

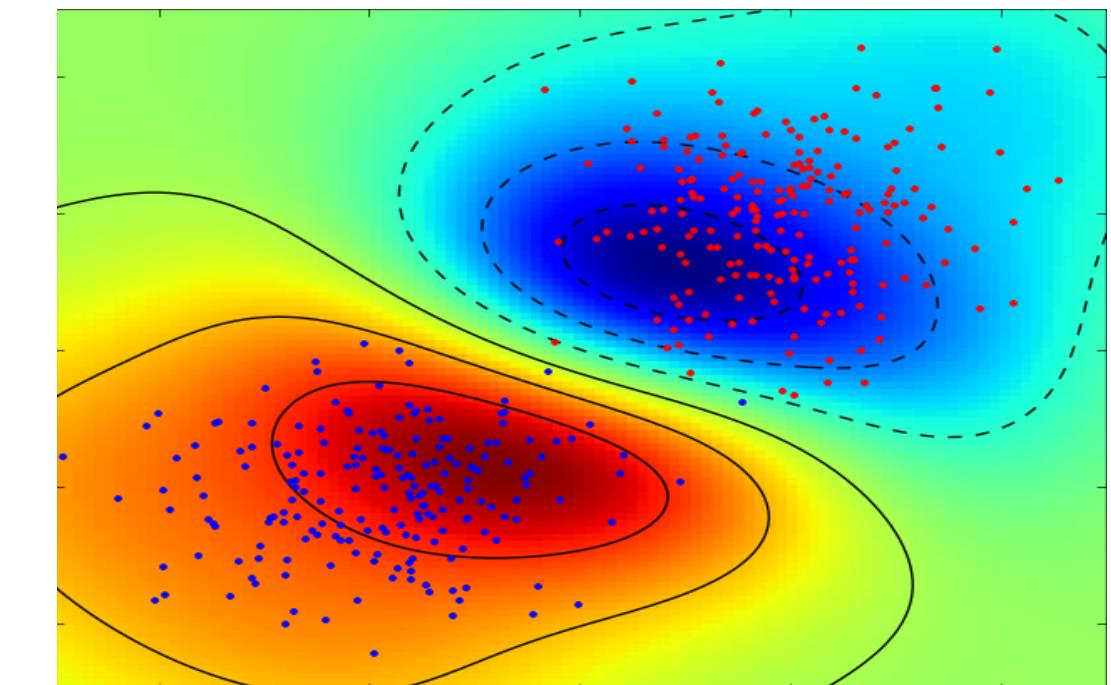
# Regularization via Optimal Transport

Soroosh Shafieezadeh Abadeh, Tepper School of Business, CMU

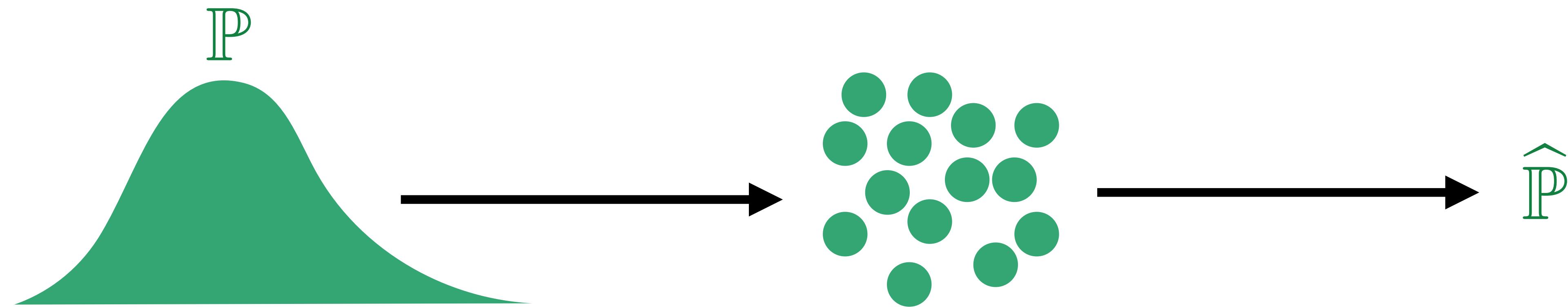


# Stochastic Programming

$$\inf_{\theta \in \Theta} \mathbf{E}_{\mathbf{P}} [\ell(\theta, \xi)]$$



# Failure Examples: Overfitting



$$\inf_{\theta \in \Theta} \mathbb{E}_{\hat{P}} [\ell(\theta, \xi)]$$

# Failure Examples: Adversarial Attack [GSS15]



$+ .007 \times$



=



Panda  
57.7% Confidence

Gibbon  
99.3% Confidence

# Failure Examples: Fake Data

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SHARE



## Fake Reviews and Inflated Ratings Are Still a Problem for Amazon

Sellers are taking advantage of the online-shopping frenzy, using old and new methods to boost ratings on products



By [Nicole Nguyen](#)

June 13, 2021 8:28 am ET



PRINT



TEXT

191



 Listen to article (10 minutes)

A charging brick recently caught my eye on [Amazon](#). **AMZN -2.96% ▼** It was a RAVPower-branded two-port [fast charger](#), and it had five stars with over 9,800 ratings. The score seemed suspect but Amazon itself

### UPCOMING EVENTS



Oct

5

2021

12:00 PM - 5:00 PM EDT

WSJ Jobs Summit

Oct

6

2021

12:30 PM - 2:00 PM EDT

The Future Of Health

# Failure Examples: Domain Change [TPJR18]



original



fog



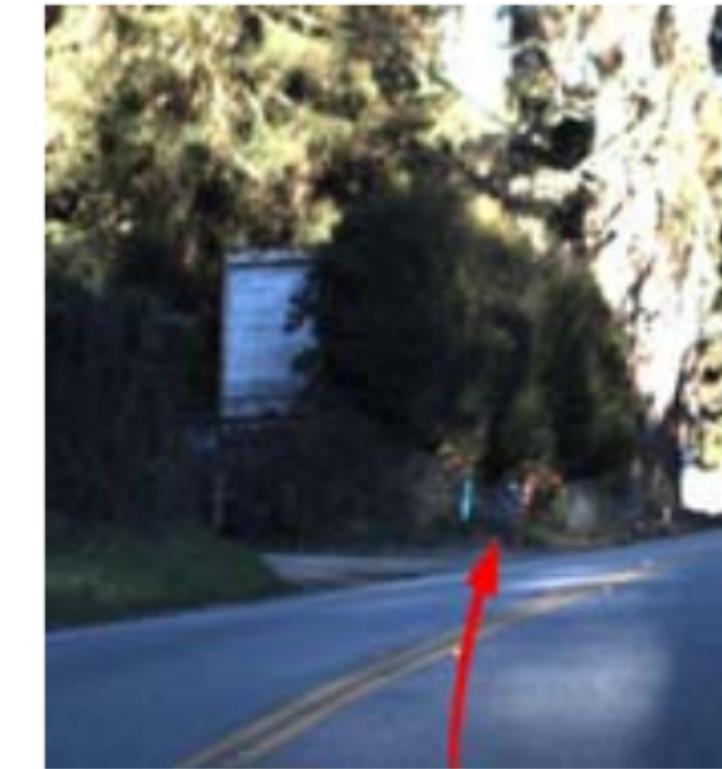
original



rain



original



shear(0.1)



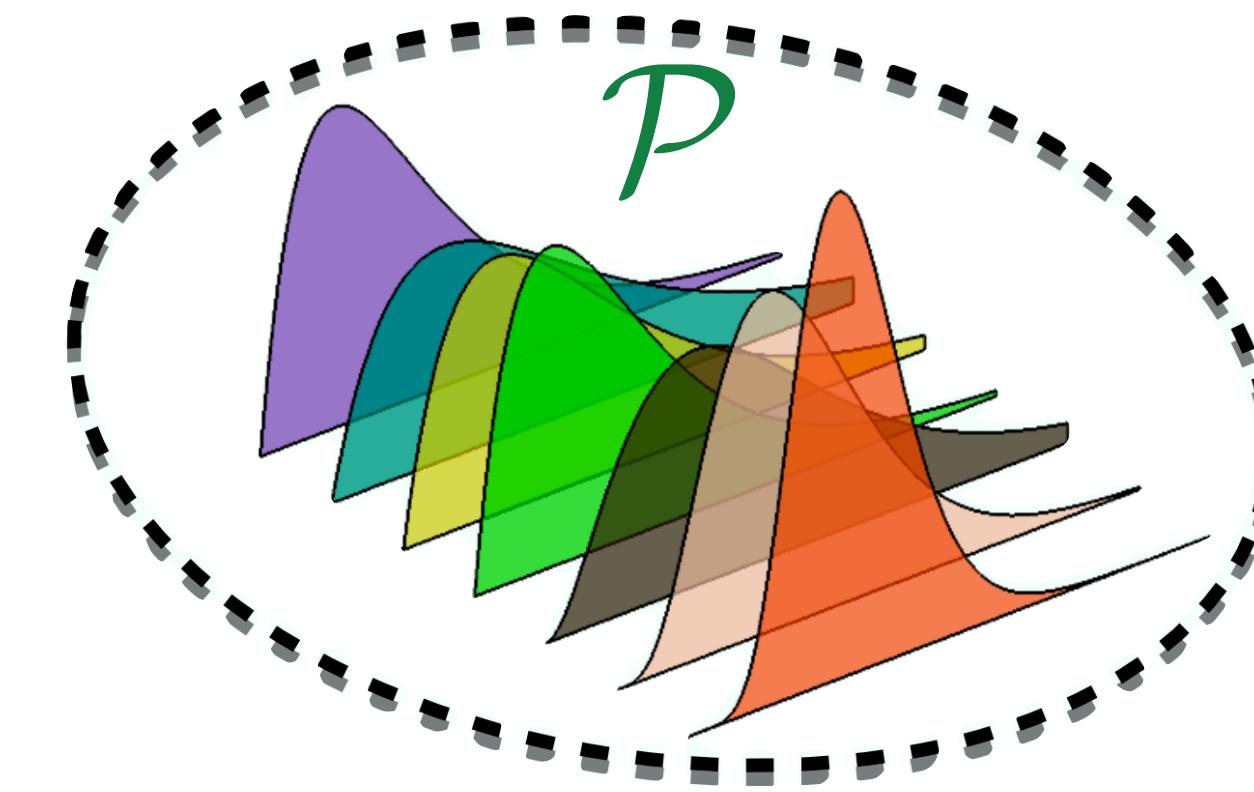
original



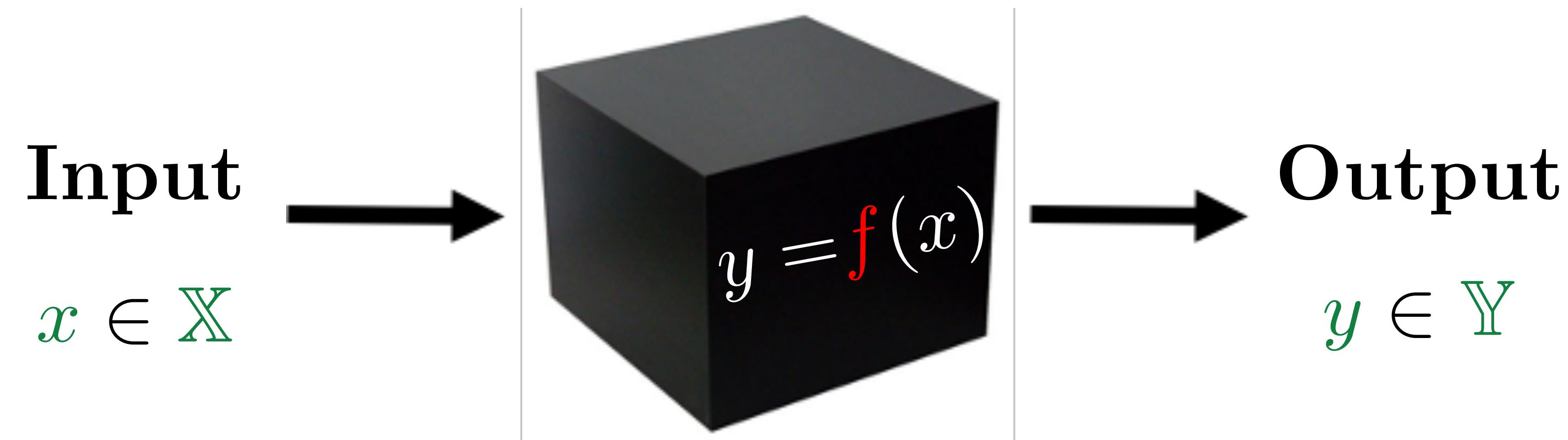
rotation(6 degree)

# Distributionally Robust Optimization

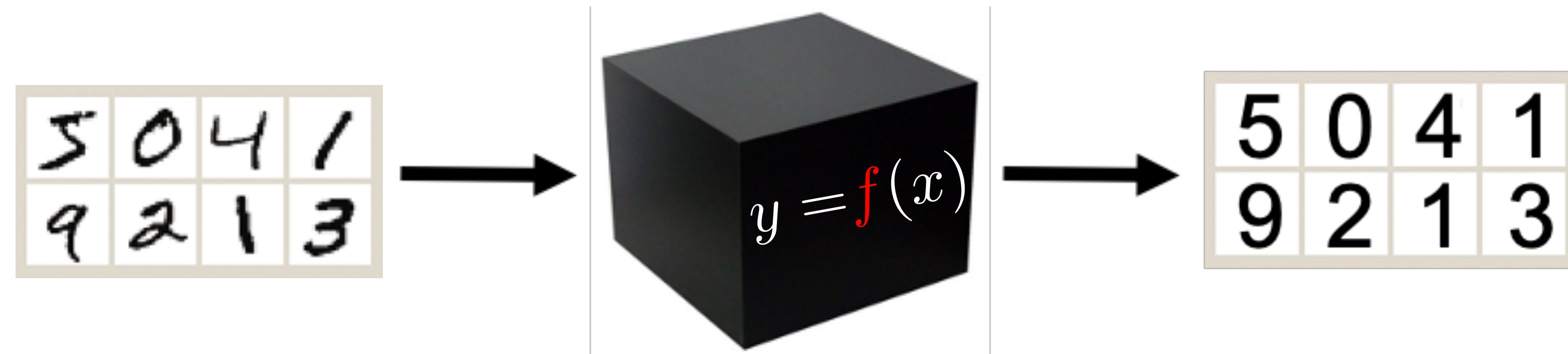
$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$



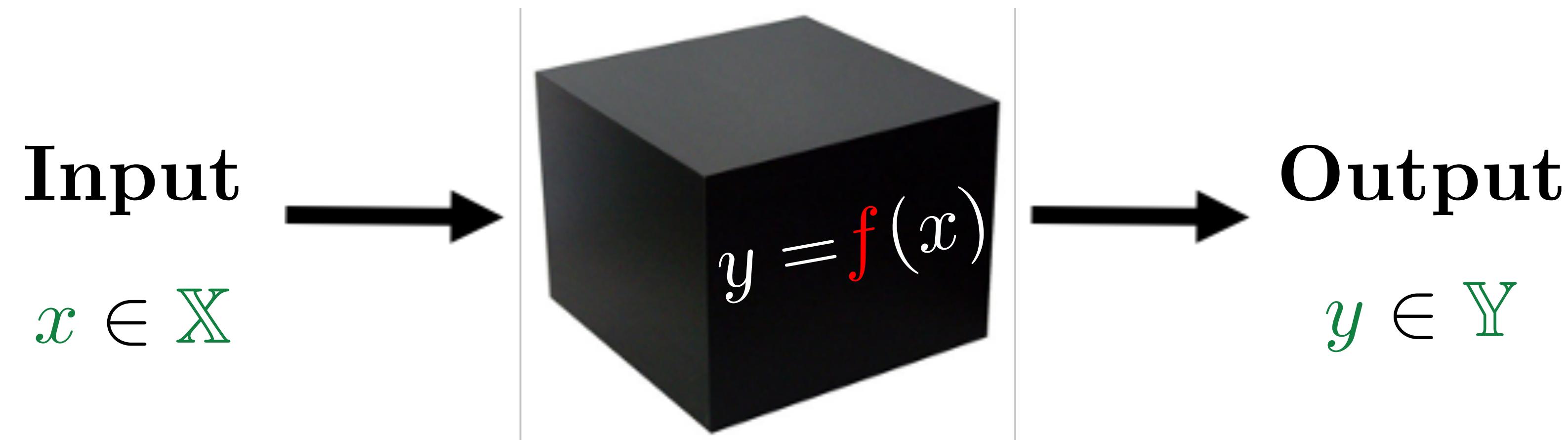
# Supervised Learning



# Supervised Learning

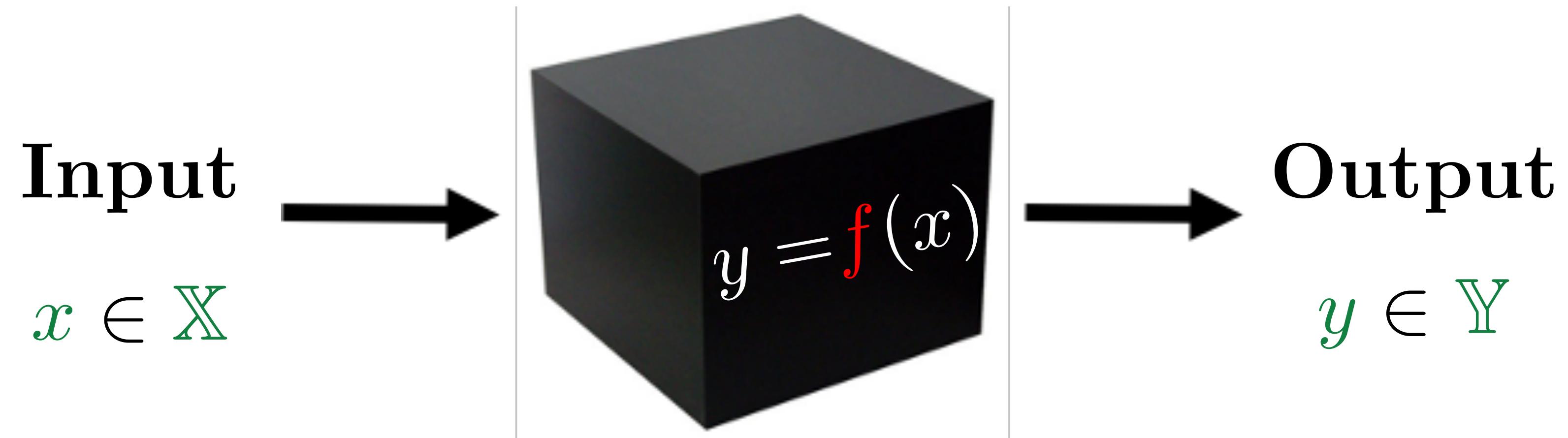


# Supervised Learning



Training data:  $\hat{\mathbb{E}}_N = (\hat{x}_1, \hat{y}_1), \dots, (\hat{x}_N, \hat{y}_N)$

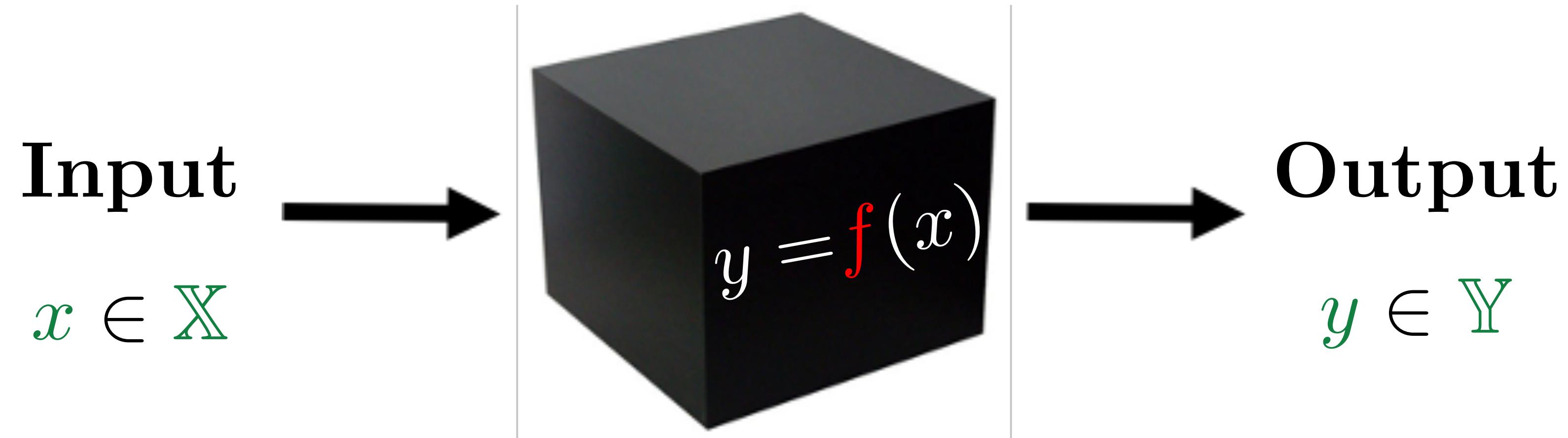
# Supervised Learning



Training data:  $\hat{\Xi}_N = (\hat{x}_1, \hat{y}_1), \dots, (\hat{x}_N, \hat{y}_N)$

Hypothesis space:  $\mathbb{H} \subseteq \{h \in \mathbb{R}^{\mathbb{X}}\}$

# Supervised Learning

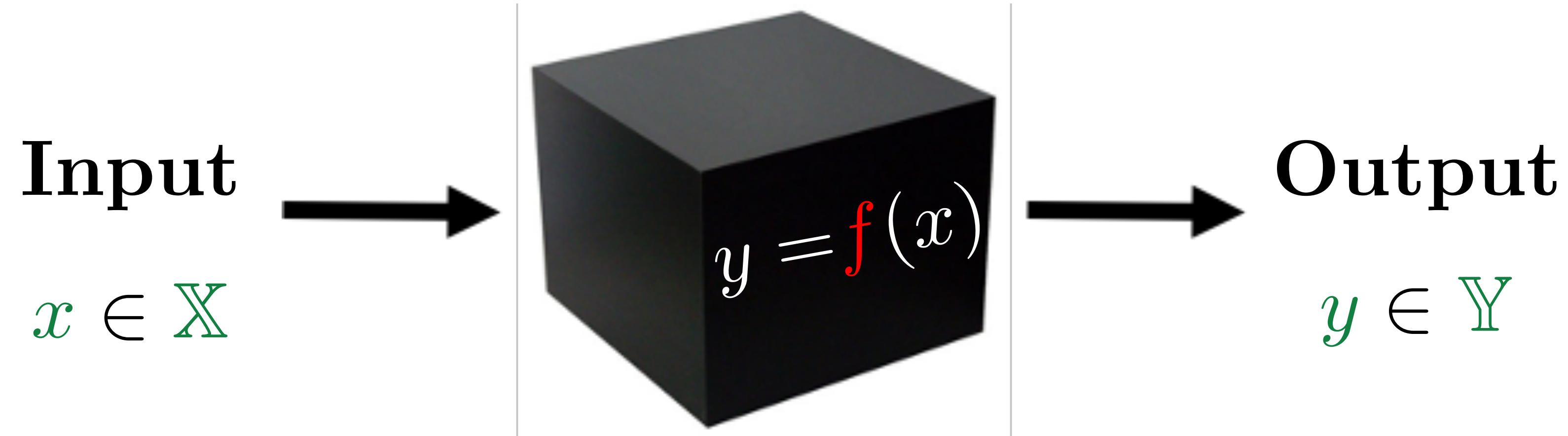


Training data:  $\hat{\Xi}_N = (\hat{x}_1, \hat{y}_1), \dots, (\hat{x}_N, \hat{y}_N)$

Hypothesis space:  $\mathbb{H} \subseteq \{h \in \mathbb{R}^{\mathbb{X}}\}$

Target function:  $f(x) \approx h(x)$

# Supervised Learning



Training data:  $\widehat{\Xi}_N = (\widehat{x}_1, \widehat{y}_1), \dots (\widehat{x}_N, \widehat{y}_N)$

Hypothesis space:  $\mathbb{H} \subseteq \{h \in \mathbb{R}^{\mathbb{X}}\}$

Target function:  $f(x) \approx h(x)$

Learning algorithm:  $\inf_{h \in \mathbb{H}} \ell(h, \widehat{\Xi}_N)$

# Regression Models

Target function:  $f(\textcolor{violet}{x}) = \textcolor{red}{h}(\textcolor{violet}{x})$

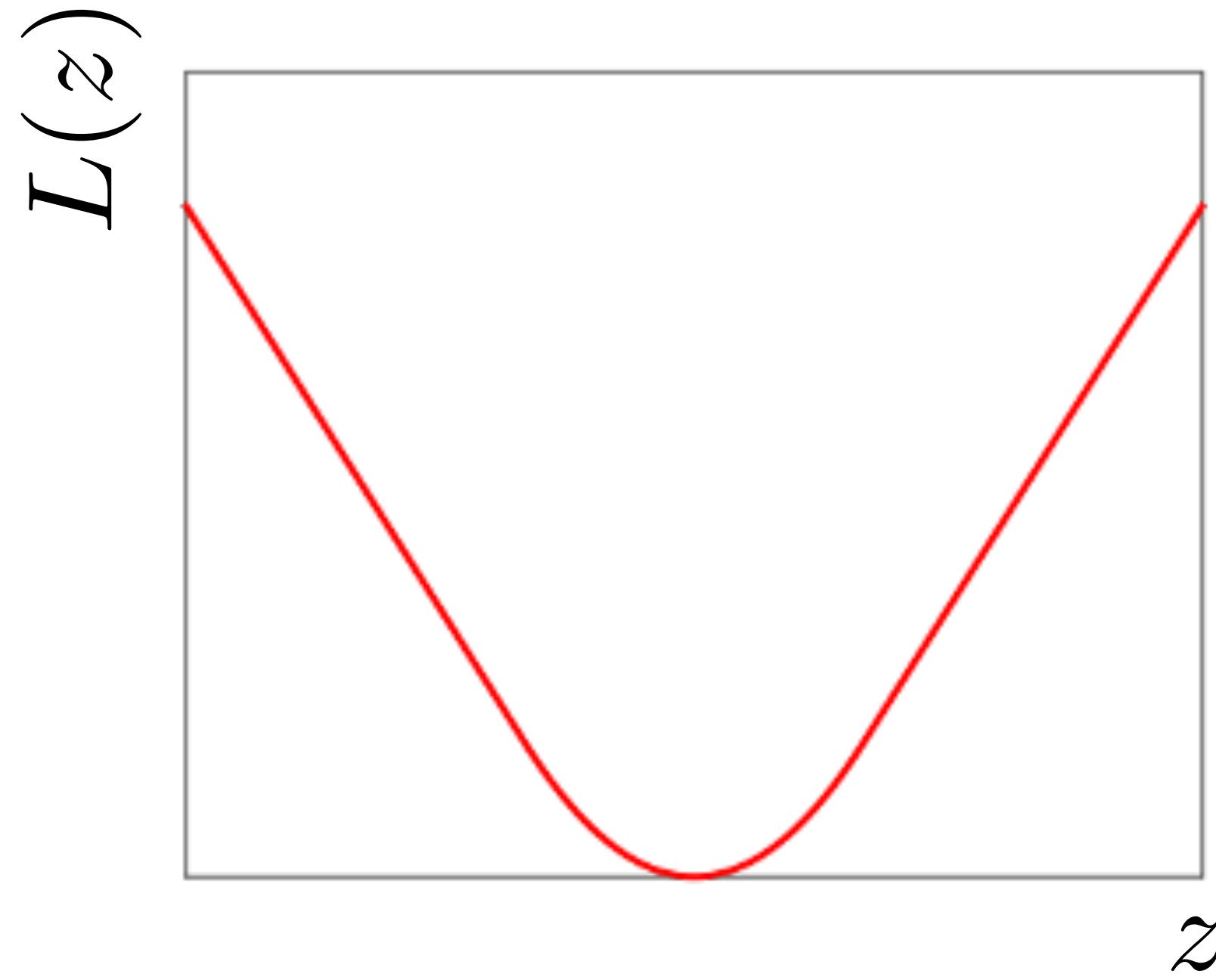
Empirical risk minimization:  $\ell(\textcolor{red}{h}, \widehat{\Xi}_N) = \frac{1}{N} \sum_{i=1}^N L(\textcolor{red}{h}(\widehat{x}_i) - \widehat{y}_i)$

# Regression Models

Target function:  $f(\textcolor{violet}{x}) = \textcolor{red}{h}(x)$

Empirical risk minimization:  $\ell(\textcolor{red}{h}, \widehat{\Sigma}_N) = \frac{1}{N} \sum_{i=1}^N L(\textcolor{red}{h}(\widehat{x}_i) - \widehat{y}_i)$

## Robust Regression



### Huber Loss:

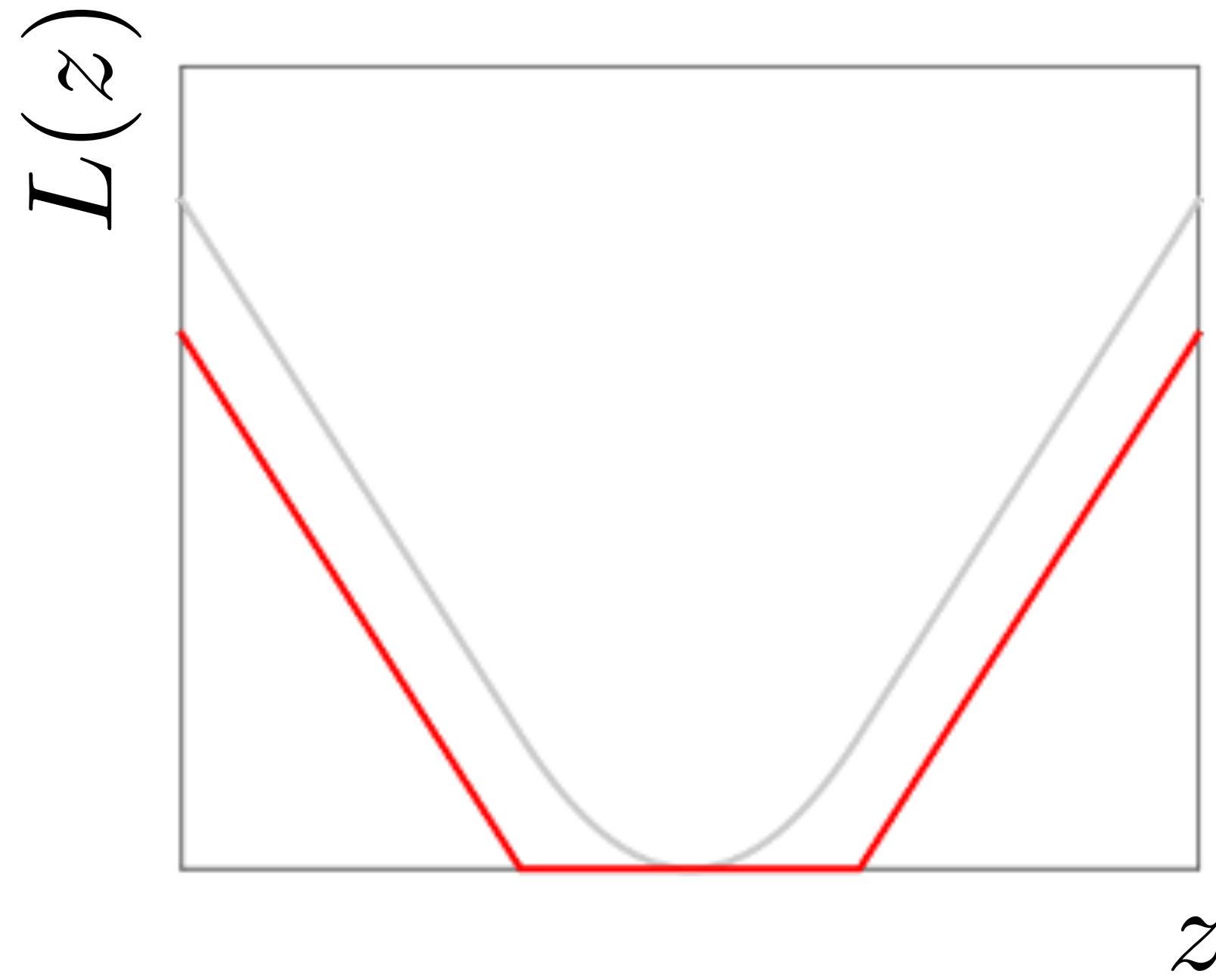
$$L(z) = \begin{cases} \frac{1}{2}z^2 & \text{if } |z| \leq \delta \\ \delta(|z| - \frac{1}{2}\delta) & \text{else} \end{cases}$$

# Regression Models

Target function:  $f(\textcolor{violet}{x}) = \textcolor{red}{h}(\textcolor{violet}{x})$

Empirical risk minimization:  $\ell(\textcolor{red}{h}, \hat{\Xi}_N) = \frac{1}{N} \sum_{i=1}^N L(\textcolor{red}{h}(\hat{x}_i) - \hat{y}_i)$

## Support Vector Regression



$\epsilon$ -insensitive Loss:

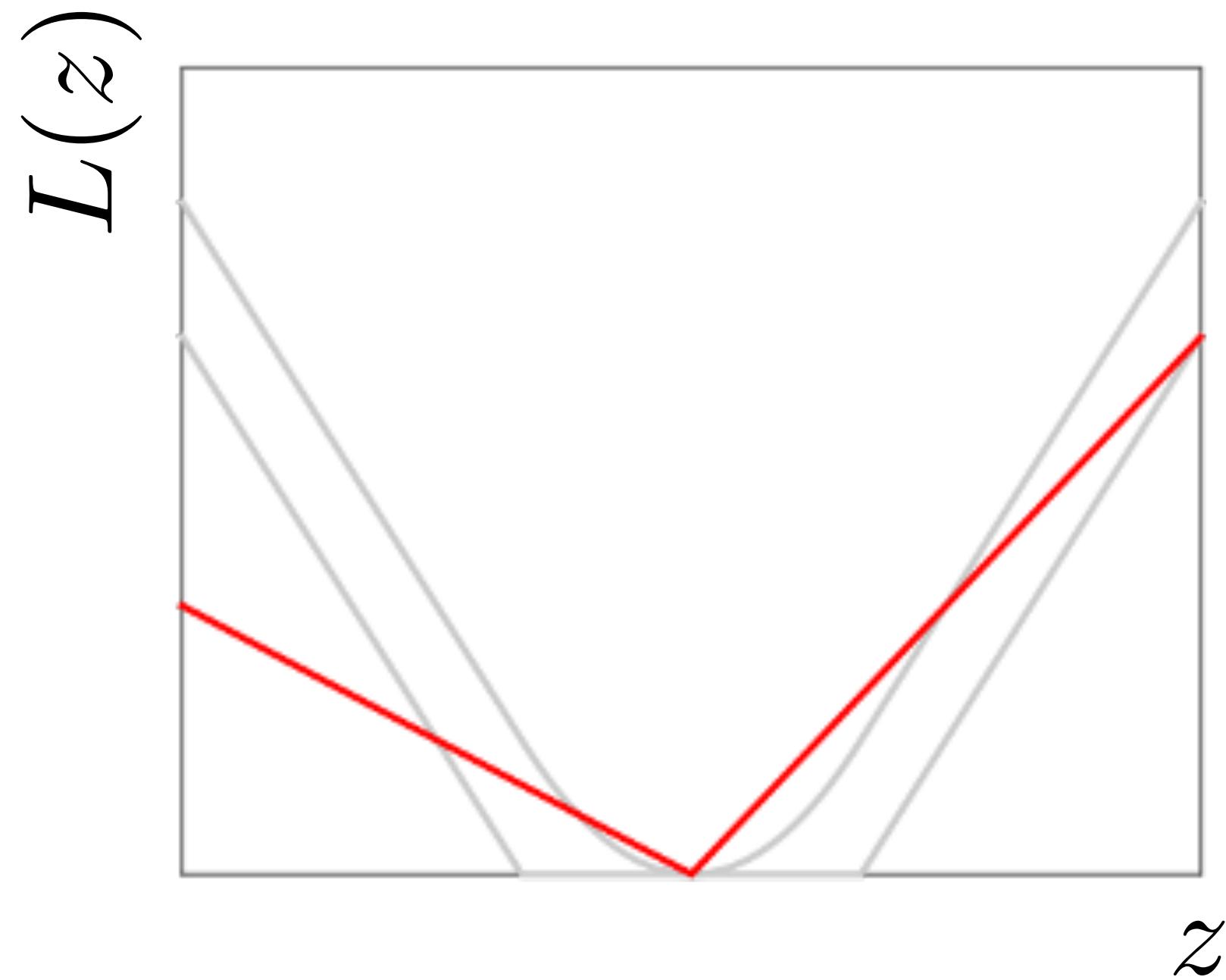
$$L(z) = \max\{0, |z| - \epsilon\}$$

# Regression Models

Target function:  $f(\mathbf{x}) = h(\mathbf{x})$

Empirical risk minimization:  $\ell(h, \hat{\Sigma}_N) = \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i) - \hat{y}_i)$

## Quantile Regression



Pinball Loss:

$$L(z) = \max\{-\tau z, (1 - \tau)z\}$$

# Classification Models

Target function:  $f(\textcolor{violet}{x}) = \operatorname{sgn}(\textcolor{red}{h}(x))$

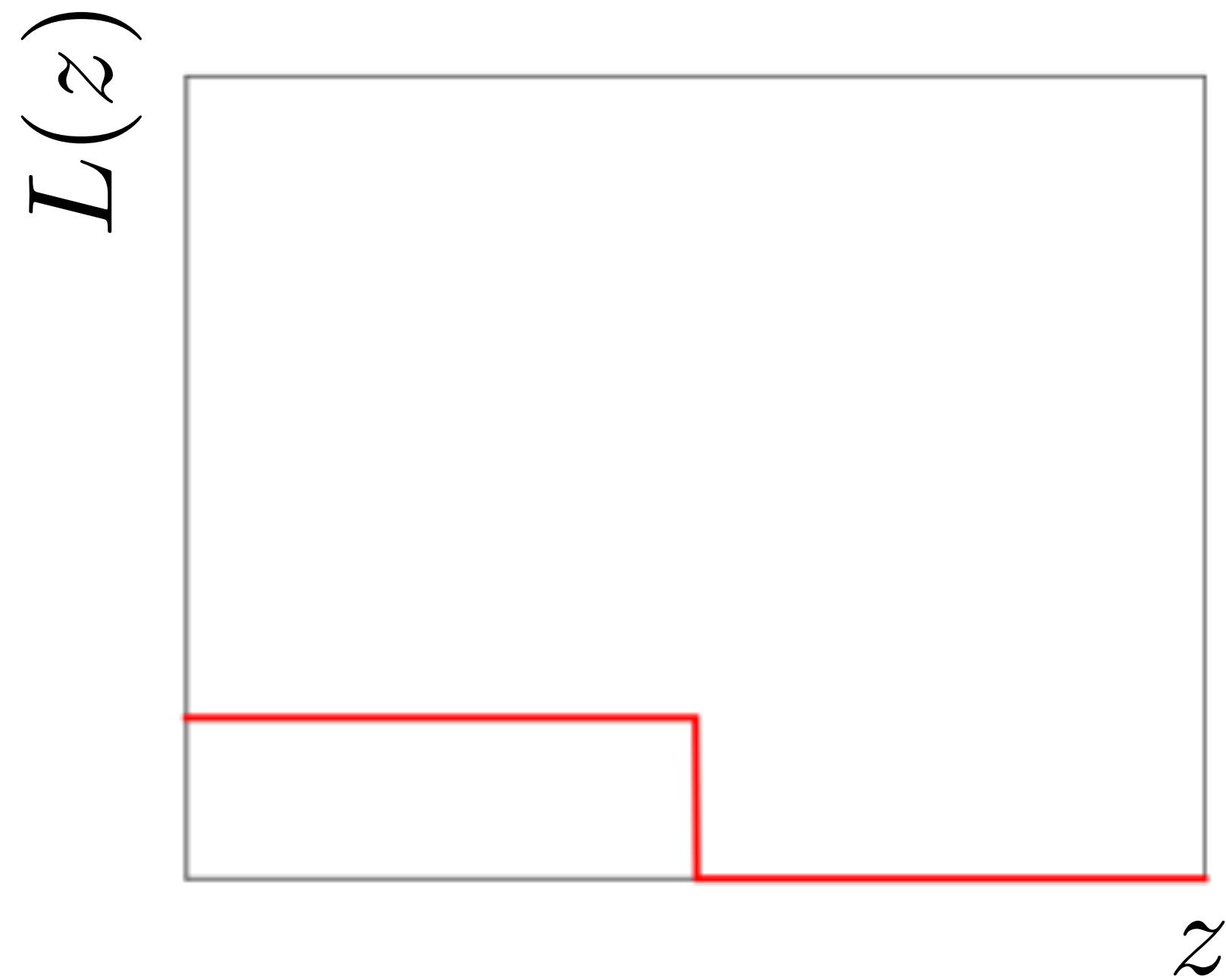
Empirical risk minimization:  $\ell(\textcolor{red}{h}, \widehat{\Xi}_{\textcolor{violet}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\widehat{y}_i \textcolor{red}{h}(\widehat{x}_i))$

# Classification Models

Target function:  $f(\textcolor{teal}{x}) = \operatorname{sgn}(h(\textcolor{red}{x}))$

Empirical risk minimization:  $\ell(\textcolor{red}{h}, \hat{\Sigma}_{\textcolor{teal}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \textcolor{red}{h}(\hat{x}_i))$

## Ideal Classification



### 0-1 Loss:

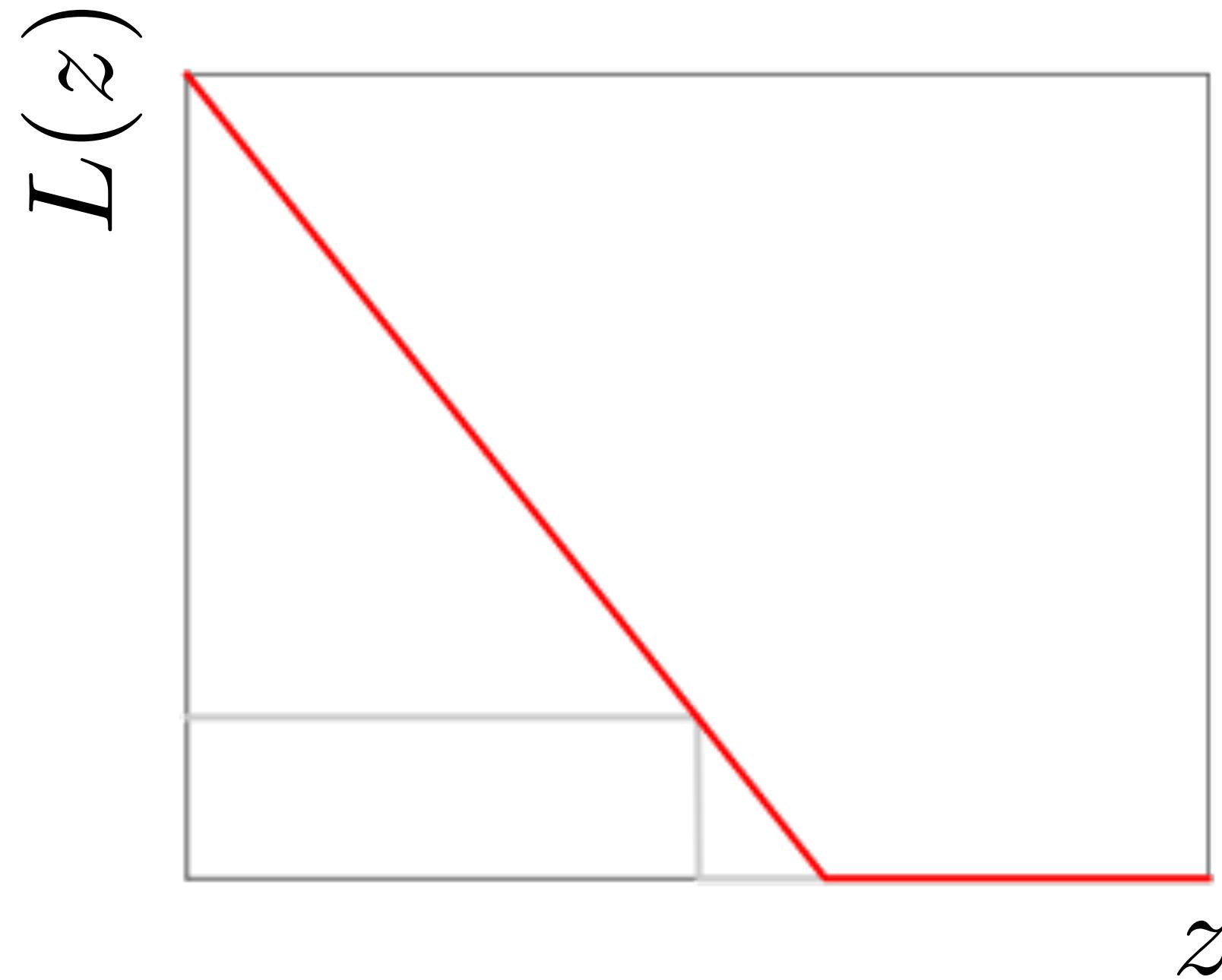
$$L(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ 0 & \text{else} \end{cases}$$

# Classification Models

Target function:  $f(\textcolor{teal}{x}) = \operatorname{sgn}(h(\textcolor{red}{x}))$

Empirical risk minimization:  $\ell(\textcolor{red}{h}, \hat{\Xi}_{\textcolor{teal}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \textcolor{red}{h}(\hat{x}_i))$

## Support Vector Machine



### Hinge Loss:

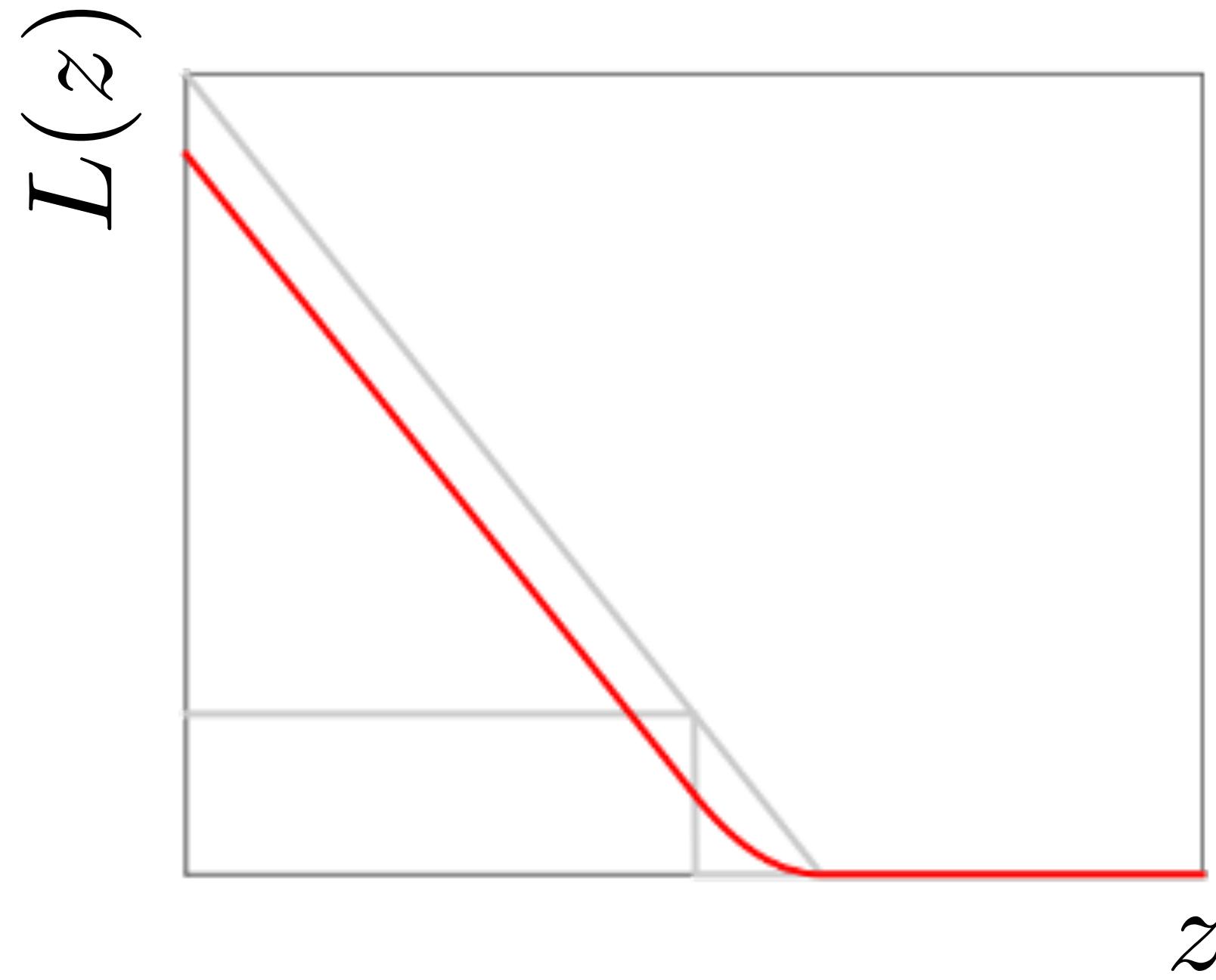
$$L(z) = \max\{0, 1 - z\}$$

# Classification Models

Target function:  $f(\mathbf{x}) = \text{sgn}(h(\mathbf{x}))$

Empirical risk minimization:  $\ell(h, \hat{\Xi}_N) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i h(\hat{x}_i))$

## Support Vector Machine II



Smooth Hinge Loss:

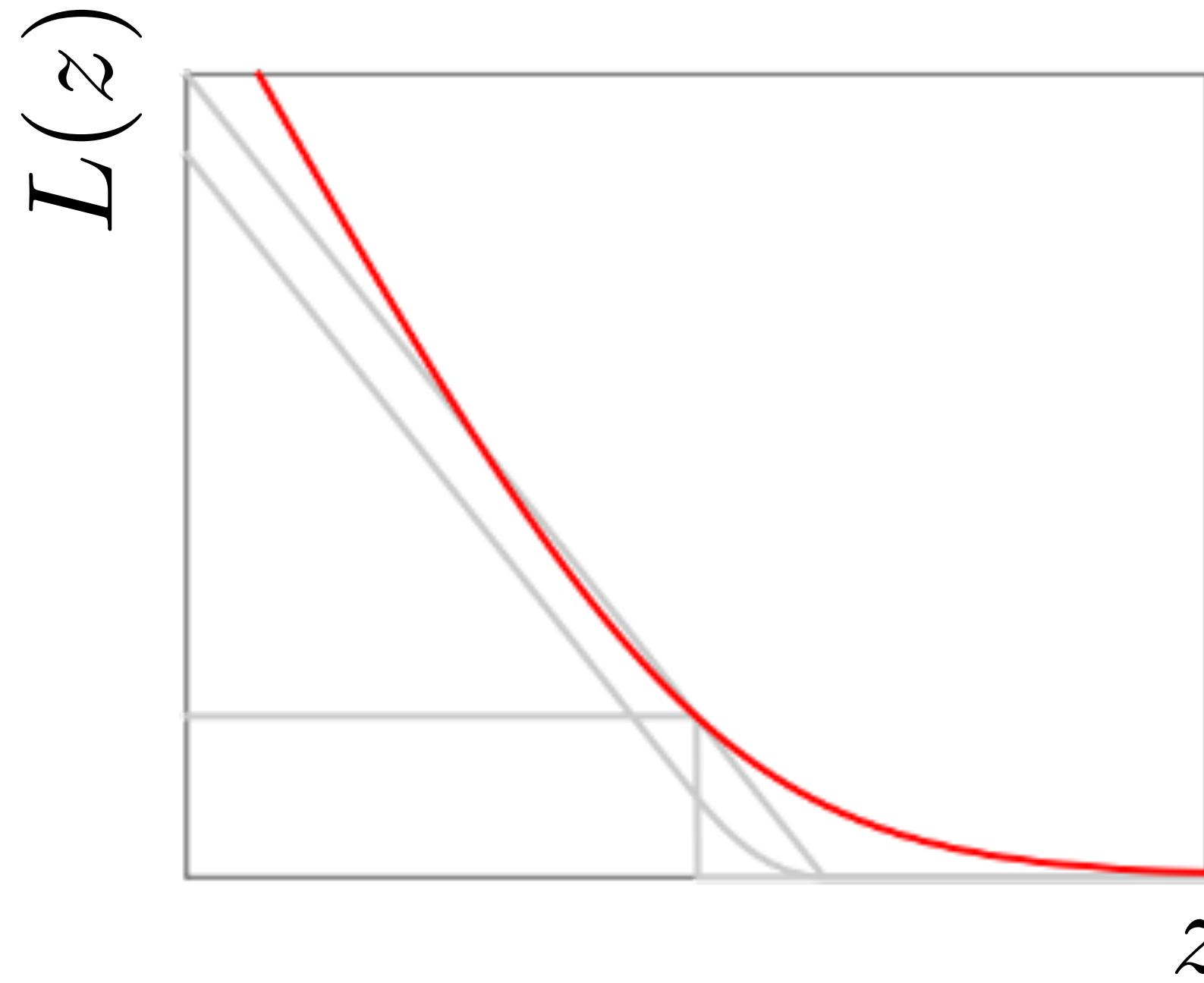
$$L(z) = \begin{cases} \frac{1}{2} - z & \text{if } z \leq 0 \\ \frac{1}{2}(1-z)^2 & \text{if } 0 < z < 1 \\ 0 & \text{else} \end{cases}$$

# Classification Models

Target function:  $f(\textcolor{teal}{x}) = \operatorname{sgn}(\textcolor{red}{h}(\textcolor{teal}{x}))$

Empirical risk minimization:  $\ell(\textcolor{red}{h}, \hat{\Sigma}_{\textcolor{teal}{N}}) = \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \textcolor{red}{h}(\hat{x}_i))$

## Logistic Regression



Logloss:

$$L(z) = \log(1 + \exp(-z))$$

# Performance of ERM

$$h_{\text{ERM}} = \operatorname{argmin}_{h \in \mathbb{H}} \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i)$$

In-Sample Loss:

$$\mathbb{E}_{\hat{\mathbb{P}}_N} [L(h(x), y)]$$

# Performance of ERM

$$h_{\text{ERM}} = \operatorname{argmin}_{h \in \mathbb{H}} \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i)$$

In-Sample Loss:

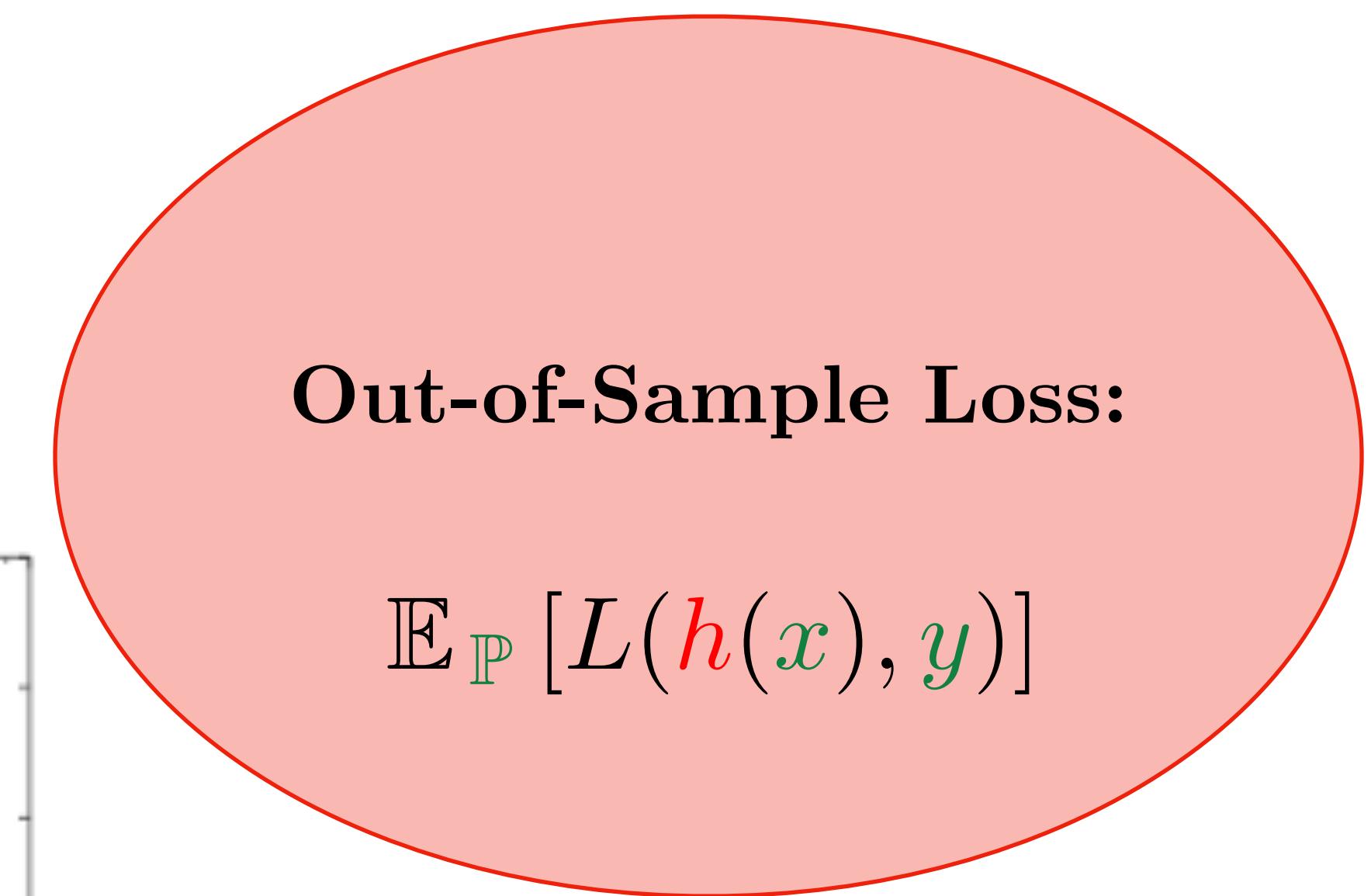
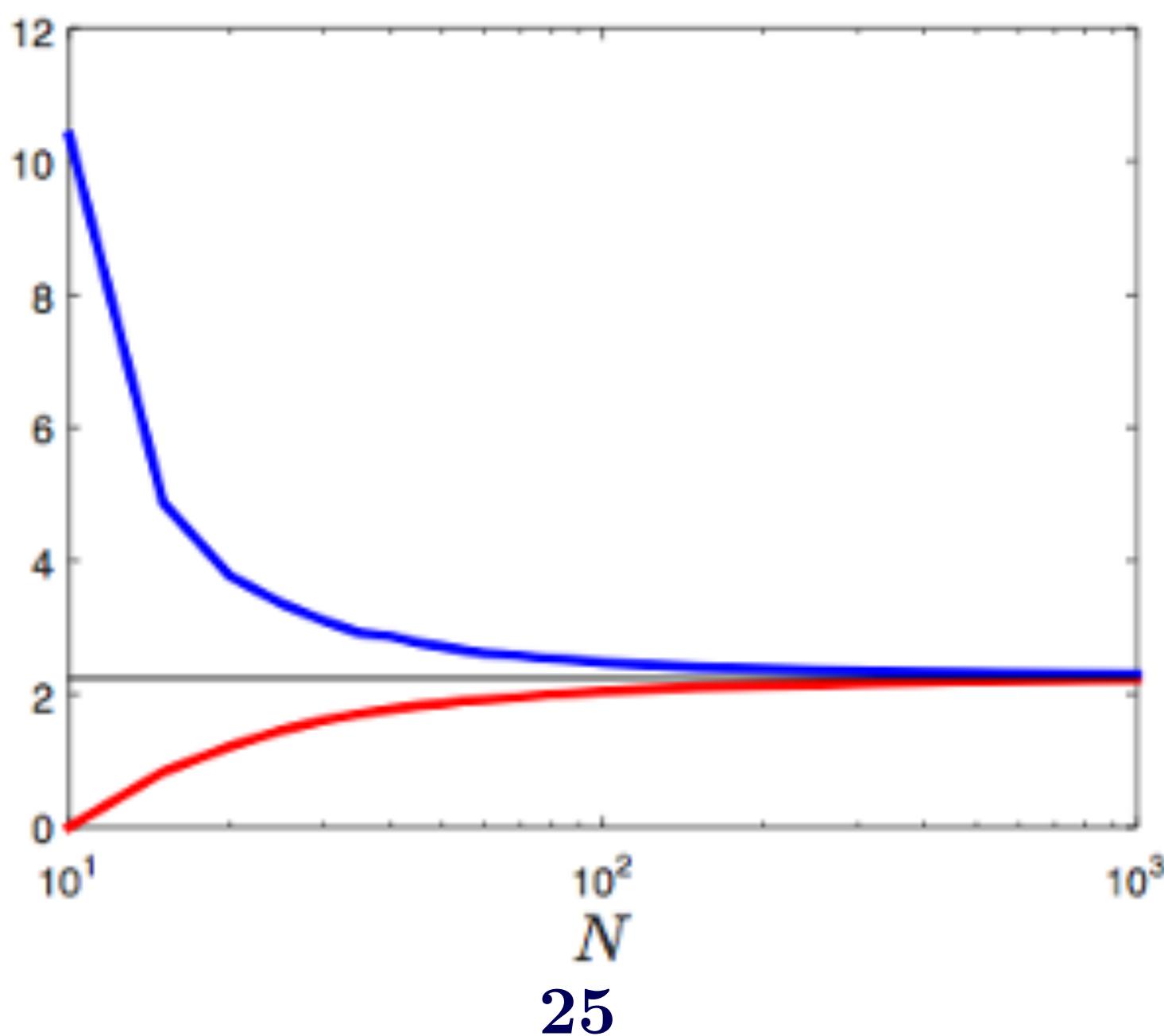
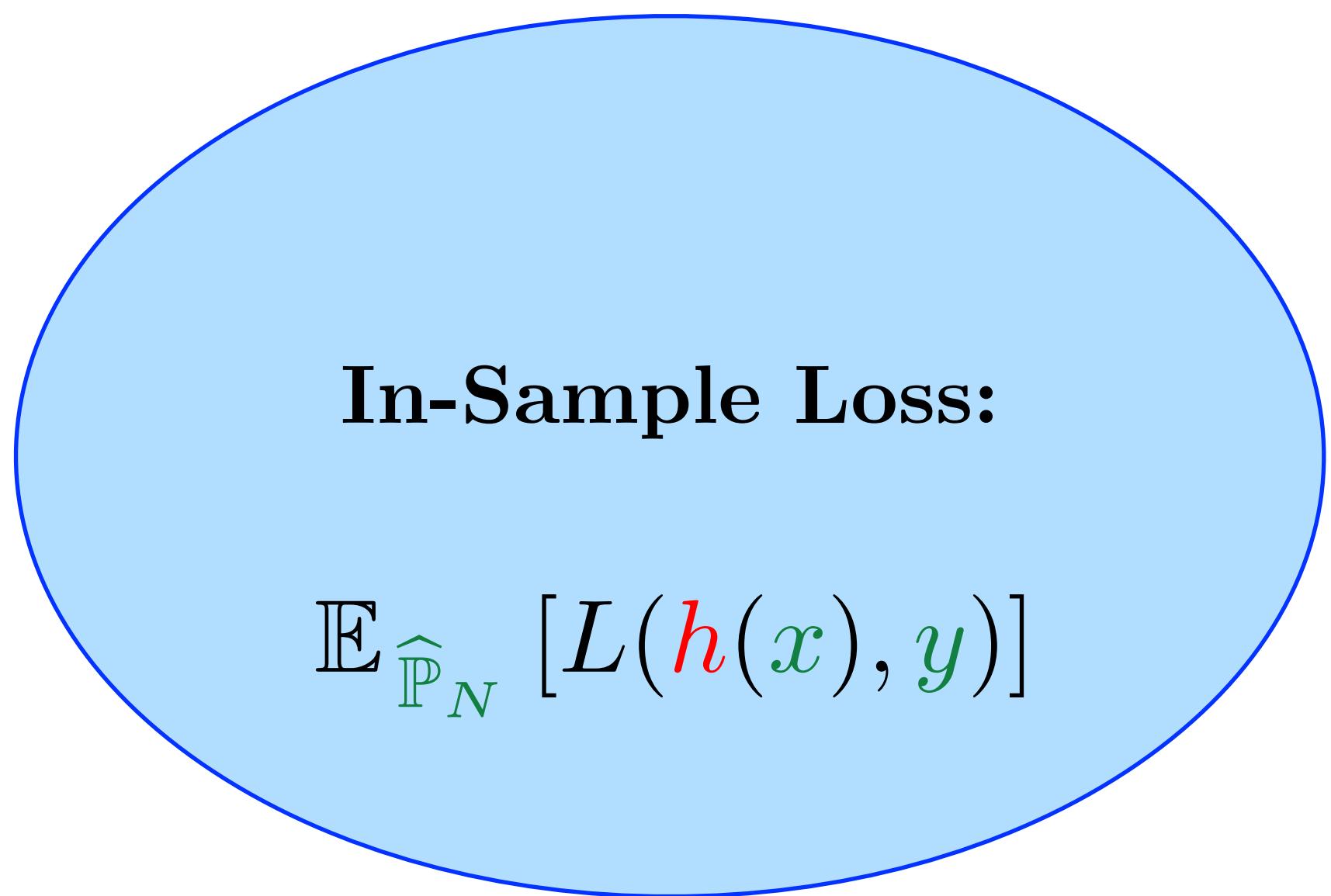
$$\mathbb{E}_{\hat{\mathbb{P}}_N} [L(h(x), y)]$$

Out-of-Sample Loss:

$$\mathbb{E}_{\mathbb{P}} [L(h(x), y)]$$

# Performance of ERM

$$h_{\text{ERM}} = \operatorname{argmin}_{h \in \mathbb{H}} \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i)$$



# Regularized ERM

$$h_{\text{REG}} = \underset{h \in \mathbb{H}}{\operatorname{argmin}} \quad \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i) + \varepsilon \Omega(h)$$

The diagram illustrates the components of the Regularized ERM formula. A pink arrow points from the term  $\varepsilon \Omega(h)$  to the text "Regularization coefficient". Another pink arrow points from the term  $\Omega(h)$  to the text "Regularization function".

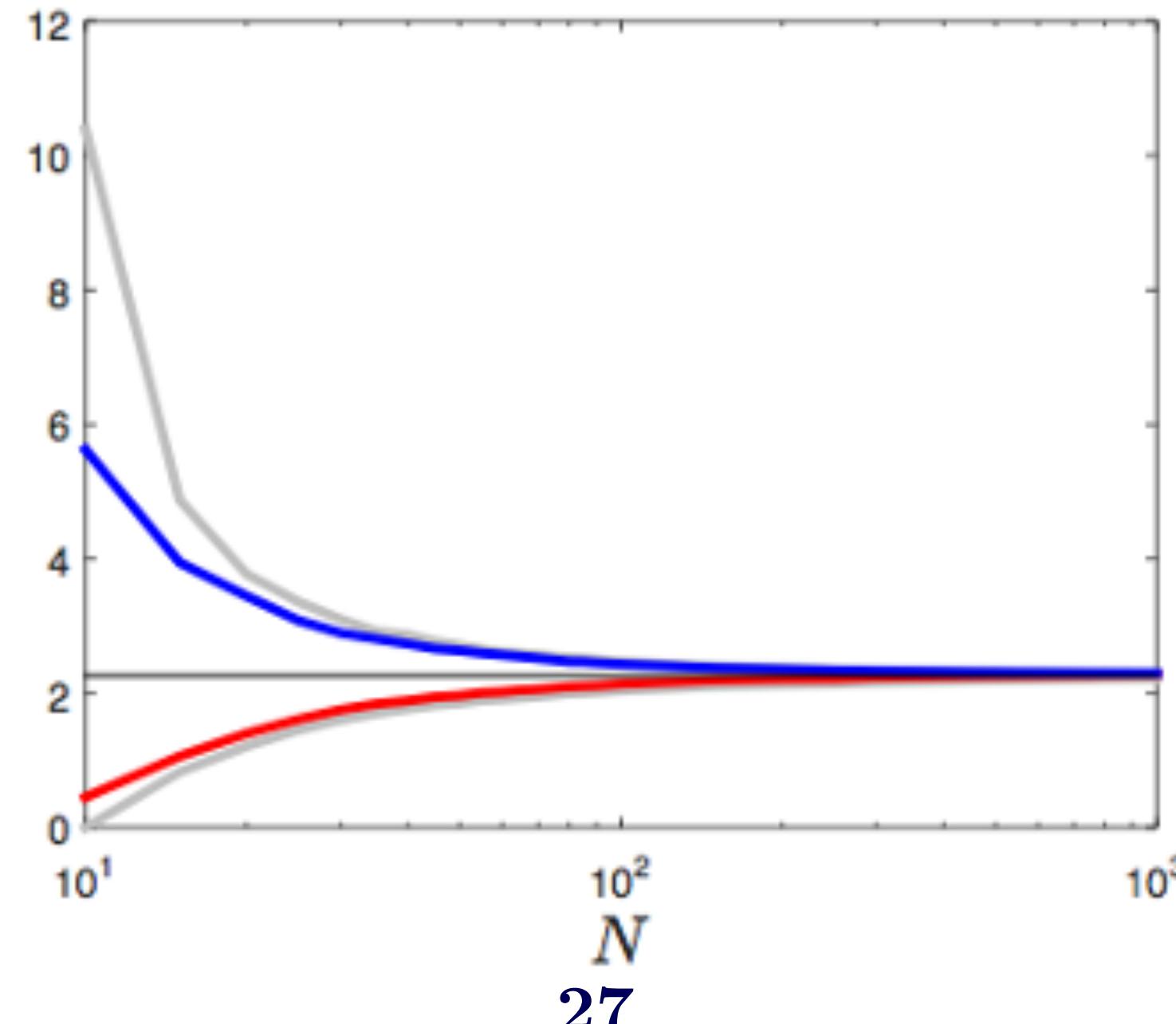
Regularization coefficient      Regularization function

# Regularized ERM

$$h_{\text{REG}} = \underset{h \in \mathbb{H}}{\operatorname{argmin}} \quad \frac{1}{N} \sum_{i=1}^N L(h(\hat{x}_i), \hat{y}_i) + \varepsilon \Omega(h)$$

Diagram illustrating the components of the regularized ERM formula:

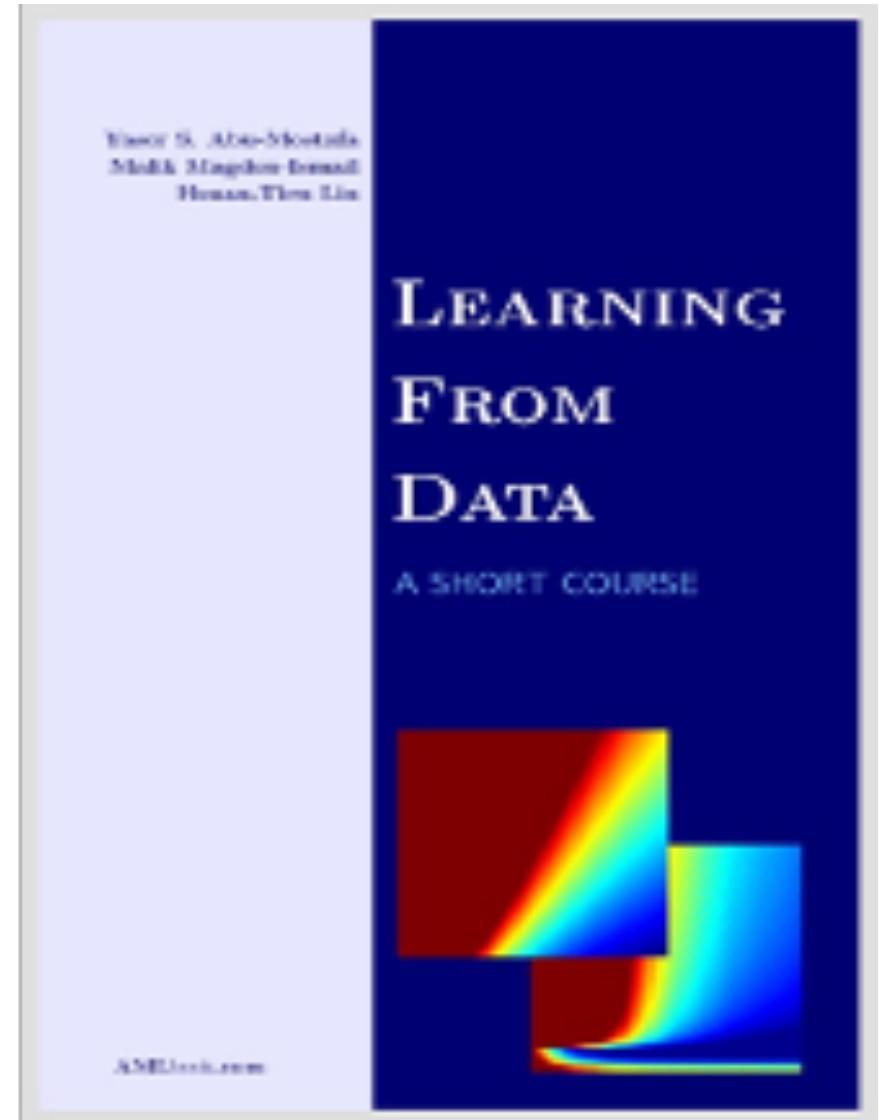
- Regularization coefficient**:  $\varepsilon$  (highlighted by a pink circle and arrow)
- Regularization function**:  $\Omega(h)$  (highlighted by a pink circle and arrow)



# Regularized ERM

“Most of the **regularization methods** used successfully in practice are **heuristic methods**.”

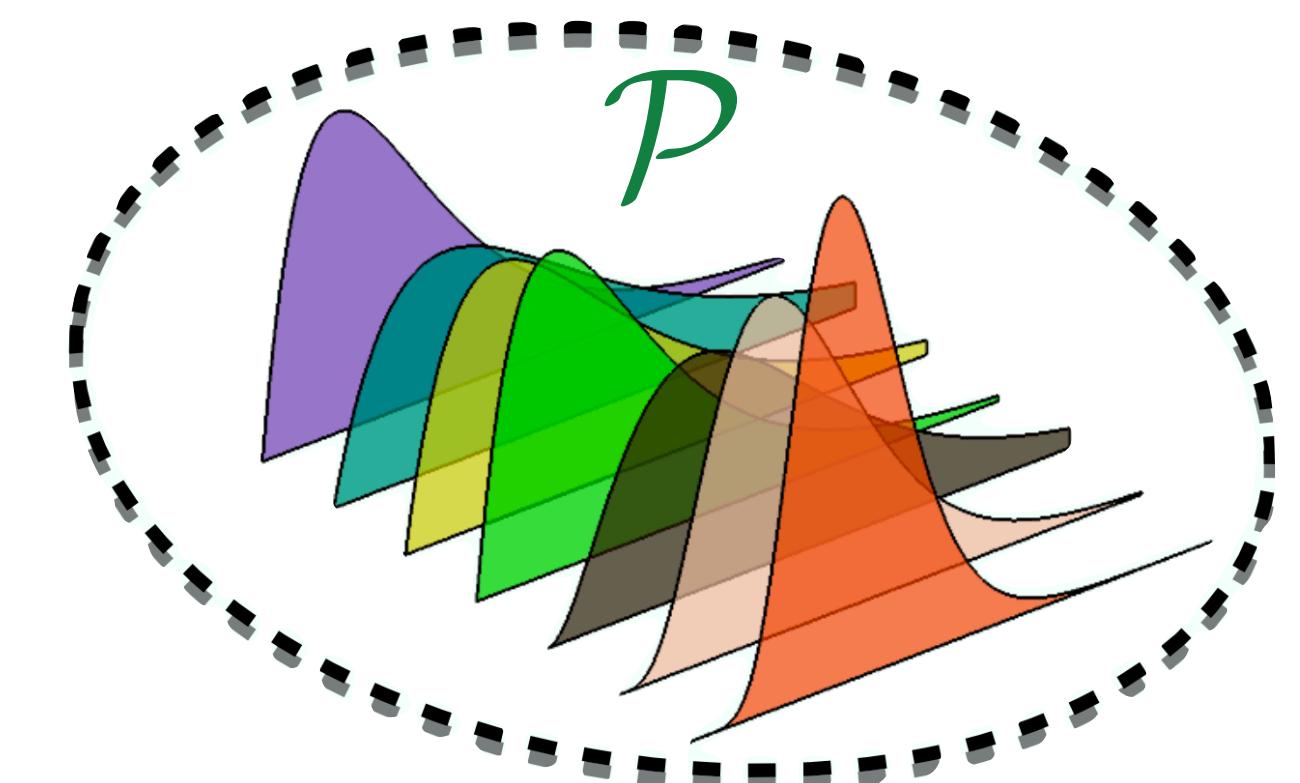
Abu-Mostafa *et al.*, 2012.



# Regularization via Optimal Transport

$$\inf_{\color{red} h \in \mathbb{H}} \sup_{\color{blue} Q \in \mathcal{P}} \mathbb{E}_{\color{blue} Q} [L(\color{red} h(x), y)]$$

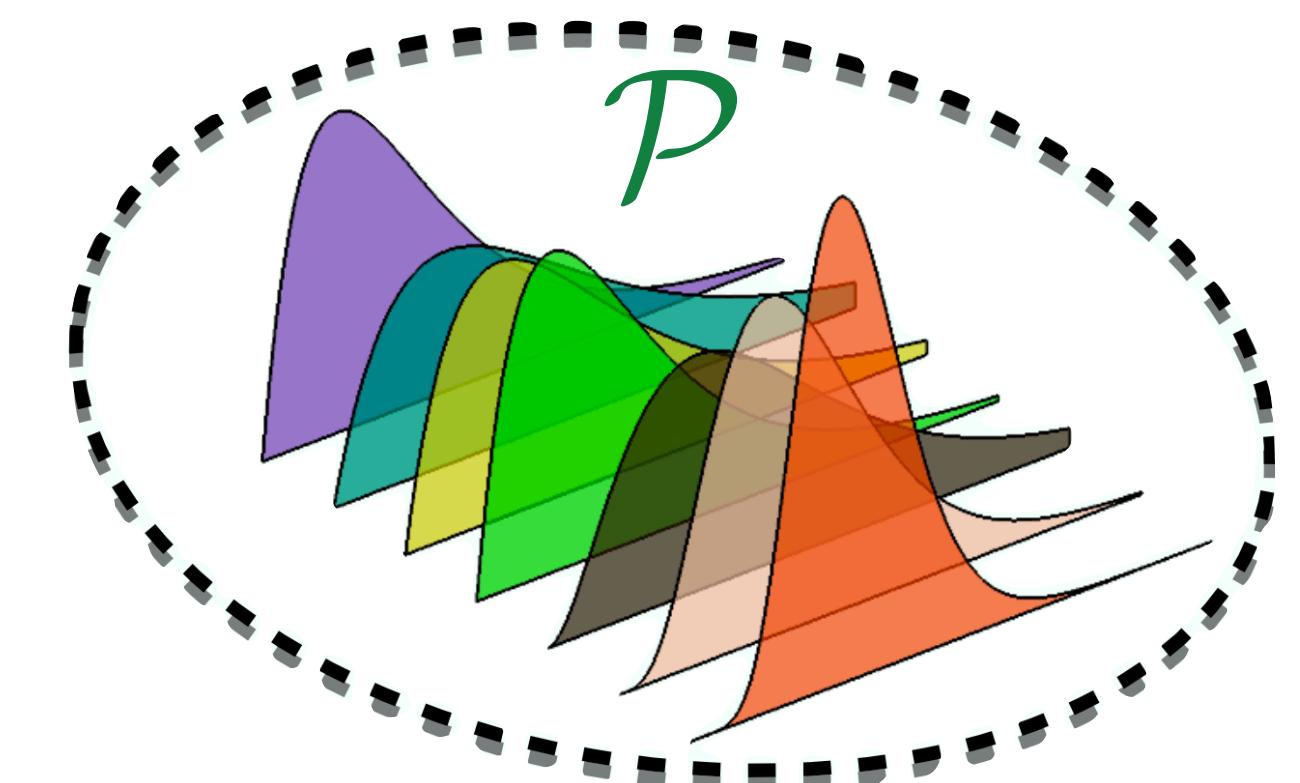
$$\mathbb{H} = \{ \color{red} h \in \mathbb{R}^X : \exists \theta \in \Theta \text{ s.t. } \color{red} h(x) = \theta^\top x \}$$



# Regularization via Optimal Transport

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x, y)]$$

$$\mathbb{H} = \{h \in \mathbb{R}^X : \exists \theta \in \Theta \text{ s.t. } h(x) = \theta^\top x\}$$



# The Real Story Behind the Success

## Regularization

[SMK15, GCK17, CP18,  
BMZ18, BKM19, SKM19]

## Statistical Guarantees

[SMK15, MK18, BKM19,  
SKM19, Gao20, BMN21]

# Optimal Transport: Old and New



Monge

Hitchcock

Kantorovich-Koopmans

Vaserstein

Brenier

Villani

Figalli

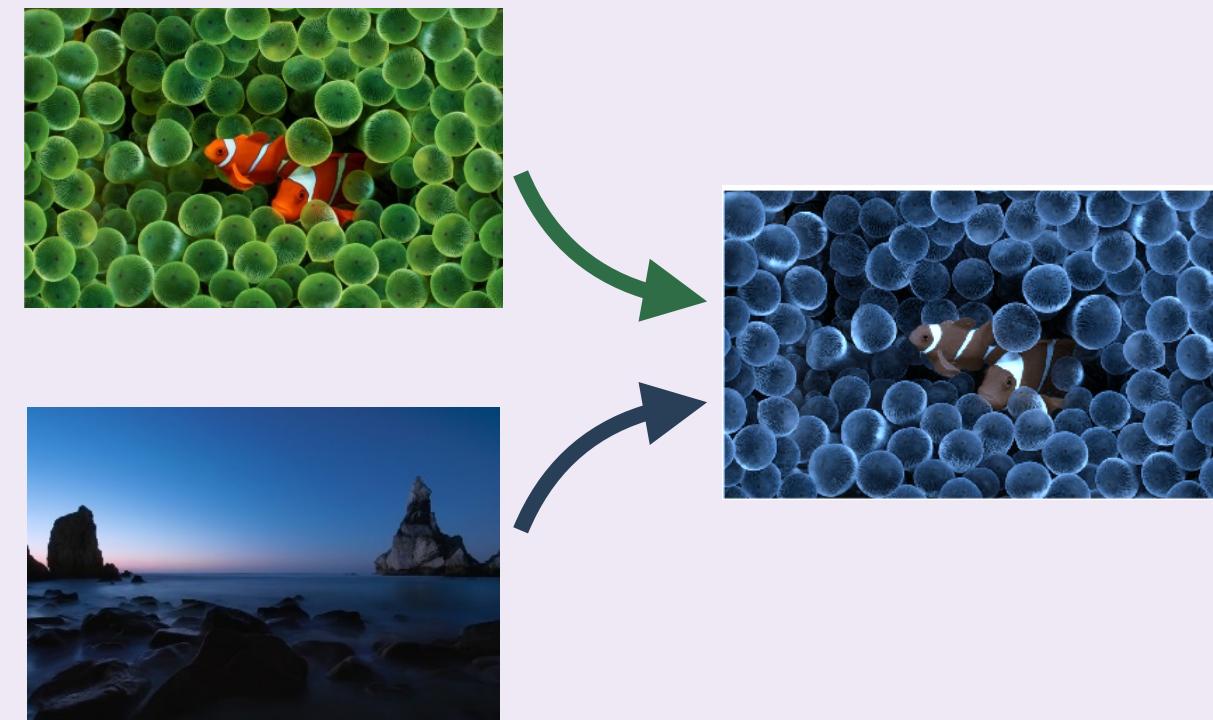
Nobel '75

Fields '10

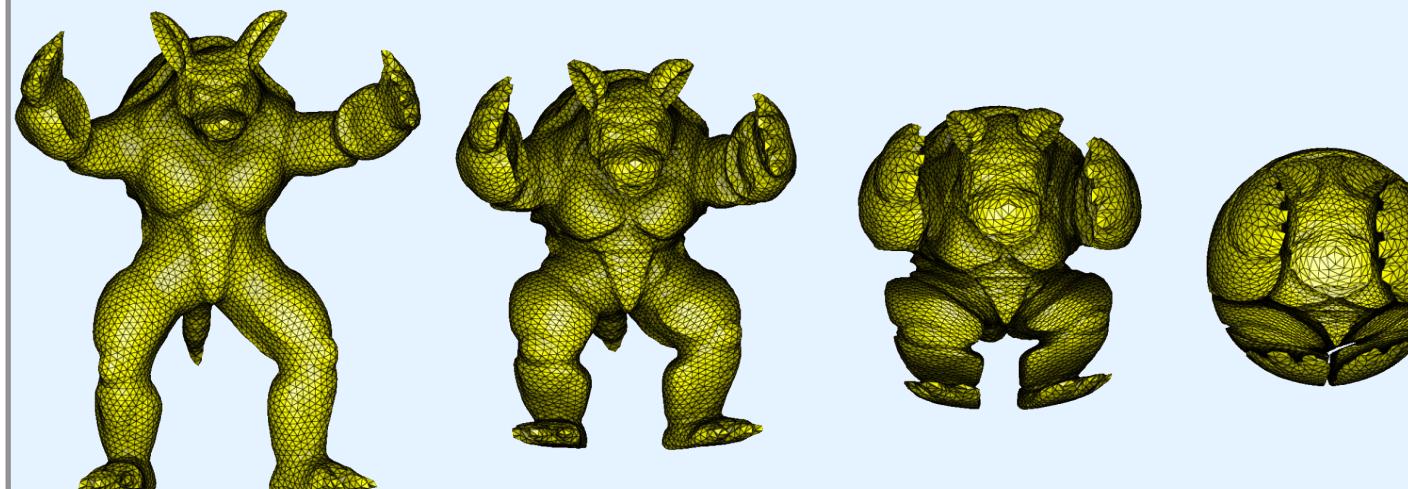
Fields '18

# Applications

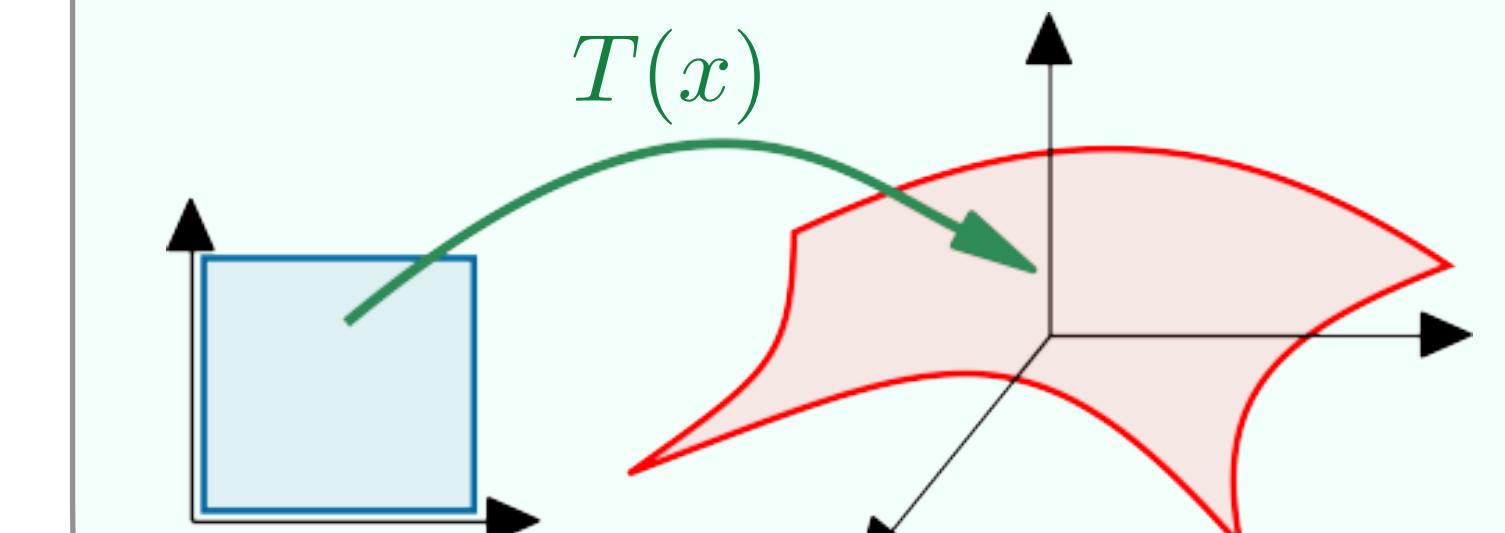
## Color Histogram [PPC10]



## Computer Graphics [LS18]



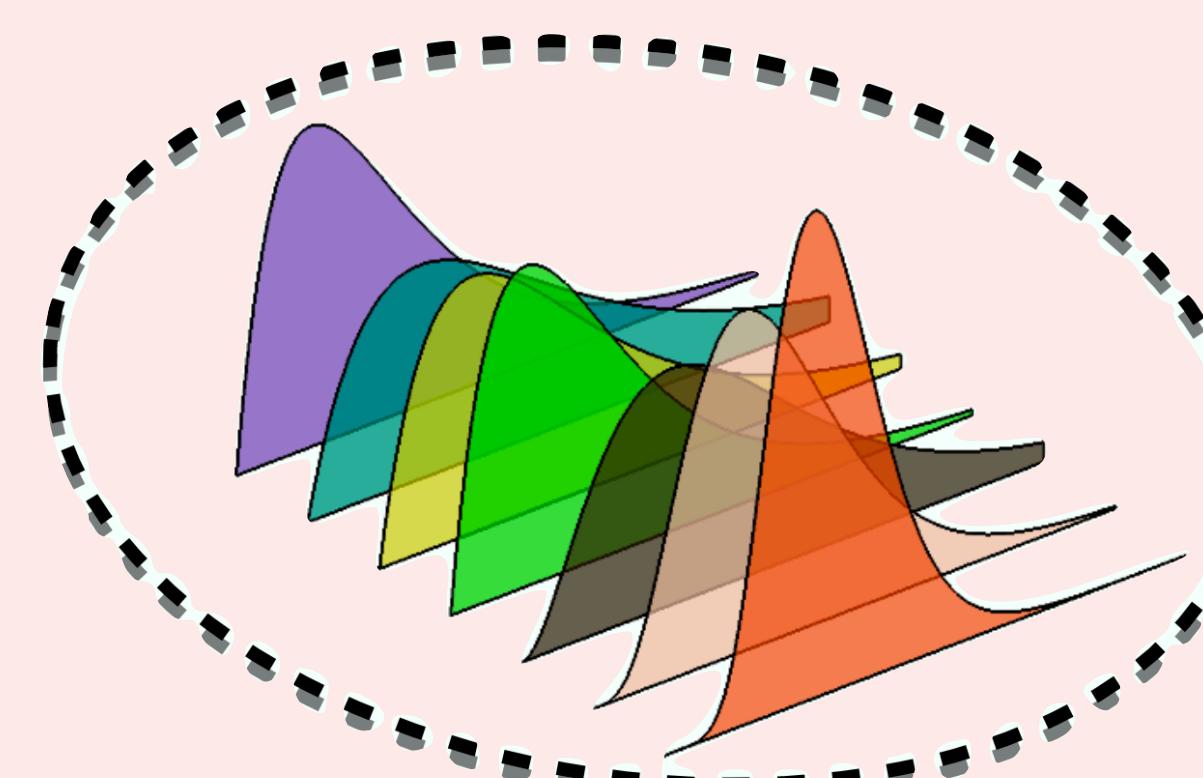
## Generative Models [ACB17]



## Finance and Economics [Gal16]



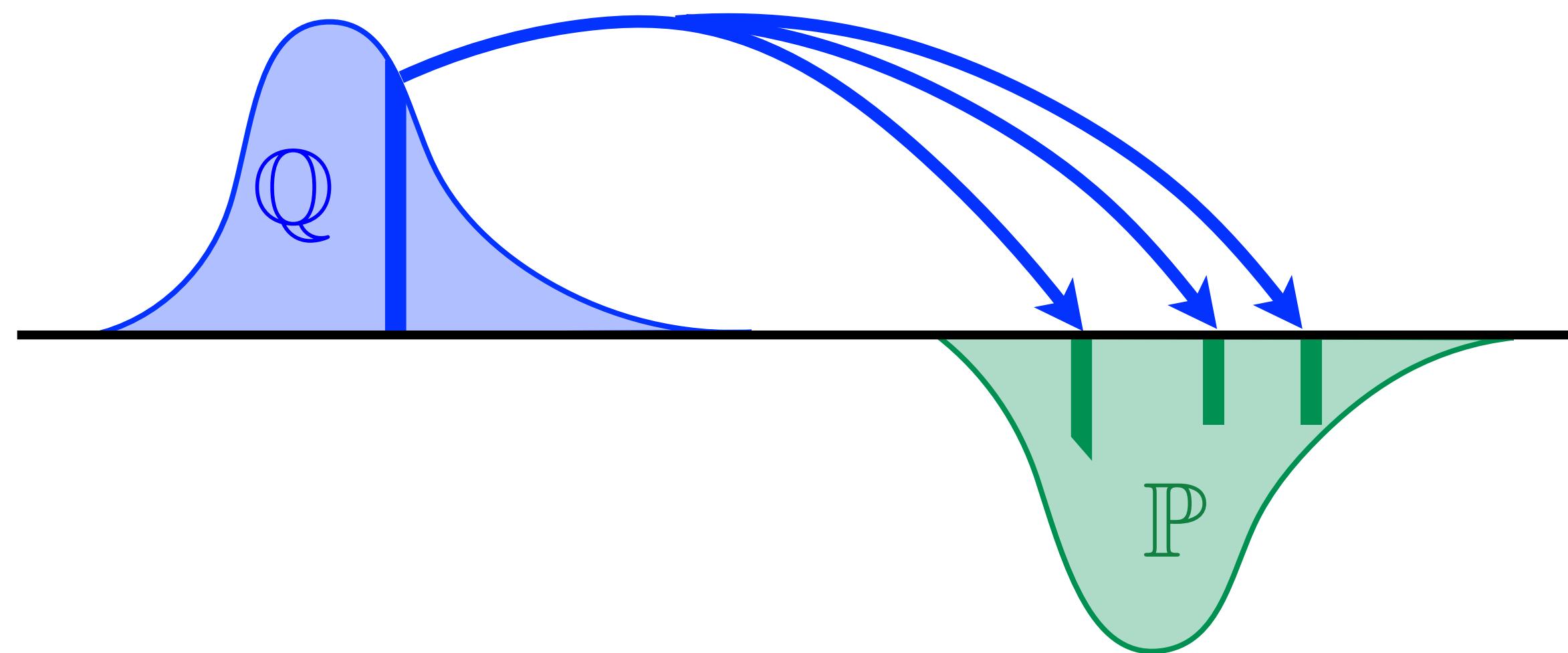
## Statistical Inference



## Text Classification [HGKSSW16]

Mazda TIFF Boone happening  
fit motherboard homosexuals  
asked automotive playoffs motorcycles animation gay  
dolphins computer atheism western  
Keith moon graphics controller  
tapped motif orbit Israel pro apple lists Rutgers hell  
talk NHL driver firebiker guns bike hockey biblical  
orbit Israeli virtual sell autos gun alt auto saint  
label ride Islamic windows circuit cute  
riding key DOD sale car rider rocket offer flight  
IDE shipping bikes mac story monitor  
Armenian card space Israeli polygon  
forsale firearms copy encryption bus warning mouse  
Turkish compatible motorcycle summarized  
powerbook electronics diamond SCSI government chip  
sunductor NASA DOS

# What is Optimal Transport?



$$W_c(Q, \mathbb{P}) = \left\{ \begin{array}{ll} \inf_{\pi \in \mathcal{M}(\Xi, \Xi)} & \mathbb{E}_{\pi} [c(\xi, \xi')] \\ \text{s.t.} & \pi \in \Pi(Q, \mathbb{P}) \end{array} \right.$$



THE DISTRIBUTION OF A PRODUCT FROM SEVERAL SOURCES TO NUMEROUS LOCALITIES

BY FRANK L. HITCHCOCK

1941

1. Statement of the problem. When several factories supply a product to a number of cities we desire the least costly manner of distribution. Due to freight rates and other matters the cost of a ton of product to a particular city will vary according to which factory supplies it, and will also vary from city to city.

OPTIMUM UTILIZATION OF THE TRANSPORTATION SYSTEM\* 1949

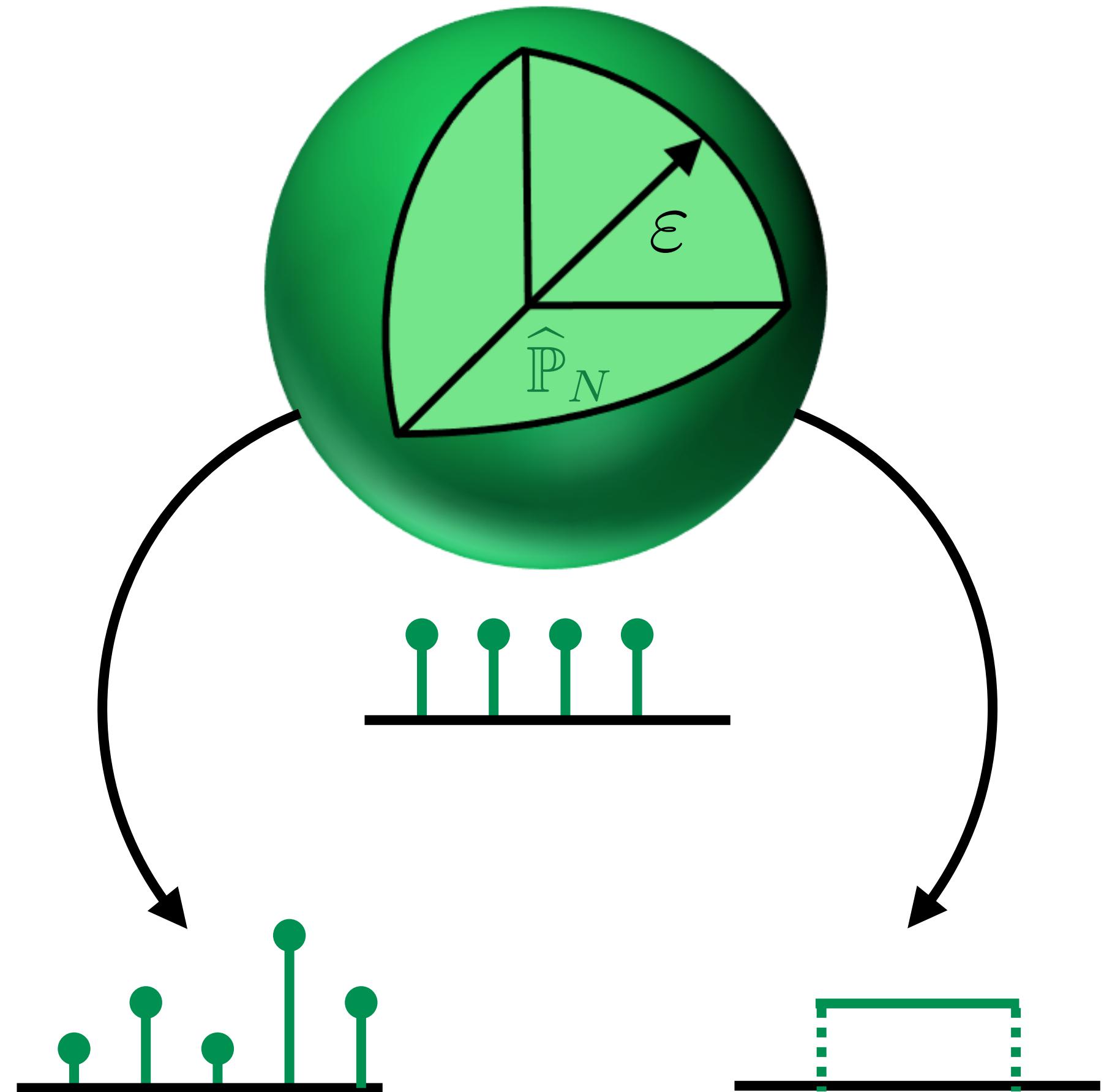
by Tjalling C. Koopmans

*Professor of Economics, The University of Chicago, and Research Associate,  
Cowles Commission for Research in Economics*

The purpose of this paper is to give an application of the theory of optimum allocation of resources to one particular industry. I shall, therefore, not speak on that theory in general. I shall use one of its

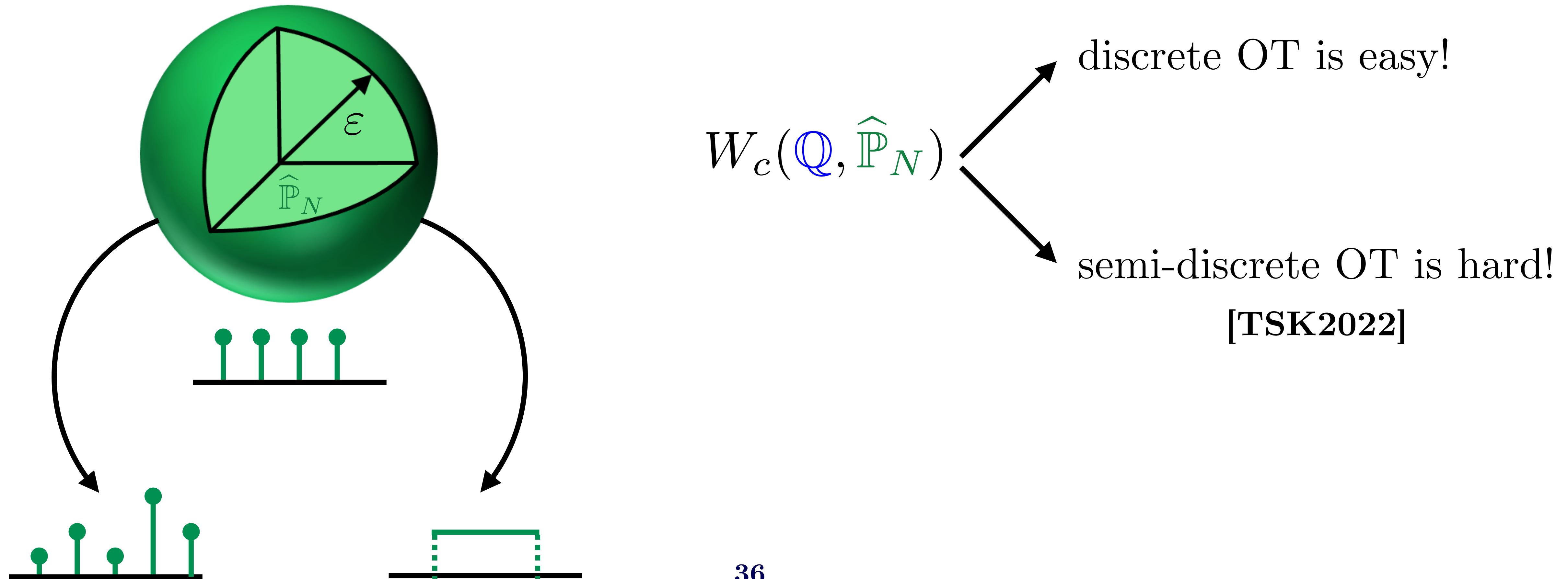
# Optimal Transport Ambiguity Set

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n \times \mathbb{Y}) : W_c(\mathbb{Q}, \hat{\mathbb{P}}_N) \leq \varepsilon\}$$



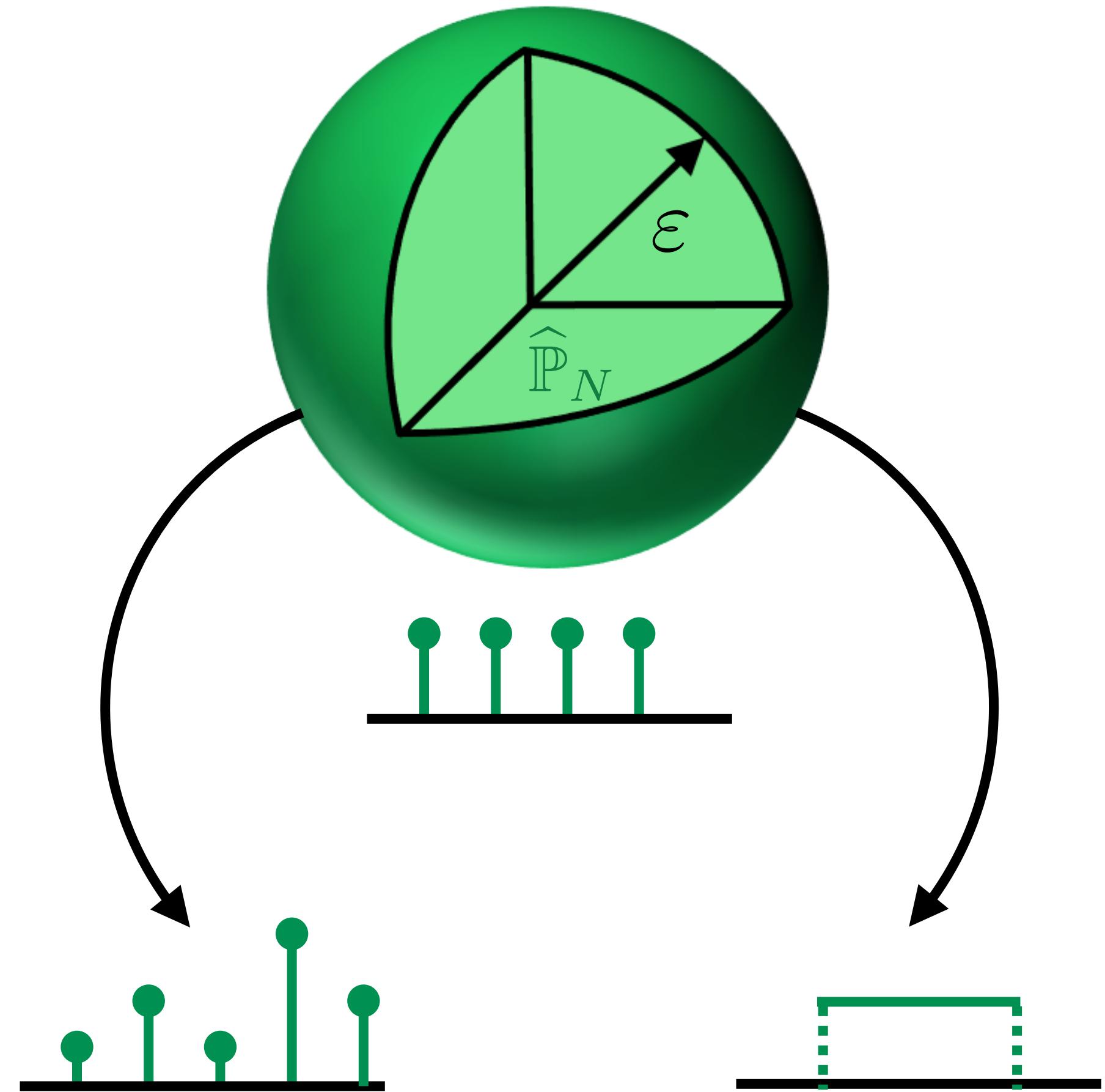
# Optimal Transport Ambiguity Set

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n \times \mathbb{Y}) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$$



# Optimal Transport Ambiguity Set

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n \times \mathbb{Y}) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$$



$$\begin{aligned} & \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x, y)] \\ \text{s.t. } & W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon \end{aligned}$$



# Tractability for Linear Regression

**Theorem 1.** Suppose that  $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$ .

If  $L$  is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$

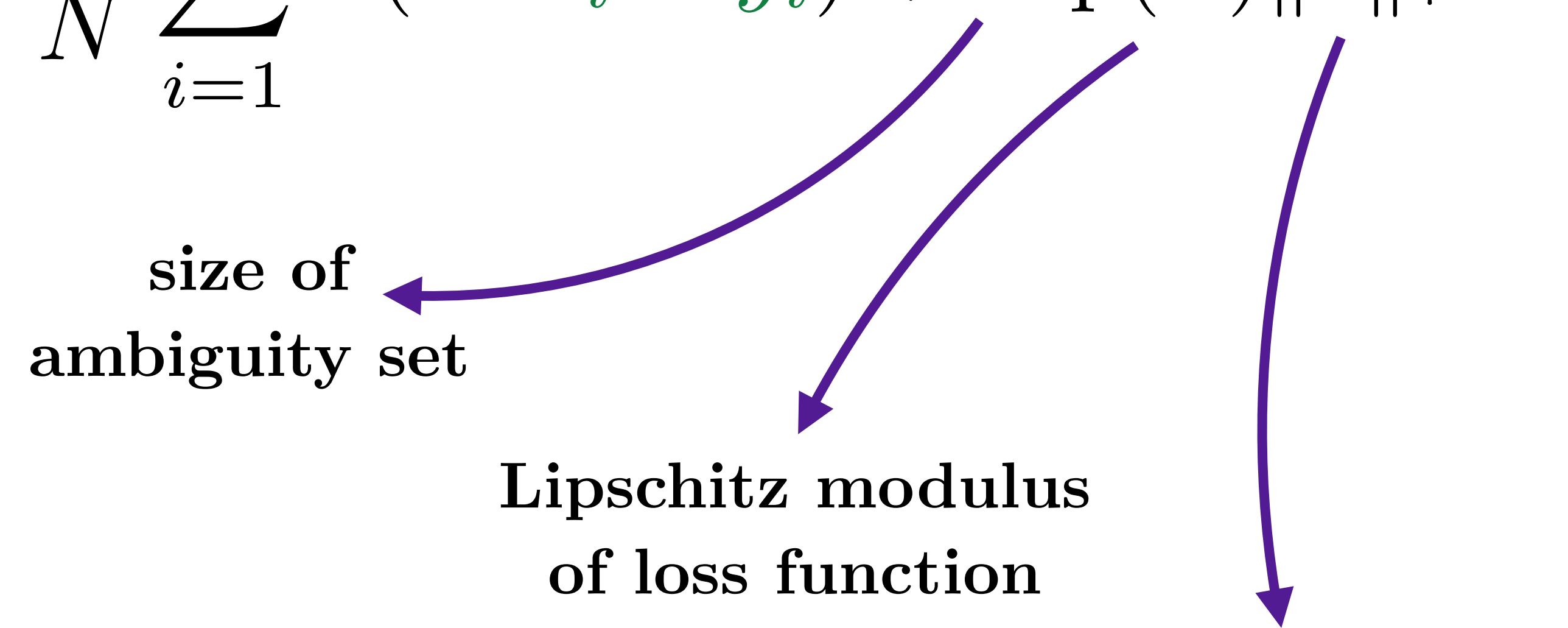
[SMK15, GCK17, SKM19, BKM19]

# Tractability for Linear Regression

**Theorem 1.** Suppose that  $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$ .

If  $L$  is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$



[SMK15, GCK17, SKM19, BKM19]

# Semi-infinite Duality

**Lemma 1.** Let  $\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\Xi) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$ . If  $c(\xi, \xi) = 0$  for all  $\xi \in \Xi$  and  $\varepsilon > 0$ , then

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[I(\xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right]$$

[MK18, ZG18, BM19, GK16, ZYG22]

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{ll} \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} \quad W_c(\mathbb{Q}, \hat{\mathbb{P}}_N) \leq \varepsilon \end{array} \right.$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{ll} \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} & \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} & W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon \end{array} \right.$$

$$= \left\{ \begin{array}{ll} \sup_{\substack{\mathbb{Q} \in \mathcal{M}(\Xi) \\ \pi \in \mathcal{M}(\Xi \times \Xi)}} & \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} & \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ & \int_{\xi \in \Xi} \int_{\xi' \in \Xi} c(\xi, \xi') \pi(\mathrm{d}\xi, \mathrm{d}\xi') \leq \varepsilon \end{array} \right.$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{l} \sup_{\mathbb{Q} \in \mathcal{M}(\Xi)} \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} \quad W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon \end{array} \right.$$

↓

$$= \left\{ \begin{array}{l} \sup_{\substack{\mathbb{Q} \in \mathcal{M}(\Xi) \\ \pi \in \mathcal{M}(\Xi \times \Xi)}} \int_{\xi \in \Xi} I(\xi) \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} \quad \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \int_{\xi \in \Xi} \int_{\xi' \in \Xi} c(\xi, \xi') \pi(\mathrm{d}\xi, \mathrm{d}\xi') \leq \varepsilon \end{array} \right.$$

$$W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) = \left\{ \begin{array}{ll} \inf_{\pi \in \mathcal{M}(\Xi, \Xi)} & \mathbb{E}_\pi [c(\xi, \xi')] \\ \text{s.t.} & \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \end{array} \right.$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \begin{cases} \sup_{\mathbb{Q}_i \in \mathcal{M}(\Xi)} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \hat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \end{cases}$$

$$\boxed{\begin{cases} \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \widehat{\mathbb{P}}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\hat{\xi}_i} \end{cases} \implies \begin{cases} \pi = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \times \delta_{\hat{\xi}_i} \\ \mathbb{Q} = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \end{cases}}$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{l} \sup_{\mathbb{Q}_i \in \mathcal{M}(\Xi)} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \end{array} \right.$$

$$\boxed{\left\{ \begin{array}{l} \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \widehat{\mathbb{P}}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\xi}_i} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \pi = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \times \delta_{\widehat{\xi}_i} \\ \mathbb{Q} = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \end{array} \right.}$$

$$= \left\{ \begin{array}{l} \sup_{\mathbb{Q}_i \geq 0} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \\ \int_{\xi \in \Xi} \mathbb{Q}_i(d\xi) = 1 \quad \forall i \in [N] \end{array} \right.$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] = \left\{ \begin{array}{l} \sup_{\mathbb{Q}_i \in \mathcal{M}(\Xi)} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \end{array} \right.$$

$\left\{ \begin{array}{l} \pi \in \Pi(\mathbb{Q}, \widehat{\mathbb{P}}_N) \\ \widehat{\mathbb{P}}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\xi}_i} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \pi = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \times \delta_{\widehat{\xi}_i} \\ \mathbb{Q} = \frac{1}{N} \sum_{i=1}^N \mathbb{Q}_i \end{array} \right.$

$$= \left\{ \begin{array}{l} \sup_{\mathbb{Q}_i \geq 0} \frac{1}{N} \sum_{i=1}^N \int_{\xi \in \Xi} I(\xi) \mathbb{Q}_i(d\xi) \\ \frac{1}{N} \int_{\xi \in \Xi} c(\xi, \widehat{\xi}_i) \mathbb{Q}_i(d\xi) \leq \varepsilon \quad (\lambda) \\ \int_{\xi \in \Xi} \mathbb{Q}_i(d\xi) = 1 \quad (s_i) \quad \forall i \in [N] \end{array} \right.$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \left\{ \begin{array}{ll} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} & \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t.} & \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{array} \right.$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \left\{ \begin{array}{ll} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} & \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t.} & \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{array} \right.$$

$$= \left\{ \begin{array}{ll} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} & \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t.} & s_i \geq \sup_{\xi \in \Xi} \frac{1}{N} I(\xi) - \frac{\lambda}{N} c(\xi, \hat{\xi}_i) \end{array} \right.$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \begin{cases} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t. } \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{cases}$$

$$= \inf_{\lambda \geq 0} \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^N \sup_{\xi \in \Xi} I(\xi) - \lambda c(\xi, \hat{\xi}_i)$$

# Proof of Lemma 1

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [I(\xi)] \leq \begin{cases} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t. } \frac{\lambda}{N} c(\xi, \widehat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{cases}$$

$$\begin{aligned} &= \inf_{\lambda \geq 0} \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^N \sup_{\xi \in \Xi} I(\xi) - \lambda c(\xi, \widehat{\xi}_i) \\ &= \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right] \end{aligned}$$

# Proof of Lemma 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [I(\xi)] \stackrel{[S01]}{\leq} \left\{ \begin{array}{ll} \inf_{\substack{\lambda \geq 0 \\ s \in \mathbb{R}^N}} & \lambda \varepsilon + \sum_{i=1}^N s_i \\ \text{s.t.} & \frac{\lambda}{N} c(\xi, \hat{\xi}_i) + s_i \geq \frac{1}{N} I(\xi) \quad \forall \xi \in \Xi \end{array} \right.$$

$$\begin{aligned} &= \inf_{\lambda \geq 0} \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^N \sup_{\xi \in \Xi} I(\xi) - \lambda c(\xi, \hat{\xi}_i) \\ &= \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\hat{\mathbb{P}}_N} \left[ \sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right] \end{aligned}$$

# Semi-infinite Duality

**Lemma 1.** Let  $\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\Xi) : W_c(\mathbb{Q}, \widehat{\mathbb{P}}_N) \leq \varepsilon\}$ . If  $c(\xi, \xi) = 0$  for all  $\xi \in \Xi$  and  $\varepsilon > 0$ , then

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[I(\xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{\zeta \in \Xi} I(\zeta) - \lambda c(\zeta, \xi) \right]$$

[MK18, ZG18, BM19, GK16, ZYG22]

# Lipschitz Envelope

**Lemma 2.** Let  $L : \mathbb{R} \rightarrow \mathbb{R}$  be a convex and Lipschitz function.

Then,

$$\sup_{\zeta \in \mathbb{R}^n} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| = \begin{cases} L(\theta^\top \xi + \theta_0) & \text{if } \text{lip}(L)\|\theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}$$

for any  $\theta, \xi \in \mathbb{R}^n, \theta_0 \in \mathbb{R}$  and  $\lambda > 0$ .

# Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

# Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

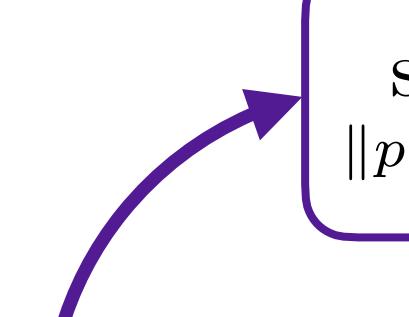
$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| = \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \lambda \|\zeta - \xi\|$$

# Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \boxed{\lambda \|\zeta - \xi\|} \\ &= \sup_{\kappa \in \mathcal{K}} \sup_{\zeta} \inf_{\|p\|_* \leq \lambda} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \end{aligned}$$

\$\sup\_{\|p\|\_\* \leq \lambda} p^\top (\zeta - \xi)\$


# Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \lambda \|\zeta - \xi\| \\ &= \sup_{\kappa \in \mathcal{K}} \sup_{\zeta} \inf_{\|p\|_* \leq \lambda} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \\ (\text{Sion's minimax}) &= \sup_{\kappa \in \mathcal{K}} \inf_{\|p\|_* \leq \lambda} \sup_{\zeta} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \end{aligned}$$

# Proof of Lemma 2

$$L(\theta^\top \zeta + \theta_0) = L^{**}(\theta^\top \zeta + \theta_0) = \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa)$$

$$\mathcal{K} = \{\kappa \in \mathbb{R} : L^*(\kappa) < \infty\}$$

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\zeta} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - \lambda \|\zeta - \xi\| \\ &= \sup_{\kappa \in \mathcal{K}} \sup_{\zeta} \inf_{\|p\|_* \leq \lambda} \kappa(\theta^\top \zeta + \theta_0) - L^*(\kappa) - p^\top (\zeta - \xi) \\ (\text{Sion's minimax}) \quad &= \sup_{\kappa \in \mathcal{K}} \inf_{\|p\|_* \leq \lambda} \kappa \theta_0 - L^*(\kappa) + p^\top \xi + \begin{cases} 0 & \text{if } \kappa \theta - p = 0 \\ +\infty & \text{else} \end{cases} \end{aligned}$$

## Proof of Lemma 2

$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| = \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}$$

## Proof of Lemma 2

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\ &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \ \forall \kappa \in \mathcal{K} \\ +\infty & \text{else} \end{cases} \end{aligned}$$

# Proof of Lemma 2

$$\begin{aligned}
 \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \ \forall \kappa \in \mathcal{K} \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \sup_{\kappa \in \mathcal{K}} \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}
 \end{aligned}$$

# Proof of Lemma 2

$$\begin{aligned}
 \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\| &= \sup_{\kappa \in \mathcal{K}} \begin{cases} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \|\kappa \theta\|_* \leq \lambda \ \forall \kappa \in \mathcal{K} \\ +\infty & \text{else} \end{cases} \\
 &= \begin{cases} \sup_{\kappa \in \mathcal{K}} \kappa(\theta^\top \xi + \theta_0) - L^*(\kappa) & \text{if } \sup_{\kappa \in \mathcal{K}} \|\kappa \theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases} \\
 \boxed{\sup_{\kappa \in \mathcal{K}} |\kappa| = \text{lip}(L)} &= \begin{cases} L(\theta^\top \xi + \theta_0) & \text{if } \text{lip}(L) \|\theta\|_* \leq \lambda \\ +\infty & \text{else} \end{cases}
 \end{aligned}$$

# Tractability for Linear Regression

**Theorem 1.** Suppose that  $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$ .

If  $L$  is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$

[SMK15, GCK17, SKM19, BKM19]

# Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

# Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$\begin{aligned} (\text{Lemma 1}) &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right] \\ &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\| \right] \end{aligned}$$

# Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \begin{cases} L(\theta^\top x - y) & \text{if } \lambda \geq \text{lip}(L) \|\theta\|_* \\ +\infty & \text{else} \end{cases} \right]$$

# Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L) \|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - y)]$$

# Proof of Theorem 1

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top \mathbf{x} - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - \mathbf{y}) - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L) \|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top \mathbf{x} - \mathbf{y})]$$

$$= \inf_{\theta \in \Theta} \varepsilon \text{lip}(L) \|\theta\|_* + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top \mathbf{x} - \mathbf{y})]$$

# Examples

## Robust Regression

$$L(z) = \begin{cases} \frac{1}{2}z^2 & \text{if } |z| \leq \delta \\ \delta(|z| - \frac{1}{2}\delta) & \text{else} \end{cases}$$

$$\text{lip}(L) = \delta$$



## Support Vector Regression

$$L(z) = \max\{0, |z| - \varepsilon\}$$

$$\text{lip}(L) = 1$$



## Quantile Regression

$$L(z) = \max\{-\tau z, (1 - \tau)z\}$$

$$\text{lip}(L) = \max\{\tau, 1 - \tau\}$$

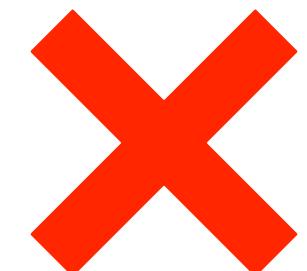


# Examples

Least Square  
Regression

$$L(z) = z^2$$

$$\text{lip}(L) = \infty$$



# Least Squares Regression

**Theorem 2.** Suppose that  $c((x, y), (x', y')) = \|x - x'\|^2 + \delta_{y=y'}.$

If  $L = z^2$ , then

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)] = \left( \inf_{\theta \in \Theta} \sqrt{\frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \theta^\top \hat{x}_i)} + \sqrt{\varepsilon} \|\theta\|_* \right)^2$$

[BKM19]

# Moreau Envelope

**Lemma 3.** Let  $L(z) = z^2$ . Then,

$$\sup_{\zeta \in \mathbb{R}^n} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 = \begin{cases} \frac{\lambda}{\lambda - \|\theta\|_*^2} L(\theta^\top \xi + \theta_0) & \text{if } \|\theta\|_*^2 < \lambda \\ +\infty & \text{else} \end{cases}$$

for any  $\theta, \xi \in \mathbb{R}^n, \theta_0 \in \mathbb{R}$  and  $\lambda > 0$ .

[BKM19, SADK22]

# Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ \boxed{\Delta \leftarrow \zeta - \xi} \quad &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \end{aligned}$$

# Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ &= \begin{cases} \sup_{\Delta, \gamma} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{cases} \end{aligned}$$

# Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ &= \sup_{\gamma} \left\{ \begin{array}{ll} \sup_{\Delta} & L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t.} & \gamma = \theta^\top \Delta \end{array} \right. \end{aligned}$$

# Proof of Lemma 3

$$\begin{aligned} & \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\ &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ &= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\ (\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \end{aligned}$$

# Proof of Lemma 3

$$\begin{aligned}
& \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\
&= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Holder inequality}) &\leq \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \|\kappa \theta\|_* \|\Delta\|
\end{aligned}$$

# Proof of Lemma 3

$$\begin{aligned}
& \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\
&= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Holder inequality}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\|\Delta\|} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \|\kappa \theta\|_* \|\Delta\|
\end{aligned}$$

$$\boxed{\|\Delta^*\| = \frac{\|\kappa \theta\|_*}{2\lambda}}$$

# Proof of Lemma 3

$$\begin{aligned}
& \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 \\
&= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Holder inequality}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\|\Delta\|} L(\gamma + \theta^\top \xi + \theta_0) - \lambda \|\Delta\|^2 - \kappa \gamma + \|\kappa \theta\|_* \|\Delta\| \\
&= \sup_{\gamma} \inf_{\kappa} L(\gamma + \theta^\top \xi + \theta_0) - \kappa \gamma + \frac{\|\kappa \theta\|_*^2}{4\lambda}
\end{aligned}$$

$\kappa^* = \frac{2\lambda\gamma}{\|\theta\|_*^2}$

# Proof of Lemma 3

$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 = \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda \gamma^2}{\|\theta\|_*^2}$$

# Proof of Lemma 3

$$\sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 = \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda \gamma^2}{\|\theta\|_*^2}$$

$$\begin{cases} \gamma^* = \frac{\theta^\top \xi}{\frac{\lambda}{\|\theta\|_*^2} - 1} & \text{if } 1 - \frac{\lambda}{\|\theta\|_*^2} < 0 \\ \text{unbounded} & \text{else} \end{cases}$$

$$= \sup_{\gamma} (\gamma + \theta^\top \xi)^2 - \frac{\lambda \gamma^2}{\|\theta\|_*^2}$$

# Proof of Lemma 3

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta + \theta_0) - \lambda \|\zeta - \xi\|^2 &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda \gamma^2}{\|\theta\|_*^2} \\ &= \sup_{\gamma} (\gamma + \theta^\top \xi)^2 - \frac{\lambda \gamma^2}{\|\theta\|_*^2} \\ &= \begin{cases} \frac{\lambda}{\lambda - \|\theta\|_*^2} (\theta^\top \xi)^2 & \text{if } \|\theta\|_*^2 < \lambda \\ +\infty & \text{else} \end{cases} \end{aligned}$$

# Least Squares Regression

**Theorem 2.** Suppose that  $c((x, y), (x', y')) = \|x - x'\|^2 + \delta_{y=y'}.$

If  $L = z^2$ , then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)] = \inf_{\theta \in \Theta} \left( \sqrt{\frac{1}{N} \sum_{i=1}^N L(\theta^\top \hat{x}_i - \hat{y}_i)} + \sqrt{\varepsilon} \|\theta\|_* \right)^2$$

[BKM19]

# Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\|^2 - \lambda \delta_{y'=\mathbf{y}} \right]$$

# Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$\begin{aligned} (\text{Lemma 1}) &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\|^2 - \lambda \delta_{y'=\mathbf{y}} \right] \\ &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - \mathbf{y}) - \lambda \|x' - \mathbf{x}\|^2 \right] \end{aligned}$$

# Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\|^2 - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\|^2 \right]$$

$$(\text{Lemma 3}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \begin{cases} \frac{\lambda}{\lambda - \|\theta\|_*^2} L(\theta^\top x - y) & \text{if } \lambda > \|\theta\|_*^2 \\ +\infty & \text{else} \end{cases} \right]$$

# Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - x\|^2 - \lambda \delta_{y' = y} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - y) - \lambda \|x' - x\|^2 \right]$$

$$(\text{Lemma 3}) = \inf_{\substack{\theta \in \Theta \\ \lambda > \|\theta\|_*^2}} \lambda \varepsilon + \frac{\lambda}{\lambda - \|\theta\|_*^2} \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - y)]$$

$$\lambda^* = \|\theta\|_*^2 + \frac{\|\theta\|_* \sqrt{\mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - y)]}}{\varepsilon}$$

# Proof of Theorem 2

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(\theta^\top x - y)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(\theta^\top x' - y') - \lambda \|x' - \mathbf{x}\|^2 - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\theta^\top x' - \mathbf{y}) - \lambda \|x' - \mathbf{x}\|^2 \right]$$

$$(\text{Lemma 3}) = \inf_{\substack{\theta \in \Theta \\ \lambda > \|\theta\|_*^2}} \lambda \varepsilon + \frac{\lambda}{\lambda - \|\theta\|_*^2} \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - \mathbf{y})]$$

$$= \inf_{\theta \in \Theta} \left( \sqrt{\varepsilon \|\theta\|_*} + \sqrt{\mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\theta^\top x - \mathbf{y})]} \right)^2$$

# Tractability for Linear Classification

**Theorem 3.** Suppose that  $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}$ .

If  $L$  is convex and Lipschitz, then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [y L(\theta^\top x)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L(\hat{y}_i \theta^\top \hat{x}_i) + \varepsilon \text{lip}(L) \|\theta\|_*$$

[SMK15, GCK17, SKM19, BKM19]

# Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - x\| - \lambda \delta_{y' = y} \right]$$

# Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$\begin{aligned} (\text{Lemma 1}) &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right] \\ &= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\mathbf{y}\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right] \end{aligned}$$

# Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\mathbf{y}\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L)\|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\mathbf{y}\theta^\top \mathbf{x})]$$

# Proof of Theorem 3

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \in \Theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(\mathbf{y}\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 2}) = \inf_{\substack{\theta \in \Theta \\ \lambda \geq \text{lip}(L)\|\theta\|_*}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\mathbf{y}\theta^\top \mathbf{x})]$$

$$= \inf_{\theta \in \Theta} \varepsilon \text{lip}(L)\|\theta\|_* + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L(\mathbf{y}\theta^\top \mathbf{x})]$$

# Examples

## Support Vector Machine

$$L(z) = \max\{0, 1 - z\}$$

$$\text{lip}(L) = 1$$



## Support Vector Machine II

$$L(z) = \begin{cases} \frac{1}{2} - z & \text{if } z \leq 0 \\ \frac{1}{2}(1 - z)^2 & \text{if } 0 < z < 1 \\ 0 & \text{else} \end{cases}$$

$$\text{lip}(L) = 1$$



## Logistic Regression

$$L(z) = \log(1 + \exp(-z))$$

$$\text{lip}(L) = 1$$

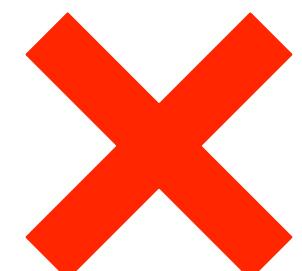


# Examples

Ideal  
Classification

$$L(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ 0 & \text{else} \end{cases}$$

nonconvex!



# Ideal Classification

**Theorem 4.** Suppose that  $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}.$

If  $L(z) = \mathbf{1}_{z \leq 0}$ , then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(\mathbf{y}\theta^\top \mathbf{x})] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L_R(\hat{y}_i \theta^\top \hat{\mathbf{x}}_i) + \varepsilon \|\theta\|_*$$

where  $L_R(z) = \max\{0, 1 - z\} + \max\{0, -z\}.$

# Ideal Classification

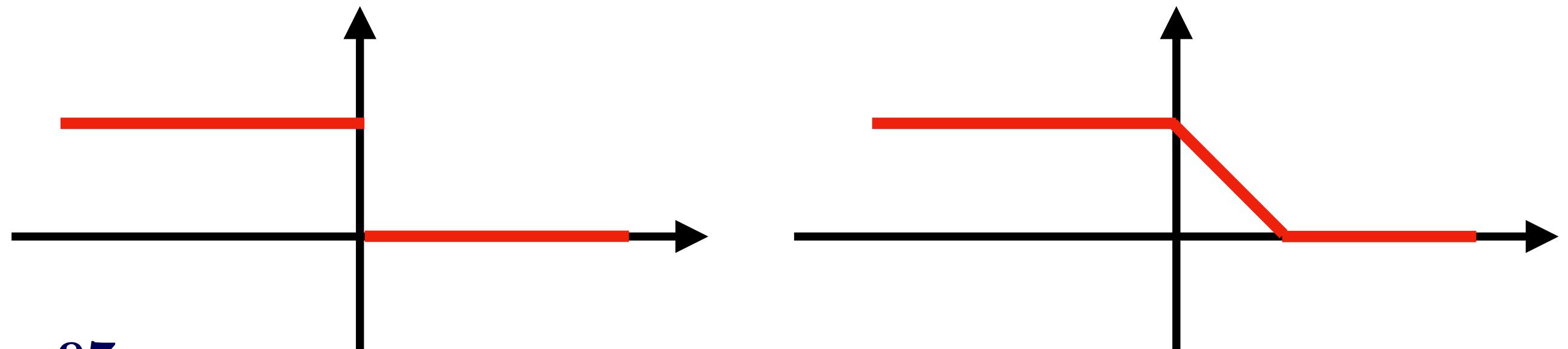
**Theorem 4.** Suppose that  $c((x, y), (x', y')) = \|x - x'\| + \delta_{y=y'}.$

If  $L(z) = \mathbf{1}_{z \leq 0}$ , then

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)] = \inf_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N L_R(\hat{y}_i \theta^\top \hat{x}_i) + \varepsilon \|\theta\|_*$$

where  $L_R(z) = \max\{0, 1 - z\} + \max\{0, -z\}.$

[H-NW22]



# Lipschitz Envelope II

**Lemma 4.** Let  $L(z) = \mathbb{1}_{z \leq 0}$ . Then,

$$\sup_{\zeta \in \mathbb{R}^n} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| = L_R\left(\lambda(\theta^\top \xi)/\|\theta\|_*\right)$$

for any  $\theta, \xi \in \mathbb{R}^n$  and  $\lambda > 0$ .

# Proof of Lemma 4

$$\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| = \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\|$$

$$\boxed{\Delta \leftarrow \zeta - \xi}$$

# Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\ &= \left\{ \begin{array}{ll} \sup_{\Delta, \gamma} & L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t.} & \gamma = \theta^\top \Delta \end{array} \right. \end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\ &= \sup_{\gamma} \left\{ \begin{array}{ll} \sup_{\Delta} & L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t.} & \gamma = \theta^\top \Delta \end{array} \right. \end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\ &= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\ (\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \boxed{\lambda \|\Delta\|} - \kappa \gamma + \kappa \theta^\top \Delta \end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^T \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^T \Delta + \theta^T \xi) - \lambda \|\Delta\| \\ &= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^T \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^T \Delta \end{array} \right. \\ (\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^T \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^T \Delta \\ &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^T \xi) - p^T \Delta - \kappa \gamma + \kappa \theta^T \Delta \end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Sion's minimax}) &= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} \sup_{\Delta} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta
\end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Sion's minimax}) &= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} \sup_{\Delta} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} L(\gamma + \theta^\top \xi) - \kappa \gamma + \begin{cases} 0 & \text{if } \kappa \theta - p = 0 \\ +\infty & \text{else} \end{cases}
\end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\Delta} L(\theta^\top \Delta + \theta^\top \xi) - \lambda \|\Delta\| \\
&= \sup_{\gamma} \left\{ \begin{array}{l} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| \\ \text{s.t. } \gamma = \theta^\top \Delta \end{array} \right. \\
(\text{Slater condition}) &= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} L(\gamma + \theta^\top \xi) - \lambda \|\Delta\| - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{\kappa} \sup_{\Delta} \inf_{\|p\|_* \leq \lambda} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
(\text{Sion's minimax}) &= \sup_{\gamma} \inf_{\substack{\kappa \\ \|p\|_* \leq \lambda}} \sup_{\Delta} L(\gamma + \theta^\top \xi) - p^\top \Delta - \kappa \gamma + \kappa \theta^\top \Delta \\
&= \sup_{\gamma} \inf_{-\lambda/\|\theta\|_* \leq \kappa \leq \lambda/\|\theta\|_*} L(\gamma + \theta^\top \xi) - \kappa \gamma
\end{aligned}$$

$\kappa^* = \frac{\lambda}{\|\theta\|_*} \text{sgn}(\gamma)$

# Proof of Lemma 4

$$\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| = \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*}$$

# Proof of Lemma 4

$$\begin{aligned}\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\ &= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma|\end{aligned}$$

$\gamma \leftarrow \gamma / \|\theta\|_*$

# Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\ &= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma| \\ &= \sup_{\gamma} \begin{cases} 1 - \lambda |\gamma| & \text{if } \gamma \|\theta\|_* + \theta^\top \xi \leq 0 \\ -\lambda |\gamma| & \text{else} \end{cases} \end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned} \sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\ &= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma| \\ &= \sup_{\gamma} \begin{cases} 1 - \lambda |\gamma| & \text{if } \gamma \|\theta\|_* + \theta^\top \xi \leq 0 \\ -\lambda |\gamma| & \text{else} \end{cases} \\ &= \begin{cases} 1 & \text{if } \theta^\top \xi \leq 0 \\ \max\{0, 1 - \lambda \frac{\theta^\top \xi}{\|\theta\|_*}\} & \text{else} \end{cases} \end{aligned}$$

# Proof of Lemma 4

$$\begin{aligned}
\sup_{\zeta} L(\theta^\top \zeta) - \lambda \|\zeta - \xi\| &= \sup_{\gamma} L(\gamma + \theta^\top \xi) - \frac{\lambda |\gamma|}{\|\theta\|_*} \\
&= \sup_{\gamma} L(\gamma \|\theta\|_* + \theta^\top \xi) - \lambda |\gamma| \\
&= \sup_{\gamma} \begin{cases} 1 - \lambda |\gamma| & \text{if } \gamma \|\theta\|_* + \theta^\top \xi \leq 0 \\ -\lambda |\gamma| & \text{else} \end{cases} \\
&= \begin{cases} 1 & \text{if } \theta^\top \xi \leq 0 \\ \max\{0, 1 - \lambda \frac{\theta^\top \xi}{\|\theta\|_*}\} & \text{else} \end{cases} \\
&= L_R(\lambda(\theta^\top \xi)/\|\theta\|_*)
\end{aligned}$$

# Proof of Theorem 4

$$\inf_{\theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

# Proof of Theorem 4

$$\inf_{\theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$\begin{aligned}
 (\text{Lemma 1}) &= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right] \\
 &= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(y\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]
 \end{aligned}$$

# Proof of Theorem 4

$$\inf_{\theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(y\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 4}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L_R(\lambda(y\theta^\top x)/\|\theta\|_*)]$$

# Proof of Theorem 4

$$\inf_{\theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(y\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 4}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L_R(\lambda(y\theta^\top x) / \|\theta\|_*)]$$

$$\begin{aligned} \theta &\leftarrow \lambda \theta / \|\theta\|_* \\ &\downarrow \\ \lambda &= \|\theta\|_* \end{aligned}$$

# Proof of Theorem 4

$$\inf_{\theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [L(y\theta^\top x)]$$

$$(\text{Lemma 1}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}} L(y'\theta^\top x') - \lambda \|x' - \mathbf{x}\| - \lambda \delta_{y'=\mathbf{y}} \right]$$

$$= \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} \left[ \sup_{x' \in \mathbb{R}^n} L(y\theta^\top x') - \lambda \|x' - \mathbf{x}\| \right]$$

$$(\text{Lemma 4}) = \inf_{\substack{\theta \\ \lambda \geq 0}} \lambda \varepsilon + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L_R(\lambda(y\theta^\top x) / \|\theta\|_*)]$$

$$\lambda \geq 0$$

$$= \inf_{\theta} \varepsilon \|\theta\|_* + \mathbb{E}_{\widehat{\mathbb{P}}_N} [L_R(y\theta^\top x)]$$

$$\boxed{\begin{array}{l} \theta \leftarrow \lambda \theta / \|\theta\|_* \\ \downarrow \\ \lambda = \|\theta\|_* \end{array}}$$

# Conclusion

Regularization = Distributional Robustness

# Take Away

$$\inf_{\theta \in \Theta} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\Xi) : W_c(\mathbb{Q}, \mathbb{P}) \leq \varepsilon\}$$

# Take Away

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{Q \in \mathcal{M}(\Xi) : W_c(Q, \mathbb{P}) \leq \varepsilon\}$$

Step 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\mathbb{P}} \left[ \sup_{\zeta \in \Xi} \ell(\theta, \zeta) - \lambda c(\zeta, \xi) \right]$$

[MK18, ZG18, BM19, GK16, ZYG22]

# Take Away

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{Q \in \mathcal{M}(\Xi) : W_c(Q, \mathbb{P}) \leq \varepsilon\}$$

Step 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\mathbb{P}} \left[ \sup_{\zeta \in \Xi} \ell(\theta, \zeta) - \lambda c(\zeta, \xi) \right]$$

Step 2

# Take Away

$$\inf_{\theta \in \Theta} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)]$$

$$\mathcal{P} = \{Q \in \mathcal{M}(\Xi) : W_c(Q, \mathbb{P}) \leq \varepsilon\}$$

Step 1

$$\sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\ell(\theta, \xi)] = \inf_{\lambda \geq 0} \lambda \varepsilon + \mathbb{E}_{\mathbb{P}} \left[ \sup_{\zeta \in \Xi} \ell(\theta, \zeta) - \lambda c(\zeta, \xi) \right]$$

Step 2



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# Optimality of Affine Hypotheses

# Minimum Mean Square Error Estimator

$$J^\star = \inf_{h \in \mathbb{H}} \mathbb{E}_{\textcolor{teal}{P}} [\| \textcolor{red}{h}(\textcolor{violet}{x}) - y \|_2^2]$$

$$\mathbb{H} = \{ h : \mathbb{R}^n \rightarrow \mathbb{R}^m \}$$

# Minimum Mean Square Error Estimator

$$J^* = \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\|h(\mathbf{x}) - y\|_2^2]$$

Optimizer:  $h^*(y) = \mathbb{E}_{\mathbb{P}}[\mathbf{x} \mid y]$

Optimal value:  $J^* = \text{Tr} (\text{COV}_{\mathbb{P}}[\mathbf{x} \mid y])$

# MMSE under Normality

$$J^\star = \inf_{\mathbf{h} \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\|\mathbf{h}(\mathbf{x}) - \mathbf{y}\|_2^2] \quad \& \quad \mathbb{P} = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Optimizer:  $\mathbf{h}^\star(\mathbf{y}) = \mathbb{E}_{\mathbb{P}}[\mathbf{x} \mid \mathbf{y}]$

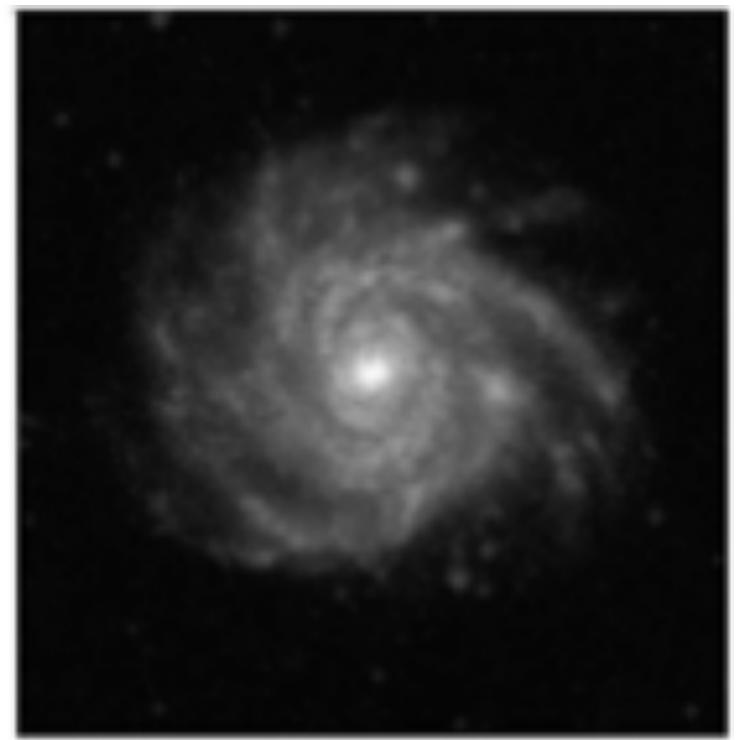
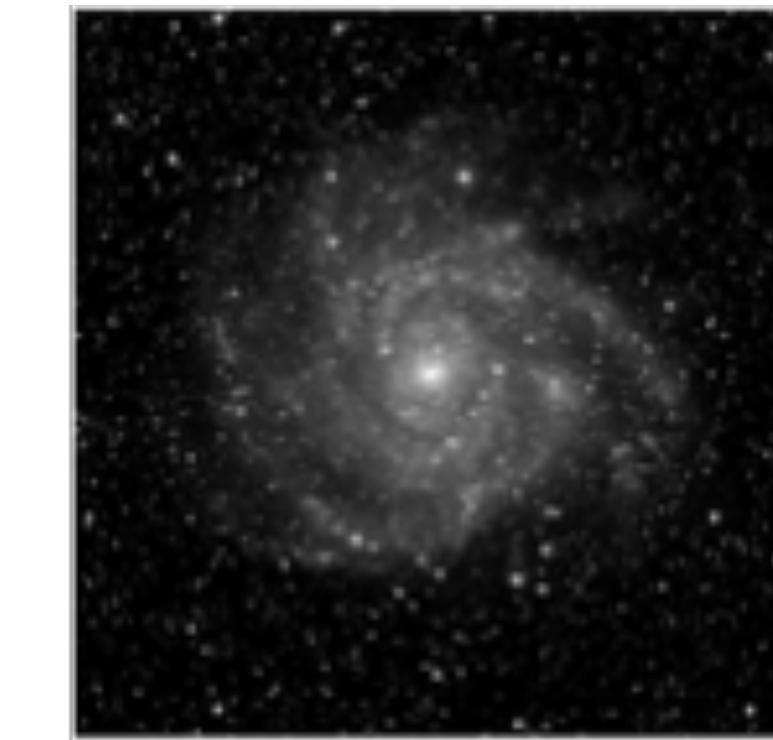
$$\mathbf{h}^\star(\mathbf{y}) = \boldsymbol{\Sigma}_{xy} \boldsymbol{\Sigma}_{yy}^{-1} (\mathbf{y} - \boldsymbol{\mu}_y) + \boldsymbol{\mu}_x$$

Optimal value:  $J^\star = \text{Tr} (\mathbb{C}\text{OV}_{\mathbb{P}}[\mathbf{x} \mid \mathbf{y}])$

$$J^\star = \text{Tr} [\boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy} \boldsymbol{\Sigma}_{yy}^{-1} \boldsymbol{\Sigma}_{yx}]$$

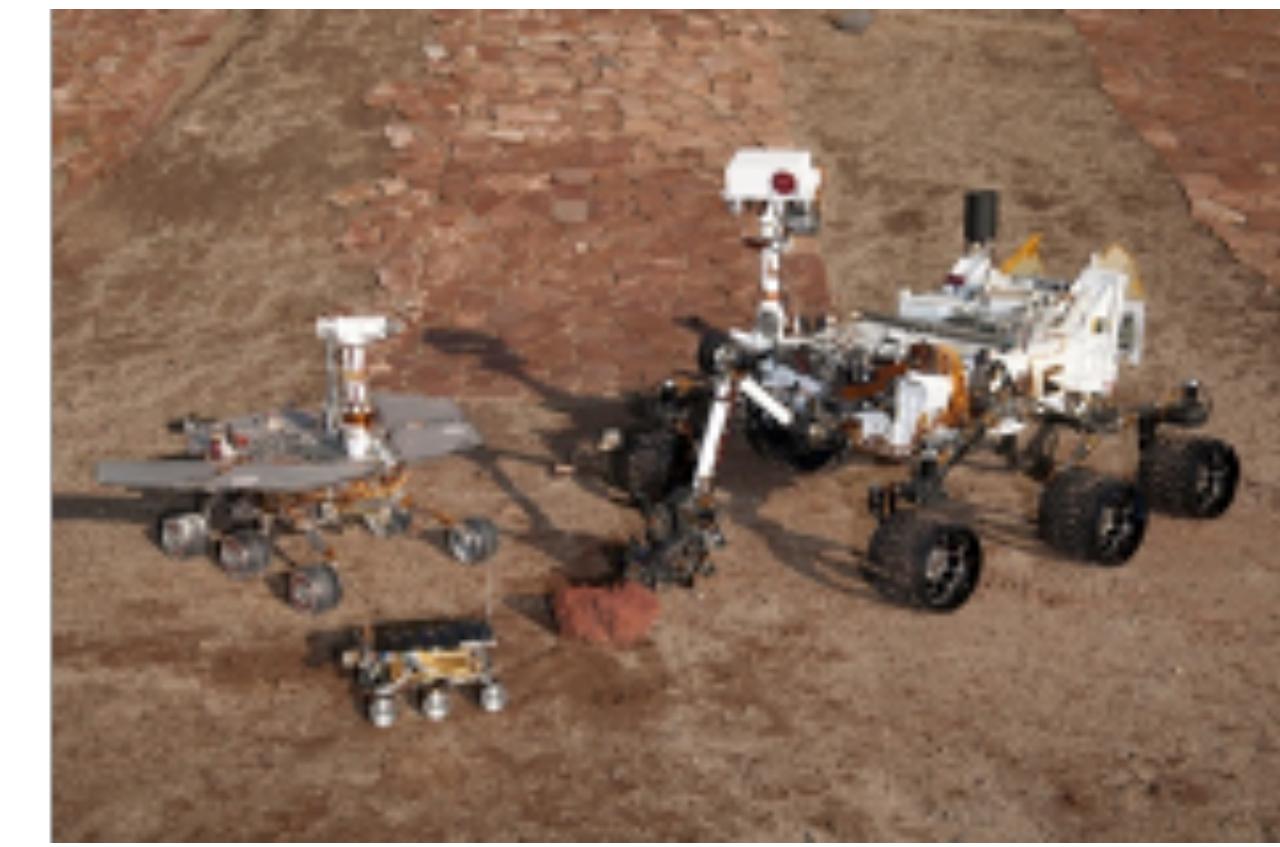
# MMSE under Normality

$$J^\star = \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\|h(x) - y\|_2^2]$$



# MMSE under Normality

$$J^{\star} = \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{P}} [\| \textcolor{red}{h}(\textcolor{violet}{x}) - y \|_2^2]$$

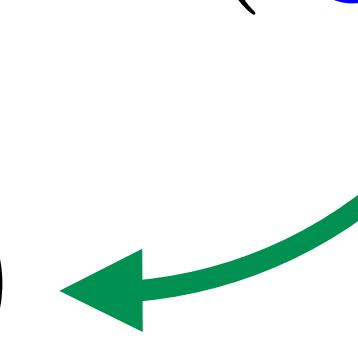


# Distributionally Robust MMSE Estimator

$$\inf_{\mathbf{h} \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|\mathbf{h}(x) - y\|_2^2]$$

$$\mathcal{P} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{X} \times \mathbb{Y}) : W_c(\mathbb{Q}, \mathbb{P}) \leq \varepsilon\}$$

$$\mathbb{P} = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

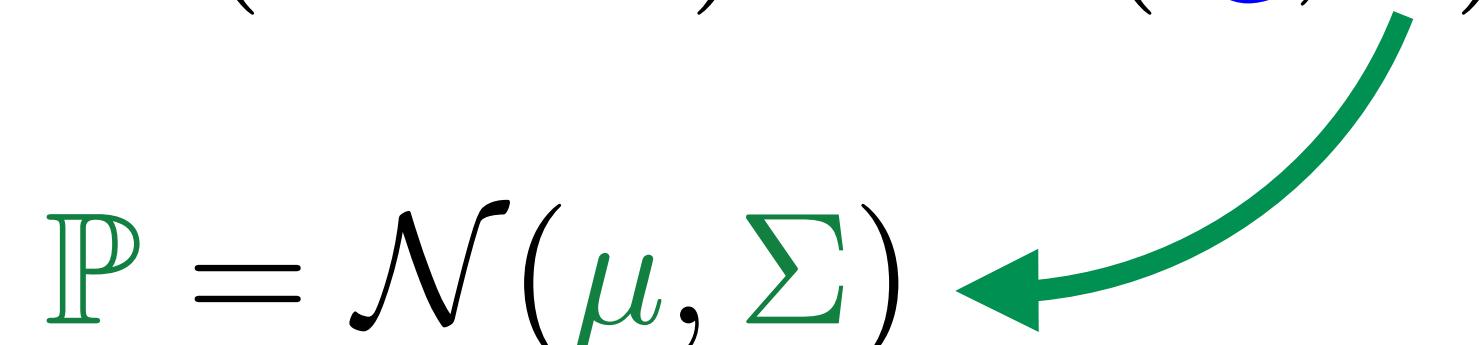


# Distributionally Robust MMSE Estimator

$$\inf_{\textcolor{red}{h} \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\textcolor{blue}{\mathbb{Q}}} [\|\textcolor{red}{h}(x) - y\|_2^2]$$

$$\mathcal{P} = \{\textcolor{blue}{\mathbb{Q}} \in \mathcal{M}(\mathbb{X} \times \mathbb{Y}) : W_c(\textcolor{blue}{\mathbb{Q}}, \textcolor{green}{\mathbb{P}}) \leq \varepsilon\}$$

$$\mathbb{P} = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$



$$c((x, y), (x', y')) = \|(x, y) - (x', y')\|_2^2$$

# Gelbrich Bound

**Lemma 5.** For any  $\mathbb{Q}_1 \sim (\mu_1, \Sigma_1)$  and  $\mathbb{Q}_2 \sim (\mu_2, \Sigma_2)$ , we have

$$W_c(\mathbb{Q}_1, \mathbb{Q}_2) \geq \|\mu_1 - \mu_2\|^2 + \text{Tr} \left[ \Sigma_1 + \Sigma_2 - 2 \left( \Sigma_2^{\frac{1}{2}} \Sigma_1 \Sigma_2^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]$$

# Gelbrich Bound

**Lemma 5.** For any  $\mathbb{Q}_1 \sim (\mu_1, \Sigma_1)$  and  $\mathbb{Q}_2 \sim (\mu_2, \Sigma_2)$ , we have

$$W_c(\mathbb{Q}_1, \mathbb{Q}_2) \geq \|\mu_1 - \mu_2\|^2 + \text{Tr} \left[ \Sigma_1 + \Sigma_2 - 2 \left( \Sigma_2^{\frac{1}{2}} \Sigma_1 \Sigma_2^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]$$

Tight under Normality.

# Gelbrich Bound

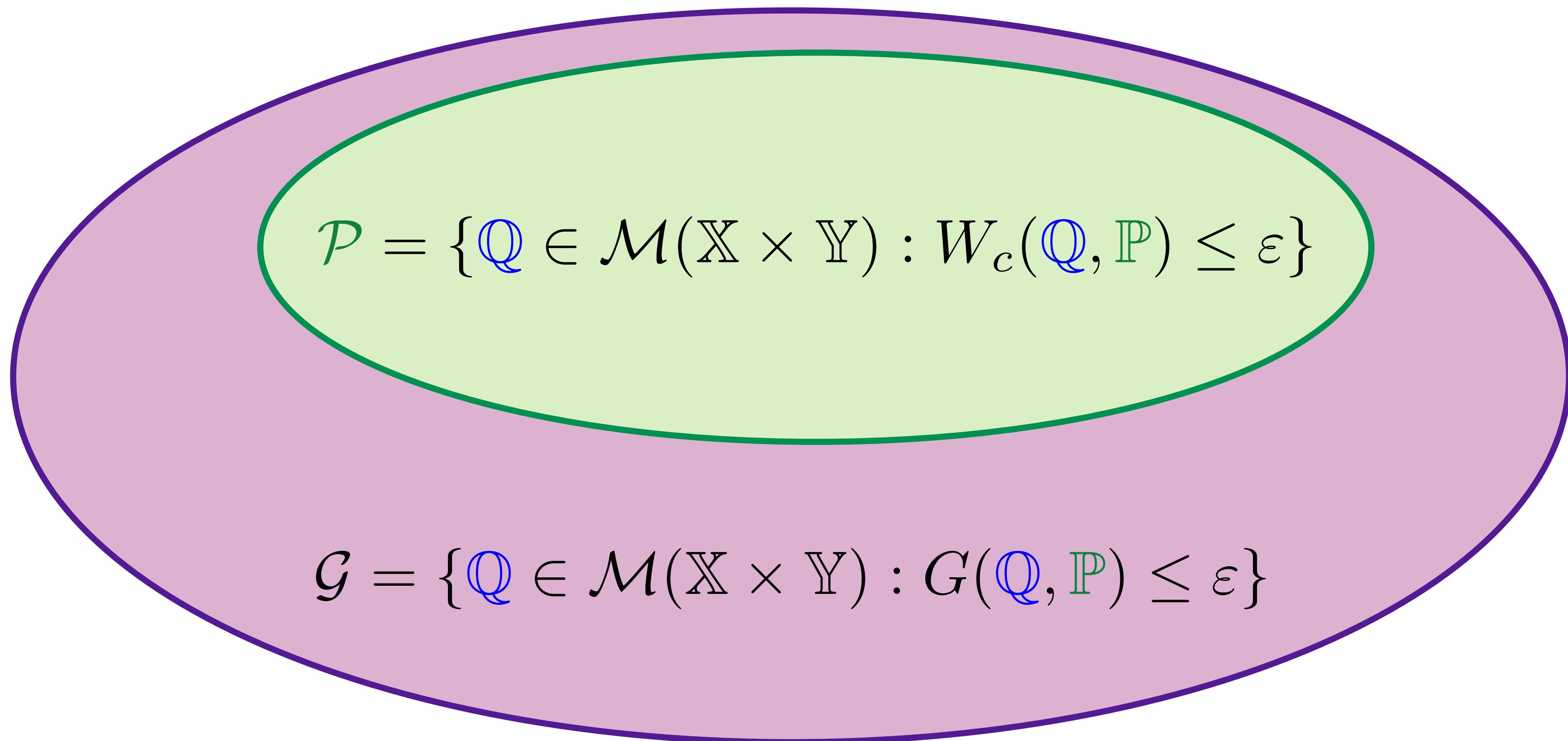
**Lemma 5.** For any  $\mathbb{Q}_1 \sim (\mu_1, \Sigma_1)$  and  $\mathbb{Q}_2 \sim (\mu_2, \Sigma_2)$ , we have

$$W_c(\mathbb{Q}_1, \mathbb{Q}_2) \geq \|\mu_1 - \mu_2\|^2 + \text{Tr} \left[ \Sigma_1 + \Sigma_2 - 2 \left( \Sigma_2^{\frac{1}{2}} \Sigma_1 \Sigma_2^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]$$

Tight under Normality.

$G(\mathbb{Q}_1, \mathbb{Q}_2)$

# Gelbrich Bound



# Useful Inequalities

$$\leq \inf_{h \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2]$$

# Useful Inequalities

$$\begin{aligned} \text{(restriction)} &\leq \inf_{\color{red} h \in \mathbb{A}} \sup_{\color{blue} Q \in \mathcal{P}} \mathbb{E}_{\color{blue} Q} [\|x - h(y)\|^2] \\ &\leq \inf_{\color{red} h \in \mathbb{A}} \sup_{\color{blue} Q \in \mathcal{G}} \mathbb{E}_{\color{blue} Q} [\|x - h(y)\|^2] \end{aligned}$$

# Useful Inequalities

$$\begin{aligned} &\leq \inf_{h \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2] \\ (\text{restriction}) \quad &\leq \inf_{h \in \mathbb{A}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2] \\ (\text{Lemma 5}) \quad &\leq \inf_{h \in \mathbb{A}} \sup_{\mathbb{Q} \in \mathcal{G}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2] \end{aligned}$$

# Useful Inequalities

$$\leq \sup_{\mathbb{Q} \in \mathcal{P}} \inf_{h \in \mathbb{H}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2]$$

(weak duality)  $\leq \inf_{h \in \mathbb{H}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2]$

(restriction)  $\leq \inf_{h \in \mathbb{A}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2]$

(Lemma 5)  $\leq \inf_{h \in \mathbb{A}} \sup_{\mathbb{Q} \in \mathcal{G}} \mathbb{E}_{\mathbb{Q}} [\|x - h(y)\|^2]$

# Useful Inequalities

(restriction)

$$\sup_{Q \in \mathcal{P} \cap \mathcal{N}} \inf_{h \in \mathbb{H}} \mathbb{E}_Q [\|x - h(y)\|^2] \leq \sup_{Q \in \mathcal{P}} \inf_{h \in \mathbb{H}} \mathbb{E}_Q [\|x - h(y)\|^2]$$

$$\text{(weak duality)} \leq \inf_{h \in \mathbb{H}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|x - h(y)\|^2]$$

$$\text{(restriction)} \leq \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|x - h(y)\|^2]$$

$$\text{(Lemma 5)} \leq \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{G}} \mathbb{E}_Q [\|x - h(y)\|^2]$$

# Sandwich Theorem

**Theorem 5.** We have



$$\sup_{Q \in \mathcal{P} \cap \mathcal{N}} \inf_{h \in \mathbb{H}} \mathbb{E}_Q [\|x - h(y)\|^2] = \inf_{h \in \mathbb{A}} \sup_{Q \in \mathcal{G}} \mathbb{E}_Q [\|x - h(y)\|^2]$$

# Optimality of Affine Estimators

$$\inf_{h \in \mathbb{H}} \sup_{Q \in \mathcal{P}} \mathbb{E}_Q [\|x - h(y)\|^2] = \sup_{Q \in \mathcal{P}} \inf_{h \in \mathbb{H}} \mathbb{E}_Q [\|x - h(y)\|^2]$$

$h^*$  is affine

$Q^*$  is Normal

$$h^* = \operatorname{argmin}_{h \in \mathbb{H}} \mathbb{E}_{Q^*} [\|h(x) - y\|_2^2]$$