} \xrightarrow{\sim} \mathcal{O}_{d_\infty}(-\nu)^{\oplus r} \to 0$$

we can induct from the case $\nu = 1$, to see that the function

$$\nu \mapsto \chi(\widehat{\mathcal{X}}, \mathscr{F}(\nu d_\infty))$$

is independent of the choice of $\mathscr{F} \in \widehat{Y}_{r,\mathbf{v}}$.

Now the modified Hilbert polynomial of $\mathscr{F}$ is given by $\nu \mapsto \chi(\widehat{\mathcal{X}}, \mathscr{F}((m\nu - 1)d_\infty))$, which thus is independent of the choice of $\mathscr{F}$. ■

Proposition C.0.5. *Under Assumption C.0.3, there is a quasiprojective scheme $\widehat{\mathbf{Y}}_{r,\mathbf{v}}$ which is a fine moduli space parametrising sheaves satisfying the conditions of Theorem C.0.2.*

Proof. By [11, Theorem 1.3], there is a quasiprojective scheme Z which is a fine moduli space of framed sheaves $(\mathscr{F}, \phi_{\mathscr{F}})$ with a given modified Hilbert polynomial P, satisfying the conditions that:

- $\phi_{\mathscr{F}} : \mathscr{F}|_{d_\infty} \to \mathcal{O}_{d_\infty}^{\oplus r}$ is an isomorphism,

- $\mathscr{F}$ is torsion-free,
- $\mathscr{F}$ is locally free in a neighbourhood of d_∞.

By Lemma C.0.4, every sheaf in $\widehat{Y}_{r,\mathbf{v}}$ has the same Hilbert polynomial, and by definition, they satisfy the first two conditions above. The third can be proved just as Lemma 5.2.11: the restriction of any such sheaf $\mathscr{F}$ to a neighbourhood U such that $d_\infty \subset U \subset \mathbb{P}^2 \smallsetminus \{o\}/\Gamma$ is the descent of a framed, torsion-free sheaf on a Γ-invariant open subscheme $V \subset \mathbb{P}^2$ such that $l_\infty \subset V$. Then we can apply Lemma 5.2.4 to show that $\mathscr{F}|_U$ descends from a sheaf locally free around l_∞, so $\mathscr{F}$ must be locally free around d_∞.

Finally, the cohomological conditions for a sheaf to lie in $\widehat{Y}_{r,\mathbf{v}}$ cut out $\widehat{\mathbf{Y}}_{r,\mathbf{v}}$ as a closed subscheme of Z.

■

Corollary C.0.6. *Under Assumption C.0.3, there is a morphism of schemes $\widehat{\mathbf{Y}}_{r,\mathbf{v}} \to \mathfrak{M}_{\theta^-}(r,\mathbf{v})$ extending the bijection from Theorem C.0.2.*

Proof. This is very similar to Theorem 6.3.1.

Consider the universal sheaf $\mathscr{U}$ on $\widehat{\mathbf{Y}}_{r,\mathbf{v}} \times \widehat{\mathcal{X}}$, and let p_1, p_2 be the projections from $\widehat{\mathbf{Y}}_{r,\mathbf{v}} \times \widehat{\mathcal{X}}$ onto the first and second factors respectively. Then, precisely as in the proof of Theorem 6.3.1 and using [53, Theorem 1.7], we find that $R^2(p_1)_*(\mathscr{U} \otimes p_2^*(\mathcal{R}^\vee \otimes \mathscr{I}_{d_\infty}))$ is a relative version of the cohomology group $H^1(\widehat{\mathcal{X}}, \mathscr{F} \otimes \mathcal{R}^\vee(-d_\infty))$. Again, the r framing arrows become r pairs of sheaf homomorphisms to and from the trivial bundle $\mathcal{O}_{\widehat{\mathbf{Y}}_{r,\mathbf{v}}}$, which together with the induced Γ-equivariant structure on $R^2(p_1)_*(\mathscr{U} \otimes p_2^*(\mathcal{R}^\vee \otimes \mathscr{I}_{d_\infty}))$ gives $\mathcal{O}_{\widehat{\mathbf{Y}}_{r,\mathbf{v}}} \oplus R^2(p_1)_*(\mathscr{U} \otimes p_2^*(\mathcal{R}^\vee \otimes \mathscr{I}_{d_\infty}))$ the structure of a flat family of θ^--stable Π-modules.

Then there is a morphism $\widehat{\mathbf{Y}}_{r,\mathbf{v}} \to \mathfrak{M}_{\theta^-}(r,\mathbf{v})$, clearly extending the bijection from Theorem C.0.2. ■

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