CHAPTER VIII*

Unlimited Sequences of Bernoulli Trials

This chapter discusses certain properties of randomness and the important law of the iterated logarithm for Bernoulli trials. A different aspect of the fluctuation theory of Bernoulli trials (at least for $p = \frac{1}{2}$) is covered in chapter III.

1. INFINITE SEQUENCES OF TRIALS

In the preceding chapter we have dealt with probabilities connected with n Bernoulli trials and have studied their asymptotic behavior as $n \to \infty$. We turn now to a more general type of problem where the events themselves cannot be defined in a finite sample space.

Example. *A problem in runs.* Let α and β be positive integers, and consider a potentially unlimited sequence of Bernoulli trials, such as tossing a coin or throwing dice. Suppose that Paul bets Peter that a run of α consecutive successes will occur before a run of β consecutive failures. It has an intuitive meaning to speak of the event that Paul wins, but it must be remembered that in the mathematical theory the term event stands for "aggregate of sample points" and is meaningless unless an appropriate sample space has been defined. The model of a finite number of trials is insufficient for our present purpose, but the difficulty is solved by a simple passage to the limit. In n trials Peter wins or loses, or the game remains undecided. Let the corresponding probabilities be x_n, y_n, z_n $(x_n + y_n + z_n = 1)$. As the number n of trials increases, the probability z_n of a tie can only decrease, and both x_n and y_n necessarily increase. Hence $x = \lim x_n$, $y = \lim y_n$, and $z = \lim z_n$ exist. Nobody would

* This chapter is not directly connected with the material covered in subsequent chapters and may be omitted at first reading.

hesitate to call them the probabilities of Peter's ultimate gain or loss or of a tie. However, the corresponding three events are defined only in the sample space of infinite sequences of trials, and this space is not discrete.

The example was introduced for illustration only, and the numerical values of x_n, y_n, z_n are not our immediate concern. We shall return to their calculation in example XIII, (8.*b*). The limits x, y, z may be obtained by a simpler method which is applicable to more general cases. We indicate it here because of its importance and intrinsic interest.

Let A be the event that *a run of* α *consecutive successes occurs before a run of* β *consecutive failures.* In the event A Paul wins and $x = \mathbf{P}\{A\}$. If u and v are the conditional probabilities of A under the hypotheses, respectively, that the first trial results in success or failure, then $x = pu + qv$ [see V, (1.8)]. Suppose first that the first trial results in success. In this case the event A can occur in α mutually exclusive ways: (1) The following $\alpha - 1$ trials result in successes; the probability for this is $p^{\alpha-1}$. (2) The first failure occurs at the νth trial where $2 \leq \nu \leq \alpha$. Let this event be H_ν. Then $\mathbf{P}\{H_\nu\} = p^{\nu-2}q$, and $\mathbf{P}\{A \mid H_\nu\} = v$. Hence (using once more the formula for compound probabilities)

$$u = p^{\alpha-1} + qv(1+p+\cdots+p^{\alpha-2}) = p^{\alpha-1} + v(1-p^{\alpha-1}). \tag{1.1}$$

If the first trial results in failure, a similar argument leads to

$$v = pu(1+q+\cdots+q^{\beta-2}) = u(1-q^{\beta-1}). \tag{1.2}$$

We have thus two equations for the two unknowns u and v and find for $x = pu + qv$

$$x = p^{\alpha-1}\frac{1-q^\beta}{p^{\alpha-1}+q^{\beta-1}-p^{\alpha-1}q^{\beta-1}}. \tag{1.3}$$

To obtain y we have only to interchange p and q, and α and β. Thus

$$y = q^{\beta-1}\frac{1-p^\alpha}{p^{\alpha-1}+q^{\beta-1}-p^{\alpha-1}q^{\beta-1}}. \tag{1.4}$$

Since $x + y = 1$, we have $z = 0$; *the probability of a tie is zero.*

For example, in tossing a coin ($p = \frac{1}{2}$) the probability that a run of two heads appears before a run of three tails is 0.7; for two consecutive heads before four consecutive tails the probability is $\frac{5}{6}$, for three consecutive heads before four consecutive tails $\frac{15}{22}$. In rolling dice there is probability 0.1753 that two consecutive aces will appear before five consecutive non-aces, etc. ▶

In the present volume we are confined to the theory of discrete sample spaces, and this means a considerable loss of mathematical elegance. The general theory considers n Bernoulli trials only as the beginning of an infinite sequence of trials. A sample point is then represented by an infinite sequence of letters S and F, and the sample space is the aggregate of all such sequences. A finite sequence, like $SSFS$, stands for the aggregate of all points with this beginning, that is, for the compound event that in an infinite sequence of trials the first four result in S, S, F, S,