last time:
we had $f_1, f_2, f_3 \in \mathbb{P}_2(t)$
we knew two things:
- $\mathbb{P}_2(t)$ has dim 3
- f_1, f_2, f_3 do not span $\mathbb{P}_2(t)$

– Could f_1, f_2, f_3 be linearly indep?

We showed if f_1, f_2, f_3 are linearly indep, then $\exists$ a subspace $W \subseteq V$ s.t. W has dim 4.
(This is ruled by prob. 7 on the homework)

So f_1, f_2, f_3 are linearly dependent (which we showed without computation.)

Why is dimension well defifd?

Def: The dimention of a vector space V is the size of any basis for V.

Def. A Basis for a vector space V is a linearly independent spanning set for V.

---

* Thm:
Every basis for V has the same size.

Our theorem will follow from this lemma:

Lemma:
Let $\{w_1, \ldots, w_k\}$ be a spanning set for V and let $\{u_1, \ldots, u_\ell\}$ be a linearly indep. set.
Then $\ell \leq k$

Lemma $\Rightarrow$ Thm:

prf:

Let $\{w_1, \ldots, w_n\}$ be a basis for V, $\{z_1, \ldots, z_m\}$ another basis.

Want to show $m = n$.

$\{z_1, \ldots z_n\}$ is lin. indep. and $\{w_1, \ldots, w_m\}$ spans, so $m \leq n$.

$\{w_1, \ldots, w_n\}$ is lin. indep and $\{z_1, \ldots z_m\}$ spans, so $n \leq m$.

(If $n \leq m$ and $m \leq n$ then $m = n$) $\square$

$\rightarrow$ Strategy of proofing of lemma:

Given a spanning set $\{w_1, \ldots, w_k\}$ and a linearly indep set $\{u_1, \ldots, u_\ell\}$, we'll add one of the u_i's to $\{w_1, \ldots, w_k\}$ to form $\{w_1, \ldots, w_k, u_i\}$

We'll show that $\{w_1, \ldots, w_k, u_i\}$ still spans, and that one of the w_i's can be removed and maintain the spanning property, i.e.

$\{w_1, \ldots, w_{j-1}, w_j, \ldots, w_k, u_i\}$ spans.

$Size(\{w_1, \ldots, w_{j-1}, w_{j+1}, \ldots w_k, u_i\}) = Size(\{w_1, \ldots, w_k\}) = k$

So we've replaced one of the w's with the u to obtain a new spanning set.

We now do this again, using $\{w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_k, u_i\}$ as our spanning set, $\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_\ell\}$ as our lin. indep. set

to get $\{w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_{j-1}, w_{j+1}, \ldots, w_k, u, u_n\}$ a new spanning set

~~A new spanning set~~

$\hookrightarrow$

↳ Applying this over and over, we'll insert all the u's into the set $\{w_1, \dots, w_k\}$ without changing its size.

In the end, we'll have $\{u_1, \dots, u_\ell\} \subseteq \{u\text{'s and } w\text{'s}\}$

↳ size of this is k.

So $\ell \leq k$.

Lemma:

Let $\{u_1, \dots, u_\ell\}$ be lin. indep, $\{w_1, \dots, w_k\}$ span.

Suppose we can form a spanning set, $\{w_1, \dots, w_{k-m}, u_1, \dots, u_m\}$ for some $m \leq \ell$. (* the size is k)

Then we can ~~insert some~~ replace w_i for som $1 \leq i \leq k-m$ with some u_j for $\ell-m \leq j \leq \ell$ and span.

Prf:

If $m = \ell$, we are done! (There is no u_i's left).

So we can assume $m < \ell$. Consider the set $\{w_1, \dots, w_{k-m}, u_1, \dots, u_m, u_{m+1}\}$

Claim: u_{m+1} can be expresed as a lin. comb of $\{w_1, \dots, w_{k-m}, u_1, \dots, u_m\}$ s.t. some coefficient of a "w" is non-trivial.

Prf of claim: $\{w_1, \dots, w_{k-m}, u_1, \dots, u_m\}$ spans V. So u_{m+1} is a lin. comb

$$u_{m+1} = \lambda_1 w_1 + \dots + \lambda_{k-m} w_{k-m} + \gamma_1 u_1 + \dots + \gamma_m u_m$$

Can $\{\lambda_1, \dots, \lambda_{k-m}, \gamma_1, \dots, \gamma_m\}$ all be zero?

NO, u_{m+1} is part of the lin. idep set $\{u_1, \dots, u_\ell\}$

So it can't be zero.

↳

Feb 13th

4

↳ Can $\{\lambda_1, \dots, \lambda_{k-m}\}$ all be zero?

No! If so we obtain the eqn $u_{m+1} = \gamma_1 u_1 + \dots + \gamma_m u_m$

But then $0 = \gamma_1 u_1 + \dots + \gamma_m u_m + (-1) u_{m+1}$ so $u_1, \dots, u_{m+1}$ are lin. dep. $\rightarrow\leftarrow$

So $\lambda_i \neq 0$ for some i.

This proves the claim.

* By re indexing, we may assume $\lambda_{k-m} \neq 0$. So we have:

$$u_{m+1} = \lambda_1 w_1 + \dots + \lambda_{k-m} w_{k-m} + \gamma_1 u_1 + \dots + \gamma_m u_m.$$

*Claim:

w_{k-m} can be replaced by u_{m+1} in the set $\{w_1, \dots, w_{k-m}, u_1, \dots, u_m\}$ to form a new spanning set of the same size.

Prf of claim:

$$-\lambda_{k-m} w_{k-m} = \lambda_1 w_1 + \dots + \lambda_{k-m-1} w_{k-m-1} + \gamma u_1 + \dots + \gamma_m u_m - u_{m+1}$$

Since $\lambda_{k-m} \neq 0$,

$$w_{k-m} = -1/\lambda_{k-m} \left(\lambda_1 w_1 + \dots + \gamma_m u_m - u_m \right)$$

i.e. $w_{k-m} \in \text{Span}\{w_1, \dots, w_{k-(m+1)}, u_1, \dots, u_{m+1}\}$

Q: Does span $\{w_1, \dots, w_{k-(m+1)}, u_1, \dots, u_{m+1}\}$ equal V?

Yes. Span $\{w_1, \dots, w_{k-m}, u_1, \dots, u_m\}$ is V.

Given $v \in V$ $\exists$ $\alpha_1, \dots, \alpha_{k-m}, \beta_1, \dots, \beta_m$ s.t.

$$v = \sum_{i=1}^{k-m} \alpha_i w_i + \sum_{i=1}^{m} \beta_i u_i = \sum_{i=1}^{k-(m+1)} \alpha_i w_i + \sum_{i=1}^{m} \beta_i u_i + \alpha_{k-m} w_{k-m}$$

We showed $w_{k-m} = \sum_{i=1}^{k-(m+1)} a_i w_i + \sum_{i=1}^{m+1} b_i u_i$

So $v = \sum_{i=1}^{k-(m+1)} \alpha_i w_i + \sum_{i=1}^{m} \beta_i u_i + \alpha_{k-m} \left(\sum_{i=1}^{k-m+1} a_i w_i + \sum_{i=1}^{m+1} b_i u_i \right)$

So v is a lin. comb of elements in $\{w_1, \ldots, w_{k-(m+1)}, u_1, \ldots, u_{m+1}\}$
Since v was arbitrary, $V =$ Span $\{w_1, \ldots, w_{k-(m+1)}, u_1, \ldots, u_{m+1}\}$

$\square$

We've shown whenever we have a set of the form $\{w_1, \ldots, w_{k-m}, u_1, \ldots u_m\}$

if there is another u_i, we can add it and remove a w.

$\{w_1, \ldots, k\} \rightsquigarrow \{w_1, \ldots, w_{k-1}, u_1\} \rightsquigarrow$ we can keep going, whenever there are u's left.

$\rightsquigarrow \{w_1, \ldots, w_{k-\ell+1}, u_1, \ldots, u_{\ell-1}\} \rightsquigarrow \{w_1, \ldots, w_{k-\ell}, u_1, \ldots, u_\ell\}$

Since all of these are sets of size k, $\ell \leq k$ $\square$