Maths 223

Lecture Notes Winter 2014

Lectures by William Cavendish
Transcribed by Hannah Lee, ...
Typeset by Calem Bendell, ...

Contents

1 Cauchy-Schwarz Inequality 3
- 1.1 Theorem . . . 3
- 1.2 Proof . . . 3

2 Triangle Inequality 3
- 2.1 Definition . . . 3
- 2.2 Proof . . . 3

3 Complex Numbers 4
- 3.1 Introduction . . . 4
- 3.2 Definition . . . 5

4 Complex Conjugation 5
- 4.1 Definitions . . . 5
- 4.2 Theorems . . . 5
- 4.3 Visualisation . . . 6

5 Complex Spaces 6
- 5.1 Definitions . . . 6
  - 5.1.1 Dot Product in $\mathbb{C}^n$ . . . 6
- 5.2 Theorems . . . 6
  - 5.2.1 $v \cdot v$ is in $\mathbb{R}^n$, and nonnegative; $v \cdot v = 0 \iff v = 0$ . . . 6

6 Matrices as Arrays 7
- 6.1 Matrix Multiplication . . . 7
- 6.2 Definitions . . . 7
- 6.3 Theorems . . . 8
  - 6.3.1 $A_{mn}I_n = A, \;\; I_nB_{nm} = B$ . . . 8
  - 6.3.2 Block Matrices . . . 8

1. Cauchy-Schwarz Inequality

1.1 Theorem

$$\forall u, v \in \mathbb{R}^n |u \cdot v| \leq \|u\| \cdot \|v\| \tag{1}$$

1.2 Proof

Let $tu + v$ be a family of vectors where t is in $\mathbb{R}$.

$$\begin{aligned}
\forall w \in \mathbb{R}^n 0 \leq w \cdot w &\quad (2)\\
0 \leq (tu + v) \cdot (tu + v) &\quad (3)\\
0 \leq tu \cdot v + 2u \cdot v + v \cdot tu + v \cdot v &\quad (4)\\
\forall a = u \cdot u, b = 2u \cdot v : 0 \leq t^2 &\quad (5)
\end{aligned}$$

//TODO CJB REREAD THE FIRST PAGE (having trouble in low light)

2. Triangle Inequality

2.1 Definition

$$\|u + v\| \leq \|u\| + \|v\| \tag{6}$$

2.2 Proof

We show that $\|u + v\|^2 \leq (\|u\| + \|v\|)^2$.

$$\begin{aligned}
\|u + v\|^2 &= (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v\dot{v} && (7)\\
&= \|u\|^2 + 2u \cdot v + \|v\|^2 && (8)\\
&\leq \|u\|^2 + 2\|u \cdot v\| + \|v\|^2 && (9)\\
&\leq \|u\|^2 + 2\|u|\ \|v\| + \|v\|^2 && \text{by Cauchy} \quad (10)\\
&\leq (\|u\| + \|v\|)^2 && (11)
\end{aligned}$$

3. Complex Numbers

3.1 Introduction

Complex numbers introduce a unit i such that $i^2 = -1$ and generate a set of numbers called $\mathbb{C}$. Usually complex numbers are represented as some combination $a + ib$ where $a, b \in \mathbb{R}$.

In complex space you can solve any polynomial equation and the number of solution is the degree [1]. In complex space //TODO return to 1/0 section.

A complex number is a pair of real numbers with operations ”+” and ”*.” The following demonstrate how complex numbers operate.

$$(a,b) + (c,d) = (a+c, b+d) \tag{12}$$
$$(a,b) * (c,d) = (ac - bd, ad + bc) \tag{13}$$
$$(0,1) * (0,1) = (-1,0) = -(1,0) \implies i^2 = 1 \tag{14}$$
$$a(1,0) + b(0,1) = (a,b) = a + ib \tag{15}$$
$$\tag{16}$$

The following demonstrates the logic of complex numbers.

$$\begin{aligned} (a+ib)(c+id) &= ac + aid + ibc + (ib)(id) &(17)\\ &= ac + i(ad+bc) + i^2 bd &(18)\\ &= (ac-bd) + i(ad+bc) &(19) \end{aligned}$$

[1] No matter what people told me I found it hard to understand the importance of complex numbers, but it turns out they become really important in numerical analysis. In stuff like predator-prey models, oscillations in population dynamics are results of complex eigenvalues. You can determine whether or not the oscillations of such a population are stable by checking if the imaginary part of the complex values are positive or negative. In black holes, everything spirals into the black hole along a circle, which can only mathematically be described using complex numbers. It’s surprisingly cool... gets a lot more cool with the stability of orbits for satellites. You want a satellite to have a slightly unstable orbit so that it doesn’t take much energy to shift the position of the satellite. -CJB

3.2 Definition

Formally:

$$\forall a, b \in \mathbb{R} : z = a + ib \tag{20}$$
$$a = realpart[z] = [Re(z)] \tag{21}$$
$$b = imaginarypart[z] = [Im(z)] \tag{22}$$

4. Complex Conjugation

The complex conjugate of some complex number z is given by $\overline{z} = a - ib$.

4.1 Definitions

- $z = a + ib, \overline{z} = (a - ib)$
- $z * \overline{z} = (a + ib)(a - ib) = a^2 + b^2$
- $|z| = \sqrt{a^2 + b^2} = sqrtz * \overline{z}$

4.2 Theorems

- $\forall z * \overline{z} : z * \overline{z} \in R$
- $\forall z * \overline{z} : z * \overline{z} \geq 0$
- $\forall z \in \mathbb{C}, z \neq 0 : \exists w \in \mathbb{C} \mid zw = 1$
  Proof:

$$z = a + ib, w = \frac{a - ib}{a^2 + b^2} \tag{23}$$
$$zw = (a + ib)\frac{a - ib}{a^2 + b^2} = \frac{a^2 - aib + aib - iib^2}{a^2 + b^2} \tag{24}$$
$$= \frac{a^2 + b^2}{a^2 + b^2} = 1 \tag{25}$$

- $|z_1 z_2| = |z_1||z_2|$
  Proof: $|z_1 z_2| = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2 \overline{z_1 z_2} = z_1 \overline{z_1} z_2 \overline{z_2} = |z_1||z_2|$

4.3 Visualisation

In the complex plane, complex conjugate is a reflection in the x-axis. //TODO generate figures for description of complex conjugate as a reflection of the complex number

5. Complex Spaces

$\mathbb{C}^n$ is the complex n space that consists of n tuples of complex numbers. Numbers in $\mathbb{C}^n$ can be added, scaled, and otherwise manipulated much like $\mathbb{R}^n$.

5.1 Definitions

5.1.1 Dot Product in $\mathbb{C}^n$

$$(z_1, \ldots, z_2) \cdot (w_1, \ldots, w_n) = z_1\overline{w_1} + \ldots + z_n\overline{w_n} \tag{26}$$

This agrees with the dot product on $\mathbb{R}^n$, as $\mathbb{R}^n$ is a subset of $\mathbb{C}^n$. For $\mathbb{R}$, $a + 0i = a - 0i$, so the conjugate makes no difference.

5.2 Theorems

5.2.1 $v \cdot v$ is in $\mathbb{R}^n$, and nonnegative; $v \cdot v = 0 \iff v = 0$

If v is $\mathbb{C}^n$ then $v \cdot v$ is in $\mathbb{R}^n$, and nonnegative; $v \cdot v = 0 \iff v = 0$

$$v = (z_1, \ldots, z_2) \tag{27}$$

$$v \cdot v = z_1\overline{z_1} + \ldots + z_n\overline{z_n} \tag{28}$$

$$z_k = a_k + ib_k \tag{29}$$

$$z_k\overline{z_k} = a_k^2 + b_k^2 \tag{30}$$

$$v \cdot v = a_1^2 + b_1^2 + \ldots + a_n^2 + b_n^2 \tag{31}$$

This implies $v \cdot v$ is a non-negative i and 0 only if it equals 0 and it equals 0 only if $a_k = 0$ and $b_k = 0$ for all k between 1 and n. In this case, v is 0.

6. Matrices as Arrays

We can look at matrices as arrays, where a matrix is an array with n rows each with m elements.

The shape of a matrix is defined by n and m. If two matrices have the same shape they can be added, where each element of the matrices are added to the corresponding elements of the other matrix.

Scaling a matrix consists of multiplying each element by some scalar.

Multiplying a matrix consists of multiplying each row by a corresponding column in the other matrix. It's a little more complicated but presumably people are familiar with this from maths 133.

The transposed matrix consists of a matrix where each row and column index has been switched.

The matrix power is a matrix multiplied by itself some n times.

6.1 Matrix Multiplication

$$(AB)C = A(BC)\text{associative} \tag{32}$$

$$A(B+C) = AB + AC\text{distributive} \tag{33}$$

$$(B+C)A = BA + CA \tag{34}$$

$$AB \neq BA \tag{35}$$

6.2 Definitions

square matrix An n by n matrix.

trace The trace of a square matrix $A = [a_{ij}]$ is given by $tr(A) = \sum_{i=1}^{n} a_{ii}$.

identity matrix The n by m identity matrix is the $I_n = [a_{ij}]$ where $\forall i = j, a_{ij} = 1 \land \forall i \neq j, a_{ij} = 0$

Matrix Power $\forall square_matrix[A] \;\; A^0 = I$
$\forall invertible[A] \;\; A^{-1} = inverse[A]$

Block Matrix A matrix partitioned in submatrices.

6.3 Theorems

6.3.1 $A_{mn}I_n = A, \; I_nB_{nm} = B$

$$\sum_{k=1}^{n} a_{ik}I_{ik} = a_{ij} \tag{36}$$

6.3.2 Block Matrices

$$\forall A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \tag{37}$$