

Maths 223
Lecture Notes Winter 2014

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1. CAUCHY-SCHWARZ INEQUALITY

1.1 Theorem

$$\forall u, v \in \mathbb{R}^n |u \cdot v| \leq \|u\| \cdot \|v\| \quad (1)$$

1.2 Proof

Let $tu + v$ be a family of vectors where t is in \mathbb{R} .

$$\forall w \in \mathbb{R}^n 0 \leq w \cdot w \quad (2)$$

$$0 \leq (tu + v) \cdot (tu + v) \quad (3)$$

$$0 \leq tu \cdot v + 2u \cdot v + v \cdot tu + v \cdot v \quad (4)$$

$$\forall a = u \cdot u, b = 2u \cdot v : 0 \leq t^2 \quad (5)$$

//TODO CJB REREAD THE FIRST PAGE (having trouble in low light)

2. TRIANGLE INEQUALITY

2.1 Definition

$$\|u + v\| \leq \|u\| + \|v\| \quad (6)$$

2.2 Proof

We show that $\|u + v\|^2 \leq (\|u\| + \|v\|)^2$.

$$\|u + v\|^2 = (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v \cdot v \quad (7)$$

$$= \|u\|^2 + 2u \cdot v + \|v\|^2 \quad (8)$$

$$\leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 \quad (9)$$

$$\leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 \quad \text{by Cauchy} \quad (10)$$

$$\leq (\|u\| + \|v\|)^2 \quad (11)$$

3. COMPLEX NUMBERS

3.1 Introduction

Complex numbers introduce a unit i such that $i^2 = -1$ and generate a set of numbers called \mathbb{C} . Usually complex numbers are represented as some combination $a + ib$ where $a, b \in \mathbb{R}$.

In complex space you can solve any polynomial equation and the number of solution is the degree ¹. In complex space //TODO return to 1/0 section.

A complex number is a pair of real numbers with operations "+" and "*." The following demonstrate how complex numbers operate.

$$(a, b) + (c, d) = (a + c, b + d) \quad (12)$$

$$(a, b) * (c, d) = (ac - bd, ad + bc) \quad (13)$$

$$(0, 1) * (0, 1) = (-1, 0) = -(1, 0) \implies i^2 = -1 \quad (14)$$

$$a(1, 0) + b(0, 1) = (a, b) = a + ib \quad (15)$$

$$(16)$$

The following demonstrates the logic of complex numbers.

$$(a + ib)(c + id) = ac + aid + ibc + (ib)(id) \quad (17)$$

$$= ac + i(ad + bc) + i^2 bd \quad (18)$$

$$= (ac - bd) + i(ad + bc) \quad (19)$$

¹No matter what people told me I found it hard to understand the importance of complex numbers, but it turns out they become really important in numerical analysis. In stuff like predator-prey models, oscillations in population dynamics are results of complex eigenvalues. You can determine whether or not the oscillations of such a population are stable by checking if the imaginary part of the complex values are positive or negative. In black holes, everything spirals into the black hole along a circle, which can only mathematically be described using complex numbers. It's surprisingly cool... gets a lot more cool with the stability of orbits for satellites. You want a satellite to have a slightly unstable orbit so that it doesn't take much energy to shift the position of the satellite. -CJB

3.2 Definition

Formally:

$$\forall a, b \in \mathbb{R} : z = a + ib \quad (20)$$

$$a = \text{realpart}[z] = [\text{Re}(z)] \quad (21)$$

$$b = \text{imaginarypart}[z] = [\text{Im}(z)] \quad (22)$$

4. COMPLEX CONJUGATION

The complex conjugate of some complex number z is given by $\bar{z} = a - ib$.

4.1 Definitions

- $z = a + ib, \bar{z} = (a - ib)$
- $z * \bar{z} = (a + ib)(a - ib) = a^2 + b^2$
- $|z| = \sqrt{a^2 + b^2} = \text{sqrt}(z * \bar{z})$

4.2 Theorems

- $\forall z * \bar{z} : z * \bar{z} \in \mathbb{R}$
- $\forall z * \bar{z} : z * \bar{z} \geq 0$
- $\forall z \in \mathbb{C}, z \neq 0 : \exists w \in \mathbb{C} \mid zw = 1$

Proof:

$$z = a + ib, w = \frac{a - ib}{a^2 + b^2} \quad (23)$$

$$zw = (a + ib) \frac{a - ib}{a^2 + b^2} = \frac{a^2 - aib + aib - iib^2}{a^2 + b^2} \quad (24)$$

$$= \frac{a^2 + b^2}{a^2 + b^2} = 1 \quad (25)$$

- $|z_1 z_2| = |z_1| |z_2|$

Proof: $|z_1 z_2| = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2 \overline{z_1 z_2} = z_1 \overline{z_1} z_2 \overline{z_2} = |z_1| |z_2|$

4.3 Visualisation

In the complex plane, complex conjugate is a reflection in the x-axis. //TODO generate figures for description of complex conjugate as a reflection of the complex number

5. COMPLEX SPACES

\mathbb{C}^n is the complex n space that consists of n tuples of complex numbers. Numbers in \mathbb{C}^n can be added, scaled, and otherwise manipulated much like \mathbb{R}^n .

5.1 Definitions

5.1.1 Dot Product in \mathbb{C}^n

$$(z_1, \dots, z_n) \cdot (w_1, \dots, w_n) = z_1 \overline{w_1} + \dots + z_n \overline{w_n} \quad (26)$$

This agrees with the dot product on \mathbb{R}^n , as \mathbb{R}^n is a subset of \mathbb{C}^n . For \mathbb{R} , $a + 0i = a - 0i$, so the conjugate makes no difference.

5.2 Theorems

5.2.1 $v \cdot v$ is in \mathbb{R}^n , and nonnegative; $v \cdot v = 0 \iff v = 0$

If v is \mathbb{C}^n then $v \cdot v$ is in \mathbb{R}^n , and nonnegative; $v \cdot v = 0 \iff v = 0$

$$v = (z_1, \dots, z_n) \quad (27)$$

$$v \cdot v = z_1 \overline{z_1} + \dots + z_n \overline{z_n} \quad (28)$$

$$z_k = a_k + ib_k \quad (29)$$

$$z_k \overline{z_k} = a_k^2 + b_k^2 \quad (30)$$

$$v \cdot v = a_1^2 + b_1^2 + \dots + a_n^2 + b_n^2 \quad (31)$$

This implies $v \cdot v$ is a non-negative i and 0 only if it equals 0 and it equals 0 only if $a_k = 0$ and $b_k = 0$ for all k between 1 and n. In this case, v is 0.

6. MATRICES AS ARRAYS

We can look at matrices as arrays, where a matrix is an array with n rows each with m elements.

The shape of a matrix is defined by n and m . If two matrices have the same shape they can be added, where each element of the matrices are added to the corresponding elements of the other matrix.

Scaling a matrix consists of multiplying each element by some scalar.

Multiplying a matrix consists of multiplying each row by a corresponding column in the other matrix. It's a little more complicated but presumably people are familiar with this from maths 133.

The transposed matrix consists of a matrix where each row and column index has been switched.

6.1 Matrix Multiplication

$$(AB)C = A(BC) \text{ associative} \quad (32)$$

$$A(B + C) = AB + AC \text{ distributive} \quad (33)$$

$$(B + C)A = BA + CA \quad (34)$$

$$AB \neq BA \quad (35)$$

6.2 Definitions

SQUARE MATRIX An n by n matrix.

TRACE The trace of a square matrix $A = [a_{ij}]$ is given by $tr(A) = \sum_{i=1}^n a_{ii}$.

IDENTITY MATRIX The n by n identity matrix is the $I_n = [a_{ij}]$ where $\forall i = j, a_{ij} = 1 \wedge \forall i \neq j, a_{ij} = 0$