Ray-Sphere collision

Each point (x, y, z) on the surface of a sphere of radius R centered at the origin can be written as:

$$x^2 + y^2 + z^2 = R^2 \tag{1}$$

We can also write an equivalent expression using vector notation where $\vec{P} = (x, y, z)$:

$$|\vec{P}| = R \Rightarrow |\vec{P}|^2 = R^2 \tag{2}$$

With the sphere instead being centered at some arbitrary point $C = (C_x, C_y, C_z)$, each point on its surface can be written as:

$$(x - C_x)^2 + (y - C_y)^2 + (z - C_z)^2 = R^2 \tag{3}$$

We can again write an equivalent expression using vector notation:

$$|\vec{P} - \vec{C}| = R \Rightarrow |\vec{P} - \vec{C}|^2 = R^2 \tag{4}$$

Each point on a ray, sent from a point $\vec{O} = (O_x, O_y, O_z)$ on the camera film, in a direction $\vec{D} = (D_x, D_y, D_z)$, can be written as follows:

$$\vec{O} + t\vec{D}, \quad t \in \mathbb{R} \tag{5}$$

Substituting expression 5 for P in equation 4 gives us:

$$|(\vec{O} + t\vec{D}) - \vec{C}|^2 = R^2 \tag{6}$$

Expanding the vector $(\vec{O} + t\vec{D}) - \vec{C}$:

$$|(O_x + tD_x - C_x, O_y + tD_y - C_y, O_z + tD_z - C_z)|^2 = R^2 \tag{7}$$

By the definition of Euclidean vector magnitude:

$$\sqrt{(O_x + tD_x - C_x)^2 + (O_y + tD_y - C_y)^2 + (O_z + tD_z - C_z)^2}^2 = R^2 \tag{8}$$

Squaring the square root:

$$(O_x + tD_x - C_x)^2 + (O_y + tD_y - C_y)^2 + (O_z + tD_z - C_z)^2 = R^2 \tag{9}$$

Computing the powers on the left side:

$$\begin{aligned} O_x^2 + 2O_x tD_x + D_x^2 - 2O_x C_x - 2D_x C_x + C_x^2 \\ +O_y^2 + 2O_y tD_y + D_y^2 - 2O_y C_y - 2D_y C_y + C_y^2 \\ +O_z^2 + 2O_z tD_z + D_z^2 - 2O_z C_z - 2D_z C_z + C_z^2 = R^2 \end{aligned} \tag{10}$$

Subtracting R^2 from both sides:

$$\begin{aligned}
&O_x^2 + 2O_xtD_x + t^2D_x^2 - 2O_xC_x - 2tD_xC_x + C_x^2 \\
&+O_y^2 + 2O_ytD_y + t^2D_y^2 - 2O_yC_y - 2tD_yC_y + C_y^2 \\
&+O_z^2 + 2O_ztD_z + t^2D_z^2 - 2O_zC_z - 2tD_zC_z + C_z^2 - R^2 = 0
\end{aligned} \tag{11}$$

Factoring:

$$\begin{aligned}
&t^2(D_x^2 + D_y^2 + D_z^2) \\
&+t(2O_xD_x + 2O_yD_y + 2O_zD_z - 2D_xC_x - 2D_yC_y - 2D_zC_z) \\
&+O_x^2 + O_y^2 + O_z^2 + C_x^2 + C_y^2 + C_z^2 - 2O_xC_x - 2O_yC_y - 2O_zC_z - R^2 = 0
\end{aligned} \tag{12}$$

And so we have a quadratic equation $at^2 + bt + c = 0$ with

$$a = D_x^2 + D_y^2 + D_z^2 \tag{13}$$

$$b = 2(O_xD_x + O_yD_y + O_zD_z - D_xC_x - D_yC_y - D_zC_z) \tag{14}$$

$$c = O_x^2 + O_y^2 + O_z^2 + C_x^2 + C_y^2 + C_z^2 - 2(O_xC_x + O_yC_y + O_zC_z) - R^2 \tag{15}$$