April 1

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Given a linear operator

$L: V \to V$, λ an eigenvalue of L

let $E_\lambda = \lambda$-eigenspace

$= \{v \in V \text{ s.t. } L(v) = \lambda v\}$

Thm: If $\lambda_1 \neq \lambda_2$, $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$

Prf:

Let v_i be an eigenvector for λ_i

$L(v_i) = \lambda_i v_i$

if $v \in E_{\lambda_1} \cap E_{\lambda_2}$

$L(v) = \lambda_1 v = \lambda_2 v$

so $\lambda_1 v - \lambda_2 v = 0 \to (\lambda_1 - \lambda_2) v = 0$

so either $v = \vec{0}$ or $\lambda_1 = \lambda_2$

$\square$

Diagonalizability of Linear Operators

Linear operators are "simple" when V can be decomposed as a direct sum of eigenspaces

Def: The geometric multiplicity of an eigenvalue λ is $\dim(E_\lambda)$

Ex:

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

the characteristic polynomial:

$\det(A - tI) = \det\left(\begin{bmatrix} 1-t & 2 \\ 0 & 1-t \end{bmatrix}\right)$

$= (1-t)^2$

so 1 is the only eigenvalue

$\to$ the algebraic multiplicity is 2

what is $\dim(E_1)$?

E_1 is the kernel of the linear transformation corresponding to $(A - 1(I))$

ie the solution set to

$\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2y \\ 0 \end{bmatrix}$$

our set $E_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t. } y = 0 \right\}$

$= \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \ a \in \mathbb{R} \right\}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \quad \checkmark$$

so $\dim(E_1) = 1$

so the geometric multiplicity is 1

Thm: The geometric multiplicity of an eigenvalue is always ≥ 1

Prf:

let λ be a eigenvalue of L, $\exists\ v$, $v \neq 0$

with $L(v) = \lambda v$

so E_λ contains a nontrivial vector

so $\dim(E_\lambda) \geq 1$

Thm: The geometric multiplicity is at most the algebraic multiplicity

Prf:

let $T: V \to V$ a linear operator, λ an eigenvalue for T

let k be the geometric multiplicity of λ

then $\dim(E_\lambda) = k$

so $\exists\ \{v_1, \dots, v_k\}$, a basis for E_λ

extend to $\underbrace{\{v_1, \dots, v_k, w_1, \dots, w_{n-k}\}}_{=S}$ for V

consider $[T]_S$

note $[v_i]_S$ is the i^{th} standard basis vector $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \leftarrow i^{th}$ position

$$\underbrace{[T]_S[v_i]_S}_{i^{th} \text{ column of } [T]_S} = [T(v_i)]_S = [\lambda v_i]_S = \lambda[v_i]_S = \begin{bmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ position}$$

so $[T]_S = \left[\begin{array}{ccc|c} \lambda & 0 & 0 & \\ 0 & \lambda & 0 & A \\ 0 & 0 & \lambda & \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \vdots & B \end{array}\right]$

the characteristic polynomial of T

$$\det\left(\left[\begin{array}{ccc|c} \lambda - t & & & \\ & \lambda - t & & A \\ & & \lambda - t \ \ddots & \\ & 0 & & B - tI \end{array}\right]\right) = \det\left(\begin{bmatrix} \lambda - t & & \\ & \lambda - t & \\ & & \ddots \end{bmatrix}\right) \cdot \det(B - tI)$$

since the first matrix is diagonal we get

char. poly$(T) = (\lambda - t)^k \cdot \det(B - tI)$

so $(\lambda - t)^k = (-1)^k (t - \lambda)^k$

$(t - \lambda)^k$ divides the characteristic polynomial

$\to$ the algebraic multiplicity is the biggest number with this property

$\square$

$\square$

Thm: The following are equivalent: given a $n \times n$ matrix A

1. A is diagonalizable (over $\mathbb{C}$), $\exists\, P$ an invertible matrix s.t. $P^{-1}AP$ is diagonal
2. the geometric multiplicity of each eigenvalue equals its algebraic multiplicity
3. $\exists$ an eigenbasis for L_A the linear transformation corresponding to A
4. the vector space V is the direct sum of its eigenspaces, ie

$$V \cong \bigoplus_{i=1}^{n} E_{\lambda_i}$$

here $V \cong K^n$

Prf:

$1 \Rightarrow 3$

if A is diagonalizable $\Rightarrow \exists$ an eigenbasis for L_A

since A is diagonalizable so $\exists\ P$ s.t. $P^{-1}AP = \mathrm{diag}(a_1, \dots, a_n) = D$

(side bar example: $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$) the standard basis vectors $\{b_1, \dots, b_n\}$ are eigenvectors for D

$\rightarrow$ So D has an eigenbasis

consider the vectors

$$\{Pb_1, Pb_2, \dots, Pb_n\}$$

$$APb_i = IAPb_i = PP^{-1}APb_i = PDb_i$$
$$= Pa_ib_i = a_iPb_i$$

so $\{Pb_1, \dots, Pb_n\}$ are eigenvectors for A

$\{Pb_1, \dots, Pb_n\}$ is a basis

$\square$

$3 \Rightarrow 1$

let $\{v_1, \dots, v_n\}$

let $P = [v_1 \mid v_2 \mid \cdots \mid v_n]$

$\rightarrow$ P is invertible because its columns are linearly independent so its rank is n

Claim: $P^{-1}AP$ is diagonal

Prf:

let b_i be the i^{th} standard basis vector if $P^{-1}APb_i = a_i b_i$

for all i, then $P^{-1}AP$ is diagonal

equivalently, we want to show that v_i is an eigenvector for all i

$$\begin{aligned} P^{-1}APb_i &= P^{-1}Av_i \\ &= P^{-1}\lambda_i v_i \quad \text{for some } \lambda_i \text{, since } v_i \text{ is an eigenvector} \\ &= \lambda_i P^{-1}v_i \\ &= \lambda_i b_i \end{aligned}$$

$\square$

How to diagonalize A (a diagonalizable matrix)

- find an eigenbasis for A (if it exists)
- build a matrix $P = [v_1 | v_2 | \dots | v_n]$

  $P^{-1}AP$ is diagonal

$2 \Leftrightarrow 3$

let $\{\lambda_1, \dots \lambda_\ell\}$ be the eigenvalues of A

suppose the geometric multiplicity of each eigenvalue equals its algebraic multiplicity

$$\sum_{i=1}^{\ell} \text{alg. mult.}(\lambda_i) = n = \sum_{i=1}^{\ell} \text{geo. mult.}(\lambda_i)$$

the dimension of the vector space

$$\sum_{i=1}^{\ell} \dim(E_{\lambda_i}) = n$$

let $\{v_j^1, v_j^2, \dots, v_j^{\ell}\}$ be a basis for E_j

(here $d_j = \dim(E_j)$)

Claim: $\bigcup_{i=1}^{\ell} \{v_j^1, v_j^2, \dots, v_j^{d_j}\}$ is a basis for V