

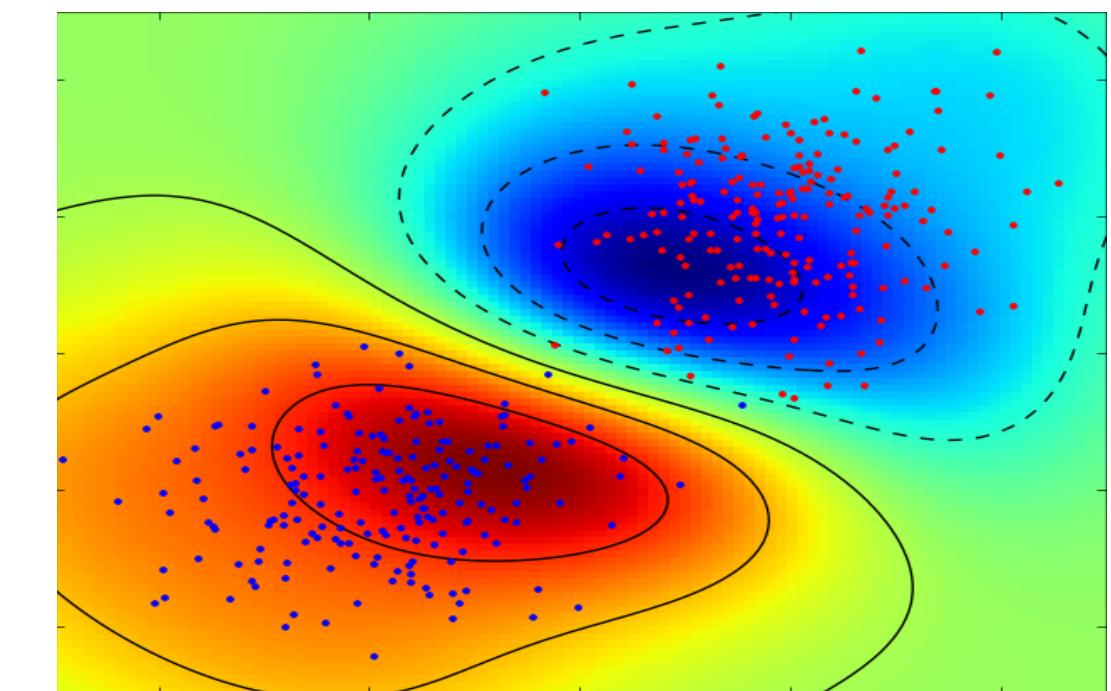
Regularization via Optimal Transport

Soroosh Shafieezadeh Abadeh, Tepper School of Business, CMU

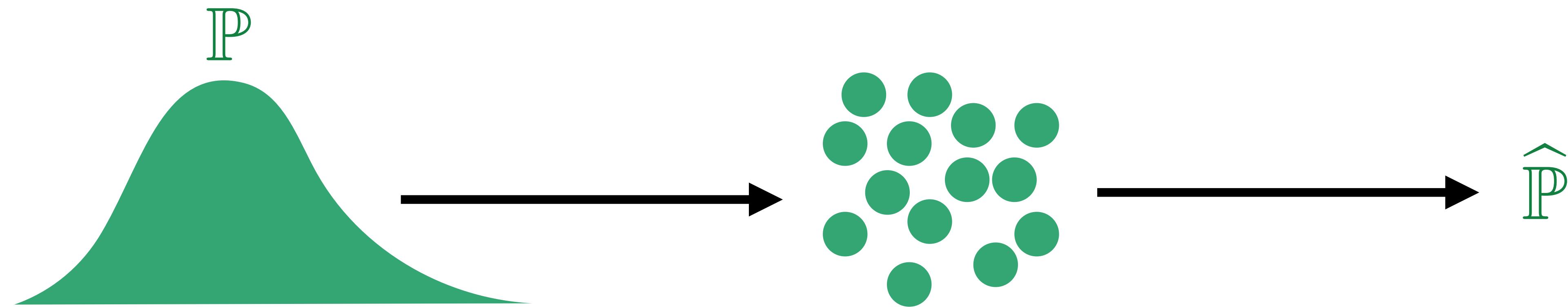


Stochastic Programming

$$\inf_{\theta \in \Theta} \mathbb{E}_{\mathbb{P}} [\ell(\theta, \xi)]$$



Failure Examples: Overfitting

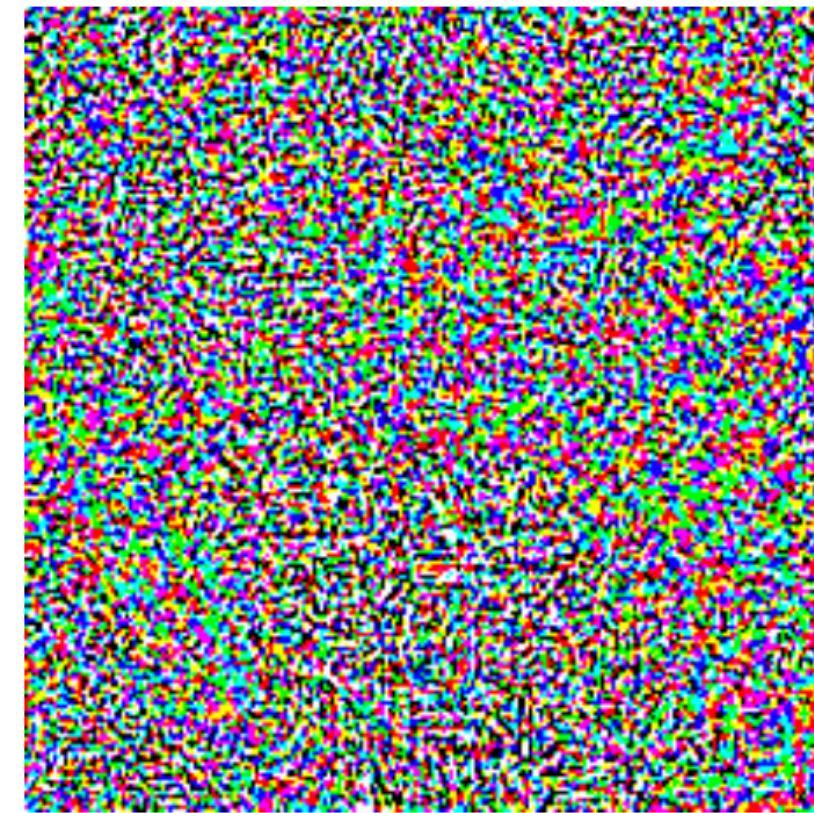


$$\inf_{\theta \in \Theta} \mathbb{E}_{\hat{P}} [\ell(\theta, \xi)]$$

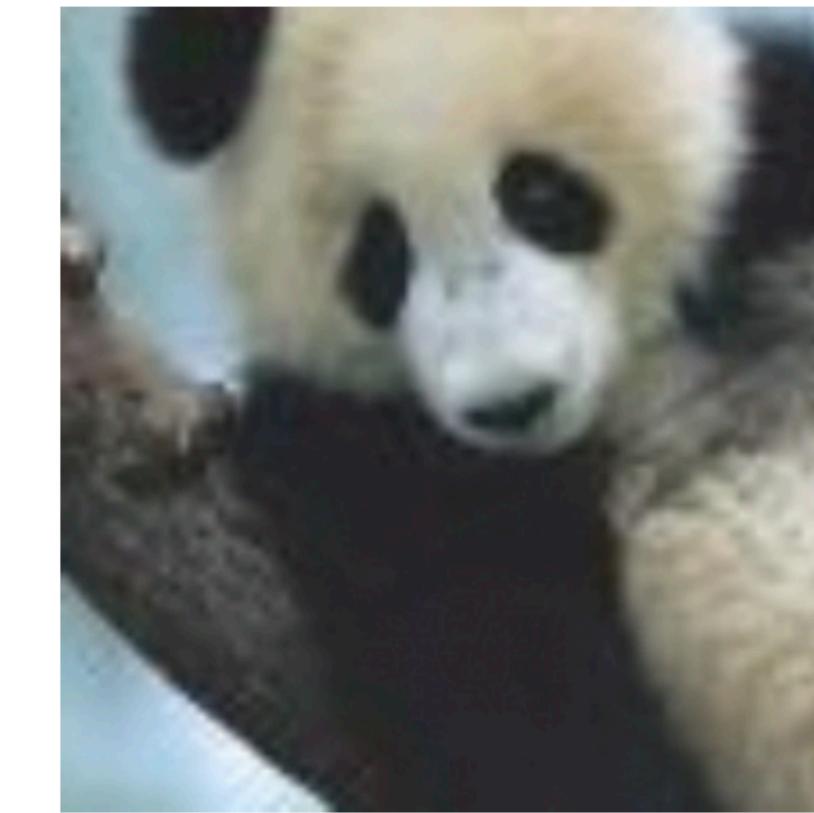
Failure Examples: Adversarial Attack [GSS15]



$+ .007 \times$



=



Panda
57.7% Confidence

Gibbon
99.3% Confidence

Failure Examples: Fake Data

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TECH | PERSONAL TECH | PERSONAL TECHNOLOGY: NICOLE NGUYEN

SHARE



Fake Reviews and Inflated Ratings Are Still a Problem for Amazon

Sellers are taking advantage of the online-shopping frenzy, using old and new methods to boost ratings on products



By [Nicole Nguyen](#)

June 13, 2021 8:28 am ET



PRINT



TEXT

191



 Listen to article (10 minutes)

A charging brick recently caught my eye on [Amazon](#). **AMZN -2.96% ▼** It was a RAVPower-branded two-port [fast charger](#), and it had five stars with over 9,800 ratings. The score seemed suspect but Amazon itself

UPCOMING EVENTS



Oct

5

2021

12:00 PM - 5:00 PM EDT

WSJ Jobs Summit

Oct

6

2021

12:30 PM - 2:00 PM EDT

The Future Of Health

Failure Examples: Domain Change [TPJR18]



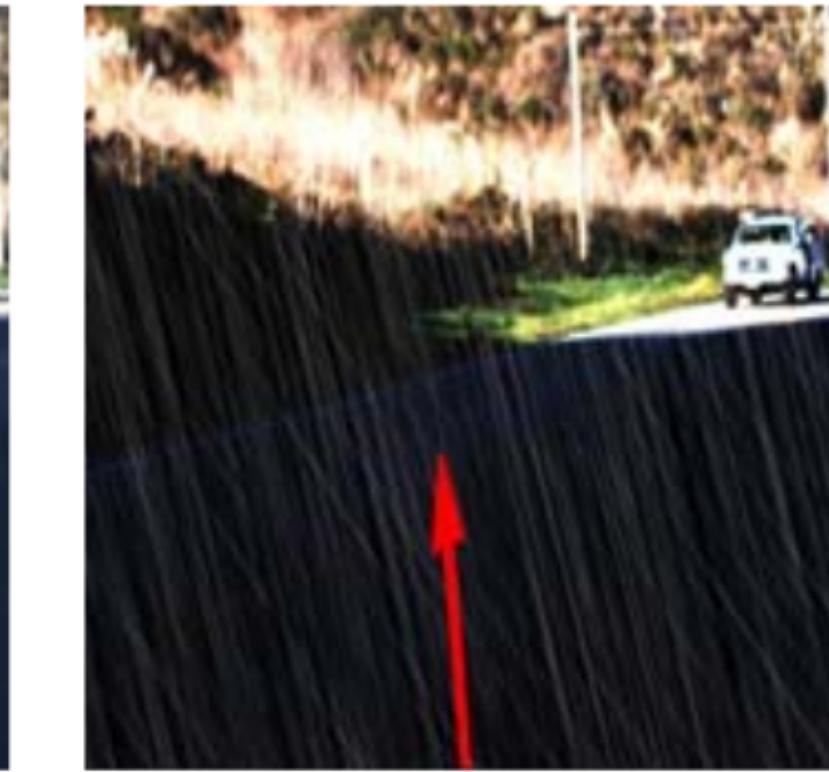
original



fog



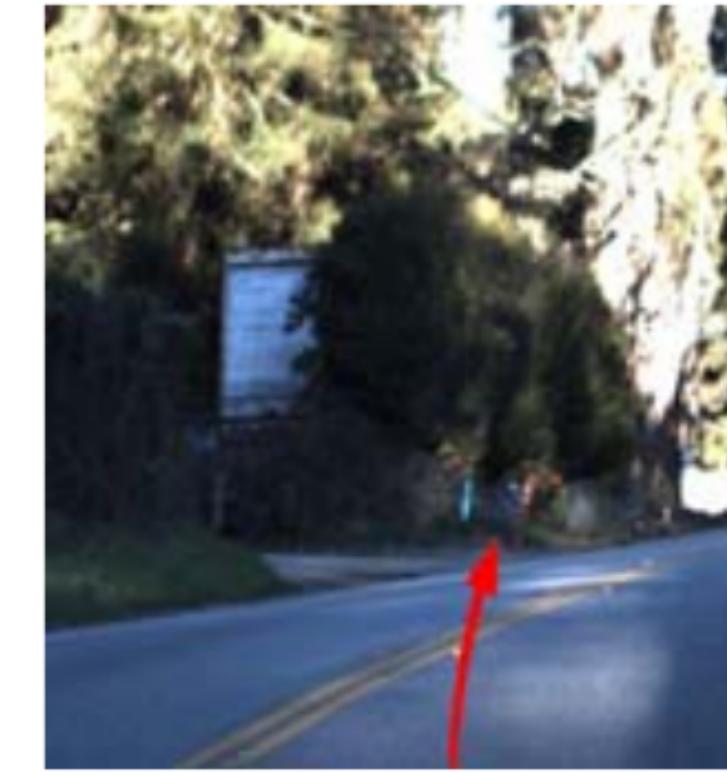
original



rain



original



shear(0.1)



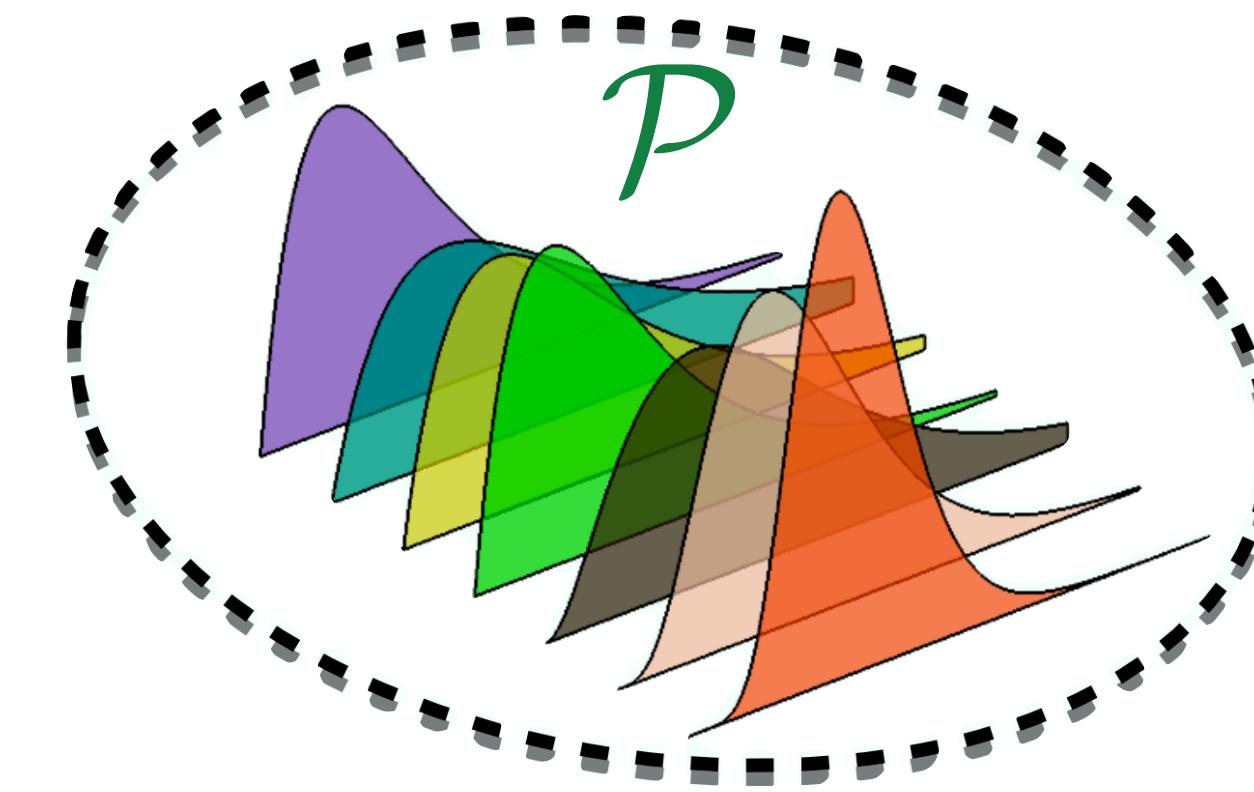
original



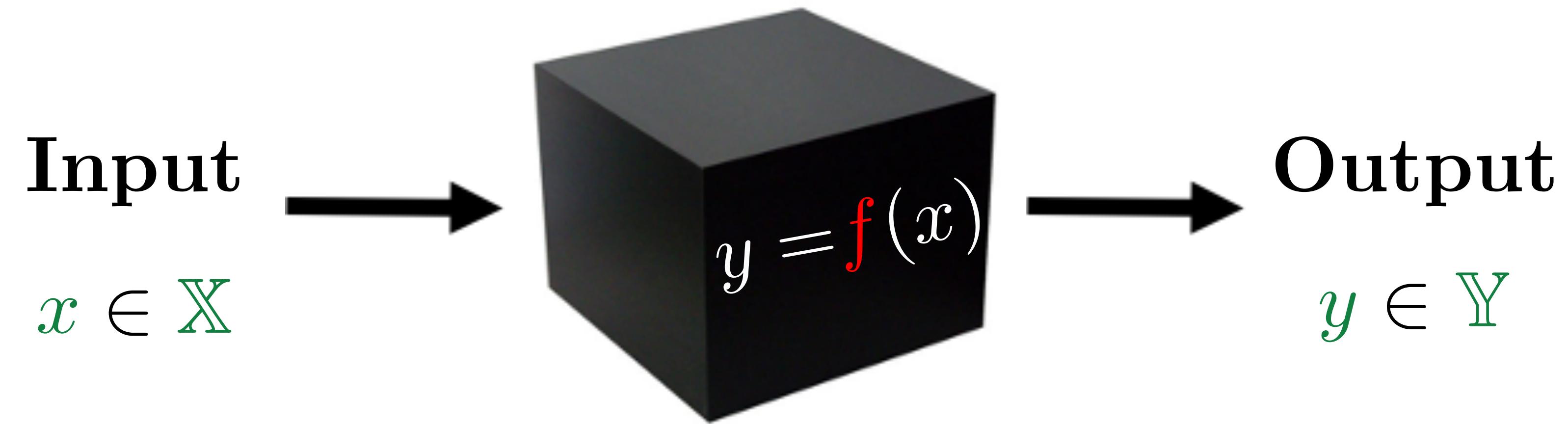
rotation(6 degree)

Distributionally Robust Optimization

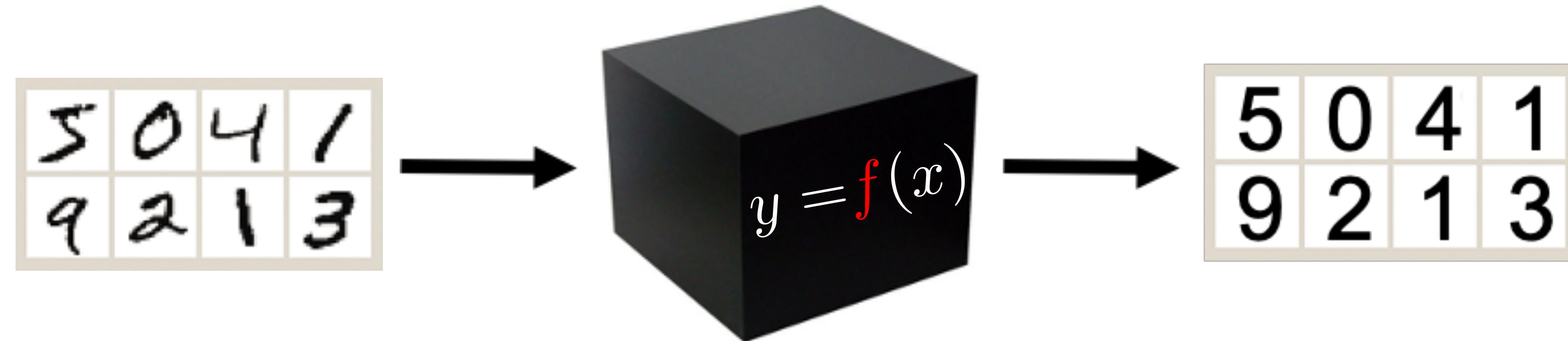
$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$



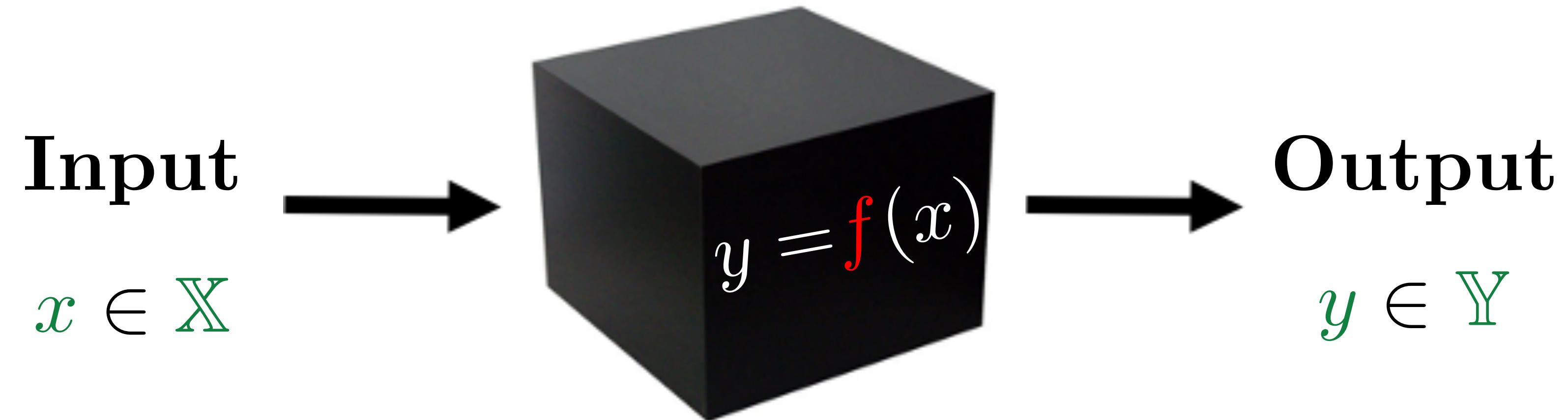
Supervised Learning



Supervised Learning

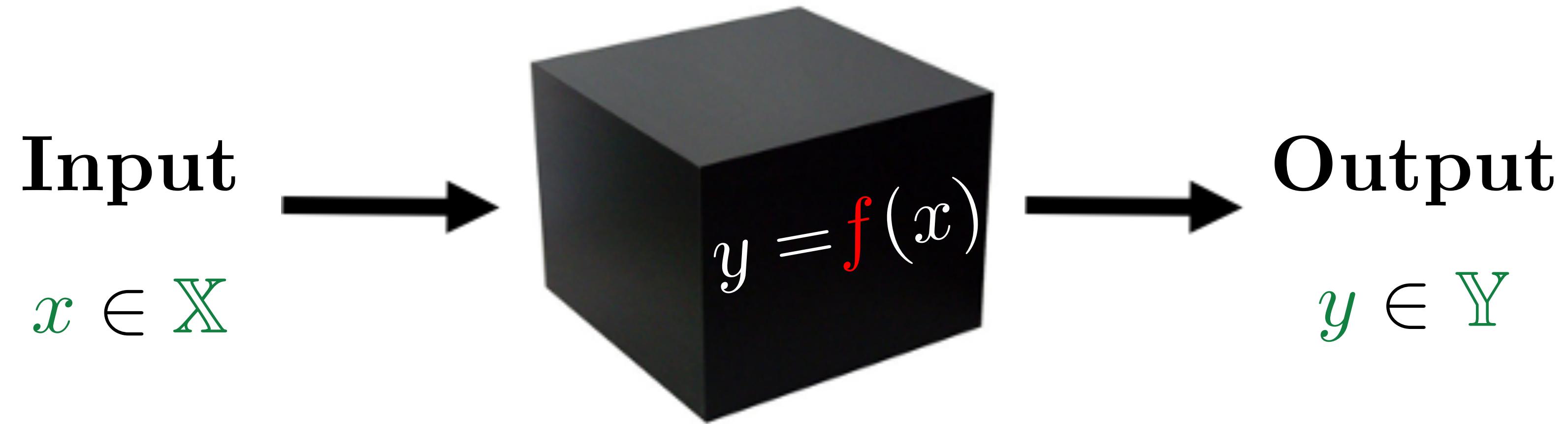


Supervised Learning



Training data: $\hat{\Xi}_N = (\hat{x}_1, \hat{y}_1), \dots, (\hat{x}_N, \hat{y}_N)$

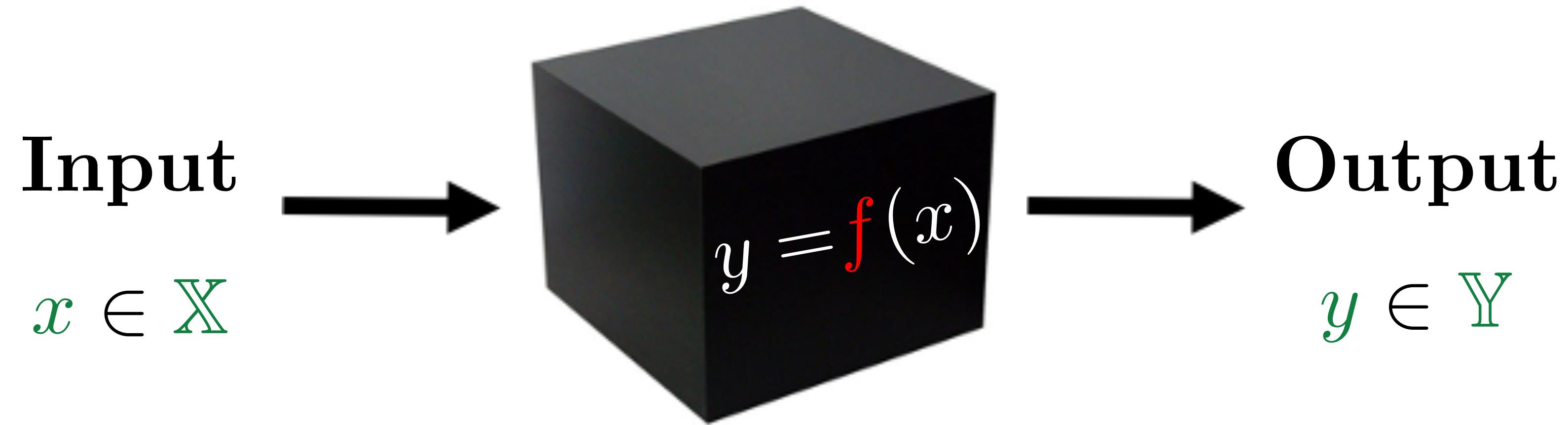
Supervised Learning



Training data: $\hat{\Xi}_N = (\hat{x}_1, \hat{y}_1), \dots (\hat{x}_N, \hat{y}_N)$

Hypothesis space: $\mathbb{H} \subseteq \{h \in \mathbb{R}^{\mathbb{X}}\}$

Supervised Learning

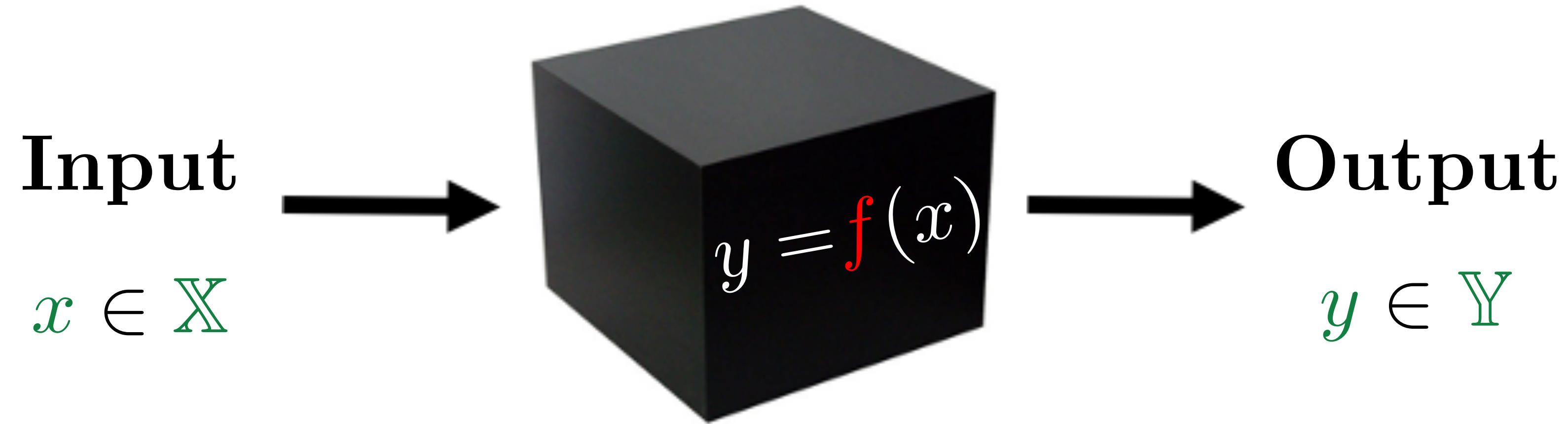


Training data: $\hat{\Xi}_N = (\hat{x}_1, \hat{y}_1), \dots, (\hat{x}_N, \hat{y}_N)$

Hypothesis space: $\mathbb{H} \subseteq \{h \in \mathbb{R}^{\mathbb{X}}\}$

Target function: $f(x) \approx h(x)$

Supervised Learning



Training data: $\widehat{\Xi}_N = (\widehat{x}_1, \widehat{y}_1), \dots (\widehat{x}_N, \widehat{y}_N)$

Hypothesis space: $\mathbb{H} \subseteq \{h \in \mathbb{R}^{\mathbb{X}}\}$

Target function: $f(x) \approx h(x)$

Learning algorithm: $\inf_{h \in \mathbb{H}} \ell(h, \widehat{\Xi}_N)$

Regression Models

Target function: $f(\textcolor{violet}{x}) = \textcolor{red}{h}(\textcolor{violet}{x})$

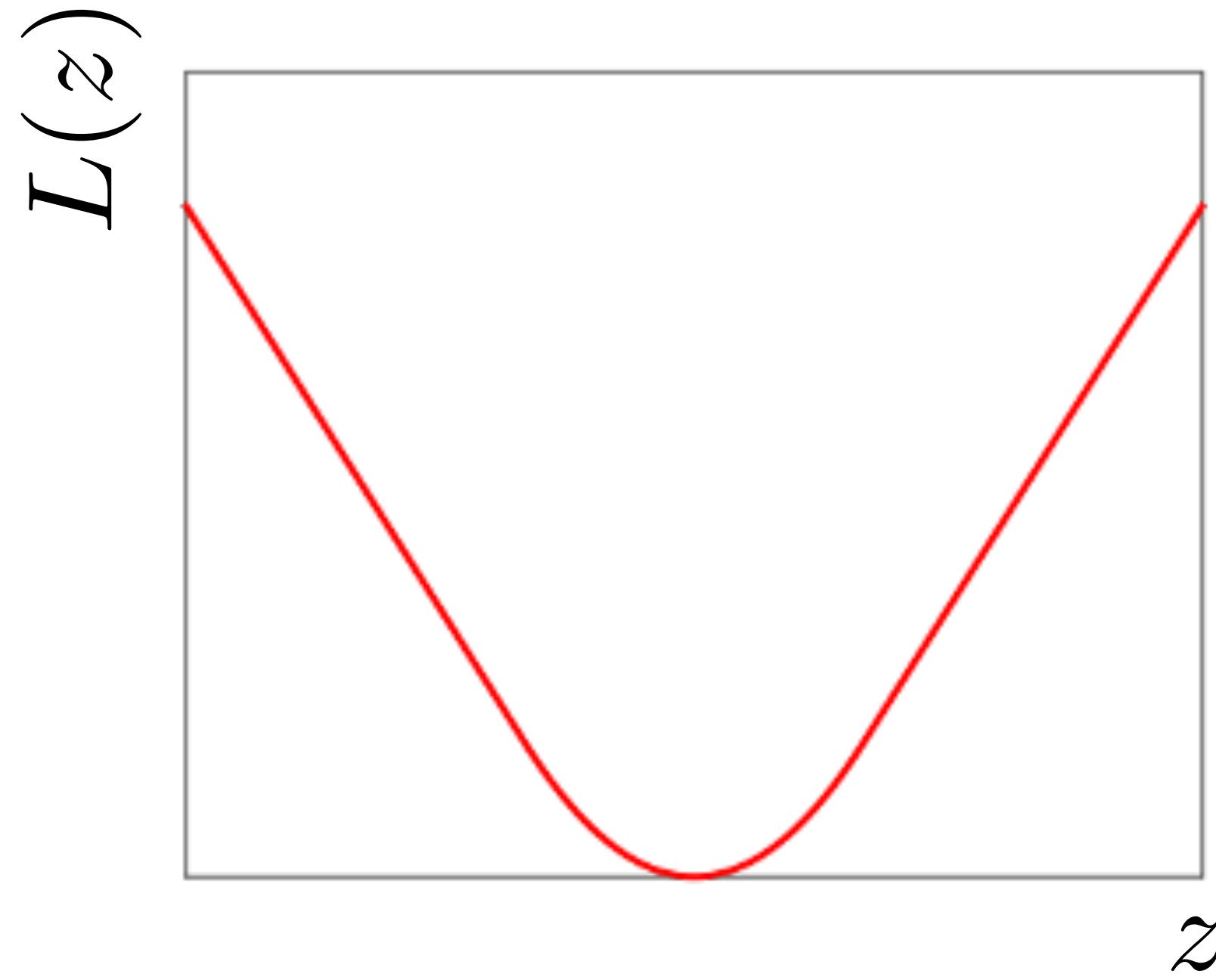
Empirical risk minimization: $\ell(\textcolor{red}{h}, \widehat{\Xi}_N) = \frac{1}{N} \sum_{i=1}^N L(\textcolor{red}{h}(\widehat{x}_i) - \widehat{y}_i)$

Regression Models

Target function: $f(\textcolor{violet}{x}) = \textcolor{red}{h}(x)$

Empirical risk minimization: $\ell(\textcolor{red}{h}, \widehat{\Sigma}_N) = \frac{1}{N} \sum_{i=1}^N L(\textcolor{red}{h}(\widehat{x}_i) - \widehat{y}_i)$

Robust Regression



Huber Loss:

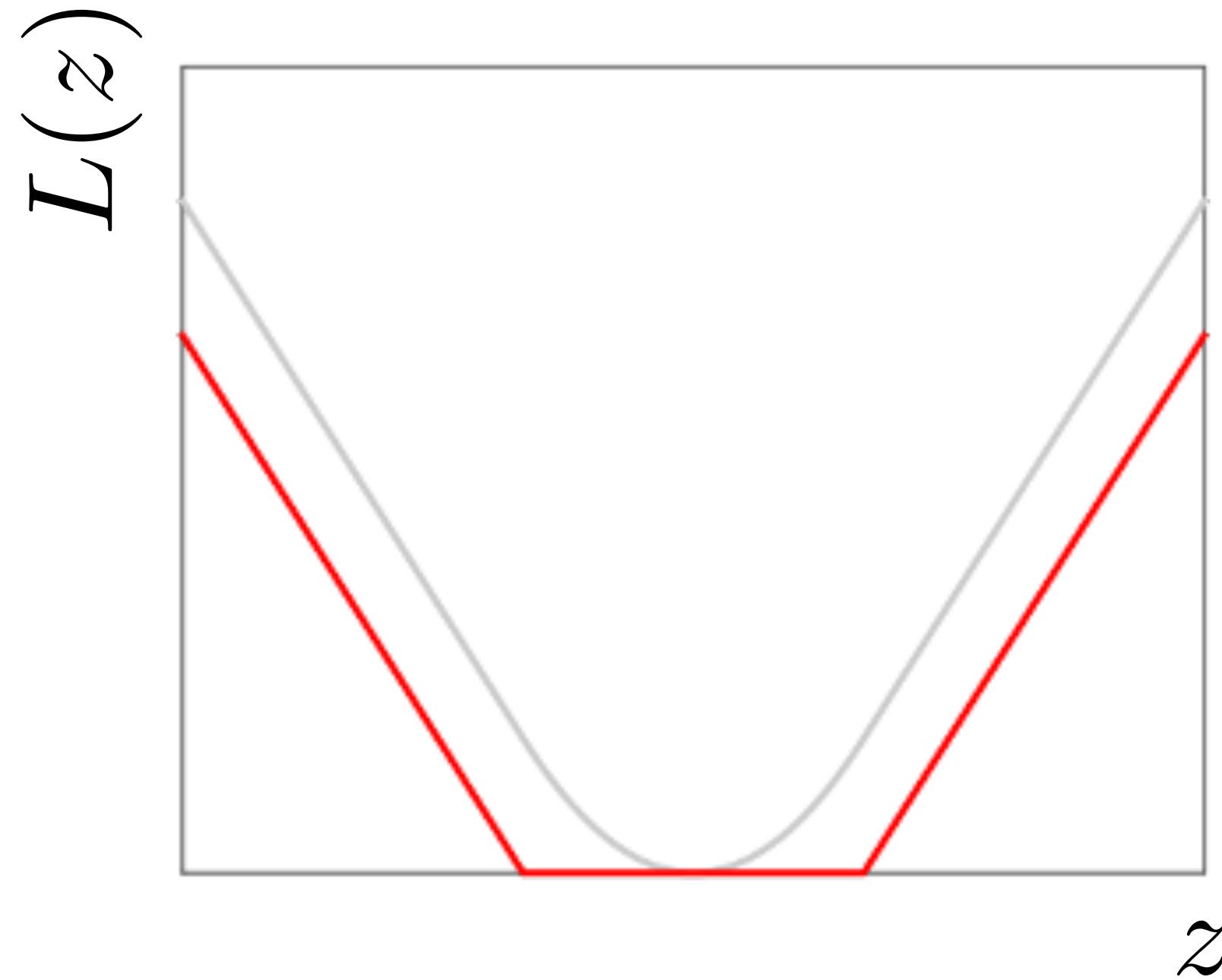
$$L(z) = \begin{cases} \frac{1}{2}z^2 & \text{if } |z| \leq \delta \\ \delta(|z| - \frac{1}{2}\delta) & \text{else} \end{cases}$$

Regression Models

Target function: $f(\textcolor{violet}{x}) = \textcolor{red}{h}(\textcolor{violet}{x})$

Empirical risk minimization: $\ell(\textcolor{red}{h}, \hat{\Xi}_N) = \frac{1}{N} \sum_{i=1}^N L(\textcolor{red}{h}(\hat{x}_i) - \hat{y}_i)$

Support Vector Regression



ε -insensitive Loss:

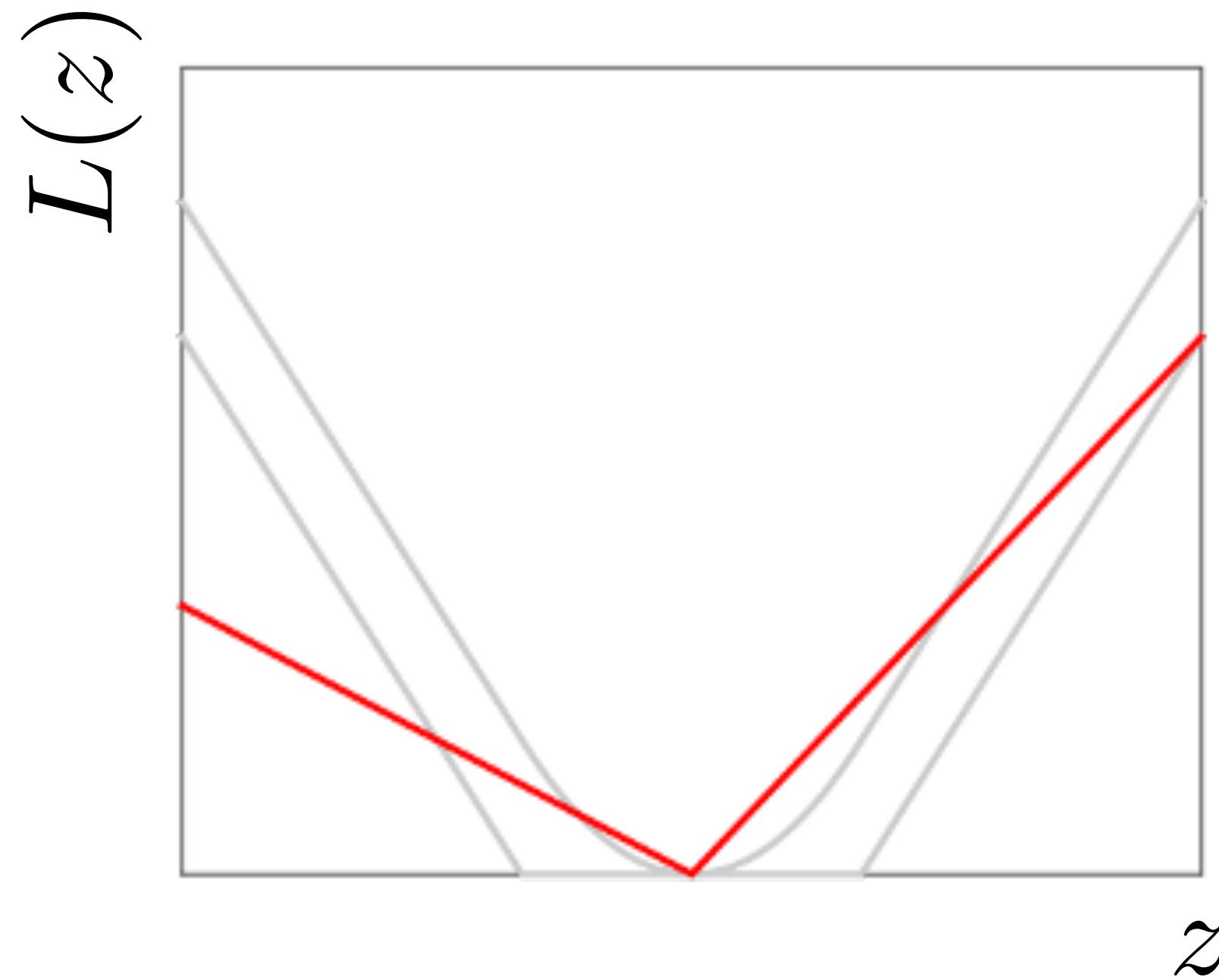
$$L(z) = \max\{0, |z| - \varepsilon\}$$

Regression Models

Target function: $f(\mathbf{x}) = h(\mathbf{x})$

Empirical risk minimization: $\ell(h, \hat{\Sigma}_N) = \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i) - \hat{y}_i)$

Quantile Regression



Pinball Loss:

$$L(z) = \max\{-\tau z, (1 - \tau)z\}$$

Classification Models

Target function: $f(\textcolor{violet}{x}) = \text{sgn}(\textcolor{red}{h}(x))$

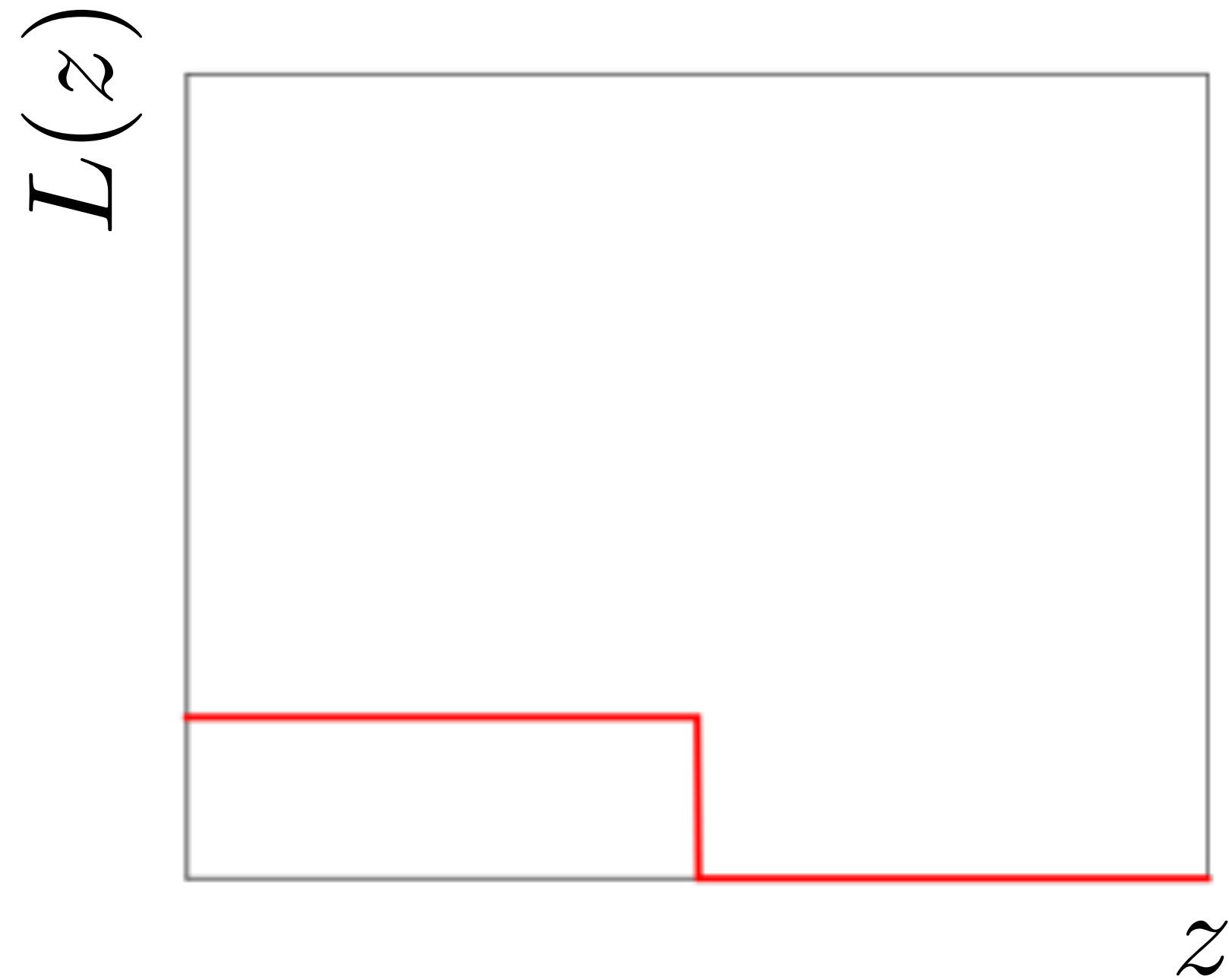
Empirical risk minimization: $\ell(\textcolor{red}{h}, \widehat{\Xi}_{\textcolor{violet}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\widehat{y}_i \textcolor{red}{h}(\widehat{x}_i))$

Classification Models

Target function: $f(\textcolor{teal}{x}) = \operatorname{sgn}(h(\textcolor{red}{x}))$

Empirical risk minimization: $\ell(\textcolor{red}{h}, \hat{\Sigma}_{\textcolor{teal}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \textcolor{red}{h}(\hat{x}_i))$

Ideal Classification



0-1 Loss:

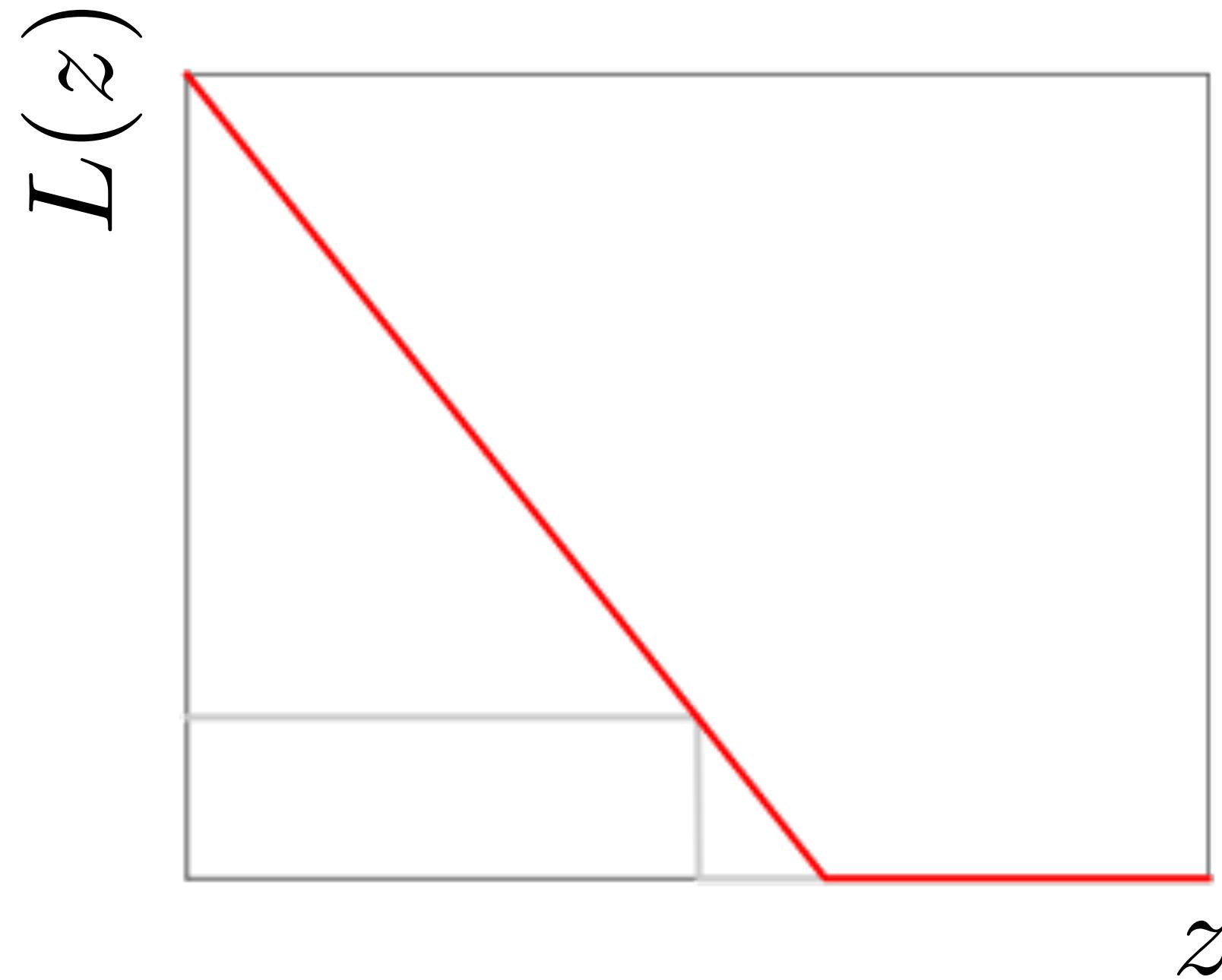
$$L(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ 0 & \text{else} \end{cases}$$

Classification Models

Target function: $f(\textcolor{teal}{x}) = \operatorname{sgn}(h(\textcolor{red}{x}))$

Empirical risk minimization: $\ell(\textcolor{red}{h}, \hat{\Xi}_{\textcolor{teal}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \textcolor{red}{h}(\hat{x}_i))$

Support Vector Machine



Hinge Loss:

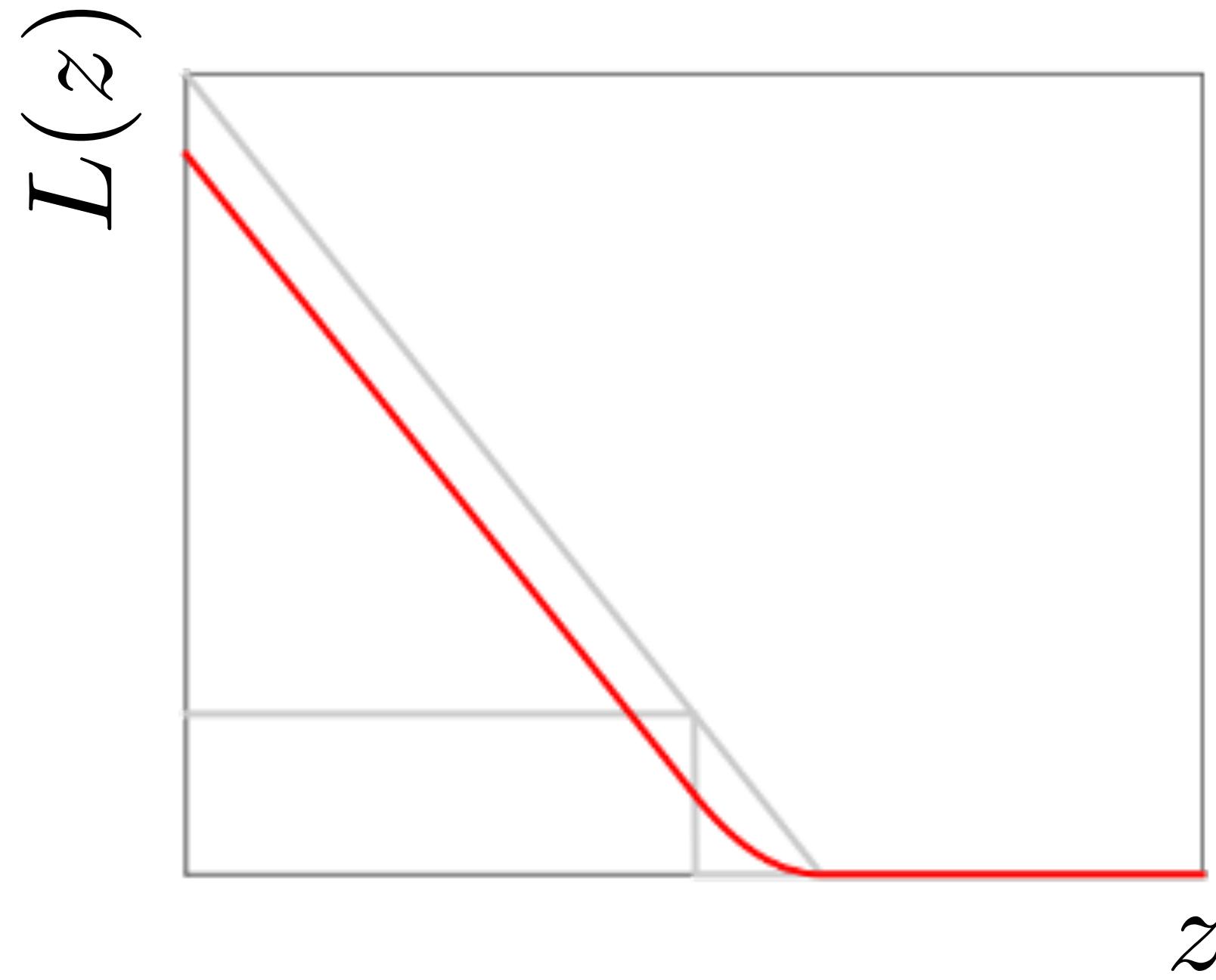
$$L(z) = \max\{0, 1 - z\}$$

Classification Models

Target function: $f(\mathbf{x}) = \text{sgn}(h(\mathbf{x}))$

Empirical risk minimization: $\ell(h, \hat{\Xi}_N) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i h(\hat{x}_i))$

Support Vector Machine II



Smooth Hinge Loss:

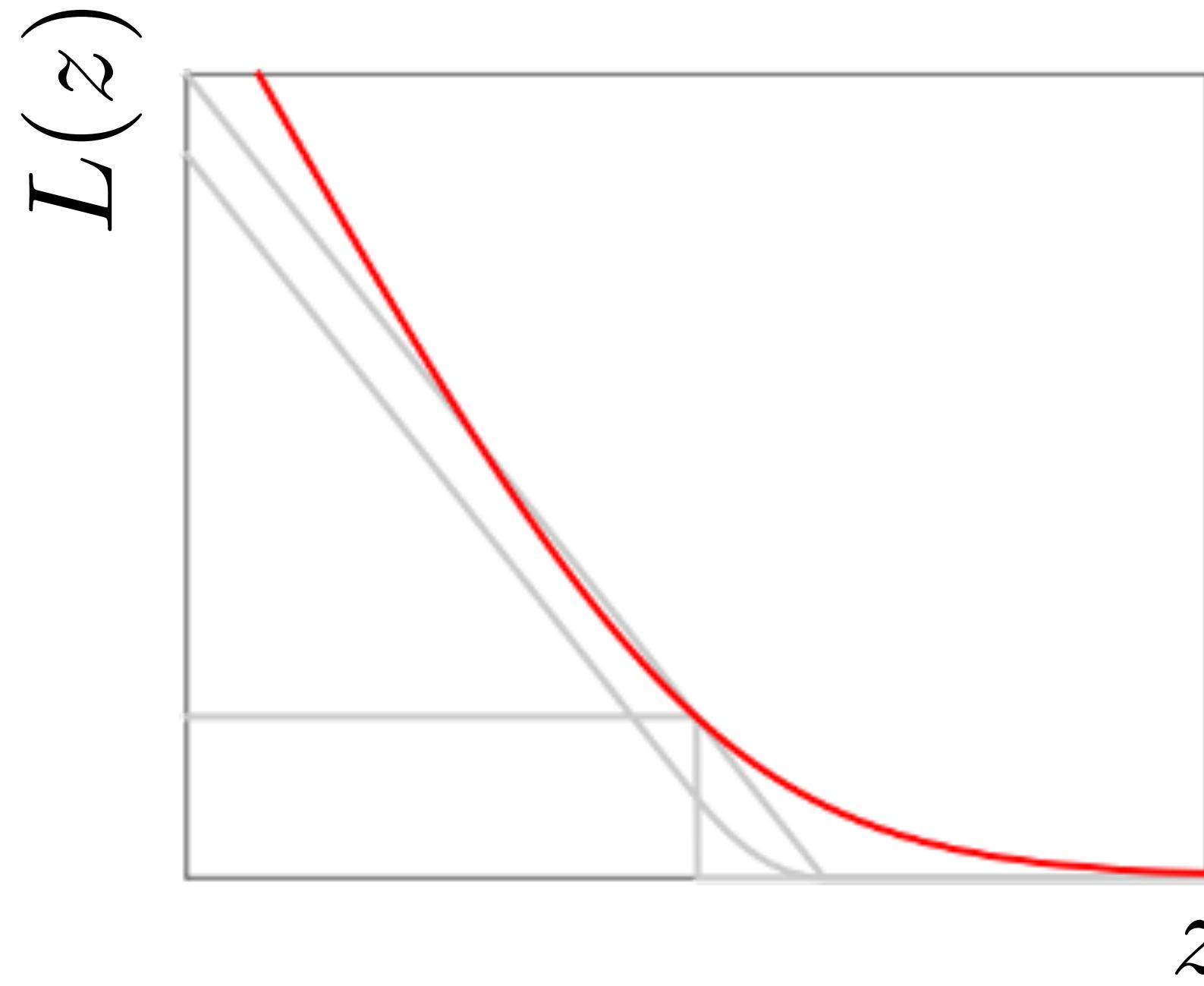
$$L(z) = \begin{cases} \frac{1}{2} - z & \text{if } z \leq 0 \\ \frac{1}{2}(1-z)^2 & \text{if } 0 < z < 1 \\ 0 & \text{else} \end{cases}$$

Classification Models

Target function: $f(\textcolor{teal}{x}) = \operatorname{sgn}(\textcolor{red}{h}(\textcolor{teal}{x}))$

Empirical risk minimization: $\ell(\textcolor{red}{h}, \hat{\Sigma}_{\textcolor{teal}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \textcolor{red}{h}(\hat{x}_i))$

Logistic Regression



Logloss:

$$L(z) = \log(1 + \exp(-z))$$

Performance of ERM

$$h_{\text{ERM}} = \operatorname{argmin}_{h \in \mathbb{H}} \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i)$$

In-Sample Loss:

$$\mathbb{E}_{\hat{\mathbb{P}}_N} [L(h(x), y)]$$

Performance of ERM

$$h_{\text{ERM}} = \operatorname{argmin}_{h \in \mathbb{H}} \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i)$$

In-Sample Loss:

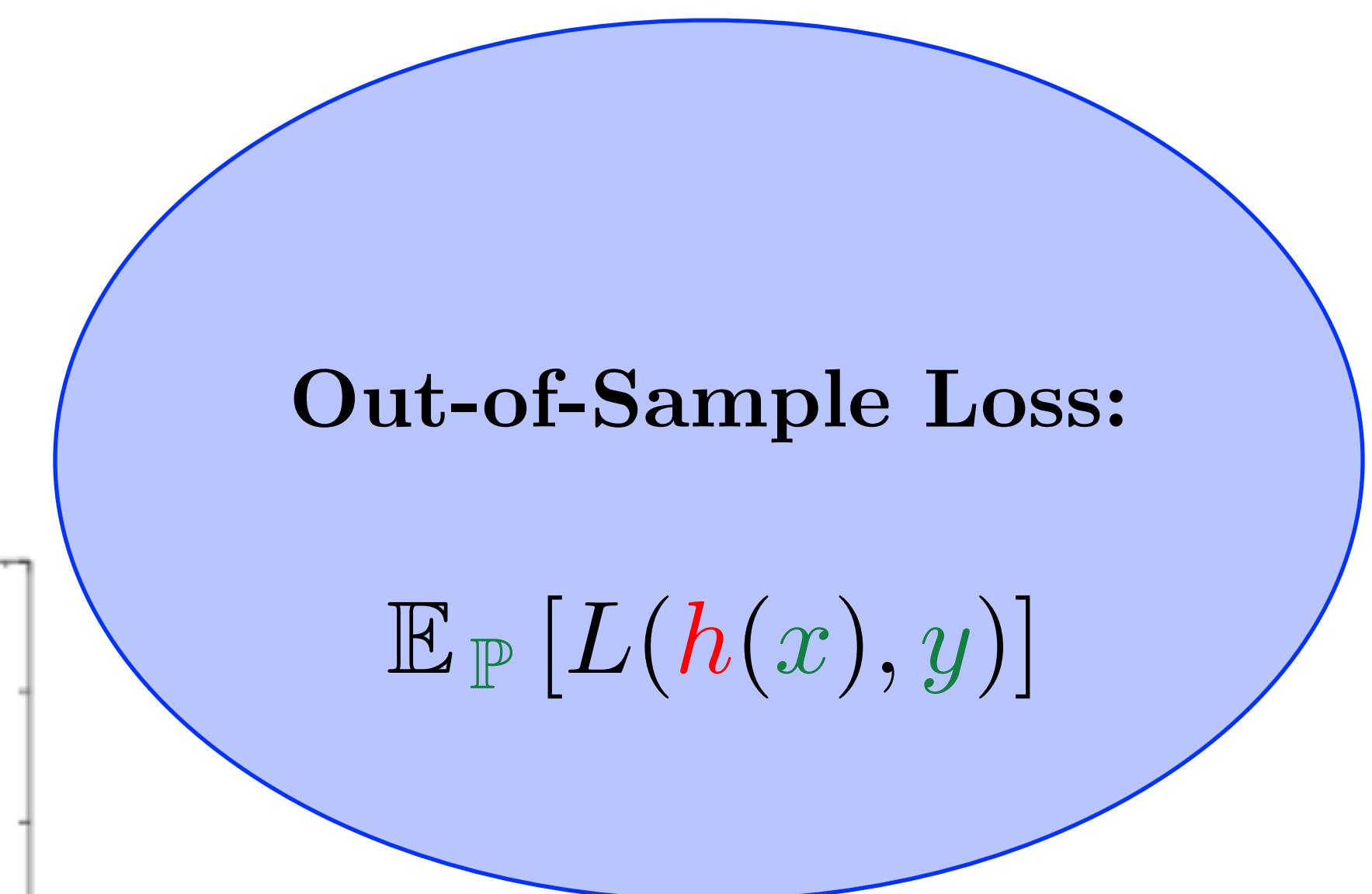
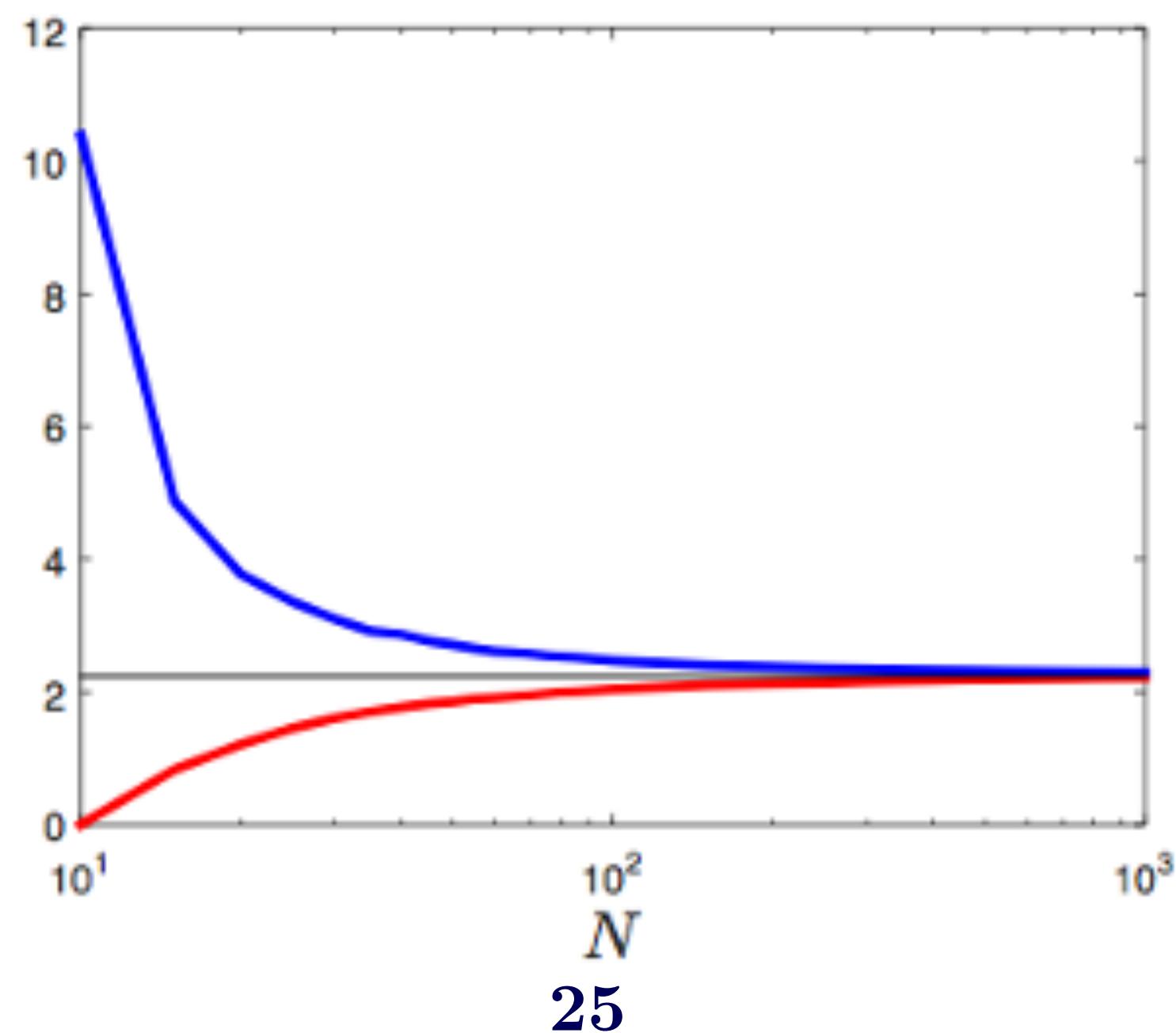
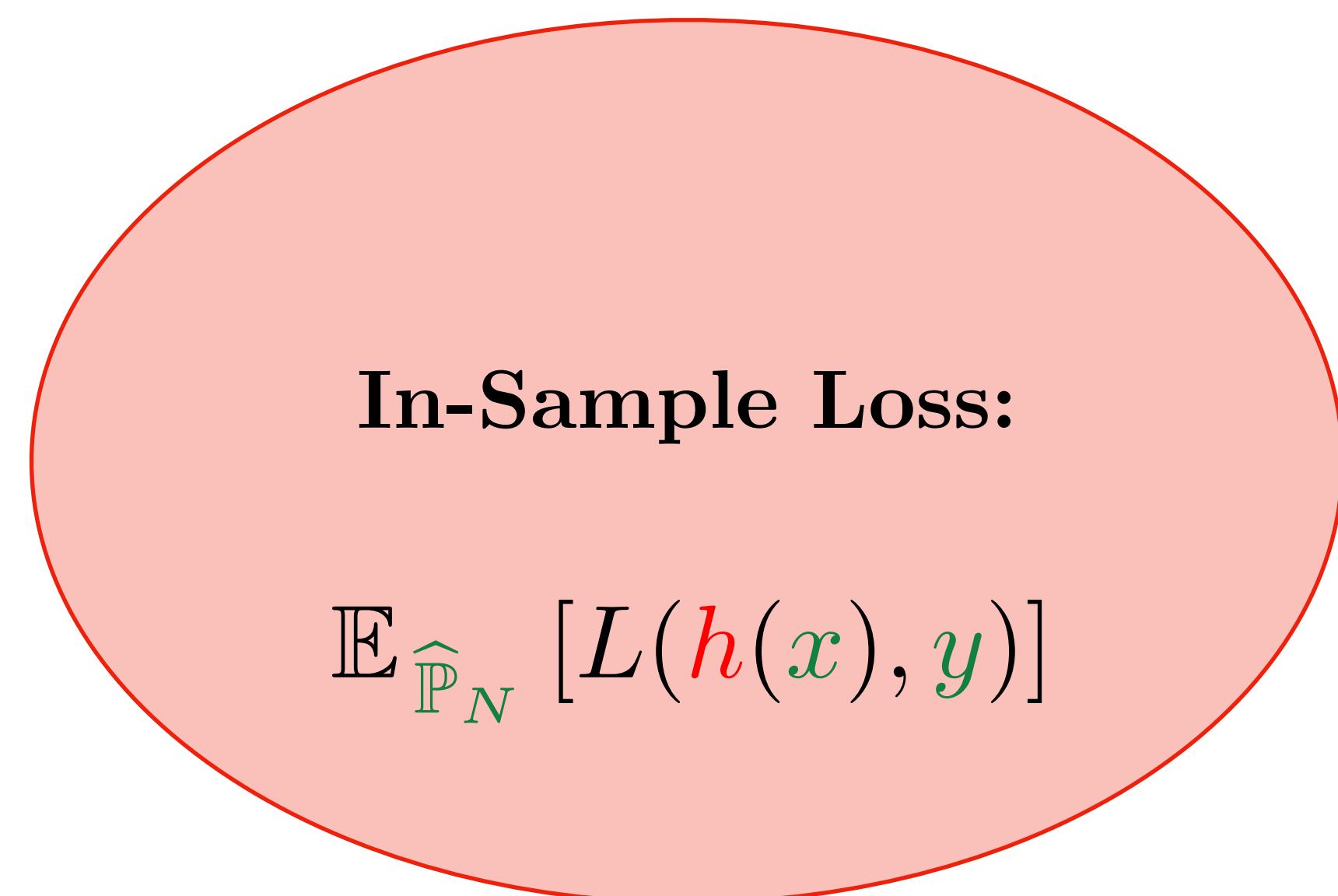
$$\mathbb{E}_{\hat{\mathbb{P}}_N} [L(h(x), y)]$$

Out-of-Sample Loss:

$$\mathbb{E}_{\mathbb{P}} [L(h(x), y)]$$

Performance of ERM

$$h_{\text{ERM}} = \operatorname{argmin}_{h \in \mathbb{H}} \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i)$$



Regularized ERM

$$h_{\text{REG}} = \underset{h \in \mathbb{H}}{\operatorname{argmin}} \quad \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i) + \varepsilon \Omega(h)$$

The diagram illustrates the components of the regularized ERM formula. A pink dotted circle highlights the regularization term $\varepsilon \Omega(h)$. A vertical pink arrow points from this circle down to the text 'Regularization coefficient'. Another pink arrow points from the same circle to the text 'Regularization function'.

Regularization coefficient

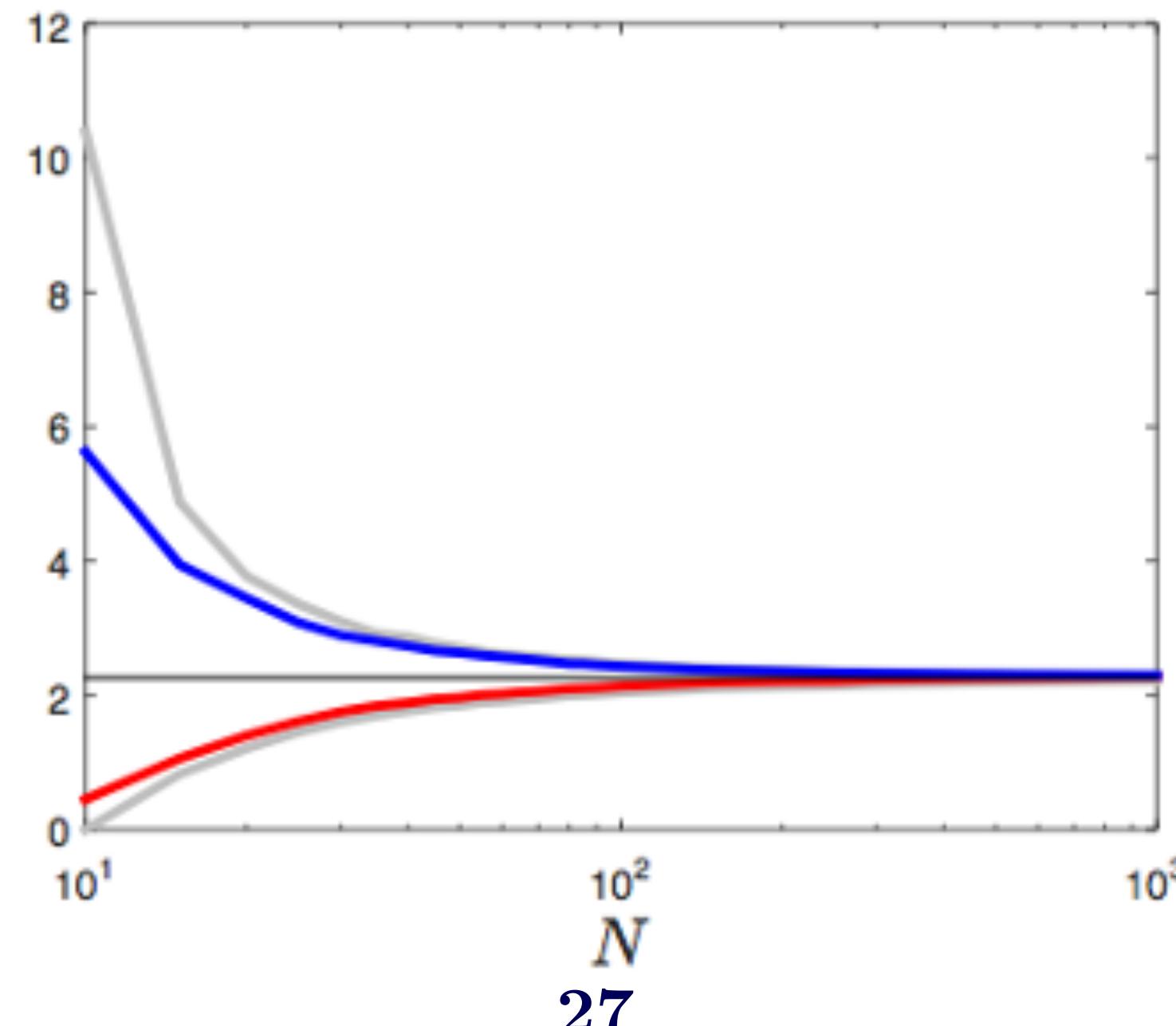
Regularization function

Regularized ERM

$$h_{\text{REG}} = \underset{h \in \mathbb{H}}{\operatorname{argmin}} \quad \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i) + \varepsilon \Omega(h)$$

Diagram illustrating the components of the regularized ERM formula:

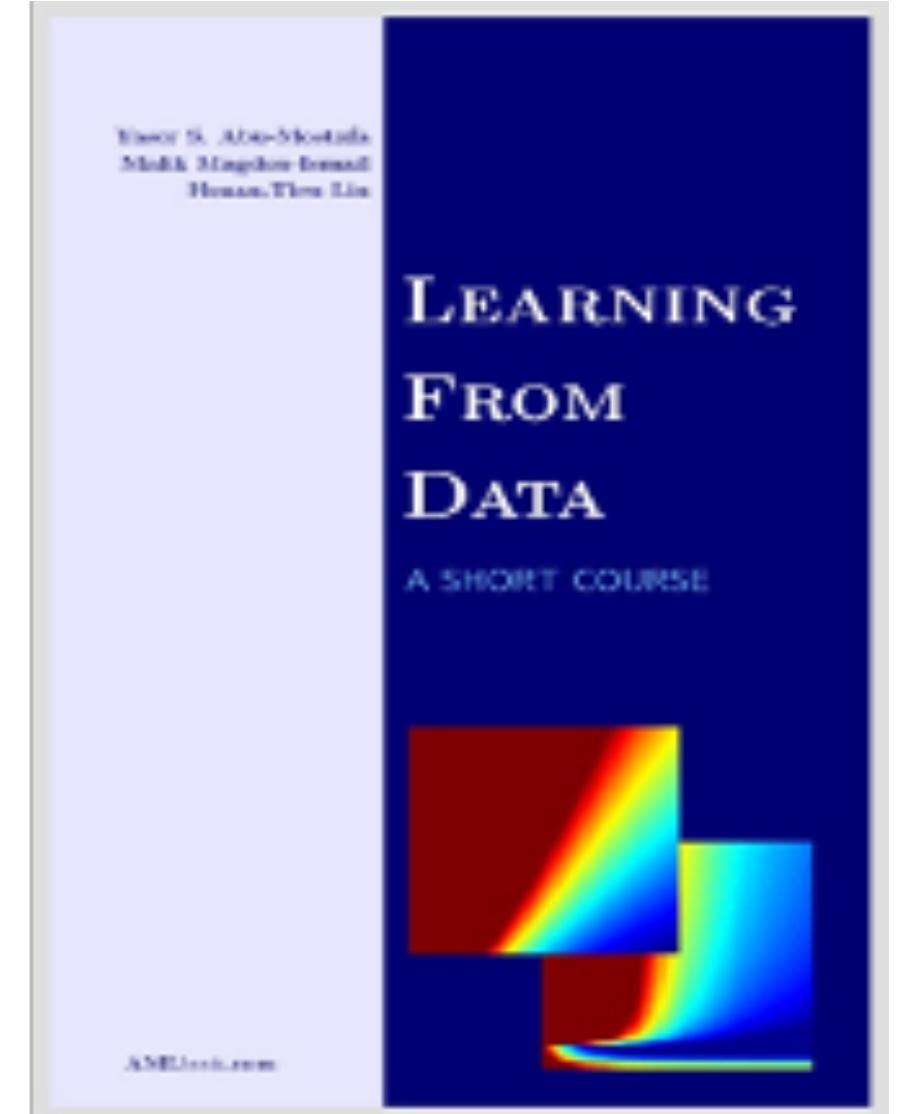
- Regularization coefficient**: ε (highlighted by a pink circle and arrow)
- Regularization function**: $\Omega(h)$ (highlighted by a pink circle and arrow)



Regularized ERM

“Most of the **regularization methods** used successfully in practice are **heuristic methods**.”

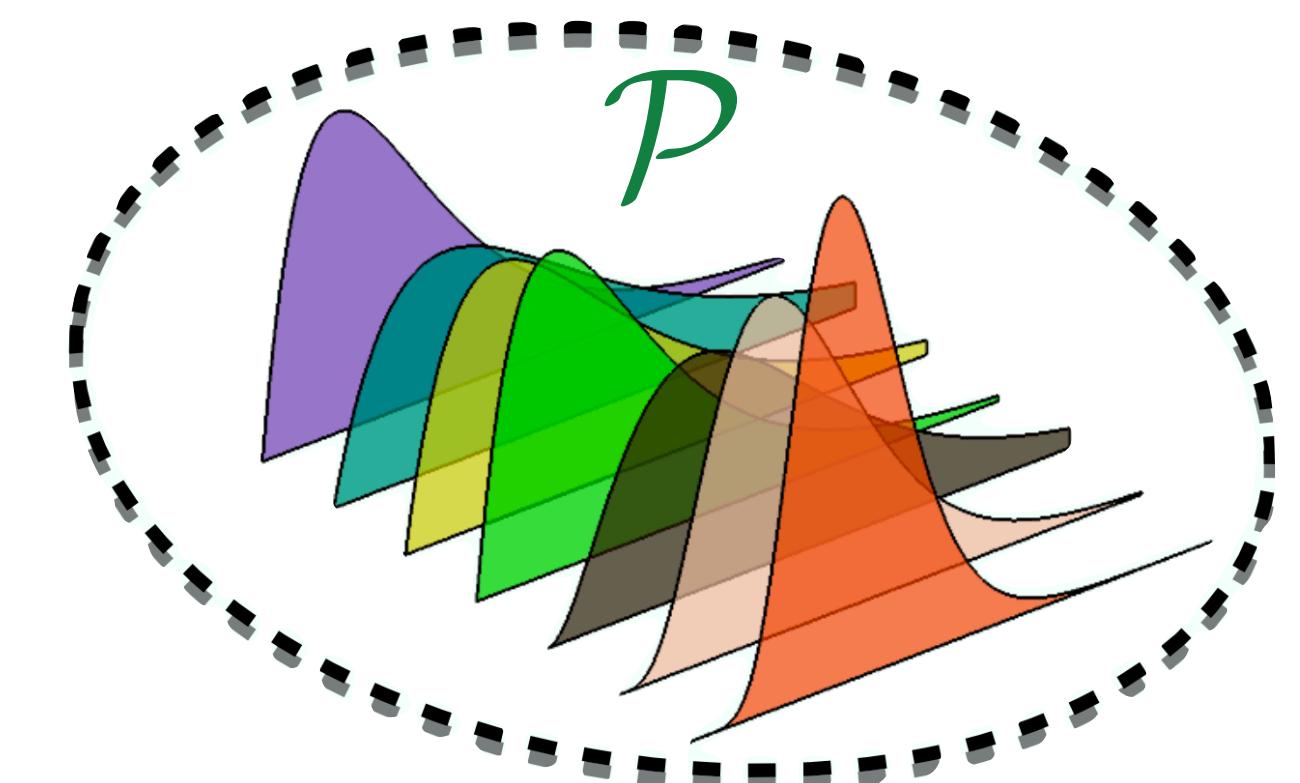
Abu-Mostafa *et al.*, 2012.



Regularization via Optimal Transport

$$\inf_{\color{red} h \in \mathbb{H}} \sup_{\color{blue} Q \in \mathcal{P}} \mathbb{E}_{\color{blue} Q} [L(\color{red} h(x), y)]$$

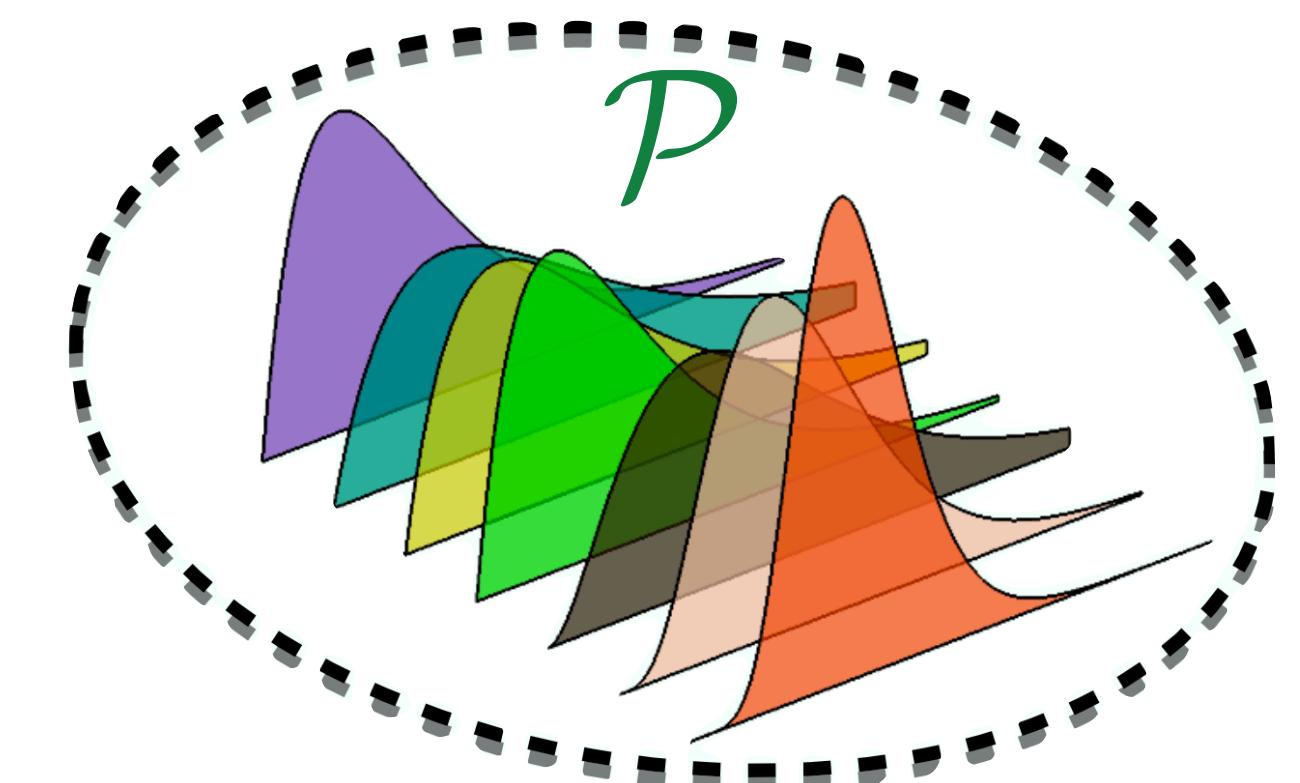
$$\mathbb{H} = \{ \color{red} h \in \mathbb{R}^X : \exists \theta \in \Theta \text{ s.t. } \color{red} h(x) = \theta^\top x \}$$



Regularization via Optimal Transport

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x, y)]$$

$$\mathbb{H} = \{h \in \mathbb{R}^X : \exists \theta \in \Theta \text{ s.t. } h(x) = \theta^\top x\}$$



The Real Story Behind the Success

Regularization

[SMK15, GCK17, CP18,
BMZ18, BKM19, SKM19]

Statistical Guarantees

[SMK15, MK18, BKM19,
SKM19, G20, BMN21]

Optimal Transport: Old and New



Monge

Hitchcock

Kantorovich-Koopmans

Vaserstein

Brenier

Villani

Figalli

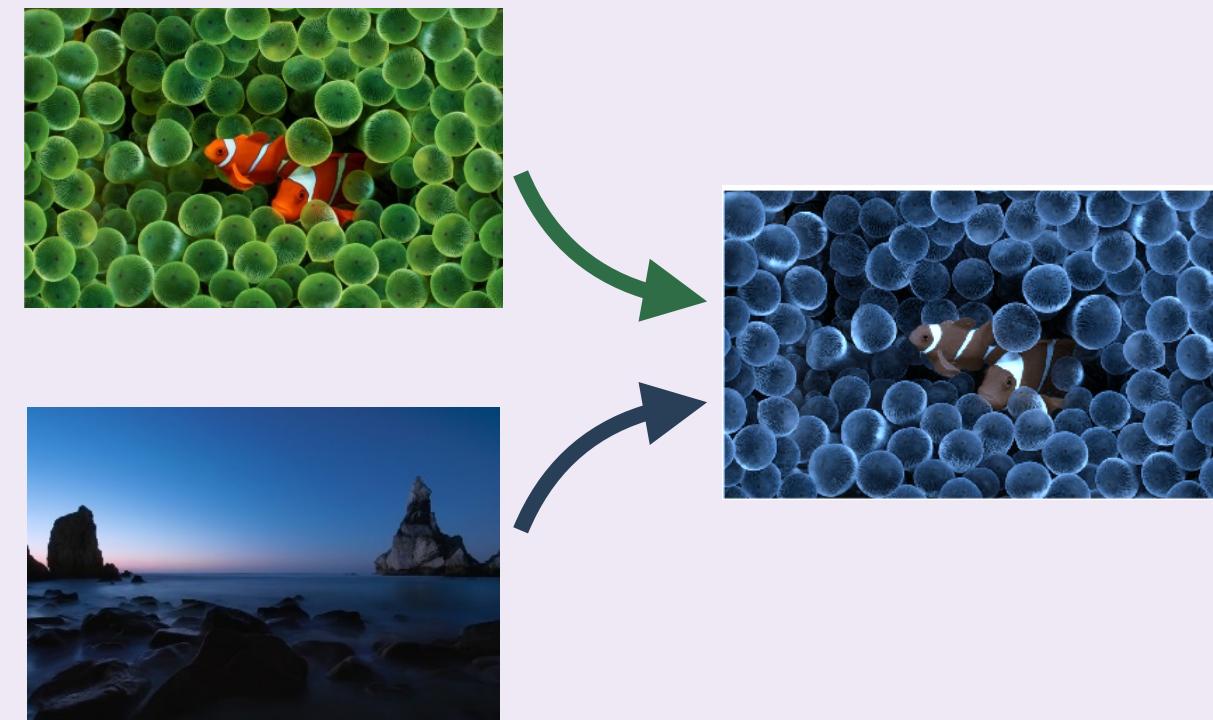
Nobel '75

Fields '10

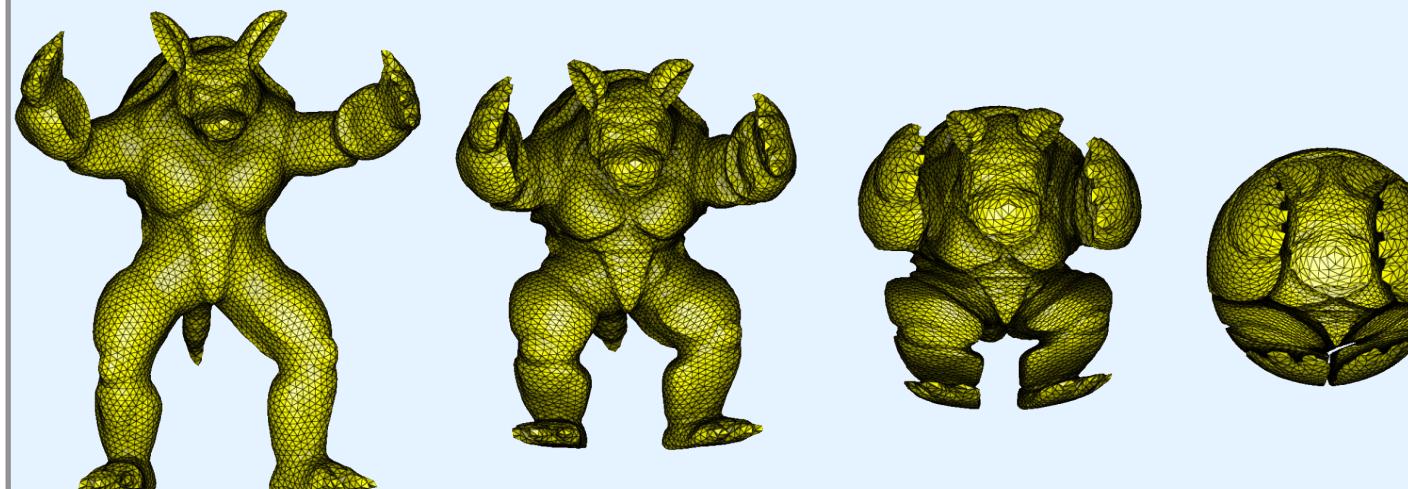
Fields '18

Applications

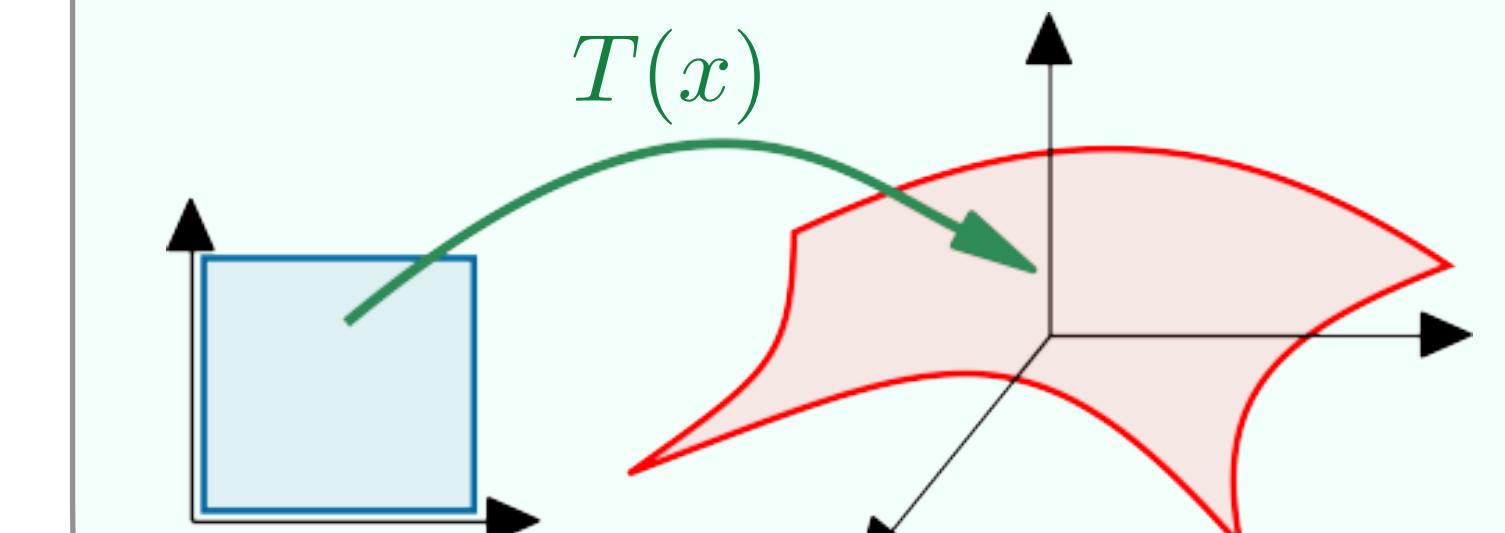
Color Histogram [PPC10]



Computer Graphics [LS18]



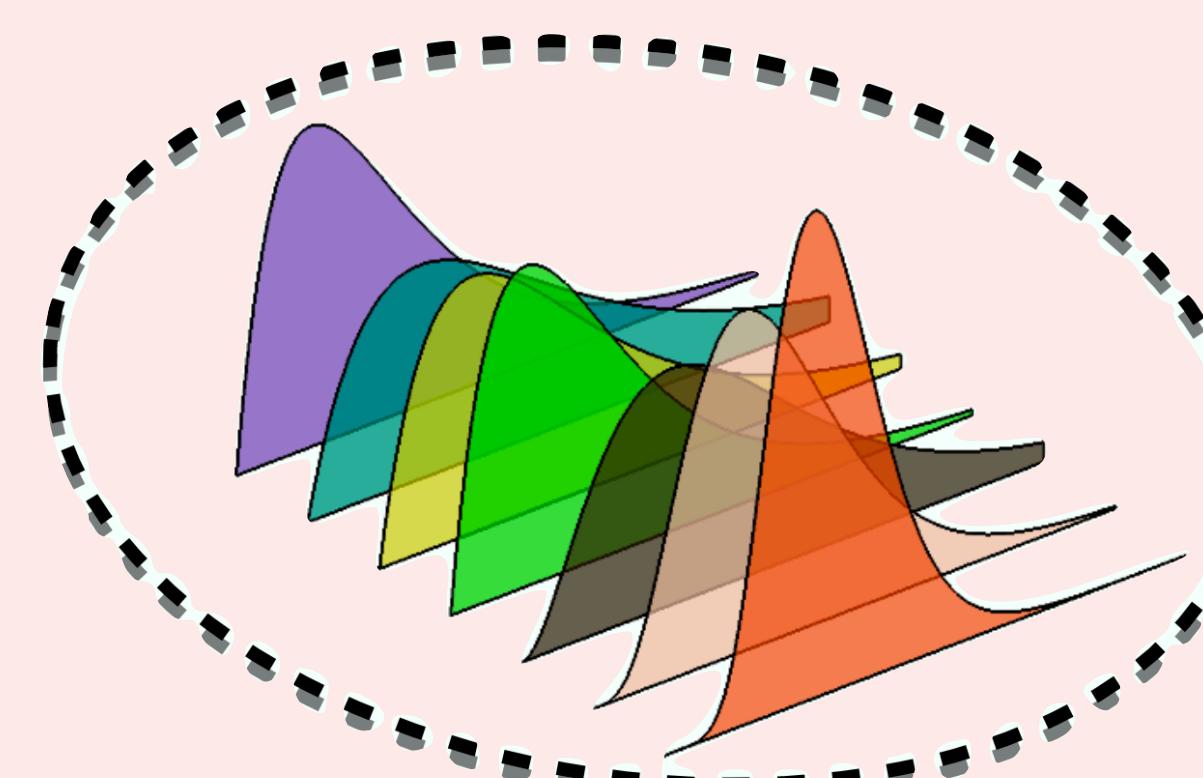
Generative Models [ACB17]



Finance and Economics [Gal16]



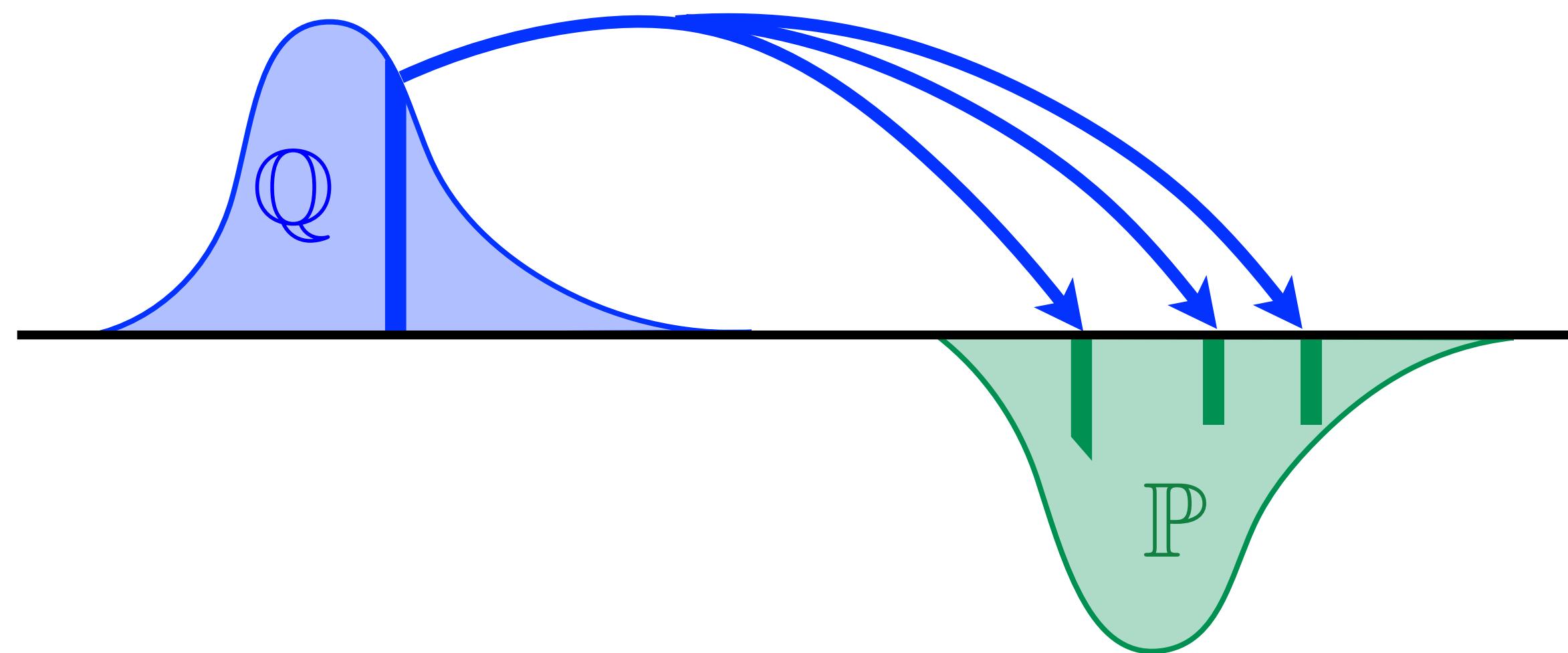
Statistical Inference



Text Classification [HGKSSW16]

Mazda Tiff Boone happening
fit motherboard homosexuals
asked automotive playoffs motorcycles animation gay
dolphins moon computer atheism western
Keith motif graphics controller
orbit Israel pro apple lists Rutgers hell
talk NHL driver guns bike biblical
fire biker virtual sell auto saint
label ride IDE key DOD sale circuit
riding shipping bikes mac rocket
Armenian card space story monitor
forsale firearms copy encryption polygon
Turkish bus compatible warning mouse
motorcycle summarized powerbook diamond
SCSI government chip SCSI
sunductor NASA DOS

What is Optimal Transport?



$$W_c(Q, \mathbb{P}) = \left\{ \begin{array}{ll} \inf_{\pi \in \mathcal{M}(\Xi, \Xi)} & \mathbb{E}_{\pi} [c(\xi, \xi')] \\ \text{s.t.} & \pi \in \Pi(Q, \mathbb{P}) \end{array} \right.$$



THE DISTRIBUTION OF A PRODUCT FROM SEVERAL SOURCES TO NUMEROUS LOCALITIES

BY FRANK L. HITCHCOCK

1941

1. Statement of the problem. When several factories supply a product to a number of cities we desire the least costly manner of distribution. Due to freight rates and other matters the cost of a ton of product to a particular city will vary according to which factory supplies it, and will also vary from city to city.

OPTIMUM UTILIZATION OF THE TRANSPORTATION SYSTEM* 1949

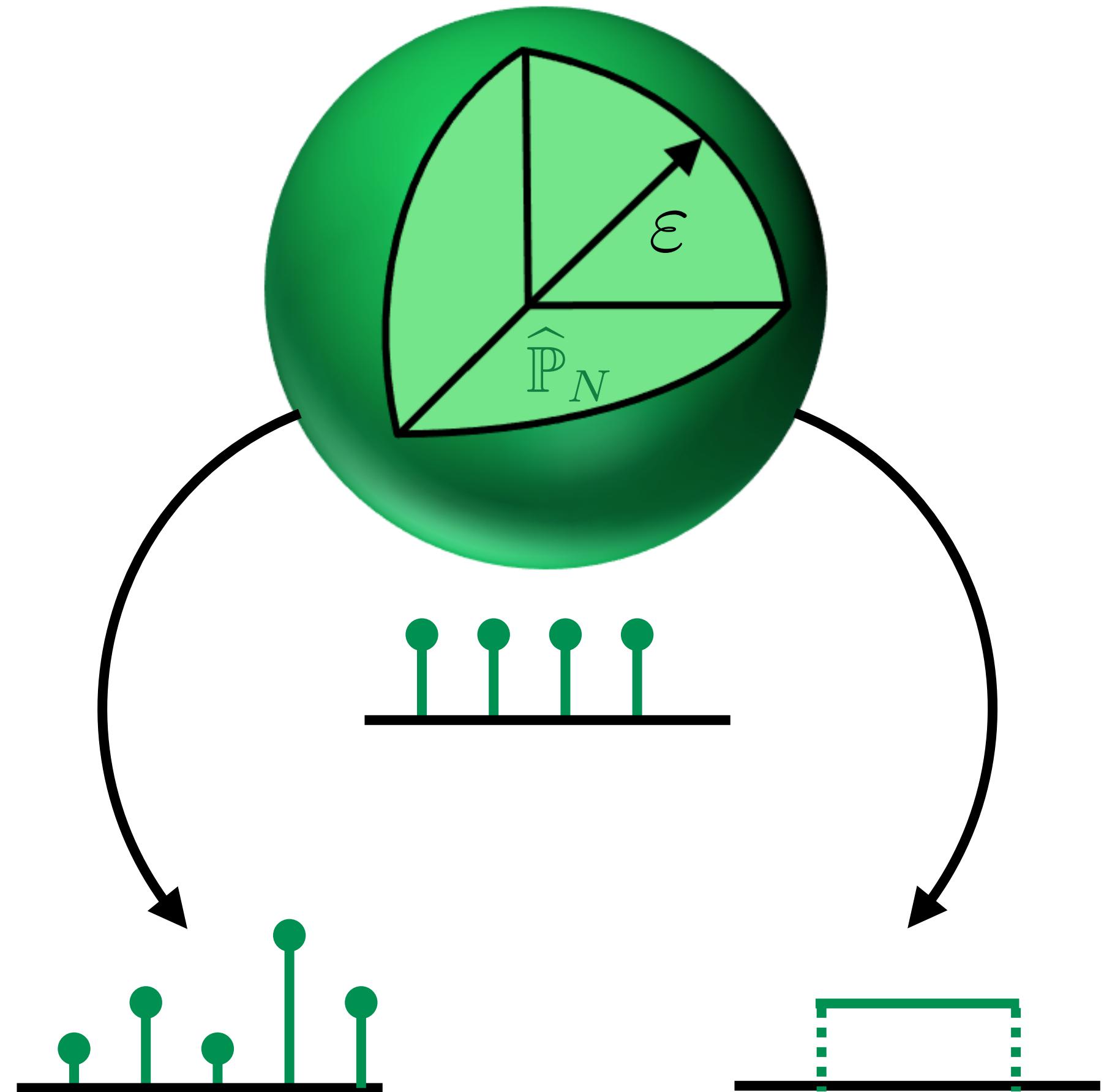
by Tjalling C. Koopmans

*Professor of Economics, The University of Chicago, and Research Associate,
Cowles Commission for Research in Economics*

The purpose of this paper is to give an application of the theory of optimum allocation of resources to one particular industry. I shall, therefore, not speak on that theory in general. I shall use one of its

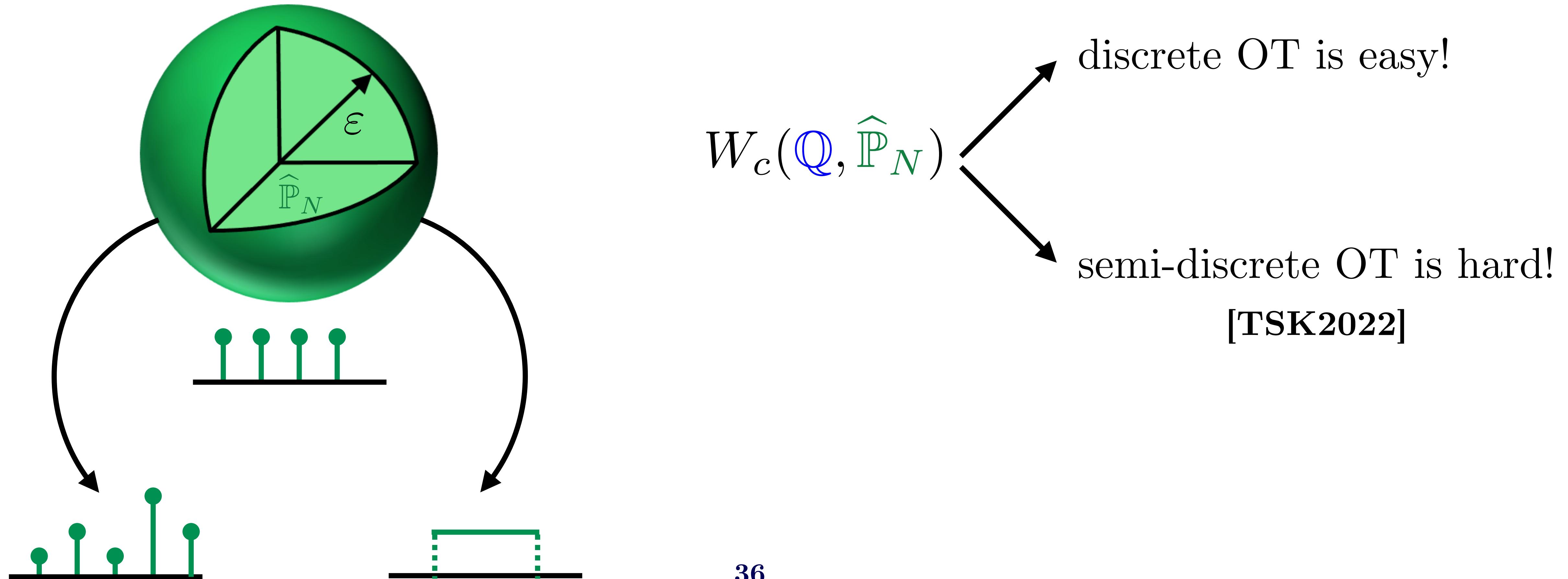
Optimal Transport Ambiguity Set

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n \times \mathbb{Y}) : W_c(\mathbb{Q}, \hat{\mathbb{P}}_N) \leq \varepsilon\}$$



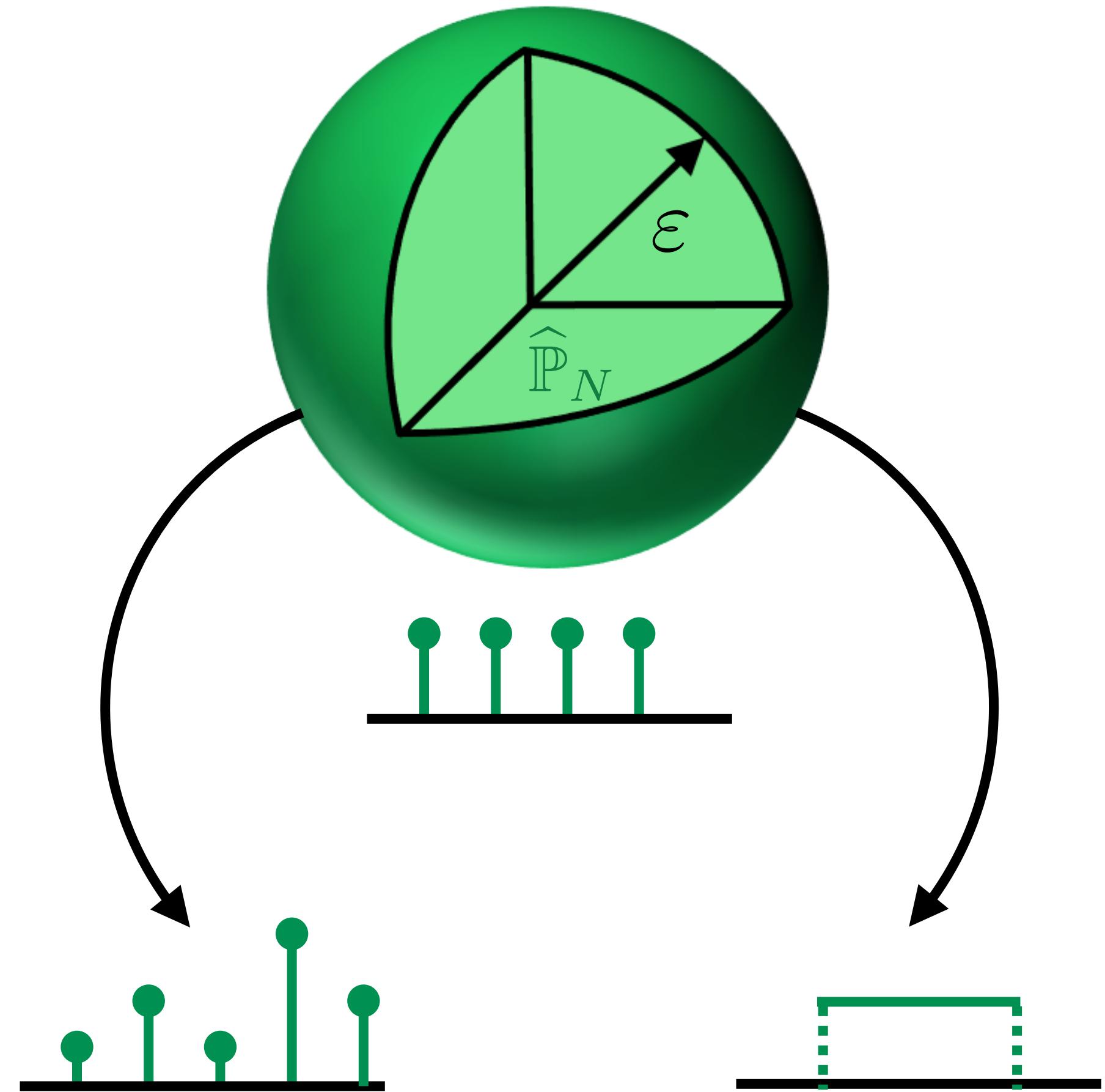
Optimal Transport Ambiguity Set

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n \times \mathbb{Y}) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$$



Optimal Transport Ambiguity Set

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n \times \mathbb{Y}) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$$



$$\begin{aligned} & \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x, y)] \\ \text{s.t. } & W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon \end{aligned}$$



Tractability for Linear Regression

Theorem 1. Suppose that $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$.

If L is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$

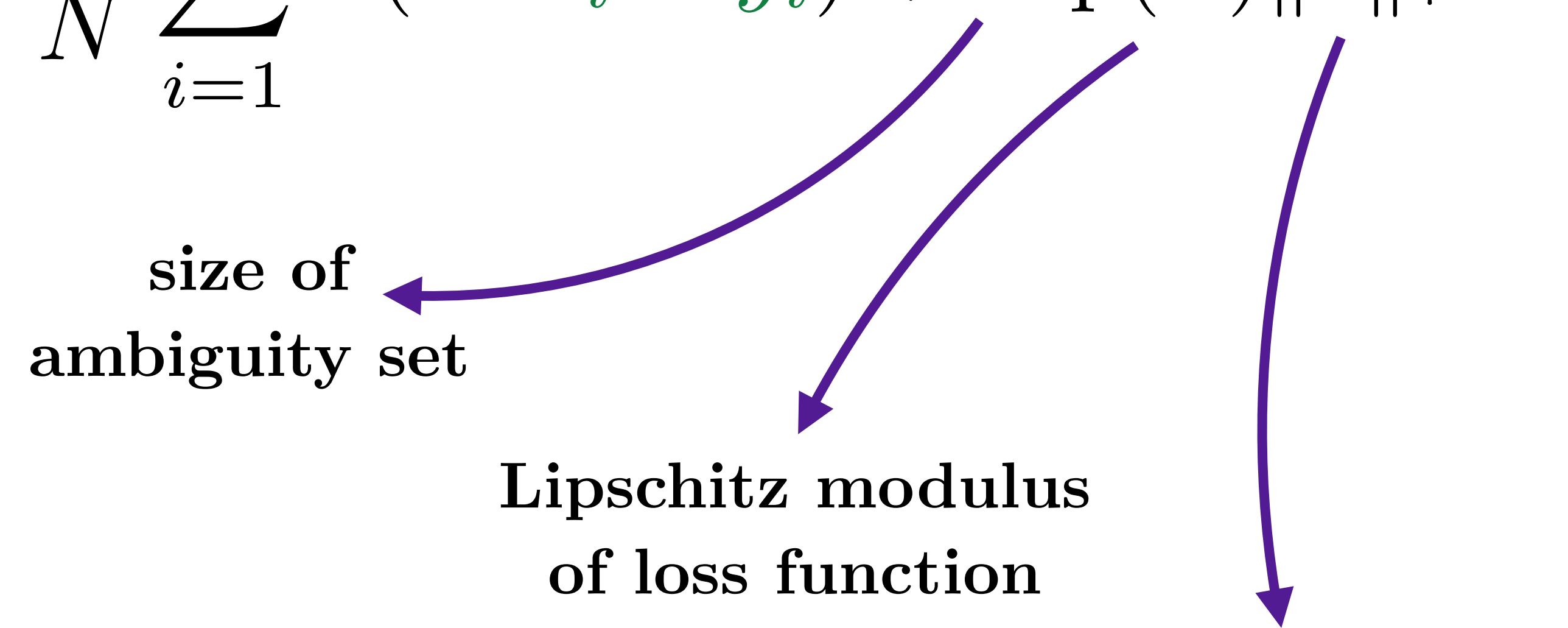
[SMK15, GCK17, SKM19, BKM19]

Tractability for Linear Regression

Theorem 1. Suppose that $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$.

If L is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$



[SMK15, GCK17, SKM19, BKM19]

Semi-infinite Duality

Lemma 1. Let $\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\Xi) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$. If $c(\xi, \xi) = 0$ for all $\xi \in \Xi$ and $\varepsilon > 0$, then

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[I(\xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right]$$

[MK18, ZG18, BM19, GK16, ZYG22]

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{ll} \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} \quad W_c(\mathbb{Q}, \hat{\mathbb{P}}_N) \leq \varepsilon \end{array} \right.$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{ll} \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} & \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} & W_c(\mathbb{Q}, \hat{\mathbb{P}}_N) \leq \varepsilon \end{array} \right.$$

$$= \left\{ \begin{array}{ll} \sup_{\substack{\mathbb{Q} \in \mathcal{M}(\Xi) \\ \pi \in \mathcal{M}(\Xi \times \Xi)}} & \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} & \pi \in \Pi(\mathbb{Q}, \hat{\mathbb{P}}_N) \\ & \int_{\xi \in \Xi} \int_{\xi' \in \Xi} c(\xi, \xi') \pi(\mathrm{d}\xi, \mathrm{d}\xi') \leq \varepsilon \end{array} \right.$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{l} \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} \quad W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon \end{array} \right.$$

↓

$$= \left\{ \begin{array}{l} \sup_{\substack{\mathbb{Q} \in \mathcal{M}(\Xi) \\ \pi \in \mathcal{M}(\Xi \times \Xi)}} \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} \quad \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \int_{\xi \in \Xi} \int_{\xi' \in \Xi} c(\xi, \xi') \pi(\mathrm{d}\xi, \mathrm{d}\xi') \leq \varepsilon \end{array} \right.$$

$$W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) = \left\{ \begin{array}{ll} \inf_{\pi \in \mathcal{M}(\Xi, \Xi)} & \mathbb{E}_\pi [c(\xi, \xi')] \\ \text{s.t.} & \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \end{array} \right.$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \begin{cases} \sup_{\mathbb{Q}_i \in \mathcal{M}(\Xi)} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \end{cases}$$

$$\boxed{\begin{cases} \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \widehat{\mathbb{P}}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\xi}_i} \end{cases} \implies \begin{cases} \pi = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \times \delta_{\widehat{\xi}_i} \\ \mathbb{Q} = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \end{cases}}$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \begin{cases} \sup_{\mathbb{Q}_i \in \mathcal{M}(\Xi)} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \end{cases}$$

$$\boxed{\begin{cases} \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \widehat{\mathbb{P}}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\xi}_i} \end{cases} \Rightarrow \begin{cases} \pi = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \times \delta_{\widehat{\xi}_i} \\ \mathbb{Q} = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \end{cases}}$$

$$= \begin{cases} \sup_{\mathbb{Q}_i \geq 0} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \\ \int_{\xi \in \Xi} \mathbb{Q}_i(d\xi) = 1 \quad \forall i \in [N] \end{cases}$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{l} \sup_{\mathbb{Q}_i \in \mathcal{M}(\Xi)} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \end{array} \right.$$

$$\left\{ \begin{array}{l} \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \widehat{\mathbb{P}}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\xi}_i} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \pi = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \times \delta_{\widehat{\xi}_i} \\ \mathbb{Q} = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \end{array} \right.$$

$$= \left\{ \begin{array}{l} \sup_{\mathbb{Q}_i \geq 0} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \quad (\lambda) \\ \int_{\xi \in \Xi} \mathbb{Q}_i(d\xi) = 1 \quad (s_i) \quad \forall i \in [N] \end{array} \right.$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \left\{ \begin{array}{ll} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} & \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t.} & \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{array} \right.$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \left\{ \begin{array}{ll} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} & \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t.} & \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{array} \right.$$

$$= \left\{ \begin{array}{ll} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} & \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t.} & s_i \geq \sup_{\xi \in \Xi} \frac{1}{N} I(\xi) - \frac{\lambda}{N} c(\xi, \hat{\xi}_i) \end{array} \right.$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \begin{cases} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t. } \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{cases}$$

$$= \inf_{\lambda \geq 0} \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^N \sup_{\xi \in \Xi} I(\xi) - \lambda c(\xi, \hat{\xi}_i)$$

Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \begin{cases} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t. } \frac{\lambda}{N} c(\xi, \widehat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{cases}$$

$$\begin{aligned} &= \inf_{\lambda \geq 0} \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^N \sup_{\xi \in \Xi} I(\xi) - \lambda c(\xi, \widehat{\xi}_i) \\ &= \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right] \end{aligned}$$

Proof of Lemma 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [I(\xi)] \stackrel{[S01]}{\leq} \left\{ \begin{array}{l} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t. } \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{array} \right.$$

$$\begin{aligned} &= \inf_{\lambda \geq 0} \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^N \sup_{\xi \in \Xi} I(\xi) - \lambda c(\xi, \hat{\xi}_i) \\ &= \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\hat{\mathbb{P}}_N} \left[\sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right] \end{aligned}$$

Semi-infinite Duality

Lemma 1. Let $\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\Xi) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$. If $c(\xi, \xi) = 0$ for all $\xi \in \Xi$ and $\varepsilon > 0$, then

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[I(\xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right]$$

[MK18, ZG18, BM19, GK16, ZYG22]

Lipschitz Envelope

Lemma 2. Let $L : \mathbb{R} \rightarrow \mathbb{R}$ be a convex and Lipschitz function.

Then,

$$\sup_{\zeta \in \mathbb{R}^n} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| = \begin{cases} L(\theta^\top \xi + \theta_0) & \text{if } \text{lip}(L)\|\theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}$$

for any $\theta, \xi \in \mathbb{R}^n, \theta_0 \in \mathbb{R}$ and $\lambda > 0$.

Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

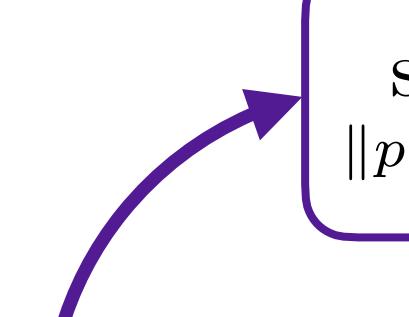
$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| = \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \lambda \|\zeta - \xi\|$$

Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \boxed{\lambda \|\zeta - \xi\|} \\ &= \sup_{\kappa \in \mathcal{K}} \sup_{\zeta} \inf_{\|p\|_* \leq \lambda} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \end{aligned}$$

\$\sup_{\|p\|_* \leq \lambda} p^\top (\zeta - \xi)\$


Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \lambda \|\zeta - \xi\| \\ &= \sup_{\kappa \in \mathcal{K}} \sup_{\zeta} \inf_{\|p\|_* \leq \lambda} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \\ (\text{Sion's minimax}) &= \sup_{\kappa \in \mathcal{K}} \inf_{\|p\|_* \leq \lambda} \sup_{\zeta} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \end{aligned}$$

Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \lambda \|\zeta - \xi\| \\ &= \sup_{\kappa \in \mathcal{K}} \sup_{\zeta} \inf_{\|p\|_* \leq \lambda} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \\ (\text{Sion's minimax}) \quad &= \sup_{\kappa \in \mathcal{K}} \inf_{\|p\|_* \leq \lambda} \kappa \theta_0 - L^*(\kappa) + p^\top \xi + \begin{cases} 0 & \text{if } \kappa \theta - p = 0 \\ +\infty & \text{else} \end{cases} \end{aligned}$$

Proof of Lemma 2

$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| = \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}$$

Proof of Lemma 2

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\ &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \ \forall \kappa \in \mathcal{K} \\ +\infty & \text{else} \end{cases} \end{aligned}$$

Proof of Lemma 2

$$\begin{aligned}
 \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \ \forall \kappa \in \mathcal{K} \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \sup_{\kappa \in \mathcal{K}} \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}
 \end{aligned}$$

Proof of Lemma 2

$$\begin{aligned}
 \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \ \forall \kappa \in \mathcal{K} \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \sup_{\kappa \in \mathcal{K}} \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\
 \boxed{\sup_{\kappa \in \mathcal{K}} |\kappa| = \text{lip}(L)} &= \begin{cases} L(\theta^\top \xi + \theta_0) & \text{if } \text{lip}(L) \|\theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}
 \end{aligned}$$

Tractability for Linear Regression

Theorem 1. Suppose that $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$.

If L is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$

[SMK15, GCK17, SKM19, BKM19]

Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$\begin{aligned} (\text{Lemma 1}) &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right] \\ &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\| \right] \end{aligned}$$

Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\begin{cases} L(\theta^\top x - y) & \text{if } \lambda \geq \text{lip}(L) \|\theta\|_* \\ +\infty & \text{else} \end{cases} \right]$$

Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L) \|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - y)]$$

Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top \mathbf{x} - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - \mathbf{y}) - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L) \|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top \mathbf{x} - \mathbf{y})]$$

$$= \inf_{\theta \in \Theta} \varepsilon \text{lip}(L) \|\theta\|_* + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top \mathbf{x} - \mathbf{y})]$$

Examples

Robust Regression

$$L(z) = \begin{cases} \frac{1}{2}z^2 & \text{if } |z| \leq \delta \\ \delta(|z| - \frac{1}{2}\delta) & \text{else} \end{cases}$$

$$\text{lip}(L) = \delta$$



Support Vector Regression

$$L(z) = \max\{0, |z| - \varepsilon\}$$

$$\text{lip}(L) = 1$$



Quantile Regression

$$L(z) = \max\{-\tau z, (1 - \tau)z\}$$

$$\text{lip}(L) = \max\{\tau, 1 - \tau\}$$

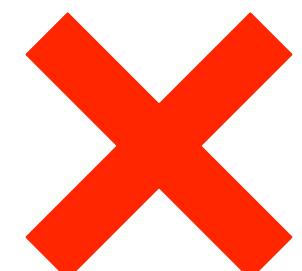


Examples

Least Square
Regression

$$L(z) = z^2$$

$$\text{lip}(L) = \infty$$



Least Squares Regression

Theorem 2. Suppose that $c((x, y), (x', y')) = \|x - x'\|^2 + \delta_{y=y'}.$

If $L = z^2$, then

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)] = \left(\inf_{\theta \in \Theta} \sqrt{\frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \theta^\top \hat{x}_i)} + \sqrt{\varepsilon} \|\theta\|_* \right)^2$$

[BKM19]

Moreau Envelope

Lemma 3. Let $L(z) = z^2$. Then,

$$\sup_{\zeta \in \mathbb{R}^n} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 = \begin{cases} \frac{\lambda}{\lambda - \|\theta\|_*^2} L(\theta^\top \xi + \theta_0) & \text{if } \|\theta\|_*^2 < \lambda \\ +\infty & \text{else} \end{cases}$$

for any $\theta, \xi \in \mathbb{R}^n, \theta_0 \in \mathbb{R}$ and $\lambda > 0$.

[BKM19, SADK22]

Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ \boxed{\Delta \leftarrow \zeta - \xi} \quad &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \end{aligned}$$

Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ &= \begin{cases} \sup_{\Delta, \gamma} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{cases} \end{aligned}$$

Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ &= \sup_{\gamma} \left\{ \begin{array}{ll} \sup_{\Delta} & L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t.} & \gamma = \theta^\top \Delta \end{array} \right. \end{aligned}$$

Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ &= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\ (\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \end{aligned}$$

Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ &= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\ (\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \\ (\text{Holder inequality}) &\leq \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \|\kappa \theta\|_* \|\Delta\| \end{aligned}$$

Proof of Lemma 3

$$\begin{aligned}
& \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\
&= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Holder inequality}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\|\Delta\|} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \|\kappa \theta\|_* \|\Delta\|
\end{aligned}$$

$$\|\Delta^*\| = \frac{\|\kappa \theta\|_*}{2\lambda}$$

Proof of Lemma 3

$$\begin{aligned}
& \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\
&= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Holder inequality}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\|\Delta\|} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \|\kappa \theta\|_* \|\Delta\| \\
&= \sup_{\gamma} \inf_{\kappa} L(\gamma + \theta^\top \xi + \theta_0) - \kappa \gamma + \frac{\|\kappa \theta\|_*^2}{4\lambda}
\end{aligned}$$

$\kappa^* = \frac{2\lambda\gamma}{\|\theta\|_*^2}$

Proof of Lemma 3

$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 = \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda \gamma^2}{\|\theta\|_*^2}$$

Proof of Lemma 3

$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 = \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda \gamma^2}{\|\theta\|_*^2}$$

$$\begin{cases} \gamma^* = \frac{\theta^\top \xi}{\frac{\lambda}{\|\theta\|_*^2} - 1} & \text{if } 1 - \frac{\lambda}{\|\theta\|_*^2} < 0 \\ \text{unbounded} & \text{else} \end{cases}$$

$$= \sup_{\gamma} (\gamma + \theta^\top \xi)^2 - \frac{\lambda \gamma^2}{\|\theta\|_*^2}$$

Proof of Lemma 3

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda \gamma^2}{\|\theta\|_*^2} \\ &= \sup_{\gamma} (\gamma + \theta^\top \xi)^2 - \frac{\lambda \gamma^2}{\|\theta\|_*^2} \\ &= \begin{cases} \frac{\lambda}{\lambda - \|\theta\|_*^2} (\theta^\top \xi)^2 & \text{if } \|\theta\|_*^2 < \lambda \\ +\infty & \text{else} \end{cases} \end{aligned}$$

Least Squares Regression

Theorem 2. Suppose that $c((x, y), (x', y')) = \|x - x'\|^2 + \delta_{y=y'}.$

If $L = z^2$, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \left(\sqrt{\frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i)} + \sqrt{\varepsilon} \|\theta\|_* \right)^2$$

[BKM19]

Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\|^2 - \lambda \delta_{y'=\mathbf{y}} \right]$$

Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$\begin{aligned} (\text{Lemma 1}) &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\|^2 - \lambda \delta_{y'=\mathbf{y}} \right] \\ &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - \mathbf{y}) - \lambda \|x' - \mathbf{x}\|^2 \right] \end{aligned}$$

Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\|^2 - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\|^2 \right]$$

$$(\text{Lemma 3}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\begin{cases} \frac{\lambda}{\lambda - \|\theta\|_*^2} L(\theta^\top x - y) & \text{if } \lambda > \|\theta\|_*^2 \\ +\infty & \text{else} \end{cases} \right]$$

Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\|^2 - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\|^2 \right]$$

$$(\text{Lemma 3}) = \inf_{\substack{\theta \in \Theta \\ \lambda > \|\theta\|_*^2}} \lambda \varepsilon + \frac{\lambda}{\lambda - \|\theta\|_*^2} \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - y)]$$

$$\lambda^* = \|\theta\|_*^2 + \frac{\|\theta\|_* \sqrt{\mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - y)]}}{\varepsilon}$$

Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\|^2 - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - \mathbf{y}) - \lambda \|x' - \mathbf{x}\|^2 \right]$$

$$(\text{Lemma 3}) = \inf_{\substack{\theta \in \Theta \\ \lambda > \|\theta\|_*^2}} \lambda \varepsilon + \frac{\lambda}{\lambda - \|\theta\|_*^2} \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - \mathbf{y})]$$

$$= \inf_{\theta \in \Theta} \left(\sqrt{\varepsilon \|\theta\|_*} + \sqrt{\mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - \mathbf{y})]} \right)^2$$

Tractability for Linear Classification

Theorem 3. Suppose that $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$.

If L is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \theta^\top \hat{x}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$

[SMK15, GCK17, SKM19, BKM19]

Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right]$$

Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$\begin{aligned}
 (\text{Lemma 1}) &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right] \\
 &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\mathbf{y}\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]
 \end{aligned}$$

Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\mathbf{y}\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L)\|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\mathbf{y}\theta^\top \mathbf{x})]$$

Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(\mathbf{y}\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L)\|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\mathbf{y}\theta^\top \mathbf{x})]$$

$$= \inf_{\theta \in \Theta} \varepsilon \text{lip}(L)\|\theta\|_* + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\mathbf{y}\theta^\top \mathbf{x})]$$

Examples

Support Vector Machine

$$L(z) = \max\{0, 1 - z\}$$

$$\text{lip}(L) = 1$$



Support Vector Machine II

$$L(z) = \begin{cases} \frac{1}{2} - z & \text{if } z \leq 0 \\ \frac{1}{2}(1 - z)^2 & \text{if } 0 < z < 1 \\ 0 & \text{else} \end{cases}$$

$$\text{lip}(L) = 1$$



Logistic Regression

$$L(z) = \log(1 + \exp(-z))$$

$$\text{lip}(L) = 1$$

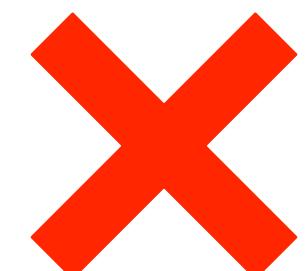


Examples

Ideal
Classification

$$L(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ 0 & \text{else} \end{cases}$$

nonconvex!



Ideal Classification

Theorem 4. Suppose that $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}.$

If $L(z) = \mathbf{1}_{z \leq 0}$, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\mathbf{y}\theta^\top \mathbf{x})] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L_R(\hat{y}_i \theta^\top \hat{\mathbf{x}}_i) + \varepsilon \|\theta\|_*$$

where $L_R(z) = \max\{0, 1 - z\} + \max\{0, -z\}.$

Ideal Classification

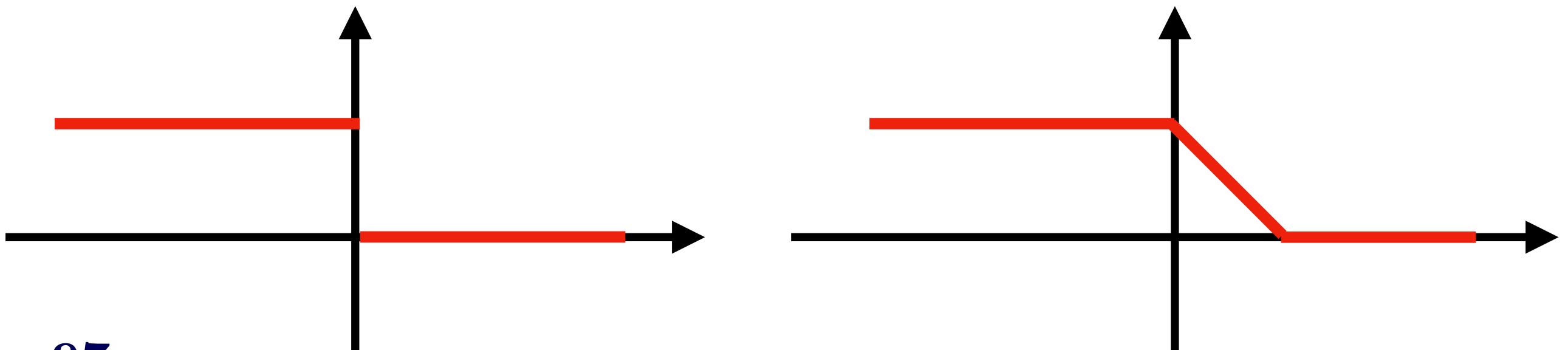
Theorem 4. Suppose that $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}.$

If $L(z) = \mathbf{1}_{z \leq 0}$, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L_R(\hat{y}_i \theta^\top \hat{x}_i) + \varepsilon \|\theta\|_*$$

where $L_R(z) = \max\{0, 1 - z\} + \max\{0, -z\}.$

[H-NW22]



Lipschitz Envelope II

Lemma 4. Let $L(z) = \mathbb{1}_{z \leq 0}$. Then,

$$\sup_{\zeta \in \mathbb{R}^n} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| = L_R\left(\lambda(\theta^\top \xi)/\|\theta\|_*\right)$$

for any $\theta, \xi \in \mathbb{R}^n$ and $\lambda > 0$.

Proof of Lemma 4

$$\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| = \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\|$$

$$\boxed{\Delta \leftarrow \zeta - \xi}$$

Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\ &= \left\{ \begin{array}{ll} \sup_{\Delta, \gamma} & L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t.} & \gamma = \theta^\top \Delta \end{array} \right. \end{aligned}$$

Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\ &= \sup_{\gamma} \left\{ \begin{array}{ll} \sup_{\Delta} & L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t.} & \gamma = \theta^\top \Delta \end{array} \right. \end{aligned}$$

Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\ &= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\ (\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \boxed{\lambda \|\Delta\|} - \kappa \gamma + \kappa \theta^\top \Delta \end{aligned}$$

Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta
\end{aligned}$$

Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Sion's minimax}) &= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} \sup_{\Delta} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta
\end{aligned}$$

Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Sion's minimax}) &= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} \sup_{\Delta} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} L(\gamma + \theta^\top \xi) - \kappa \gamma + \begin{cases} 0 & \text{if } \kappa \theta - p = 0 \\ +\infty & \text{else} \end{cases}
\end{aligned}$$

Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Sion's minimax}) &= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} \sup_{\Delta} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{-\lambda/\|\theta\|_* \leq \kappa \leq \lambda/\|\theta\|_*} L(\gamma + \theta^\top \xi) - \kappa \gamma
\end{aligned}$$

$\kappa^* = \frac{\lambda}{\|\theta\|_*} \text{sgn}(\gamma)$

Proof of Lemma 4

$$\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| = \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*}$$

Proof of Lemma 4

$$\begin{aligned}\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\ &= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma|\end{aligned}$$

$\gamma \leftarrow \gamma / \|\theta\|_*$

Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\ &= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma| \\ &= \sup_{\gamma} \begin{cases} 1 - \lambda |\gamma| & \text{if } \gamma \|\theta\|_* + \theta^\top \xi \leq 0 \\ -\lambda |\gamma| & \text{else} \end{cases} \end{aligned}$$

Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\ &= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma| \\ &= \sup_{\gamma} \begin{cases} 1 - \lambda |\gamma| & \text{if } \gamma \|\theta\|_* + \theta^\top \xi \leq 0 \\ -\lambda |\gamma| & \text{else} \end{cases} \\ &= \begin{cases} 1 & \text{if } \theta^\top \xi \leq 0 \\ \max\{0, 1 - \lambda \frac{\theta^\top \xi}{\|\theta\|_*}\} & \text{else} \end{cases} \end{aligned}$$

Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\
&= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma| \\
&= \sup_{\gamma} \begin{cases} 1 - \lambda |\gamma| & \text{if } \gamma \|\theta\|_* + \theta^\top \xi \leq 0 \\ -\lambda |\gamma| & \text{else} \end{cases} \\
&= \begin{cases} 1 & \text{if } \theta^\top \xi \leq 0 \\ \max\{0, 1 - \lambda \frac{\theta^\top \xi}{\|\theta\|_*}\} & \text{else} \end{cases} \\
&= L_R(\lambda(\theta^\top \xi)/\|\theta\|_*)
\end{aligned}$$

Ideal Classification

Theorem 4. Suppose that $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}.$

If $L(z) = \mathbf{1}_{z \leq 0}$, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\mathbf{y}\theta^\top \mathbf{x})] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L_R(\hat{y}_i \theta^\top \hat{\mathbf{x}}_i) + \varepsilon \|\theta\|_*$$

where $L_R(z) = \max\{0, 1 - z\} + \max\{0, -z\}.$

Proof of Theorem 4

$$\inf_{\theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

Proof of Theorem 4

$$\inf_{\theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)]$$

$$\begin{aligned} (\text{Lemma 1}) &= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right] \\ &= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(y\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right] \end{aligned}$$

Proof of Theorem 4

$$\inf_{\theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(y\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 4}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L_R(\lambda(y\theta^\top x) / \|\theta\|_*)]$$

Proof of Theorem 4

$$\inf_{\theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

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$$\begin{aligned} \theta &\leftarrow \lambda \theta / \|\theta\|_* \\ &\Downarrow \\ \lambda &= \|\theta\|_* \end{aligned}$$

Proof of Theorem 4

$$\inf_{\theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[\sup_{x' \in \mathbb{R}^n} L(y\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 4}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L_R(\lambda(y\theta^\top x) / \|\theta\|_*)]$$

$$\lambda \geq 0$$

$$= \inf_{\theta} \varepsilon \|\theta\|_* + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L_R(y\theta^\top x)]$$

$$\begin{aligned} \theta &\leftarrow \lambda \theta / \|\theta\|_* \\ &\Downarrow \\ \lambda &= \|\theta\|_* \end{aligned}$$

Conclusion

Regularization = Distributional Robustness

Take Away

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\Xi) : W_c(\mathbb{Q}, \mathbb{P}) \leq \varepsilon\}$$

Take Away

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{Q \in \mathcal{M}(\Xi) : W_c(Q, \mathbb{P}) \leq \varepsilon\}$$

Step 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\mathbb{P}} \left[\sup_{\zeta \in \Xi} \ell(\theta, \zeta) - \lambda c(\zeta, \xi) \right]$$

[MK18, ZG18, BM19, GK16, ZYG22]

Take Away

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{Q \in \mathcal{M}(\Xi) : W_c(Q, \mathbb{P}) \leq \varepsilon\}$$

Step 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\mathbb{P}} \left[\sup_{\zeta \in \Xi} \ell(\theta, \zeta) - \lambda c(\zeta, \xi) \right]$$

Step 2

Take Away

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{Q \in \mathcal{M}(\Xi) : W_c(Q, \mathbb{P}) \leq \varepsilon\}$$



Step 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\mathbb{P}} \left[\sup_{\zeta \in \Xi} \ell(\theta, \zeta) - \lambda c(\zeta, \xi) \right]$$

Step 2

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Optimality of Affine Hypotheses

Minimum Mean Square Error Estimator

$$J^\star = \inf_{h \in \mathbb{H}} \mathbb{E}_{\textcolor{teal}{P}} [\| \textcolor{red}{h}(\textcolor{violet}{x}) - y \|_2^2]$$

$$\mathbb{H} = \{ h : \mathbb{R}^n \rightarrow \mathbb{R}^m \}$$

Minimum Mean Square Error Estimator

$$J^\star = \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\|h(x) - y\|_2^2]$$

Optimizer: $h^\star(x) = \mathbb{E}_{\mathbb{P}}[y \mid x]$

Optimal value: $J^\star = \text{Tr}(\text{COV}_{\mathbb{P}}[y \mid x])$

MMSE under Normality

$$J^\star = \inf_{\mathbf{h} \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\|\mathbf{h}(x) - y\|_2^2] \quad \& \quad \mathbb{P} = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Optimizer: $\mathbf{h}^\star(x) = \mathbb{E}_{\mathbb{P}}[y \mid x]$

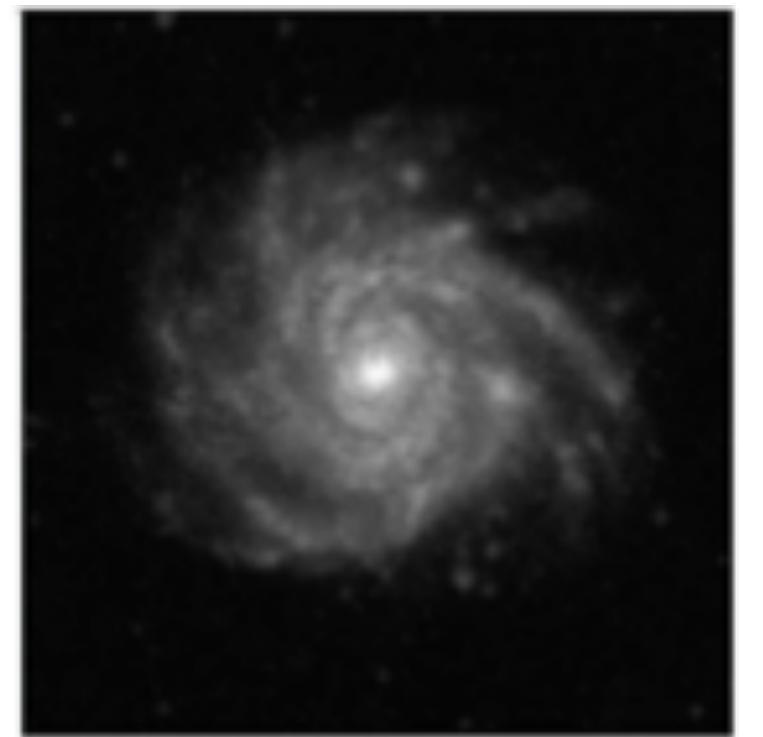
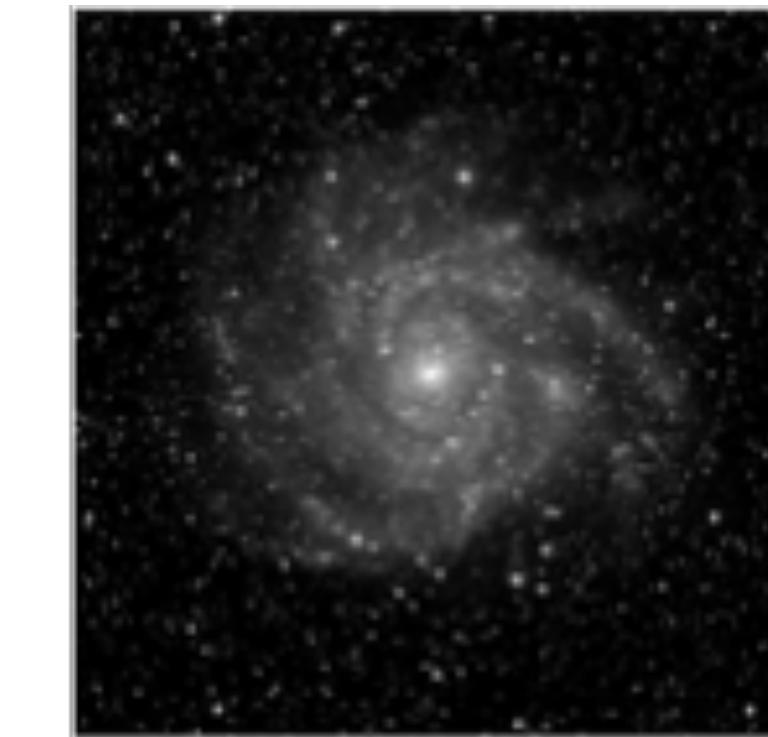
$$\mathbf{h}^\star(x) = \boldsymbol{\Sigma}_{yx} \boldsymbol{\Sigma}_{xx}^{-1} (x - \boldsymbol{\mu}_x) + \boldsymbol{\mu}_y$$

Optimal value: $J^\star = \text{Tr} (\text{COV}_{\mathbb{P}}[y \mid x])$

$$J^\star = \text{Tr} [\boldsymbol{\Sigma}_{yy} - \boldsymbol{\Sigma}_{yx} \boldsymbol{\Sigma}_{xx}^{-1} \boldsymbol{\Sigma}_{xy}]$$

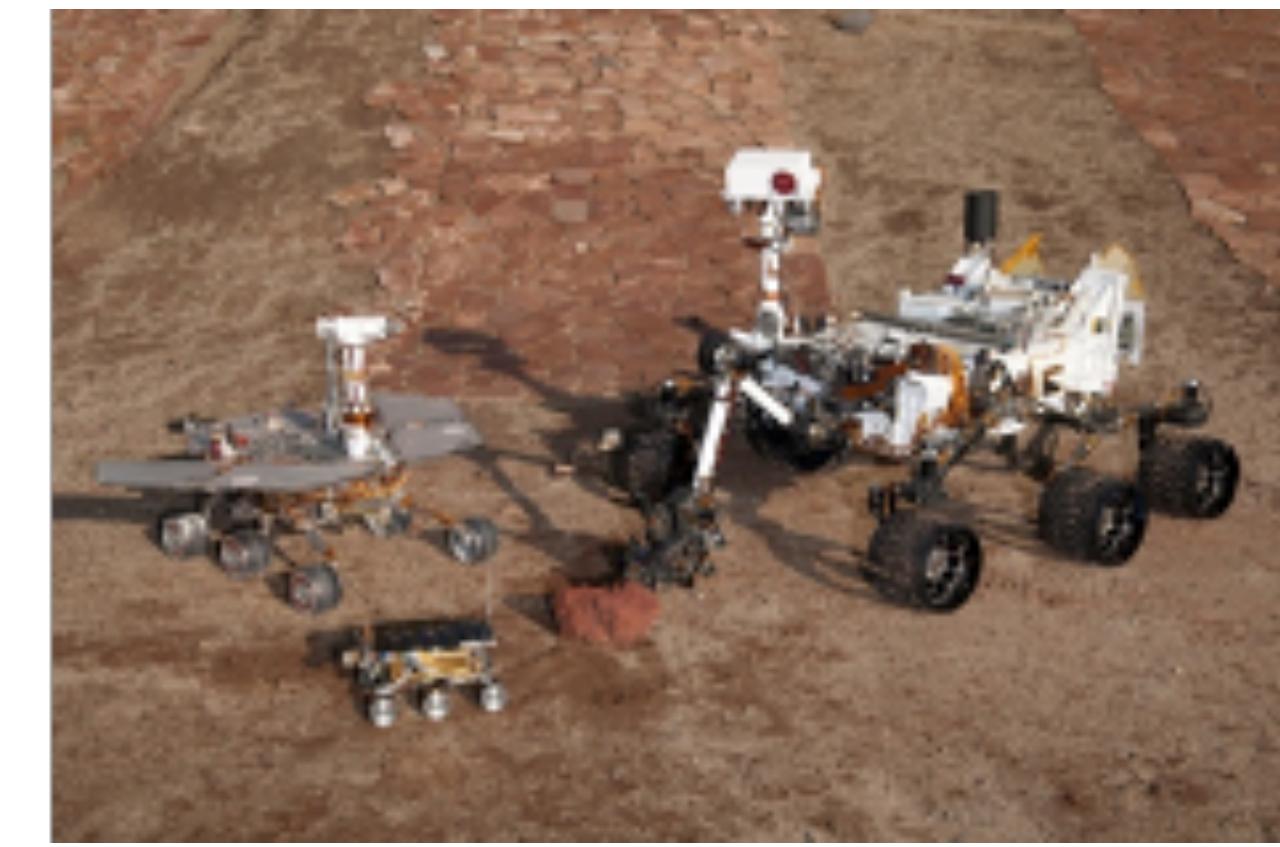
MMSE under Normality

$$J^\star = \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\|h(\textcolor{red}{x}) - y\|_2^2]$$



MMSE under Normality

$$J^{\star} = \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\| \textcolor{red}{h}(\textcolor{violet}{x}) - y \|_2^2]$$

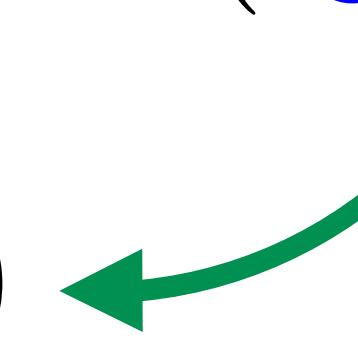


Distributionally Robust MMSE Estimator

$$\inf_{\mathbf{h} \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|\mathbf{h}(x) - y\|_2^2]$$

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{X} \times \mathbb{Y}) : W_c(\mathbb{Q}, \mathbb{P}) \leq \varepsilon\}$$

$$\mathbb{P} = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

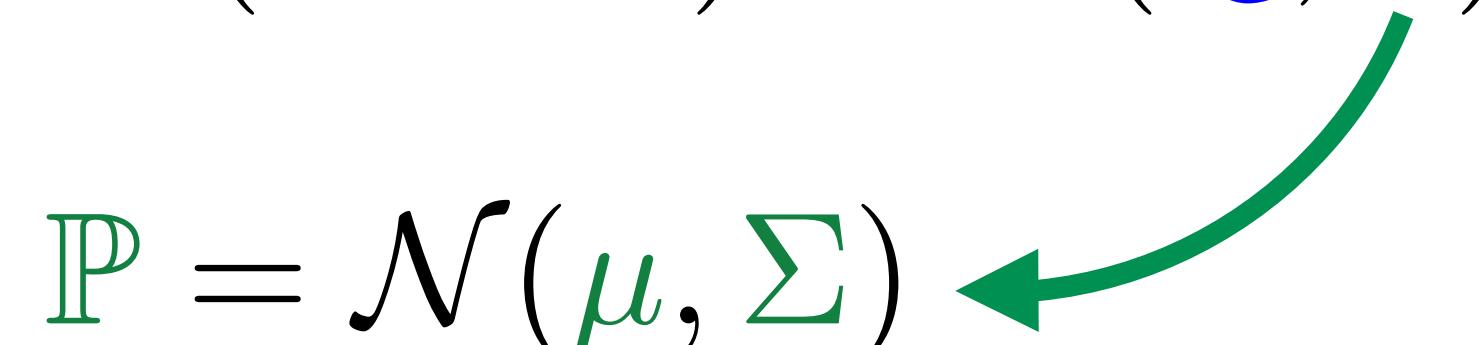


Distributionally Robust MMSE Estimator

$$\inf_{\textcolor{red}{h} \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\textcolor{blue}{\mathbb{Q}}} [\|\textcolor{red}{h}(x) - y\|_2^2]$$

$$\mathcal{P} = \{\textcolor{blue}{\mathbb{Q}} \in \mathcal{M}(\mathbb{X} \times \mathbb{Y}) : W_c(\textcolor{blue}{\mathbb{Q}}, \textcolor{green}{\mathbb{P}}) \leq \varepsilon\}$$

$$\mathbb{P} = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$



$$c((x, y), (x', y')) = \|(x, y) - (x', y')\|_2^2$$

Gelbrich Bound

Lemma 5. For any $\mathbb{Q}_1 \sim (\mu_1, \Sigma_1)$ and $\mathbb{Q}_2 \sim (\mu_2, \Sigma_2)$, we have

$$W_c(\mathbb{Q}_1, \mathbb{Q}_2) \geq \|\mu_1 - \mu_2\|^2 + \text{Tr} \left[\Sigma_1 + \Sigma_2 - 2 \left(\Sigma_2^{\frac{1}{2}} \Sigma_1 \Sigma_2^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]$$

[G90]

Gelbrich Bound

Lemma 5. For any $\mathbb{Q}_1 \sim (\mu_1, \Sigma_1)$ and $\mathbb{Q}_2 \sim (\mu_2, \Sigma_2)$, we have

$$W_c(\mathbb{Q}_1, \mathbb{Q}_2) \geq \|\mu_1 - \mu_2\|^2 + \text{Tr} \left[\Sigma_1 + \Sigma_2 - 2 \left(\Sigma_2^{\frac{1}{2}} \Sigma_1 \Sigma_2^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]$$

Tight under Normality.

[G90]

Gelbrich Bound

Lemma 5. For any $\mathbb{Q}_1 \sim (\mu_1, \Sigma_1)$ and $\mathbb{Q}_2 \sim (\mu_2, \Sigma_2)$, we have

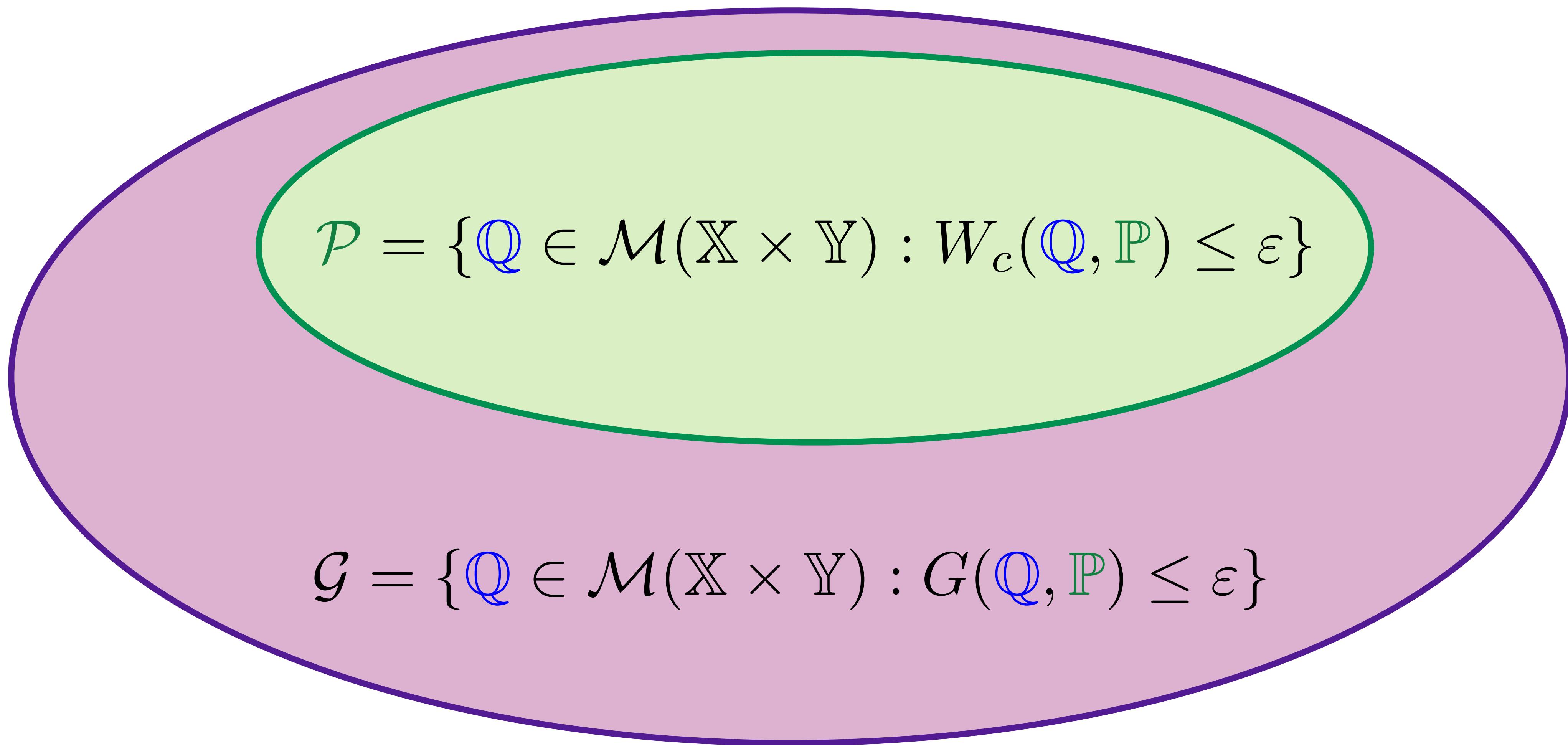
$$W_c(\mathbb{Q}_1, \mathbb{Q}_2) \geq \boxed{\|\mu_1 - \mu_2\|^2 + \text{Tr} \left[\Sigma_1 + \Sigma_2 - 2 \left(\Sigma_2^{\frac{1}{2}} \Sigma_1 \Sigma_2^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]}$$

Tight under Normality.

$G(\mathbb{Q}_1, \mathbb{Q}_2)$

[G90]

Gelbrich Ambiguity Set



Useful Inequalities

$$\inf_{\mathbf{h} \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|\mathbf{h}(\mathbf{x}) - \mathbf{y}\|_2^2]$$

Useful Inequalities

$$\begin{aligned} & \inf_{h \in \mathbb{H}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|h(x) - y\|_2^2] \\ (\text{restriction}) \quad & \leq \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|h(x) - y\|_2^2] \end{aligned}$$

Useful Inequalities

$$\begin{aligned} & \inf_{h \in \mathbb{H}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|h(x) - y\|_2^2] \\ (\text{restriction}) \quad & \leq \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|h(x) - y\|_2^2] \\ (\text{Lemma 5}) \quad & \leq \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{G}} \mathbb{E}_Q [\|h(x) - y\|_2^2] \end{aligned}$$

Useful Inequalities

$$\sup_{\mathbb{Q} \in \mathcal{P}} \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{Q}} [\|h(\textcolor{blue}{x}) - y\|_2^2]$$

$$(\text{weak duality}) \leq \inf_{h \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|h(\textcolor{blue}{x}) - y\|_2^2]$$

$$(\text{restriction}) \leq \inf_{h \in \mathbb{A}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|h(\textcolor{blue}{x}) - y\|_2^2]$$

$$(\text{Lemma 5}) \leq \inf_{h \in \mathbb{A}} \sup_{\mathbb{Q} \in \mathcal{G}} \mathbb{E}_{\mathbb{Q}} [\|h(\textcolor{blue}{x}) - y\|_2^2]$$

Useful Inequalities

(restriction)

$$\sup_{Q \in \mathcal{P} \cap \mathcal{N}} \inf_{h \in \mathbb{H}} \mathbb{E}_Q [\|h(x) - y\|_2^2] \leq \sup_{Q \in \mathcal{P}} \inf_{h \in \mathbb{H}} \mathbb{E}_Q [\|h(x) - y\|_2^2]$$

$$(\text{weak duality}) \leq \inf_{h \in \mathbb{H}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|h(x) - y\|_2^2]$$

$$(\text{restriction}) \leq \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|h(x) - y\|_2^2]$$

$$(\text{Lemma 5}) \leq \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{G}} \mathbb{E}_Q [\|h(x) - y\|_2^2]$$

Sandwich Theorem

Theorem 5. We have



$$\sup_{Q \in \mathcal{P} \cap \mathcal{N}} \inf_{h \in \mathbb{H}} E_Q [\|h(x) - y\|_2^2] = \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{G}} E_Q [\|h(x) - y\|_2^2]$$

[NSKM21]

Optimality of Affine Estimators

$$\inf_{h \in \mathbb{H}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|h(x) - y\|_2^2] = \sup_{Q \in \mathcal{P}} \inf_{h \in \mathbb{H}} \mathbb{E}_Q [\|h(x) - y\|_2^2]$$

h^\star is affine

Q^\star is Normal

$$h^\star = \operatorname{argmin}_{h \in \mathbb{H}} \mathbb{E}_{Q^\star} [\|h(x) - y\|_2^2]$$