# Maths 223
## Lecture Notes Winter 2014

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# Contents

# 1. Cauchy-Schwarz Inequality

## 1.1 Theorem

$$\forall u, v \in \mathbb{R}^n |u \cdot v| \leq \|u\| \cdot \|v\| \tag{1}$$

## 1.2 Proof

Let $tu + v$ be a family of vectors where t is in $\mathbb{R}$.

$$\begin{align}
\forall w \in \mathbb{R}^n 0 \leq w \cdot w \tag{2}\\
0 \leq (tu + v) \cdot (tu + v) \tag{3}\\
0 \leq tu \cdot v + 2u \cdot v + v \cdot tu + v \cdot v \tag{4}\\
\forall a = u \cdot u, b = 2u \cdot v : 0 \leq t^2 \tag{5}
\end{align}$$

//TODO CJB REREAD THE FIRST PAGE (having trouble in low light)

# 2. Triangle Inequality

## 2.1 Definition

$$\|u + v\| \leq \|u\| + \|v\| \tag{6}$$

## 2.2 Proof

We show that $\|u + v\|^2 \leq (\|u\| + \|v\|)^2$.

$$\begin{align}
\|u + v\|^2 &= (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v\dot{v} \tag{7}\\
&= \|u\|^2 + 2u \cdot v + \|v\|^2 \tag{8}\\
&\leq \|u\|^2 + 2\|u \cdot v\| + \|v\|^2 \tag{9}\\
&\leq \|u\|^2 + 2\|u|\ \|v\| + \|v\|^2 && \text{by Cauchy} \tag{10}\\
&\leq (\|u\| + \|v\|)^2 \tag{11}
\end{align}$$

# 3. Complex Numbers

## 3.1 Introduction

Complex numbers introduce a unit $i$ such that $i^2 = -1$ and generate a set of numbers called $\mathbb{C}$. Usually complex numbers are represented as some combination $a + ib$ where $a, b \in \mathbb{R}$.

In complex space you can solve any polynomial equation and the number of solution is the degree [1]. In complex space //TODO return to 1/0 section.

A complex number is a pair of real numbers with operations "+" and "*." The following demonstrate how complex numbers operate.

$$(a,b) + (c,d) = (a+c, b+d) \tag{12}$$
$$(a,b) * (c,d) = (ac - bd, ad + bc) \tag{13}$$
$$(0,1) * (0,1) = (-1,0) = -(1,0) \implies i^2 = 1 \tag{14}$$
$$a(1,0) + b(0,1) = (a,b) = a + ib \tag{15}$$
$$\tag{16}$$

The following demonstrates the logic of complex numbers.

$$(a+ib)(c+id) = ac + aid + ibc + (ib)(id) \tag{17}$$
$$= ac + i(ad + bc) + i^2 bd \tag{18}$$
$$= (ac - bd) + i(ad + bc) \tag{19}$$

## 3.2 Definition

Formally:

$$\forall a, b \in \mathbb{R} : z = a + ib \tag{20}$$
$$a = realpart[z] = [Re(z)] \tag{21}$$
$$b = imaginarypart[z] = [Im(z)] \tag{22}$$

[1] No matter what people told me I found it hard to understand the importance of complex numbers, but it turns out they become really important in numerical analysis. In stuff like predator-prey models, oscillations in population dynamics are results of complex eigenvalues. You can determine whether or not the oscillations of such a population are stable by checking if the imaginary part of the complex values are positive or negative. In black holes, everything spirals into the black hole along a circle, which can only mathematically be described using complex numbers. It's surprisingly cool... gets a lot more cool with the stability of orbits for satellites. You want a satellite to have a slightly unstable orbit so that it doesn't take much energy to shift the position of the satellite. -CJB

# 4. Complex Conjugation

The complex conjugate of some complex number z is given by $\overline{z} = a - ib$.

## 4.1 Definitions

- $z = a + ib, \overline{z} = (a - ib)$
- $z * \overline{z} = (a + ib)(a - ib) = a^2 + b^2$
- $|z| = \sqrt{a^2 + b^2} = sqrtz * \overline{z}$

## 4.2 Theorems

- $\forall z * \overline{z} : z * \overline{z} \in R$
- $\forall z * \overline{z} : z * \overline{z} \geq 0$
- $\forall z \in \mathbb{C}, z \neq 0 : \exists w \in \mathbb{C} \mid zw = 1$
  Proof:

$$z = a + ib, w = \frac{a - ib}{a^2 + b^2} \tag{23}$$

$$zw = (a + ib)\frac{a - ib}{a^2 + b^2} = \frac{a^2 - aib + aib - iib^2}{a^2 + b^2} \tag{24}$$

$$= \frac{a^2 + b^2}{a^2 + b^2} = 1 \tag{25}$$

- $|z_1 z_2| = |z_1||z_2|$
  Proof: $|z_1 z_2| = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2 \overline{z_1}\,\overline{z_2} = z_1 \overline{z_1} z_2 \overline{z_2} = |z_1||z_2|$

## 4.3 Visualisation

In the complex plane, complex conjugate is a reflection in the x-axis. //TODO generate figures for description of complex conjugate as a reflection of the complex number

# 5. Complex Spaces

$\mathbb{C}^n$ is the complex n space that consists of n tuples of complex numbers. Numbers in $\mathbb{C}^n$ can be added, scaled, and otherwise manipulated much like $\mathbb{R}^n$.

### 5.1 Definitions

#### 5.1.1 Dot Product in $\mathbb{C}^n$

$$(z_1, \ldots, z_2) \cdot (w_1, \ldots, w_n) = z_1\overline{w_1} + \ldots + z_n\overline{w_n} \tag{26}$$

This agrees with the dot product on $\mathbb{R}^n$, as $\mathbb{R}^n$ is a subset of $\mathbb{C}^n$. For $\mathbb{R}$, $a + 0i = a - 0i$, so the conjugate makes no difference.

### 5.2 Theorems

#### 5.2.1 $v \cdot v$ is in $\mathbb{R}^n$, and nonnegative; $v \cdot v = 0 \iff v = 0$

If $v$ is $\mathbb{C}^n$ then $v \cdot v$ is in $\mathbb{R}^n$, and nonnegative; $v \cdot v = 0 \iff v = 0$

$$v = (z_1, \ldots, z_2) \tag{27}$$

$$v \cdot v = z_1\overline{z_1} + \ldots + z_n\overline{z_n} \tag{28}$$

$$z_k = a_k + ib_k \tag{29}$$

$$z_k\overline{z_k} = a_k^2 + b_k^2 \tag{30}$$

$$v \cdot v = a_1^2 + b_1^2 + \ldots + a_n^2 + b_n^2 \tag{31}$$

This implies $v \cdot v$ is a non-negative i and 0 only if it equals 0 and it equals 0 only if $a_k = 0$ and $b_k = 0$ for all k between 1 and n. In this case, v is 0.

## 6. Matrices as Arrays

We can look at matrices as arrays, where a matrix is an array with n rows each with m elements.

The shape of a matrix is defined by n and m. If two matrices have the same shape they can be added, where each element of the matrices are added to the corresponding elements of the other matrix.

Scaling a matrix consists of multiplying each element by some scalar.

Multiplying a matrix consists of multiplying each row by a corresponding column in the other matrix. It's a little more complicated but presumably people are familiar with this from maths 133.

The transposed matrix consists of a matrix where each row and column index has been switched.

The matrix power is a matrix multiplied by itself some $n$ times.

### 6.1 Matrix Multiplication

$$(AB)C = A(BC)\text{associative} \tag{32}$$
$$A(B+C) = AB + AC\text{distributive} \tag{33}$$
$$(B+C)A = BA + CA \tag{34}$$
$$AB \neq BA \tag{35}$$

### 6.2 Definitions

SQUARE MATRIX An n by n matrix.

TRACE The trace of a square matrix $A = [a_{ij}]$ is given by $tr(A) = \sum_{i=1}^{n} a_{ii}$.

IDENTITY MATRIX The n by m identity matrix is the $I_n = [a_{ij}]$ where $\forall i = j, a_{ij} = 1 \wedge \forall i \neq j, a_{ij} = 0$

MATRIX POWER $\forall square_m atrix[A]$ $A^0 = I$
$\forall invertible[A]$ $A^{-1} = inverse[A]$

BLOCK MATRIX A matrix partitioned in submatrices.

LINEAR EQUATION A linear equation with unknowns is an expression of the form $a_1x_1 + \ldots + a_nx_n = b$ where some vector $a_n$ and the value $b$ are in some number system. A solution vector $x_n$ exists that satisfies this equation.

### 6.3 Theorems

#### 6.3.1 $A_{mn}I_n = A$, $I_nB_{nm} = B$

$$\sum_{k=1}^{n} a_{ik}I_{ik} = a_{ij} \tag{36}$$

#### 6.3.2 $A_{ik}B_{kj}$

$$\forall A_{ik}B_{kj} \tag{37}$$
$$AB = [C_{ij}] \tag{38}$$
$$C_{ij} = \sum_{k=i}^{n} A_{ik}B_{kj} \tag{39}$$

### 6.4 Note on Excluded Material

I have not included some theorems on addition and multiplication and the like. I also have not typeset notes on how to do proofs. The assignments seem to do an excellent job of covering this material. Should I find the time (probably will), these will be typeset after the midterm to complete the notes properly.

# 7. Linear Equations

A linear equation over with unknowns is represented by $\forall\{a_1,\ldots,a_n\}, b \in K,\ \ a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. A system of linear equations is a set of $L$'s, where $L$ is a linear equation as above. Linear equations form a vector space that can be scaled.

Given a system of linear equations with unknowns, a solution vector $u = (a_1,\ldots,a_n)$ is something that simultaneously satisfies all of the linear equations.

## 7.1 Definitions

Augmented Matrix Given a system of linear equations defined above, the augmented matrix is:

$$M = \left[\begin{array}{cc|c} a_{11} & \ldots & b_1 \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & b_m \end{array}\right]$$

Equivalent Systems of Linear Equations Two system of linear equations are equivalent if they have the same system. More precisely, $M_1$ and $M_2$ are equivalent when the rows of one are linear combinations of the rows of the other. This means that systems of linear equations are equivalen when they are row equivalent.

Elementary Row Operation (ERO) An ERO is an operation of either interchanging two rows, replacing a row with a multiple of itself, or replacing a row with the addition of itself and another row.

Row Echelon Form Row echelon form consists of a matrix such that each leftmost nonzero entry of a row is to the right of the leftmost nonzero entry of each proceeding row, all zero rows are at the bottom of the matrix.

Row Canonical Form (Reduced Row Echelon) (Fully Reduced) Row canonical form or full reduced matrix form is a matrix that satisfies row echelon form where every first entry in each row is 1.

Pivot A pivot is the first nonzero entry in a row of a matrix in row ecehlon form.

Gaussian Elimination A matrix can be converted into reduced row echelon form by performing row separations to put it into echelon form, dividing throughout so that every pivot is 1, and then using the 1 to zero out every space above pivots.

## 7.2 Theorems

### 7.2.1 Solution to Linear Equation

If $u$ is the solution to a linear equation where $L_1 : x_1 + x_2 = 0; L_2 = x_1 - x_2 = 3; L_1 + L_2 : 2x_1 = 3$ then $x_1 = 3/2, x_2 = -3/2$ by back substitution.

### 7.2.2 Non-Invertible Matrix

A matrix is not invertible if there exists some nonzero vector such that the matrix multiplied by the vector is 0.

### 7.2.3 Products of Invertible Matrices

A product of matrices is invertible if and only if each of the individual matrices are invertible.

### 7.2.4 A matrix is row equivalent to a unique matrix in RCF

Wellp... the subsection name sums it up. A proof is further below.

## 7.3 Proofs

### 7.3.1 Every ERO has an inverse operation

BLURRY PAGE, GO BACK!

### 7.3.2 A product of a set of matrices is invertible if and only if each of the set of matrices is invertible

$$\forall invertible[A], \exists v \neq 0 \;\; s.t. \;\; Av = 0 \tag{40}$$

$$B = A^{-1} \tag{41}$$

$$B(Av) = B \cdot [0] = [0] \tag{42}$$

$$(BA)v = (A^{-1}A)v = Iv = v \implies v = 0 \tag{43}$$

In words, since this proof can be somewhat confusing (or at least it confused me): if A is invertible there must be some nonzero vector such that Av is 0. Creating some matrix B that is the inverse of the vector A, should you multiply Av (0) by B it should be 0... or it could be v, depending on how you frame the problem. So v must be 0 and thus the matrix must be invertible.

Another proof. Suppose a set of matrices A are all invertible and their product is B. If you repeatedly multiply both sides of this equality by a corresponding inverse of A to replace the set of matrices with the identity matrix, you will get $I = (A_n^{-1} \ldots A_1^{-1})B$. This can be reduced to

$(A_n^{-1} \ldots A_1^{-1}) = B^{-1}$ by multiplying both sides by the inverse of B. Therefore, the inverse of B exists and the product of a set of invertible matrices is invertible.

We can use this to show each element is invertible. If $A_1 A_2 \ldots A_n$ is invertible so is $(A_1 \ldots A_{n-1})A_n$ by associativity. The expression has two terms and the term in paranetheses is invertible and $A_n$ is invertible if we know the second term.

### 7.3.3 A matrix is row equivalent to a unique matrix in RCF

We know that $M_1$ and $M_2$ represent equivalent linear systems if and only if these matrices are row equivalent, thus there is a sequence of EROs that can convert matrices to each other. $M_2$ can be transformed into a matrix $N$ in RCF by some sequence of EROS. $M_1$ can be transformed into a matrix $N$ in RCF by some sequence of EROS. All three of these matrices are equivalent.