Seminar 10

Monday, December 4, 2023 2:06 PM

Spații vectoriale. Subspații vectoriale

Def: Fie $(K, +, \cdot)$ un corp comutativ

Un grup abelian $(V, +)$ împreună cu o operație externă $\cdot : K \times V \to V$ a.î.:
s.n. K spațiu vectorial

(SV1) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

(SV2) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$

(SV3) $(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$

(SV4) $1 \cdot x = x$

$\forall \alpha, \beta \in K$ scalari

$\forall x, y \in V$ vectori

$\boxed{T}$ Caracterizarea subspațiilor vectoriale

Fie V un K-sp vectorial și $U \subseteq V$

(i) $U \le_K V$

(ii) $0 \in U$ (0 el neutru al lui V)

$\forall x, y \in U \Rightarrow x + y \in U$

$\forall \alpha \in K$ și $x \in U \Rightarrow \alpha \cdot x \in U$

(iii) $0 \in U$

$\left.\begin{array}{l}\forall \alpha, \beta \in K \\ \forall x, y \in U\end{array}\right\} \Rightarrow \alpha \cdot x + \beta \cdot y \in U$

Def: Fie V un K-sp vectorial și $X \subseteq V$

Subsp. generat de X este:

$$\langle X \rangle = \bigcap_{X \subseteq U \le_K V}$$

$\langle X \rangle = \bigcap\limits_{X \subseteq U \leq_K V}$

sau = cel mai mic subsp. vectorial al lui V care conține mulțimea X

sau = $\{\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\}$ pentru $n \in \mathbb{N}$

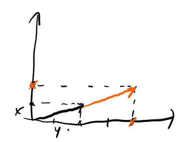
combinație liniară $\quad \alpha_1, \alpha_2, \ldots \alpha_n \in K$

$x_1, x_2, \ldots, x_n \in X$

3.1.31 $(\mathbb{R}_+^*, \boxplus) \xrightarrow{?}$ este grup abelian (imediat deoarece avem $(\mathbb{R}_+^*, \cdot)$)

adunarea vectorilor

• Verificăm axiomele spațiului vectorial:



$2 \cdot (x+y) = 2x + 2y$

(SV_1) $\alpha \boxdot (x \boxplus y) \overset{?}{=} (\alpha \boxdot x) \boxplus (\alpha \boxdot y)$

dem: $\alpha \boxdot (x \boxplus y) = \alpha \boxdot (x \cdot y) = (x \cdot y)^\alpha = x^\alpha \cdot y^\alpha = (\alpha \boxdot x) \cdot (\alpha \boxdot y)$

(SV_2) $(\alpha + \beta) \boxdot x = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$

dem: $(\alpha+\beta) \boxdot x = x^{\alpha+\beta} = x^\alpha \cdot x^\beta = (\alpha \boxdot x) \cdot (\beta \boxdot x) = (\ ) \boxplus (\ )$

(SV_3) $(\alpha \cdot \beta) \boxdot x \overset{?}{=} \alpha \boxdot (\beta \boxdot x)$

dem: $(\alpha \cdot \beta) \boxdot x = x^{\alpha \cdot \beta} = (x^\beta)^\alpha = (\beta \boxdot x)^\alpha = \alpha \boxdot (\beta \boxdot x)$

(SV_4) $1 \boxdot x \overset{?}{=} x$

dem: $1 \boxdot x = x^1 = x$

Deci $\mathbb{R}_+^*$ este un $\mathbb{R}$-sp. vectorial

3.1.32 $(\mathbb{R}, +, \cdot)$ este un corp scalar

$x \boxplus y = \sqrt[5]{x^5 + y^5}$

$\alpha \boxdot x = \sqrt[5]{\alpha} \cdot x$

Verificăm $(\mathbb{R}, \boxplus) \overset{?}{-}$ grup abelian TEMĂ

• SV_1 $\quad \alpha \boxdot (x \boxplus \beta) \overset{?}{=} (\alpha \boxdot x) \boxplus (\beta \boxdot y)$

$\alpha \boxdot (x \boxplus y) = \alpha \boxdot \sqrt{x^5 + y^5} = \sqrt[5]{\alpha} \cdot \sqrt[5]{x^5 + y^5} = \sqrt[5]{\alpha \cdot x^5 + \alpha \cdot y^5}$

$= \sqrt[5]{\left(\sqrt[5]{\alpha \cdot x^5}\right)^5 + \left(\sqrt[5]{\alpha \cdot y^5}\right)^5} = \sqrt[5]{(\sqrt[5]{\alpha} \cdot x)^5 + (\sqrt[5]{\alpha} \cdot y)^5} = (\sqrt[5]{\alpha}\, x) + (\sqrt[5]{\alpha} \cdot y) = (\alpha \boxdot x) \boxplus (\alpha \boxdot y)$

• SV_2 $\quad (\alpha + \beta) \boxdot x = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$

$(\alpha + \beta) \boxdot x = \sqrt[5]{\alpha + \beta} \cdot x = \sqrt[5]{(\alpha + \beta) \cdot x^5} = \sqrt[5]{\alpha \cdot x^5 + \beta \cdot x^5} = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$

• SV_3 $\quad ?$

$$\ldots = \sqrt[5]{\alpha+\beta} \cdot x = \sqrt[5]{(\alpha+\beta)\cdot x^5} = \sqrt[5]{\alpha \cdot x^5 + \beta \cdot x^5} = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$$

- SV_3 $\quad (\alpha \cdot \beta) \boxdot x \overset{?}{=} \alpha \boxdot (\beta \boxdot x)$

$$(\alpha \cdot \beta) \boxdot x = \sqrt[5]{\alpha \cdot \beta} \cdot x = \sqrt[5]{\alpha} \cdot (\sqrt[5]{\beta} \cdot x) = \alpha \boxdot (\beta \boxdot x)$$

- $SV4$ $\quad 1 \boxdot x \overset{?}{=} x$

$$1 \boxdot x = \sqrt[5]{1} \cdot x = 1 \cdot x = x$$

$\Rightarrow \mathbb{R}$ este $\mathbb{R}$-sp. vectorial

Reamintim

$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$+ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \qquad (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$\cdot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \qquad \alpha \cdot (x_1, x_2) = (\alpha \cdot x_1, \alpha \cdot x_2)$

$\mathbb{R}^2$ este un $\mathbb{R}$-sp. vectorial

3.1.33

$0 \notin B \Rightarrow B \not\leq_{\mathbb{R}} \mathbb{R}^3$ (B nu e subspațiu)

$0 \notin E \Rightarrow E \not\leq_{\mathbb{R}} \mathbb{R}^3$

- $0 \in A$

Fie $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in A \qquad x + y \overset{?}{\in} A$

$2x_1 + x_2 - x_3 = 0 \qquad 2y_1 + y_2 - y_3 = 0$

$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \overset{?}{\in} A$

$$2(x_1 + y_1) + (x_2 + y_2) - (x_3 + y_3) = \underbrace{2x_1 + x_2 - x_3}_{0} + \underbrace{2y_1 + y_2 - y_3}_{0} = 0 + 0 = 0$$

$\Rightarrow x + y \in A$

Fie $\alpha \in \mathbb{R}$, $x \in A \qquad \alpha \cdot x \overset{?}{\in} A$

$(\alpha x_1, \alpha x_2, \alpha x_3) \overset{?}{\in} A$

$$2(\alpha x_1) + \alpha x_2 - \alpha x_3 = \alpha(2x_1 + x_2 - x_3) = \alpha \cdot 0 = 0 \Rightarrow \alpha \cdot x \in A$$

$\Rightarrow A \leq \mathbb{R}^3$

- $0 = 0 = 0 \Rightarrow (0, 0, 0) \in C$

Fie $\alpha, \beta \in \mathbb{R}$; $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in C$

$\alpha x_1 + \beta y_1 = \alpha x_2 + \beta y_2 = \alpha x_3 + \beta y_3 \Rightarrow \alpha x + \beta y \in C$

$\Rightarrow C \leq_{\mathbb{R}} \mathbb{R}^3$

$\alpha x_1 + \beta y_1 = \alpha x_2 + \beta y_2 = \alpha x_3 + \beta y_3 \Rightarrow \alpha x + \beta y \in C$

$\Rightarrow C \leq_{\mathbb{R}} \mathbb{R}^3$

- $0 \in D$ poate fi oricât

Fie $x = (1, -1, 0) \in D$, $y = (2, -4, 1) \in D$ | Dar $x + y = (3, -5, 1) \notin D \Rightarrow D \not\leq_{\mathbb{R}} \mathbb{R}^3$

$3^2 - 5 = 0 \Leftrightarrow 9 - 5 = 0 \Leftrightarrow 4 = 0$ fals

- $0 \in F$

$2x_1 + x_2 - x_3 \neq 0$

$x = (1, 1, 1) \in F$, $y = (1, 1, 5) \in F$ | Dar $x + y = (2, 2, 6) \neq 0$

$2 \cdot 2 + 2 - 6 = 0 \Rightarrow x + y \notin \mathbb{R}^3 \setminus A$

$\Rightarrow x + y \notin F \Rightarrow F \leq_{\mathbb{R}} \mathbb{R}^3$

<u>3.1.35</u>

$S = \langle (1, 2, -1) \rangle$

$x \in S \Leftrightarrow x = t \cdot (1, 2, -1)$ pt $t \in \mathbb{R}$

$(x_1, x_2, x_3) = (t, 2t, -t)$

$$\Leftrightarrow \begin{cases} x_1 = t \\ x_2 = 2t \\ x_3 = -t \end{cases} \Leftrightarrow \begin{array}{l} x_2 = 2x_1 \\ x_3 = -x_1 \end{array}$$

$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 2x_1,\ x_3 = -x_1\}$

$T = \langle [1, 2, 1], [-2, 1, -3] \rangle$

$\Rightarrow x \in T \Leftrightarrow x = \alpha \cdot (1, 2, 1) + \beta \cdot (-2, 1, -3)$

$\Leftrightarrow (x_1, x_2, x_3) = (\alpha - 2\beta, 2\alpha + \beta, \alpha - 3\beta)$

$$\begin{cases} x_1 = \alpha - 2\beta \\ x_2 = 2\alpha + \beta \\ x_3 = \alpha - 3\beta \end{cases} \Leftrightarrow \begin{cases} \alpha - 2\beta = x_1 \Rightarrow \alpha = x_1 + 2\beta \\ 2\alpha + \beta = x_2 \\ \alpha - 3\beta = x_3 \end{cases}$$

$$\Rightarrow \begin{cases} 2x_1 + 5\beta = x_2 \\ x_1 - \beta = x_3 \Rightarrow \beta = x_1 - x_3 \end{cases} \Leftrightarrow 7x_1 - 5x_3 = x_2$$

$\Rightarrow T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 7x_1 - x_2 - 5x_3 = 0\}$

<u>3.1.36</u>

$S = \{ \text{— // —} \quad x_1 - x_2 - x_3 = 0\}$

$x_1 = t$, $x_2 = s \Rightarrow x_3 = t - s$

$(x_1, x_2, x_3) = (t, s, t - s) = (t, 0, t) + (0, s, -s)$

$(x_1, x_2, x_3) = (t, s, t-s) = (t, 0, t) + (0, s, -s)$

$= t(1, 0, 1) + s(0, 1, -1) \in \langle (1, 0, 1), (0, 1, -1) \rangle$

$T = \{ -\text{"}- \quad x_1 - x_2 = x_2 - x_3 = x_3 - x_1 \}$

$\begin{cases} x_1 - x_2 = x_2 - x_3 \\ x_2 - x_3 = x_3 - x_1 \end{cases} \Leftrightarrow \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$

$x_3 = t \quad \begin{cases} x_1 - 2x_2 = t \\ x_1 + x_2 = 2t \end{cases}$

$3x_2 = 3t \Leftrightarrow x_2 = t \qquad x_1 = t$

$\Rightarrow (x_1, x_2, x_3) = (t, t, t) = t \cdot (1, 1, 1) \in \langle (1, 1, 1) \rangle$

$T = \langle (1, 1, 1) \rangle$