Maths 223
Lecture Notes Winter 2014

Lectures by William Cavendish
Transcribed by Hannah Lee, ...
Typeset by Calem Bendell, ...

Contents

1. Cauchy-Schwarz Inequality

1.1 Theorem

$$\forall u, v \in \mathbb{R}^n |u \cdot v| \leq \|u\| \cdot \|v\| \tag{1}$$

1.2 Proof

Let $tu + v$ be a family of vectors where t is in $\mathbb{R}$.

$$\begin{align}
\forall w \in \mathbb{R}^n 0 \leq w \cdot w \tag{2}\\
0 \leq (tu + v) \cdot (tu + v) \tag{3}\\
0 \leq tu \cdot v + 2u \cdot v + v \cdot tu + v \cdot v \tag{4}\\
\forall a = u \cdot u, b = 2u \cdot v : 0 \leq t^2 \tag{5}
\end{align}$$

//TODO CJB REREAD THE FIRST PAGE (having trouble in low light)

2. Triangle Inequality

2.1 Definition

$$\|u + v\| \leq \|u\| + \|v\| \tag{6}$$

2.2 Proof

We show that $\|u + v\|^2 \leq (\|u\| + \|v\|)^2$.

$$\begin{align}
\|u + v\|^2 &= (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v\dot{v} \tag{7}\\
&= \|u\|^2 + 2u \cdot v + \|v\|^2 \tag{8}\\
&\leq \|u\|^2 + 2\|u \cdot v\| + \|v\|^2 \tag{9}\\
&\leq \|u\|^2 + 2\|u|\ \|v\| + \|v\|^2 && \text{by Cauchy} \tag{10}\\
&\leq (\|u\| + \|v\|)^2 \tag{11}
\end{align}$$

3. Complex Numbers

3.1 Introduction

Complex numbers introduce a unit i such that $i^2 = -1$ and generate a set of numbers called $\mathbb{C}$. Usually complex numbers are represented as some combination $a + ib$ where $a, b \in \mathbb{R}$.

In complex space you can solve any polynomial equation and the number of solution is the degree [1]. In complex space //TODO return to 1/0 section.

A complex number is a pair of real numbers with operations "+" and "*." The following demonstrate how complex numbers operate.

$$(a,b) + (c,d) = (a+c, b+d) \tag{12}$$

$$(a,b) * (c,d) = (ac - bd, ad + bc) \tag{13}$$

$$(0,1) * (0,1) = (-1,0) = -(1,0) \implies i^2 = 1 \tag{14}$$

$$a(1,0) + b(0,1) = (a,b) = a + ib \tag{15}$$

$$\tag{16}$$

The following demonstrates the logic of complex numbers.

$$\begin{aligned} (a+ib)(c+id) &= ac + aid + ibc + (ib)(id) && (17)\\ &= ac + i(ad+bc) + i^2 bd && (18)\\ &= (ac - bd) + i(ad + bc) && (19) \end{aligned}$$

[1] No matter what people told me I found it hard to understand the importance of complex numbers, but it turns out they become really important in numerical analysis. In stuff like predator-prey models, oscillations in population dynamics are results of complex eigenvalues. You can determine whether or not the oscillations of such a population are stable by checking if the imaginary part of the complex values are positive or negative. In black holes, everything spirals into the black hole along a circle, which can only mathematically be described using complex numbers. It's surprisingly cool... gets a lot more cool with the stability of orbits for satellites. You want a satellite to have a slightly unstable orbit so that it doesn't take much energy to shift the position of the satellite. -CJB

3.2 Definition

Formally:

$$\forall a, b \in \mathbb{R} : z = a + ib \tag{20}$$

$$a = realpart[z] = [Re(z)] \tag{21}$$

$$b = imaginarypart[z] = [Im(z)] \tag{22}$$

4. Complex Conjugation

The complex conjugate of some complex number z is given by $\overline{z} = a - ib$.

4.1 Definitions

- $z = a + ib, \overline{z} = (a - ib)$
- $z * \overline{z} = (a + ib)(a - ib) = a^2 + b^2$
- $|z| = \sqrt{a^2 + b^2} = sqrtz * \overline{z}$

4.2 Theorems

- $\forall z * \overline{z} : z * \overline{z} \in R$
- $\forall z * \overline{z} : z * \overline{z} \geq 0$
- $\forall z \in \mathbb{C}, z \neq 0 : \exists w \in \mathbb{C} \mid zw = 1$
  Proof:

$$z = a + ib, w = \frac{a - ib}{a^2 + b^2} \tag{23}$$

$$zw = (a + ib)\frac{a - ib}{a^2 + b^2} = \frac{a^2 - aib + aib - iib^2}{a^2 + b^2} \tag{24}$$

$$= \frac{a^2 + b^2}{a^2 + b^2} = 1 \tag{25}$$

- $|z_1 z_2| = |z_1||z_2|$
  Proof: $|z_1 z_2| = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2 \overline{z_1 z_2} = z_1 \overline{z_1} z_2 \overline{z_2} = |z_1||z_2|$