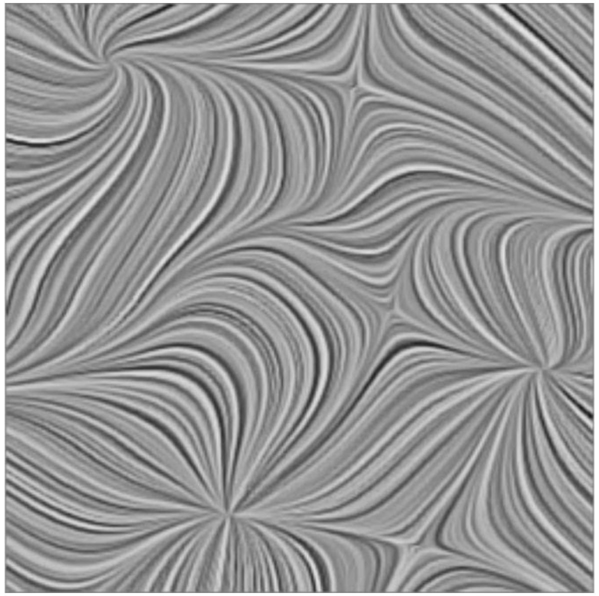
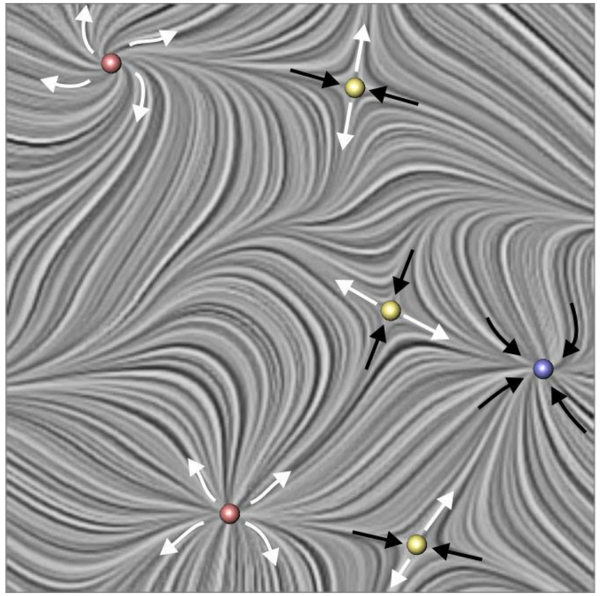
Topological Structures

steady 2D vector field

$$\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix}$$

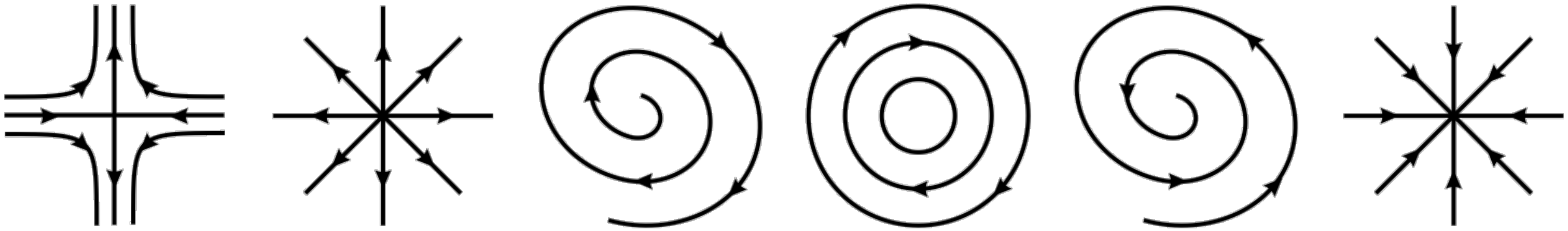


stream lines



critical points

$$\mathbf{v}(\mathbf{x}_0) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_0 \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$$

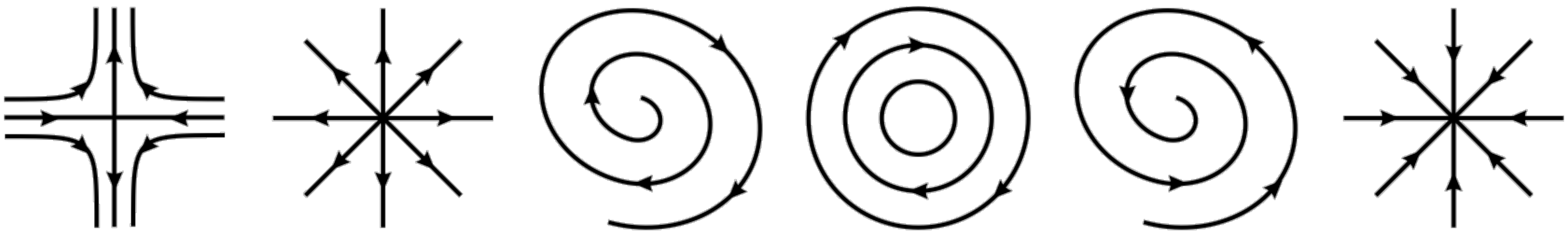


Saddle point	Repelling node	Repelling focus	Center	Attracting focus	Attracting node
$R_1 < 0, R_2 > 0$	$R_1, R_2 > 0$	$R_1 = R_2 > 0$	$R_1 = R_2 = 0$	$R_1 = R_2 < 0$	$R_1, R_2 < 0$
$I_1 = I_2 = 0$	$I_1 = I_2 = 0$	$I_1 = -I_2 \neq 0$	$I_1 = -I_2 \neq 0$	$I_1 = -I_2 \neq 0$	$I_1 = I_2 = 0$

Visualization, Tino Weinkauf, KTH Stockholm

- **First order 2D critical points**
  - The point $\mathbf{x}_0$ is a first order critical of the vector field **v** iff
  - 1) $\mathbf{x}_0$ is a critical point of **v**, and
  - 2) $\det(\mathbf{J}_\mathbf{v}(\mathbf{x}_0)) \neq 0$
  - The different types can be classified by the eigenvalues of the Jacobian:
    - R_1, R_2 ➔ real part of the eigenvalue

      I_1, I_2 ➔ imaginary part of the eigenvalue



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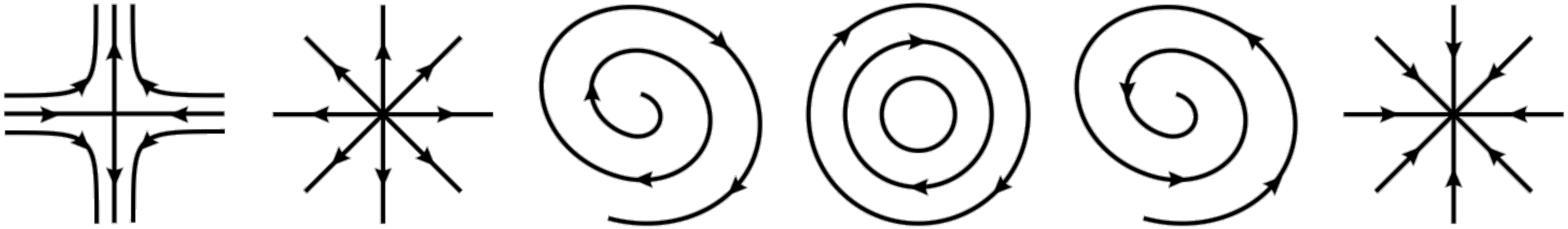

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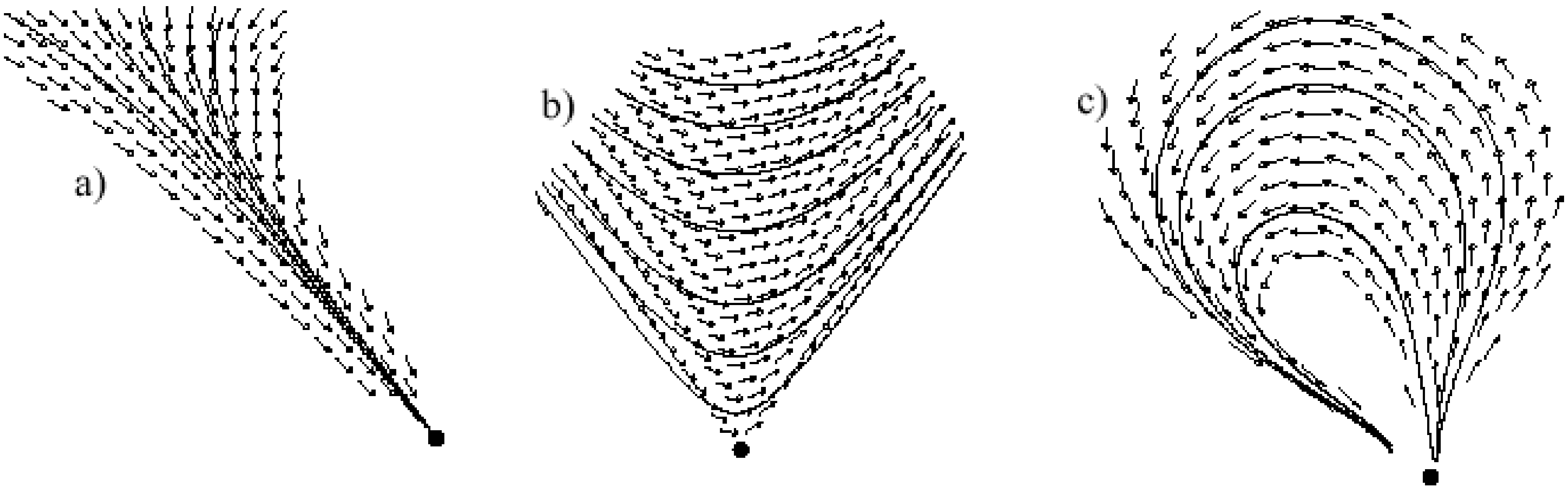
$$e_{1,2} = \frac{(u_x + v_y)}{2} \pm \sqrt{\frac{(u_x + v_y)^2}{4} - (u_x v_y - u_y v_x)}$$

$$T = u_x + v_y \quad \text{trace of Jacobian}$$

$$D = u_x v_y - u_y v_x \quad \text{determinant of Jacobian}$$

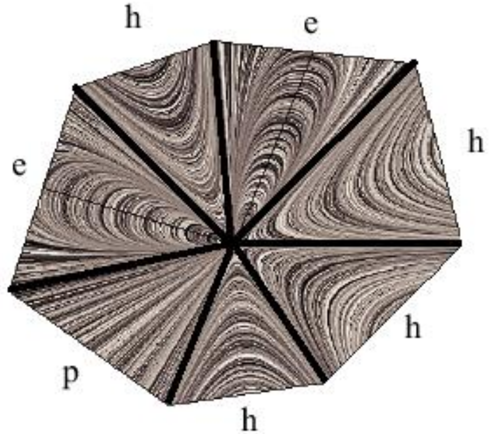
$$e_{1,2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$$

- **General classification of 2D critical points:**
  - Distinguish regions of different flow behavior around a critical point.
    3 cases are possible:
    - In a ***parabolic sector*** either all tangent curves end, or all tangent curves originate, in the critical point.
    - In a ***hyperbolic sector*** all tangent curves go by the critical point, except for two tangent curves making the boundaries of the sector. One of these two tangent curves ends in the critical point while the other one originates in it.
    - In an ***elliptic sector*** all tangent curves originate and end in the critical point.



a) parabolic sector; b) hyperbolic sector; c) elliptic sector

critical point consisting of 7 sectors



- **Poincaré-Index of a critical point:**
  - place small closed curve around critical point
  - index: number of counterclockwise rotations of the vectors of **v** while traveling counterclockwise on the closed curve
  - ➔ index is an (possibly negative) integer

- **Poincaré-Index of a critical point:**
  - For general critical point: index can be obtained by counting different sectors:

$$\text{index} = 1 + \frac{n_e - n_h}{2}$$

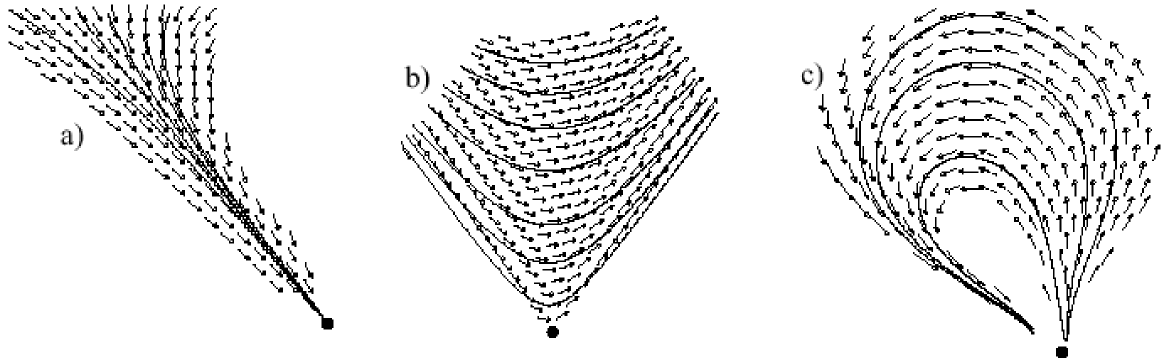
  - n_e : number of elliptic sectors
  - n_h : number of hyperbolic sectors

$$\text{index} = 1 + \frac{n_e - n_h}{2}$$

n_e : number of elliptic sectors

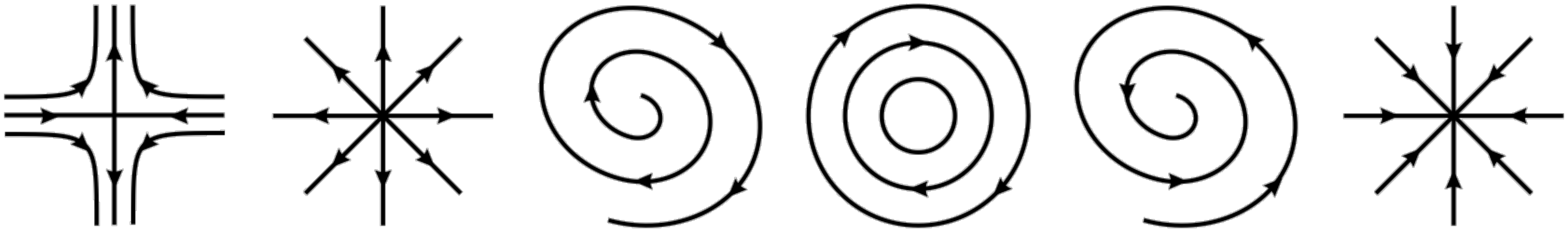
n_h : number of hyperbolic sectors

a) parabolic sector
b) hyperbolic sector
c) elliptic sector



index(saddle) = -1

index(source/sink/center) = +1



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- **Poincaré-Index of a first-order critical point:**
  - first order critical points: have index +1 or –1
  - index(saddle) = -1
  - index(source/sink/center) = +1

- **Index-theorem:**

  - index(area) = Σ index(cp)

  i.e., sum over all indices of the critical points in the area