CONTENTS

UNIT1: PARTIAL DIFFERENTIATION

1. Introduction

Any function of two or more variables may be differentiated partially with respect to one variable treating other variables as constants; for instance, the function $f(x,y)$ may be differentiated with respect to x taking y as constant and similarly we can perform partial differentiation with respect to y keeping x constant. Here partial derivatives of $f(x,y)$ with respect to x and y are denoted as $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ respectively; also the partial derivatives of $\frac{\partial f}{\partial x}$ with respect to x and y are denoted as $\frac{\partial^2 f}{\partial x^2}$ or f_{xx} and $\frac{\partial^2 f}{\partial x\,\partial y}$ or f_{xy} respectively.

Again, if '$z = f(x,y)$ ' is a function of two independent variables 'x' and 'y', let us use the following notations for the partial derivatives of 'z':

$$\frac{\partial z}{\partial x} \equiv p\ ,\ \frac{\partial z}{\partial y} \equiv q\ ,\ \frac{\partial^2 z}{\partial x^2} \equiv r,\quad \frac{\partial^2 z}{\partial x\,\partial y} \equiv s,\ \frac{\partial^2 z}{\partial y^2} \equiv t$$

Remarks

- **If $z = f(x)$ be a function of single independent variable x then $\frac{\partial z}{\partial x} = \frac{dz}{dx}$**
- **Order of partial differentiation is commutative i.e. $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$, if $f(x,y)$ possesses continuous partial derivatives.**
- **Geometrical interpretation: $\frac{\partial f}{\partial x}$ gives the slope of tangent to the curve $z = f(x,y)$ at the point of intersection with the plane $y = 0$.**

Example1 Find all the first order partial order derivatives for the function

$$t = x^3yz + 5\sqrt{y}\,z + \sin\frac{1}{y}e^{x^2y-2y^2}$$

Solution: $\frac{\partial t}{\partial x} = 3x^2yz + \sin\frac{1}{y}e^{x^2y-2y^2}(2xy)$

$\frac{\partial t}{\partial y} = x^3z + \frac{5z}{2\sqrt{y}} + \sin\frac{1}{y}e^{x^2y-2y^2}(x^2-4y) + \cos\frac{1}{y}\left(-\frac{1}{y^2}\right)e^{x^2y-2y^2}$

$\frac{\partial t}{\partial z} = x^3y + 5\sqrt{y}$

Example2 If $v = (x^2+y^2+z^2)^{-\frac{1}{2}}$, show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

Solution: $\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - \frac{3x}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+z^2)^3} = -\frac{(x^2+y^2+z^2)^{\frac{1}{2}}(x^2+y^2+z^2-3x^2)}{(x^2+y^2+z^2)^3} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Similarly $\frac{\partial^2 v}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$ and $\frac{\partial^2 v}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

Example3 If $u = f(r)$ and $x = r\cos\theta, y = r\sin\theta$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

Solution: $\frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2} = f''(r).\left(\frac{\partial r}{\partial x}\right)^2 + f'(r).\frac{\partial^2 r}{\partial x^2}$

Similarly $\frac{\partial^2 u}{\partial y^2} = f''(r).\left(\frac{\partial r}{\partial y}\right)^2 + f'(r).\frac{\partial^2 r}{\partial y^2}$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r).\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r).\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right] \quad \ldots (1)$$

We have $x = r\cos\theta, y = r\sin\theta$ then $x^2 + y^2 = r^2$ ...(2)

$\Rightarrow r = \sqrt{x^2 + y^2}$

$$\therefore \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}.2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \text{ and } \frac{\partial^2 r}{\partial x^2} = \frac{r.1 - x.\frac{\partial r}{\partial x}}{r^2} = \frac{r - \frac{x^2}{r}}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \quad \text{(using (2))}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$

Substituting the values of $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial^2 r}{\partial x^2}, \frac{\partial^2 r}{\partial y^2}$ in equation (1), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r).\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right] + f'(r).\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r)$$

Example4 If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$?

Solution: $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial \theta}{\partial r} = t^n.e^{-r^2/4t}\left(\frac{-2r}{4t}\right) = -\frac{r}{2}t^{n-1}e^{-r^2/4t}$$

$$\therefore \left(r^2\frac{\partial \theta}{\partial r}\right) = -\frac{r^3}{2}t^{n-1}e^{-r^2/4t}$$

and $\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = -\frac{3r^2}{2}t^{n-1}e^{-r^2/4t} - \frac{r^3}{2}t^{n-1}e^{-r^2/4t}\left(\frac{-2r}{4t}\right)$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \left(-\frac{3}{2}t^{n-1} - \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$$

Also $\frac{\partial \theta}{\partial t} = nt^{n-1}e^{-r^2/4t} + t^n e^{-r^2/4t}.\frac{r^2}{4t^2} = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$

As $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

$\Rightarrow \left(-\frac{3}{2}t^{n-1} - \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t} = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$

$\therefore n = \frac{-3}{2}$

2. Composite Functions

- If $z = f(x,y)$, such that $x = g(t)$ and $y = h(t)$, z is called a composite function of single variable t.
- $z = f(x,y)$, such that $x = g(u,v)$ and $y = h(u,v)$, z is called a composite function of two variables u and v.

2.1 Differentiation of Composite Functions (Chain Rule)

- If $z = f(x,y)$, where $x = g(t)$ and $y = h(t) \Rightarrow \frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$

  where $\frac{dz}{dt}$ is called the total derivative of z.
- If $z = f(x,y)$, where $x = g(u,v)$ and $y = h(u,v)$

  $\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$
- If $z = f(x,y)$, where $y = g(x) \Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}.\frac{dy}{dx}$

Example5 If $z = f(x,y)$, $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$,

Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution: Here $z = f(x,y)$, such that $x = g(u,v)$ and $y = h(u,v)$

$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u}$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}e^v$

$\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u} + \frac{\partial z}{\partial x}e^{-v} + \frac{\partial z}{\partial y}e^v$

$= (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Example6 If $z = f(x,y)$, $x = \log u$ and $y = \log v$,

Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = u^2\frac{\partial^2 z}{\partial u^2} + v^2\frac{\partial^2 z}{\partial v^2} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

Solution: Here $z = f(x,y)$, such that $x = g(u)$ and $y = h(v)$

$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} \because \frac{\partial y}{\partial u} = 0$

$= \frac{\partial z}{\partial x}\frac{dx}{du} = \frac{\partial z}{\partial x}.\frac{1}{u}$

$\Rightarrow \frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left[\frac{\partial z}{\partial u}\right] = \frac{\partial z}{\partial x}.\left(-\frac{1}{u^2}\right) + \frac{1}{u}\frac{\partial^2 z}{\partial u \partial x}$

Similarly $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y}.\frac{1}{v}$ and $\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial y}.\left(-\frac{1}{v^2}\right) + \frac{1}{v}\frac{\partial^2 z}{\partial v \partial y}$

$$\therefore u^2\frac{\partial^2 z}{\partial u^2} + v^2\frac{\partial^2 z}{\partial v^2} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} + u\frac{\partial^2 z}{\partial u \partial x} - \frac{\partial z}{\partial y} + v\frac{\partial^2 z}{\partial v \partial y} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

$$= u\frac{\partial^2 z}{\partial u \partial x} + v\frac{\partial^2 z}{\partial v \partial y} = u\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial u}\right] + v\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial v}\right]$$

$$= u\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}.\frac{1}{u}\right] + v\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}.\frac{1}{v}\right] = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

Example7 If $u = x\log xy$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$

Solution: Here $u = f(x,y)$, such that $y = f(x)$

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx} \ldots (1)$$

Now $\frac{\partial u}{\partial x} = x.\frac{y}{xy} + \log xy = 1 + \log xy \ldots (2)$

Also $\frac{\partial u}{\partial y} = x.\frac{x}{xy} = \frac{x}{y} \ldots (3)$

Further to find $\frac{dy}{dx}$, differentiating $x^3 + y^3 + 3xy = 1$ with respect to x

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2+y}{y^2+x} \ldots (4)$$

Using (2), (3), (4) in (1), we get

$$\frac{du}{dx} = 1 + \log xy - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

Example8 If $u = \sin^{-1}(x-y), x = 3t, y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Solution: Here $u = f(x,y)$, such that $x = g(t)$ and $y = h(t)$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}.3 + \frac{1}{\sqrt{1-(x-y)^2}}(-1).12t^2$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}} = \frac{3(1-4t^2)}{\sqrt{1-t^2-8t^2+16t^4+8t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4)-t^2(1-8t^2-16t^6)}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

Example9 If $x(u) = 1 + au$ and $y(u) = bu^3$, find the rate of change of $f(x,y) = xe^{-y}$ with respect to u.

Solution: $\frac{df}{du} = \frac{\partial f}{\partial x}.\frac{dx}{du} + \frac{\partial f}{\partial y}.\frac{dy}{du} = e^{-y}.a + (-xe^{-y}).3bu^2$

which on substitution of x and y gives

$\frac{df}{du} = e^{-bu^3}.a + (-1 - au)e^{-bu^3}.3bu^2$

$= e^{-bu^3}[a - 3bu^2 - 3abu^3]$

Example10 Find the total derivative of $f(x, y) = x^2 + 3xy$ with respect to x given that $y = \sin^{-1} x$.

Solution: $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx}$

$= (2x + 3y) + 3x.\frac{1}{\sqrt{1-x^2}}$

which on substitution of y gives

$\frac{df}{dx} = 2x + 3\sin^{-1} x + \frac{3x}{\sqrt{1-x^2}}$

Example11 If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$$

Solution: Let $p = x^2 + 2yz$, $q = y^2 + 2zx$

$\therefore u = f(p, q)$, such that $p = g(x, y, z), q = h(x, y, z)$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}.\frac{\partial q}{\partial x}$

$= \frac{\partial u}{\partial p}.2x + \frac{\partial u}{\partial q}.2z \; ... (1)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}.\frac{\partial q}{\partial y}$

$= \frac{\partial u}{\partial p}.2z + \frac{\partial u}{\partial q}.2y \; ... (2)$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}.\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}.\frac{\partial q}{\partial z}$

$= \frac{\partial u}{\partial p}.2y + \frac{\partial u}{\partial q}.2x \; ... (3)$

Substituting the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ from (1), (2), (3) respectively,

we get $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

Example12 If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution: Let $l = e^{y-z}$, $m = e^{z-x}, n = e^{x-y}$

$\therefore u = f(l, m, n)$, such that $l = f(x, y, z), m = g(x, y, z), n = h(x, y, z)$

$$u = u(e^{y-z}, e^{z-x}, e^{x-y})$$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}.\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}.\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}.\frac{\partial n}{\partial x} = -m\frac{\partial u}{\partial m} + n\frac{\partial u}{\partial n}$ ...①

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}.\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}.\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}.\frac{\partial n}{\partial y} = l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n}$ ...②

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}.\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}.\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}.\frac{\partial n}{\partial y} = l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n}$ ...③

Substituting the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ from ①, ②, ③ respectively,

we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

3. Differentiation of Implicit Functions

If $f(x, y) = 0$ is an implicit relation between x and y

$\Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$, also $\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$

Note: We can evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for implicit functions by direct differentiation in most cases.

Example13 If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, Find $\frac{dy}{dx}$ using differential formula of implicit functions. Also verify the result by direct differentiation.

Solution: For implicit functions, $\frac{dy}{dx} = -\frac{f_x}{f_y}$

Here $f_x = 3x^2 + 6xy + 6y^2$, $f_y = 3x^2 + 12xy + 3y^2$

$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x^2+2xy+2y^2}{x^2+4xy+y^2}$

Verification: Differentiating the equation $x^3 + 3x^2y + 6xy^2 + y^3 = 1$ w.r.t. x

$\Rightarrow 3x^2 + 3x^2\frac{dy}{dx} + 6xy + 12xy\frac{dy}{dx} + 6y^2 + 3y^2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x^2+2xy+2y^2}{x^2+4xy+y^2}$

Example14 If $x^n + y^n = a^n$, Find $\frac{d^2y}{dx^2}$

Solution: For implicit functions $\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$

Here $f_x = nx^{n-1}$, $f_{xx} = n(n-1)x^{n-2}$, $f_x^2 = n^2x^{2n-2}$, $f_{xy} = 0$

$f_y = ny^{n-1}$, $f_{yy} = n(n-1)y^{n-2}$, $f_y^2 = n^2y^{2n-2}$, $f_y^3 = n^3y^{3n-3}$

$\therefore \frac{d^2y}{dx^2} = -\frac{n(n-1)x^{n-2}.n^2y^{2n-2} - 2nx^{n-1}ny^{n-1}.0 + n(n-1)y^{n-2}.n^2x^{2n-2}}{n^3y^{3n-3}}$

$$= -\frac{n^3(n-1)x^{n-2}y^{2n-2}+n^3(n-1)y^{n-2}x^{2n-2}}{n^3y^{3n-3}}$$

$$= -n^3(n-1)x^{n-2}y^{n-2}\frac{y^n+x^n}{n^3y^{3n-3}}$$

$$= -\frac{(n-1)x^{n-2}a^n}{y^{2n-1}}$$

Example15 If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$

Solution: Given $x^2 + y^2 + z^2 - 2xyz - 1 = 0 \ldots$ ①

Now, if $f(x, y, z) = 0$ is an implicit relation between x, y, z

Then, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$

$\Rightarrow (2x - 2yz)dx + (2y - 2xz)dy + (2z - 2xy)dz = 0$

$\Rightarrow (x - yz)dx + (y - xz)dy + (z - xy)dz = 0$

$\Rightarrow \frac{dx}{(y-xz)(z-xy)} + \frac{dy}{(x-yz)(z-xy)} + \frac{dz}{(x-yz)(y-xz)} = 0 \ldots$ ②

Now $(y - xz)^2 = y^2 + x^2z^2 - 2xyz$

$= x^2z^2 + (y^2 - 2xyz) = x^2z^2 + 1 - x^2 - z^2$

$\because (y^2 - 2xyz) = 1 - x^2 - z^2$ using ①

$\Rightarrow (y - xz)^2 = (1 - x^2)(1 - z^2)$

$\because (1 - a)(1 - b) = 1 - a - b + ab$

$\Rightarrow (y - xz) = \sqrt{1 - x^2}\sqrt{1 - z^2} \ldots$ ③

Similarly $(z - xy) = \sqrt{1 - x^2}\sqrt{1 - y^2} \ldots$ ④

$(x - yz) = \sqrt{1 - y^2}\sqrt{1 - z^2} \ldots$ ⑤

Using ③, ④, ⑤ in ②, we get

$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$

4. Exact Differentials

A differential of the form $df = f_1(x, y)dx + f_2(x, y)dy$ is said to be exact or perfect, if $\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial f_2(x,y)}{\partial x}$

Example16 Given the differential $df = (x^2 + 4xy + 3y^2)dx + (2x^2 + 6xy)dy$, check whether it is exact or not.

Solution: Given differential is of the form $df = f_1(x, y)dx + f_2(x, y)dy$,

$f_1(x, y) = x^2 + 4xy + 3y^2$, $f_2(x, y) = 2x^2 + 6xy$

We have $\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial(x^2+4xy+3y^2)}{\partial y} = 4x + 6y \ldots ①$

Also, $\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial(2x^2+6xy)}{\partial x} = 4x + 6y \ldots ②$

From ① and ② $\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial f_2(x,y)}{\partial x}$

$\therefore df = (x^2 + 4xy + 3y^2)dx + (2x^2 + 6xy)dy$ is an exact differential.

5. Maxima and Minima of Functions of Two Variables

A function $z = f(x, y)$ has extreme points, where both the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are zero. An extreme point of a function of two variables variable is either a local maximum, a local minimum, or a saddle point.

Saddle Point: A point (x_i, y_i)on the curve $z = f(x, y)$, where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are zero, but the value of function $z = f(x, y)$ on L.H.S. and R.H.S. of point (x_i, y_i) are of different signs. Here the function does not have any maximum or minimum value, but a point of inflection.

Algorithm to find maximum or minimum values of a function $z = f(x, y)$

1. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
2. Solve the equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, simultaneously to obtain the extreme points as (x_i, y_i).
3. Find the differentials $\frac{\partial^2 z}{\partial x^2} \equiv r$, $\frac{\partial^2 z}{\partial x\, \partial y} \equiv s$ and $\frac{\partial^2 z}{\partial y^2} \equiv t$
4. Evaluate the expression $(rt - s^2)$ at extreme points (x_i, y_i).
    i. If $[rt - s^2]_{(x_i,y_i)} > 0$, check the value of $[r]_{(x_i,y_i)}$
        a. $[r]_{(x_i,y_i)} > 0 \Rightarrow z = f(x, y)$ has a minimum value at (x_i, y_i)
        b. $[r]_{(x_i,y_i)} < 0 \Rightarrow z = f(x, y)$ has a maximum value at (x_i, y_i)
    ii. If $[rt - s^2]_{(x_i,y_i)} = 0$, check the value of $z = f(x_i, y_i)$ in the immediate vicinity on both sides. If the value of function $z = f(x, y)$ on L.H.S. and R.H.S. of point (x_i, y_i) are of different signs, it is a saddle point.
    iii. If $[rt - s^2]_{(x_i,y_i)} < 0 \Rightarrow z = f(x, y)$ has no extreme value at (x_i, y_i), i.e., it is a saddle point.

Example17 Locate the stationary points of $u = x^3 - 3x^2 - 4y^2 + 1$ and determine their nature.

Solution: For stationary points $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 6x = 0 \quad \ldots (1)$$

$$\frac{\partial u}{\partial y} = -8y = 0 \quad \ldots (2)$$

From (1), we get $x = 0$ or $x = 2$

From (2), we get $y = 0$

$\therefore$ Stationary points are (0,0) and (2,0)

Now $\frac{\partial^2 u}{\partial x^2} \equiv r = 6x - 6,\ \frac{\partial^2 u}{\partial x\,\partial y} \equiv s = 0,\ \frac{\partial^2 u}{\partial y^2} \equiv t = -8$

$\therefore rt - s^2 = -8(6x - 6) - 0 = -48x + 48$

Again, $[rt - s^2]_{(0,0)} = 48 > 0$ and $[r]_{(0,0)} < 0$

$\therefore$ (0, 0) is a point of maxima and maximum value = 1

Also $[rt - s^2]_{(2,0)} = -48 < 0$

$\therefore$ (2,0) is a saddle point

Example18 Locate the stationary points of $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature.

Solution: For stationary points $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \ldots (1)$$

$$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \ldots (2)$$

Adding (1) and (2), we get

$x^3 + y^3 = 0 \quad \Rightarrow (x + y)(x^2 - xy + y^2) = 0$

or

$\Rightarrow (x + y) = 0$ $\Rightarrow y = -x \quad \ldots (3)$	$(x^2 - xy + y^2) = 0$ $x = 0,\ y = 0$ is the only solution for $(x^2 - xy + y^2) = 0$

Using (3) in (1), we get $x^3 - x - x = 0$

$\Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$

$\Rightarrow x = 0, \pm\sqrt{2}$

$\therefore$ The extreme points using (3) are (0,0), $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

Now $\frac{\partial^2 u}{\partial x^2} \equiv r = 12x^2 - 4,\ \frac{\partial^2 u}{\partial x\,\partial y} \equiv s = 4,\ \frac{\partial^2 u}{\partial y^2} \equiv t = 12y^2 - 4$

$\therefore rt - s^2 = 4(3x^2-1)4(3y^2-1) - 16 = 16[(3x^2-1)(3y^2-1) - 1]$

Again, $[rt - s^2]_{(0,0)} = 0$, checking the values of the function $u = f(x,y)$ in the immediate vicinity on both sides.

$[u]_{(-1,1)} = 1+1-2-4-2 < 0$, $[u]_{(1,1)} = 1+1-2+4-2 > 0$

$\therefore$ (0,0) is a saddle point

Again $[rt - s^2]_{(\sqrt{2},-\sqrt{2})} = 16(5)(5) - 16 > 0$, and $[r]_{(\sqrt{2},-\sqrt{2})} > 0$

$\therefore \left(\sqrt{2},-\sqrt{2}\right)$ is a point of minima

Also $[rt - s^2]_{(-\sqrt{2},\sqrt{2})} > 0$, and $[r]_{(-\sqrt{2},\sqrt{2})} > 0$

$\therefore \left(-\sqrt{2},\sqrt{2}\right)$ is also a point of minima

Example19 Locate and determine the nature of the stationary points of the function

$\sin x + \sin y + \sin(x+y)$, where $x, y \in [0,\pi]$.

Solution: Let $f(x,y) = \sin x + \sin y + \sin(x+y)$

For stationary points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\Rightarrow \frac{\partial f}{\partial x} = \cos x + \cos(x+y) = 0 \ldots ①$

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0 \ldots ②$

Subtracting ② from ①, we get

$\cos x - \cos y = 0$

$\Rightarrow \cos x = \cos y$

$\Rightarrow x = y \because x, y \in [0,\pi] \ldots ③$

Using ③ in ①, we get

$\cos x + \cos 2x = 0$

$\Rightarrow \cos 2x = -\cos x$

$\Rightarrow \cos 2x = \cos(\pi - x)$

$\Rightarrow 2x = 2n\pi \pm (\pi - x)$, $n \in z$

$\Rightarrow 3x = (2n+1)\pi$ or $x = (2n-1)\pi$, $n \in z$

$\Rightarrow x = \frac{\pi}{3}, \pi$ for $n = 0$ and $n = 1$ respectively $\quad \because x \in [0,\pi]$

$\therefore$ The extreme points using ③ are $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and (π,π) respectively.

Now $\frac{\partial^2 f}{\partial x^2} \equiv r = -\sin x - \sin(x+y)$

$\frac{\partial^2 f}{\partial x\, \partial y} \equiv s = -\sin(x+y)$

$\frac{\partial^2 f}{\partial y^2} \equiv t = -\sin y - \sin(x+y)$

Now $[r]_{\left(\frac{\pi}{3},\frac{\pi}{3}\right)} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3} < 0$

$[s]_{\left(\frac{\pi}{3},\frac{\pi}{3}\right)} = -\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$[t]_{\left(\frac{\pi}{3},\frac{\pi}{3}\right)} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore [rt - s^2]_{\left(\frac{\pi}{3},\frac{\pi}{3}\right)} = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \left(-\frac{\sqrt{3}}{2}\right)^2 > 0$

$\Rightarrow [rt - s^2]_{\left(\frac{\pi}{3},\frac{\pi}{3}\right)} > 0$ and $[r]_{\left(\frac{\pi}{3},\frac{\pi}{3}\right)} < 0$

$\therefore f(x,y)$ has a maximum value at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

Also $f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$\therefore$ Maximum value of the function is $\frac{3\sqrt{3}}{2}$ at the point $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

Also, $[r]_{(\pi,\pi)} = 0$

$[s]_{(\pi,\pi)} = 0$

$[t]_{(\pi,\pi)} = 0$

$\therefore [rt - s^2]_{(\pi,\pi)} = 0$

Checking the values of the function $f(x,y)$ in the immediate vicinity on both sides.

$[f]_{\left(\frac{5\pi}{6},\frac{5\pi}{6}\right)} = \sin\frac{5\pi}{6} + \sin\frac{5\pi}{6} + \sin\frac{10\pi}{6} = \frac{1}{2} + \frac{1}{2} - \frac{\sqrt{3}}{2} > 0$

Also, $[f]_{\left(\frac{7\pi}{6},\frac{7\pi}{6}\right)} = \sin\frac{7\pi}{6} + \sin\frac{7\pi}{6} + \sin\frac{14\pi}{6} = -\frac{1}{2} - \frac{1}{2} + \frac{\sqrt{3}}{2} < 0$

$\therefore (\pi,\pi)$ is a saddle point

6. Method of Lagrange Multipliers for Finding Extreme Values

If we want to find extreme values of a function whose variables are connected by a relation, we can use the method of Lagrange Multipliers.

Algorithm to find maximum or minimum values of a function $f(x,y,z)$ using Lagrange Multipliers

1. Consider the function: Maximize/Minimize $f(x,y,z)$ subject to $g(x,y,z) = k$

2. Form an auxiliary function $F(x,y,z,\lambda) = f(x,y,z) + \lambda[g(x,y,z) - k]$, where λ is the Lagrange multiplier.
3. Solve the equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and $g(x,y,z) = k$ by the method of grouping or method of multipliers or both.

***Method of multipliers**: Consider a fraction $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$
Taking x, y , z as multipliers, each fraction $= \frac{1\times x}{2\times x} = \frac{2\times y}{4\times y} = \frac{3\times z}{6\times z}$

Example20 Find the extreme values of $f(x,y) = 3x^2 + 4y^2$, given that $x + y = 35$

Solution: Let $F(x,y,\lambda) = 3x^2 + 4y^2 + \lambda(x + y - 35)$

$$\Rightarrow F(x,y,\lambda) = 3x^2 + 4y^2 + \lambda x + \lambda y - 35\lambda$$

Differentiating $F(x,y,\lambda)$ with respect to $x\&y$, and equating to zero

$$\frac{\partial F}{\partial x} = 6x + \lambda = 0 \Rightarrow x = -\frac{\lambda}{6} \ldots (1)$$

$$\frac{\partial F}{\partial y} = 8y + \lambda = 0 \Rightarrow y = -\frac{\lambda}{8} \ldots (2)$$

Given $x + y = 35 \ldots (3)$

Using (1) and (2) in (3), we get

$$-\frac{\lambda}{6} - \frac{\lambda}{8} = 35 \Rightarrow -\frac{7\lambda}{24} = 35 \therefore \lambda = -120$$

Substituting the value of λ in (1) and (2)

$$x = -\frac{\lambda}{6} = \frac{120}{6} = 20 \text{ , } y = -\frac{\lambda}{8} = \frac{120}{8} = 15$$

$\therefore$ The extreme value of the function is $f(20,15) = 3(20)^2 + 4(15)^2 = 2100$

Checking the value at nearby point $f(19,16) = 3(19)^2 + 4(16)^2 = 2107$

$\Rightarrow$ (20,15) is the minima of the function $f(x,y) = 3x^2 + 4y^2$, subject to $x + y = 35$

Example21 Find the extreme values of $f(x,y,z) = xyz$, subject to $x^2 + y^2 + z^2 = 3$

Solution: Let $F(x,y,z,\lambda) = xyz + \lambda(x^2 + y^2 + z^2 - 3)$

$$\Rightarrow F(x,y,z,\lambda) = xyz + \lambda x^2 + \lambda y^2 + \lambda z^2 - 3\lambda$$

Differentiating $F(x,y,z,\lambda)$ with respect to x, y, z and equating to zero

$$\frac{\partial F}{\partial x} = yz + 2\lambda x = 0 \Rightarrow \lambda = -\frac{yz}{2x} \ldots (1)$$

$$\frac{\partial F}{\partial y} = xz + 2\lambda y = 0 \Rightarrow \lambda = -\frac{xz}{2y} \ldots (2)$$

$\frac{\partial F}{\partial z} = xy + 2\lambda z = 0 \Rightarrow \lambda = -\frac{xy}{2z} \ldots (3)$

From (1), (2) and (3), we get: $\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$

Taking x, y , z as multipliers $\Rightarrow \frac{xyz}{2x^2} = \frac{xyz}{2y^2} = \frac{xyz}{2z^2} \Rightarrow x^2 = y^2 = z^2 \ldots (4)$

Given $x^2 + y^2 + z^2 = 3 \ldots (5)$

Using (4) in (5), we get

$3x^2 = 3 \Rightarrow x = \pm 1, y = \pm 1 \ , z = \pm 1$

$\therefore$ $(1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(1,-1,-1)$, $(-1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, $(-1,-1,-1)$ are the eight points of maxima or minima.

Clearly $f(x, y, z) = 1$ at $(1,1,1)$, $(1,-1,-1)$, $(-1,-1,1)$ and $(-1,1,-1)$

$\therefore$ $(1,1,1)$, $(1,-1,-1)$, $(-1,-1,1)$ and $(-1,1,-1)$ are the points of maxima

Also $f(x, y, z) = -1$ at $(1,-1,1)$, $(1,1,-1)$, $(-1,1,1)$ and $(-1,-1,-1)$

$\therefore$ $(1,-1,1)$, $(1,1,-1)$, $(-1,1,1)$ and $(-1,-1,-1)$ are the points of minima

Example22 Find the dimensions of the box with maximum volume if the total surface area is $24cm^2$.

Solution: Let $V(x, y, z) = xyz$, given that $2xy + 2yz + 2zx = 24$

Let $F(x, y, z, \lambda) = xyz + \lambda(xy + yz + zx - 12)$

$\Rightarrow F(x, y, z, \lambda) = xyz + \lambda xy + \lambda yz + \lambda zx - 12\lambda$

Differentiating $F(x, y, z, \lambda)$ with respect to x, y, z and equating to zero

$\frac{\partial F}{\partial x} = yz + \lambda y + \lambda z = 0 \Rightarrow \lambda = -\frac{yz}{y+z} \ldots (1)$

$\frac{\partial F}{\partial y} = zx + \lambda x + \lambda z = 0 \Rightarrow \lambda = -\frac{xz}{x+z} \ldots (2)$

$\frac{\partial F}{\partial z} = xy + \lambda y + \lambda x = 0 \Rightarrow \lambda = -\frac{xy}{x+y} \ldots (3)$

From (1), (2) and (3), we get

$\frac{yz}{y+z} = \frac{xz}{x+z} = \frac{xy}{x+y}$

Taking x, y , z as multipliers $\Rightarrow \frac{xyz}{x(y+z)} = \frac{xyz}{y(x+z)} = \frac{xyz}{z(x+y)}$

From first two equations, we get: $x(y + z) = y(x + z)$

$\Rightarrow xy + xz = xy + yz \Rightarrow x = y$ ... ④

Similarly from last two equations, we get $y = z$ ... ⑤

From ④ and ⑤, we get $x = y = z$ ... ⑥

Given $xy + yz + zx = 12$ ... ⑦

Using ⑥ in ⑦, we get

$3x^2 = 12 \Rightarrow x^2 = 4$

$\therefore x = 2, y = 2, z = 2 \because x = y = z$ and $x, y, z > 0$

Hence, maximum volume of the cube is $V(x, y, z) = xyz = 8$ cubic units

Example23 Find the shortest and longest distances from the point $(1,2,-1)$ to the sphere $x^2 + y^2 + z^2 = 24$.

Solution: Let f be the square of distance of $A(1,2,-1)$ from any point (x, y, z)on sphere,

Then, $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2$

Let $F(x, y, z, \lambda) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$

$\Rightarrow F(x, y, z, \lambda) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda x^2 + \lambda y^2 + \lambda z^2 + 24\lambda$

Differentiating $F(x, y, z, \lambda)$ with respect to x, y, z and equating to zero

$\frac{\partial F}{\partial x} = 2(x-1) + 2\lambda x = 0 \Rightarrow \lambda = -\frac{(x-1)}{x}$ ... ①

$\frac{\partial F}{\partial y} = 2(y-1) + 2\lambda y = 0 \Rightarrow \lambda = -\frac{(y-2)}{y}$ ... ②

$\frac{\partial F}{\partial z} = 2(z-1) + 2\lambda z = 0 \Rightarrow \lambda = -\frac{(z+1)}{z}$ ... ③

From ①, ② and ③, we get

$$-\frac{(x-1)}{x} = -\frac{(y-2)}{y} = -\frac{(z+1)}{z}$$

Taking x, y , z as multipliers $\Rightarrow \frac{(x^2-x)}{x^2} = \frac{(y^2-2y)}{y^2} = \frac{(z^2+z)}{z^2}$

From first two equations, we get $(x^2 - x)y^2 = x^2(y^2 - 2y)$

$\Rightarrow x^2y^2 - xy^2 = x^2y^2 - 2x^2y \Rightarrow y = 2x$ ... ④

Similarly from last two equations, we get $y = -2z$ ... ⑤

From ④ and ⑤, we get $x = -z$ ... ⑥

Given $x^2 + y^2 + z^2 = 24$. ...⑦

Using ⑤ and ⑥ in ⑦, we get

$(-z^2) + (-2z)^2 + z^2 = 24 \Rightarrow z^2 + 4z^2 + z^2 = 24$

$\Rightarrow 6z^2 = 24 \Rightarrow z = \pm 2$

At $z = 2$, we get $x = -2, y = -4$ (Using ⑤ and ⑥)

At $z = -2$, we get $x = 2, y = 4$ (Using ⑤ and ⑥)

Thus, we get two points $P(2,4,-2)$and $Q(-2,-4,2)$

The sphere which are at a shortest and longest distances from the point $A(1,2,-1)$

Now $AP = \sqrt{(1-2)^2 + (2-4)^2 + (-1+2)^2} = \sqrt{1+4+1} = \sqrt{6}$

$AQ = \sqrt{(1+2)^2 + (2+4)^2 + (-1-2)^2} = \sqrt{9+36+9} = 3\sqrt{6}$

$\therefore$ At $P(2,4,-2)$, shortest distance $=\sqrt{6}$

At $Q(-2,-4,2)$, longest distance $=3\sqrt{6}$

7. Differentiation Under Integral Sign

Differentiation under the integral sign is a rule which is used to compute some specific integrals with more than one variable. The variable other than variable of integration is taken as a parameter, and the integral is differentiated with respect to the parameter.

Leibnitz's Rule for Differentiation Under Integral Sign

If $I = \int_{a(t)}^{b(t)} f(x,t)dx$, then $\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}$

Deduction: If the limits a and b are constants (independent of t), $\frac{dI(t)}{dt} = \int_a^b \frac{\partial f(x,t)}{\partial t} dx$

Note: Here $\boldsymbol{t}$ is the parameter because integration is being performed with respect to $\boldsymbol{x}$.

Example24 If $I = \int_0^{a^2} \tan^{-1}\frac{x}{a} dx$, prove that $\frac{dI}{da} = -\frac{1}{2}\log(1+a^2) + 2a\tan^{-1} a$

Solution: Given $I = \int_0^{a^2} \tan^{-1}\frac{x}{a} dx$ ...①

Differentiating ① with respect to parameter a

$\frac{dI}{da} = \int_0^{a^2} \frac{\partial}{\partial a}\left(\tan^{-1}\frac{x}{a}\right) dx + f(a^2,a)\frac{da^2}{da} - f(0,a)\frac{d(0)}{da}$ by Leibnitz's Rule

$\Rightarrow \frac{dI}{da} = \int_0^{a^2} \frac{1}{1+\frac{x^2}{a^2}}\left(-\frac{x}{a^2}\right) dx + \tan^{-1}\frac{a^2}{a}(2a) - 0$

$= \int_0^{a^2} -\frac{x}{a^2+x^2} dx + 2a\tan^{-1} a$

$= -\frac{1}{2}\int_0^{a^2} \frac{2x}{a^2+x^2} dx + 2a\tan^{-1} a$

$= -\frac{1}{2}[\log(a^2+x^2)]_0^{a^2} + 2a\tan^{-1} a \because \int \frac{f'(t)}{f(t)} dx = \log f(t)$

$= -\frac{1}{2}[\log(a^2+a^4) - \log a^2] + 2a\tan^{-1} a$

$= -\frac{1}{2}\log\frac{a^2+a^4}{a^2} + 2a\tan^{-1} a \because \log m - \log n = \log\frac{m}{n}$

$\Rightarrow \frac{dI}{da} = -\frac{1}{2}\log(1+a^2) + 2a\tan^{-1} a$ **Hence proved**

Example25 If $I = \int_{x^2}^{x^3} \frac{1}{\log t} dt$, prove that $\frac{dI}{dx} = \frac{x(x-1)}{\log x}$

Solution: Given $I = \int_{x^2}^{x^3} \frac{1}{\log t} dt$ ...①

Differentiating ① with respect to parameter x

$\frac{dI}{dx} = \int_{x^2}^{x^3} \frac{\partial}{\partial x}\left(\frac{1}{\log t}\right) dt + f(x^3, x)\frac{dx^3}{dx} - f(x^2, x)\frac{dx^2}{dx}$ by Leibnitz's Rule

$\Rightarrow \frac{dI}{dx} = \int_{x^2}^{x^3} 0\, dt + \frac{1}{\log x^3}(3x^2) - \frac{1}{\log x^2}(2x)$

$= 0 + \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = \frac{x^2}{\log x} - \frac{x}{\log x} \because \log m^n = n\log m$

$\Rightarrow \frac{dI}{dx} = \frac{x(x-1)}{\log x}$ **Hence proved**

Example26 Prove that $\int_0^\infty \frac{e^{-ax}\sin\lambda x}{x} dx = \tan^{-1}\frac{\lambda}{a}$, hence deduce that $\int_0^\infty \frac{\sin\lambda x}{x} dx = \frac{\pi}{2}$

Solution: Let $I = \int_0^\infty \frac{e^{-ax}\sin\lambda x}{x} dx$ ...①

Differentiating ① with respect to parameter λ

$\frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda}\left(\frac{e^{-ax}\sin\lambda x}{x}\right) dx \because$ limits a and b are independent of parameter λ

$\Rightarrow \frac{dI}{d\lambda} = \int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial\lambda}(\sin\lambda x) dx = \int_0^\infty \frac{e^{-ax}}{x}(x\cos\lambda x) dx$

$= \int_0^\infty e^{-ax}\cos\lambda x\, dx = \left[\frac{e^{-ax}}{a^2+\lambda^2}(-a\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$

$\because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$\Rightarrow \frac{dI}{d\lambda} = \frac{a}{a^2+\lambda^2} \because e^{-\infty} = 0$

$\Rightarrow dI = \frac{a}{a^2+\lambda^2} d\lambda$

Integrating both sides with respect to λ

$\Rightarrow I = a.\frac{1}{a}\tan^{-1}\frac{\lambda}{a} + c = \tan^{-1}\frac{\lambda}{a} + c$

If $\lambda = 0 \Rightarrow I = 0$,$\therefore c = 0$ using ①

$\Rightarrow I = \tan^{-1}\frac{\lambda}{a}$

$\Rightarrow \int_0^\infty \frac{e^{-ax}\sin\lambda x}{x}dx = \tan^{-1}\frac{\lambda}{a}$ ...② **Hence Proved.**

Putting $a = 0$ in ② , we get

$\int_0^\infty \frac{\sin\lambda x}{x}dx = \frac{\pi}{2} \because \tan^{-1}\infty = \frac{\pi}{2}$

Example27 Prove that $\int_0^{\frac{\pi}{2}} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx = \pi(\sqrt{1+y} - 1)$

Solution: Let $I = \int_0^{\frac{\pi}{2}} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx$ ...①

Differentiating ① with respect to parameter y

$\frac{dI}{dy} = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial y}\left(\frac{\log(1+y\sin^2 x)}{\sin^2 x}\right)dx \because$ limits a and b are independent of parameter y

$\Rightarrow \frac{dI}{dy} = \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\frac{\partial}{\partial y}(\log(1 + y\sin^2 x))dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x}.\frac{\sin^2 x}{(1+y\sin^2 x)}dx = \int_0^{\frac{\pi}{2}} \frac{1}{(1+y\sin^2 x)}dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{(\sin^2 x+\cos^2 x+y\sin^2 x)}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x+\sin^2 x(1+y)}dx = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{1+tan^2 x(1+y)}dx$

$\Rightarrow \frac{dI}{dy} = \int_0^\infty \frac{1}{1+t^2(1+y)}dt$, by putting $tan\, x = t$, $\sec^2 x\, dx = dt$

$= \frac{1}{1+y}\int_0^\infty \frac{1}{\frac{1}{(1+y)}+t^2}dt = \frac{1}{1+y}\int_0^\infty \frac{1}{\left(\frac{1}{\sqrt{1+y}}\right)^2+t^2}dt = \frac{\sqrt{1+y}}{1+y}\left[\tan^{-1}\frac{t}{\frac{1}{\sqrt{1+y}}}\right]_0^\infty$

$\Rightarrow \frac{dI}{dy} = \frac{1}{\sqrt{1+y}}(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{2}\frac{1}{\sqrt{1+y}} \because \tan^{-1}\infty = \frac{\pi}{2}$

$\Rightarrow dI = \frac{\pi}{2}\frac{1}{\sqrt{1+y}}dy$

Integrating both sides with respect to y

$\Rightarrow I = \frac{\pi}{2}.2\sqrt{1+y} + c = \pi\sqrt{1+y} + c$ ...②

If $y = 0 \Rightarrow I = 0$, by using ①

$\therefore 0 = \pi + c \Rightarrow c = -\pi$, by using ②

$\Rightarrow I = \pi\sqrt{1+y} - \pi = \pi(\sqrt{1+y} - 1)$

Hence Proved.

Example28 Evaluate $\int_0^a \frac{log(1+ax)}{1+x^2} dx$ and hence show that $\int_0^1 \frac{log(1+x)}{1+x^2} dx = \frac{\pi}{8} log2$

Solution: Given $I = \int_0^a \frac{log(1+ax)}{1+x^2} dx$ ...①

Differentiating ① with respect to parameter a

$$\frac{dI}{da} = \int_0^a \frac{\partial}{\partial a}\left(\frac{log(1+ax)}{1+x^2}\right) dx + f(a,a)\frac{da}{dx} - f(a,a)\frac{d0}{da} \quad \text{by Leibnitz's Rule}$$

$$\Rightarrow \frac{dI}{da} = \int_0^a \frac{1}{1+x^2} \cdot \frac{x}{1+ax} dx + \frac{log(1+a^2)}{1+a^2}$$

$$\therefore \frac{dI}{da} = \int_0^a \frac{x}{(1+x^2)(1+ax)} . dx + \frac{log(1+a^2)}{1+a^2} \quad \cdots ②$$

Let $\frac{x}{(1+x^2)(1+ax)} = \frac{-1/a}{\left(1+\frac{1}{a^2}\right)(1+ax)} + \frac{Ax+B}{1+x^2} = \frac{-a}{(1+a^2)(1+ax)} + \frac{Ax+B}{1+x^2}$

$$\Rightarrow x = \frac{-a}{1+a^2}(1+x^2) + (Ax+B)(1+ax)$$

Equate constant and coefficient of x^2, we get

$$0 = \frac{-a}{1+a^2} + B \Rightarrow \mathrm{B} = \frac{\mathrm{a}}{1+\mathrm{a}^2}$$

$$0 = \frac{-a}{1+a^2} + aA \Rightarrow \mathrm{A} = \frac{1}{1+\mathrm{a}^2}$$

$$\therefore \frac{x}{(1+x^2)(1+ax)} = \frac{1}{1+a^2}\left(\frac{x+a}{1+x^2} - \frac{a}{1+ax}\right)$$

$$\Rightarrow \int_0^a \frac{x}{(1+x^2)(1+ax)} . dx = \frac{1}{1+a^2}\left[-log(1+ax) + \frac{1}{2} log(1+x^2) + a\, tan^{-1} x\right]_0^a$$

$$= \frac{1}{1+a^2}\left(-log(1+a^2) + \frac{1}{2} log(1+a^2) + a\, tan^{-1} a\right)$$

$$= \frac{1}{1+a^2}\left(\frac{-1}{2} log(1+a^2) + a \tan^{-1} a\right)$$

From ②, we get $\frac{dI}{da} = \frac{1}{1+a^2}\left(\frac{-1}{2} log(1+a^2) + a \tan^{-1} a\right) + \frac{log(1+a^2)}{1+a^2}$

$$= \frac{1}{1+a^2}\left(\frac{1}{2} log(1+a^2) + a \tan^{-1} a\right)$$

$$\Rightarrow dI = \frac{1}{1+a^2}\left(\frac{1}{2} log(1+a^2) + a \tan^{-1} a\right) da$$

Integrating both sides with respect to a

$$\Rightarrow I = \frac{1}{2}\int log(1+a^2) . \frac{1}{1+a^2} da + \int \frac{a \tan^{-1} a}{1+a^2} da + c$$

$$= \frac{1}{2}\left[log(1+a^2) . \tan^{-1} a - \int \frac{2a}{1+a^2} \tan^{-1} a\, da\right] + \int \frac{a \tan^{-1} a}{1+a^2} da + c$$

$$I = \frac{1}{2} log(1+a^2) . \tan^{-1} a + c$$

If $a = 0 \Rightarrow I = 0$, by using ①

$\therefore 0 = 0 + c \Rightarrow c = 0$, by using ②

$\Rightarrow \boldsymbol{I = \frac{1}{2} log(1 + a^2). \tan^{-1} a}$

Putting $a = 1$, we get $\int_0^1 \frac{log(1+x)}{1+x^2} dx = \frac{\pi}{8} log2$

8. Jacobians

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u , v with respect to x, y and is denoted by the symbol $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$. Similarly If u, v, w are functions of x, y, z, then the Jacobian of u, v, w with respect to x, y, z is $J\left(\frac{u,v,w}{x,y,z}\right)$ or $\frac{\partial(u,v,w)}{\partial(x,y,z)}$and is given by: $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Chain Rule forJacobians

If $u = f(r,s)$, $v = g(r,s)$ and $r = \phi(x,y)$, $s = \psi(x,y)$, then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} . \frac{\partial(r,s)}{\partial(x,y)}$

Example29 If (x, y, z) and (ρ, θ, ϕ) are respectively cartesian and spherical coordinates of a point, find the value of $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$.

Solution: Sperical coordinates are given by:

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$$\therefore \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

Taking out $\rho\sin\phi$ from second column and ρ from third column, we get

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{vmatrix}$$

$$= \rho^2\sin\phi \begin{bmatrix} \sin\phi\cos\theta\,(-\sin\phi\cos\theta) - \sin\phi\sin\theta\,(\sin\theta\sin\phi) \\ + \cos\phi\,(-sin^2\theta\cos\phi - cos^2\theta\cos\phi) \end{bmatrix}$$

$$= \rho^2\sin\phi\,[-sin^2\phi\, cos^2\theta - sin^2\phi\, sin^2\theta - sin^2\theta\, cos^2\phi - cos^2\theta\, cos^2\phi]$$

$$= \rho^2\sin\phi\,[-sin^2\phi\,(cos^2\theta + sin^2\theta) - cos^2\phi\,(sin^2\theta + cos^2\theta)\ ]$$

$$= \rho^2\sin\phi\,[-sin^2\phi - cos^2\phi] = -\rho^2\sin\phi$$

Jacobians to Determine Functional Dependence: Two functions $f(x, y)$ and $g(x, y)$ are called functionally dependent if they are functions of each other. In terms of Jacobians, if $f(x, y)$ and $g(x, y)$ are functionally dependent, then $\frac{\partial(f,g)}{\partial(x,y)} = 0$

Example30 If $u = \sin^{-1} x + \sin^{-1} y,\ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, find if u and v are functionally dependent, if so then find the relation between them.

Solution: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} + \frac{1}{2}\frac{y(-2x)}{\sqrt{1-x^2}} & \sqrt{1-x^2} + \frac{1}{2}\frac{x(-2y)}{\sqrt{1-y^2}} \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} & \sqrt{1-x^2} \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ \frac{-xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} \end{vmatrix} = 1 - 1 + 0 = 0$$

$\therefore$ u and v are functionally dependent

Also $\sin^{-1} v = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \sin^{-1} x + \sin^{-1} y = u$

$\therefore \sin^{-1} v = u$ or $v = \sin u$

Jacobians of Implicit Functions

Let u, v be implicit functions of variables x, y and connected by the relations $f_1(u,v,x,y) = 0$ and $f_2(u,v,x,y) = 0$, then $\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$

Partial Derivatives of Implicit Functions Using Jacobians

- $\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}},\ \frac{\partial u}{\partial y} = -\frac{\frac{\partial(f_1,f_2)}{\partial(y,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$
- $\frac{\partial v}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,x)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}},\ \frac{\partial v}{\partial y} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$

Again, if u, v, w be implicit functions of variables x, y, z and connected by the relations

$f_1(u,v,w,x,y,z) = 0,\ f_2(u,v,w,x,y,z) = 0$ and $f_3(u,v,w,x,y,z) = 0$

then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$

Note: If $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$, then $\frac{1}{J} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

Result: If u and v are functions of two independent variables x and y, such that $J = \frac{\partial(u,v)}{\partial(x,y)},\ J' = \frac{\partial(x,y)}{\partial(u,v)}$, then $JJ' = 1$, where $J' = \frac{1}{J}$

Proof: Let $u = u(x,y)$ ...①

$v = v(x,y)$ ...②

Differentiating ① with respect to u and v

$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial u}$ and $\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial v}$

Differentiating ② with respect to u and v

$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial u}$ and $\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial v}$

Now $JJ' = \frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(u,v)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{, Hence Proved}$$

Example31 For the transformation $x = e^u \cos v$, $y = e^u \sin v$, show that $\frac{\partial(u,v)}{\partial(x,y)} = e^{-2u}$

Solution: Let $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$

$$= e^u . e^u \begin{vmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{vmatrix} = e^{2u}(1)$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = e^{-2u}$$

Example32 If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$, show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2-y^2}{u^2+y^2}$

Solution: We have $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$

$\Rightarrow f_1(u, v, x, y) = 0$ and $f_2(u, v, x, y) = 0$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} \Bigg/ \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = \frac{2x^2-2y^2}{2u^2+2v^2} = \frac{x^2-y^2}{u^2+v^2}$$

Example32 If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: We have $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

$\Rightarrow 3\lambda^3 - 3(x + y + z)\lambda^2 + 3(x^2 + y^2 + z^2)\lambda - (x^3 + y^3 + z^3) = 0$

$\Rightarrow f_1(u, v, x, y) = 0$ and $f_2(u, v, x, y) = 0$

Since u, v, w are its roots, therefore,

$$u + v + w = x + y + z$$
$$uv + vw + wu = x^2 + y^2 + z^2$$
$$uvw = \tfrac{1}{3}(x^3 + y^3 + z^3)$$

$\Rightarrow f_1(u, v, w, x, y, z) = 0$, $f_2(u, v, w, x, y, z) = 0$ and $f_3(u, v, w, x, y, z) = 0$

$\Rightarrow f_1(u, v, w, x, y, z) = u + v + w - x - y - z = 0$

$f_2(u, v, w, x, y, z) = uv + vw + wu - x^2 - y^2 - z^2 = 0$

$f_3(u, v, w, x, y, z) = uvw - \tfrac{1}{3}(x^3 + y^3 + z^3) = 0$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

Now $\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} = (-1)(-2)(-3)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \end{vmatrix} \quad \text{Using, } c_2 \to c_2 - c_1,\ c_3 \to c_3 - c_1$$

$$= -2(y-x)(z-x)(z-y) = -2(x-y)(y-z)(z-x)$$

and $\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & (u-v)w & (u-w)v \end{vmatrix} \quad \text{Using } c_2 \to c_2 - c_1,\ c_3 \to c_3 - c_1$$

$$= (u-v)(v-w)(u-w)$$

Hence, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(u-w)} = \frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

Example33 If $u = x + y^2$, $v = y + z^2$ and $w = z + x^2$, show that $\frac{\partial x}{\partial u} = \frac{1}{1+8xyz}$

Solution: Here $f_1 = u - x - y^2 = 0$, $f_2 = v - y - z^2 = 0$, $f_3 = w - z - x^2 = 0$

$$\therefore \frac{\partial x}{\partial u} = -\frac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)} \Big/ \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

$$= -\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & -2y & 0 \\ 0 & -1 & -2z \\ 0 & 0 & -1 \end{vmatrix} \Bigg/ \begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix}$$

$$\Rightarrow \frac{\partial x}{\partial u} = -\frac{1(1)}{-1(1)-2x(4yz)} = \frac{1}{1+8xyz}$$

9. Transformations of Coordinates

We have been using Cartesian coordinate system in most of the applications, where the position of each data point is defined by (x, y)coordinates in two dimensions or by (x, y, z)coordinates in three dimensions. However, for some applications involving curved lines or surfaces, system derived from circular shapes, such as polar, cylindrical or spherical coordinate systems are better used to specify the positions.

9.1 Polar Coordinates: A polar coordinate system is a two dimensional system where every point in the plane is determined by its distance r from the origin and an angle θ. Here θ is the angle by which we rotate about the positive direction of x-axis to get to the point. We can establish a relation between Cartesian and polar coordinates using Figure1.

Polar Coordinates

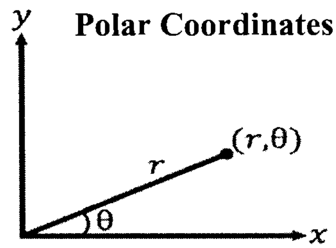


Figure 1

Clearly $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

We can convert from Cartesian coordinates to polar and vice-versa using above relations.

Jacobian of Transformations of Cartesian to Polar Coordinates

We know that $x = r\cos\theta$, $y = r\sin\theta$ then

$$\text{Jacobian of Transformation} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\,sin^2\,\theta = r$$

9.2 Cylindrical Coordinates: In cylindrical coordinate system, xy plane is replaced by the polar plane and the vertical z -axis remains the same (Figure 2).

Here, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$\Rightarrow r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$, $z = z$

We can convert from Cartesian coordinates to cylindrical and vice- versa using above relations.

Cylindrical Coordinates

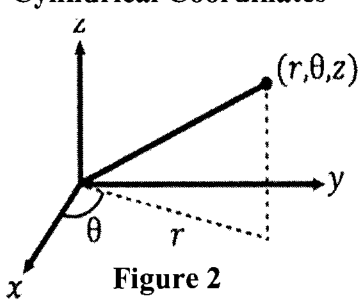


Figure 2

Jacobian of Transformations of Cartesian to Cylindrical Coordinates

We know that $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ then

$$\text{Jacobian of Transformation} = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\,sin^2\,\theta = r$$

9.3 Spherical Polar Coordinates: In spherical coordinate system, we define the position of a point by three coordinates ρ, θ and ϕ (Figure 3). Here ρ ($\rho > 0$) is the distance of the point from the origin (similar to r in polar coordinates), θ is same as that of polar or cylindrical coordinates, and ϕ ($0 \le \phi \le \pi$) is the angle that we need to rotate down from the positive z-axis to get to the point.

Spherical Coordinates

Here $r = \rho\sin\phi$, $z = \rho\cos\phi$, $\rho = \sqrt{r^2 + z^2}$ ...①
Also, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ ...② from cylindrical coordinates

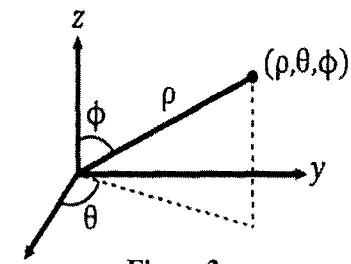


Figure3

Using ① in ②, we get $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$\Rightarrow \rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}\frac{y}{x}$, $\phi = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$

We can convert from Cartesian coordinates to spherical and vice-versa using above relations.

Note: Some authors define spherical polar coordinates in terms of r, θ and ϕ

Then $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Jacobian of Transformations of Cartesian to Spherical Coordinates

We know that $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ then

$$\text{Jacobian of Transformation} = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi)$$
$$+ r\sin\theta(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi)$$
$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

Example34 Convert the point$\left(\sqrt{3}, \frac{\pi}{4}, 1\right)$ from cylindrical to spherical coordinates.

Solution: We have $\rho^2 = r^2 + z^2 \Rightarrow \rho^2 = 3 + 1 = 4$, $\therefore \rho = 2$

Also, $\theta = \frac{\pi}{4}$ $\because$ θ is same in polar, cylindrical and spherical coordinates

Again, $z = \rho\cos\phi \Rightarrow 1 = 2\cos\phi \Rightarrow \cos\phi = \frac{1}{2} \Rightarrow \phi = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

$\therefore$The point$\left(\sqrt{3}, \frac{\pi}{4}, 1\right)$ in cylindrical coordinates is same as the point the point$\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ in spherical coordinates.

Example35 If $z = f(x, y)$, by changing the independent variables u and v to x and y by means of relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$,

Show that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ is transformed to $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$.

Solution: Here $z = f(x, y)$, let $x = g(u, v)$ and $y = h(u, v)$

Now $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}$

$\Rightarrow \frac{\partial}{\partial u}(z) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)(z)$

$\Rightarrow \frac{\partial}{\partial u} = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)$

Again $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}$

$$\Rightarrow \frac{\partial}{\partial v}(z) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)(z) \ \Rightarrow \frac{\partial}{\partial v} = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)$$

$$\therefore \frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right)$$

$$= cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + sin^2\alpha\frac{\partial^2 z}{\partial y^2} \dots (1)$$

Also, $\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)$

$$= sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + cos^2\alpha\frac{\partial^2 z}{\partial y^2} \dots (2)$$

Adding (1) and (2), we get $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

Example36 Transform $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar coordinates.

Solution: Let $x = r\cos\theta$, $y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)}\left(-\frac{y}{x^2}\right)\frac{\partial u}{\partial \theta}$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial u}{\partial \theta}$$

$$= \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta} \because r = \sqrt{x^2+y^2}$$

$$= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \because \ x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\Rightarrow \frac{\partial}{\partial x}(u) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(u)$$

$$\Rightarrow \frac{\partial}{\partial x} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

Again, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{1}{2}\cdot\frac{2y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)}\left(\frac{1}{x}\right)\frac{\partial u}{\partial \theta}$

$$= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial u}{\partial \theta} = \frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta} \qquad \because r = \sqrt{x^2+y^2}$$

$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \because \ x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\Rightarrow \frac{\partial}{\partial y}(u) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)(u)$$

$$\Rightarrow \frac{\partial}{\partial y} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= cos^2\theta\frac{\partial^2 u}{\partial r^2} - \cos\theta\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r}\right) + \frac{sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= cos^2\theta\frac{\partial^2 u}{\partial r^2} - \cos\theta\left(\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} - \frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 u}{\partial r\partial \theta} - \sin\theta\frac{\partial u}{\partial r}\right) + \frac{sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \dots (1)$$

Also, $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$

$$= sin^2\theta\frac{\partial^2 u}{\partial r^2} + \sin\theta\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial r}\right) + \frac{cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= sin^2\theta\frac{\partial^2 u}{\partial r^2} + \sin\theta\left(\frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} - \frac{\cos\theta}{r^2}\frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 u}{\partial r\partial \theta} + \cos\theta\frac{\partial u}{\partial r}\right) + \frac{cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \ldots (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$ is the required transformation.

Test Your Knowledge

1. **(a)** Find $\frac{\partial u}{\partial x}$, if $u = \tan^{-1}\frac{x^2+y^2}{x+y}$

   (b) Find $\frac{\partial u}{\partial y}$, if $u = \cos^{-1}\frac{x}{y}$

2. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$
3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$
4. If $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, and $v = r^m$, show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m(m+1)r^{m-2}$
5. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
6. If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^3$, find the value of $\frac{dz}{dx}$ when $x = y = a$
7. If $f(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$
8. Find the total differential of the function $f(x, y) = ye^{(x+y)}$
9. If $u = f(y - z, z - x, x - y)$,show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$
10. Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$
11. In a plane triangle, find the maximum value of $\cos A \cos B \cos C$
12. Show that the function $f(x, y) = x^3e\text{^}(-x^2 - y^2)$ has a maximum at the point $\left(\sqrt{\frac{3}{2}}, 0\right)$, a minimum at $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and a stationary point at the origin.
13. Find the maximum and minimum distances of the points (3,4,12) from the sphere $x^2 + y^2 + z^2 = 4$
14. Find the minimum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$
15. Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.
16. Evaluate the integral $\int_0^1 \frac{x^\alpha - 1}{\log x}dx$ by applying differentation under the integral sign $(\alpha \geq 0)$.
17. Find the derivative with respect to x of the integral $F(x) = \int_x^{x^2} \frac{\sin xt}{t}dt$
18. . If $I = \int_1^2 \frac{\cos tx}{x}dx$, prove that $\frac{dI}{dt} = \frac{\cos 2t - \cos t}{t}$

19. Find the value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$ where $u = a(x+y), v = b(x-y)$ and $x = r^2 \cos 2\theta \, , y = r^2 \sin 2\theta$

20. If $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$, find $\frac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)}$

21. Show that $u = x + 2y + z, v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$ are functionally dependent. Find the relation between them.

22. If $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}$, evaluate $\frac{\partial(r,\theta)}{\partial(x,y)}$

23. Using Jacobians, find $\frac{\partial u}{\partial x}$ if $u^2 + xy^2 - xy = 0$ and $u^2 + uvx + v^2 = 0$

24. If $x + y = 2e^{\theta} \cos \phi$ and $x - y = 2e^{\theta} \sin \phi$, show that $\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}$

25. If $z = f(x,y), x^2 = uv, y^2 = \frac{u}{v}$, then change the independent variables to u and v in the equation $x^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + 2y \frac{\partial z}{\partial y} = 0$

Answers

1. (a) $\frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2}$ (b) $\frac{x}{y\sqrt{y^2-x^2}}$

5. $\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}, \frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}$

6. 0

8. $df = \left(ye^{(x+y)}\right)dx + \left(e^{(x+y)}(y+1)\right)dy$

10. (0,0) saddle point and $\left(\frac{1}{2}, \frac{1}{3}\right)$ maxima and maximum value $= \frac{1}{432}$

11. f is maximum at $A = B = C = \frac{\pi}{3}$, maximum value= $\frac{1}{8}$

13. 14, 12

14. $p^2/a^2 + b^2 + c^2$

16. $log(1+\alpha)$

17. $\frac{1}{x}(3 \sin x^3 - 2 \sin x^2)$

19. $-8abr^3$

20. $1/4$

21. $u^2 - v^2 = 4w$

22. $\frac{1}{\sqrt{x^2+y^2}}$

23. $\frac{u}{2x}$

UNIT2: ORDINARY DIFFERENTIAL EQUATIONS

1. Introduction: Relationship between variables and their rate of changes gives rise to differential equations. Mathematical formulation of most of the physical and engineering problems leads to differential equations. It is very important for scientists and engineers to know the inception and solving of differential equations. These are of two types:

1) Ordinary Differential Equations (ODE)
2) Partial Differential Equations (PDE)

An Ordinary Differential Equation (ODE) involves the derivatives of a dependent variable with respect to a single independent variable whereas a partial differential equation (PDE) contains the derivatives of a dependent variable with respect to two or more independent variables. In this chapter we will confine our studies to ordinary differential equations.

Important Results

- Integration by parts when first function vanishes after a finite number of differentiations: If u and v are both differentiable functions of x, such that u vanishes finitely, then

  $\int \boldsymbol{u.v\,dx = uv_1 - u^{(1)}v_2 + u^{(2)}v_3 - u^{(3)}v_4 + \cdots}$

  Here $\boldsymbol{u^{(n)}}$is derivative of $\boldsymbol{u^{(n-1)}}$ and $\boldsymbol{v_n}$isintegral of $\boldsymbol{v_{n-1}}$

  For example:

  $$\int \boldsymbol{x^2.\sin nx\;dx} = (x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{sinnx}{n^2}\right) + (2)\left(\frac{cosnx}{n^3}\right)$$

  $$= -\frac{x^2\cos nx}{n} + \frac{2x\,sinnx}{n^2} + \frac{2cosnx}{n^3}$$

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

2. Order and Degree of Ordinary Differential Equations (ODE)

A general ordinary differential equation of n^{th}order can be represented in the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$.Order of an ordinary differential equation is that of the highest derivative occurring in it and the degree is the power of highest derivative after it has been freed from all radical signs.

The differential equation$\left(\frac{d^2y}{dx^2} + 2y\right)^3 + \frac{d^3y}{dx^3} + y = 0$ is having order 3 and degree 1, whereas $\left(\frac{d^3y}{dx^3} + 2y\right)^3 + \frac{d^2y}{dx^2} + y = 0$ is of order 3 and degree 3.

The order and degree of the differential equation$\sqrt{\frac{d^2y}{dx^2}} = \left(\frac{dy}{dx}\right)^3 + y$ are 2 and 1 respectively.

3. Geometric Meaning of First and Second Order Differential Equations

The order of a differential equation depends upon the number of arbitrary constants present in the original equation. For instance, the equation $y = mx$ has only one arbitrary constant, therefore the corresponding differential equation will be of first order; while the equation

$y = mx + c$ has two arbitrary constants, hence it will lead to a second order differential equation. Now since for any first order differential equation, m can take infinite values, hence the locus of the equation is made up of single infinity of curves. Also, for a second order differential equation, m can take infinite values and c can take infinite values at the same time, therefore the general solution can be said to have double infinity of curves. Hence, we can conclude that any n^{th} order differential equation has n^{th} infinity of curves as its general solution.

4. Approximate Solutions of Differential Equations: In some cases, where analytical methods are tedious to apply, we can find approximate solutions of first order differential equations using graphical or numerical methods.

4.1 Approximating the Curve Using Directional Fields or Slope Fields

With the help of direction fields (slope fields), we can approximate the general solution of a first-order differential equation of the type $\frac{dy}{dx} = f(x,y)$ by drawing isoclines (lines having same slopes). Here we use the fact that $\frac{dy}{dx}$ at any given point (x,y) on the curve gives the slope of the tangent (or gradient) to the curve at (x,y).

Algorithm to plot the curve using slope fields:

Step1: Arrange the given first order differential equation in the form $\frac{dy}{dx} = f(x,y)$, where $\frac{dy}{dx} = m$ is the slope if the tangent to the curve at any point (x,y) on the curve.

Step2: Draw the isoclines corresponding to different values of m like -1, -2, 0, 1, 2 etc. Here the isoclines corresponding to $m = 0$, known as null clines (the tangents parallel to x-axis), provide the positioning of the curve about the x-axis.

Step3: Plot the family of the curves by estimating the direction with the help of isoclines plotted on the direction field.

Example1 Find the family of curves for the equation $\frac{dy}{dx} = x$ using slope fields.

Solution: Let $\frac{dy}{dx} = x = m = \tan\theta$

The isoclines corresponding to different values of m are as computed as given below:

$m = \tan\theta$	θ	x
-2	116.57^0	-2
-1	135^0	-1
0	0^0	0
1	45^0	1
2	63.43^0	2

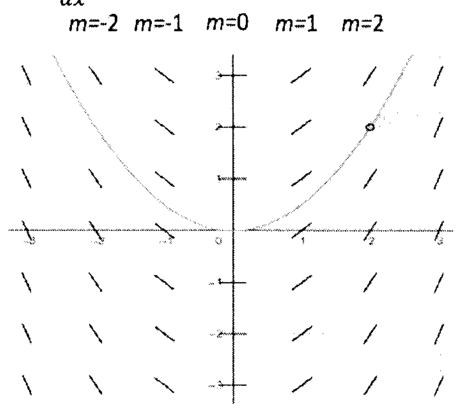


Figure 1

Here θ is the angle made by the tangent to the curve at the point $(x,y)$with positive direction of x-axis. The table shows that the null clines are placed about the line $x = 0$, i.e. y-axis.

Also slopes of all the tangents to the family of curves about the line $x = 1$ are one, i.e. tangents to the curves make an angle $\frac{\pi}{4}$, when the curve passes through the line $x = 1$, and similar interpretations for remaining values of m in the table.

It is evident that infinite number of curves can be drawn in the given direction field. Figure 1 and Figure 2 show the two curves in the family of curves.

Note: Analytic solution of the given differential equation using variable separable method may be computed as

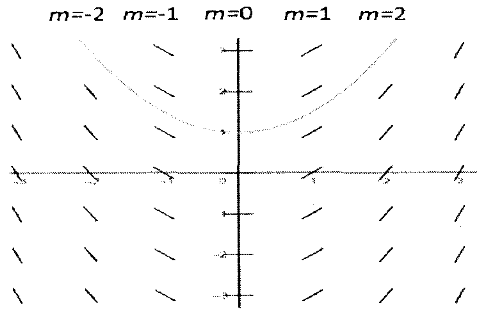


Figure 2

$y = \frac{x^2}{2} + c$. Clearly Figure1 shows a particular solution for $c = 0$ and Figure2 shows the solution corresponding to $c = 1$.

Example2 Find the family of curves for the equation $\frac{dy}{dx} = y$ using slope fields.

Solution: Let $\frac{dy}{dx} = y = m = \tan\theta$

The isoclines corresponding to different values of m are as computed as given below:

$m = \tan\theta$	θ	y
-2	116.57^0	-2
-1	135^0	-1
0	0^0	0
1	45^0	1
2	63.43^0	2

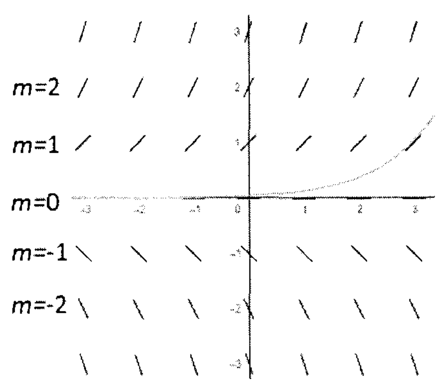


Figure 3

Here θ is the angle made by the tangent to the curve at the point $(x, y)$with positive direction of x-axis. The table shows that the null clines are placed about the line $y = 0$, i.e. x-axis. Also slopes of all the tangents to the family of curves about the line $y = 1$ are one, i.e. tangents to the curves make an angle $\frac{\pi}{4}$, when the curve passes through the line $y = 1$, and similar interpretations for remaining values of m in the table. It is evident that infinite number of curves can be drawn in the given direction field. Figures 3 and 4 show the two curves in the family of curves.

Note: Analytic solution of the given differential equation using variable separable method may be computed as $y = ce^x$. Clearly Figure 3 depicts a particular solution for a positive value of arbitrary constant c and Figure 4 shows a solution corresponding to a negative value of c.

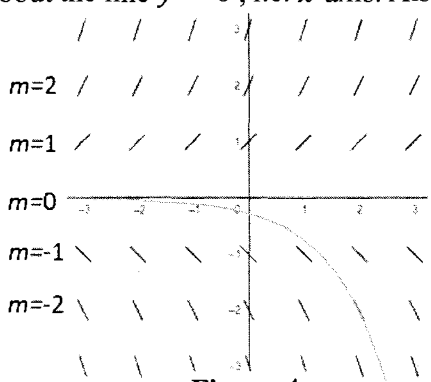


Figure 4

4.2 Numerical Methods to Find an Approximate solution

The first-order differential equation and a given initial value constitute a first-order initial value problem given as: $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$. An approximate solution can be foundusing numerical methods; Euler's method is one of them.

4.2.1 Euler's Method

Euler's Method provides us with a numerical solution of the initial value problem $\frac{dy}{dx} = f(x, y);\ \ y(x_0) = y_0 \cdots (1)$, by joining multiple small line segments A_0A_1 , A_1A_2, $A_2A_3, \cdots$, and making an approximation of the actual curve, as shown in the adjoining figure. Thus if$[x_0, x_1]$ is the small interval, where $x_1 = x_0 + h$, we approximate the curve by the tangent drawn to curve at the point A_0, having coordinates(x_0, y_0), whose equation is given by $y - y_0 = m(x - x_0)$, where m is slope of tangent at the point(x_0, y_0)

Also $m = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} = f(x_0, y_0)$ from (1)

$$\Rightarrow \quad y = y_0 + f(x_0, y_0)\,(x - x_0)$$

$$\Rightarrow y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0) \because y(x_1) = y_1$$

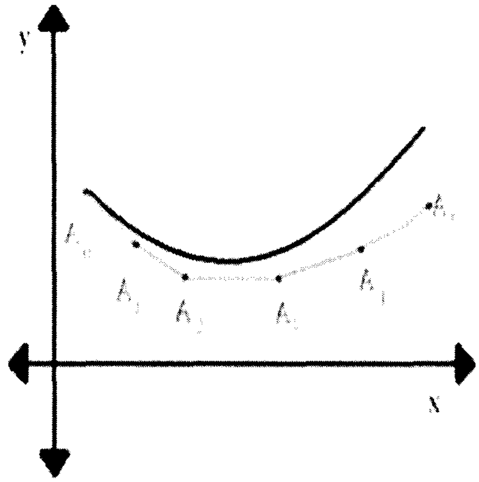
$$\Rightarrow y_1 = y_0 + hf(x_0, y_0) \ \because\ x_1 - x_0 = h$$

Similarly for range$[x_1, x_2]$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$\vdots$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$



It is evident from the given figure that h has to be kept small to avoid the approximations diverging away from the curve. As a result, this method is very slow and needs to be improved.

Example3 Using Euler's method, Compute $y(0.12)$for the initial value problem:

$$\frac{dy}{dx} = x^3 + y\ ;\ y(0) = 1, \text{ taking } h = 0.02 .$$

Solution: Given $f(x, y) = x^3 + y$,$x_0 = 0$, $y_0 = 1$, $x_n = x_{n-1} + h$, $h = 0.02$

$\therefore x_1 = 0.02$, $x_2 = 0.04$, $x_3 = 0.06$, $x_4 = 0.08$, $x_5 = 0.1$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$$\Rightarrow y_n = y_{n-1} + h({x_{n-1}}^3 + y_{n-1}) \cdots (1)$$

Putting $n = 1$ in (1), $y_1 = y(0.02) = y_0 + h\,({x_0}^3 + y_0)$

$\therefore\ y_1 = 1 + 0.02(\,0 + 1) = 1.02$ n

Putting $n = 2$ in (1), $y_2 = y(0.04) = y_1 + h\,({x_1}^3 + y_1)$

$\therefore\ y_2 = 1.02 + 0.02((0.02)^3 + 1.02) = 1.04040016$

Putting $n = 3$ in (1), $y_3 = y(0.06) = y_2 + h\,({x_2}^3 + y_2)$

$\therefore\ y_3 = 1.04040016 + 0.02((0.04)^3 + 1.04040016) = 1.061209443$

Putting $n = 4$ in (1), $y_4 = y(0.08) = y_3 + h\,({x_3}^3 + y_3)$

$\therefore\ y_4 = 1.061209443 + 0.02((0.06)^3 + 1.061209443) = 1.082437952$

Putting $n = 5$ in ①, $y_5 = y(0.1) = y_4 + h\,({x_4}^3 + y_4)$

$\therefore\ y_5 = 1.082437952 + 0.02((0.08)^3 + 1.082437952) = 1.104096951$

Putting $n = 6$ in ①, $y_6 = y(0.12) = y_5 + h\,({x_5}^3 + y_5)$

$\therefore\ y_6 = 1.104096951 + 0.02((0.1)^3 + 1.104096951) = 1.126198890$

Thus at $x = 0.12,\ y = 1.126198890 \Rightarrow y(0.12) = 1.126198890$

Example4 Using Euler's method, solve $\frac{dy}{dx} = \frac{x-y}{2}$; $y(0) = 1$, over the interval $[0,2]$, taking the step size 0.5.

Solution: Given $f(x,y) = \frac{x-y}{2}$, $x_0 = 0,\ y_0 = 1,\ x_n = x_{n-1} + h\,,\ h = 0.5$

$\therefore x_1 = \frac{1}{2} = 0.5\,,\ x_2 = 1\,,\ x_3 = \frac{3}{2} = 1.5\,,\ x_4 = 2$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$$\Rightarrow\ y_n = y_{n-1} + \frac{h}{2}(x_{n-1} - y_{n-1})$$

or $y_n = y_{n-1} + 0.25(x_{n-1} - y_{n-1}) \cdots$ ①

Putting $n = 1$ in①, $y_1 = y\left(\frac{1}{2}\right) = y_0 + 0.25(x_0 - y_0)$

$\therefore\ y_1 = 1 + 0.25(0 - 1) = 0.75$

Putting $n = 2$ in①, $y_2 = y(1) = y_1 + 0.25(x_1 - y_1)$

$\therefore\ y_2 = 0.75 + 0.25(0.5 - 0.75) = 0.6875$

Putting $n = 3$ in①, $y_3 = y\left(\frac{3}{2}\right) = y_2 + 0.25(x_2 - y_2)$

$\therefore\ y_3 = 0.6875 + 0.25(1 - 0.6875) = 0.765625$

Putting $n = 4$ in①, $y_4 = y(2) = y_3 + 0.25(x_3 - y_3)$

$\therefore\ y_4 = 0.765625 + 0.25(1.5 - 0.765625) = 0.94921875$

5. Separable Ordinary Differential Equations

Any Separable differential equation can be arranged in the form $N(y)\frac{dy}{dx} = M(x)$, and can be solved by integrating both sides with respect to x as shown:

$\int N(y)\frac{dy}{dx}dx = \int M(x)dx \Rightarrow \int N(y)dy = \int M(x)dx$

Example5 Solve the differential equation $\frac{dy}{dx} = x$

Solution: Arranging the given differential equation in variable separable form

$\Rightarrow dy = xdx$

Integrating both sides, we have $\int dy = \int xdx$

$\Rightarrow y = \frac{x^2}{2} + c$ is the required solution of the given differential equation.

Example6 Solve the differential equation $\frac{dy}{dx} = y$

Solution: Arranging the given differential equation in variable separable form

$\Rightarrow \frac{dy}{y} = dx$

Integrating both sides, we have: $\int \frac{dy}{y} = \int dx$

$\Rightarrow \log y = x + \log c$

$\Rightarrow \log \frac{y}{c} = x \Rightarrow e^{\log \frac{y}{c}} = e^x \Rightarrow \frac{y}{c} = e^x$

$\Rightarrow y = ce^x$ is the required solution of the given differential equation.

Example7 Solve the differential equation $\frac{dy}{dx} = e^{x+y} \sec y$

Solution: Arranging the given differential equation in variable separable form

$\Rightarrow e^{-y} \cos y \, dy = e^x dx$

Integrating both sides, we have

$\int e^{-y} \cos y \, dy = \int e^x dx$

$\Rightarrow e^{-y} \sin y + \int e^{-y} \sin y \, dy \ = e^x + c$

$\Rightarrow e^{-y} \sin y + [- e^{-y} \cos y - \int e^{-y} \cos y \, dy] = e^x + c$

$\Rightarrow e^{-y} \sin y - e^{-y} \cos y - I = e^x + c$, if $\int e^{-y} \cos y \, dy = I$ say

$\Rightarrow \frac{e^{-y}}{2} (\sin y - \cos y) = e^x + c$ is the required solution

Example8 Solve the differential equation $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$

Solution: Arranging the given differential equation in variable separable form

$\Rightarrow (1 + x)\frac{dy}{dx} = y - y^2$

$\Rightarrow \frac{dy}{y(1-y)} = \frac{dx}{(1+x)} \Rightarrow \frac{dy}{y} + \frac{dy}{1-y} = \frac{dx}{(1+x)}$

Integrating both sides, we have

$\log|y| - \log|1 - y| = \log|1 + x| + \log c$

$\Rightarrow \log \frac{y}{1-y} = \log c(1 + x)$

$\Rightarrow \frac{y}{(1-y)(1+x)} = c$

$\Rightarrow (1 + x)(1 - y) = Ay$, where $A = \frac{1}{c}$ is an arbitrary constant

6. Differential Equations Reducible to Separable Form

In some cases, a differential equation can be reduced to separable form by the substitution $f(x, y) = t$ and can be easily solved thereafter.

Example9 Solve the differential equation $(x + y)^2 \frac{dy}{dx} = 1$

Solution: Putting $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ Given differential equation can be rewritten as $t^2 \left(\frac{dt}{dx} - 1\right) = 1$

$\Rightarrow \frac{dt}{dx} = \frac{1}{t^2} + 1 = \frac{1+t^2}{t^2}$

$\Rightarrow \frac{t^2}{1+t^2} dt = dx$

Integrating both sides, we have : $\int \frac{t^2}{1+t^2} dt = \int dx$

$\Rightarrow \int \frac{1+t^2-1}{1+t^2} dt = \int dx$

$\Rightarrow \int \left(1 - \frac{1}{1+t^2}\right) dt = \int dx$

$\Rightarrow\ t - \tan^{-1} t = x + c$

$\Rightarrow x + y - \tan^{-1}(x + y) = x + c$

$\Rightarrow y - \tan^{-1}(x + y) = c$, where c is an arbitrary constant

Example10 Solve the differential equation $\cos(x + y)\frac{dy}{dx} = 1$

Solution: Putting $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$Given differential equation can be rewritten as $\cos t \left(\frac{dt}{dx} - 1\right) = 1$

$\Rightarrow \frac{dt}{dx} = \frac{1}{\cos t} + 1 = \frac{1+\cos t}{\cos t}$

$\Rightarrow \frac{\cos t}{1+\cos t} dt = dx$

Integrating both sides, we have

$\int \frac{\cos t}{1+\cos t} dt = \int dx$

$\Rightarrow \int \frac{1+\cos t-1}{1+\cos t} dt = \int dx$

$\Rightarrow \int \left(1 - \frac{1}{1+\cos t}\right) dt = \int dx$

$\Rightarrow \int \left(1 - \frac{1}{2} sec^2 \frac{t}{2}\right) dt = \int dx$

$\Rightarrow\ t - \tan\frac{t}{2} = x + c$

$\Rightarrow x + y - \tan\frac{(x+y)}{2} = x + c$

$\Rightarrow y - \tan\frac{(x+y)}{2} = c$, where c is an arbitrary constant

7. Exact Differential Equations of First Order

A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be exact if it can be directly obtained from its primitive by differentiation.

Theorem: The necessary and sufficient condition for the equation $M(x, y)dx + N(x, y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working Rule to Solve an Exact Differential Equation:

1. For the equation $M(x, y)dx + N(x, y)dy = 0$, check the condition for exactness i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Solution of the given differential equation is given by $I_1 + I_2 = c$

Where $I_1 = \int M\, dx$, taking y as constant

$I_2 = \int N_y\, dy$, Here N_y denotes terms in $N(x, y)$ which do not contain x

or $\int\limits_{y\ constant} M dx + \int N_y dy = C$

Example11 Solve the differential equation:

$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$ .....①

Solution: $M = (e^y + 1)\cos x$, $N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x$, $\frac{\partial N}{\partial x} = e^y \cos x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of the differential equation ① is given by:

$\int\limits_{y\ constant} (e^y + 1)\cos x\ dx + \int 0\, dy = C$

$\Rightarrow (e^y + 1)\sin x = C$

Example12 Solve the differential equation:

$(\sec x \tan x\ \tan y - e^x)dx + (\sec x\, sec^2 y)dy = 0$ .....①

Solution: $M = \sec x \tan x\ \tan y - e^x$, $N = \sec x\, sec^2 y$

$\frac{\partial M}{\partial y} = \sec x \tan x\, sec^2 y$, $\frac{\partial N}{\partial x} = \sec x \tan x\, sec^2 y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$\int\limits_{y\ constant} (\sec x \tan x\ \tan y - e^x)\ dx + \int 0\, dy = C$

$\Rightarrow \sec x \tan y - e^x = C$

Example13 Solve the differential equation:

$\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]dy = 0$ .....①

Solution: $M = y\left(1 + \frac{1}{x}\right) + \cos y$, $N = x + \log x - x\sin y$

$\frac{\partial M}{\partial y} = \left(1 + \frac{1}{x}\right) - \sin y$, $\frac{\partial N}{\partial x} = \left(1 + \frac{1}{x}\right) - \sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$\int \left[\underset{y\ constant}{y\left(1 + \frac{1}{x}\right) + \cos y)}\right] dx + \int 0\, dy = C$

$\Rightarrow y\,(x + \log x) + x\cos y = C$

Example14 Solve the differential equation:

$x\, dx + y\, dy = \frac{a^2(x\, dy - y\, dx)}{x^2+y^2}$ .....①

Solution: ①$\Rightarrow \left(x + \frac{a^2 y}{x^2+y^2}\right) dx + \left(y - \frac{a^2 x}{x^2+y^2}\right) dy = 0$

$M = x + \frac{a^2 y}{x^2+y^2}$, $N = y - \frac{a^2 x}{x^2+y^2}$

$\frac{\partial M}{\partial y} = \frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2}$, $\frac{\partial N}{\partial x} = \frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is exact.

Solution of ① is given by:

$$\underset{y\ constant}{\int}\left(x + \frac{a^2\,y}{x^2+y^2}\right) dx \; + \int y\, dy = C$$

$$\Rightarrow \frac{x^2}{2} + a^2 tan^{-1}\frac{x}{y} + \frac{y^2}{2} = \mathrm{C}$$

$$\Rightarrow x^2 + 2a^2 tan^{-1}\frac{x}{y} + y^2 = \mathrm{D}\ \ , D = 2C$$

8. Equations Reducible to Exact Differential Equations:

Sometimes a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is not exact i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. It can be made exact by multiplying the equation by some function of x and y known as integrating factor (IF).

8.1 Integrating Factor (IF) Found by Inspection

Some non-exact differential equations can be grouped or rearranged and solved directly by integration, after multiplying by an integrating factor (IF) which can be found just by inspection as shown below:

Term	IF	Result
$xdy + ydx$	1. $\frac{1}{xy}$	$\frac{xdy+ydx}{xy} = \frac{1}{y}\mathrm{d}y + \frac{1}{x}\,\mathrm{d}x = d[\log(xy)]$
	2. $\frac{1}{(xy)^n}, n \neq 1$	$\frac{xdy+ydx}{(xy)^n} = \frac{\mathrm{d}(xy)}{(xy)^n} = -d\left[\frac{1}{(n-1)(xy)^{n-1}}\right]$
	3. $\cos(xy)$	$\cos(xy)\,(x\,dy + \ y\,dx\,) = d\,[\sin(xy)]$
	4. $\sin(xy)$	$\sin(xy)\,(x\,dy + \ y\,dx\,) = -d\,[\cos(xy)]$
$xdy - ydx$	1. $\frac{1}{x^2}$	$\frac{xdy-\,ydx}{x^2} = d\left[\frac{y}{x}\right]$
	2. $\frac{1}{y^2}$	$\frac{xdy-\,ydx}{y^2} = -d\left[\frac{x}{y}\right]$
	3. $\frac{1}{xy}$	$\frac{xdy-\,ydx}{xy} = d\left[log\frac{y}{x}\right]$
	4. $\frac{1}{x^2+y^2}$	$\frac{xdy-\,ydx}{x^2+y^2} = d\left[tan^{-1}\frac{y}{x}\right]$

	5. $\frac{1}{x\sqrt{x^2-y^2}}$	$\frac{xdy-ydx}{x\sqrt{x^2-y^2}} = d\left[sin^{-1}\frac{y}{x}\right]$
$xdx + ydy$	1. $\frac{1}{x^2+y^2}$	$\frac{xdx+ydy}{x^2+y^2} = \frac{1}{2}d[log(x^2+y^2)]$
	2. $\frac{1}{(x^2+y^2)^n}, n \neq 1$	$\frac{xdx+ydy}{(x^2+y^2)^n} = \frac{1}{2}d\left[\frac{(x^2+y^2)^{-n+1}}{-n+1}\right]$

Example15 Solve the differential equation:
$x\,dy - y\,dx + 2x^3dx = 0$ .....①
Solution: ① $\Rightarrow (-y + 2x^3)dx + xdy = 0$
$M = -y + 2x^3$, $N = x$
$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$
$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.
Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(x\,dy - y\,dx)$
① may be rewritten as : $\frac{xdy-ydx}{x^2} + 2x\,dx = 0$
$\Rightarrow d\left[\frac{y}{x}\right] + 2x\,dx = 0$......②
Integrating ②, solution is given by : $\frac{y}{x} + x^2 = C$
$\Rightarrow y + x^3 = Cx$
Example16 Solve the differential equation:
$y\,dx - x\,dy + (1+x^2)dx + x^2\cos y\,dy = 0$ .....①
Solution: $\Rightarrow (y + 1 + x^2)dx + (x^2\cos y - x)dy = 0$
$M = y + 1 + x^2$, $N = x^2\cos y - x$
$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 2x\cos y - 1$
$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.
Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(y\,dx - x\,dy)$
① may be rewritten as : $\frac{ydy-xdx}{x^2} + \left(\frac{1}{x^2}+1\right)dx + \cos y\,dy = 0$
$\Rightarrow -d\left[\frac{y}{x}\right] + \left(\frac{1}{x^2}+1\right)dx + \cos y\,dy = 0$......②
Integrating ②, solution is given by : $-\frac{y}{x} + \left(-\frac{1}{x} + x\right) + \sin y = C$
$\Rightarrow x^2 - y - 1 + x\sin y = Cx$
Example17 Solve the differential equation:

$x\,dx + y\,dy = a(x^2 + y^2)dy$ .....①

Solution: $\Rightarrow xdx + (y - a(x^2 + y^2))dy = 0$

$M = x\,,\ N = y - a(x^2 + y^2)$

$\frac{\partial M}{\partial y} = 0\,,\ \frac{\partial N}{\partial x} = -2ax$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2 + y^2}$ as integrating factor due to presence of the term $(x\,dx + y\,dy)$

① may be rewritten as: $\frac{xdx + ydy}{x^2 + y^2} - a\,dy = 0$

$\Rightarrow \frac{1}{2}d[log(x^2 + y^2)] - a\,dy = 0$

$\Rightarrow d[log(x^2 + y^2)] - 2\,a\,dy = 0$......②

Integrating ②, solution is given by: $log(x^2 + y^2) - 2ay = C$, Cis an arbitrary constant

Example18 Solve the differential equation:

$a(x\,dy + 2y\,dx) = xy\,dy$ .....①

Solution: ① $\Rightarrow 2aydx + (ax - xy)dy = 0$

$M = 2ay\,,\ N = ax - xy$

$\frac{\partial M}{\partial y} = 2a\,,\ \frac{\partial N}{\partial x} = a - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ①as $a(x\,dy + y\,dx) + ay\,dx = xy\,dy$ ......②

Taking $\frac{1}{xy}$ as integrating factor due to presence of the term $(x\,dy + y\,dx)$

②may be rewritten as : $a\frac{xdy + ydx}{xy} + \frac{a}{x}\,dx - dy = 0$

$\Rightarrow ad[\log(xy)] + \frac{a}{x}\,dx - dy = 0$......③

Integrating ③ solution is given by: $a\log(xy) + a\log x - y = C$

$\Rightarrow a\log(x^2y) - y = C$, Cis an arbitrary constant

Example19 Solve the differential equation:

$x^4\frac{dy}{dx} + x^3y + \sec(xy) = 0$ .....①

Solution: ① $\Rightarrow (x^3y + \sec(xy))dx + x^4dy = 0$

$M = x^3y + \sec(xy),\ N = x^4$

$\frac{\partial M}{\partial y} = x^3 + x\sec(xy)\tan(xy)\,,\ \frac{\partial N}{\partial x} = 4x^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as: $x^3(x\,dy + y\,dx) + \sec(xy)\,dx = 0$

Using $\cos(xy)$ as integrating factor

$\Rightarrow \cos(xy)(x\,dy + y\,dx) + x^{-3}dx = 0$

$\Rightarrow \cos(xy)\,(x\,dy + y\,dx) + x^{-3}dx = 0$

$\Rightarrow d\,[\sin(xy)] + x^{-3}dx = 0$ ......②

Integrating ②, we get the required solution as:

$\sin(xy) - \frac{x^{-2}}{2} = C$

$\Rightarrow 2\,x^2\sin(xy) - 1 = Cx^2$

8.2 Integrating Factor (IF) of a Non-Exact Homogeneous Equation

If the equation $Mdx + Ndy = 0$ is a homogeneous equation, then the integrating factor (IF) will be $\frac{1}{Mx+Ny}$, provided $Mx + Ny \neq 0$

Example20 Solve the differential equation:

$(x^3 + y^3)dx - xy^2\,dy = 0$ .....①

Solution: $M = x^3 + y^3$, $N = -xy^2$

$\frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Since ① is a homogeneous equation, ∴ IF $= \frac{1}{Mx+Ny} = \frac{1}{x^4 + xy^3 - xy^3} = \frac{1}{x^4}$

∴ ① may be rewritten as : $\left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx - \frac{y^2}{x^3} dy = 0$......②

New $M = \frac{1}{x} + \frac{y^3}{x^4}$, New $N = -\frac{y^2}{x^3}$

$\frac{\partial M}{\partial y} = \frac{3y^2}{x^4}$, $\frac{\partial N}{\partial x} = \frac{3y^2}{x^4}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴②is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx + \int 0\,dy = C$$

$\Rightarrow \log x - \frac{1}{3}\left(\frac{y}{x}\right)^3 = \mathrm{C}$

Example21 Solve the differential equation:

$(3y^4 + 3x^2y^2)dx + (x^3y - 3xy^3)\,dy = 0$ .....①

Solution: $M = 3y^4 + 3x^2y^2$, $N = x^3y - 3xy^3$

$\frac{\partial M}{\partial y} = 12y^3 + 6x^2y$, $\frac{\partial N}{\partial x} = 3x^2y - 3y^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Since ① is a homogeneous equation

∴ IF$= \frac{1}{Mx+Ny} = \frac{1}{3xy^4+3x^3y^2+x^3y^2-3xy^4} = \frac{1}{4x^3y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{3y^2}{4x^3}+\frac{3}{4x}\right)dx+\left(\frac{1}{4y}-\frac{3y}{4x^2}\right)dy=0\ldots\ldots②$

New $M=\frac{3y^2}{4x^3}+\frac{3}{4x}$, New $N=\frac{1}{4y}-\frac{3y}{4x^2}$

$\frac{\partial M}{\partial y}=\frac{6y}{4x^3}=\frac{3y}{2x^3},\ \frac{\partial N}{\partial x}=\frac{3y}{2x^3}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, ∴② is an exact differential equation.

Solution of ② is given by:

$\int\limits_{y\ constant}\left(\frac{3y^2}{4x^3}+\frac{3}{4x}\right)dx+\int\frac{1}{4y}dy=C$

$\Rightarrow\frac{-3y^2}{8x^2}+\frac{3}{4}\log x+\frac{1}{4}\log y=\mathrm{C}$

$\Rightarrow\log x^3y-\frac{3y^2}{2x^2}=\mathrm{D}$, $\mathrm{D}=4\mathrm{C}$

8.3 Integrating Factor of a Non-Exact Differential Equation of the Form

$\boldsymbol{yf_1(xy)dx+xf_2(xy)\,dy=0}$ **:** If the equation $Mdx+Ndy=0$ is of the given form, then the integrating factor (IF) will be $\frac{1}{Mx-Ny}$ provided $Mx-Ny\neq 0$

Example22 Solve the differential equation:

$y(1+xy)dx+x(1-xy)\,dy=0\ \ldots..①$

Solution: $M=y+xy^2,\ N=x-x^2y$

$\frac{\partial M}{\partial y}=1+2xy,\ \frac{\partial N}{\partial x}=1-2xy$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx+xf_2(xy)dy=0$,

$\therefore\ \mathrm{IF}=\frac{1}{Mx-Ny}=\frac{1}{xy+x^2y^2-xy+x^2y^2}=\frac{1}{2x^2y^2}$

∴① may be rewritten after multiplying by IF as:

$\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy=0\ldots\ldots②$

New $M=\frac{1}{2x^2y}+\frac{1}{2x}$, New $N=\frac{1}{2xy^2}-\frac{1}{2y}$

$\frac{\partial M}{\partial y}=\frac{-1}{2x^2y^2}$, $\frac{\partial N}{\partial x}=\frac{-1}{2x^2y^2}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, ∴② is an exact differential equation.

Solution of ② is given by:

$\int\limits_{y\ constant}\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int-\frac{1}{2y}dy=0$

$\Rightarrow\frac{-1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=\mathrm{C}$

$\Rightarrow\log\frac{x}{y}-\frac{1}{xy}=\mathrm{D},\ \mathrm{D}=2\mathrm{C}$

Example23 Solve the differential equation:

$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)\,dy = 0$ .....①

Solution: $M = xy^2 + 2x^2y^3,\ N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2,\ \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As equation ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ IF $= \frac{1}{Mx - Ny} = \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{3x^3y^3}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$......②

New $M = \frac{1}{x^2y} + \frac{2}{x}$, New $N = \frac{1}{xy^2} - \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{-1}{x^2y^2},\ \frac{\partial N}{\partial x} = \frac{-1}{x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴②is an exact differential equation.

Solution of ② is given by:

$\underset{y\ constant}{\int\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx} + \int -\frac{1}{y}dy = C$

$\Rightarrow \frac{-1}{xy} + 2\log x - \log y = \mathrm{C}$

$\Rightarrow \log\frac{x^2}{y} - \frac{1}{xy} = \mathrm{C}$

8.4 Integrating Factor (IF) of a Non-Exact Differential Equation

$Mdx + Ndy = 0$ in which $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are connected in a specific way as shown:

i. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x alone, then IF = $e^{\int f(x)dx}$

ii. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a function of y alone, then IF = $e^{\int -g(y)dy}$

Example24 Solve the differential equation:

$(x^3 + y^2 + x)dx + xy\,dy = 0$ .....①

Solution: $M = x^3 + y^2 + x,\ N = xy$

$\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = y$

Clearly $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$ say

$\therefore$ IF = $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$\therefore$① may be rewritten after multiplying by IF as:

$$(x^4 + xy^2 + x^2)dx + x^2y\,dy = 0 \ \ldots\ldots②$$

New $M = x^4 + xy^2 + x^2$, New $N = x^2y$

$$\frac{\partial M}{\partial y} = 2xy\ ,\ \frac{\partial N}{\partial x} = 2xy$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$②is an exact differential equation.

Solution of ②is given by:

$$\underset{y\ constant}{\int (x^4 + xy^2 + x^2)\,dx} + \int 0\,dy = C$$

$$\Rightarrow \frac{x^5}{5} + \frac{x^2y^2}{2} + \frac{x^3}{3} = \mathrm{C}$$

Example25 Solve the differential equation:

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ .....①

Solution: $M = y^4 + 2y,\ N = xy^3 + 2y^4 - 4x$

$$\frac{\partial M}{\partial y} = 4y^3 + 2\,,\ \frac{\partial N}{\partial x} = y^3 - 4$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^3 + 6$

Clearly $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M} = \frac{3y^3+6}{y^4+2y} = \frac{3}{y} = g(y)$ say

$\therefore$ IF $= e^{\int -g(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = \frac{1}{y^3}$

$\therefore$ ① may be rewritten after multiplying by IF as:

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0\ldots\ldots②$$

New $M = y + \frac{2}{y^2}$, New $N = x + 2y - \frac{4x}{y^3}$

$$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}\ ,\ \frac{\partial N}{\partial x} = 1 - \frac{4}{y^3}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int \left(y + \frac{2}{y^2}\right) dx} + \int 2y dy = C$$

$$\Rightarrow \left(y + \frac{2}{y^2}\right)x + y^2 = \mathrm{C}$$

Example26 Solve the differential equation:

$(x^2 - y^2 + 2x)dx - 2y\,dy = 0$ .....①

Solution: $M = x^2 - y^2 + 2x,\ N = -2y$

$\frac{\partial M}{\partial y} = -2y,\ \frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y}{-2y} = 1 = f(x)$ say

∴ IF $= e^{\int f(x)dx} = e^{\int 1dx} = e^x$

∴ ① may be rewritten after multiplying by IF as:

$e^x(x^2 - y^2 + 2x)dx - 2e^x y\,dy = 0$.......②

New $M = e^x(x^2 - y^2 + 2x)$, New $N = -2e^x y$

$\frac{\partial M}{\partial y} = -2e^x y,\ \frac{\partial N}{\partial x} = -2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴②is an exact differential equation.

Solution of ② is given by:

$\int e^x(x^2 - y^2 + 2x)\,dx + \int 0\,dy = C$

y constant

$\Rightarrow (x^2 - y^2 + 2x)e^x - (2x + 2)e^x + (2)e^x = C$

$\Rightarrow (x^2 - y^2)e^x = C$, C is an arbitrary constant

Example27 Solve the differential equation:

$2ydx + (2x\log x - xy)\,dy = 0$ .....①

Solution: $M = 2y,\ N = 2x\log x - xy$

$\frac{\partial M}{\partial y} = 2,\ \frac{\partial N}{\partial x} = 2(1 + \log x) - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x)$ say

∴ IF $= e^{\int f(x)dx} = e^{\int \frac{-1}{x}dx} = e^{\log x^{-1}} = \frac{1}{x}$

∴ ① may be rewritten after multiplying by IF as:

$\frac{2y}{x}dx + (2\log x - y)\,dy = 0$ ......②

New $M = \frac{2y}{x}$, New $N = 2\log x - y$

$\frac{\partial M}{\partial y} = \frac{2}{x} = \frac{\partial N}{\partial x} = \frac{2}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴② is an exact differential equation.

Solution of ②is given by:

$\int_{y\ constant} \left(\frac{2y}{x}\right) dx + \int -ydy = C \Rightarrow 2y\log x - \frac{y^2}{2} = C$

8.5 Integrating Factor (IF) of a Non-Exact Differential Equation

$\boldsymbol{x^a y^b(m_1 ydx + n_1 xdy) + x^c y^d(m_2 ydx + n_2 xdy) = 0}$, where $a, b, c, d,$

m_1, n_1, m_2, n_2 are constants, then IF is given by $x^\alpha y^\beta$, where α and β are connected by the relation $\frac{a+\alpha+1}{m_1} = \frac{b+\beta+1}{n_1}$ and $\frac{c+\alpha+1}{m_2} = \frac{d+\beta+1}{n_2}$

Example28 Solve the differential equation:

$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ .....①

Solution: $M = y^2 + 2x^2y,\ N = 2x^3 - xy$

$\frac{\partial M}{\partial y} = 2y + 2x^2 ,\ \frac{\partial N}{\partial x} = 6x^2 - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as $x^2y^0(2ydx + 2xdy) + x^0y^1(ydx - xdy) = 0$.......②

Comparing with standard form $a = 2,\ b = 0,\ c = 0,\ d = 1,$

$m_1 = 2,\ n_1 = 2,\ m_2 = 1,\ n_2 = -1$

$\therefore \frac{2+\alpha+1}{2} = \frac{0+\beta+1}{2}$ and $\frac{0+\alpha+1}{1} = \frac{1+\beta+1}{-1}$

$\Rightarrow \alpha - \beta = -2$ and $\alpha + \beta = -3$

Solving we get $\alpha = \frac{-5}{2}$ and $\beta = \frac{-1}{2}$

$\therefore$ IF $= x^\alpha y^\beta = x^{\frac{-5}{2}}y^{\frac{-1}{2}}$

∴ ① may be rewritten after multiplying by IF as:

$x^{\frac{-5}{2}}y^{\frac{-1}{2}}(y^2 + 2x^2y)dx + x^{\frac{-5}{2}}y^{\frac{-1}{2}}(2x^3 - xy)dy = 0$.......②

$\Rightarrow \left(x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right)dx + \left(2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}\right)dy = 0$

New $M = x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}$, New $N = 2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}$

$\frac{\partial M}{\partial y} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$, $\frac{\partial N}{\partial x} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int} \left(x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right) dx + \int 0\, dy = 0$$

$$\Rightarrow\ 4(xy)^{\frac{1}{2}} - \frac{2}{3}\left(\frac{y}{x}\right)^{\frac{3}{2}} = C$$, C is an arbitrary constant

9. Linear Differential Equations

A differential equation of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$in which the dependent variable y and its derivatives viz. $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc occur in first degree and are not multiplied together is called a Linear Differential Equation.

9.1 First Order Linear Differential Equations (Leibnitz's Linear Equations)

A first order linear differential equation is of the form$\frac{dy}{dx} + Py = Q$, ……. Ⓐ

where P and Q are functions of x alone or constants. To solve Ⓐ, multiplying throughout by $e^{\int P\,dx}$ (here $e^{\int P\,dx}$ is known as Integrating Factor (IF)), we get

$$\frac{dy}{dx}e^{\int P\,dx} + Py\, e^{\int P\,dx} = Q\, e^{\int P\,dx}$$

$$\Rightarrow \mathrm{d}\left(y\, e^{\int P\,dx}\right) = Q\, e^{\int P\,dx}$$

$$\Rightarrow y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx} dx + C$$

Algorithm to solve a first order linear differential equation (Leibnitz's Equation)

1. Write the given equation in standard form i.e., $\frac{dy}{dx} + Py = Q$
2. Find the integrating factor (IF) = $e^{\int P\,dx}$
3. Solution is given by y. IF = $\int Q$ .IF dx + C, C is an arbitrary constant

Note: If the given equation is of the type $\frac{dx}{dy} + Px = Q$,

then IF = $e^{\int P\,dy}$and the solution is given by x. IF = $\int Q$ .IF $dy + C$

Example29 Solve the differential equation: $\frac{dy}{dx} = \frac{x + y\sin x}{1+cosx}$

Solution: The given equation may be written as:

$$\frac{dy}{dx} - \frac{\sin x}{1+cosx}y = \frac{x}{1+cosx} \ldots\ldots ①$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{\sin x}{1+cosx}$ and $Q = \frac{x}{1+cosx}$

$$\text{IF } = e^{\int P\,dx} = e^{\int -\frac{\sin x}{1+cosx}dx} = e^{log|1+\cos x|} = 1 + \cos x$$

∴ Solution of ① is given by

$$y.\,(1 + \cos x) = \int \frac{x}{1+cosx}(1 + \cos x)dx + C$$

$\Rightarrow y(1+\cos x) = \frac{x^2}{2} + C$

Example30 Solve the differential equation: $\frac{dy}{dx} = (1+x) + (1-y)$

Solution: The given equation may be written as:

$\frac{dy}{dx} + y = 2 + x$ .... ①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = 1$ and $Q = 2 + x$

IF $= e^{\int P\,dx} = e^{\int dx} = e^x$

∴ Solution of ①is given by

$y.\,e^x = \int (2+x)e^x\,dx + C$

$\Rightarrow y = 1 + x + Ce^{-x}$

Example31 Solve the differential equation: $(x + y + 1)\frac{dy}{dx} = 1$

Solution: The given equation may be written as:

$\frac{dx}{dy} = x + y + 1 \Rightarrow \frac{dx}{dy} - x = y + 1$ ......①

This is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$

Where $P = -1$ and $Q = y + 1$

IF $= e^{\int P\,dy} = e^{\int -dy} = e^{-y}$

∴ Solution of ① is given by

$x.\,e^{-y} = \int (y+1)e^{-y}\,dy + C$

$\Rightarrow xe^{-y} = -(y+2)e^{-y} + C$

$\Rightarrow x = -(y+2) + C\,e^{y}$

Example32 Solve the differential equation: $x\log x\,\frac{dy}{dx} + y = 2\log x$

Solution: The given equation may be written as:

$\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$ ....①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = \frac{1}{x\log x}$ and $Q = \frac{2}{x}$

IF $= e^{\int P\,dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

∴ Solution of ① is given by

$y.\,\log x = \int \frac{2}{x}\log x\,\,dx + C$

$\Rightarrow y\log x = (\log x)^2 + C$, Cis an arbitrary constant

Example33 Solve the differential equation: $\frac{dy}{dx} = \frac{e^{2\sqrt{x}} + y}{\sqrt{x}}$

Solution: The given equation may be written as:

$\frac{dy}{dx} - \frac{1}{\sqrt{x}} y = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$ ......①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{1}{\sqrt{x}}$ and $Q = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$

IF $= e^{\int P\,dx} = e^{\int \frac{-1}{\sqrt{x}}dx} = e^{-2\sqrt{x}}$

∴ Solution of ① is given by

$y.\, e^{-2\sqrt{x}} = \int \frac{e^{2\sqrt{x}}}{\sqrt{x}} e^{-2\sqrt{x}}\, dx + C$

$\Rightarrow y.\, e^{-2\sqrt{x}} = \int \frac{1}{\sqrt{x}} dx + C$

$\Rightarrow y.\, e^{-2\sqrt{x}} = 2\sqrt{x} + C$

$\Rightarrow y = 2\sqrt{x}e^{2\sqrt{x}} + Ce^{2\sqrt{x}}$

9.2 Equations Reducible to Leibnitz's Equations (Bernoulli's Equations)

Differential equation of the form $\frac{dy}{dx} + Pf(y) = Qg(y)$, ... Ⓑ

where P and Q are functions of x alone or constant, is called Bernoulli's equation. Dividing both sides of Ⓑ by $g(y)$, we get $\frac{1}{g(y)}\frac{dy}{dx} + P\frac{f(y)}{g(y)} = Q$. Now putting $\frac{f(y)}{g(y)} = t$, Ⓑ reduces to Leibnitz's equation.

Example34 Solve the differential equation: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ ......①

Solution: The given equation may be written as:

$e^{-y}\frac{dy}{dx} + \frac{1}{x}e^{-y} = \frac{1}{x^2}$ ......②

Putting $e^{-y} = t,\ -e^{-y}\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2}$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x^2}$

IF $= e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

∴ Solution of ④is given by

$t.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}dx + \mathrm{C}$

$\Rightarrow t.\frac{1}{x} = \frac{1}{2x^2} + \mathrm{C}$

Substituting $t = e^{-y}$

$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + \mathrm{C}$

$\Rightarrow 2x = e^y(2cx^2 + 1)$

Example35 Solve the differential equation: $\tan y\frac{dy}{dx} + \tan x = \cos y cos^3 x$ ......①

Solution: The given equation may be written as:

$$\frac{\tan y}{\cos y}\frac{dy}{dx}+\frac{\tan x}{\cos y}=cos^3x$$

$$\Rightarrow \sec y\tan y\frac{dy}{dx}+\sec y\tan x=cos^3x \text{ ......②}$$

Putting $\sec y=t,\ \sec y\tan y\frac{dy}{dx}=\frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx}+(\tan x)\,t=cos^3x$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx}+Pt=Q$

Where $P=\tan x$ and $Q=cos^3x$

$$\text{IF }=e^{\int P\,dx}=e^{\int \tan x dx}=e^{\log|\sec x|}=\sec x$$

∴ Solution of ④ is given by

$t.\sec x\ =\int cos^3x.\sec x\,dx+\mathrm{C}$

$\Rightarrow$ t. $\sec x\ =\int cos^2x\,dx+\mathrm{C}$

$\Rightarrow$ t. $\sec x\ =\int\frac{1+\cos 2x}{2}dx+\mathrm{C}$

$\Rightarrow$t. $\sec x=\frac{x}{2}+\frac{\sin 2x}{4}+\mathrm{C}$

Substituting t $=\sec y$,

$\Rightarrow \sec x\sec y=\frac{x}{2}+\frac{\sin 2x}{4}+\mathrm{C}$

Example36 Solve the differential equation: $\frac{dy}{dx}=\frac{y}{x+\sqrt{xy}}$ ......①

Solution: The given equation may be written as:

$$\frac{dx}{dy}=\frac{x+\sqrt{xy}}{y}$$

$$\Rightarrow \frac{dx}{dy}-\frac{1}{y}x=\sqrt{\frac{x}{y}}$$

Dividing throughout by $\sqrt{x}$

$\Rightarrow \frac{1}{\sqrt{x}}\frac{dx}{dy}-\frac{1}{y}\sqrt{x}=\frac{1}{\sqrt{y}}$ ......②

Putting $\sqrt{x}=t,\ \frac{1}{2\sqrt{x}}\frac{dx}{dy}=\frac{dt}{dy}$ ......③

Using ③ in ②, we get $\frac{dt}{dy}-\frac{1}{2y}t=\frac{1}{2\sqrt{y}}$ ....④

④ is a linear differential equation of the form $\frac{dt}{dy}+Pt=Q$

Where $P=-\frac{1}{2y}$ and $Q=\frac{1}{2\sqrt{y}}$

$\text{IF}=e^{\int P\,dy}=e^{\int\frac{-1}{2y}dy}=e^{\frac{-1}{2}\log y}=e^{\log\frac{1}{\sqrt{y}}}=\frac{1}{\sqrt{y}}$

∴ Solution of ④is given by

$t.\frac{1}{\sqrt{y}}=\int\frac{1}{2\sqrt{y}}.\frac{1}{\sqrt{y}}dy+\mathrm{C}$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \int \frac{1}{2y} dy + \mathrm{C}$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \frac{1}{2}\, log\, y + \mathrm{C}$

Substituting $t = \sqrt{x}$

$\sqrt{\frac{x}{y}} = log\sqrt{y} + C$

Example37 Solve the differential equation: $x\frac{dy}{dx} + y = y^2 \log x$ .....①

Solution: The given equation may be written as:

$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x}\log x$

Dividing throughout by y^2

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$......②

Putting $\frac{1}{y} = t$, $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}\log x$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x}\log x$

$\mathrm{IF} = e^{\int P\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

∴ Solution of ④ is given by

$t.\frac{1}{x} = \int -\frac{1}{x}\log x.\frac{1}{x}dx + C$

$\Rightarrow t.\frac{1}{x} = \int -\frac{1}{x^2}\log x\, dx + C$

Putting $\log x = u, \frac{1}{x}dx = du$, also $x = e^u$

$\Rightarrow t.\frac{1}{x} = -\int ue^{-u}\,du + C$

$\Rightarrow t.\frac{1}{x} = -[u(-e^{-u}) - 1(e^{-u})] + C$

$\Rightarrow t.\frac{1}{x} = e^{-u}(u+1) + C$

$\Rightarrow t.\frac{1}{x} = \frac{1}{x}(log\, x + 1) + C$

Substituting $t = \frac{1}{y}$

$\Rightarrow \frac{1}{xy} = \frac{1}{x}(\log x + 1) + C$

$\Rightarrow \frac{1}{y} = (\log x + 1) + Cx$, Cis an arbitrary constant

9.3 Higher Order Linear Differential Equations with Constant Coefficients

A general linear differential equation of n^{th} order with constant coefficients is given by:

$k_0\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_{n-1}\frac{dy}{dx} + k_n y = F(x)$....①

where $k's$ are constant and $F(x)$ is a function of x alone or constant.

$\Rightarrow (k_0 D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n)y = F(x)$

Or $f(D)y = F(x)$, where $D^n \equiv \frac{d^n}{dx^n}$, $D^{n-1} \equiv \frac{d^{n-1}}{dx^{n-1}}$, ..., $D \equiv \frac{d}{dx}$ are called differential operators. If $F(x) = 0$, ① is called a **Homogeneous Linear Differential Equation** with Constant Coefficients.

9.3.1 Solving Homogeneous and Non- Homogeneous Linear Differential Equations with Constant Coefficients

Complete solution of equation $f(D)y = F(x)$ is given by $y =$ C.F + P.I.

where C.F. denotes complimentary function and P.I. is the particular integral.

When $F(x) = 0$, it is a homogeneous linear differential equation with constant coefficients and the solution of equation $f(D)y = 0$ is given by $y =$ C.F

Rules for Finding Complimentary Function (C.F.)

Consider the equation $f(D)y = F(x)$

$\Rightarrow (k_0 D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n)y = F(x)$

Step 1: Put $D = m$, auxiliary equation (A.E) is given by $f(m) = 0$

$\Rightarrow k_0 m^n + k_1 m^{n-1} + \cdots + k_{n-1} m + k_n = 0$ ......③

Step 2: Solve the auxiliary equation given by ③

I. If the n roots of A.E. are real and distinct say $m_1, m_2, \ldots m_n$

C.F. $= c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

II. If two or more roots are equal i.e., $m_1 = m_2 = \ldots = m_k,\ k \leq n$

C.F. $= (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_1 x} + \cdots + c_n e^{m_n x}$

III. If A.E. has a pair of imaginary roots i.e., $m_1 = \alpha + i\,\beta$, $m_2 = \alpha - i\,\beta$

C.F. $= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

IV. If 2 pairs of imaginary roots are equal i.e., $m_1 = m_2 = \alpha + i\,\beta$, $m_3 = m_4 = \alpha - i\,\beta$

C.F. $= e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + \cdots + c_n e^{m_n x}$

Example38 Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution: Given: $(D^2 - 8D + 15)y = 0$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3, 5$

C.F. $= c_1 e^{3x} + c_2 e^{5x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{3x} + c_2 e^{5x}$

Example39 Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Solution: Given: $(D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$ ....①

By hit and trial $(m-2)$ is a factor of ①

$\therefore$ ① may be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

C.F. $= c_1e^x + c_2e^{2x} + c_3e^{3x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x}$

Example40 Solve $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$ ....①

By hit and trial $(m-1)$ is a factor of ①

$\therefore$ ① may be rewritten as

$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$

$\Rightarrow m^3(m-1) - 9m^2(m-1) + 26m(m-1) - 24(m-1) = 0$

$\Rightarrow (m-1)(m^3 - 9m^2 + 26m - 24) = 0$ ......②

By hit and trial $(m-2)$ is a factor of ②

$\therefore$②May be rewritten as

$(m-1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$

$\Rightarrow (m-1)[m^2(m-2) - 7m(m-2) + 12(m-2)] = 0$

$\Rightarrow (m-1)(m^2 - 7m + 12)(m-2) = 0$

$\Rightarrow (m-1)(m-3)(m-4)(m-2) = 0$

$\Rightarrow m = 1,2,3,4$

C.F. $= c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Example41 Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: Given: $(D^3 + 2D^2 + D)y = 0$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m+1)^2 = 0$

$\Rightarrow m = 0, -1, -1$

C.F. $= c_1 + (c_2 + c_3x)e^{-x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 + (c_2 + c_3x)e^{-x}$

Example42 Solve the differential equation: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$

Solution: Given: $(D^4 - 2D^2 + 1)y = 0$

Auxiliary equation is: $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2 - 1)^2 = 0$

$\Rightarrow (m+1)^2(m-1)^2 = 0$

$\Rightarrow m = -1, -1, 1, 1$

C.F. $= (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$

Example43 Solve the differential equation: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

Solution: Given: $(D^3 - 2D + 4)y = 0$

Auxiliary equation is: $m^3 - 2m + 4 = 0$ ....(1)

By hit and trial $(m+2)$ is a factor of (1)

$\therefore$ (1) may be rewritten as

$m^3 + 2m^2 - 2m^2 - 4m + 2m + 4 = 0$

$\Rightarrow m^2(m+2) - 2m(m+2) + 2(m+2) = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm i$

C.F. $= c_1e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Example44 Solve the differential equation: $(D^2 - 2D + 5)^2y = 0$

Solution: Auxiliary equation is: $(m^2 - 2m + 5)^2$ ...(1)

Solving (1), we get $m = 1 \pm 2i, 1 \pm 2i$

C.F. $= e^x[(c_1 + c_2\,x) \cos 2x + (c_3 + c_4x) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = e^x[(c_1 + c_2\,x) \cos 2x + (c_3 + c_4x) \sin 2x]$

Example45 Solve the differential equation: $(D^2 + 4)^3y = 0$

Solution: Auxiliary equation is: $(m^2 + 4)^3 = 0$ ...(1)

Solving (1), we get $m = \pm 2i, \pm 2i, \pm 2i$

C.F. $= (c_1 + c_2x + c_3\,x^2) \cos 2x + (c_4 + c_5x + c_6x^2) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2x + c_3\,x^2) \cos 2x + (c_4 + c_5x + c_6x^2) \sin 2x$

Shortcut Rules for Finding Particular Integral (P.I.)

Consider the non- homogeneous linear differential equation

$f(D)y = F(x), F(x) \neq 0$

$\Rightarrow (k_0 D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n)y = F(x)$

Then P.I $= \frac{1}{f(D)} F(x)$, Clearly P.I. = 0 if $F(x) = 0$

Case I: When $F(x) = e^{ax}$

Use the rule P.I $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, $f(a) \neq 0$

In case of failure i.e., if $f(a) = 0$

P.I $= x \frac{1}{f'(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$, $f'(a) \neq 0$

If $f'(a) = 0$, P.I. $= x^2 \frac{1}{f''(a)} e^{ax}$, $f''(a) \neq 0$ and so on

Example46 Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^{2x}$

Solution: Given: $(D^2 - 2D + 10)y = e^{2x}$

Auxiliary equation is: $m^2 - 2m + 10 = 0$

$\Rightarrow m = 1 \pm 3i$

C.F. $= e^x(c_1 \cos 3x + c_2 \sin 3x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} e^{2x} = \frac{1}{f(2)} e^{2x}$, by putting $D = 2$

$= \frac{1}{2^2 - 2(2) + 10} e^{2x} = \frac{1}{10} e^{2x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10} e^{2x}$

Example47 Solve the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Solution: Given: $(D^2 + D - 2)y = e^x$

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2, 1$

C.F. $= c_1 e^{-2x} + c_2 e^x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} e^x$, putting $D = 1, f(1) = 0$

$\therefore$P.I. $= x \frac{1}{f'(D)} e^x$ $\because$ P.I. $= x \frac{1}{f'(a)} e^{ax}$if $f(a) = 0$

$\Rightarrow$ P.I. $= x \frac{1}{2D+1} e^x = x \frac{1}{f'(1)} e^x$, $f'(1) \neq 0$

$\Rightarrow$ P.I. $= \frac{xe^x}{3}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-2x} + c_2 e^x + \frac{xe^x}{3}$

Example48 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = \sinh(2x + 1) + 4^x$

Solution: $\Rightarrow (D^2 - 4)y = \sinh(2x+1) + 4^x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. $= c_1e^{2x} + c_2e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x)$

$= \frac{1}{f(D)}(\sinh(2x+1) + 4^x)$

$= \frac{1}{D^2-4}\left(\frac{e^{(2x+1)} - e^{-(2x+1)}}{2}\right) + \frac{1}{D^2-4}\left(e^{x\log 4}\right) \quad \because \sinh x = \frac{e^x - e^{-x}}{2}$ and $4^x = e^{x\log 4}$

$= \frac{e}{2}\frac{1}{D^2-4}e^{2x} - \frac{e^{-1}}{2}\frac{1}{D^2-4}e^{-2x} + \frac{1}{D^2-4}e^{x\log 4}$

Putting $D = 2, -2$ and $\log 4$ in the three terms respectively

$f(2) = 0$ and $f(-2) = 0$ for first two terms

$\therefore$ P.I. $= \frac{e}{2}x\frac{1}{2D}e^{2x} - \frac{e^{-1}}{2}x\frac{1}{2D}e^{-2x} + \frac{1}{(log4)^2-4}e^{x\log 4}$

$\because$ P. I. $= x\frac{1}{f'(a)}e^{ax}$ if $f(a) = 0$

Now putting $D = 2, -2$ in first two terms respectively

$\Rightarrow$P.I. $= \frac{ex}{8}e^{2x} + \frac{e^{-1}x}{8}e^{-2x} + \frac{4^x}{(log4)^2-4} \because e^{x\log 4} = 4^x$

$\Rightarrow$P.I. $= \frac{x}{4}\left(\frac{e^{(2x+1)} + e^{-(2x+1)}}{2}\right) + \frac{4^x}{(log4)^2-4}$

$\Rightarrow$P.I. $= \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2-4} \because \cosh x = \frac{e^x + e^{-x}}{2}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2-4}$

Case II: When $F(x) = Sin\ (ax+b) or\ Cos\ (ax+b)$

If $F(x) = Sin\ (ax+b)\ or\ Cos\ (ax+b)$, put $D^2 = -a^2$,

$D^3 = D^2D = -a^2D, D^4 = (D^2)^2 = a^4, \ldots\ldots$

This will form a linear expression in D in the denominator. Now rationalize the denominator to substitute $D^2 = -a^2$. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

In case of failure i.e., if $f(-a^2) = 0$

P.I. $= x\frac{1}{f'(-a^2)} Sin\ (ax+b)\ or\ Cos\ (ax+b), f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, P.I.$= x^2\frac{1}{f''(-a^2)} Sin\ (ax+b)\ or\ Cos\ (ax+b), f''(-a^2) \neq 0$ and so on.

Example49 Solve the differential equation: $(D^2 + D - 2)y = \sin x$

Solution: Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m+2)(m-1) = 0$

$\Rightarrow m = -2, 1$

C.F. $= c_1e^{-2x} + c_2e^{x}$

P.I. $= \frac{1}{f(D)}\ F(x) = \frac{1}{f(D)}\sin x = \ \frac{1}{D^2+D-2}\sin x$

putting $D^2\ = -1^2\ = -1$

P.I $= \frac{1}{D-3}\sin x = \frac{D+3}{D^2-9}\sin x$, Rationalizing the denominator

$= \frac{(D+3)\sin x}{-10}$, Putting $D^2\ = -1$

$\therefore$ P. I. $= \frac{-1}{10}(D\ sin\ x\ +3\ \sin x)$

$= \frac{-1}{10}(\cos x\ +3\ \sin x)$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ \ c_1e^{-2x} + c_2e^{x} - \frac{1}{10}(\cos x\ +3\ \sin x)$

Example50 Solve the differential equation: $(D^2 + 2D + 1)y = cos^2x$

Solution: Auxiliary equation is: $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0 \Rightarrow\ m = -1, -1$

C.F. $= e^{-x}(c_1 + c_2x)$

P.I. $= \frac{1}{f(D)}\ F(x) = \frac{1}{f(D)}cos^2x = \ \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{D^2+2D+1}e^{0x} + \frac{1}{2}\frac{1}{D^2+2D+1}\cos 2x$

Putting $D\ = 0$ in the 1st term and $D^2\ = -2^2\ = -4$ in the 2nd term

P.I $= \frac{1}{2} + \ \frac{1}{2}\frac{1}{2D-3}\cos 2x$

$= \frac{1}{2} + \ \frac{1}{2}\frac{2D+3}{4D^2-3^2}\cos 2x$, Rationalizing the denominator

$= \frac{1}{2} + \ \frac{1}{2}\frac{(2D+3)\cos 2x}{-25}$, Putting $D^2\ = -4$

$\therefore$ P. I. $= \frac{1}{2} - \ \frac{1}{50}\ (-4\sin 2x + 3\cos 2x)$

Now $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ \ e^{-x}(c_1 + c_2x) + \frac{1}{2} - \ \frac{1}{50}\ (-4\sin 2x + 3\cos 2x)$

Example51 Solve the differential equation: $(D^2 + 9)y = \sin 2x\cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow\ m = \pm 3i$

C.F. $=\ c_1\cos 3x + c_2\sin 3x$

P.I. $= \frac{1}{f(D)}\ F(x) = \frac{1}{f(D)}\sin 2x\cos x = \ \frac{1}{2}\frac{1}{D^2+9}(\sin 3x + \sin x)$

$= \frac{1}{2}\frac{1}{D^2+9}\sin 3x + \ \frac{1}{2}\frac{1}{D^2+9}\sin x$

Putting $D^2\ = -9$ in the 1st term and $D^2\ = -1$ in the 2nd term

We see that $f(D^2\ = -9)\ = 0$ for the 1st term

$\therefore$ P.I. $\ = \frac{1}{2}\ x\frac{1}{2D}\sin 3x + \ \frac{1}{2}\frac{1}{8}\sin x$

$\because$P.I. $= x\frac{1}{f'(-a^2)}\, Sin\,(ax+b), f'(-a^2) \neq 0$

$\Rightarrow$ P.I. $= -\frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1\cos 3x + c_2\sin 3x - \frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Case III: When $F(x) = x^n$, n is a positive integer

P.I. $= \frac{1}{f(D)}\, F(x) = \frac{1}{f(D)}x^n$

1. Take the lowest degree term common from $f(D)$ to get an expression of the form $[1 \pm \phi(D)]$ in the denominator and take it to numerator to become $[1 \pm \phi(D)]^{-1}$
2. Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to n^{th} degree as $(n+1)^{th}$derivative of x^n is zero
3. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

Following expansions will be useful to expand $[1 \pm \phi(D)]^{-1}$

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

Example52 Solve the differential equation: $\frac{d^2y}{dx^2} - y = 5x - 2$

Solution: $\Rightarrow (D^2-1)y = 5x - 2$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. $= c_1e^x + c_2e^{-x}$

P.I. $= \frac{1}{f(D)}\, F(x) = \frac{1}{D^2-1}(5x-2)$

$= \frac{1}{-(1-D^2)}(5x-2)$

$= -(1-D^2)^{-1}(5x-2)$

$= -[1 + D^2 + \cdots](5x-2)$

$= -(5x-2)$

$\therefore$P.I. $= -5x + 2$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^x + c_2e^{-x} - 5x + 2$

Example53 Solve the differential equation: $(D^4 + 4D^2)y = x^2 + 1$

Solution: Auxiliary equation is: $m^4 + 4m^2 = 0$

$\Rightarrow m^2(m^2+4) = 0$

$\Rightarrow m = 0, 0, \pm 2i$

C.F. $= (c_1 + c_2x) + (c_3\cos 2x + c_4\sin 2x)$

P.I. $= \frac{1}{f(D)}\, F(x) = \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}(x^2+1)$

$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1}(x^2+1)$

$= \frac{1}{4D^2}\left[1-\frac{D^2}{4}+\cdots\right](x^2+1)$

$= \frac{1}{4D^2}\left(x^2+1-\frac{1}{2}\right)$

$= \frac{1}{4D^2}\left(x^2+\frac{1}{2}\right)$

$= \frac{1}{4D}\int\left(x^2+\frac{1}{2}\right)dx$

$= \frac{1}{4D}\left(\frac{x^3}{3}+\frac{x}{2}\right)$

$= \frac{1}{4}\int\left(\frac{x^3}{3}+\frac{x}{2}\right)dx$

$\therefore$P.I $= \frac{1}{4}\left(\frac{x^4}{12}+\frac{x^2}{4}\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1+c_2x)+(c_3\cos 2x+c_4\sin 2x)+\frac{1}{4}\left(\frac{x^4}{12}+\frac{x^2}{4}\right)$

Example54 Solve the differential equation: $(D^2-6D+9)y = 1+x+x^2$

Solution: Auxiliary equation is: $m^2-6m+9=0$

$\Rightarrow (m-3)^2 = 0 \Rightarrow m = 3,3$

C.F. $= e^{3x}(c_1+c_2x)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-6D+9}(1+x+x^2)$

$= \frac{1}{9\left(1-\frac{2D}{3}+\frac{D^2}{9}\right)}(1+x+x^2)$

$= \frac{1}{9}\left(1-\left(\frac{2D}{3}-\frac{D^2}{9}\right)\right)^{-1}(1+x+x^2)$

$= \frac{1}{9}\left[1+\left(\frac{2D}{3}-\frac{D^2}{9}\right)+\left(\frac{2D}{3}-\frac{D^2}{9}\right)^2+\cdots\right](1+x+x^2)$

$= \frac{1}{9}\left[1+\frac{2D}{3}-\frac{D^2}{9}+\frac{4D^2}{9}+\cdots\right](1+x+x^2)$

$= \frac{1}{9}\left[1+\frac{2D}{3}+\frac{D^2}{3}+\cdots\right](1+x+x^2)$

$= \frac{1}{9}\left(1+x+x^2+0+\frac{2}{3}+\frac{4x}{3}+0+0+\frac{2}{3}\right)$

$\therefore$ P.I $= \frac{1}{9}\left(\frac{7}{3}+\frac{7x}{3}+x^2\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = e^{3x}(c_1+c_2x)+\frac{1}{9}\left(\frac{7}{3}+\frac{7x}{3}+x^2\right)$

Case IV: When $F(x) = e^{ax}g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)}e^{ax}g(x) = e^{ax}\left(\frac{1}{f(D+a)}g(x)\right)$

Example55 Solve the differential equation: $(D^2+2)y = x^2e^{3x}$

Solution: Auxiliary equation is: $m^2+2=0$

$\Rightarrow m^2 = -2 \Rightarrow m = \pm\sqrt{2}i$

C.F. $= (c_1\cos(\sqrt{2}\,x) + c_2\sin(\sqrt{2}\,x))$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2+2}x^2e^{3x}$

$= e^{3x}\frac{1}{(D+3)^2+2}x^2$

$= e^{3x}\frac{1}{D^2+6D+11}x^2$

$= \frac{e^{3x}}{11}\frac{1}{\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)}x^2$

$= \frac{e^{3x}}{11}\left(1+\left(\frac{6D}{11}+\frac{D^2}{11}\right)\right)^{-1}x^2$

$= \frac{e^{3x}}{11}\left[1-\left(\frac{6D}{11}+\frac{D^2}{11}\right)+\left(\frac{6D}{11}+\frac{D^2}{11}\right)^2+\cdots\right]x^2$

$= \frac{e^{3x}}{11}\left[1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}+\cdots\right]x^2$

$= \frac{e^{3x}}{11}\left[1-\frac{6D}{11}+\frac{25D^2}{121}+\cdots\right]x^2$

$= \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$

$\therefore P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1\cos(\sqrt{2}\,x) + c_2\sin(\sqrt{2}\,x)) + \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$

Example56 Solve the differential equation: $(D^3+1)y = e^{2x}\sin x$

Solution: Auxiliary equation is: $m^3+1=0$

$\Rightarrow m^3 = -1$

$\Rightarrow m = -1, \frac{1\pm\sqrt{3}i}{2}$

C.F. $= c_1e^{-x} + e^{\frac{x}{2}}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^3+1}e^{2x}\sin x$

$= e^{2x}\frac{1}{(D+2)^3+1}\sin x$

$= e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x$

$= e^{2x}\frac{1}{-D-6+12D+9}\sin x$, Putting $D^2 = -1$

$= e^{2x}\frac{1}{11D+3}\sin x$

$= e^{2x}\frac{11D-3}{121D^2-9}\sin x$, Rationalizing the denominator

$= -\frac{e^{2x}}{130}(11D-3)\sin x$, Putting $D^2 = -1$

$\therefore$ P.I $= -\frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right) - \frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Example57 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = x\ \sinh x$

Solution: $\Rightarrow (D^2 - 4)y = x\ \sinh x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. $= c_1e^{2x} + c_2e^{-2x}$

P.I. $= \frac{1}{f(D)}F(x)$

$= \frac{1}{f(D)}(x\ \sinh x)$

$= \frac{1}{D^2-4}\left(x\frac{e^x-e^{-x}}{2}\right) \qquad \because \sinh x = \frac{e^x-e^{-x}}{2}$

$= \frac{1}{D^2-4}\left(x\frac{e^x}{2} - x\frac{e^{-x}}{2}\right)$

$= \frac{e^x}{2}\frac{1}{(D+1)^2-4}x - \frac{e^{-x}}{2}\frac{1}{(D-1)^2-4}x$

$= \frac{e^x}{2}\frac{1}{(D^2+2D-3)}x - \frac{e^{-x}}{2}\frac{1}{D^2-2D-3}x$

$= \frac{e^x}{2}\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{2D}{3}\right)}x - \frac{e^{-x}}{2}\frac{1}{-3\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)}x$

$= -\frac{e^x}{6}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1}x + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]^{-1}x$

$= -\frac{e^x}{6}\left(1+\frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1-\frac{2D}{3}\right)x$

$= -\frac{e^x}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$

$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right)$

$\therefore$ P.I.$= -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Example58 Solve the differential equation: $(D^2+1)y = x^2\sin 2x$

Solution: Auxiliary equation is: $m^2+1=0$

$\Rightarrow m^2=-1$

$\Rightarrow m=\pm i$

C.F. $= c_1\cos x+\ c_2\sin x$

P.I. $=\frac{1}{f(D)}\ F(x)=\frac{1}{D^2+1}x^2\sin 2x$

$=$ Imaginary part of $\frac{1}{D^2+1}x^2e^{i2x}$

Now $\frac{1}{D^2+1}x^2e^{i2x}=e^{i2x}\frac{1}{(D+2i)^2+1}x^2$

$$=e^{i2x}\frac{1}{D^2+4i^2+\ 4iD+1}x^2$$

$$=e^{i2x}\frac{1}{D^2+\ 4iD-3}x^2$$

$$=e^{i2x}\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{4iD}{3}\right)}x^2$$

$$=\frac{-e^{i2x}}{3}\left[1-\left(\frac{D^2}{3}+\frac{4iD}{3}\right)\right]^{-1}x^2$$

$$=\frac{-e^{i2x}}{3}\left[1+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)^2\right]x^2$$

$$=\frac{-e^{i2x}}{3}\left[1+\frac{D^2}{3}+\frac{4iD}{3}+\frac{16i^2D^2}{9}\right]x^2$$

$$=\frac{-e^{i2x}}{3}\left[1-\frac{13D^2}{9}+\frac{4iD}{3}\right]x^2$$

$$=\frac{-e^{i2x}}{3}\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$$=-\frac{1}{3}(\cos 2x+i\sin 2x)\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$\therefore$ P.I. $=$ Imaginary part of $\frac{1}{D^2+1}x^2e^{i2x}=-\frac{1}{3}\left(\frac{8x}{3}\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right)$

$=-\frac{8x}{9}\cos 2x+\frac{1}{27}(26-9x^2)\sin 2x$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =c_1\cos x+\ c_2\sin x\ -\frac{8x}{9}\cos 2x+\frac{1}{27}(26-9x^2)\sin 2x$

Example59 Solve the differential equation: $(D^2-4D+4)y=x^2e^{2x}\sin 2x$

Solution: Auxiliary equation is: $m^2-4m+4=0$

$\Rightarrow (m-2)^2$

$\Rightarrow m=2,2$

C.F. $=(c_1+c_2x)e^{2x}$

P.I. $=\frac{1}{f(D)}\ F(x)=\frac{1}{D^2-4D+4}x^2e^{2x}\sin 2x$

$=e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}x^2\sin 2x$

$=e^{2x}\frac{1}{D^2}x^2\sin 2x$

$= e^{2x}\frac{1}{D}\int x^2 \sin 2x\,dx$

$= e^{2x}\frac{1}{D}\left[(x^2)\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right)\right]$

$= e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]$

$= e^{2x}\left[-\frac{1}{2}\int x^2\cos 2x + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{4}\int \cos 2x\,dx\right]$

$= e^{2x}\left[-\frac{1}{2}\left[(x^2)\left(\frac{\sin 2x}{2}\right) - (2x)\left(\frac{-\cos 2x}{4}\right) + (2)\left(\frac{-\sin 2x}{8}\right)\right] + \frac{1}{2}\left[(x)\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right] + \frac{1}{4}\left(\frac{\sin 2x}{2}\right)\right]$

$\therefore$ P.I. $= e^{2x}\left[\frac{-x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ = (c_1 + c_2x)e^{2x} + e^{2x}\left[\frac{-x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$

Case V: When $F(x) = x\,g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)}\left(x\,g(x)\right) = x\frac{1}{f(D)}g(x) + \left(\frac{d}{dD}\frac{1}{f(D)}\right)g(x)$

Example60 Solve the differential equation: $(D^2+9)y = x\cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow m^2 = -9$

$\Rightarrow m = \pm 3i$

C.F. $= (c_1\cos 3x + c_2\sin 3x)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2+9}x\cos x$

$= x\frac{1}{D^2+9}\cos x + \frac{-2D}{(D^2+9)^2}\cos x$

$= x\frac{1}{-1+9}\cos x + \frac{-2D}{(-1+9)^2}\cos x,$ Putting $D^2 = -1$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$\therefore$ P.I. $= \frac{x\cos x}{8} + \frac{\sin x}{32}$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ = c_1\cos 3x + c_2\sin 3x + \frac{x\cos x}{8} + \frac{\sin x}{32}$

Example61 Solve the differential equation: $(D^2-1)y = x\sin x + (1+x^2)e^x$

Solution: Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. $= c_1e^x + c_2e^{-x}$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-1}(x\sin x + (1+x^2)e^x)$

$= \frac{1}{D^2-1} x \sin x + \frac{1}{D^2-1}(1+x^2)e^x$

Now, $\frac{1}{D^2-1} x \sin x = x \frac{1}{D^2-1} \sin x + \frac{-2D}{(D^2-1)^2} \sin x$

$= x \frac{1}{-1-1} \sin x + \frac{-2D}{(-1-1)^2} \sin x,$ Putting $D^2 = -1$

$= -\frac{1}{2}(x \sin x + \cos x)$

Also, $\frac{1}{D^2-1}(1+x^2)e^x = e^x \frac{1}{(D+1)^2-1}(1+x^2)$

$= e^x \frac{1}{D^2+2D}(1+x^2)$

$= e^x \frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$

$= e^x \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$

$= e^x \frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{4}\right](1+x^2)$

$= e^x \frac{1}{2D}\left[1+x^2-x+\frac{1}{2}\right]$

$= e^x \frac{1}{2D}\left[x^2-x+\frac{3}{2}\right]$

$= \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

$\therefore$ P.I. $= -\frac{1}{2}(x \sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Case VI: When $F(x)$ is any general function of x not covered in shortcut methods I to V above

Resolve $f(D)$ into partial fractions and use the rule:

$\frac{1}{D+a} F(x) = e^{-ax} \int e^{ax} F(x)\, dx$

Example62 Solve the differential equation: $(D^2+3D+2)y = e^{e^x}$

Solution: Auxiliary equation is: $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0$

$\Rightarrow m = -1, -2$

C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+3D+2} e^{e^x}$

$= \frac{1}{(D+1)(D+2)} e^{e^x}$

$= \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{e^x}$

$= e^{-x} \int e^x e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} dx$

$= e^{-x} \int D e^{e^x} dx - e^{-2x} \int e^x D e^{e^x} dx$

$= e^{-x}e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int e^x e^{e^x} dx\right]$, Integrating 2nd term by parts

$= e^{-x}e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int De^{e^x} dx\right]$

$= e^{-x}e^{e^x} - e^{-2x}\left[e^x e^{e^x} - e^{e^x}\right]$

$\therefore$ P.I. $= e^{-2x}e^{e^x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}e^{e^x}$

Example63 Solve the differential equation: $(D^2 + 4)y = \tan 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+4} \tan 2x$

$= \frac{1}{(D-2i)(D+2i)} \tan 2x$

$= \frac{1}{4i}\left(\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right) \tan 2x$

P.I. $= \frac{1}{4i}\left(\frac{1}{D-2i} \tan 2x\right) - \frac{1}{4i}\left(\frac{1}{D+2i} \tan 2x\right) \quad \cdots ①$

Now $\frac{1}{D-2i} \tan 2x = e^{2ix} \int e^{-2ix} \tan 2x \, dx$

$= e^{2ix} \int (\cos 2x - i \sin 2x) \tan 2x \, dx$

$= e^{2ix} \int (\sin 2x - i \frac{sin^2 2x}{cos 2x}) \, dx$

$= e^{2ix} \int \left(\sin 2x - i \frac{1-cos^2 2x}{cos 2x}\right) dx$

$= e^{2ix} \int (\sin 2x - i \sec 2x + i \cos 2x) \, dx$

$= e^{2ix} \left(-\frac{1}{2}\cos 2x - \frac{i}{2} log|\sec 2x + \tan 2x| + \frac{i}{2}\sin 2x\right)$

$\therefore \frac{1}{D-2i} \tan 2x = e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2} log|\sec 2x + \tan 2x|\right) \ldots ②$

Replacing i by $-i$

$\frac{1}{D+2i} \tan 2x = e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2} log|\sec 2x + \tan 2x|\right) \ldots ③$

Using ②and ③ in ①

P.I. $= \frac{1}{4i}\left[e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2} log|\sec 2x + \tan 2x|\right)\right]$

$-\frac{1}{4i}\left[e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2} log|\sec 2x + \tan 2x|\right)\right]$

$= \frac{1}{4i}\left[-\frac{1}{2} - \frac{i}{2}e^{2ix} log|\sec 2x + \tan 2x| + \frac{1}{2} - \frac{i}{2}e^{-2ix} log|\sec 2x + \tan 2x|\right]$

$= \frac{1}{4i}\left[-i\frac{e^{2ix} + e^{-2ix}}{2} log|\sec 2x + \tan 2x|\right]$

$\therefore$ P.I. $= -\frac{1}{4}[\cos 2x \, log|\sec 2x + \tan 2x|]$

Complete solution is: $y =$ C.F. + P.I.

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$

9.4 Differential Equations Reducible to Linear Form with Constant Coefficients

Some special type of homogenous and non-homogeneous linear differential equations with variable coefficients after suitable substitutions can be reduced to linear differential equations with constant coefficients.

9.4.1 Euler–Cauchy Differential Equation

The differential equation of the form:

$k_0 x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_{n-1} x \frac{dy}{dx} + k_n y = F(x)$

is called Euler–Cauchy Equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$x = e^t \Rightarrow \log x = t$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} = Dy$, where $D \equiv \frac{d}{dt}$

Similarly, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$, $x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Example64 Solve the differential equation:

$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 13\, cos(log\, x), x > 0$ ....①

Solution: This is a Euler–Cauchy Equation with variable coefficients.

Putting $x = e^t \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y$ and $x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$

$\therefore$ ① may be rewritten as

$(D(D-1)(D-2) + 3D(D-1) + D + 8)y = 13 \cos t$

$\Rightarrow (D^3 + 8)y = 13 \cos t\,, D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^3 + 8 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm \sqrt{3}i$

C.F. $= c_1 e^{-2t} + e^t\left(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t\right)$

$= \frac{c_1}{x^2} + x\left(c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)\right)$

P.I. $= \frac{1}{f(D)} F(x) = 13\frac{1}{D^3+8}\cos t$

$= 13\frac{1}{-D+8}\cos t$, Putting $D^2 = -1$

$= 13\frac{(8+D)}{64-D^2}\cos t = 13\frac{(8+D)}{65}\cos t$ Putting $D^2 = -1$

$\therefore$ P. I. $= \frac{1}{5}(8\cos t + D\cos t)$

$= \frac{1}{5}(8\cos t - \sin t)$

$= \frac{1}{5}(8\cos(\log x) - \sin(\log x))$

Complete solution is: $y \;=\;$ C.F. + P.I

$$\Rightarrow y \;=\; \frac{c_1}{x^2} + x\big(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x\big) + \frac{1}{5}(8\cos(\log x) - \sin(\log x))$$

Example65 Solve the differential equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y \;= \frac{x^3}{1+x^2}$ ....①

Solution: This is a Euler–Cauchy Equation with variable coefficients.

Putting $x = e^t \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ ① may be rewritten as

$(D(D-1) + D - 1)y = \frac{e^{3t}}{1+e^{2t}}$

$\Rightarrow (D^2 - 1)y = \frac{e^{3t}}{1+e^{2t}},\; D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\; m = \pm 1$

C.F. $= c_1e^{-t} + c_2e^t \;=\; \frac{c_1}{x} + c_2x$

P.I. $= \frac{1}{f(D)}\,F(x) \;=\; \frac{1}{D^2-1}\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{(D-1)(D+1)}\frac{e^{3t}}{1+e^{2t}} = \frac{1}{2}\left(\frac{1}{(D-1)} - \frac{1}{(D+1)}\right)\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{2}\left(\frac{1}{(D-1)}\frac{e^{3t}}{1+e^{2t}} - \frac{1}{(D+1)}\frac{e^{3t}}{1+e^{2t}}\right)$

$= \frac{1}{2}\left(e^t\int e^{-t}\frac{e^{3t}}{1+e^{2t}}dt - e^{-t}\int e^t\frac{e^{3t}}{1+e^{2t}}dt\right) \because \frac{1}{D+a}\,F(x) = e^{-ax}\int e^{ax}F(x)\,dx$

$= \frac{1}{2}\left(e^t\int\frac{e^{2t}}{1+e^{2t}}dt - e^{-t}\int\frac{e^{4t}}{1+e^{2t}}dt\right)$

Put $e^{2t} = u \;\Rightarrow 2e^{2t}dt = du$

$\therefore$ P.I $= \frac{1}{4}\left(e^t\int\frac{1}{1+u}du - e^{-t}\int\frac{u}{1+u}du\right)$

$= \frac{1}{4}\left(e^t\log(1+u) - e^{-t}\int\frac{1+u-1}{1+u}du\right)$

$= \frac{1}{4}\left(e^t\log(1+u) - e^{-t}\int\left(1 - \frac{1}{1+u}\right)du\right)$

$= \frac{1}{4}(e^t\log(1+u) - e^{-t}(u - \log(1+u))$

$= \frac{1}{4}(e^t\log(1+e^{2t}) - e^{-t}(e^{2t} - \log(1+e^{2t}))$

$= \frac{1}{4}\left(x\log(1+x^2) - \frac{1}{x}(x^2 - \log(1+x^2)\right)$

$= \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

Complete solution is: $y \;=\;$ C.F. + P.I

$$\Rightarrow y \;=\; \frac{c_1}{x} + c_2 x + \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$$

$$\Rightarrow y \;=\; \frac{c_1}{x} + c_3 x + \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1+x^2) \;\; , c_3 = c_2 - \frac{1}{4}$$

Example66 Solve the differential equation:

$x^2D^2 - 2xD - 4y \;=\; x^2 + 2\log x\,, \quad x > 0 \qquad ....①$

Solution: This is a Euler–Cauchy Equation with variable coefficients.

Putting $x = e^t \therefore \log x = t$

$\Rightarrow xD = \delta y, x^2D^2 \;=\; \delta(\delta-1)\mathrm{y} \;\; , \delta \equiv \frac{d}{dt}$

$\therefore$ ① may be rewritten as

$(\delta(\delta-1) - 2\delta - 4)y = \; e^{2t} + 2t$

$\Rightarrow (\delta^2 - 3\delta - 4)y = e^{2t} + 2t$

Auxiliary equation is: $m^2 - 3m - 4 = 0$

$\Rightarrow (m+1)(m-4) = 0 \qquad \therefore \; m = -1, 4$

C.F. $= c_1 e^{-t} + c_2 e^{4t} = \frac{c_1}{x} + c_2 x^4$

P.I. $= \frac{1}{f(\delta)}\, F(x) \;=\; \frac{1}{\delta^2 - 3\delta - 4}(e^{2t} + 2t)$

$= \frac{1}{\delta^2-3\delta-4}e^{2t} + \frac{1}{\delta^2-3\delta-4}2t$

$= \frac{1}{-6}e^{2t} + 2\frac{1}{-4\left(1-\frac{\delta^2}{4}+\frac{3\delta}{4}\right)}t$ Putting $\delta = 2$ in the 1st term

$$= \frac{-e^{2t}}{6} - \frac{1}{2}\left(1 - \left(\frac{\delta^2}{4} - \frac{3\delta}{4}\right)\right)^{-1} t$$

$$= \frac{-e^{2t}}{6} - \frac{1}{2}\left[1 + \frac{\delta^2}{4} - \frac{3\delta}{4} + \cdots\right]t \;\; = \frac{-e^{2t}}{6} - \frac{1}{2}\left[t - \frac{3}{4}\right]$$

$\therefore$ P. I. $= \frac{-x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$

Complete solution is: $y \;=\;$ C.F. + P.I

$$\Rightarrow y \;= \frac{c_1}{x} + c_2 x^4 - \frac{x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

9.5 Method of Variation of Parameters for Finding Particular Integral

Method of Variation of Parameters enables us to find the solution of 2^{nd} and higher order differential equations with constant coefficients as well as equations with variable coefficients.

Working rule: Consider a 2^{nd} order linear differential equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy \;= F(x) \;.......①$$

1. Find complimentary function given as: C.F. $= c_1 y_1 + c_2 y_2$, where y_1 and y_2 are two linearly independent solutions of ①
2. Calculate $W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$, W is called Wronskian of y_1 andy_2

3. Compute $u_1 = -\int \frac{y_2 F(x)}{W} dx,\ u_2 = \int \frac{y_1 F(x)}{W} dx$
4. Find P.I. $= u_1 y_1 + u_2 y_2$
5. Complete solution is given by: $y = \text{C.F.} + \text{P.I}$

Note: Method is commonly used to solve 2^{nd} order differential equations, but it can be extended to solve differential equations of higher orders.

Example67 Solve the differential equation: $\frac{d^2y}{dx^2} + y = cosec\, x$ using method of variation of parameters.

Solution: $\Rightarrow (D^2+1)y = \operatorname{cosec} x$

Auxiliary equation is: $(m^2+1) = 0 \quad \therefore\ m = \pm i$

C.F. $= c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$

$\therefore\ y_1 = \cos x$ and $y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \sin x \operatorname{cosec} x\ dx = -\int 1 dx = -x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \cos x \operatorname{cosec} x\ dx = \int \cot x\, dx = log|\sin x|$$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$= -x\cos x + \sin x\ log|\sin x|$

Complete solution is: $y = \text{C.F.} = \text{P.I}$

$\Rightarrow y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x\ log|\sin x|$

Example68 Solve the differential equation: $(D^2 - 2D + 1)y = e^x$ using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - 2m + 1) = 0 \quad \therefore\ m = 1,1$

C.F. $= (c_1 + c_2 x)e^x = c_1 e^x + c_2 x\, e^x = c_1 y_1 + c_2 y_2$

$\therefore\ y_1 = e^x$ and $y_2 = x\, e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x\, e^x \\ e^x & x\, e^x + e^x \end{vmatrix} = e^{2x}$$

$$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{xe^x e^x}{e^{2x}} dx = -\int x dx = -\frac{x^2}{2}$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x}{e^{2x}} dx = \int 1 dx = x$$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2 = -\frac{x^2}{2} e^x + x^2 e^x = \frac{x^2}{2} e^x$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$\Rightarrow y = (c_1 + c_2 x)e^x + \frac{x^2}{2} e^x$

Example69 Solve the differential equation: $\frac{d^2y}{dx^2} + 4y = x\, sin\, 2x$ using method of variation of parameters.

Solution: $\Rightarrow (D^2+4)y = x \sin 2x$

Auxiliary equation is: $(m^2+4)=0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\frac{1}{2}\int x \sin^2 2x \ dx = -\frac{1}{4}\int x(1-\cos 4x)\, dx$

$= -\frac{1}{4}\left[\frac{x^2}{2} - \left[(x)\left(\frac{\sin 4x}{4}\right) - (1)\left(-\frac{\cos 4x}{16}\right)\right]\right] = \left[-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64}\right]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2}\int x \sin 2x \cos 2x \ dx = \frac{1}{4}\int x \sin 4x \ dx$

$= \frac{1}{4}\left[(x)\left(-\frac{\cos 4x}{4}\right) - (1)\left(-\frac{\sin 4x}{16}\right)\right] = \left[-\frac{x \cos 4x}{16} + \frac{\sin 4x}{64}\right]$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$= \left[-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64}\right] + \sin 2x \left[-\frac{x \cos 4x}{16} + \frac{\sin 4x}{64}\right]$

$= \frac{x}{16}(\sin 4x \cos 2x - \cos 4x \sin 2x) + \frac{1}{64}(\cos 4x \cos 2x + \sin 4x \sin 2x) - \frac{x^2}{8}\cos 2x$

$= \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Example70 Solve the differential equation:$(D^2 - D - 2)y = e^{(e^x + 3x)}$ using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - m - 2) = 0 \ \therefore \ m = -1, 2$

C.F. $= c_1 e^{-x} + c_2 e^{2x} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{-x}$ and $y_2 = e^{2x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{e^{2x} e^{(e^x+3x)}}{3e^x} dx = -\int \frac{e^{2x} e^{e^x} e^{3x}}{3e^x} dx$

$= -\frac{1}{3}\int e^{4x} e^{e^x} dx$, Putting $e^x = t \Rightarrow e^x dx = t dt$

$u_1 = -\frac{1}{3}\int t^3 e^t dt = -\frac{1}{3}[(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t)]$

$\Rightarrow u_1 = -\frac{e^{e^x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{-x} e^{(e^x+3x)}}{3e^x} dx = \int \frac{e^{-x} e^{e^x} e^{3x}}{3e^x} dx \ \frac{1}{3}\int e^x e^{e^x} dx = \frac{e^{e^x}}{3}$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$= -\frac{e^{e^x} e^{-x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6] + \frac{e^{e^x} e^{2x}}{3}$

$= \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1e^{-x} + c_2e^{2x} + \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Example71 Given that $y_1 = x$ and $y_2 = \frac{1}{x}$ are two linearly independent solutions of the differential equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x, x \neq 0$. Find the particular integral and general solution using method of variation of parameters.

Solution: Rewriting the equation as: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$

Given that $y_1 = x$ and $y_2 = \frac{1}{x}$

$\therefore$ C.F. $= c_1y_1 + c_2y_2 = c_1x + \frac{c_2}{x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}$$

$u_1 = -\int \frac{y_2F(x)}{W}dx = \int \frac{1}{x}.\frac{1}{x}.\frac{x}{2}\,dx = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}\log x$

$u_2 = \int \frac{y_1F(x)}{W}dx = -\int x.\frac{1}{x}.\frac{x}{2}\,dx = -\frac{x^2}{4}$

$\therefore$ P.I. $= u_1y_1 + u_2y_2 = \frac{x}{2}\log x - \frac{x}{4}$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1x + \frac{c_2}{x} + \frac{x}{2}\log x - \frac{x}{4}$

Example72 Solve the differential equation: $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2\log x$ using method of variation of parameters.

Solution: This is a Euler-Cauchy linear differential equation with variable coefficients.

Putting $x = e^t \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Given differential equation may be rewritten as

$(D(D-1) - 4D + 6)y = te^{2t}$

$\Rightarrow (D^2 - 5D + 6)y = te^{2t}$

Auxiliary equation is: $(m-2)(m-3) = 0 \therefore m = 2,3$

C.F. $= c_1e^{2t} + c_2e^{3t} = c_1y_1 + c_2y_2$

$\therefore y_1 = e^{2t}$ and $y_2 = e^{3t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$u_1 = -\int \frac{y_2F(t)}{W}dt = -\int \frac{e^{3t}te^{2t}}{e^{5t}}\,dt = -\int t\,dt = -\frac{t^2}{2}$

$u_2 = \int \frac{y_1 F(t)}{W} dt = \int \frac{e^{2t} t e^{2t}}{e^{5t}} dt = \int t e^{-t} dt = [(t)(-e^{-t}) - (1)(e^{-t})] = -te^{-t} - e^{-t}$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2 = -\frac{t^2}{2} e^{2t} - (te^{-t} + e^{-t}) e^{3t}$

$= -\frac{t^2}{2} e^{2t} - te^{2t} - e^{2t} = -e^{2t}\left(\frac{t^2}{2} + t + 1\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 e^{2t} + c_2 e^{3t} - e^{2t}\left(\frac{t^2}{2} + t + 1\right)$

or $y = c_1 x^2 + c_2 x^3 - x^2\left(\frac{(\log x)^2}{2} + \log x + 1\right)$

$\Rightarrow y = c_3 x^2 + c_2 x^3 - \frac{x^2}{2}(\log x)^2 - x^2 \log x, \ c_3 = c_1 - 1$

10. Population Dynamics

A Population is the group of individuals of same species, and population dynamics is the study of population changes over time, which can be estimated by mathematical modelling.

Here we confine our study to two most practical growth models viz. Exponential and Logistic patterns.

I. Exponential Model

An exponential growth model is possible only if there are unlimited resources and the population can reproduce to its maximum capacity. This is generally not feasible under natural phenomena due to limited available resources. Exponential growth can be achieved if all favorable conditions are provided in a specific environment, for example, bacteria culture in a laboratory. Exponential growth can be represented by a J shaped curve as shown in *Figure4*. If $N(t)$ denotes the size of the population at any time t, then under normal circumstances rate of change of population is directly proportional to population itself i.e., $\frac{dN}{dt} \propto$ N .

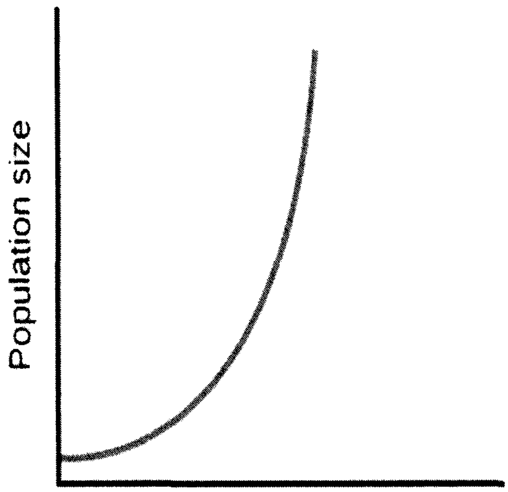


Figure 4

$\Rightarrow \frac{dN}{dt}$ = rN , where r is the relative growth rate

$\Rightarrow \frac{dN}{N} = rdt$

Integrating both sides, we have

$\int \frac{dN}{N} = \int rdt$

$\Rightarrow \log N = rt + \log c$

$\Rightarrow \log \frac{N}{c} = rt \Rightarrow e^{\log \frac{N}{c}} = e^{rt} \Rightarrow \frac{N}{c} = e^{rt}$

$\Rightarrow N = ce^{rt}$ is the required solution of the given differential equation.

If N_0 be the initial population at $t = 0$, then population at any time t is given by

$N(t) = N_0 e^{rt}$

Example73 The population of rabbits in a zoo by the end of year 2010 is 1125 and by the end of the year 2011 is 1242. Assuming population growth to follow exponential model by providing all necessary resources,

(a) Determine the relative growth rate

(b) Write the general equation for population dynamics in exponential model

(c) Calculate the time required for the population to be doubled

(d) Estimate the number of rabbits by the end of the year 2021

Solution: If N denotes the size of the population at any time t, r is the relative growth rate and $N_0(at\ t = 0)$ be the initial population by the end of year 2010. Then the general equation for population dynamics in exponential model is given by $N(t) = N_0e^{rt}$

$\Rightarrow N(t) = 1125e^{rt} \because N_0 = 1125$

(a) Population by the end of year 2011, i.e., at $t = 1$ is 1242

$\Rightarrow N(1) = 1242 = 1125e^r$

$\Rightarrow e^r = \frac{1242}{1125} = 1.104$

$\Rightarrow r = \ln 1.104 = 0.0989$

(b) The general equation for population dynamics is given by:

$N(t) = 1125e^{0.0989t}$

(c) For population to be doubled, i.e., $N(t) = 2250$

$\therefore 2250 = 1125e^{0.0989t}$

$\Rightarrow e^{0.0989t} = \frac{2250}{1125} = 2$

$\Rightarrow 0.0989t = \ln 2 = 0.6931$

$\Rightarrow t = \frac{0.6931}{0.0989} = 7.008$ years approximately

(d) We have $N(t) = 1125e^{0.0989t}$

$\therefore$ The population by the end of the year 2021, i.e., at $t = 11$ is given by

$N(11) = 1125e^{1.0879} = 1125\ (2.96803) = 3339$ approx.

Example74 A sample culture has initially P_0 bacteria. After five hours, the number of bacteria is measured to be $5P_0$. Determine the time required for number of bacteria to be ten times as of initial number.

Solution: The general equation for bacteria growth is given by

$P(t) = P_0e^{rt}$, where $P(0) = P_0$

Given $5P_0 = P_0e^{5r}$

$\Rightarrow e^{5r} = 5$

$\Rightarrow 5r = \ln 5 \Rightarrow r = \frac{1}{5}\ln 5$

$\therefore$The general equation for population dynamics is given by: $P(t) = P_0e^{\frac{t}{5}\ln 5}$

Now for bacteria to grow ten times of initial number

$$10P_0 = P_0e^{\frac{t}{5}\ln 5}$$

$\Rightarrow e^{\frac{t}{5}\ln 5} = 10$

$\Rightarrow \frac{t}{5}\ln 5 = \ln 10$

$\Rightarrow t = \frac{5\ln 10}{\ln 5} = 7.1534$ hours approx.

II. **Logistic Growth:** Logistic population growth is more practical approach under limited resources. Logistic growth takes place under all-natural phenomenon, when a population becomes almost constant, as it approaches a maximum quantity imposed by limited resources. Logistic growth produces an S-shaped curve (*Figure5*), where L is the carrying capacity of the system. If $N(t)$ denotes the size of the population at any time t and$r$ is the relative growth rate in the logistic growth model, then the rate of growth of population may be defined by the differential equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{L}\right) \ldots Ⓐ$$

$$\Rightarrow \frac{dN}{dt} = rN - \frac{r}{L}N^2 \ldots Ⓑ$$

Equation Ⓑ implies that if L is very large compared to N, $\frac{r}{L}N^2 \to 0$ and hence $\frac{dN}{dt} \approx rN$

Thus, any population follows exponential growth model for small population number. This is called as Malthus's law.

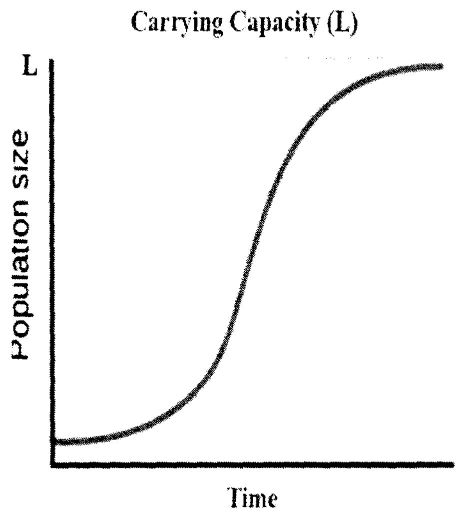


Figure 5

Now Ⓐ $\Rightarrow \frac{dN}{N\left(1-\frac{N}{L}\right)} = rdt$

$\Rightarrow \frac{LdN}{N(L-N)} = rdt$

Integrating both sides

$\int \frac{L}{N(L-N)} dN = \int rdt$

$\Rightarrow -\int \left(\frac{1}{N-L} - \frac{1}{N}\right) dN = \int rdt$

$\Rightarrow \ln(N) - \ln(N-L) = rt + \ln c,$

c is an arbitrary constant

$\Rightarrow \ln \frac{N}{c(N-L)} = rt$

$\Rightarrow \frac{N}{(N-L)} = ce^{rt}$

$\Rightarrow (N-L) = -Nbe^{-rt}$ putting $\frac{1}{c} = -b$

$\Rightarrow L = N + Nbe^{-rt} = N(1 + be^{-rt})$

$\Rightarrow N(t) = \frac{L}{(1+be^{-rt})}$

Example75 100 fishes of an exotic species were released in a large fish aquarium of a museum having a maximum capacity of 1100 fishes. After seven months, there were 220 fishes in the aquarium. Assuming logistic growth,

(a)Write a general equation that describes the population $N(t)$at time t.

(b) How many fishes will be there in the aquarium after one year?

(c) In how many months the fish population can reach 500?

Solution: (a) The general equation that describes the population $N(t)$at time t, assuming logistic growth is $N(t) = \frac{L}{(1+be^{-rt})}$ ...①

Here $N(0) = 100$ and $L = 1100$

Putting $t = 0$, ① $\Rightarrow 100 = \frac{1100}{(1+b)}$

$\Rightarrow (1 + b) = 11 \Rightarrow \text{b} = 10$

$\Rightarrow N(t) = \frac{1000}{(1+10e^{-rt})}$ ...②

Again, given that $N(7) = 220$

Putting $t = 7$, ② $\Rightarrow 220 = \frac{1100}{(1+10e^{-7r})}$

$\Rightarrow (1 + 10e^{-7r}) = 5$

$\Rightarrow e^{-7r} = 0.4 \Rightarrow -7r = \ln 0.4$

$\Rightarrow r = \frac{\ln 0.4}{-7} = \frac{-0.9163}{-7} = 0.1309$

∴The general equation that describes the population $N(t)$at time t is given by:

$N(t) = \frac{1100}{(1+10e^{-0.1309t})}$

(b) Number of fishes in the aquarium after 12 months is given by

$N(12) = \frac{1100}{(1+10e^{-0.1309(12)})} = 357.2834$, i.e., 357 fishes approximately

(c) Here $N(t) = 500$, $t =?$

We have $N(t) = \frac{1100}{(1+10e^{-0.1309t})}$

$\Rightarrow 500 = \frac{1100}{(1+10e^{-0.1309t})}$

$\Rightarrow (1 + 10e^{-0.1309t}) = 2.2$

$\Rightarrow e^{-0.1309t} = 0.11$

$\Rightarrow -0.1309t = \ln 0.11$

$\Rightarrow t = \frac{\ln 0.11}{-0.1309} = \frac{-2.2073}{-0.1309} = 16.8625$ months, i.e., nearly 17 months

11. Orthogonal Trajectories

Orthogonal trajectories are the curves that are perpendicular to a given family of curves. Let the family of curves $F(x, y, c)$ be the solution of a given differential equation $\frac{dy}{dx} = f(x, y)$; then the family of curves $G(x, y, d)$ represents the orthogonal trajectory of $F(x, y, c)$, if every curve of $G(x, y, d)$ is orthogonal (perpendicular) to each curve of the family $F(x, y, c)$.

Example76 Find the orthogonal trajectories of the families of parabolas $x = ky^2$

Solution: Given family of parabolas is $x = ky^2$ ...①

Differentiating both sides of ① with respect to x

$$1 = 2ky\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2ky} = \frac{y}{2x} = m\ , \ \because k = \frac{x}{y^2}$$

where m is the slope of the family of parabolas $x = ky^2$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = -\frac{2x}{y}$

Hence the differential equation of orthogonal trajectories is given by

$\frac{dy}{dx} = -\frac{2x}{y}$

$\Rightarrow ydy = -2xdx$

Integrating both sides, we have

$\int ydy = \int -2xdx$

$\Rightarrow \frac{y^2}{2} = -x^2 + c\ ,$ c is an arbitrary constant

$\Rightarrow 2x^2 + y^2 = d$, $d = 2c$ is an arbitrary constant

$\therefore 2x^2 + y^2 = d$

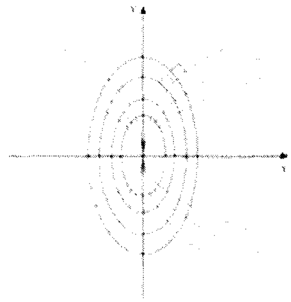


Figure 6

Figure 6 shows the required orthogonal trajectory, which is a family of ellipse.

Example77 Find the orthogonal trajectories of the family of circles $x^2 + y^2 = r^2$

Solution: Given family of circles is: $x^2 + y^2 = r^2$ ... 1

Differentiating both sides of 1 with respect to x

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} = m\ ,$

where m is the slope of the family of circles $x^2 + y^2 = r^2$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = \frac{y}{x}$

Hence the differential equation of orthogonal trajectories is given by

$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$

Integrating both sides, we have: $\int \frac{dy}{y} = \int \frac{dx}{x}$

$\Rightarrow \log y = \log x + \log c\ ,$ c is an arbitrary constant

$\Rightarrow \log y = \log cx \Rightarrow y = cx$

$\therefore y = cx$ is the required orthogonal trajectory

Example78 Find the orthogonal trajectories of the family of curves $y = e^{ax}$

Solution: Given family of curves is: $y = e^{ax}$ ... 1

Differentiating both sides of 1 with respect to x

$$\frac{dy}{dx} = ae^{ax} = m \ldots 2$$

where m is the slope of the family of curves $y = e^{ax}$

To remove the constant 'a' from m.

Taking natural log on both sides of equation 1

$\Rightarrow \ln y = ax \Rightarrow a = \frac{\ln y}{x}$

Substituting $e^{ax} = y$ and $a = \frac{\ln y}{x}$ in ②

$\Rightarrow m = \frac{y \ln y}{x}$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = -\frac{x}{y \ln y}$

Hence the differential equation of orthogonal trajectories is given by

$\frac{dy}{dx} = -\frac{x}{y \ln y}$

$\Rightarrow y \ln y \, dy = -x dx$

Integrating both sides, we have

$\int y \ln y \, dy = -\int x dx$ ...③

Now let $I = \int y \ln y \, dy$

$\Rightarrow I = y(y \ln y - y) - \left(I - \frac{y^2}{2}\right) + c_1$, $\because \int \ln y \, dy = y \ln y - y$

$\Rightarrow 2I = y(y \ln y - y) + \frac{y^2}{2} + c_1 \Rightarrow I = \frac{y}{2}(y \ln y - y) + \frac{y^2}{4} + c_1$

$\Rightarrow I = \frac{y^2 \ln y}{2} - \frac{y^2}{2} + \frac{y^2}{4} + c_1 \Rightarrow I = \frac{y^2 \ln y}{2} - \frac{y^2}{4} + c_1$ ...④

Using ④ in ③ $\Rightarrow \frac{y^2 \ln y}{2} - \frac{y^2}{4} + c_1 = -\frac{x^2}{2} + c_2$, c_1 and c_2 are arbitrary constants

$\therefore 2x^2 - y^2 + 2y^2 \ln y = c$ is the required orthogonal trajectory.

12. Modeling of Free Oscillations of a Mass-Spring System

Consider an undamped (unaffected by any external forces like air or friction) mass-spring system as shown in Figure7. Assume that the spring can resist both extension and compression with stiffness constant 'K'. The system is purely theoretical because it neglects damping forces resulting in uninterrupted free oscillations, which is impracticable. Any practical model will always have damping forces resulting in oscillations to stop eventually.

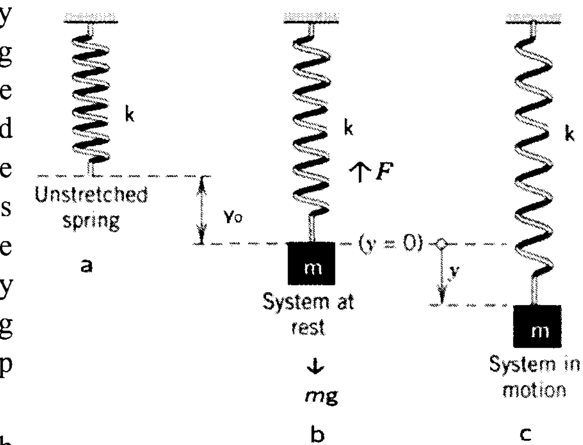


Figure 7

Suppose we have an elastic spring with stiffness constant 'K' hanging from a fixed surface (Figure7a).

We attach an object with mass 'm', resulting the string to stretch by a length 'y_0' after the system attains its rest position (Figure7b).

While in rest position, the gravitational force on the system acting in downward direction is mg, and an upwards restoring force 'F' also acts on the system due to initial displacement 'y_0'.

Let's define a reference frame, where the downward direction is the positive y-direction, primarily because gravitational forces pull the spring in the downwards direction.

Also, let $y = 0$ be the equilibrium position of the top surface of the suspended mass 'm' and upward is the negative y-direction.

Now, the upward restoring force 'F' on the system caused due to string stiffness is directly proportional to the initial displacement 'y_0', i.e., $F \propto y_0$

$\Rightarrow F = -ky_0 \ldots$ 1

Note that we have used negative sign because upward displacement is in negative y-direction as per our reference system.

Again, the downward gravitational force on the system 'mg', and the restoring force 'F' balance each other, so that the system is at rest in its equilibrium position.

$\Rightarrow F + mg = 0 \ldots$ 2

From 1 and 2, $-ky_0 + mg = 0$

$\Rightarrow k = \frac{mg}{y_0} \ldots$ 3, k is the spring stiffness measure.

Now, when we pull the mass 'm' in downwards direction (Figure7c) by a distance y,

as per Hooke's Law, the system produces an upward restoring force 'F' to resist the displacement 'y', such that $F \propto y$.

$\Rightarrow F = -ky$

$\Rightarrow m\frac{d^2y}{dt^2} = -ky, \because Force = mass \times$ acceleration

$\Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m}y = 0$

$\Rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0$, putting $\frac{k}{m} = \omega^2 \ldots$ 4

is the required differential equation of the mass-spring system.

Clearly, 4 is a homogeneous linear equation with constant coefficients.

$\Rightarrow (D^2 + \omega^2)y = 0$, $D = \frac{d}{dt}$

Auxiliary equation is: $p^2 + \omega^2 = 0 \Rightarrow p = \pm i\omega$

C.F. $= c_1 \cos \omega t + c_2 \sin \omega t$

$\therefore$ Solution of the mass-spring equation $\frac{d^2y}{dx^2} + \frac{k}{m}y = 0$is given by

$y = c_1 \cos \omega t + c_2 \sin \omega t,\ \omega^2 = \frac{k}{m}$

Clearly 4 is the differential equation of the Simple Harmonic Motion (SHM).

Also, period of oscillation (T) is given by $\frac{2\pi}{\omega}$ and frequency (ρ) is $\frac{1}{T}$

$\therefore T = 2\pi\sqrt{\frac{m}{k}}$ and $\rho = \frac{1}{2\pi} = \sqrt{\frac{k}{m}}$

Example 78 Solve the mass-spring equation $\frac{d^2y}{dt^2} + 64y = 0, y(0) = 4, y'(0) = 0.$

Also, interpret the initial value problem and find the period and the frequency of the simple harmonic motion.

Solution: Given the mass spring equation $\frac{d^2y}{dt^2} + 64y = 0 \ldots$ 1

Comparing with the equation $\frac{d^2y}{dt^2} + \frac{k}{m}y = 0, \frac{k}{m} = \omega^2 \ldots$ 2

$\Rightarrow \frac{k}{m} = \omega^2 = 64$

Solution of ② is given by $y = c_1 \cos \omega\, t + c_2 \sin \omega t,\ \omega^2 = \frac{k}{m}$

$\therefore$ Solution of ① is $y(t) = c_1 \cos 8\, t + c_2 \sin 8t$ ...③

Also, given $y(0) = 4$, i.e. ...④

Initial condition ④ implies that the mass is displaced downwards by 4 units to initiate the simple harmonic motion in the mass-spring system.

Given $y'(0) = 0$ ...⑤

Also, ⑤ implies that no initial velocity is given to the system.

Using ④ in ③ $\Rightarrow y(0) = 4 = c_1 \cos 0 + c_2 \sin 0 \Rightarrow c_1 = 4$

$\therefore$ ③ $\Rightarrow y(t) = 4 \cos 8\, t + c_2 \sin 8t$ ...⑥

Differentiating equation ⑥,$y'(t) = -32 \sin 8\, t + 8c_2 \cos 8t$ ...⑦

Using ⑤in ⑦ $\Rightarrow y'(0) = 0 = -32 \sin 0 + 8c_2 \cos 0 \Rightarrow c_2 = 0$

$\therefore y(t) = 4 \cos 8t$ is the required solution of the given mass -spring equation.

Also, the Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

Frequency $\rho = \frac{1}{T} = \frac{4}{\pi}$

Example79 An object of weight 4lbs stretches a string by 12 inches. Find the equation of motion if the spring is released from its equilibrium position with an upward velocity of 10 ft/s. Find the frequency and the period of the motion.

Solution: Let the equation of mass-spring system be given by $\frac{d^2y}{dt^2} + \frac{k}{m} y = 0$

Given that $W = mg = 4$ lbs, initial displacement $y_0 = 1$ft

Also, $y(0) = 0,\ y'(0) = -10$ ft/s

Now, $k = \frac{mg}{y_0} = \frac{4}{1} = 4$

Also, $W = mg$

$\Rightarrow 4 = 32m \because g = 32.14$ft/s^2 in British system

$\Rightarrow m = 1/8$

The differential equation of mass spring- system be given is given by

$\frac{d^2y}{dt^2} + \frac{k}{m} y = 0 \Rightarrow \frac{d^2y}{dt^2} + 32y = 0$ ...① $\because k = 4$ and $m = 1/8$

Solution of ① is $y = c_1 \cos \omega\, t + c_2 \sin \omega t,\ \omega^2 = \frac{k}{m} = 32$

$\Rightarrow y = c_1 \cos \sqrt{32}\, t + c_2 \sin \sqrt{32} t$ ... ②

Given $y(0) = 0,\ y'(0) = -10$, substituting in ②, we get $c_1 = 0$ and $c_2 = -\frac{10}{\sqrt{32}}$

$\Rightarrow y = -\frac{10}{\sqrt{32}} c_2 \sin \sqrt{32} t$ is the required equation of motion.

Also, the Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{32}} = \frac{2\pi}{2\sqrt{8}} = \frac{\pi}{\sqrt{8}}$

Frequency $\rho = \frac{1}{T} = \frac{\sqrt{8}}{\pi}$

13. Series Solutions and Special Functions

Consider the linear differential equation $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$, whose solution is given by

$y = c_1e^{3x} + c_2e^{5x} = c_1\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} + c_2\sum_{n=0}^{\infty}\frac{(5x)^n}{n!}$, which is a series solution.

Some differential equations with variable coefficients cannot be solved by usual methods, and we need to employ series solution method to find their solutions in terms of infinite convergent series.

13.1 Power Series

An infinite series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$, is called a power series about the point x_0; a_0, a_1, a_2 ... are arbitrary constants. The point $x = x_0$ is called center of the power series. The power series about the origin ($x_0 = 0$), is called standard power series and is given as: $\sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$

13.2 Series Solutions

Consider a second order linear differential equation:

$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \ldots (1)$

where $P(x)$, $Q(x)$ and $R(x)$ are functions of x or constants.

Ordinary and Singular Points

I. The point $x = x_0$ is called an ordinary point of equation (1), if $P(x_0) \neq 0$
II. The point $x = x_0$ is called a singular point of equation (1), if $P(x_0) = 0$
   a) If both $\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}$ and $\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$ are finite, then the point $x = x_0$ is called a regular singular point of equation (1).
   b) If either or both of $\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}$ and $\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$ are non-finite, then the point $x = x_0$ is called an irregular singular point of equation (1).

Note: Series solution does not exist if $x = x_0$ is an irregular singular point of a differential equation.

Example80 Find the ordinary points, regular singular, and irregular singular points of the differential equation $x^2(x-1)(x-2)\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + 2xy = 0$

Solution: Given $x^2(x-1)(x-2)\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + 2xy = 0 \ldots (1)$

Comparing with the differential equation $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

$P(x) = x^2(x-1)(x-2)$, $Q(x) = (x-1)$ and $R(x) = 2x$,

Now for x_0 to be an ordinary point, $P(x_0) \neq 0$

$\Rightarrow x^2(x-1)(x-2) \neq 0 \Rightarrow x_0 \in R - \{0,1,2\}$

$\therefore$ all real numbers except 0, 1 and 2 are ordinary points of differential equation①

Again, for singular points$P(x_0) = 0$

i.e., $x^2(x-1)(x-2) = 0 \Rightarrow x_0 \in \{0,1,2\}$

$\therefore$ 0, 1 and 2 are singular points of differential equation ①

Now$\lim\limits_{x\to 0}(x-0)\frac{(x-1)}{x^2(x-1)(x-2)} = \lim\limits_{x\to 0}\frac{1}{x(x-2)} = \infty$

$\Rightarrow x = 0$ is an irregular singular point of the differential equation①

Again, $\lim\limits_{x\to 1}(x-1)\frac{(x-1)}{x^2(x-1)(x-2)} = \lim\limits_{x\to 1}\frac{(x-1)}{x^2(x-2)} = 0$ i.e., finite

And $\lim\limits_{x\to 1}(x-1)^2\frac{2x}{x^2(x-1)(x-2)} = \lim\limits_{x\to 1}\frac{2(x-1)}{x(x-2)} = 0$, i.e., finite

$\Rightarrow x = 1$is a regular singular point of the differential equation①

Also, $\lim\limits_{x\to 2}(x-2)\frac{(x-1)}{x^2(x-1)(x-2)} = \lim\limits_{x\to 2}\frac{1}{x^2} = \frac{1}{4}$ i.e., finite

And $\lim\limits_{x\to 2}(x-2)^2\frac{2x}{x^2(x-1)(x-2)} = \lim\limits_{x\to 2}\frac{2(x-2)}{x(x-1)} = 0$, i.e., finite

$\Rightarrow x = 2$ is a regular singular point of the differential equation①

13.3 Algorithm to find series solution

Algorithm to find series solution when $x = 0$ is an ordinary point of equation ①, i.e., $P(0) \neq 0$

Step1: Assume the solution of equation ① as

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$ ...②

Step2: Differentiate ② with respect to x to find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Step3: Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the differential equation ①

Step4: As R.H.S. is zero, equate to zero the coefficients of different powers of x, particularly x^r in most cases to find a recurrence relation between the coefficients.

Step5: Substitute the values of a_0, a_1, a_2, a_3 in ② to get the required solution.

Example81 Find the power series solution about $x = 0$ for the differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Solution: Given $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ ...①

Let the solution of equation ① be given as

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$ ...②

Differentiating ② with respect to x, $\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$ ...③

Again differentiating ③ with respect to x, $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$ ...④

Substituting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ②, ③ and ④ in equation ①

$\Rightarrow (1-x^2)[\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}] - 2x[\sum_{r=1}^{\infty} a_r r x^{r-1}] + 2[\sum_{r=0}^{\infty} a_r x^r] = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r - 2\sum_{r=0}^{\infty} a_r r x^r + 2\sum_{r=0}^{\infty} a_r x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 - r + 2r - 2]\,x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 + r - 2]\,x^r = 0$ ...⑤

Equating to zero the coefficient of x^r in equation ⑤

$\Rightarrow a_{r+2}(r+2)(r+1) - a_r(r^2 + r - 2) = 0$

$\Rightarrow a_{r+2} = \frac{(r^2+r-2)}{(r+2)(r+1)} a_r = \frac{(r+2)(r-1)}{(r+2)(r+1)} a_r = \frac{(r-1)}{(r+1)} a_r$

$\Rightarrow a_{r+2} = \frac{(r-1)}{(r+1)} a_r$ ...⑥ is the required recurrence relation

Putting $r = 0$ in ⑥, $a_2 = -a_0$

Putting $r = 1$ in ⑥, $a_3 = 0$

Putting $r = 2$ in ⑥, $a_4 = \frac{1}{3}a_2 = -\frac{1}{3}a_0 \because a_2 = -a_0$

Putting $r = 3$ in ⑥, $a_5 = \frac{1}{2}a_3 = 0 \because a_3 = 0$

Putting $r = 4$ in ⑥, $a_6 = \frac{3}{5}a_4 = \frac{3}{5}\left(-\frac{1}{3}a_0\right) = -\frac{1}{5}a_0 \because a_4 = -\frac{1}{3}a_0$

Similarly, all the coefficients can be found using the recurrence relation ⑥

Substituting the values of $a_2, a_3, a_4, a_5, \ldots$ in equation ②

$\Rightarrow y = a_0 + a_1 x + (-a_0)x^2 + \left(-\frac{a_0}{3}\right)x^4 + \left(-\frac{a_0}{5}\right)x^6 + \cdots$

$\Rightarrow y = a_1 x + a_0\left(1 - x^2 - \frac{x^4}{3} - -\frac{x^6}{5} - \cdots\right)$ is the required series solution of equation①.

Example82 Solve in series the equation $\frac{d^2y}{dx^2} + xy = 0$

Solution: Given $\frac{d^2y}{dx^2} + xy = 0$ ... ①

Let the solution of equation ① be given as

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$ ...②

Differentiating ② with respect to x, $\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$ ...③

Again differentiating ③ with respect to x, $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$ ...④

Substituting values of y and $\frac{d^2y}{dx^2}$ from ② and ④ in equation ①

$\Rightarrow [\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}] + x[\sum_{r=0}^{\infty} a_r x^r] = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=0}^{\infty} a_r x^{r+1} = 0$ ...⑤

Equating to zero the coefficient of x^r in equation ⑤

$\Rightarrow a_{r+2}(r+2)(r+1) + a_{r-1} = 0$

$\Rightarrow a_{r+2} = \frac{-1}{(r+2)(r+1)} a_{r-1}$ ...⑥ is the required recurrence relation

Putting $r = 0$ in ⑥, $a_2 = \frac{-1}{(0+2)(0+1)} a_{-1} = 0$

Putting $r = 1$ in ⑥, $a_3 = \frac{-1}{(1+2)(1+1)} a_0 = \frac{-1}{(3)(2)} a_0 = \frac{-1}{6} a_0$

Putting $r = 2$ in ⑥, $a_4 = \frac{-1}{(2+2)(2+1)} a_1 = \frac{-1}{(4)(3)} a_1 = \frac{-1}{12} a_1$

Putting $r = 3$ in ⑥, $a_5 = \frac{-1}{(3+2)(3+1)} a_2 = \frac{-1}{(5)(4)} a_2 = 0 \quad \because a_2 = 0$

Putting $r = 4$ in ⑥, $a_6 = \frac{-1}{(4+2)(4+1)} a_3 = \frac{-1}{(6)(5)}\left(\frac{-1}{6} a_0\right) = \frac{1}{180} a_0 \quad \because a_3 = \frac{-1}{6} a_0$

Similarly, all the coefficients can be found using the recurrence relation ⑥

Substituting the values of $a_2, a_3, a_4, a_5, \ldots$ in equation ②

$\Rightarrow y = a_0 + a_1 x + 0 + \left(\frac{-a_0}{6}\right)x^3 + \left(\frac{-a_1}{12}\right)x^4 + 0 + \left(\frac{a_0}{180}\right)x^6 + \cdots$

$\Rightarrow y = a_0\left(1 - \frac{x^3}{6} + \frac{x^6}{180} - \cdots\right) + a_1\left(x - \frac{x^4}{12} + \cdots\right)$ is the series solution of equation ①.

14. Legendre's Equation

Another important differential equation used in problems showing spherical symmetry is Legendre's equation given by $\boldsymbol{(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0}$ ...... ①

Here n is a real number, though in most practical applications only non-negative integral values are required. Solving equation ① about the point $x = 0$, which is an ordinary point

Let the solution of equation ① be given as

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots a_r x^r + \cdots$ ...②

Differentiating ② with respect to x, $\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$ ...③

Again differentiating ③ with respect to x, $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$ ...④

Substituting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ②, ③ and ④ in equation ①

$\Rightarrow (1-x^2)[\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}] - 2x[\sum_{r=1}^{\infty} a_r r x^{r-1}] + n(n+1)[\sum_{r=0}^{\infty} a_r x^r] = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r - 2\sum_{r=0}^{\infty} a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 - r + 2r - n(n+1)]\, x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 + r - n(n+1)]\, x^r = 0$ ...⑤

Equating to zero the coefficient of x^r in equation ⑤

$\Rightarrow a_{r+2}(r+2)(r+1) - a_r(r^2 + r - n(n+1)) = 0$

$\Rightarrow a_{r+2} = \frac{r(r+1)-n(n+1)}{(r+2)(r+1)} a_r$ ...⑥ is the required recurrence relation

Putting $r = 0$ in ⑥, $a_2 = \frac{-n(n+1)}{2!} a_0$

Putting $r = 1$ in ⑥, $a_3 = \frac{2-n(n+1)}{6} a_1 = \frac{-(n-1)(n+2)}{3!} a_1$

Putting $r = 2$ in ⑥, $a_4 = \frac{6-n(n+1)}{12} a_2 = \frac{-(n-2)(n+3)}{12}\left(\frac{-n(n+1)}{2}\right) a_0 = \frac{(n-2)n(n+1)(n+3)}{4!} a_0$

$\because a_2 = \frac{-n(n+1)}{2} a_0$

Putting $r = 3$ in ⑥, $a_5 = \frac{12-n(n+1)}{20} a_3 = \frac{-(n-3)(n+4)}{20}\left(\frac{-(n-1)(n+2)}{3!}\right) a_1$

$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 \qquad \because a_3 = \frac{-(n-1)(n+2)}{3!} a_1$

Similarly, all the coefficients can be found using the recurrence relation ⑥

Substituting the values of $a_2, a_3, a_4, a_5, \ldots$ in equation ②

$$y = a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 x^2 - \frac{(n-1)(n+2)}{3!} a_1 x^3 + \frac{(n-2)n(n+1)(n+3)}{4!} a_0 x^4 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 x^5 + \cdots$$

$$\Rightarrow y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots\right] + a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots\right]$$

is the required series solution of Legendre's equation given in ①

∴ Series solution of ① in terms of Legendre's function $P_n(x)$ and $Q_n(x)$ is given by

$y = a_0 P_n(x) + a_1 Q_n(x)$,

Here $P_n(x)$ is called Legendre polynomial and $Q_n(x)$ is called Legendre function of 2^{nd} kind.

14.1 Generating function

Generating function for $P_n(x)$: The function $(1 - 2xz + z^2)^{-\frac{1}{2}}$ is called the generating function of Legendre's polynomials as $\boldsymbol{(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) z^n}$

Results: (i) $\boldsymbol{P_n(1) = 1}$

(ii) $\boldsymbol{P_n(-1) = (-1)^n}$

Example83 Prove that $P_n(-x) = (-1)^n P_n(x)$

Solution: Generating function of $P_n(x)$ is given by

$(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) z^n$.....①

Replacing x by $-x$ on both sides of ①

$(1+2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(-x) z^n$ ..... ②

Again replacing z by $-z$ on both sides of ①

$(1+2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)(-z)^n = \sum_{n=0}^{\infty}(-1)^n P_n(x) z^n$ ⋯③

Comparing ②and ③

$\sum_{n=0}^{\infty} P_n(-x) z^n = \sum_{n=0}^{\infty}(-1)^n P_n(x) z^n$

$\Rightarrow P_n(-x) = (-1)^n P_n(x)$

14.2 Recurrence Relations of Legendre's Polynomials

Recurrence Relations of Legendre's Polynomials $\boldsymbol{P_n(x)}$

(1) $\boldsymbol{(n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)}$

Proof: From generating function $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$......①

Differentiating both sides of ①partially with respect to z, we get

$-\frac{1}{2}(1-2xz+z^2)^{-\frac{3}{2}}(-2x+2z) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$

$\Rightarrow (x-z)\,(1-2xz+z^2)^{-\frac{1}{2}-1} = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$

$\Rightarrow (x-z)(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$

$\Rightarrow (x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$ using ①

Equating coefficient of z^n on both sides

$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$

$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)$

(2) $\boldsymbol{P_n(x) = P'_{n+1}(x) - 2x\, P'_n(x) + P'_{n-1}(x)}$

Differentiating both sides of ① partially with respect to x, we get

$-\frac{1}{2}(1-2xz+z^2)^{-1-\frac{1}{2}}(-2z) = \sum_{n=0}^{\infty} z^n\, P'_n(x)$

$\Rightarrow z(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\, P'_n(x)$

$\Rightarrow z\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\, P'_n(x)$ using ①

Equating coefficient of z^{n+1} on both sides

$P_n(x) = P'_{n+1}(x) - 2x\, P'_n(x) + P'_{n-1}(x)$

(3) $\boldsymbol{nP_n(x) = xP'_n(x) - P'_{n-1}(x)}$

Differentiating recurrence relation (1) partially with respect to x, we get

$(n+1)P'_{n+1}(x) = (2n+1)x\,P'_n(x) + (2n+1)P_n(x) - nP'_{n-1}(x)$.....②

Also from recurrence relation (2)

$P'_{n+1}(x) = P_n(x) + 2x\,P'_n(x) - P'_{n-1}(x)$ ......③

Using③ in ②, we get

$(n+1)[P_n(x) + 2x\,P'_n(x) - P'_{n-1}(x)] = (2n+1)x\,P'_n(x) + (2n+1)P_n(x) - nP'_{n-1}(x)$

$\Rightarrow nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

(4) $(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$

Adding recurrence relations (2) and (3), we get

$(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$

(5) $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Adding recurrence relations (3) and (4), we get

$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

(6) $(1-x^2)\,P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$

Replacing n by $(n-1)$ in recurrence relation (4)

$nP_{n-1}(x) = P'_n(x) - xP'_{n-1}(x)$ ...... ④

Also multiplying recurrence relation (3) by x

$n\,xP_n(x) = x^2P'_n(x) - xP'_{n-1}(x)$ ....⑤

Subtracting⑤ from ④

$(1-x^2)\,P'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]$

(7) $(1-x^2)\,P'_n(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]$

Replacing n by $(n+1)$ in recurrence relation (3)

$\Rightarrow (n+1)P_{n+1}(x) = xP'_{n+1}(x) - P'_n(x)$ ......⑥

Also multiplying recurrence relation (4) by x

$(n+1)xP_n(x) = xP'_{n+1}(x) - x^2P'_n(x)$ ....⑦

Subtracting ⑥from ⑦, we get

$(1-x^2)\,P'_n(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]$

Example84 Prove that $\int P_n(x)dx + C = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$

Solution: From recurrence relation (5)

$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

$\Rightarrow P_n(x) = \frac{P'_{n+1}(x) - P'_{n-1}(x)}{(2n+1)}$

Integrating both sides, we get

$\int P_n(x)dx + C = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$

Example85 Show that $P'_{n+1} + P'_{n-1} = P_0 + 3P_1 + \cdots + (2n+1)P_n$

Solution: From recurrence relation (5)

$(2n+1)P_n = P'_{n+1} - P'_{n-1}$

Putting $n = 1,2,3 \dots . n$, we get

$3P_1 = P'_2 - P'_0$

$5P_2 = P'_3 - P'_1$

$7P_3 = P'_4 - P'_2$

... .

$(2n-1)P_{n-1} = P'_n - P'_{n-2}$

$(2n+1)P_n = P'_{n+1} - P'_{n-1}$

Adding all these relations, we get

$3P_1 + 5P_2 + 7P_3 + \dots + (2n+1)P_n = -P'_0 - P'_1 + P'_n + P'_{n+1}$

$= 0 - P_0 + P'_n + P'_{n+1}$

$\because P'_0 = 0, P'_1 = 1 = P_0$

$\Rightarrow P'_{n+1} + P'_{n-1} = P_0 + 3P_1 + \dots + (2n+1)P_n$

14.3 Rodrigue's Formula

Rodrigue's formula is helpful in producing Legendre's polynomials of various orders and is given by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Proof: Let $y = (x^2-1)^n$

$\therefore \frac{dy}{dx} = n(x^2-1)^{n-1}2x = 2nx\frac{(x^2-1)^n}{(x^2-1)}$

$\Rightarrow y_1(x^2-1) - 2nxy = 0\ ,\ y_1 \equiv \frac{dy}{dx}$ ....②

Differentiating ② $(n+1)$ times using Leibnitz's theorem:

$\Rightarrow y_{n+2}(x^2-1) + (n+1)y_{n+1}(2x) + \frac{(n+1).n}{2!}y_n(2) - 2n[y_{n+1}(x) + (n+1)y_n(1)] = 0$

$\Rightarrow y_{n+2}(x^2-1) + 2xy_{n+1} - (n^2+n)y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0$ ....③

Putting $y_n = V$, so that $y_{n+1} = \frac{dV}{dx}$ and $y_{n+2} = \frac{d^2V}{dx}$

③$\Rightarrow (1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + n(n+1)V = 0$

which is Legendre's equation with the solution $V = AP_n(x) + BQ_n(x)$

But since $V = y_n = \frac{d^n}{dx^n}(x^2-1)^n$ contains only positive powers of x, solution can only be a constant multiple of $P_n(x)$.

$\therefore P_n(x) = CV = Cy_n$

$= C\frac{d^n}{dx^n}(x^2-1)^n$ .... ④

$= CD^n[(x-1)^n(x+1)^n]\ ,\frac{d^n}{dx^n} \equiv D^n$

$= CD^n[(x-1)^n(x+1)^n]$

$= C\left[D^n(x-1)^n(x+1)^n + n_{C_1}D^{n-1}(x-1)^n n(x+1)^{n-1} + \cdots + (x-1)^n D^n(x+1)^n\right]$

$= C[n!\,(x+1)^n + n.n(n-1)\ldots 3.2.(x-1)n(x+1)^{n-1} + \cdots + (x-1)^n n!]$

Taking $x = 1$ on both sides

$\Rightarrow 1 = Cn!\,2^n + 0 \because P_n(1) = 1$

$\Rightarrow C = \frac{1}{2^n n!}$ ………⑤

Using ⑤ in ④, we get

$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Putting $n = 0$, $P_0(x) = 1$

Putting $n = 1$, $P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1)^1 = \frac{1}{2}2x = x$

Putting $n = 2$, $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1)$

Putting $n = 3$, $P_3(x) = \frac{1}{2}(5x^3-3x)$

Putting $n = 4$, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

Putting $n = 5$, $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$ etc…

Example86 Expand the following functions in series of Legendre's polynomials.

(i) $(1+2x-x^2)$

(ii) (x^3-5x^2+x+1)

Solution: $1 = P_0(x)$, $x = P_1(x)$,

$P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow x^2 = \frac{1}{3}(2P_2(x)+1) = \frac{1}{3}\left(2P_2(x)+P_0(x)\right)$

$P_3(x) = \frac{1}{2}(5x^3-3x) \Rightarrow x^3 = \frac{1}{5}(2P_3(x)+3x) = \frac{1}{5}\left(2P_3(x)+3P_1(x)\right)$

(i) Let $E = (1+2x-x^2)$

Substituting values of 1, x and x^2 in terms of Legendre's polynomials, we get

$$E = \left(P_0(x) + 2P_1(x) - \frac{1}{3}\left(2P_2(x)+P_0(x)\right)\right)$$
$$= \frac{1}{3}\left(3P_0(x)+6P_1(x)-2P_2(x)-P_0(x)\right)$$
$$= \frac{2}{3}\left(P_0(x)+3P_1(x)-P_2(x)\right)$$

(ii) Let $F = (x^3-5x^2+x+1)$

Substituting values of 1, x, x^2 and x^3in terms of Legendre's polynomials, we get

$$F = \left[\frac{1}{5}\left(2P_3(x)+3P_1(x)\right) - \frac{5}{3}\left(2P_2(x)+P_0(x)\right) + P_1(x) + P_0(x)\right]$$
$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) - \frac{2}{3}P_0(x)$$

Example87 Prove that (i) $P_n'(1) = \frac{n(n+1)}{2}$ (ii) $P_n'(-1) = (-1)^{(n+1)}\frac{n(n+1)}{2}$

Solution: $P_n(x)$ is the solution of Legendre's equation given by:

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ ....①

$\therefore y = P_n(x)$ will satisfy equation ①

$\Rightarrow (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$ ....②

Putting $x = 1$ in ② we get : $-2P_n'(1) + n(n+1)P_n(1) = 0$

$\Rightarrow P_n'(1) = \frac{n(n+1)}{2} \because P_n(1) = 1$

Putting $x = -1$ in ② we get: $2P_n'(-1) + n(n+1)P_n(-1) = 0$

$\Rightarrow P_n'(-1) = -\frac{n(n+1)}{2}P_n(-1)$

$= (-1)^{(n+1)}\frac{n(n+1)}{2} \because P_n(-1) = (-1)^n$

14.4 Orthogonality of Legendre's polynomial

Orthogonality property of Legendre's polynomials is given by the relations

$\int_{-1}^{1} P_m(x)P_n(x)dx = 0$, when $m \neq n$

and $\int_{-1}^{1} P_n{}^2(x)dx = \frac{2}{2n+1}$, when $m = n$

where m and n are positive integers

Example88 Evaluate

(i) $\int_{-1}^{1} P_5^2(x)\,dx$ (ii) $\int_{-1}^{1} P_4(x)P_5(x)\,dx$

Solution: By Orthogonal property of Legendre's polynomial

(i) $\int_{-1}^{1} P_5^2(x)\,dx = \frac{2}{(2\times5+1)} = \frac{2}{11} \because \int_{-1}^{1} P_n{}^2(x)dx = \frac{2}{2n+1}$, when $m = n$

(ii) $\int_{-1}^{1} P_4(x)P_5(x)\,dx = 0 \quad \because \int_{-1}^{1} P_m(x)P_n(x)dx = 0$, when $m \neq n$

15. Gamma Function

We define Gamma function as:

$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$

Important results

1. *i.* $\boldsymbol{\Gamma 1 = 1}$

Proof: $\Gamma 1 = \int_0^\infty e^{-x}x^0dx = -[e^{-x}]_0^\infty = 1$

ii. $\boldsymbol{\Gamma \frac{1}{2} = \sqrt{\pi}}$

Proof: $\Gamma\frac{1}{2} = \int_0^\infty e^{-x}x^{-\frac{1}{2}}dx = \int_0^\infty e^{-t^2}t^{-1}2t\,dt$, By putting $x = t^2$

$= 2\int_0^\infty e^{-t^2}dt = \sqrt{\pi}, \qquad \because \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

$\therefore \Gamma\frac{1}{2} = \Gamma(0.5) = \sqrt{\pi} = 1.772$

2. **Reduction formula for $\boldsymbol{\Gamma n}$: $\boldsymbol{\Gamma(n+1) = n\Gamma n}$**

We have $\Gamma(n+1) = \int_0^\infty e^{-x}\, x^n dx$

$$= -[x^n e^{-x}]_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= 0 + n\Gamma n$$

$\therefore \Gamma(n+1) = n\Gamma n$

3. $\int_0^\infty e^{-kx}\, x^{n-1} dx = \frac{\Gamma n}{k^n}$

Proof: We have $\Gamma n = \int_0^\infty e^{-t}\, t^{n-1} dt$

Putting $t = kx \Rightarrow dt = kdx$

$\therefore \Gamma n = \int_0^\infty e^{-kx}\,(kx)^{n-1} k dx = k^n \int_0^\infty e^{-kx}\, x^{n-1} dx$

$\Rightarrow \int_0^\infty e^{-kx}\, x^{n-1} dx = \frac{\Gamma n}{k^n}$

Extension of Gamma function from factorial notation

Case i. When n is a positive integer

We have $\Gamma(n+1) = n\Gamma n$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$\vdots$$

$$= n(n-1)(n-2)\cdots 3.2.1\Gamma 1 = n!$$

$\therefore \Gamma 2 = 1!\,, \Gamma 3 = 2!\,, \Gamma 4 = 3!$ etc.

case ii. When n is a positive rational number

$\Gamma n = (n-1)(n-2)\cdots$ upto a positive number in Γ function

Illustration: $\Gamma\frac{7}{2} = \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2} = \frac{15\sqrt{\pi}}{8}$

Also, $\Gamma\frac{11}{4} = \frac{7}{4}.\frac{3}{4}\,\Gamma\frac{3}{4}$

Now value of $\Gamma\frac{3}{4}$ can be obtained from table of gamma function.

case iii. When n is a negative rational number

Using $\Gamma(n+1) = n\Gamma n$

$\Rightarrow \Gamma n = \frac{\Gamma(n+1)}{n} = \frac{(n+1)\Gamma(n+1)}{n(n+1)}$

$$= \frac{\Gamma(n+2)}{n(n+1)}$$

$$= \frac{\Gamma(n+3)}{n(n+1)(n+2)}$$

$$\vdots$$

Continuing in this manner, we get $\Gamma n = \frac{\Gamma(n+k+1)}{n(n+1)\ldots(n+k)}$,

where k is the least positive integer such that $(n+k+1) > 0$

Illustration: $\Gamma(-3.4) = \frac{\Gamma(-3.4+k+1)}{(-3.4)(-2.4)\ldots(-3.4+k)}$, $(-3.4+k+1) > 0$

$\Rightarrow k > 2.4 \Rightarrow k = 3$

$\therefore \Gamma(-3.4) = \frac{\Gamma(-3.4+4)}{(-3.4)(-2.4)(-1.4)(-0.4)} = \frac{\Gamma 0.6}{(-3.4)(-2.4)(-1.4)(-0.4)}$

$\Gamma 0.6$ can be found using tables.

Also, to evaluate $\Gamma(-2.5)$

$\Gamma(-2.5) = \frac{\Gamma(-2.5+k+1)}{(-2.5)(-1.5)\ldots(-2.5+k)}, (-2.5 + k + 1) > 0$

$\Rightarrow k > 1.5 \Rightarrow k = 2$

$\therefore \Gamma(-2.5) = \frac{\Gamma(-2.5+3)}{(-2.5)(-1.5)(-0.5)} = \frac{\Gamma 0.5}{(-2.5)(-1.5)(-0.5)} = -\frac{1.772}{1.875} = -0.945$

case *iv*. Γn is not defined when $n = 0$ or a negative integer

We know $\Gamma n = \frac{\Gamma(n+k+1)}{n(n+1)\ldots(n+k)}, n = 0, -1, -2, \ldots$

For all $n = 0, -1, -2, \ldots$, we will have a zero in the denominator

For instance, $\Gamma 0 = \frac{\Gamma(0+k+1)}{0(1)\ldots(0+k)}, \Gamma(-1) = \frac{\Gamma(-1+k+1)}{(-1)(0)\ldots(-1+k)}, \ldots$

Hence, we can conclude that gamma function cannot be defined for zero or negative integers.

Few More Results

1. $\int_0^{\frac{\pi}{2}} sin^p\theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$
2. $\Gamma n \, \Gamma(1 - n) = \frac{\pi}{\sin n\pi}, 0 < n < 1$

Example89 If n is a positive integer, show that

$2^n\Gamma\left(n + \frac{1}{2}\right) = 1.3.5 \ldots (2n - 1)\sqrt{\pi}$

Solution: $\Gamma\left(n + \frac{1}{2}\right) = \Gamma\left(n - \frac{1}{2} + 1\right)$

$= \left(n - \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right) \quad \because \Gamma(n + 1) = n\Gamma n$

$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\Gamma\left(n - \frac{3}{2}\right)$

$\vdots$

$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right)\ldots\frac{3}{2}.\frac{1}{2}.\Gamma\frac{1}{2}$

$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\ldots\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$

$\Rightarrow 2^n\Gamma\left(n + \frac{1}{2}\right) = 1.3.5 \ldots (2n - 1)\sqrt{\pi}$

Example90 Evaluate the following integrals

$i. \int_0^\infty e^{-x^2} x^{2n-1} dx, n > 1 \quad ii. \int_0^\infty e^{-\sqrt{x}} x^{\frac{1}{4}} dx$

$iii. \int_0^\infty \frac{x^a}{a^x} dx \quad iv. \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx, n > 0$

Solution: $i.$ We have $\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt, \ldots 1$

Putting $t = x^2$ in 1, we get

$$\Gamma n = \int_0^\infty e^{-x^2}\, x^{2n-2}.\, 2x dx$$

$$\Rightarrow \int_0^\infty e^{-x^2} x^{2n-1} dx = \frac{\Gamma n}{2}$$

ii. Putting $t = \sqrt{x}$in ①, we get

$$\Gamma n = \int_0^\infty e^{-\sqrt{x}}\, x^{\frac{n}{2}-\frac{1}{2}}.\frac{1}{2} x^{-\frac{1}{2}} dx = \frac{1}{2}\int_0^\infty e^{-\sqrt{x}}\, x^{\frac{n}{2}-1} dx$$

Substituting $\frac{n}{2} - 1 = \frac{1}{4}$, i.e. $n = \frac{5}{2}$, we get

$$\Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\int_0^\infty e^{-\sqrt{x}}\, x^{\frac{1}{4}} dx$$

$$\therefore \int_0^\infty e^{-\sqrt{x}} x^{\frac{1}{4}} dx = 2\Gamma\left(\frac{5}{2}\right) = 2.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{2}$$

iii. Putting $a^x = e^t$ or $x \log a = t \Rightarrow dx = \frac{dt}{\log a}$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\, dx = \int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^a \frac{dt}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t} t^{(a+1)-1}\, dt = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

iv. We have $\Gamma n = \int_0^\infty e^{-t}\, t^{n-1} dt$

Putting $t = \log\frac{1}{x} \Rightarrow -t = \log x \Rightarrow e^{-t} = x$

Also $dt = -\frac{1}{x} dx$ and $t = 0 \Rightarrow x = 1$, $t = \infty \Rightarrow x = 0$

$$\therefore \Gamma n = \int_1^0 x\left(\log\frac{1}{x}\right)^{n-1}\left(-\frac{1}{x}\right) dx = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$$

$$\Rightarrow \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \Gamma n$$

Example91 Prove that $\int_0^{\frac{\pi}{2}} sin^3\theta \cos^{\frac{5}{2}}\theta\, d\theta = \frac{8}{77}$

Solution: $\int_0^{\frac{\pi}{2}} sin^3\theta \cos^{\frac{5}{2}}\theta\, d\theta = \dfrac{\Gamma(2)\ \Gamma\left(\frac{7}{4}\right)}{2\Gamma\left(\frac{3+\frac{5}{2}+2}{2}\right)} = \dfrac{1.\ \Gamma\left(\frac{7}{4}\right)}{2.\frac{11}{4}.\frac{7}{4}.\Gamma\left(\frac{7}{4}\right)} = \frac{8}{77}$

$$\because \int_0^{\frac{\pi}{2}} sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Example92 Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\ d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

Solution: $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^{\frac{-1}{2}}\theta\ d\theta$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}.\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \qquad \cdots ①$$

$$\because \Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi},\ 0 < n < 1$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\tan\left(\frac{\pi}{2}-\theta\right)}\ d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\ d\theta \qquad \cdots ②$$

From ① and ②, we get: $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\ d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}$

16. Bessel's Equation

The differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ ......①

is known as Bessel's equation of order n and its solutions are called Bessel's functions.

Note that $x = 0$ is a regular singular point of Bessel's equation.

Series solution of ① in terms of Bessel's functions $J_n(x)$ and $J_{-n}(x)$ is given by

$y = AJ_n(x) + BJ_{-n}(x)$

where $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$

Proposition If n is any integer then $J_{-n}(x) = (-1)^n J_n(x)$

Proof: Case I: n is a positive integer

$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$

If n is a positive integer, values of r from 0 to $(n-1)$ will give gamma function of $-ve$ integers in the denominator, which being infinite all such terms will vanish.

$\therefore J_{-n}(x) = \sum_{r=n}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$

Putting $r = n + k$, we get

$J_{-n}(x) = \sum_{k=0}^{\infty}(-1)^{n+k} \frac{1}{(n+k)!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{n+2k}$

$= (-1)^n \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!\,\Gamma(n+k+1}\left(\frac{x}{2}\right)^{n+2k} = (-1)^n J_n(x)$

Case II: $n = 0$

$J_{-0}(x) = (-1)^0 J_0(x)$

or $J_0(x) = J_0(x)$, which is true

Case III: n is a negative integer

Let $n = -p$, where p is a positive integer

From case I $J_p(x) = (-1)^{-p}J_{-p}(x) \Rightarrow J_{-n}(x) = (-1)^n J_n(x)$

16.1 Some Important Expansions

Expansions of $J_0(x)$, $J_1(x)$, $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$

We have $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}$

1. $J_0(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{r!\,\Gamma(r+1)} = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{(r!)^2}$

$\because \Gamma(r+1 = r!$ when r is a positive integer

$$\Rightarrow J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

2. $J_1(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{1+2r} \frac{1}{r!\,\Gamma(r+2)} = \frac{x}{2}\sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{r!\,(r+1)!}$

$\because \Gamma(r + 2 = (r + 1)!$ when r is a positive integer

$$\Rightarrow J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 + \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

3. $J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \frac{1}{r!\,\Gamma\left(r+\frac{3}{2}\right)}$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma\frac{3}{2}} - \frac{1}{1!\,\Gamma\frac{5}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\frac{7}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma\frac{9}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\frac{1}{2}\Gamma\frac{1}{2}} - \frac{1}{1!\,\frac{3\,1}{2\,2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5\,3\,1}{2\,2\,2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7\,5\,3\,1}{2\,2\,2\,2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma(n + 1 = n\Gamma n$

$$= \sqrt{\frac{x}{2\pi}}\left[\frac{1}{\frac{1}{2}} - \frac{1}{1!\,\frac{3\,1}{2\,2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5\,3\,1}{2\,2\,2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7\,5\,3\,1}{2\,2\,2\,2}}\left(\frac{x}{2}\right)^6 + \cdots\right] \because \Gamma\frac{1}{2} = \sqrt{\pi}$$

$$= \sqrt{\frac{x}{2\pi}}\left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \frac{2x^6}{7!} + \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\,\frac{2}{x}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

4. $J_{-\frac{1}{2}}(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{-\frac{1}{2}+2r} \frac{1}{r!\,\Gamma\left(r+\frac{1}{2}\right)}$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma\frac{1}{2}} - \frac{1}{1!\,\Gamma\frac{3}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\frac{5}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma\frac{7}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma\frac{1}{2}} - \frac{1}{1!\,\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{3\,1}{2\,2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{5\,3\,1}{2\,2\,2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma(n + 1 = n\Gamma n$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{1}{1!\,\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{3\,1}{2\,2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{5\,3\,1}{2\,2\,2}}\left(\frac{x}{2}\right)^6 + \cdots\right] \qquad \because \Gamma\frac{1}{2} = \sqrt{\pi}$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] = \sqrt{\frac{2}{\pi x}}\cos x$$

16.2 Generating function

Generating function for $J_n(x)$ is $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

16.3 Recurrence Relations of Bessel's Function

(1) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ **or** $\int x^n J_{n-1}(x)dx = x^n J_n(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$\Rightarrow x^n J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2n+2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$

$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty}(-1)^r \frac{2(n+r)x^{2n+2r-1}}{2^{n+2r}} \frac{1}{r!\,(n+r)\Gamma(n+r)}$

$\because \Gamma(n+r+1) = (n+r)\Gamma(n+r)$

$= x^n \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{r!\,\Gamma((n-1)+r+1)} = x^n J_{n-1}(x)$

(2) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$ **or** $\int x^{-n}J_{n+1}(x)\,dx = -\frac{d}{dx}[x^{-n}J_n(x)]$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$\Rightarrow x^{-n}J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$

$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = \sum_{r=1}^{\infty}(-1)^r \frac{2r\,x^{2r-1}}{2^{n+2r}} \frac{1}{(r-1)!\ r\,\Gamma(n+r+1)}$

$= x^{-n}\sum_{r=1}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{(r-1)!\ \Gamma(n+r+1)}$

$= -x^{-n}\sum_{k=0}^{\infty}(-1)^k \left(\frac{x}{2}\right)^{(n+1)+2k} \frac{1}{k!\ \Gamma((n+1)+k+1)}$

Putting $r = k+1$

$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

(3) $J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$

Proof: From recurrence relation (1)

$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$\Rightarrow x^n J_n{}'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$

Dividing by x^n, we get

$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$

$\Rightarrow J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$

(4) $J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$

Proof: From recurrence relation (2)

$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$\Rightarrow x^{-n}J_n{}'(x) - n x^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$

Dividing by x^{-n}, we get

$J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$

(5) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Proof: Adding recurrence relations (3) and (4), we get

$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

(6) $\boldsymbol{2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]}$

Proof: Subtracting recurrence relations (3) from (4), we get

$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

$\Rightarrow\ 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

16.4 Orthogonality of Bessel's Function

If α and β be roots of the equation $J_n(x) = 0$, then

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0 & if\ \beta \neq \alpha \\ \frac{1}{2}J_{n+1}^2(\alpha) & if\ \beta = \alpha \end{cases}$$

Example93 Evaluate $J_{\frac{3}{2}}(x), J_{-\frac{3}{2}}(x)$, $J_{\frac{5}{2}}(x)$ and $J_{-\frac{5}{2}}(x)$

Solution: From recurrence relation (6)

$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ ....①

Putting $n = \frac{1}{2}$ in ①

$\Rightarrow \frac{1}{x}J_{\frac{1}{2}}(x) = J_{\frac{-1}{2}}(x) + J_{\frac{3}{2}}(x)$

$\Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{\frac{-1}{2}}(x)$

$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)\quad \because J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Putting $n = -\frac{1}{2}$ in ①

$\Rightarrow -\frac{1}{x}J_{\frac{-1}{2}}(x) = J_{\frac{-3}{2}}(x) + J_{\frac{1}{2}}(x)$

$\Rightarrow J_{\frac{-3}{2}}(x) = -\frac{1}{x}J_{\frac{-1}{2}}(x) - J_{\frac{1}{2}}(x)$

$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)\because J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Putting $n = \frac{3}{2}$ in ①

$\Rightarrow \frac{3}{x}J_{\frac{3}{2}}(x) = J_{\frac{1}{2}}(x) + J_{\frac{5}{2}}(x)$

$\Rightarrow J_{\frac{5}{2}}(x) = \frac{3}{x}J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$

$= \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) - \sqrt{\frac{2}{\pi x}}\sin x = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$

Putting $n = -\frac{3}{2}$ in ①

$\Rightarrow -\frac{3}{x}J_{\frac{-3}{2}}(x) = J_{\frac{-5}{2}}(x) + J_{\frac{-1}{2}}(x)$

$$\Rightarrow J_{\frac{-5}{2}}(x) = -\frac{3}{x}J_{\frac{-3}{2}}(x) - J_{\frac{-1}{2}}(x)$$

$$= \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right) - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\cos x + \frac{3}{x}\sin x\right]$$

Example94 Show that:

(i) $J_0'(x) = -J_1(x)$

(ii) $J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4)J_{n+4}(x)$

(iii) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$

(iv) $\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^{-n}J_n(x)] = (-1)^m \frac{1}{x^{n+m}}J_{n+m}(x)$

(v) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x\left[J_n{}^2(x) - J_{n+1}{}^2(x)\right]$

(vi) $J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0$

Solution: (i) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$\Rightarrow \frac{d}{dx}[J_0(x)] = -J_1(x)$, Putting $n = 0$

or $J_0'(x) = -J_1(x)$

(ii) From recurrence relation (6)

$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$ ....①

Replacing n by $(n+4)$ in ①

$2(n+4)J_n(x) = x[J_{n+3}(x) + J_{n+5}(x)]$

$\Rightarrow J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4)J_{n+4}(x)$

(iii) From recurrence relation (5)

$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$ ....①

$\Rightarrow J_n''(x) = \frac{1}{2}[J_{n-1}'(x) - J_{n+1}'(x)]$ ....②

Replacing $(n-1)$ in place of n in ①

$\Rightarrow J_{n-1}'(x) = \frac{1}{2}[J_{n-2}(x) - J_n(x)]$ ....③

Replacing $(n+1)$ in place of n in ①

$\Rightarrow J_{n+1}'(x) = \frac{1}{2}[J_n(x) - J_{n+2}(x)]$ ....④

Using ③ and ④ in ②

$$J_n''(x) = \frac{1}{2}\left[\frac{1}{2}[J_{n-2}(x) - J_n(x)] - \frac{1}{2}[J_n(x) - J_{n+2}(x)]\right]$$

$$= \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

(iv) From recurrence relation (2)

$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

Multiplying both sides by $\frac{1}{x}$

$\frac{1}{x}\frac{d}{dx}\left[\frac{1}{x^n}J_n(x)\right] = -\frac{1}{x^{n+1}}J_{n+1}(x)$ ....①

Multiplying both sides of ① by $\left(\frac{1}{x}\frac{d}{dx}\right)$

$\left(\frac{1}{x}\frac{d}{dx}\right)^2[x^{-n}J_n(x)] = -\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x^{n+1}}J_{n+1}(x)\right]$

$= (-1)^2\frac{1}{x^{n+2}}J_{n+2}(x)$ using ① ......②

Again multiplying both sides of ② by $\left(\frac{1}{x}\frac{d}{dx}\right)$

$\left(\frac{1}{x}\frac{d}{dx}\right)^3[x^{-n}J_n(x)] = (-1)^2\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x^{n+2}}J_{n+2}(x)\right]$

$= (-1)^3\frac{1}{x^{n+3}}J_{n+3}(x)$ again using ①

Continuing in this manner m times, we get

$\left(\frac{1}{x}\frac{d}{dx}\right)^m[x^{-n}J_n(x)] = (-1)^m\frac{1}{x^{n+m}}J_{n+m}(x)$

(v) $\frac{d}{dx}[x\,J_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x\,J_n^{\prime(x)J_{n+1}(x)} + x\,J_n(x)J_{n+1}^{\prime(x)}$ ...①

From recurrence relation (4)

$J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$ ...②

Also from recurrence relation (3)

$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$

$\Rightarrow J_{n+1}'(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x)$ ...③

Using ② and ③in ① we get

$\frac{d}{dx}[x\,J_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x\left[-J_{n+1}(x) + \frac{n}{x}J_n(x)\right]J_{n+1}(x)$

$+ x\,J_n(x)\left[J_n(x) - \frac{n+1}{x}J_{n+1}(x)\right]$

$= J_n(x)J_{n+1}(x) - xJ_{n+1}{}^2(x) + nJ_n(x)J_{n+1}(x) + xJ_n{}^2(x) - (n+1)J_n(x)J_{n+1}(x)$

$= x\left[J_n{}^2(x) - J_{n+1}{}^2(x)\right]$

(vi) $J_0'(x) = -J_1(x)$ from (i)

$\Rightarrow J_0''(x) = -J_1'(x) = -\frac{1}{2}[J_0(x) - J_2(x)]$ Using $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Differentiating again we get

$J_0'''(x) = -\frac{1}{2}[J_0'(x) - J_2'(x)]$

$= -\frac{1}{2}\left[J_0'(x) - \frac{1}{2}\left(J_1(x) - J_3(x)\right)\right]$ Using $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

$= -\frac{1}{2}\left[J_0'(x) + \frac{1}{2}J_0'(x) + \frac{1}{2}J_3(x)\right]$ Using $-J_1(x) = J_0'(x)$

$= -\frac{1}{4}[3J_0'(x) + J_3(x)]$

$\Rightarrow J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0$

Example95 Show that:

(i) $\int xJ_0{}^2(x)dx = \frac{1}{2}x^2\left[J_0{}^2(x) + J_1{}^2(x)\right]$

(ii) $\int_0^{\frac{\pi}{2}} \sqrt{\pi x}\, J_{\frac{1}{2}}(2x)dx = 1$

(iii) $\int J_3(x)\, dx = -\frac{2}{x}J_1(x) - J_2(x)$

(iv) $\int x^{-1}J_4(x)\, dx = -\frac{1}{x}J_3(x) - \frac{2}{x^2}J_2(x)$

(v) $\int J_5(x)\, dx = -J_4(x) - \frac{4}{x}J_3(x) - \frac{8}{x^2}J_2(x)$

Solution: (i) $\int xJ_0{}^2(x)\, dx = J_0{}^2(x).\frac{x^2}{2} - \int 2J_0(x)J_0'(x).\frac{x^2}{2}dx$

$= J_0{}^2(x).\frac{x^2}{2} + \int x^2J_0(x)J_1(x)dx \qquad \because J_0'(x) = -J_1(x)$

$= J_0{}^2(x).\frac{x^2}{2} + \int\; xJ_1(x)\; xJ_0(x)dx$

$= J_0{}^2(x).\frac{x^2}{2} + \int xJ_1(x)\frac{d}{dx}[xJ_1(x)]dx$

$\because\; xJ_0(x) = \frac{d}{dx}[xJ_1(x)]$from recurrence relation (1) by putting $n = 1$

$\therefore \int xJ_0{}^2(x)\, dx = J_0{}^2(x).\frac{x^2}{2} + \frac{(xJ_1(x))^2}{2} = \frac{1}{2}x^2\left[J_0{}^2(x) + J_1{}^2(x)\right]$

(ii) $\int_0^{\frac{\pi}{2}} \sqrt{\pi x}\, J_{\frac{1}{2}}(2x)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^{\pi} \sqrt{t}\, J_{\frac{1}{2}}(t)dt$ Putting $2x = t$

$= \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^{\pi}\left(\sqrt{t}\sqrt{\frac{2}{\pi t}}\sin t\right)dt \qquad \because J_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\sin t$

$= \int_0^{\pi} \sin t\, dt = 1$

(iii) $\int J_3(x)\, dx = \int x^2x^{-2}J_3(x)dx$

$= -\int x^2\frac{d}{dx}[x^{-2}J_2(x)]dx \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -x^2[x^{-2}J_2(x)] + \int 2x[x^{-2}J_2(x)]dx$ Integrating by parts

$= -J_2(x) + 2\int[x^{-1}J_2(x)]\, dx$

$= -J_2(x) - 2\int\frac{d}{dx}[x^{-1}J_1(x)]\, dx \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -J_2(x) - \frac{2}{x}J_1(x)$

(iv) $\int x^{-1}J_4(x)\, dx = \int x^2(x^{-3}J_4(x))\, dx$

$= -\int x^2\frac{d}{dx}(x^{-3}J_3(x))\, dx \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -[x^2x^{-3}J_3(x) - 2\int x\, x^{-3}J_3(x)dx]$ Integrating by parts

$= -\frac{1}{x}J_3(x) + 2\int x^{-2}J_3(x)dx$

$= -\frac{1}{x}J_3(x) - 2\int \frac{d}{dx}(x^{-2}J_2(x))dx$

$= -\frac{1}{x}J_3(x) - \frac{2}{x^2}J_2(x)$

(v) $\int J_5(x)\,dx = \int x^4x^{-4}J_5(x)dx$

$= -\int x^4\frac{d}{dx}[x^{-4}J_4(x)]dx \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -x^4[x^{-4}J_4(x)] + \int 4x^3[x^{-4}J_4(x)]dx$ Integrating by parts

$= -J_4(x) + 4\int[x^2x^{-3}J_4(x)]\,dx$

$= -J_4(x) - 4\int x^2\frac{d}{dx}[x^{-3}J_3(x)]dx \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -J_4(x) - 4x^2x^{-3}J_3(x) + 8\int x^{-2}J_3(x)dx$ Integrating by parts

$= -J_4(x) - \frac{4}{x}J_3(x) - 8\int\frac{d}{dx}[x^{-2}J_2(x)]dx$

$\because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -J_4(x) - \frac{4}{x}J_3(x) - \frac{8}{x^2}J_2(x)$

Test Your Knowledge

1. Determine the order and degree, if defined, of the following differential equations. State also if these are linear or non-linear
   (a) $y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
   (b) $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2$
   (c) $\tan\left(\frac{d^2s}{dt^2}\right) + 3\left(\frac{ds}{dt}\right)^3 + 4t = 0$
2. Find the family of curves for the equation for the following equations using slope fields
   (a) $\frac{dy}{dx} = -y + 2$
   (b) $\frac{dy}{dx} = x - y$
3. Find $y(2.2)$ using Euler's method from the equation $\frac{dy}{dx} = -xy^2$ with $y(2) = 1$
4. Solve the differential equation
   (a) $\sec^2 x\tan y\,dx + \sec^2 y\tan x\,dy = 0$
   (b) $\frac{dy}{dx} = -2xy, y(0) = 1.8$
5. Solve the differential equation
   (a) $\left(y^2e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$
   (b) $\frac{dy}{dx} + \frac{y\cos x+\sin y+y}{\sin x+x\cos y+x} = 0$

6. Solve the differential equation
   (a) $y(2xy + e^x)dx = e^x dy$
   (b) $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$
   (c) $[xy\sin(xy) + \cos(xy)]ydx + [xy\sin(xy) - \cos(xy)]xdy = 0$
   (d) $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2ydy = 0$
   (e) $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$
   (f) $(2x^2y^2 + y)dx + (3x - x^3y)dy = 0$
7. Solve the differential equation
   (a) $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$
   (b) $x\frac{dy}{dx} + y = y^2x^3\cos x$
8. Solve the differential equation
   (a) $(D^2 + D - 2)y = 0$
   (b) $(D^3 + D^2 + 4D + 4)y = 0$
   (c) $(D^4 + 4)y = 0$
9. Solve the differential equation
   (a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$
   (b) $(D^2 + D + 1)y = \sin 2x$
   (c) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x$
   (d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x$
   (e) $(D^2 - 5D + 6)y = x\cos 2x$
   (f) $(D^2 + a^2)y = \tan ax$
10. Solve the differential equation
    (a) $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$
    (b) $x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0$
    (c) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$
    (d) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^4\sin x$
    (e)
11. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ by the method of variation of parameters.
12. Solve by method of variation of parameters $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$
13. Show that the two functions $\sin 2x, \cos 2x$ are independent solutions of $y'' + 4y = 0$
14. Find orthogonal trajectories of
    (a) $y = ax^2$
    (b) $xy = c$
15. Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2} + \frac{y^2}{a^2+\beta} = 1$, where β is the parameter.

16. Solve in series $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$
17. Expand the following functions in series of Legendre's polynomials
    (a) $f(x) = 10x^3 + 6x^2 - 2x + 1$
    (b) $f(x) = 4x^3 + 6x^2 + 7x + 2$
18. Using Rodrigues formula, prove the recurrence relation $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$
19. Prove that
    (a) $P_n'(x) - P_{n-2}'(x) = (2n-1)P_{n-1}(x)$
    (b) $(2n+1)(1-x^2)P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$
20. Prove that $\int_{-1}^{1} x^2\, P_{n+1}(x)P_{n-1}(x)\, dx = \frac{2n\,(n+1)}{(2n-1)(2n+1)(2n+3)}$
21. Prove that
    (a) $J_2'(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x)$
    (b) $\int J_3(x)dx + J_2(x) + \frac{2}{x}J_1(x) = 1$
22. Show that
    (a) $\int xJ_0^2(x)dx = \frac{1}{2}x^2[J_0^2(x) + J_1^2(x)]$
    (b) $J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0$
23. Show that
    (a) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\ d\theta, n$ being an integer
    (b) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\varphi)d\varphi$
24. Find the value of $\Gamma\left(\frac{-5}{2}\right)$ and $\Gamma\left(\frac{-12}{5}\right)$ given that $\Gamma\left(\frac{8}{5}\right) = 0.8935$
25. Evaluate $\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} dx, m > 0, n > 0$

Answers

1. (a) 1,2, non-linear (b) 1,2, non-linear (c) 2, undefined, non-linear
3. $y(2.2) = 0.68692$
4. (a) $\tan x \tan y = k$ (b) $y = 1.8\,e^{-x^2}$
5. (a) $e^{xy^2} + x^4 - y^3 = c$ (b) $y\sin x + (\sin y + y)x = c$
6. (a) $\frac{e^x}{y} + x^2 = c$ (b) $\frac{x}{y} - 2\log x + 3\log y = c$ (c) $x\sec(xy) = cy$
   (d) $\frac{1}{3}e^{\frac{1}{x^3}} - \frac{1}{2}\frac{y^2}{x^2} = c$ (e) $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$ (f) $(4x^2y-5)^7 = cx^4y^{12}$
7. (a) $y(\sin x^2 - x^2) + x = c$ (b) $\frac{1}{xy} = -x\sin x - \cos x + c$
8. (a) $y = c_1e^{-2x} + c_2e^x$ (b) $y = c_1e^{-x} + (c_2\cos 2x + c_3\sin 2x)$
   (c) $y = e^{-t}(c_1\cos t + c_2\sin t) + e^t(c_3\cos t + c_4\sin t)$
9. (a) $(c_1 + c_2x + 3x^2)e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$
   (b) $y = e^{\frac{-x}{2}}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] - \frac{1}{13}(2\cos 2x + 3\sin 2x)$

(c) $c_1 + (c_2 + c_3x)e^{-x} + \frac{1}{18}e^{2x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$

(d) $y = [c_1 + (c_2 - \sin x)x - 2\cos x]e^x$

(e) $y = c_1e^{2x} + c_2e^{3x} + \frac{x}{52}(\cos 2x - 5\sin 2x) - \frac{1}{1352}(40\cos 2x + 73\sin 2x)$

(f) $y = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}[\cos ax\, log|\sec ax + \tan ax|]$

10. (a) $y = c_1 + c_2\log x + c_3(\log x)^2$

(b) $y = (c_1 + c_2\log x)x^3$

(c) $y = c_1\cos(\log x) + C_2\sin(\log x)$

(d) $y = c_1x^2 + c_2x^3 - x^2\sin x$

11. $y = c_1\cos 2x + c_2\sin 2x - \cos 2x\log|\sec 2x + \tan 2x|$

12. $y = (c_1 + c_2x)e^{3x} - e^{3x}(1 + \log x)$

14. (a) $\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1$ (b) $x^2 - y^2 = c$

15. $x^2 + y^2 = 2a^2\log x + c'$

16. $y = a_0(1 - 2x^2) + a_1\left(x - \frac{x^3}{2} - \frac{x^5}{8} - \cdots\right)$

17. (a) $4P_3(x) + 4P_2(x) + 4P_1(x) + 3P_0(x)$

(b) $\frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x)$

24. $\frac{-8}{15}\sqrt{\pi}$, $\frac{-625}{504} \times 0.8935$

25. $\frac{\Gamma n}{m^n}$

UNIT-3 MATRICES

1. Introduction

A rectangular array of $m * n$ numbers consisting of m rows and n columns is termed as a matrix of order $m \times n$ and given as:

$$A = \begin{pmatrix} a_{11} a_{12} & \cdots & a_{1n} \\ a_{21} a_{22} & \cdots & a_{2n} \\ \cdots \quad \cdots & \cdots & \cdots \\ a_{m1} a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ or } A = \begin{bmatrix} a_{11} a_{12} & \cdots & a_{1n} \\ a_{21} a_{22} & \cdots & a_{2n} \\ \cdots \quad \cdots & \cdots & \cdots \\ a_{m1} a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It may also be denoted as $A = [a_{ij}],\ i = 1 \dots m, j = 1 \dots n$

Null Matrix: A matrix with all zero elements is known as a null matrix or zero matrix.

Square matrix: A matrix having equal number of rows and columns is called a square matrix.

$$A = \begin{pmatrix} a_{11} a_{12} & \cdots & a_{1n} \\ a_{21} a_{22} & \cdots & a_{2n} \\ \cdots \quad \cdots & \cdots & \cdots \\ a_{n1} a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ is a square matrix of order } n \times n$$

Sum of all elements in the principal diagonal of a square matrix A is known as '**Trace** $\boldsymbol{A}$' or '**Spur** $\boldsymbol{A}$'. $\therefore$**Trace** $\boldsymbol{A = a_{11} + a_{22} + \cdots + a_{nn}}$

Identity or Unit Matrix: A square matrix having all principal diagonal elements unity and non-diagonal elements zero is called an identity matrix.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is an identity matrix of order 3}$$

Triangular Matrix: A square matrix in which all elements above or below principal diagonal are zero is called a triangular matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 2 & 6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Lower Triangular Matrix　　　　Upper Triangular Matrix

Diagonal Matrix: A square matrix having all non-diagonal elements zero is called a diagonal matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ is a diagonal matrix of order 3}$$

Scalar Matrix: A diagonal matrix with all equal elements is called a scalar matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ is a scalar matrix of order 3}$$

Singular Matrix: If the determinant of a square matrix is zero i.e., $|A| = 0$, then it is known as a singular matrix.

$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -3 \\ -2 & 1 & 3 \end{pmatrix}$ is a singular matrix of order 3

Transpose: The matrix A' or A^T obtained by interchanging rows and columns of a matrix A is known as its transpose.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ 0 & 2 & 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

Symmetric and Skew-Symmetric Matrices:

A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$ or $a_{ij} = a_{ji}\ \forall\ i,j$ and skew-symmetric if $A^T = -A$ or $a_{ij} = -a_{ji}\ \forall\ i,j$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Symmetric Matrix Skew- Symmetric Matrix

Results: 1. Diagonal elements of a skew-symmetric matrix are all zero as

$$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$$

2. Any real matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix as $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, where $\frac{1}{2}(A + A^T)$ is symmetric, while $\frac{1}{2}(A - A^T)$ is skew-symmetric.

Hermitian and Skew-Hermitian Matrices:

A square matrix $A = [a_{ij}]$ is said to be Hermitian iff $A^\theta = A$ or $a_{ij} = \overline{a_{ji}}$ or $\bar{A} = A^T$, $\forall\ i,j$ and Skew-Hermitian iff $A^\theta = -A$ or $a_{ij} = -\overline{a_{ji}}$ or $\bar{A} = -A^T$, $\forall\ i,j$

where $A^\theta = (\bar{A})^T$

$$A = \begin{pmatrix} 1 & 4-5i & 2i \\ 4+5i & 2 & 7 \\ -2i & 7 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 4-5i & 2i \\ -4-5i & 2i & 7 \\ 2i & -7 & -4i \end{pmatrix}$$

Hermitian Matrix Skew- Hermitian Matrix

1. Diagonal elements of a Hermitian matrix are real.
2. Diagonal elements of a Skew-Hermitian matrix are zero or purely imaginary.
3. Any square matrix A can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix as $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$, where $\frac{1}{2}(A + A^\theta)$ is Hermitian, while $\frac{1}{2}(A - A^\theta)$ is skew-hermitian.

Orthogonal Matrix

A square matrix $A = [a_{ij}]$ is said to be orthogonal if $AA^T = I = A^TA$, i.e., $A^{-1} = A^T$

Two matrices A_1 and A_2 are orthogonal if $A_1^T A_2 = A_1 A_2^T = 0$ (for real matrix)

and $A_1^\theta A_2 = A_1 A_2^\theta = 0$ (for complex matrix)

Result: If A and B are two orthogonal matrices, then AB is also an orthogonal matrix.

Proof: $(AB)(AB)^T = (AB)B^TA^T \because (AB)^T = B^TA^T$

$= A(BB^T)A^T$

$= AIA^T \because B$ is an orthogonal matrix

$= AA^T = I \because A$ is an orthogonal matrix

Similarly, we can prove $(AB)^T(AB) = I$

Hence, (AB) is an orthogonal matrix

Unitary Matrix

A square matrix $A = [a_{ij}]$ is said to be unitary if $AA^\theta = I = A^\theta A$, i.e., $A^{-1} = A^\theta$

where $A^\theta = (\bar{A})^T$

2. Algebra of Matrices

Addition and Subtraction of Matrix: Addition or subtraction can be performed on two matrices if and only if they are of same order.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\text{Then } A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{pmatrix}$$

Multiplication of Matrix by a Scalar: If we multiply a matrix A by a scalar k, then each element of the matrix is multiplied by k

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$$

Multiplication of Two Matrices: Matrix product AB is possible only if number of columns in matrix$A$ are same as number of rows in matrix$B$.

$$A = \begin{pmatrix} a_{11}\, a_{12}\, a_{13} \\ a_{21}\, a_{22}\, a_{23} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$C = AB = \begin{pmatrix} c_{11} = a_{11}b_{11} + a_{12}\, b_{21} + a_{13}b_{31}c_{12} = a_{11}b_{12} + a_{12}\, b_{22} + a_{13}b_{32} \\ c_{21} = a_{21}b_{11} + a_{22}\, b_{21} + a_{23}b_{31}c_{22} = a_{21}b_{12} + a_{22}\, b_{22} + a_{23}b_{32} \end{pmatrix}$$

Note that: (i)$A_{m\times n}B_{n\times k} = C_{m\times k}$

(ii) $AB \neq BA$ in general

(iii) $AB = 0$ does not necessarily imply that $A = 0$ or $B = 0$

(iv) $AB = 0$ does not necessarily imply that $BA = 0$

For example,$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Example1 If$A = \begin{pmatrix} \sin x \cos x \\ \sin x \cos x \end{pmatrix} B = \begin{pmatrix} \sin x \sin x \\ \cos x \cos x \end{pmatrix}$, find $AB$and BA

Solution: $AB = \begin{pmatrix} \sin^2 x + \cos^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \sin^2 x + \cos^2 x \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$BA = \begin{pmatrix} 2\sin^2 x & \sin 2x \\ \sin 2x & 2\cos^2 x \end{pmatrix}$

Example2 Express the matrix $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ 1 & 2 & 3 \end{pmatrix}$ as the sum of symmetric and skew-symmetric matrices.

Solution: $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ 1 & 2 & 3 \end{pmatrix}$ $\qquad A^T = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 5 & 4 & 3 \end{pmatrix}$

$\frac{1}{2}(A + A^T) = \begin{pmatrix} 1 & \frac{5}{2} & 3 \\ \frac{5}{2} & -1 & 3 \\ 3 & 3 & 3 \end{pmatrix}, \qquad \frac{1}{2}(A - A^T) = \begin{pmatrix} 0 & \frac{1}{2} & 2 \\ \frac{-1}{2} & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$

$$\therefore A = \underbrace{\begin{pmatrix} 1 & \frac{5}{2} & 3 \\ \frac{5}{2} & -1 & 3 \\ 3 & 3 & 3 \end{pmatrix}}_{\text{Symmetric}} + \underbrace{\begin{pmatrix} 0 & \frac{1}{2} & 2 \\ \frac{-1}{2} & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}}_{\text{Skew-Symmetric}}$$

Example3 If $A = \begin{pmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{pmatrix}$, show that AA^θ is a Hermitian matrix

Solution: We have: $A^\theta = (\bar{A})^T = \begin{pmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{pmatrix}$

$$AA^\theta = \begin{pmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{pmatrix} \begin{pmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{pmatrix}$$

$= \begin{pmatrix} 24 & -20+2i \\ -20-2i & 46 \end{pmatrix}$, which is a Hermitian matrix

Example4 Prove that the matrix $\begin{pmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{pmatrix}$ is unitary

Solution: Let $A = \begin{pmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{pmatrix}$

$A^\theta = (\bar{A})^T = \begin{pmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{pmatrix}$ n

$$AA^\theta = \begin{pmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{pmatrix} \begin{pmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}\big((1+i)(1-i)+(-1+i)(-1-i)\big) & \big((1+i)(1-i)+(-1+i)(1+i)\big)\\ \big((1+i)(1-i)+(1-i)(-1-i)\big) & \big((1+i)(1-i)+(1-i)(1+i)\big)\end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$, which is a unitary matrix

3. Minors, Cofactors, Determinants and Adjoint of a matrix

Minors associated with elements of a square matrix

A minor of each element of a square matrix is the unique value of the determinant associated with it, which is obtained after eliminating the row and column in which the element exists.

For a 2×2 matrix $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$

$M_{11}=a_{22}$, $M_{12}=a_{21}$, $M_{21}=a_{12}$, $M_{22}=a_{11}$

For a 3×3 matrix $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$

$M_{11}=\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}$, $M_{12}=\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}$, ..., $M_{33}=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$

Cofactors associated with elements of a square matrix

The cofactor of each element is obtained on multiplying its minor by $(-1)^{i+j}$.

$C_{ij}=(-1)^{i+j}M_{ij}$

Determinant of a square matrix

Every square matrix is associated with a determinant and is denoted by det (A) or $|A|$.

$$\det(A)=|A|=\begin{vmatrix}a_{11}a_{12} & \dots & a_{1n}\\ a_{21}a_{22} & \dots & a_{2n}\\ \dots\ \dots & \dots & \dots\\ a_{n1}a_{n2} & \dots & a_{nn}\end{vmatrix}$$

Determinant of order n can be expanded by any one row or column using the formula $|A|=\sum_{j=1}^{n}a_{ij}\,C_{ij}$, where C_{ij} is the cofactor corresponding to the element a_{ij}.

A determinant of order 2 is evaluated as: $|A|=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

A determinant of order 3 is evaluated as:

$$|A|=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=\sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij}$$

$$=a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

$$=a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{21}a_{32}-a_{31}a_{22})$$

Note: A determinant may be evaluated using any row or column, value remains the same.

Properties of Determinants

- Value of a determinant remains unchanged if rows and columns are interchanged i.e. $|A|=|A^T|$

- If any two rows or columns are interchanged, the value of determinant is multiplied by (-1)
- The value of determinant remains unchanged if $k$times elements of a row (column) is added to another row (column).
- If elements in any row (column) in a determinant are multiplied by a scalar k, then value of determinant is multiplied by k. Thus, if each element in the determinant is multiplied by k, value of determinant of order n multiplies by k^ni.e.,$|kA| = k^n|A|$
- If A and B are square matrices of same order, then $|AB| = |A||B|$

Adjoint of a square matrix

The adjoint of a square matrix A of order n is the transpose of the matrix of cofactors of each element. If C_{11}, C_{12}, C_{13},...,C_{nn} be the cofactors of elements a_{11}, a_{12}, a_{13},...,a_{nn} of the matrix A. Then adjoint of A is given by

$$adj(A) = \begin{pmatrix} C_{11}C_{12} & \dots & C_{1n} \\ C_{21}C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots \\ C_{n1}C_{n2} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11}C_{21} & \dots & C_{n1} \\ C_{12}C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots \\ C_{1n}C_{2n} & \dots & C_{nn} \end{pmatrix}$$

4. Inverse of a Matrix

The inverse of a square matrix A of order n, denoted by A^{-1} is such that

$AA^{-1} = A^{-1}A = I_n$ where I_n is an identity matrix of order n.

A matrix is invertible if and only if matrix is non-singular i.e., $|A| \neq 0$. There are many methods to find inverse of a square matrix.

4.1 Inverse of a matrix using Adjoint Method

Working rule to find inverse of a matrix using adjoint:

1. Calculate$|A|$
   i. If $|A| = 0$, inverse does not exist
   ii. if $|A| \neq 0$, go to step 2
2. Find $adj(A)$ and compute the inverse using the formula $A^{-1} = \frac{adj(A)}{|A|}$

Example5 Find inverse of the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

$$|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1(7) - 3(1) + 3(-1) = 1$$

$$adj(A) = \begin{pmatrix} C_{11}C_{12} & \dots & C_{1n} \\ C_{21}C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots \\ C_{n1}C_{n2} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11}C_{21} & \dots & C_{n1} \\ C_{12}C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots \\ C_{1n}C_{2n} & \dots & C_{nn} \end{pmatrix}$$

$C_{11} = (-1)^2(16-9) = 7$ $C_{12} = (-1)^3(4-3) = -1$ $C_{13} = (-1)^4(3-4) = -1$

$C_{21} = (-1)^3(12-9) = -3$ $C_{22} = (-1)^4(4-3) = 1$ $C_{23} = (-1)^5(3-3) = 0$

$C_{31} = (-1)^4(9-12) = -3$ $C_{32} = (-1)^5(3-3) = 0$ $C_{33} = (-1)^6(4-3) = 1$

$$\therefore adj(A) = \begin{pmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{1}\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

4.2 Inverse of a matrix by using Gauss-Jordan method

To find the inverse of a matrix using Gauss-Jordan method, we take an augmented matrix $(A : I)$ and transform it into another augmented matrix $(I : A)$ using elementary row (column) transformations.

Elementary row (column) transformations: As the name suggests, row (columns) operations are executed on matrices according to certain set of rules such that the transformed matrix is equivalent to the original matrix. These rules are:

- Any two rows (columns) are interchangeable, i.e., $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
- All the elements of any row (column) can be multiplied by any non-zero number k, i.e., $R_i \rightarrow kR_i$
- All the elements of a row (column) can be added one to one to corresponding scalar multiples of another row (column), i.e., $R_i \rightarrow R_i + kR_j$

Working rule to find inverse of a matrix using Gauss-Jordan method:

1. Prepare an augmented matrix $(A : I)$
2. Using elementary row transformations, make element (1,1) of the augmented matrix as one, and using this make all other elements in the 1st column zero.
3. Now make the element (2,2) as one, using row transformations and remaining elements in the 2nd column zero.
4. Continue the process until the augmented matrix is transformed to $(I : A^{-1})$

Note: (i) Do not apply row and column transformations to the same matrix while using Gauss-Jordan method.

(ii) While using column transformations, make element (1,1) of the augmentedmatrix as one, and using this make all other elements in the 1st row zero and similarly proceed for other rows.

(iii) In the process of forming identity matrix, ensure that previously formed zeros and ones are not altered while applying row (column) transformations. For this while making element (2,2) as one, do not use R_1 (C_1) and while making element (3,3) as one neither use R_1 (C_1) nor R_2 (C_2).

Example6 Find the inverse of the following matrices using Gauss-Jordan method

(i) $\begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$

Solution: (i) Let $A = \begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 3 \end{pmatrix}$

Augmented matrix $(A : I)$ is $\begin{pmatrix} 3 & 1 & 3 & : 1 & 0 & 0 \\ 3 & 1 & 4 & : 0 & 1 & 0 \\ 4 & 1 & 3 & : 0 & 0 & 1 \end{pmatrix}$

Transforming element at (1,1) position to one

$R_1 \to -R_1 + R_3 \quad \begin{pmatrix} 1 & 0 & 0 & :-1 & 0 & 1 \\ 3 & 1 & 4 & : \ 0 & 1 & 0 \\ 4 & 1 & 3 & : \ 0 & 0 & 1 \end{pmatrix}$

Making element at (2,1) and (3,1) positions as zero

$R_2 \to R_2 - 3R_1 \,, R_3 \to R_3 - 4R_1 \quad \begin{pmatrix} 1 & 0 & 0 & :-1 & 0 & 1 \\ 0 & 1 & 4 & : \ 3 & 1 & -3 \\ 0 & 1 & 3 & : \ 4 & 0 & -3 \end{pmatrix}$

Elements at (2,2) and (1,2) are already one and zero, so making element at (3,2) position to zero

$R_3 \to R_3 - R_2 \quad \begin{pmatrix} 1 & 0 & 0 & :-1 & 0 & 1 \\ 0 & 1 & 4 & : \ 3 & 1 & -3 \\ 0 & 0 & -1 & : \ 1 & -1 & 0 \end{pmatrix}$

Now transforming element at (3,3) position to one

$R_3 \to -R_3 \quad \begin{pmatrix} 1 & 0 & 0 & : -1 & 0 & 1 \\ 0 & 1 & 4 & : \ 3 & 1 & -3 \\ 0 & 0 & 1 & : -1 & 1 & 0 \end{pmatrix}$

Element at (1,3) is zero, so transforming element at (2,3) position to zero

$R_2 \to R_2 - 4R_3 \begin{pmatrix} 1 & 0 & 0 & : -1 & 0 & 1 \\ 0 & 1 & 0 & : \ 7 & -3 & -3 \\ 0 & 0 & 1 & : -1 & 1 & 0 \end{pmatrix} = (I : A^{-1})$

$\therefore \ A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 7 & -3 & -3 \\ -1 & 1 & 0 \end{pmatrix}$

(ii) Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$

Augmented matrix $(A : I)$ is $\begin{pmatrix} 3 & -1 & 2 & : 1 & 0 & 0 \\ 2 & -1 & 3 & : 0 & 1 & 0 \\ -1 & 1 & 1 & : 0 & 0 & 1 \end{pmatrix}$

Transforming element at (1,1) position as one

$R_1 \to R_1 - R_2 \quad \begin{pmatrix} 1 & 0 & -1 & : 1 & -1 & 0 \\ 2 & -1 & 3 & : 0 & 1 & 0 \\ -1 & 1 & 1 & : 0 & 0 & 1 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as zero

$R_2 \to R_2 - 2R_1 \,, R_3 \to R_3 + R_1 \begin{pmatrix} 1 & 0 & -1 & : \ 1 & -1 & 0 \\ 0 & -1 & 5 & : -2 & 3 & 0 \\ 0 & 1 & 0 & : \ 1 & -1 & 1 \end{pmatrix}$

Transforming element at (2,2) position as one

$$R_2 \to -R_2 \quad \begin{pmatrix} 1 & 0 & -1 & : 1 & -1 & 0 \\ 0 & 1 & -5 & : 2 & -3 & 0 \\ 0 & 1 & 0 & : 1 & -1 & 1 \end{pmatrix}$$

Element at (1,2) is zero, so transforming element at (3,2) position to zero

$$R_3 \to R_3 - R_2 \quad \begin{pmatrix} 1 & 0 & -1 & : 1 & -1 & 0 \\ 0 & 1 & -5 & : 2 & -3 & 0 \\ 0 & 0 & 5 & :-1 & 2 & 1 \end{pmatrix}$$

Now making element at (3,3) position as one

$$R_3 \to \frac{1}{5} R_3 \quad \begin{pmatrix} 1 & 0 & -1 & : 1 & -1 & 0 \\ 0 & 1 & -5 & : 2 & -3 & 0 \\ 0 & 0 & 1 & :-\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Now transforming elements at (1,3) and (2,3) positions to zero

$$R_1 \to R_1 + R_3\,,\, R_2 \to R_2 + 5R_3 \begin{pmatrix} 1 & 0 & 0: \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \\ 0 & 1 & 0: 1 & -1 & 1 \\ 0 & 0 & 1: -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix} = (I : A^{-1})$$

$$\therefore\ A^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \\ 1 & -1 & 1 \\ -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

5. Rank of a Matrix

The rank of a matrix$A$ is the order of the highest order non-zero minor in A. It is denoted by$\rho(A)$.

Example7 Find the rank of the following matrices:

(i) $A = \begin{pmatrix} 1 & 4 \\ 3 & 7 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

Solution: (i) Here$\begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = -5 \neq 0 \therefore\ \rho(A) = 2$

(ii) Here$\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \quad \therefore\ \rho(A) = 1$

(iii) Here $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{vmatrix} = 0 \ \therefore\ \rho(A) \neq 3$

Next consider $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \neq 0 \ \therefore\ \rho(A) = 2$

5.1 Rank of a matrix using Normal form

The normal form of a matrix is one of the following:

I_n, $\begin{pmatrix} I_n \\ 0 \end{pmatrix}$,$(I_n \quad 0)$or $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$where $I_n$is the identity matrix of order n.

Changing to normal form, n is the rank of the given matrix.

Equivalent Matrix Two matrices A and B are equivalent if they are of same order and of same rank. We write $A \sim B$.

Example8 Find the rank of the following matrices by reducing them to normal form:

(i) $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

Solution: (i) Transforming element at (1,1) position to one

Applying $R_1 \to \frac{1}{3}R_1$, we get $A \sim \begin{pmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (2,1) position to zero

$R_2 \to R_2 - 2R_1$ $A \sim \begin{pmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (1,2) and (1,3) positions to zero

$C_2 \to C_2 + C_1$, $C_3 \to C_3 - \frac{4}{3}C_1$ $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (2,2) position to one

$R_2 \to -R_2$ $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (3,2) position to zero

$R_3 \to R_3 + R_2$ $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & -1/3 \end{pmatrix}$

Transforming element at (2,3) position to zero

$C_3 \to C_3 + \frac{4}{3}C_2$ $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$

Transforming element at (3,3) position to one

$C_3 \to -3C_3$ $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Hence rank of the given matrix is 3.

(ii) Here $A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions to zero

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get $\quad A \sim \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Transforming element at (1,2), (1,3) and (1,4) position to zero

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 4C_1$, $C_4 \to C_4 - 5C_1 \quad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Making element at (2,2) position as one

$R_2 \to -R_2 \quad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Making element at (3,2) position as zero

$R_3 \to R_3 - 2R_2 \quad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Making element at (2,3) and (2,4) positions as zero

$C_3 \to C_3 + 2C_2$, $C_4 \to C_4 + 2C_2 \quad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Hence rank of the given matrix is 2.

5.2 Rank of a matrix using Echelon form

Echelon Form: A matrix is said to be in Echelon form if:

(i) The number of zeros in succeeding row are greater than previous row

(ii) The first non-zero entry in each non-zero row is equal to unity.

Working rule: Transform the matrix to echelon form. The number of non-zero rows in echelon form becomes the rank of the matrix.

Example9 Find the rank of the following matrices by reducing them to echelon form:

(i) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

Solution: (i) Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ we get

$$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Making element at (3,3) position as zero

$R_3 \to R_3 - R_2 \quad A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Now the matrix is reduced to echelon form. Since the number of non-zero rows is 2, hence the rank of the given matrix is 2.

(ii) Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$, we get

$$A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & -2 & 1 \end{pmatrix}$$

Making element at (2,2) position as one

$$R_2 \to -R_2 \qquad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -2 & 1 \end{pmatrix}$$

Making element at (3,2) position as zero

$$R_3 \to R_3 + R_2 \qquad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

Making element at (3,3) position as one

$$R_3 \to -R_3 \qquad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Now the matrix is reduced to echelon form. Since the number of non-zero rows is 3, hence the rank of the given matrix is 3.

6. Solutions of Linear System of Equations

Consider $a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m$$

This is the system of m equations in n unknowns and it can be written in the form $AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ a_{31} & a_{32} \ldots\ldots & a_{3n} \\ & \vdots & \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$

Here if $b_i = 0\ \forall i$ then system of equations is said to be homogeneous otherwise it is non-homogeneous.

The matrix $C = [A\!:\!B] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} : b_1 \\ a_{21} & a_{22} \ldots\ldots & a_{2n} : b_2 \\ a_{31} & a_{32} \ldots\ldots & a_{3n} : b_3 \\ & \vdots & \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} : b_m \end{bmatrix}$ is called augmented matrix.

6.1 Existence and Uniqueness of Solution

A linear system of equations is consistent (has a solution), if the matrix $A$and the augmented matrix $C = [A\!:\!B]$ are having the same rank, i.e., $\rho(A) = \rho(C)$.

If $\rho(A) \neq \rho(C)$, the system is inconsistent, i.e., it does not have any solution.

Uniqueness: The solution of a linear system of equations is unique, if $\rho(A) = \rho(C) = n$, where n is the number of variables in the system.

Infinitely Many Solutions: The system of linear equations has infinitely many solutions, if $\rho(A) = \rho(C) < n$, where n is the number of variables in the system.

For a homogeneous system of equations, it has a unique or trivial solution (0,0,..0) if $\rho(A) = n$ and infinitely many solutions or non- trival solution if $\rho(A) < n$.

Example10 Show that the following system of equations is inconsistent.

$$x + 2y + z = 2$$
$$3x + y - 2z = 1$$
$$4x - 3y - z = 3$$
$$2x + 4y + 2z = 5$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ , B = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

Forming the augmented matrix $C = [A\colon B]$

$$C = [A\colon B] = \begin{pmatrix} 1 & 2 & 1 & : & 2 \\ 3 & 1 & -2 & : & 1 \\ 4 & -3 & -1 & : & 3 \\ 2 & 4 & 2 & : & 5 \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get

$$C = [A\colon B] \sim \begin{pmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -5 & -5 & : & -5 \\ 0 & -11 & -5 & : & -5 \\ 0 & 0 & 0 & : & 1 \end{pmatrix}$$

$\therefore$ Rank of matrix $C = [A\colon B]$ is 4, i.e., $\rho(C) = 4$

By similar row transformations, Rank of matrix A is 3, i.e., $\rho(A) = 3$

$\rho(A) \neq \rho(C)$, Hence the given system of equations is inconsistent.

Example11 Solve the given system of equations, if consistent.

$$2x_1 - x_2 + 3x_3 \qquad = 3$$
$$x_1 + 2x_2 - x_3 - 5x_4 = 4$$
$$x_1 + 3x_2 - 2x_3 - 7x_4 = 5$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 2 & -1 & 3 & 0 \\ 1 & 2 & -1 & -5 \\ 1 & 3- & 2 & -7 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\ , B = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Forming the augmented matrix $C = [A\colon B]$

$$C = [A\colon B] = \begin{pmatrix} 2 & -1 & 3 & 0 & : & 3 \\ 1 & 2 & -1 & -5 & : & 4 \\ 1 & 3- & 2 & -7 & : & 5 \end{pmatrix}$$

Clearly $\rho(C) \leq 3$, applying the transformation $R_1 \leftrightarrow R_2$

$$C \sim \begin{pmatrix} 1 & 2 & -1 & -5 & : & 4 \\ 2 & -1 & 3 & 0 & : & 3 \\ 1 & 3 & -2 & -7 & : & 5 \end{pmatrix}$$

Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, we get

$$C \sim \begin{pmatrix} 1 & 2 & -1 & -5 & : & 4 \\ 0 & -5 & 5 & 10 & : & -5 \\ 0 & 1 & -1 & -2 & : & 1 \end{pmatrix}$$

Applying $R_2 \to R_2/-5$

$$C \sim \begin{pmatrix} 1 & 2 & -1 & -5 & : & 4 \\ 0 & 1 & -1 & -2 & : & 1 \\ 0 & 1 & -1 & -2 & : & 1 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$C \sim \begin{pmatrix} 1 & 2 & -1 & -5 & : & 4 \\ 0 & 1 & -1 & -2 & : & 1 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \ldots (1)$$

Matrix C is in echelon form with two non-zero rows.

$\therefore$ Rank of matrix C is 2, i.e., $\rho(C) = 2$

By similar row transformations, Rank of matrix A is 2, i.e., $\rho(A) = 2$

$\rho(A) = \rho(C) = 2 < n$, where $n = 4$ is the number of variables in the system

$\therefore$ the given system of equations is consistent and having infinitely many solutions.

Using the augmented matrix given by (1)

$x_1 + 2x_2 - x_3 - 5x_4 = 4 \ldots (2)$

$x_2 - x_3 - 2x_4 = 1 \ldots (3)$

Putting $x_3 = k_1$ and $x_4 = k_2$ in (3), we get

$x_2 = 1 + k_1 + 2k_2 \ldots (4)$

Using (4) in (2), we get

$x_1 + 2(1 + k_1 + 2k_2) - k_1 - 5k_2 = 4$

$\Rightarrow x_1 = -k_1 + k_2 + 2$

Putting arbitrary values of k_1 and k_2, we can find the infinite solutions of the given system of equations.

For instance, if $k_1 = 0, k_2 = 1$

$x_3 = 0, x_4 = 1, x_2 = 3, x_1 = 3$

Example12 Determine the values of $a$and b for which the system

$$\begin{pmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 3 \\ -1 \end{pmatrix}$$

has a (i) unique solution (ii) no solution (iii) infinitely many solutions

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; , B = \begin{pmatrix} b \\ 3 \\ -1 \end{pmatrix}$$

Forming the augmented matrix $C = [A:B]$

$$C = [A:B] = \begin{pmatrix} 3 & -2 & 1 & : b \\ 5 & -8 & 9 & : 3 \\ 2 & 1 & a & : -1 \end{pmatrix}$$

Clearly $\rho(C) \leq 3$

Transforming element (1,1) position to one

Applying $R_1 \to R_1 - R_3$ $C \sim \begin{pmatrix} 1 & -3 & 1-a & : b+1 \\ 5 & -8 & 9 & : 3 \\ 2 & 1 & a & : -1 \end{pmatrix}$

Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$, we get

$$C \sim \begin{pmatrix} 1 & -3 & 1-a & : b+1 \\ 0 & 7 & 4+5a & : -5b-2 \\ 0 & 7 & -2+3a & : -2b-3 \end{pmatrix}$$

Making element at (3,2) position as zero

Applying$R_3 \to R_3 - R_2$, $C \sim \begin{pmatrix} 1 & -3 & 1-a & : b+1 \\ 0 & 7 & 4+5a & : -5b-2 \\ 0 & 0 & -6-2a & : 3b-1 \end{pmatrix}$

Matrix C is in echelon form

(i) For Unique solution: $\rho(A) = \rho(C) = 3 = n$,where n is the number of variables in the system then $-6 - 2a \neq 0$
$a \neq -3$ and b may have any value

(ii) For no solution: $\rho(A) \neq \rho(C)$
$-6 - 2a = 0, 3b - 1 \neq 0$
$$a = -3, b \neq \frac{1}{3}$$

(iii) For infinitely many solutions: $\rho(A) = \rho(C) = 2 = n$,where n is the number of variables in the system then
$-6 - 2a = 0, 3b - 1 = 0$
$$a = -3, b = \frac{1}{3}$$

6.2 Direct Methods for Solving a System of Linear Equations.

6.2.1 Matrix Method

Working rule to solve a system of equations using matrix method

1. Write the system of equations as $AX = B$
2. Calculate $|A|$
   i. If $|A| = 0$, system of equations can not be solved using matrix method
   ii. if $|A| \neq 0$, go to step 3
3. Find $adj(A)$ and compute the inverse using the formula $A^{-1} = \frac{adj(A)}{|A|}$
4. Solution of the system of equations is given by $X = A^{-1}B$

Example13 Solve the system of equations using matrix method

$$x + 3y + 2z = 5$$

$2x + 4y - 6z = -4$
$x + 5y + 3z = 10$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

$|A| = 1(12 + 30) - 2(9 - 10) + 1(-18 - 8) = 18 \neq 0$

$C_{11} = (-1)^2(12 + 30) = 42 \quad C_{12} = (-1)^3(6 + 6) = -12 \quad C_{13} = (-1)^4(10 - 4) = 6$

$C_{21} = (-1)^3(9 - 10) = 1 \quad C_{22} = (-1)^4(3 - 2) = 1 \quad C_{23} = (-1)^5(5 - 3) = -2$

$C_{31} = (-1)^4(-18 - 8) = -26 \quad C_{32} = (-1)^5(-6 - 4) = 10 \quad C_{33} = (-1)^6(4 - 6) = -2$

$$\therefore adj(A) = \begin{pmatrix} 42 & -12 & 6 \\ 1 & 1 & -2 \\ -26 & 10 & -2 \end{pmatrix}^T = \begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{18}\begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{18}\begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

$$= \frac{1}{18}\begin{pmatrix} 42(5) + 1(-4) - 26(10) \\ -12(5) + 1(-4) + 10(10) \\ 6(5) - 2(-4) - 2(10) \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -54 \\ 36 \\ 18 \end{pmatrix}$$

$\therefore x = -3, y = 2, z = 1$

6.2.2 Cramer's Rule Method

Working rule to solve system of equations using Cramer's Rule method

1. Write the system of equations as $AX = B$
2. Find the $det(A)$and represent it by D
3. Find the determinants D_1, D_2 and D_3 where $D_i$is the determinant of matrix A whose i^{th} column is replaced by the column matrix B.
4. We divide each of these determinants by D to find the values of the corresponding variables, i.e. $x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$

This system of equations has a unique solution only when $D \neq 0$

Remark: when $D = 0$, then the system of equations has no solution or infinite number of solutions.

Example14 Solve the system of equations using Cramer's Rule method

$x + 3y + 2z = 5$
$2x + 4y - 6z = -4$
$x + 5y + 3z = 10$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix}, \ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ , B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

By Cramer's rule, we get

$$D = det(A) = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{vmatrix} = 1(12+30) - 2(9-10) + 1(-18-8) = 18 \neq 0$$

$$D_1 = \begin{vmatrix} 5 & 3 & 2 \\ -4 & 4 & -6 \\ 10 & 5 & 3 \end{vmatrix} = 5(12+30) + 4(9-10) + 10(-18-8) = -54$$

$$D_2 = \begin{vmatrix} 1 & 5 & 2 \\ 2 & -4 & -6 \\ 1 & 10 & 3 \end{vmatrix} = 1(-12+60) - 2(15-20) + 1(-30+8) = 36$$

$$D_3 = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & -4 \\ 1 & 5 & 10 \end{vmatrix} = 1(40+20) - 2(30-25) + 1(-12-20) = 18$$

Hence, $x = \frac{D_1}{D} = \frac{-54}{18} = -3, \ y = \frac{D_2}{D} = \frac{36}{18} = 2, z = \frac{D_3}{D} = \frac{18}{18} = 1$

6.2.3 Gauss Elimination Method

Working rule to solve system of equations using Gauss Elimination method

1. Write the system of equations as $AX = B$
2. Write the matrix in augmented form as $C = (A{:}B)$
3. Reduce matrix A in $C = (A{:}B)$to echelon form using row transformations
4. Solve the system of equations $AX = B$ by backward substitution method.

Example 15 Solve the system of equations using Gauss Elimination method

$$x + 3y + 2z = 5\text{n}$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix}, \ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ , B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

Augmented matrix $C = \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 2 & 4 & -6 & : & -4 \\ 1 & 5 & 3 & : & 10 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as zero

$$R_2 \to R_2 - 2R_1 \, , R_3 \to R_3 - R_1 \begin{pmatrix} 1 & 3 & 2 & : 5 \\ 0 & -2 & -10 & : -14 \\ 0 & 2 & 1 & : 5 \end{pmatrix}$$

Transforming element at (2,2) position to one

$$R_2 \to R_2/-2 \qquad \begin{pmatrix} 1 & 3 & 2 & : 5 \\ 0 & 1 & 5 & : 7 \\ 0 & 2 & 1 & : 5 \end{pmatrix}$$

Transforming element at (3,2) to zero

$$R_3 \to R_3 - 2R_2 \qquad \begin{pmatrix} 1 & 3 & 2 & : 5 \\ 0 & 1 & 5 & : 7 \\ 0 & 0 & -9 & :-9 \end{pmatrix}$$

$\therefore$ Corresponding system of equations is given as
$x + 3y + 2z = 5 \ldots ①$, $y + 5z = 7 \ldots ②$, $-9z = -9 \ldots ③$
Solving by back substitution
③$\Rightarrow z = 1$, using $z = 1$ in ② $\Rightarrow y = 2$, using $y = 2,\ z = 1$ in ①$\Rightarrow x = -3$
$\therefore x = -3, y = 2, z = 1$ is the required solution of given system of equations

6.2.4 Gauss Jordan Elimination Method

Working rule to solve system of equations using Gauss Jordan Elimination method

1. Write the system of equations as $AX = B$
2. Write the matrix in augmented form as $C = (A\colon B)$
3. Apply elementary row transformations to reduce the matrix A in $C = (A\colon B)$ to unit matrix
4. Last column of the transformed augmented matrix gives vector X.

Example16 Solve the system of equations using Gauss Jordan Elimination method

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

Augmented matrix $C = \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 2 & 4 & -6 & : & -4 \\ 1 & 5 & 3 & : & 10 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as zero

$$R_2 \to R_2 - 2R_1\ ,\ R_3 \to R_3 - R_1 \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & -2 & -10 & : & -14 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming element at (2,2) position to one

$$R_2 \to R_2/-2 \qquad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & 1 & 5 & : & 7 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming elements at (1,2) and (3,2) to zero

$$R_1 \to R_1 - 3R_2, R_3 \to R_3 - 2R_2 \begin{pmatrix} 1 & 0 & -13 & : & -16 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & -9 & : & -9 \end{pmatrix}$$

Transforming element at (3,3) to one

$$R_3 \to R_3/-9 \qquad \begin{pmatrix} 1 & 0 & -13 & : & -16 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Transforming elements at (1,3) and (2,3) to zero

$$R_1 \to R_1 + 13R_3, R_2 \to R_2 - 5R_3 \begin{pmatrix} 1 & 0 & 0 & : & -3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

$\therefore x = -3, y = 2, z = 1$ is the required solution of given system of equations

7. Linear Dependence and Independence of Vectors

The set of vectors $X_1, X_2, X_3, \dots, X_n$ is said to be linearly dependent if there exist scalars $C_1, C_2, C_3, \dots, C_n$ not all zero, such that $C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n = 0$, and they are linearly independent if $C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n = 0 \Rightarrow C_i = 0\ \forall i = 1,2,3,\dots,n$

Example17 Examine the following system of vectors for linear dependence. If dependent find the relation between them:

(i) $X_1 = [1 \quad -1 \quad 1]$, $X_2 = [2 \quad 1 \quad 1]$ and $X_3 = [3 \quad 0 \quad 2]$

(ii) $X_1 = [1 \quad 2 \quad 3]$ and $X_2 = [2 \quad -2 \quad 6]$

Solution: (i) Consider $C_1X_1 + C_2X_2 + C_3X_3 = 0 \dots ①$

$$\Rightarrow C_1[1 \quad -1 \quad 1] + C_2[2 \quad 1 \quad 1] + C_3[3 \quad 0 \quad 2] = 0$$

$$\Rightarrow C_1 + 2C_2 + 3C_3 = 0$$
$$-C_1 + C_2 + 0C_3 = 0$$
$$C_1 + C_2 + 2C_3 = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \to R_3 + \frac{1}{3}R_2$, we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow C_1 + 2C_2 + 3C_3 = 0$$
$$0C_1 + 3C_2 + 3C_3 = 0$$

Let $C_3 = k \Rightarrow C_2 = -k$ and $C_1 = -k$

Hence the given vectors are linearly dependent.

Putting these values in ①, we get $-kX_1 - kX_2 + kX_3 = 0$

$$\Rightarrow -k(X_1 + X_2 - X_3) = 0$$
$$\Rightarrow X_1 + X_2 - X_3 = 0$$

which is the required relation between them.

(ii) Consider $C_1X_1 + C_2X_2 = 0 \dots ①$

$$\Rightarrow C_1[1 \quad 2 \quad 3] + C_2[2 \quad -2 \quad 6] = 0$$

$$\Rightarrow C_1 + 2C_2 = 0$$
$$2C_1 - 2C_2 = 0$$
$$3C_1 + 6C_2 = 0$$

$\Rightarrow C_1 = 0$ and $C_2 = 0$, hence the given vectors are linearly independent.

Example18 Find the value of α so that the vectors $[1 \quad 2 \quad 9 \quad 8]$, $[2 \quad 0 \quad 0 \quad \alpha]$, $[\alpha \quad 0 \quad 0 \quad 8]$ and $[0 \quad 0 \quad 1 \quad 0]$ are linearly dependent

Solution: Since the vectors are dependent, so there exists a non-trivial solution.

Consider $C_1X_1 + C_2X_2 + C_3X_3 + C_4X_4 = 0$

$\Rightarrow C_1[1 \quad 2 \quad 9 \quad 8] + C_2[2 \quad 0 \quad 0 \quad \alpha] + C_3[\alpha \quad 0 \quad 0 \quad 8] + C_4[0 \quad 0 \quad 1 \quad 0] = 0$

$$\Rightarrow C_1 + 2C_2 + \alpha C_3 + 0C_4 = 0$$
$$2C_1 + 0C_2 + 0C_3 + 0C_4 = 0$$
$$9C_1 + 0C_2 + 0C_3 + C_4 = 0$$
$$8C_1 + \alpha C_2 + 8C_3 + 0C_4 = 0$$

Hence, these homogeneous equations have non- trival solution i.e., rank is less than 4

$$\Rightarrow \begin{vmatrix} 1 & 2 & \alpha & 0 \\ 2 & 0 & 0 & 0 \\ 9 & 0 & 0 & 1 \\ 8 & \alpha & 8 & 0 \end{vmatrix} = 0$$

Using row two for finding determinant

$$\Rightarrow -2 \begin{vmatrix} 2 & \alpha & 0 \\ 0 & 0 & 1 \\ \alpha & 8 & 0 \end{vmatrix} = 0$$

Again, using row two for finding determinant

$$\Rightarrow 2 \begin{vmatrix} 2 & \alpha \\ \alpha & 8 \end{vmatrix} = 0$$

$$\Rightarrow 16 - \alpha^2 = 0$$

$$\Rightarrow \alpha = \pm 4$$

8. Eigen Values and Eigen Vectors

For a square matrix $A = [a_{ij}]$ of order $n \times n$, the equation $|A - \lambda I| = 0$,

i.e. $\begin{vmatrix} a_{11-\lambda} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22-\lambda} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn-\lambda} \end{vmatrix} = 0$ is called the **Characteristic equation** of matrix A.

The roots of the Characteristic equation $|A - \lambda I| = 0$ are called the **Characteristic roots** or **Latent roots** or **Eigen values** of matrix A. The set of Eigen values is called the **Spectrum** of matrix A.

Also, the vector 'X' found by solving the equation $AX = \lambda X = 0$, i.e. $(A - \lambda I)X = 0$, is called the **Characteristic vector** or **Eigen vector** of the matrix A.

An Eigen basis is a basis of $\mathbb{R}^n$ consisting of eigenvectors of the matrix A.

Geometrical Interpretation of Eigen values and Eigenvectors: A non-zero Eigen vector X when multiplied to the matrix A is scaled to the vector λX, where λ is the eigen value of the matrix A.

Note that the direction of vector λX is in the same as that of vector X andthe direction is reversed if the eigenvalue is negative.

8.1 Properties of Eigen Values

- The sum of Eigen values of a matrix is equal to '**Trace** $[A]$', where **Trace** $[A]$is the sum of principal diagonal elements of the matrix A.
- The product of the Eigen values of a matrix A is equal to determinant of A.
- Eigen values of the matrix A and matrix A^T are same, where $A^T$is the transpose of matrix A.
- If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of a matrix A of order $n \times n$, then
  - the Eigen values of matrix kA are $k\lambda_1, k\lambda_2, ..., k\lambda_n$
  - the Eigen values of matrix A^p are $\lambda_1{}^p, \lambda_2{}^p, ..., \lambda_n{}^p$
  - if $|A| \neq 0$, Eigen values of the matrix A^{-1} are$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$

8.2 Properties of Eigen Vectors

- The Eigen vector X of a matrix A is not unique.
- If $\lambda_1, \lambda_2, ..., \lambda_n$ be the distinct eigen values of a matrix A, then the corresponding eigen vectors $X_1, X_2, ..., X_n$ are linearly independent.
- If two or more Eigen values of a matrix A are equal, in some cases we may not get linearly independent vectors corresponding to the equal Eigen values.
- Eigen vectors of the symmetric matrix A are pairwise orthogonal.

For practical purposes, while finding Eigen values and vectors of a matrix, two cases arise.

1. Distinct Eigen values 2. Repeated Eigen values

8.3 Eigen Vectors for Matrices having All Distinct Eigen Values

Working Rule for finding Eigen values and vectors, when all the Eigen values are distinct.

Step 1: Find the Eigen values by solving the characteristic equation $|A - \lambda I| = 0$.

Step 2: Solve for the Eigen vectors distinctly for each Eigen valueλ using the equation $(A - \lambda I)X = 0$

Example19 (a) Find the Eigen values of the matrix $A = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$. Hence find the eigenvalues of the matrices A^{46} and $A - 2I$.

Solution: The characteristic equation is given by $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (2 - \lambda)(-1 - \lambda) = 0 \Rightarrow \lambda = -1, 2$

$\therefore$ The required Eigen values of the matrix A are -1 and 2.

Also, the Eigen values of the matrix A^{46} are $(-1)^{46}$ and 2^{46}, i.e.,-1 and 2^{46}.

The Eigen values of the matrix$A - 2I$ are $-1 - 2$ and $2 - 2$, i.e.,-3 and 0.

(b) Find the product of Eigen values of $A = \begin{pmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$

Solution: Product of Eigen values of $A = |A| = \begin{vmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{vmatrix}$

$$= 7(1 - 4) - 2(6 - 12) + 2(-12 + 6) = -21$$

(c) Two Eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 and 1.

Find the Eigen values of A^{-1}.

Solution: Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of the given matrix then $\lambda_1 = \lambda_2 = 1$

But $\lambda_1 + \lambda_2 + \lambda_3 = 7 \because$ sum of Eigen values =Trace $[A]$

$\Rightarrow \lambda_3 = 7 - (\lambda_1 + \lambda_2) = 7 - 2 = 5$

$\therefore$ Eigen values of A are 1,1,5

Hence, Eigen values of A^{-1} are $1,1,\frac{1}{5}$

Example20 Find all the Eigen values and Eigen vectors of the matrix $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

Also find the Eigen values the matrix $A^2 + 4A + 2I$.

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\Rightarrow \lambda = 2, 3, 5$$

$\therefore$ The required Eigen values are 2, 3 and 5.

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\lambda = 2$, $(A - 2I)X = 0 \Rightarrow \begin{pmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2 , x_3

Taking first two equations:

$x_1 + x_2 + 4x_3 = 0$

$0x_1 + 0x_2 + 6x_3 = 0$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{6} = \frac{-x_2}{6} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0} = k_1, \text{say}$$

$$\Rightarrow x_1 = k_1 \ , x_2 = -k_1 \ , x_3 = 0$$

$\therefore$The required Eigen vector corresponding to $\lambda = 2$ is $\begin{pmatrix} k_1 \\ -k_1 \\ 0 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $k_1 \in R - \{0\}$

$$\therefore \boldsymbol{X_1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{0} \end{pmatrix}$$

For $\boldsymbol{\lambda = 3}$, $(A-3I)X = 0 \Rightarrow \begin{pmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking first two equations:

$0\,x_1 + x_2 + 4x_3 = 0$

$0x_1 - x_2 + 6x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} 1 & 4 \\ -1 & 6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{10} = \frac{-x_2}{0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = k_2$, say

$\Rightarrow x_1 = k_2\ , x_2 = 0\ , x_3 = 0$

$\therefore$The required Eigen vector corresponding to $\lambda = 3$ is $\begin{pmatrix} k_2 \\ 0 \\ 0 \end{pmatrix} = k_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $k_2 \in R - \{0\}$

$$\therefore \boldsymbol{X_2} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

For $\boldsymbol{\lambda = 5}$, $(A-5I)X = 0 \Rightarrow \begin{pmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking first two equations:

$-2\,x_1 + x_2 + 4x_3 = 0$

$0x_1 - 3x_2 + 6x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} 1 & 4 \\ -3 & 6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{18} = \frac{-x_2}{-12} = \frac{x_3}{6} \Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = k_3$, say

$\Rightarrow x_1 = 3k_3\ , x_2 = 2k_3\ , x_3 = k_3$

$\therefore$The required Eigen vector corresponding to $\lambda = 5$ is $\begin{pmatrix} 3k_3 \\ 2k_3 \\ k_3 \end{pmatrix} = k_3\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $k_3 \in R - \{0\}$

$\therefore X_3 = \begin{pmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix}$

Again to find the Eigen values the matrix $A^2 + 4A + 2I$

The Eigen values matrix A^2 are 2^2, 3^2 and 5^2, i.e.,4, 9 and 25

The Eigen values matrix $4A$ are 8, 12 and 20

The Eigen values matrix $2I$ are 2, 2 and 2

$\therefore$ The Eigen values the matrix $A^2 + 4A + 2I$ are 14, 23 and 47.

Short -cut method for finding Eigen Values

- For a 2×2 matrix, the characteristic equation $|A - \lambda I| = 0$ is given by:

$$\lambda^2 - (trace\ [A])\lambda + |A| = 0$$

- For a 3×3 matrix, the characteristic equation $|A - \lambda I| = 0$ is given by:

$$\lambda^3 - (trace\ [A])\lambda^2 + (Sum\ of\ minors\ of\ principal\ diagonal\ elements)\lambda - |A| = 0$$

Example 21 Find all the Eigen values and Eigen vectors of the matrix

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - (8 + 7 + 3)\lambda^2 + ([21 - 16] + [24 - 4] + [56 - 36])\lambda - |A| = 0$$

Also$|A| = 8[21 - 16] + 6[-18 + 8] + 2[24 - 14] = 0$

$\therefore$ Characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$

$\Rightarrow \lambda(\lambda - 3)(\lambda - 15) = 0$

$\Rightarrow \lambda = 0, 3, 15$

$\therefore$ The required Eigen values are 0, 3 and 15.

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\boldsymbol{\lambda = 0}$, $(A - 0I)X = 0 \Rightarrow \begin{pmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2 , x_3

Taking last two equations:

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

$\Rightarrow \dfrac{x_1}{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{21-16} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14}$

$\Rightarrow \frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k_1$, say

$\Rightarrow x_1 = k_1 \ \ ,x_2 = 2k_1 \ \ , \ x_3 = 2k_1$

$\therefore$The required Eigen vector corresponding to $\lambda = 0$ is $\begin{pmatrix} k_1 \\ 2k_1 \\ 2k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $k_1 \in R - \{0\}$

$\therefore \boldsymbol{X_1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{pmatrix}$

For $\boldsymbol{\lambda = 3}$, $(A - 3I)X = 0 \Rightarrow \begin{pmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2 , x_3

Taking last two equations:

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 0x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8}$

$\Rightarrow \frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{2}$

$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k_2$, say

$x_1 = 2k_2 \ \ , \ \ x_2 = k_2 \ , \ x_3 = -2k_2$

$\therefore$The required Eigen vector corresponding to $\lambda = 3$ is $\begin{pmatrix} 2k_2 \\ k_2 \\ -2k_2 \end{pmatrix} = k_2 \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $k_2 \in R - \{0\}$

$\therefore \boldsymbol{X_2} = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{-2} \end{pmatrix}$

Note: 1. We can find the Eigen vectors using any two equations simultaneously.

2. We can also solve the above set of equations by backward substitution as:

Let $x_2 = k_2 \Rightarrow x_1 = 2k_2$

$\therefore 5(2k_2) - 6k_2 + 2x_3 = 0 \Rightarrow 4k_2 + 2x_3 = 0 \Rightarrow x_3 = -2k_2$

For $\boldsymbol{\lambda = 15}$, $(A - 15I)X = 0 \Rightarrow \begin{pmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking last two equations:

$$3x_1 + 4x_2 + 2x_3 = 0$$

$$x_1 - 2x_2 - 6x_3 = 0$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} 4 & 2 \\ -2 & -6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 2 \\ 1 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-24+4} = \frac{-x_2}{-18-2} = \frac{x_3}{-6-4}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{20} = \frac{x_3}{-10} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k_3, \text{ say}$$

$$x_1 = 2k_3 \ , \ x_2 = -2k_3 \ , \ x_3 = k_3$$

$\therefore$The required Eigen vector corresponding to $\lambda = 15$ is $\begin{pmatrix} 2k_3 \\ -2k_3 \\ k_3 \end{pmatrix} = k_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, k_3 \in R - \{0\}$

$$\therefore \boldsymbol{X_3} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-2} \\ \mathbf{1} \end{pmatrix}$$

8.4 Eigen Vectors for Matrices having Repeated Eigen Values

If two or more Eigen values of a matrix are equal, in some cases we may not get linearly independent vectors corresponding to the equal Eigen values. In this particular case, while solving the equation $(A - \lambda I)X$for a repeated eigen value λ, if we get the same equation against all the rows in the matrix, then the two Eigen vectors obtained are linearly independent, else the vectors are linearly dependent.

- **Note: In case of symmetric matrices, while finding Eigen vectors for a repeated Eigen value λ, we use the property: Eigen vectors of the symmetric matrix A are pairwise orthogonal.**

Example22 Find all the Eigen values and Eigen vectors of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - (2+3+4)\lambda^2 + ([12-6] + [8-3] + [6-2])\lambda - |A| = 0$$

Also $|A| = 2[12-6] - 2[4-3] + 3[2-3] = 7$

$\therefore$ Characteristic equation is $\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0 \ldots (1)$

Clearly $(\lambda - 1)$ is a factor, $\therefore$ rewriting (1) as:

$$\lambda^3 - \lambda^2 - 8\lambda^2 + 8\lambda + 7\lambda - 7 = 0$$

$$\Rightarrow \lambda^2(\lambda - 1) - 8\lambda(\lambda - 1) + 7(\lambda - 1) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 8\lambda + 7) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 1)(\lambda - 7) = 0$$

$\Rightarrow \lambda = 1, 1, 7 \;\therefore$ The required Eigen values are 1, 1 and 7.

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\lambda = 7$, $(A - 7I)X = 0 \Rightarrow \begin{pmatrix} 2-7 & 1 & 1 \\ 2 & 3-7 & 2 \\ 3 & 3 & 4-7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking last two equations:

$x_1 - 2x_2 + x_3 = 0$

$x_1 + x_2 - x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{2-1} = \frac{-x_2}{-1-1} = \frac{x_3}{1+2}$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3} = k_1$, say

$x_1 = k_1 \;,\; x_2 = 2k_1 \;,\; x_3 = 3k_1$

$\therefore$The required Eigen vector corresponding to $\lambda = 7$ is $\begin{pmatrix} k_1 \\ 2k_1 \\ 3k_1 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $k_1 \in R - \{0\}$

$\therefore \boldsymbol{X_1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$

For $\lambda = 1$, $(A - I)X = 0 \Rightarrow \begin{pmatrix} 2-1 & 1 & 1 \\ 2 & 3-1 & 2 \\ 3 & 3 & 4-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 0$

Let $x_1 = k_1, x_2 = k_2 \Rightarrow x_3 = -k_1 - k_2$

$\therefore X = \begin{pmatrix} k_1 \\ k_2 \\ -k_1 - k_2 \end{pmatrix} = X_2 + X_3$, say

Here X_2 and X_3 are two linearly independent vectors.

For X_2, putting $k_1 = 0$ in the vector X

$\Rightarrow X_2 = \begin{pmatrix} 0 \\ k_2 \\ -k_2 \end{pmatrix} = k_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $k_2 \in R - \{0\} \;\therefore \boldsymbol{X_2} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$

For X_3, putting $k_2 = 0$ in the vector X

$$\Rightarrow X_3 = \begin{pmatrix} k_1 \\ 0 \\ -k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, k_1 \in R - \{0\} \therefore \boldsymbol{X_3} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$$

Example 23 Find all the Eigen values and Eigen vectors of the matrix $A = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix}$

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - (5+0+5)\lambda^2 + ([0-4] + [25-0] + [0+4])\lambda - |A| = 0$$

Also $|A| = 5[0-4] - 1[-20-0] + 0 = 0$

$\therefore$ Characteristic equation is $\lambda^3 - 10\lambda^2 + 25\lambda = 0$

$\Rightarrow (\lambda)(\lambda^2 - 10\lambda + 25) = 0$

$\Rightarrow \lambda(\lambda - 5)(\lambda - 5) = 0$

$\Rightarrow \lambda = 0, 5, 5$

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\boldsymbol{\lambda = 0}$, $(A - 0I)X = 0 \Rightarrow \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We may take any two equations for finding relative values of x_1, x_2 , x_3

Taking last two equations:

$x_1 + 0x_2 + 2x_3 = 0$

$0x_1 + 2x_2 + 5x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} 0 & 2 \\ 2 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{0-4} = \frac{-x_2}{5-0} = \frac{x_3}{2-0}$

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{5} = \frac{x_3}{-2} = k_1$, say

$x_1 = 4k_1$, $x_2 = 5k_1$, $x_3 = -2k_1$

$\therefore$The required Eigen vector corresponding to $\lambda = 0$ is $\begin{pmatrix} 4k_1 \\ 5k_1 \\ -2k_1 \end{pmatrix} = k_1 \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$, $k_1 \in R - \{0\}$

$$\therefore \boldsymbol{X_1} = \begin{pmatrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{-2} \end{pmatrix}$$

For $\boldsymbol{\lambda = 5}$, $(A - 5I)X = 0 \Rightarrow \begin{pmatrix} 5-5 & -4 & 0 \\ 1 & 0-5 & 2 \\ 0 & 2 & 5-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives $0x_1 - 4x_2 + 0x_3 = 0$

$x_1 - 5x_2 + 2x_3 = 0$, $0x_1 + 2x_2 + 0x_3 = 0$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking last two equations:

$\Rightarrow x_2 = 0, x_1 + 2x_3 = 0$

Let $x_3 = k_2$, $\Rightarrow x_1 = -2k_1$

$\therefore X_2 = \begin{pmatrix} -2k_2 \\ 0 \\ k_2 \end{pmatrix} = k_1 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, k_2 \in R - \{0\}$

Hence, for the Eigen value $\lambda = 5$, $\boldsymbol{X_2} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$

In this particular case, the equations obtained against each row of the matrix $(A - 5I)X = 0$ are not same, hence the other eigen vector corresponding to repeated Eigen value 5, is a linearly dependent vector toX_2.

Example24 Find all the Eigen values and Eigen vectors of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ and also verify that the Eigen vectors are pairwise orthogonal.

Solution: Here$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$is a symmetric matrix.

The characteristic equation is given by $|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - (6 + 3 + 3)\lambda^2 + ([9 - 1] + [18 - 4] + [18 - 4])\lambda - |A| = 0$

Also $|A| = 6[9 - 1] + 2[-6 + 2] + 2[2 - 6] = 32$

$\therefore$ Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \ldots (1)$

Clearly $(\lambda - 2)$ is a factor, $\therefore$ rewriting (1) as:

$\lambda^3 - 2\lambda^2 - 10\lambda^2 + 20\lambda + 16\lambda - 32 = 0$

$\Rightarrow \lambda^2(\lambda - 2) - 10\lambda(\lambda - 2) + 16(\lambda - 2) = 0$

$\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$

$\Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$

$\Rightarrow \lambda = 2, 2, 8$

$\therefore$ The required Eigen values are 2, 2 and 8.

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\boldsymbol{\lambda = 8}$, $(A - 8I)X = 0 \Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking last two equations:

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{25-1} = \frac{-x_2}{10+2} = \frac{x_3}{2+10}$

$\Rightarrow \frac{x_1}{24} = \frac{x_2}{-12} = \frac{x_3}{12} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k_1$ say

$x_1 = 2k_1 \quad , \quad x_2 = -k_1 \quad , \quad x_3 = k_1$

$\therefore$The required Eigen vector corresponding to $\lambda = 8$ is $\begin{pmatrix} 2k_1 \\ -k_1 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $k_1 \in R - \{0\}$

$\therefore \ \boldsymbol{X_1} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-1} \\ \mathbf{1} \end{pmatrix}$

For $\boldsymbol{\lambda = 2}$, $(A - 2I)X = 0 \Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives $\quad 2x_1 - x_2 + x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

All the three equations are same.

Let $x_1 = k_1$, $x_2 = k_2 \Rightarrow x_3 = -2k_1 + k_2$

$\therefore X = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 + k_2 \end{pmatrix}$, Putting $k_1 = 0$, $X = \begin{pmatrix} 0 \\ k_2 \\ k_2 \end{pmatrix} = k_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $k_2 \in R - \{0\}$

$\therefore \ \boldsymbol{X_2} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$

Now for the symmetric matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$, we have a repeated eigen value $\lambda = 2$.

Using the property: 'Eigen vectors of the symmetric matrix A are pairwise orthogonal', the third Eigen vector X_3 is orthogonal to the vectors X_1 and X_2.

$\therefore 2x_1 - x_2 + x_3 = 0$

and $\quad 0x_1 + x_2 + x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{-1-1} = \frac{-x_2}{2-0} = \frac{x_3}{2-0}$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{-2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1} = k_3$, say

$x_1 = k_3 \quad , \quad x_2 = k_3 \quad , \quad x_3 = -k_3$

$\therefore$The second Eigen vector corresponding to $\lambda = 2$ is $\begin{pmatrix} k_3 \\ k_3 \\ -k_3 \end{pmatrix} = k_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $k_3 \in R - \{0\}$

$\therefore \; \boldsymbol{X_3} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$

Verification: $X_1^T X_2 = (2 \quad -1 \quad 1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$

$X_2^T X_3 = (0 \quad 1 \quad 1)\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

$X_3^T X_1 = (1 \quad 1 \quad -1)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$

Hence, Eigen vectors are pair wise orthogonal

9. Diagonalization of a Matrix

A square matrix having all non-diagonal elements zero is called a diagonal matrix.

Preposition1: A square matrix is said to be diagonalizable if there exists a non-singular matrix P, such that $P^{-1}AP$ is a diagonal matrix. The matrix P is termed as Modal matrix.

Preposition2: A square matrix of order n is diagonalizable if and only if it has n linearly independent vectors.

Preposition3: The diagonal matrix $D = P^{-1}AP = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the n Eigen values of the matrix A. The matrix D is termed as Spectral matrix

Algorithm to find the matrix P, such that $P^{-1}AP$ is a diagonal matrix.

Step1: Find the Eigen values and the Eigen vectors of the given matrix A.

Step2: Construct a matrix P, whose columns are the set of independent eigen vectors, $X_1, X_2, \dots, X_n$.

Step3: Check if the matrix P is invertible, i.e.,$|P| \neq 0$ and hence find P^{-1}.

Step4: Compute$D = P^{-1}AP$, such that D is a diagonal matrix.

Example25 For the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, find the matrix P such that $P^{-1}AP$ is a diagonal matrix. Also find $D = P^{-1}AP$.

Solution: For finding the matrix P, first let us compute the eigen vectors of the matrix A.

The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - (3 + 5 + 3)\lambda^2 + ([15 - 1] + [9 - 1] + [15 - 1])\lambda - |A| = 0$$

Also $|A| = 3[15 - 1] + 1[-3 + 1] + 1[1 - 5] = 36$

$\therefore$ Characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$ ... (1)

Clearly $(\lambda - 2)$ is a factor, $\therefore$ rewriting (1) as:

$\lambda^3 - 2\lambda^2 - 9\lambda^2 + 18\lambda + 18\lambda - 36 = 0$

$\Rightarrow \lambda^2(\lambda - 2) - 9\lambda(\lambda - 2) + 18(\lambda - 2) = 0$

$\Rightarrow (\lambda - 2)(\lambda^2 - 9\lambda + 18) = 0$

$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$

$\Rightarrow \lambda = 2, 3, 6$

$\therefore$ The required Eigen values are 2, 3 and 6.

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\lambda = 2$, $(A - 2I)X = 0 \Rightarrow \begin{pmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking last two equations:

$-x_1 + 3x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{3-1} = \frac{-x_2}{-1+1} = \frac{x_3}{1-3}$

$\Rightarrow \frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k_1$, say

$x_1 = k_1$, $x_2 = 0$, $x_3 = -k_1$

$\therefore$The required Eigen vector corresponding to $\lambda = 2$ is $\begin{pmatrix} k_1 \\ 0 \\ -k_1 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $k_1 \in R - \{0\}$

$\therefore$ $\boldsymbol{X_1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{-1} \end{pmatrix}$

For $\lambda = 3$, $(A - 3I)X = 0 \Rightarrow \begin{pmatrix} 3-3 & -1 & 1 \\ -1 & 5-3 & -1 \\ 1 & -1 & 3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking last two equations:

$-x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 + 0x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{0-1} = \frac{-x_2}{0+1} = \frac{x_3}{1-2}$

$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k_2$, say

$x_1 = k_2 \quad , \quad x_2 = k_2 \quad , \quad x_3 = k_2$

$\therefore$The required Eigen vector corresponding to $\lambda = 3$ is $\begin{pmatrix} k_2 \\ k_2 \\ k_2 \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $k_2 \in R - \{0\}$

$\therefore \ \boldsymbol{X_2} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$

For $\boldsymbol{\lambda = 6}$, $(A - 6I)X = 0 \Rightarrow \begin{pmatrix} 3-6 & -1 & 1 \\ -1 & 5-6 & -1 \\ 1 & -1 & 3-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking last two equations:

$-x_1 - x_2 - x_3 = 0$

$x_1 - x_2 - 3x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} -1 & -1 \\ -1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -1 \\ 1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix}}$

$\Rightarrow \frac{x_1}{3-1} = \frac{-x_2}{3+1} = \frac{x_3}{1+1}$

$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1} = k_3$, say

$x_1 = k_3 \quad , \quad x_2 = -2k_3 \quad , \quad x_3 = k_3$

$\therefore$The required Eigen vector corresponding to $\lambda = 6$ is $\begin{pmatrix} k_3 \\ -2k_3 \\ k_3 \end{pmatrix} = k_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $k_3 \in R - \{0\}$

$\therefore \ \boldsymbol{X_3} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-2} \\ \mathbf{1} \end{pmatrix}$

Now the modal matrix $P = (X_1 \quad X_2 \quad X_3)$

Hence $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$ and $P^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

$\therefore$ Spectral matrix $D = P^{-1}AP = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Example26 The Eigen vectors of a matrix A corresponding to Eigen values $7, -5$ and 1 are $[4 \quad -1 \quad -3]^T, [-3 \quad 1 \quad 3]^T$ and $[6 \quad -2 \quad -5]^T$ respectively. Find the matrix A and also calculate A^2.

Solution: We have

$$P = \begin{pmatrix} 4 & -3 & 6 \\ -1 & 1 & -2 \\ -3 & 3 & -5 \end{pmatrix}, D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, $|P| = 4(-5+6) + 1(15-18) - 3(6-6) = 1 \neq 0$

$C_{11} = (-1)^2(-5+6) = 1 \;\; C_{12} = (-1)^3(5-6) = 1 \;\; C_{13} = (-1)^4(-3+3) = 0$

$C_{21} = (-1)^3(15-18) = 3 \;\; C_{22} = (-1)^4(-20+18) = -2 \;\; C_{23} = (-1)^5(12-9) = -3$

$C_{31} = (-1)^4(6-6) = 0 \;\; C_{32} = (-1)^5(-8+6) = 2 \;\; C_{33} = (-1)^6(4-3) = 1$

$$\therefore adj(P) = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -2 & -3 \\ 0 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 0 \\ 1 & -2 & 2 \\ 0 & -3 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{adj(P)}{|P|} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & -2 & 2 \\ 0 & -3 & 1 \end{pmatrix}$$

We know that $P^{-1}AP = D$ such that $A = PDP^{-1}$

$$\therefore \mathrm{A} = \begin{pmatrix} 4 & -3 & 6 \\ -1 & 1 & -2 \\ -3 & 3 & -5 \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 1 & -2 & 2 \\ 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 43 & 36 & 36 \\ -12 & -5 & -12 \\ -36 & -18 & -35 \end{pmatrix}$$

Now, $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$ $\qquad [as\ P^{-1}P = I]$

$$= \begin{pmatrix} 4 & -3 & 6 \\ -1 & 1 & -2 \\ -3 & 3 & -5 \end{pmatrix} \begin{pmatrix} 49 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 1 & -2 & 2 \\ 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 121 & 720 & -144 \\ -24 & -191 & 48 \\ -72 & -576 & 145 \end{pmatrix}$$

Note: We can also compute A^4 which is equal to PD^4P^{-1} and so on.

10. Quadratic Forms of a Matrix

A quadratic form in n variables is a polynomial in which every term has degree two.

For example: $x^2 - 2xy + y^2, 3y^2 + z^2 + 2xy + 4xz - 8yz$ etc.

Associated Symmetric Matrix

The symmetric matrix$A = [a_{ij}]_{n \times n}$ determines a quadratic form Q in n variables, such that

$$Q(x_1, x_2 \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j = X^T A X, \text{ where } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Above preposition leads to following results:

1. Matrix A of the quadratic form $Q(x)$ is a symmetric matrix.

2. Any quadratic form can be expressed as $Q(x) = X^T AX$, where$A$ is the matrix of the quadratic form $Q(x)$.
3. Coefficients of x_1^2, $x_2^2, \dots, x_n^2$ in the quadratic form $Q(x)$ are the principal diagonal elements $a_{11}, a_{22} \dots, a_{nn}$ of the matrix A.
4. The elementsa_{12} and $a_{21}$of the Matrix$A$are half the coefficient of x_1x_2 in the quadratic form $Q(x)$; theelements a_{13} and $a_{31}$of the Matrix are half the coefficient of x_1x_3; similarly the elements a_{23} and a_{32} of the Matrix are half the coefficient of $x_2x_3$and so on.

Example27 Write the matrix of the quadratic form $2x^2 + 6xy + 3y^2$.

Solution: For the quadratic form $Q(x) = 2x^2 + 6xy + 3y^2$, if x be taken as x_1, and y be taken as x_2, the matrix of the quadratic form$Q(x)$is $A = \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$.

Example28 Write the matrix of the quadratic form $x^2 + 3z^2 + 2xy + 6xz - 2yz$.

Solution: For the quadratic form $Q(x) = x^2 + 3z^2 + 2xy + 6xz - 2yz$, if x be taken as x_1 , y be taken as x_2, and z be taken as x_3, the matrix of the quadratic form $Q(x)$ is

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix}$$

Example29 Write the quadratic form of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & -3 \\ 4 & -3 & 3 \end{bmatrix}$

Solution: If x, y, z are the orderly elements x_1, x_2, $x_3$of the quadratic form $Q(x)$, the coefficients of x^2, y^2, z^2 are the principal diagonal elements of the matrix A.

Also, $a_{12} = a_{21} = \frac{1}{2}$ (the coefficient of xy)

$a_{13} = a_{31} = \frac{1}{2}$ (the coefficient of xz)

$a_{23} = a_{32} = \frac{1}{2}$ (the coefficient of yz)

$\therefore Q(x) = x^2 + 2y^2 + 3z^2 + 4xy + 8xz - 6yz$

10.1 Nature, Index and Signature of Quadratic Forms

The nature, index and signature of a quadratic form $Q(x) = X^T AX$ can be determined by computing the eigen values of the associated matrix A.

Nature

- A real quadratic form $Q(x) = X^T AX$ is said to be positive definite if all the Eigen values of matrix A are positive, and it is positive semi-definite if the Eigen values are non-negative, i.e. either positive or zero.
- A real quadratic form $Q(x) = X^T AX$ is said to be negative definite if all the Eigen values of matrix A are negative, and it is negative semidefinite if the eigenvalues are either negative or zero.
- A real quadratic form $Q(x) = X^T AX$ is indefinite if the Eigen values of matrix A are both positive and negative.

Index

- The index of the quadratic form is equal to the number of positive Eigen values of the matrix of quadratic form

Signature

- The signature of the quadratic form is equal to the difference between the number of positive Eigen values and the number of negative Eigen values of the matrix of quadratic form

Example30 Determine the nature, index and signature of the quadratic form

$$6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$$

Solution: For the quadratic form $Q(x) = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$, if x be taken as x_1 , y be taken as x_2, and z be taken as x_3, the matrix of the quadratic form $Q(x)$ is given by: $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - (6+3+3)\lambda^2 + ([9-1] + [18-4] + [18-4])\lambda - |A| = 0$

Also, $|A| = 6[9-1] + 2[-6+2] + 2[2-6] = 32$

$\therefore$ Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \ldots (1)$

Clearly $(\lambda - 2)$ is a factor, $\therefore$ rewriting (1) as:

$\lambda^3 - 2\lambda^2 - 10\lambda^2 + 20\lambda + 16\lambda - 32 = 0$

$\Rightarrow \lambda^2(\lambda - 2) - 10\lambda(\lambda - 2) + 16(\lambda - 2) = 0$

$\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$

$\Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$

$\Rightarrow \lambda = 2, 2, 8$

$\therefore$ The required Eigen values are 2, 2 and 8.

Since the Eigen values of the associated matrix are only positive, hence the nature of quadratic form is positive definite.

Index:3 (number of positive Eigen values)

Signature: $3 - 0 = 3$ (difference between number of positive and negative Eigen values)

10.2 Reduction of Quadratic form to Canonical form

A quadratic form $Q(x)$ is transformed to canonical form by converting the associated matrix A to a diagonal matrix. Here the diagonal matrix $D = N^T A N = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where N is the normalized matrix of eigenvectors, i.e. normalized modal matrix P, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n Eigen values of the matrix A

Algorithm to Reduce a Quadratic Form $Q(x)$ to Canonical form

Step1: Write the matrix A of the quadratic form $Q(x)$.

Step2: Find the Eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$and the eigen vectors$X_1, X_2, \ldots, X_n$ of thematrix A.

Step3: Construct the modal matrix P, whose columns are the set of independent Eigen vectors, $X_1, X_2, \dots, X_n$.

Step4: Write the normalized matrix N by normalizing each vector $X_1, X_2, \dots, X_n$ of matrix P.

Step5: Compute$D = N^T AN$, such that D is a diagonal matrix.

Step6: The canonical form of $Q(x)$ is given by $d_{11}x_1^2 + d_{22}x_2^2 + \cdots + d_{nn}x_n^2$, where d_{11}, d_{22}.., d_{nn} are the diagonal elements of the matrix D.

Example31 Reduce the quadratic form $2xy + 2yz + 2zx$ into the canonical form and discuss the nature.

Solution: For the quadratic form $Q(x) = 2xy + 2yz + 2zx$, if x be taken as x_1 , y be taken as x_2, and z be taken as x_3, the matrix of the quadratic form $Q(x)$ is given by:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is given by $|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - (0 + 0 + 0)\lambda^2 + ([0 - 1] + [0 - 1] + [0 - 1])\lambda - |A| = 0$

Also $|A| = 0 - 1[0 - 1] + 1[1 - 0] = 2$

$\therefore$ Characteristic equation is $\lambda^3 - 3\lambda - 2 = 0$ ... ①

Clearly $(\lambda + 1)$ is a factor, $\therefore$ rewriting ① as:

$\lambda^3 + \lambda^2 - \lambda^2 - \lambda - 2\lambda - 2 = 0$

$\Rightarrow \lambda^2(\lambda + 1) - \lambda(\lambda + 1) - 2(\lambda + 1) = 0$

$\Rightarrow (\lambda + 1)(\lambda^2 - \lambda - 2) = 0$

$\Rightarrow (\lambda + 1)(\lambda - 2)(\lambda + 1) = 0$

$\Rightarrow \lambda = 2, -1, -1$

$\therefore$ The required Eigen values are 2, -1 and -1.

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\lambda = 2$, $(A - 2I)X = 0 \Rightarrow \begin{pmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of x_1, x_2 , x_3

Taking last two equations:

$x_1 - 2x_2 + x_3 = 0$

$x_1 + x_2 - 2x_3 = 0$

$\Rightarrow \frac{x_1}{4-1} = \frac{-x_2}{-2-1} = \frac{x_3}{1+2}$

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k_1$ say

$x_1 = k_1$, $x_2 = k_1$, $x_3 = k_1$

$\therefore$The required Eigen vector corresponding to $\lambda = 2$ is $\begin{pmatrix} k_1 \\ k_1 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $k_1 \in R - \{0\}$

$\therefore \ \boldsymbol{X_1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$

For $\boldsymbol{\lambda = -1}$

$(A + I)X = 0 \Rightarrow \begin{pmatrix} 0+1 & 1 & 1 \\ 1 & 0+1 & 1 \\ 1 & 1 & 0+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

This gives $x_1 + x_2 + x_3 = 0$

Let $x_1 = k_1$, $x_2 = k_2 \Rightarrow x_3 = -k_1 - k_2$

$\therefore X = \begin{pmatrix} k_1 \\ k_2 \\ -k_1 - k_2 \end{pmatrix}$, Putting $k_1 = 0$, $X = \begin{pmatrix} 0 \\ k_2 \\ -k_2 \end{pmatrix} = k_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $k_2 \in R - \{0\}$

$\therefore \ \boldsymbol{X_2} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$

Now for the symmetric matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, we have a repeated eigen value $\lambda = -1$.

Using the property: 'Eigen vectors of the symmetric matrix A are pairwise orthogonal', the third Eigen vector X_3 is orthogonal to the vectors X_1 and X_2.

$\therefore x_1 + x_2 + x_3 = 0$

and $0x_1 + x_2 - x_3 = 0$

$\Rightarrow \frac{x_1}{-1-1} = \frac{-x_2}{-1-0} = \frac{x_3}{1-0}$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{1} = k_3$ say

$x_1 = -2k_3$, $x_2 = k_3$, $x_3 = k_3$

$\therefore$The second Eigen vector corresponding to $\lambda = -1$ is $\begin{pmatrix} -2k_3 \\ k_3 \\ k_3 \end{pmatrix} = k_3 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $k_3 \in R - \{0\}$

$\therefore \ \boldsymbol{X_3} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$

Hence the modal matrix $P = (X_1 \quad X_2 \quad X_3)$

$\therefore P = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Again the normalized matrix $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $N^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$$\therefore D = N^T A N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Hence the required canonical form is $\lambda_1 x^2 - \lambda_2 y^2 - \lambda_3 z^2$

i.e. $2x^2 - y^2 - z^2$

Also since the Eigen values of the associated matrix are both positive and negative, hence the nature of quadratic form is indefinite.

Example32 Reduce the quadratic form $x^2 + 2y^2 + z^2 - 2xy + 2yz$ into the canonical form

Solution: For the quadratic form $Q(x) = x^2 + 2y^2 + z^2 - 2xy + 2yz$, if x be taken as x_1 , y be taken as x_2, and z be taken as x_3, the matrix of the quadratic form $Q(x)$ is given by:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The characteristic equation is given by $|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - (1+2+1)\lambda^2 + ([2-1] + [1-0] + [2-1])\lambda - |A| = 0$

Also, $|A| = 1[2-1] + 1[-1-0] + 0 = 0$

$\therefore$ Characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda = 0 \ldots (1)$

Clearly $(\lambda - 1)$ is a factor, $\therefore$ rewriting (1) as:

$\lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda = 0$

$\Rightarrow \lambda^2(\lambda - 1) - 3\lambda(\lambda - 1) = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 3\lambda) = 0$

$\Rightarrow (\lambda - 1)\lambda(\lambda - 3) = 0$

$\Rightarrow \lambda = 0,1,3$ $\quad\therefore$ The required Eigen values are 0, 1 and 3.

To find the Eigen vectors corresponding to the Eigen values, use the equation $(A - \lambda I)X = 0$

For $\lambda = 0$, $(A - 0I)X = 0 \Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We may take any two equations for finding relative values of x_1, x_2, x_3

Taking first two equations:

$x_1 - x_2 + 0x_3 = 0$

$-x_1 + 2x_2 + x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}} = k_1$, say

$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{1} = \frac{x_3}{1} = k_1$

$\Rightarrow x_1 = -k_1 \ , x_2 = -k_1 \, , x_3 = k_1$

$\therefore$The required Eigen vector corresponding to $\lambda = 2$ is

$$\begin{pmatrix} -k_1 \\ -\ k_1 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ -\ 1 \\ 1 \end{pmatrix} or \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad k_1 \in R - \{0\}$$

$$\therefore \boldsymbol{X_1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$$

For $\lambda = 1$, $(A - I)X = 0 \Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of $x_1, x_2\ , x_3$

Taking first two equations:

$0\,x_1 - x_2 + 0x_3 = 0$

$-x_1 + x_2 + x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = k_2$, say

$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{-1} = k_2$

$\Rightarrow x_1 = -k_2 \ , x_2 = 0 \, , x_3 = -k_2$

$\therefore$The required Eigen vector corresponding to $\lambda = 2$ is

$$\begin{pmatrix} -k_2 \\ 0 \\ -k_2 \end{pmatrix} = k_2 \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} or \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, k_2 \in R - \{0\}\, , \therefore \boldsymbol{X_2} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

For $\lambda = 3$, $(A - I)X = 0 \Rightarrow \begin{pmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may take any two equations for finding relative values of $x_1, x_2\ , x_3$

Taking first two equations:

$-2x_1 - x_2 + 0x_3 = 0$ and $-x_1 - x_2 + x_3 = 0$

$\Rightarrow \frac{x_1}{\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -1 \end{vmatrix}} = k_3$, say

$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{-2} = \frac{x_3}{1} = k_3$

$\Rightarrow x_1 = -k_3 \ , x_2 = 2k_3 \ , x_3 = k_3$

$\therefore$The required Eigen vector corresponding to $\lambda = 2$ is $\begin{pmatrix} -k_3 \\ 2k_3 \\ k_3 \end{pmatrix} = k_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $k_3 \in R - \{0\}$

$\therefore \boldsymbol{X_3} = \begin{pmatrix} \mathbf{-1} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix}$

Hence the modal matrix $P = (X_1 \quad X_2 \quad X_3)$

$\therefore P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

Again the normalized matrix $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $N^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$\therefore D = N^T A N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

Hence the required canonical form is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$, i.e. $y^2 + 3z^2$

11. Cayley- Hamilton Theorem

Every square matrix satisfies its own characteristic equation, i.e. if the characteristic equation of the n^{th} order square matrix is $|A - \lambda I| = (-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = 0$, then: $(-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0$

Example33 Verify Cayley- Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

and hence find $A^{-1}$and A^4

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - (1 - 1 + 1)\lambda^2 + \lambda([-1 - 0] + [1 - 0] + [-1 - 4]) - |A| = 0$

$|A| = 1(-1 - 0) - 2(2 - 0) + 0 = -5$

$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$

By Cayley-Hamilton Theorem, $A^3 - A^2 - 5A + 5I = 0$

Verification:

$\boldsymbol{A^2 = A.A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 - A^2 - 5A + 5I = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^{-1}, by Cayley- Hamilton theorem$A^3 - A^2 - 5A + 5I = 0$

Multiplying both sides by A^{-1} , we get $A^2 - A - 5I + 5A^{-1} = 0$

$$A^{-1} = \frac{1}{5}(-A^2 + A + 5I)$$

$$= \frac{1}{5}\left(\begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

To find A^4, by Cayley- Hamilton theorem $A^3 - A^2 - 5A + 5I = 0$

Multiplying both sides by A , we get $A^4 - A^3 - 5A^2 + 5A = 0$

$$A^4 = A^3 + 5A^2 - 5A = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example34 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, use Cayley- Hamilton theorem to express

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - (1+3)\lambda + |A| = 0$$

$$|A| = 3 + 2 = 5$$

$$\therefore \lambda^2 - 4\lambda + 5 = 0$$

By Cayley-Hamilton Theorem, $A^2 - 4A + 5I = 0 \Rightarrow A^2 = 4A - 5I$

$\therefore A^3 = 4A^2 - 5A$, $A^4 = 4A^3 - 5A^2$, $A^5 = 4A^4 - 5A^3$, $A^6 = 4A^5 - 5A^4$

$$\Rightarrow A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = (4A^5 - 5A^4) - 4A^5 + 8A^4 - 12A^3 + 14A^2$$

$$= 3A^4 - 12A^3 + 14A^2$$

$$= 3(4A^3 - 5A^2) - 12A^3 + 14A^2$$

$$= -A^2 = -4A + 5I$$

Test Your Knowledge

1) Find the values of x, y, z and α which satisfy the matrix equation

$$\begin{pmatrix} x+3 & 2y+x \\ x-1 & 4\alpha-6 \end{pmatrix} = \begin{pmatrix} 0 & -7 \\ 3 & 2\alpha \end{pmatrix}$$

2) Express the matrix $A = \begin{pmatrix} 2 & 1 & 5 \\ -1 & 2 & 6 \\ 3 & 2 & 9 \end{pmatrix}$ as the sum of symmetric and skew-symmetric matrix.

3) Express the matrix $A = \begin{pmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{pmatrix}$ as the sum of Hermitian and skew-Hermitian matrices.

4) Verify if the following matrices are orthogonal and hence find their inverse

a) $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

b) $A = \begin{bmatrix} cos\emptyset & -sin\emptyset & 0 \\ sin\emptyset & cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5) Find l, m, n and A^{-1} if $A = \begin{pmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{pmatrix}$ is orthogonal.

6) Prove that the matrix $A = \frac{1}{2}\begin{pmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is unitary and hence find A^{-1}

7) Use Gauss-Jordan method to find the inverse of the matrix $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

8) Find the rank of the matrix

a) $\begin{pmatrix} 2\ 3\ 4\ -1 \\ 5\ 2\ 0\ -1 \\ -4 5 1 2 -1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 1\ 1 \\ 1 & 3 & -21 \\ 2 & 0 & -32 \\ 3 & 3 & 0\ 3 \end{pmatrix}$

9) Reduce the following matrix into normal form and hence find its rank

a) $\begin{pmatrix} 1232 \\ 2351 \\ 1345 \end{pmatrix}$

b) $\begin{pmatrix} 8 & 1 & 3\ 6 \\ 0 & 3 & 2\ 2 \\ -8 & -1 & -34 \end{pmatrix}$

10) Find the values of a and b such that the rank of matrix $A = \begin{bmatrix} 1-2\ 3\ 1 \\ 2\ 1\ -12 \\ 6-2\ a\ b \end{bmatrix}$ is 2

11) For what value of λ and μ, the system of equations $x + y + z = 6,\ x + 2y + 3z = 10,$ $x + 2y + \lambda z = \mu$ have

a) no solution

b) unique solution

c) more than one solution

12) Investigate whether the set of equations is consistent or not, and, if consistent, solve it

$2x - y - z = 2, x + 2y + z = 2, 4x - 7y - 5z = 2$

13) Show that the vectors $(1 \quad 2 \quad 4), (2 \quad -1 \quad 3), (0 \quad 1 \quad 2)$ and $(-3 \quad 7 \quad 2)$ are linearly dependent and find the relation between them.

14) Find the Eigen values and Eigen vectors of the matrix

a) $\begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

15) Obtain the Eigen values and Eigen vectors for the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and verify that their Eigen vectors are orthogonal.

16) Verify Cayley- Hamilton theorem for the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

17) Verify Cayley- Hamilton theorem for the matrix$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. Also express

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ as quadratic polynomial in A.

18) Find the characteristic equation for the given matrix and find its inverse $A = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{pmatrix}$

19) Find the Modal matrix and Spectral matrix for the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

20) The eigen vectors of a matrix A corresponding to eigen values 1,1 and 3 are $[1 \quad 0 \quad -1]^T, [0 \quad 1 \quad -1]^T$ and $[1 \quad 1 \quad 0]^T$ respectively. Find the matrix A

21) Diagonalize the matrix $A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$

22) Write down the quadratic form corresponding to the following matrices

a) $\begin{pmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$

23) Write down the matrix of the following quadratic form

a) $2x^2 + 3y^2 + 6xy$

b) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

24) Reduce the quadratic form $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2$ to canonical form.

25) Reduce $2x^2 + 2y^2 - z^2 - 4yz + 4xz - 8xy$ to canonical form. Also find the rank, index, signature and value class (nature)

Answers

1) $x = -3, y = -2, z = 4, \alpha = 3$

2) $A = P + Q$, where $P = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 4 \\ 4 & 4 & 9 \end{pmatrix}$ is symmetric matrix and $Q = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}$ is skew-symmetric matrix.

3) $A = P + Q$, where $P = \begin{pmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{pmatrix}$ is hermitian matrix and $Q = \begin{pmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{pmatrix}$ is skew-hermitian matrix.

4) **a)** $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ **b)** $A = \begin{bmatrix} cos\emptyset & sin\emptyset & 0 \\ -sin\emptyset & cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5) $l = \pm\frac{1}{\sqrt{2}}, m = \pm\frac{1}{\sqrt{6}}, n = \pm\frac{1}{\sqrt{3}}$ and $A^{-1} = \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{2}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} & \mp\frac{1}{\sqrt{6}} \\ \pm\frac{1}{\sqrt{3}} & \mp\frac{1}{\sqrt{3}} & \pm\frac{1}{\sqrt{3}} \end{pmatrix}$

6) $A^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$

7) $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$

8) **a)** 2 **b)** 3

9) **a)** $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, 2 **b)** $[I_3 \quad 0]$, 3

10) $a = 4, b = 6$

11) **a)** $\lambda = 3, \mu \neq 10$ **b)** $\lambda \neq 3, \mu$ may have any value **c)** $\lambda = 3, \mu = 10$

12) consistent, $x = \frac{6+k}{5}, y = \frac{2-3k}{5}, z = k$ where $k \in R$

13) linearly dependent, $9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$

14) a) 3,4 ; $[3 \quad 2]^T, [1 \quad 4]^T$ **b)** 1,2,4 ; $[0 \quad 1 \quad -1]^T, [1 \quad 1 \quad -1]^T, [3 \quad 5 \quad 1]^T$

c) 1,1,3 ; $[1 \quad -3 \quad 2]^T, [1 \quad -1 \quad 2]^T, [1 \quad 1 \quad 0]^T$

15) 1,2,3; $[1 \quad 0 \quad -1]^T, [0 \quad 1 \quad 0]^T, [1 \quad 0 \quad 1]^T$, vectors are orthogonal

16) verified, $A^3 - 6A^2 + 9A - 4I = 0$

17) verified, $A^2 + A + I$

18) $A^3 - 3A^2 - A - I = 0, A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$

19) modal= $\begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, spectral= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

20) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

21) $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

22) a) $2x^2 + 3y^2 + z^2 + 8xy + 2yz + 10zx$ **b)** $x_1^2 - 4x_2^2 + 6x_3^2 + 2x_4^2 + 2x_1x_2 - 4x_1x_3 - 6x_3x_4$

23) a) $\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -1/2 \\ 4 & -1/2 & -6 \end{bmatrix}$

24) $2y_1^2 + 3y_2^2 + 6y_3^2$

25) $-2y_1^2 - 2y_2^2 + 7y_3^2$; 3, 1, −1, indefinite

UNIT-4: VECTOR CALCULUS

1. Scalar and Vector Fields

Let $P(x, y, z)$ be any point in the domain of definition 'R'. Then, we define a scalar point function $\phi(P) = \phi(x, y, z)$, which depends on the point 'P' in the space; parallelly, we define the vector point function $\vec{V}(P) = \vec{V}(x, y, z)$, whose values are vectors. Also, corresponding region 'R', where the functions are defined, are the scalar or vector fields.

Derivative: A vector $\vec{V}(t)$ is said to be differentiable at a point 't', if $\lim\limits_{\delta t \to 0} \frac{\vec{V}(t+\delta t)-\vec{V}(t)}{\delta t}$ exists.

Here $\vec{V}'(t) = \lim\limits_{\delta t \to 0} \frac{\vec{V}(t+\delta t)-\vec{V}(t)}{\delta t}$ is called the derivative of $\vec{V}(t)$.

In the component form: $\vec{V}'(t) = \left[\vec{V_1}'(t), \vec{V_2}'(t), \vec{V_3}'(t)\right]$; therefore, $\vec{V}'(t)$ is obtained by differentiating each component separately.

Important Properties:

1. $(\vec{u}.\vec{v})' = \vec{u}.\vec{v}' + \vec{u}'.\vec{v}$
2. $(\vec{u} \times \vec{v})' = \vec{u} \times \vec{v}' + \vec{u}' \times \vec{v}$
3. $(c\vec{u})' = c\vec{u}'$
4. $(\vec{u} + \vec{v})' = \vec{u}' + \vec{v}'$

Example1 If $\vec{a} = 5t^2\hat{\imath} + t\hat{\jmath} - t^3\hat{k}$ and $\vec{b} = \sin t\,\hat{\imath} - \cos t\,\hat{\jmath}$, find $\frac{d}{dt}(\vec{a}.\vec{b})$ and $\frac{d}{dt}(\vec{a} \times \vec{b})$

Solution: $\frac{d}{dt}(\vec{a}.\vec{b}) = \vec{a}.\vec{b}' + \vec{a}'.\vec{b}$

$= (5t^2\hat{\imath} + t\hat{\jmath} - t^3\hat{k}).(\cos t\,\hat{\imath} + \sin t\hat{\jmath}) + (10t\hat{\imath} + \hat{\jmath} - 3t^2\hat{k}).(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath})$

$= 5t^2\cos t + t\sin t + 10t\sin t - \cos t = 5t^2\cos t + 11t\sin t - \cos t$

Also, $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \vec{b}' + \vec{a}' \times \vec{b}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{\imath}(t^3\sin t) - \hat{\jmath}(t^3\cos t) + \hat{k}(5t^2\sin t - t\cos t) +$$
$$\hat{\imath}(-3t^2\cos t) - \hat{\jmath}(3t^2\sin t) + \hat{k}(-10t\cos t - \sin t)$$
$$= \hat{\imath}(t\sin t - 3\cos t)t^2 - \hat{\jmath}(t\cos t + 3\sin t)t^2 + \hat{k}\left((5t^2 - 1)\sin t - 11t\cos t\right)$$

Note: Derivatives can be found directly by performing dot and cross products first and differentiating component-wise.

2. Representation of curves in vector form

A curve 'C' may be represented in vector form taking 't' as a parameter is shown as:

C: $\vec{r}(t) \equiv \langle t_1, t_2, t_3\rangle \equiv t_1\hat{\imath} + t_2\hat{\jmath} + t_3\hat{k}$

For example: 1. $x^2 + y^2 = 4$, $z = 0$ is a circle with centre at $(0,0)$ and radius 2.

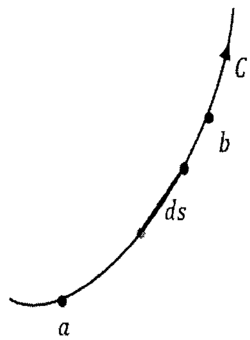
$$\Rightarrow \vec{r}(t) \equiv \langle 2\cos t, 2\sin t, 0\rangle \equiv 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} + 0\hat{k}$$

2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$

$\Rightarrow \vec{r}(t) \equiv \langle a\cos t, b\sin t, 0\rangle \equiv a\cos t\,\hat{\imath}+b\sin t\,\hat{\jmath}+0\hat{k}$

3. Arc Length

Let a curve 'C' be defined by the vector function $\vec{r}(t) \equiv \langle x(t),\ y(t),\ z(t)\rangle$ in the interval $a \le t \le b$, then the arc length $S=\int_a^b dS$ ...①



Now, $dS=\sqrt{(dx)^2+(dy)^2+(dz)^2}$ ...②

Substituting dS from ② in ① and multiplying by $\frac{dt}{dt}$, we get

$$S=\int_a^b \sqrt{(dx)^2+(dy)^2+(dz)^2}\,\frac{dt}{dt}$$

$$\Rightarrow S=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt \ \text{...③}$$

Also, $\vec{r}(t)=x(t)\hat{\imath}+y(t)\hat{\jmath}+z(t)\hat{k}$

$$\Rightarrow \vec{r}'(t)=\frac{dx}{dt}\hat{\imath}+\frac{dy}{dt}\hat{\jmath}+\frac{dz}{dt}\hat{k}$$

$$\therefore |\vec{r}'(t)|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} \ \text{...④}$$

Using ④ in ③, we get: $S=\int_a^b |\vec{r}'(t)|\,dt$ ...⑤

From ① and ⑤, we have: $\boldsymbol{S=\int_a^b dS=\int_a^b |\vec{r}'(t)|\,dt}$

$\Rightarrow dS=|\vec{r}'(t)|dt$ or $\frac{ds}{dt}=|\vec{r}'(t)|$

Example2 Find arc length of the vector function $\vec{r}(t)=\sqrt{2}t\hat{\imath}+e^t\hat{\jmath}+e^{-t}\hat{k}, 0\le t\le 1$

Solution: We have $S=\int_0^1 |\vec{r}'(t)|\,dt$

Here, $\vec{r}(t)=\langle\sqrt{2}t, e^t, e^{-t}\rangle$

$\Rightarrow \vec{r}'(t)=\langle\sqrt{2}, e^t, -e^{-t}\rangle$

$\Rightarrow |\vec{r}'(t)|=\sqrt{2+e^{2t}+e^{-2t}}$

$$\therefore S=\int_0^1 |\vec{r}'(t)|\,dt=\int_0^1 \sqrt{2+e^{2t}+e^{-2t}}\,dt=\int_0^1 \sqrt{(e^t+e^{-t})^2}\,dt$$

$$=\int_0^1 (e^t+e^{-t})\,dt=[e^t-e^{-t}]_0^1=e-\frac{1}{e}=\frac{e^2-1}{e}$$

Example3 Find the arc length of the position of the curve $x=\cos t, y=\sin t, z=\sqrt{3}t$ from the point (3,0,0) to $(-3,0,\sqrt{3}\pi)$

Solution: Conveniently taking limits on z, $S=\int_0^\pi |\vec{r}'(t)|\,dt$

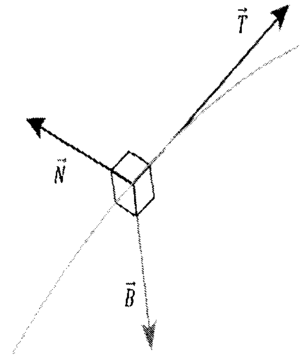
$\because \sqrt{3}t=0 \Rightarrow t=0$ and $\sqrt{3}t=\sqrt{3}\pi \Rightarrow t=\pi$

Here, $\vec{r}(t)=\langle\cos t,\ \sin t, \sqrt{3}t\rangle$

$\Rightarrow \vec{r}'(t)=\langle-\sin t,\ \cos t, \sqrt{3}\rangle \ \Rightarrow |\vec{r}'(t)|=\sqrt{\sin^2 t+\cos^2 t+3}=\sqrt{4}=2$

$\therefore S=\int_0^\pi |\vec{r}'(t)|\,dt=\int_0^\pi 2\,dt=2\pi$

4. Tangent, Normal and Binormal Vectors



For a position vector $\vec{r}(t)$, the tangent vector is $\vec{r}'(t)$ and the unit tangent vector $\vec{T}(t)$ is given by: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

The unit normal vector $\vec{N}(t)$ is orthogonal to unit tangent vector and is defined as : $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$

The unit binormal vector $\vec{B}(t)$ is perpendicular to both $\vec{T}(t)$ and $\vec{N}(t)$ and is given by: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Example4 If a position vector $\vec{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$, find the unit tangent, unit normal and binormal vectors.

Solution: Here, $\vec{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$

$\Rightarrow \vec{r}'(t) = \langle 3\cos t, -3\sin t, 4\rangle$

$|\vec{r}'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{25} = 5$

Therefore, the unit tangent vector $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$

Again, $\vec{T}'(t) = \langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0\rangle$

$|\vec{T}'(t)| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t + 0} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Therefore, the unit normal vector $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\sin t, -\cos t, 0\rangle$

Now, the binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix}$

$\Rightarrow \vec{B}(t) = \hat{\imath}\left(\frac{4}{5}\cos t\right) - \hat{\jmath}\left(\frac{4}{5}\sin t\right) + \hat{k}\left(-\frac{3}{5}\cos^2 t - \frac{3}{5}\sin^2 t\right)$

$= \langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\rangle$

5. Curvature

Curvature is the measure how fast a curve is changing its direction at a point.

Note: Curvature is defined only for smooth curves, i.e. the curves which do not have any corners or cusps.

By definition, the curvature '$k(S)$' is the magnitude of the rate of change of unit tangent vector with respect to arc length. $\therefore\ k(S) = \left|\frac{d\vec{T}}{dS}\right| = \left|\frac{d\vec{T}/dt}{dS/dt}\right| = \left|\frac{\vec{T}'(t)}{\vec{r}'(t)}\right|$, $\because \frac{ds}{dt} = |\vec{r}'(t)|$

Alternate formulae for curvature:

1. $k(S) = \left|\frac{d\vec{T}}{dS}\right|$, where $\vec{T}$ is the unit tangent vector
2. $k(t) = \left|\frac{\vec{T}'(t)}{\vec{r}'(t)}\right|$
3. $k(t) = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}$

Remarks:
1. Curvature is zero at any point on a straight line.
2. Radius of curvature (ρ) at any point on the curve is reciprocal of Curvature, i.e. $\rho = \frac{1}{k}$

Example5 Find the curvature for the curve: $\vec{r}(t) = \langle 2t, \ t^2, -\frac{t^3}{3}\rangle$

Solution: Here, $\vec{r}(t) = \langle 2t, \ t^2, \ -\frac{t^3}{3}\rangle$

$\Rightarrow \vec{r}'(t) = \langle 2, \ 2t, -t^2\rangle$

$|\vec{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$

Therefore, the unit tangent vector $\vec{T}(t) = \frac{\vec{r}\prime(t)}{|\vec{r}\prime(t)|} = \langle \frac{2}{t^2+2}, \frac{2t}{t^2+2}, \frac{-t^2}{t^2+2}\rangle$

Again, $\vec{T}'(t) = \langle \frac{-4t}{(t^2+2)^2}, \frac{-2t^2+4}{(t^2+2)^2}, \frac{-4t}{(t^2+2)^2}\rangle$

$\Rightarrow |\vec{T}'(t)| = \sqrt{\frac{16t^2+4t^4+16-16t^2+16t^2}{(t^2+2)^4}} = \sqrt{\frac{4(t^2+2)^2}{(t^2+2)^4}} = \frac{2}{t^2+2}$

$\therefore k(t) = \left|\frac{\vec{T}\prime(t)}{\vec{r}\prime(t)}\right| = \frac{2/t^2+2}{t^2+2} = \frac{2}{(t^2+2)^2}$

6. Torsion

The arc rate of rotation of the binormal vector is called Torsion and is denoted by τ.

Thus, $\frac{d\vec{B}}{ds} = -\tau\vec{N} \Rightarrow \frac{d\vec{B}/dt}{dS/dt} = -\tau\vec{N}$ or $\frac{\vec{B}\prime(t)}{|\vec{r}\prime(t)|} = -\tau\vec{N}$,$\because \frac{ds}{dt} = |\vec{r}'(t)|$

We can also find torsion(τ) using the formula $\tau = \frac{[\vec{r}\prime(t)\vec{r}\prime\prime(t)\vec{r}\prime\prime\prime(t)]}{|\vec{r}\prime(t)\times\vec{r}\prime\prime(t)|^2}$, where $[\vec{r}'(t)\vec{r}''(t)\vec{r}'''(t)]$ represents scalar triple product of $\vec{r}'(t)$, $\vec{r}''(t)$ and $\vec{r}'''(t)$.

Example6 Find the curvature and torsion of the curve:

$$x = a\cos t, y = a\sin t \ , \ z = bt$$

Solution: Here, $\vec{r}(t) = \langle a\cos t, \ a\sin t, \ bt\rangle$

$\Rightarrow \vec{r}'(t) = \langle -a\sin t, \ a\cos t, b\rangle$

$|\vec{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$

Therefore, the unit tangent vector $\vec{T}(t) = \frac{\vec{r}\prime(t)}{|\vec{r}\prime(t)|} = \langle \frac{-a\sin t}{\sqrt{a^2+b^2}}, \frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\rangle$

Again, $\vec{T}'(t) = \langle \frac{-a\cos t}{\sqrt{a^2+b^2}}, \frac{-a\sin t}{\sqrt{a^2+b^2}}, 0\rangle$

$\Rightarrow |\vec{T}'(t)| = \sqrt{\frac{a^2\cos^2 t+a^2\sin^2 t}{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}$

$\therefore k(t) = \left|\frac{\vec{T}\prime(t)}{\vec{r}\prime(t)}\right| = \frac{a/\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$

Also, $\vec{N}(t) = \frac{\vec{T}\prime(t)}{|\vec{T}\prime(t)|} = \langle -\cos t, \ -\sin t, 0\rangle$

Now, the binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a\sin t & a\cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$\Rightarrow \vec{B}(t) = \frac{1}{\sqrt{a^2+b^2}}\left[\hat{\imath}(b\sin t) - \hat{\jmath}(b\cos t) + \hat{k}(a\sin^2 t + a\cos^2 t)\right]$

$= \frac{1}{\sqrt{a^2+b^2}}\left[\hat{\imath}(b\sin t) + \hat{\jmath}(-b\cos t) + \hat{k}a\right]$

$\therefore \vec{B}(t) = \langle \frac{b\sin t}{\sqrt{a^2+b^2}}, \frac{-b\cos t}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}} \rangle$

$\vec{B}'(t) = \langle \frac{b\cos t}{\sqrt{a^2+b^2}}, \frac{b\sin t}{\sqrt{a^2+b^2}}, 0 \rangle$

Now, $\frac{\vec{B}'(t)}{|\vec{r}'(t)|} = -\tau\vec{N}$

$\Rightarrow \frac{b\cos t}{a^2+b^2}\hat{\imath} + \frac{b\sin t}{a^2+b^2}\hat{\jmath} + 0\hat{k} = -\tau(-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + 0\hat{k})$

$\Rightarrow \frac{b}{a^2+b^2}(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}) = \tau(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath})$

Therefore, the torsion $\tau = \frac{b}{a^2+b^2}$

Example7 Find the curvature and torsion of the curve $x = a\cos t, y = a\sin t, \ z = at\cot\alpha$

Solution: Here, $\vec{r}(t) = \langle a\cos t, \ a\sin t, \ at\cot\alpha \rangle$

$\Rightarrow \vec{r}'(t) = \langle -a\sin t, \ a\cos t, \ a\cot\alpha \rangle$

$|\vec{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2\cot^2\alpha} = \sqrt{a^2(1+\cot^2\alpha)}$

$= \sqrt{a^2\operatorname{cosec}^2\alpha} = a\operatorname{cosec}\alpha$

Therefore, the unit tangent vector $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{a\operatorname{cosec}\alpha}\langle -a\sin t, \ a\cos t, \ a\cot\alpha \rangle$

Again, $\vec{T}'(t) = \frac{1}{a\operatorname{cosec}\alpha}\langle -a\cos t, \ -a\sin t, 0 \rangle = \sin\alpha\,\langle -\cos t, \ -\sin t, 0 \rangle$

$\Rightarrow |\vec{T}'(t)| = \sqrt{\sin^2\alpha\,(\cos^2 t + \sin^2 t)} = \sin\alpha$

$\therefore k(t) = \left|\frac{\vec{T}'(t)}{\vec{r}'(t)}\right| = \frac{\sin\alpha}{a\operatorname{cosec}\alpha} = \frac{1}{a}\sin^2\alpha$

Also, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, \ -\sin t, 0 \rangle$

Now, the binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{a\operatorname{cosec}\alpha}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a\sin t & a\cos t & a\cot\alpha \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$\Rightarrow \vec{B}(t) = \frac{1}{a\operatorname{cosec}\alpha}\left[\hat{\imath}(a\cot\alpha\sin t) - \hat{\jmath}(a\cot\alpha\cos t) + \hat{k}(a\sin^2 t + a\cos^2 t)\right]$

$= \frac{\sin\alpha}{a}\left[\hat{\imath}(a\cot\alpha\sin t) + \hat{\jmath}(-a\cot\alpha\cos t) + \hat{k}a\right]$

$\therefore \vec{B}(t) = \langle \cos\alpha\sin t, -\cos\alpha\cos t, \ \sin\alpha \rangle$

$\vec{B}'(t) = \langle \cos\alpha\cos t, \ \cos\alpha\sin t, 0 \rangle$

Now, $\frac{\vec{B}'(t)}{|\vec{r}'(t)|} = -\tau\vec{N}$

$\Rightarrow \frac{\cos\alpha\cos t}{a\operatorname{cosec}\alpha}\hat{\imath} + \frac{\cos\alpha\sin t}{a\operatorname{cosec}\alpha}\hat{\jmath} + 0\hat{k} = -\tau(-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + 0\hat{k})$

$\Rightarrow \frac{\cos\alpha}{a\operatorname{cosec}\alpha}(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}) = \tau(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath})$

Therefore, the torsion $\tau = \frac{1}{a}\sin\alpha\cos\alpha$

Example8 Find the curvature and torsion of the curve

$x = (3t - t^3)a, y = 3at^2, z = a(3t + t^3)$

Solution: Here, $\vec{r}(t) = \langle (3t - t^3)a, 3at^2, a(3t + t^3)\rangle$

$\Rightarrow \vec{r}'(t) = \langle (3 - 3t^2)a,\ 6at,\ a(3 + 3t^2)\rangle$

$$|\vec{r}'(t)| = \sqrt{(3 - 3t^2)^2a^2 + 36a^2t^2 + (3 + 3t^2)^2a^2}$$
$$= 3a\sqrt{(1 - t^2)^2 + 4t^2 + (1 + t^2)^2}$$
$$= 3a\sqrt{2t^4 + 4t^2 + 2} = 3\sqrt{2}a\sqrt{(t^2 + 1)^2} = 3\sqrt{2}a(t^2 + 1)$$

The unit tangent vector $\vec{T}(t) = \frac{\vec{r}\prime(t)}{|\vec{r}\prime(t)|} = \frac{1}{3\sqrt{2}a(t^2+1)}\langle (3 - 3t^2)a,\ 6at,\ a(3 + 3t^2)\rangle$

$$= \frac{1}{\sqrt{2}}\langle \frac{1-t^2}{t^2+1},\ \frac{2t}{t^2+1}, 1\rangle$$

Again, $\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle \frac{(t^2+1)(-2t)-(1-t^2)2t}{(t^2+1)^2},\ \frac{(t^2+1)2-2t(2t)}{(t^2+1)^2}, 0\rangle$

$$= \frac{1}{\sqrt{2}}\langle \frac{-4t}{(t^2+1)^2},\ \frac{2-2t^2}{(t^2+1)^2}, 0\rangle$$

$$\Rightarrow |\vec{T}'(t)| = \frac{1}{\sqrt{2}}\sqrt{\frac{16t^2}{(t^2+1)^4} + \frac{4+4t^4-8t^2}{(t^2+1)^4}} = \frac{2}{\sqrt{2}}\sqrt{\frac{1+t^4+2t^2}{(t^2+1)^4}} = \sqrt{2}\sqrt{\frac{(t^2+1)^2}{(t^2+1)^4}} = \frac{\sqrt{2}}{t^2+1}$$

$$\therefore k(t) = \left|\frac{\vec{T}\prime(t)}{\vec{r}\prime(t)}\right| = \frac{\sqrt{2}/(t^2+1)}{3\sqrt{2}a(t^2+1)} = \frac{1}{3a(t^2+1)^2}$$

Also, $\vec{N}(t) = \frac{\vec{T}\prime(t)}{|\vec{T}\prime(t)|} = \frac{t^2+1}{2}\langle \frac{-4t}{(t^2+1)^2},\ \frac{2-2t^2}{(t^2+1)^2}, 0\rangle$

$$\Rightarrow \vec{N}(t) = \langle \frac{-2t}{t^2+1},\ \frac{1-t^2}{t^2+1}, 0\rangle$$

Now, the binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}(t^2+1)}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{1-t^2}{t^2+1} & \frac{2t}{t^2+1} & 1 \\ -2t & 1 - t^2 & 0 \end{vmatrix}$

$$= \frac{1}{\sqrt{2}(t^2+1)^2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 - t^2 & 2t & t^2 + 1 \\ -2t & 1 - t^2 & 0 \end{vmatrix}$$

$$\Rightarrow \vec{B}(t) = \frac{1}{\sqrt{2}(t^2+1)^2}\left[\hat{\imath}\big((t^2 - 1)(t^2 + 1)\big) - \hat{\jmath}\big(2t(t^2 + 1)\big) + \hat{k}((1 - t^2)^2 + 4t^2)\right]$$
$$= \frac{1}{\sqrt{2}(t^2+1)^2}\left[\hat{\imath}\big((t^2 - 1)(t^2 + 1)\big) + \hat{\jmath}\big(-2t(t^2 + 1)\big) + \hat{k}(t^2 + 1)^2\right]$$

$$\therefore \vec{B}(t) = \frac{1}{\sqrt{2}(t^2+1)}\left[\hat{\imath}\big((t^2 - 1)\big) + \hat{\jmath}(-2t) + \hat{k}(t^2 + 1)\right]$$

$$\therefore \vec{B}(t) = \frac{1}{\sqrt{2}}\langle \frac{t^2-1}{t^2+1},\ \frac{-2t}{t^2+1},\ 1\rangle$$

$$\vec{B}'(t) = \frac{1}{\sqrt{2}}\langle \frac{(t^2+1)(2t)-(t^2-1)2t}{(t^2+1)^2},\ \frac{(t^2+1)(-2)+2t(2t)}{(t^2+1)^2}, 0\rangle$$

$= \sqrt{2}\langle \frac{2t}{(t^2+1)^2}, \ \frac{t^2-1}{(t^2+1)^2}, 0\rangle = \frac{\sqrt{2}}{(t^2+1)}\langle \frac{2t}{(t^2+1)}, \frac{t^2-1}{(t^2+1)}, 0\rangle$

Now, $\frac{\vec{B}'(t)}{|\vec{r}'(t)|} = -\tau\vec{N}$

$\Rightarrow \frac{\sqrt{2}}{3\sqrt{2}a(t^2+1)^2}\left(\frac{2t}{(t^2+1)}\hat{\imath} + \frac{t^2-1}{(t^2+1)}\hat{\jmath} + 0\hat{k}\right) = -\tau\left(\frac{-2t}{t^2+1}\hat{\imath}, \ \frac{1-t^2}{t^2+1}\hat{\jmath} + 0\hat{k}\right)$

$\Rightarrow \frac{1}{3a(t^2+1)^2}\left(\frac{2t}{(t^2+1)}\hat{\imath} + \frac{t^2-1}{(t^2+1)}\hat{\jmath} + 0\hat{k}\right) = \tau\left(\frac{2t}{t^2+1}\hat{\imath}, \ \frac{t^2-1}{t^2+1}\hat{\jmath} + 0\hat{k}\right)$

Therefore, the torsion $\tau = \frac{1}{3a(t^2+1)^2}$

Hence, $k(t) = \tau = \frac{1}{3a(t^2+1)^2}$

Alternatively, we can solve Example 8 as shown here:

We have $k(t) = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}$ and $\tau = \frac{[\vec{r}'(t)\vec{r}''(t)\vec{r}'''(t)]}{|\vec{r}'(t)\times\vec{r}''(t)|^2}$

Here, $\vec{r}(t) = \langle(3t - t^3)a, \ 3at^2, \ a(3t + t^3)\rangle$

$\Rightarrow \vec{r}'(t) = \langle(3 - 3t^2)a, \ 6at, \ a(3 + 3t^2)\rangle$

$\vec{r}''(t) = \langle -6at, \ 6a, \ 6at\rangle$

$\vec{r}'''(t) = \langle -6a, \ 0, \ 6a\rangle$

$$\therefore \vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (3-3t^2)a & 6at & (3+3t^2)a \\ -6at & 6a & 6at \end{vmatrix} = 6a^2\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (3-3t^2) & 6t & (3+3t^2) \\ -t & 1 & t \end{vmatrix}$$

$$= 6a^2\left[\hat{\imath}(6t^2 - 3 - 3t^2) - \hat{\jmath}(3t - 3t^3 + 3t + 3t^3) + \hat{k}(3 - 3t^2 + 6t^2)\right]$$

$$= 18a^2\left[\hat{\imath}(t^2 - 1) - \hat{\jmath}(2t) + \hat{k}(1 + t^2)\right]$$

$|\vec{r}'(t)\times\vec{r}''(t)| = 18a^2\sqrt{(t^2-1)^2 + 4t^2 + (1+t^2)^2}$

$= 18a^2\sqrt{2t^4 + 4t^2 + 2} = 18\sqrt{2}a^2\sqrt{(t^2+1)^2}$

$\Rightarrow |\vec{r}'(t)\times\vec{r}''(t)| = 18\sqrt{2}a^2(t^2+1) \ldots (1)$

Also, $|\vec{r}'(t)| = 3a\sqrt{(1-t^2)^2 + 4t^2 + (1+t^2)^2}$

$= 3a\sqrt{2t^4 + 4t^2 + 2} = 3\sqrt{2}a\sqrt{(t^2+1)^2} = 3\sqrt{2}a(t^2+1)$

$\therefore k(t) = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{18\sqrt{2}a^2(t^2+1)}{54\sqrt{2}a^3(t^2+1)^3} = \frac{1}{3a(t^2+1)^2}$

$$\text{Again, } [\vec{r}'(t)\vec{r}''(t)\vec{r}'''(t)] = \begin{vmatrix} (3-3t^2)a & 6at & (3+3t^2)a \\ -6at & 6a & 6at \\ -6a & 0 & 6a \end{vmatrix}$$

$$= 108a^3\begin{vmatrix} (1-t^2) & 2t & (1+t^2) \\ -t & 1 & t \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 108a^3[(1-t^2)(1) - 2t(-t+t) + (1+t^2)(1)] = 216a^3$$

Also, $|\vec{r}'(t)\times\vec{r}''(t)|^2 = \left(18\sqrt{2}a^2(t^2+1)\right)^2 = 648a^4(t^2+1)^2$ using (1)

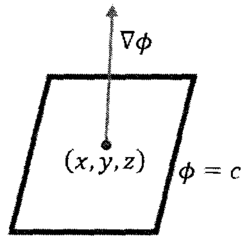
Therefore, torsion $\tau = \frac{[\vec{r}'(t)\vec{r}''(t)\vec{r}'''(t)]}{|\vec{r}'(t)\times\vec{r}''(t)|^2} = \frac{216a^3}{648a^4(t^2+1)^2} = \frac{1}{3a(t^2+1)^2}$

7. Gradient of a Scalar Field

Vector Differential Operator ∇ (Del):$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

The vector function $\nabla\phi$ is defined as the gradient of the scalar point function ϕ and is written asgradϕ.

Thus grad$\phi = \nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$



Physical Interpretation:

1. $\nabla\phi$ is the vector normal to the surface $\phi = c$
2. $\nabla\phi$ is a vector quantity with magnitude as the maximum rate of change of function ϕ and direction along the maximum rate of change.

Example 9 Find grad ϕ, where $\phi(x, y, z) = x^2y + y^2x + z^2$ at point $(1,1,1)$

Solution: Here $\phi = x^2y + y^2x + z^2$

$\therefore$ grad $\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \hat{\imath}(2xy + y^2) + \hat{\jmath}(x^2 + 2xy) + \hat{k}(2z)$

$\Rightarrow$ grad $\phi]_{(1,1,1)} = \hat{\imath}(2 + 1) + \hat{\jmath}(1 + 2) + \hat{k}(2) = 3\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$

Example 10 Find the values of a and b such that the surfaces $ax^2 - byz = (a + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at the point $(1, -1, 2)$.

Solution: Let$f_1 = ax^2 - byz - (a + 2)x$

$f_2 = 4x^2y + z^3 - 4$

Then,$\nabla f_1 = \hat{\imath}\frac{\partial f_1}{\partial x} + \hat{\jmath}\frac{\partial f_1}{\partial y} + \hat{k}\frac{\partial f_1}{\partial z}$

$= \hat{\imath}\big(2ax - (a + 2)\big) + \hat{\jmath}(-bz) + \hat{k}(-by)$

$\Rightarrow \nabla f_1]_{(1,-1,2)} = \hat{\imath}(a - 2) + \hat{\jmath}(-2b) + \hat{k}(b)$

$\nabla f_2 = \hat{\imath}\frac{\partial f_2}{\partial x} + \hat{\jmath}\frac{\partial f_2}{\partial y} + \hat{k}\frac{\partial f_2}{\partial z}$

$= \hat{\imath}(8xy) + \hat{\jmath}(4x^2) + \hat{k}(3z^2)$

$\Rightarrow \nabla f_2]_{(1,-1,2)} = \hat{\imath}(-8) + \hat{\jmath}(4) + \hat{k}(12)$

Now, sincef_1 and f_2 cut orthogonally at the point $(1, -1, 2)$

$\therefore \nabla f_1]_{(1,-1,2)} . \nabla f_1]_{(1,-1,2)} = 0$

$\Rightarrow -8(a - 2) + 4(-2b) + 12(b) = 0$

$\Rightarrow -8a + 4b + 16 = 0$

$\Rightarrow 2a - b - 4 = 0 \ldots (1)$

Also, since the point $(1, -1, 2)$ lies on the surfaces f_1 and f_2

$a(1)^2 - b(-1)(2) - (a + 2)(1) = 0 \Rightarrow 2b - 2 = 0 \Rightarrow b = 1$

Putting $b = 1$ in (1) $\Rightarrow a = \frac{5}{2}$

Example 11 Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Solution: Let $f_1=x^2+y^2+z^2-9$

$$f_2=x^2+y^2-z-3$$

Then the angle between surfaces f_1 and f_2 at the point $(2,-1,2)$ is same as the angle between the normal to the surfaces at the point $(2,-1,2)$

Now, $\nabla f_1=\hat{\imath}\frac{\partial f_1}{\partial x}+\hat{\jmath}\frac{\partial f_1}{\partial y}+\hat{k}\frac{\partial f_1}{\partial z}$

$$=2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}$$

$\Rightarrow \nabla f_1]_{(2,-1,\ 2)}=4\hat{\imath}-2\hat{\jmath}+4\hat{k}=\vec{n_1}$, say

$$\nabla f_2=\hat{\imath}\frac{\partial f_2}{\partial x}+\hat{\jmath}\frac{\partial f_2}{\partial y}+\hat{k}\frac{\partial f_2}{\partial z}$$

$$=2x\hat{\imath}+2y\hat{\jmath}-\hat{k}$$

$\Rightarrow \nabla f_2]_{(2,-1,\ 2)}=4\hat{\imath}-2\hat{\jmath}-\hat{k}=\vec{n_2}$, say

If θ be the angle between ∇f_1 and ∇f_2 at the point $(2,-1,2)$

Then, $\cos\theta=\frac{\vec{n_1}.\ \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}=\frac{(4\hat{\imath}-2\hat{\jmath}+4\hat{k}).(4\hat{\imath}-2\hat{\jmath}-\hat{k})}{\sqrt{16+4+16}\sqrt{16+4+1}}=\frac{4(4)-2(-2)+4(-1)}{6\sqrt{(21)}}=\frac{16}{6\sqrt{(21)}}$

$\therefore \theta=\cos^{-1}\frac{16}{6\sqrt{(21)}}$

Example12 Find $f(r)$ such that $\nabla \mathrm{f}=\frac{\vec{\mathrm{r}}}{\mathrm{r}^5}$ and $f(1)=0$

Solution: $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}\Rightarrow r=\sqrt{x^2+y^2+z^2}\ \Rightarrow r^2=x^2+y^2+z^2$

$$\therefore 2r\frac{\partial r}{\partial x}=2x\ \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$

Now, $\nabla \mathrm{f}=\hat{\imath}\frac{\partial \mathrm{f}}{\partial x}+\hat{\jmath}\frac{\partial \mathrm{f}}{\partial y}+\hat{k}\frac{\partial \mathrm{f}}{\partial z}=\sum\hat{\imath}\frac{\partial \mathrm{f}}{\partial x}=\frac{\vec{\mathrm{r}}}{\mathrm{r}^5}=\sum\frac{x\hat{\imath}}{r^5}$

$$=\sum\hat{\imath}\ \frac{1}{r^4}\frac{x}{r}=\sum\hat{\imath}\ \frac{1}{r^4}\frac{\partial r}{\partial x}\qquad \because \frac{\partial r}{\partial x}=\frac{x}{r}$$

$$=\sum\hat{\imath}\frac{\partial}{\partial x}\left(-\frac{1}{3r^3}\right)$$

$$\therefore f=-\frac{1}{3r^3}+c$$

But $f(1)=0\ \ \therefore 0=-\frac{1}{3}+c\Rightarrow c=\frac{1}{3}$

Hence, $f(r)=\frac{1}{3}-\frac{1}{3r^3}$

Example13 Prove that $\nabla \mathrm{r}^{\mathrm{n}}=\mathrm{nr}^{\mathrm{n}-2}\vec{\mathrm{r}}$, where $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

Solution: $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}\Rightarrow r=\sqrt{x^2+y^2+z^2}\ \Rightarrow r^2=x^2+y^2+z^2$ and

$$r^n=(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$\therefore \frac{\partial r^n}{\partial x}=\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}.2x=nxr^{n-2}$$

Similarly, $\frac{\partial r}{\partial y} = nyr^{n-2}$ and $\frac{\partial r}{\partial z} = nzr^{n-2}$

Now, $\nabla r^n = \hat{\imath}\frac{\partial r^n}{\partial x} + \hat{\jmath}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z} = \sum \hat{\imath}\frac{\partial r^n}{\partial x} = \sum \hat{\imath}\, nxr^{n-2}$

$= nr^{n-2}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = nr^{n-2}\vec{r}$

8. Directional Derivative

The directional derivative of a scalar function ϕ at a point $P(x, y, z)$ in the direction of vector $\vec{\alpha}$ is defined as the component of $\nabla\phi$ in the direction of $\vec{\alpha}$. It is given by dot product of $\nabla\phi$ at point $P(x, y, z)$ with the unit vector $\hat{\alpha}$, i.e., $\nabla\phi]_{(x,y,z)}.\hat{\alpha}$, where $\hat{\alpha} = \frac{\vec{\alpha}}{|\vec{\alpha}|}$

Physically, directional derivative of ϕ at a point P is the rate of change of f at $P(x, y, z)$ in the direction of α.

Note: Directional derivative is maximum in its own direction.

Example14 Find the directional derivative of the function $\phi = 2xy + z^2$ at point $(1, -1, 3)$ in the direction of the vector $\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Solution: Here $\phi = 2xy + z^2$

$\therefore\ \nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \hat{\imath}(2y) + \hat{\jmath}(2x) + \hat{k}(2z)$

$\Rightarrow \nabla\phi]_{(1,-1,\ 3)} = -2\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

Again, let $\vec{\alpha} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\Rightarrow \hat{\alpha} = \frac{\vec{\alpha}}{|\vec{\alpha}|} = \frac{\hat{\imath}+2\hat{\jmath}+2\hat{k}}{\sqrt{1+4+4}} = \frac{1}{3}(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

Therefore, the directional derivative of ϕ in the direction of $\vec{\alpha}$is $\nabla\phi]_{(x,y,z)}.\hat{\alpha}$

i.e., $(-2\hat{\imath} + 2\hat{\jmath} + 6\hat{k}).\frac{1}{3}(\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) = \frac{1}{3}[(-2)(1) + (2)(2) + (6)(2)] = \frac{14}{3}$

Example15 Find the directional derivative of the function $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

Solution: Here $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$

$\therefore\ \nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \hat{\imath}\left(10xy + \frac{5}{2}z^2\right) + \hat{\jmath}(5x^2 - 10yz) + \hat{k}(-5y^2 + 5zx)$

$\Rightarrow \nabla\phi]_{(1,1,1)} = \left(10 + \frac{5}{2}\right)\hat{\imath} + (5 - 10)\hat{\jmath} + (-5 + 5)\hat{k} = \frac{25}{2}\hat{\imath} - 5\hat{\jmath}$

Again, the direction ratios of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ are 2, -2, 1

Therefore, let $\vec{\alpha} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$

$\Rightarrow \hat{\alpha} = \frac{\vec{\alpha}}{|\vec{\alpha}|} = \frac{2\hat{\imath}-2\hat{\jmath}+\hat{k}}{\sqrt{4+4+1}} = \frac{1}{3}(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$

Thus, the directional derivative of ϕin the direction of $\vec{\alpha}$is $\nabla\phi]_{(x,y,z)}.\hat{\alpha}$

i.e., $\left(\frac{25}{2}\hat{\imath} - 5\hat{\jmath}\right).\frac{1}{3}(2\hat{\imath} - 2\hat{\jmath} + \hat{k}) = \frac{1}{3}\left[\left(\frac{25}{2}\right)(2) + (-5)(-2)\right] = \frac{35}{3}$

Example16 What is the directional derivative of $\emptyset = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$

Solution: : Here $\emptyset = xy^2 + yz^3$

$\therefore\ \nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \hat{\imath}(y^2) + \hat{\jmath}(2xy + z^3) + \hat{k}(3yz^2)$

$\Rightarrow \nabla\phi]_{(2,-1,1)} = \hat{\imath} - 3\hat{\jmath} - 3\hat{k}$

Again, let $\alpha = x \log z - y^2 + 4$, then

$\nabla\alpha = \hat{\imath}\frac{\partial\alpha}{\partial x} + \hat{\jmath}\frac{\partial\alpha}{\partial y} + \hat{k}\frac{\partial\alpha}{\partial z} = \hat{\imath}(\log z) + \hat{\jmath}(-2y) + \hat{k}\left(\frac{x}{z}\right)$

$\Rightarrow \nabla\alpha]_{(-1,2,1)} = -4\hat{\jmath} - \hat{k} = \vec{\alpha}$, say

$\therefore \hat{\alpha} = \frac{\vec{\alpha}}{|\vec{\alpha}|} = \frac{-4\hat{\jmath}-\hat{k}}{\sqrt{16+1}} = \frac{1}{\sqrt{17}}\left(-4\hat{\jmath} - \hat{k}\right)$

Therefore, the directional derivative of ϕ in the direction of $\vec{\alpha}$ is $\nabla\phi]_{(x,y,z)}.\hat{\alpha}$

i.e., $\left(\hat{\imath} - 3\hat{\jmath} - 3\hat{k}\right).\frac{1}{\sqrt{17}}\left(-4\hat{\jmath} - \hat{k}\right) = \frac{1}{\sqrt{17}}[(1)(0) + (-3)(-4) + (-3)(-1)] = \frac{15}{\sqrt{17}}$

9. Divergence of a Vector Point Function

The divergence of a vector point function $\vec{V}$ is denoted by div$\vec{V}$ and is defined as

$$\text{div}\vec{V} = \nabla.\vec{V} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\vec{V}$$

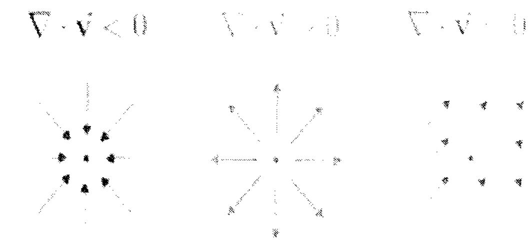
$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

where V_1 , V_2 , V_3 are the components of $\vec{V}$

The divergence of a vector point function is a scalar quantity.

Physical Interpretation: It is the quantitative measure of how much a vector field diverges.



Negative Divergence Positive Divergence Zero Divergence

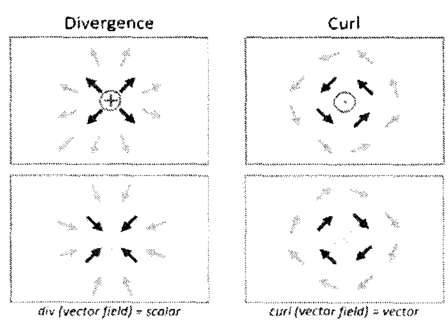
In particular, if div$\vec{V} = 0$, the vector field is called solenoidal and $\vec{V}$ is called solenoidal function.

10. Curl of a Vector Point Function

The curl (or rotation) of a vector point function $\vec{V}$ is denoted by Curl$\vec{V}$ and is defined as:

$$\text{Curl}\vec{V} = \nabla \times \vec{V} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$



Physical Interpretation: The curl of a vector point function is a vector quantity. It signifies how much a vector quantity curls or twists.

In particular, if Curl$\vec{V} = 0$, the vector $\vec{V}$ is called irrotational or conservative.

Remarks: grad(scalar point function) gives a vector quantity

div(vector point function) gives a scalar quantity

Curl(vector point function) gives a vector quantity

Example17 Find the divergence and Curl at the point $(2,-1,1)$of the vector

$\vec{F} = (xyz)\hat{\imath} + (3x^2y)\hat{\jmath} + (xz^2 - y^2z)\hat{k}$

Solution: $\text{div}\vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$

$= yz + 3x^2 + 2xz - y^2$

$\therefore \text{div}\vec{F}\big]_{(2,-1,1)} = -1 + 12 + 4 - 1 = 14$

$$\text{Also, Curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$= (-2yz - 0)\hat{\imath} - (z^2 - xy)\hat{\jmath} + (6xy - xz)\hat{k}$

$\therefore \text{Curl}\vec{F}\big]_{(2,-1,1)} = 2\hat{\imath} - 3\hat{\jmath} - 14\hat{k}$

Example18 Find div$\vec{V}$ and Curl$\vec{V}$, where $\vec{V} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

Solution: Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$\vec{V} = \text{grad}\,\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$

$= (3x^2 - 3yz)\hat{\imath} + (3y^2 - 3xz)\hat{\jmath} + (3z^2 - 3xy)\hat{k}$

$\therefore \text{div}\vec{V} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$

$= 6x + 6y + 6z = 6(x + y + z)$

$$\text{Also, Curl}\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$= (-3x + 3x)\hat{\imath} - (-3y + 3y)\hat{\jmath} + (-3z + 3z)\hat{k} = 0$

Since, Curl$\vec{V} = 0$, the vector $\vec{F}$ is irrotational.

Example19 A vector field is given by$\vec{F} = (x^2 + xy^2)\hat{\imath} + (y^2 + x^2y)\hat{\jmath}$.

Show that $\vec{F}$ is irrotational.

$$\textbf{Solution:}\quad \text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$= (0 - 0)\hat{\imath} - (0 - 0)\hat{\jmath} + (2xy - 2xy)\hat{k} = 0$

Since, Curl$\vec{F} = 0$, the vector $\vec{F}$ is irrotational.

Example20 If the vector $\vec{F} = (ax^2y + yz)\hat{\imath} + (xy^2 - xz^2)\hat{\jmath} + (2xyz - 2x^2y^2)\hat{k}$ is solenoidal. Find the value of a. Also find the Curl of this solenoidal vector.

Solution: $\text{div}\vec{F} = \frac{\partial}{\partial x}\big((ax^2y + yz)\big) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2)$

$$= 2axy + 2xy + 2xy = 2axy + 4xy = 2(a+2)xy$$

Since $\vec{F}$ is solenoidal, $\text{div}\vec{F} = 0$

$\Rightarrow\ 2(a+2)xy = 0 \quad \therefore \text{a} = -2$

Now, $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x^2y + yz & xy^2 - xz^2 & 2xyz - 2x^2y^2 \end{vmatrix}$

$$= (2xz - 4x^2y + 2xz)\hat{\imath} - (2yz - 4xy^2 - y)\hat{\jmath} + (y^2 - z^2 + 2x^2 - z)\hat{k}$$

$$= 4x(z - xy)\hat{\imath} + (y + 4xy^2 - 2yz)\hat{\jmath} + (2x^2 + y^2 - z^2 - z)\hat{k}$$

Example21 For what values of a and b will $\vec{F} = (y^2 + 2bzx)\hat{\imath} + y(ax + bz)\hat{\jmath} + (y^2 + bx^2)\hat{k}$ be irrotational? Find the scalar ϕ such that $\vec{F} = \nabla\phi$

Solution: $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2bzx & y(ax + bz) & y^2 + bx^2 \end{vmatrix}$

$$= (2y - by)\hat{\imath} - (2bx - 2bx)\hat{\jmath} + (ay - 2y)\hat{k}$$

Since $\vec{F}$ is irrotational, therefore, $\text{Curl}\vec{F} = 0$

$\Rightarrow a = 2, b = 2$

Now, $\vec{F} = (y^2 + 2bzx)\hat{\imath} + y(ax + bz)\hat{\jmath} + (y^2 + bx^2)\hat{k} = \nabla\phi$

$$= \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\therefore d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= (y^2 + 4zx)dx + (2xy + 2yz)dy + (y^2 + 2x^2)dz$$

$$= d(xy^2 + 2x^2z + y^2z)$$

Integrating both sides, we get: $\phi = xy^2 + 2x^2z + y^2z$

Properties of Divergence and Curl

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. In general, divergence explains how the field behaves towards or away from a point. Similarly, curl is used to measure the rotational extent of the field about a particular point.

1. For a constant vector $\vec{a}$, div $\vec{a} = 0$, Curl $\vec{a} = 0$
2. $\text{div}(\vec{A} + \vec{B}) = \text{div}\vec{A} + \text{div}\vec{B}$ or $\nabla.(\vec{A} + \vec{B}) = \nabla.\vec{A} + \nabla.\vec{B}$
3. $\text{Curl}(\vec{A} + \vec{B}) = \text{Curl}\vec{A} + \text{Curl}\vec{B}$ or $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function, then div $\phi\vec{A} = \phi$ div$\vec{A}$ + (gradϕ)$\vec{A}$ or $\nabla.(\phi\vec{A}) = \phi\,(\nabla.\vec{A}) + (\nabla\phi).\vec{A}$
5. Curl$(\phi\vec{A})$ = (grad ϕ) $\times \vec{A} + \phi$ Curl$\vec{A}$ or $\nabla \times (\phi\vec{A}) = (\nabla\phi) \times \vec{A} + \phi\,(\nabla \times \vec{A})$
6. div$(\vec{A} \times \vec{B}) = \vec{B}.$Curl$\vec{A} - \vec{A}\,.$Curl$\vec{B}$ or $\nabla.(\vec{A} \times \vec{B}) = \vec{B}.(\nabla \times \vec{A}) - \vec{A}\,.(\nabla \times \vec{B})$

Repeated Operations by ∇

1. div(gradϕ) $= \nabla^2\phi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
2. Curl(gradϕ) $= \nabla \times \nabla\,\phi = \vec{0}$
3. div$(\text{Curl}\vec{V}) = \nabla.(\nabla \times \vec{V}) = 0$
4. Curl$(\text{Curl}\vec{V})$ = grad div$\vec{V} - \nabla^2\vec{V} = \nabla(\nabla.\vec{V}) - \nabla^2\vec{V}$

Example22 Prove that Curl(grad ϕ) $= \vec{0}$

Solution: Curl(grad ϕ) $= \nabla \times (\nabla\phi) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right)\hat{\imath} - \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right)\hat{\jmath} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)\hat{k} = \vec{0}\text{v}$$

Example23 For a solenoidal field $\vec{F}$, prove that Curl Curl Curl Curl $\vec{F} = \nabla^4\vec{F}$

Solution: We know that for a solenoidal field $\vec{F}, \nabla.\vec{F} = 0$ ... ①

Now, Curl $\vec{F} = \nabla \times \vec{F}$

$\Rightarrow$ Curl Curl $\vec{F} = \nabla \times (\nabla \times \vec{F}) = (\nabla.\vec{F})\nabla - (\nabla.\nabla)\vec{F}$

$$\because \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$\Rightarrow$ Curl Curl $\vec{F} = 0 - \nabla^2\vec{F} = -\nabla^2\vec{F}$ $\quad \because$ using ①, $\nabla.\vec{F} = 0$

Again, Curl Curl Curl Curl $\vec{F} = \nabla \times [\nabla \times (-\nabla^2\vec{F})]$ $\because$ Curl Curl $\vec{F} = -\nabla^2\vec{F}$

$$= \left(\nabla.(-\nabla^2\vec{F})\right)\nabla - (\nabla.\nabla)(-\nabla^2\vec{F})\text{v}$$

$$= 0 + \nabla^4\vec{F} = \nabla^4\vec{F}\text{v}$$

Example24 Prove that div(grad r^n) $= \nabla^2(r^n) = n(n+1)\,r^{n-2}$, where $r = \sqrt{(x^2+y^2+z^2)}$

Solution: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \Rightarrow r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$and

$r^n = (x^2+y^2+z^2)^{\frac{n}{2}}$

$\therefore \frac{\partial r^n}{\partial x} = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}.2x = nxr^{n-2}$

Similarly, $\frac{\partial r}{\partial y} = nyr^{n-2}$ and $\frac{\partial r}{\partial z} = nzr^{n-2}$

Now, $\nabla \text{r}^\text{n} = \hat{\imath}\frac{\partial \text{r}^\text{n}}{\partial x} + \hat{\jmath}\frac{\partial \text{r}^\text{n}}{\partial y} + \hat{k}\frac{\partial \text{r}^\text{n}}{\partial z} = \sum \hat{\imath}\frac{\partial \text{r}^\text{n}}{\partial x} = \sum \hat{\imath}\,nxr^{n-2}$

$= nr^{n-2}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = nr^{n-2}\vec{r}$

Now, $\nabla^2(r^n) = \nabla.(\nabla r^n) = \sum \hat{\imath}.\frac{\partial}{\partial x}(nr^{n-2}\vec{r})$

$= \sum \hat{\imath}.\left(nr^{n-2}\hat{\imath} + n(n-2)r^{n-3}\frac{\partial r}{\partial x}\vec{r}\right)$

$= 3nr^{n-2} + \sum n(n-2)r^{n-3}\frac{x\hat{\imath}}{r}.\vec{r}$

$= 3nr^{n-2} + n(n-2)r^{n-4}\vec{r}.\vec{r}$

$= 3nr^{n-2} + n(n-2)r^{n-2} = n(n+1)r^{n-2}$

11. Line Integral

An integral which is evaluated along a curve is known as Line integral.

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a vector point function in a region R and C be a regular curve from point A to point B. Then the line integral of $\vec{F}$ is defined as:

$\int_A^B \vec{F}.d\vec{r}$ ...①, where $d\vec{r} = d(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$ or $d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$

$\Rightarrow \int_A^B \vec{F}.d\vec{r} = \int_A^B (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz) = \int_A^B (F_1dx + F_2dx + F_3dx)$ ...②

Remarks: 1. Line integral ① or ② gives the total work done by the force $\vec{F}$ along A to B.

2. If $x = x(t),\ y = y(t), z = z(t)$ are the parametric equations of the curve C, then $\int_A^B \vec{F}.d\vec{r} = \int_{t_1}^{t_2}\left[F_x(t)\frac{dx}{dt} + F_y(t)\frac{dy}{dt} + F_z(t)\frac{dz}{dt}\right]$

3. If $\oint \vec{F}.d\vec{r} = 0$, no work is done and the energy is conserved, i.e. $\vec{F}$ is conservative

Theorem 1: If $\vec{F} = \nabla\phi$, the work done in moving a particle from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ in the force field $\vec{F}$, is independent of the path joining A and B.

Proof: Work done by the force $\vec{F}$ along A to B is:

$\int_A^B \vec{F}.d\vec{r} = \int_A^B \nabla\phi.d\vec{r}$, $\because \vec{F} = \nabla\phi$

$= \int_A^B \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$

$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$

$= \int_A^B d\phi = [\phi]_A^B = \phi(B) - \phi(A)$

Thus in a conservative (irrotational) field, the work done depends upon the end points A and B and is independent of the path joining A and B.

Remark: If $\vec{F}$ is conservative, i.e., Curl$\vec{F} = 0 \Leftrightarrow \vec{F} = \nabla\phi$, where ϕ is the scalar potential of the field.

Example25 Evaluate the line integral $\int_C \left(\frac{y+z}{x}\right)ds$, where C is the arc of the circle $x^2 + y^2 = 4, z = 0$ from $(2,0,0)$ and $(\sqrt{2}, \sqrt{2}, 0)$ in the counterclockwise direction.

Solution: Equation of the circle is written as $x = 2\cos t, y = 2\sin t, z = 0; 0 \le t \le 2\pi$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$

Now, the point $(2,0,0) \Rightarrow t = 0$and $\left(\sqrt{2}, \sqrt{2}, 0\right) \Rightarrow t = \frac{\pi}{4}$

Hence, $\int_C \left(\frac{y+z}{x}\right) ds = \int_C \left(\frac{y+z}{x}\right) \frac{ds}{dt} . dt = \int_0^{\frac{\pi}{4}} \left(\frac{2\sin t}{2\cos t}\right) . 2dt = 2\int_0^{\frac{\pi}{4}} \tan t\, dt$

$$= 2[log[\sec t]]_0^{\frac{\pi}{4}} = 2\left(\log\sqrt{2} - 0\right) = \log 2$$

Example26 Show that $\vec{F} = (2xy + z^3)\hat{\imath} + (x^2)\hat{\jmath} + (3z^2x)\hat{k}$ is a conservative field. Find its scalar potential and work done in moving a particle from $(1, -2, 1)$ to $(3,1,4)$.

Solution: $\text{Curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3z^2x \end{vmatrix}$

$$= (0 - 0)\hat{\imath} - (3z^2 - 3z^2)\hat{\jmath} + (2x - 2x)\hat{k} = 0$$

Since, $\text{Curl}\vec{F} = 0$, the vector $\vec{F}$ is conservative(irrotational).

$\therefore \vec{F} = \nabla\phi \Rightarrow \vec{F} = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$\Rightarrow (2xy + z^3)\hat{\imath} + (x^2)\hat{\jmath} + (3z^2x)\hat{k} = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3, \ \frac{\partial\phi}{\partial y} = x^2, \ \frac{\partial\phi}{\partial z} = 3z^2x$

Also, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$= (2xy + z^3)dx + (x^2)dy + (3z^2x)dz$$
$$= (2xydx + x^2dy) + (z^3dx + 3z^2xdz)$$
$$= d(x^2y) + d(z^3x)$$
$$= d(x^2y + z^3x)$$

Therefore, the scalar potential $\phi = x^2y + z^3x + c$

Also, the work done in moving a particle from $A(1, -2, 1)$ to $B(3,1,4)$ is:

$\int_A^B \vec{F}.d\vec{r} = [\phi]_A^B = [x^2y + z^3x + c]_{(1,-2,1)}^{(3,1,4)} = (9 + 192) - (-2 + 1) = 202$ units

Example27 For the vector field $\vec{F} = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}$, calculate $\int_C \vec{F}.d\vec{r}$, from origin to $A(1,1,1)$ along the path C

i. $x = t\, , y = t^2, z = t^3$ ***ii.*** The straight line from origin to $A(1,1,1)$

Solution: ***i.*** Along the path $x = t\, , y = t^2, z = t^3$, we have:

$$\vec{r} = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{\jmath} \text{ and } d\vec{r} = \left(\hat{\imath} + 2t\hat{\jmath} + 3t^2\hat{k}\right)dt$$

And, $\vec{F} = (3t^2 + 6t^2)\hat{\imath} - 14t^5\hat{\jmath} + 20t^7\hat{k} = 9t^2\hat{\imath} - 14t^5\hat{\jmath} + 20t^7\hat{k}$

$\therefore \vec{F}.d\vec{r} = [1(9t^2) + 2t(-14t^5) + 3t^2(20t^7)]dt = [9t^2 - 28t^6 + 60t^9]dt$

Now, the point $(0,0,0) \Rightarrow t = 0$ and $(1,1,1) \Rightarrow t = 1$

Hence, $\int_C \vec{F}.d\vec{r} = \int_0^1 [9t^2 - 28t^6 + 60t^9]dt$

$= [3t^3 - 4t^7 + 6t^{10}]_0^1 = [3 - 4 + 6] = 5$

$ii.$ On the line OA , $x = t$, $y = t, z = t$

Therefore, the equation of line OA is:$\vec{r} = t\hat{\imath} + t\hat{\jmath} + t\hat{k}$

$\Rightarrow d\vec{r} = (\hat{\imath} + \hat{\jmath} + \hat{k})dt$

And, $\vec{F} = (3t^2 + 6t)\hat{\imath} - 14t^2\hat{\jmath} + 20t^3\hat{k}$

$\therefore \vec{F}.d\vec{r} = [3t^2 + 6t - 14t^2 + 20t^3]dt = [20t^3 - 11t^2 + 6t]dt$

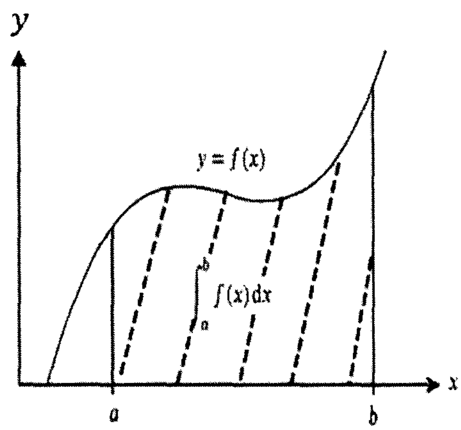
Now, the point $(0,0,0) \Rightarrow t = 0$ and $(1,1,1) \Rightarrow t = 1$

Hence, $\int_C \vec{F}.d\vec{r} = \int_0^1 [20t^3 - 11t^2 + 6t]dt$

$= \left[5t^4 - \frac{11}{3}t^3 + 3t^2\right]_0^1 = \left[5 - \frac{11}{3} + 3\right] = \frac{13}{3}$

12. Double Integral

A definite integral $\int_a^b f(x)\,dx$ represents the area bounded by the curve $y = f(x)$, x- axis and the ordinates $x = a$ and $x = b$. A Double integral denoted by $\iint_R f(x,y)dxdy$, is also used to determine the surface area of a two dimensional figure.



It is usually defined for the following three cases.

1. **When the limits of integration are constants in $x\&y$**

The double integral over a region R, where R is the rectangle described by $x = a, x = b, y = c, y = d$, $a \le x \le b$ and $c \le y \le d$, is given by:

$$\iint_R f(x,y)dxdy = \int_c^d \left\{\int_a^b f(x,y)dx\right\} dy$$
$$= \int_a^b \left\{\int_c^d f(x,y)dy\right\} dx$$

Here, the limits of integration are constants and the integral can be evaluated by taking any of the limits first.

2. **When the limits of integration are constant with respect to 'x' and variable with respect to 'y'**

   Here, $\iint_R f(x,y)dxdy = \int_a^b \left\{\int_{y=g(x)}^{y=h(x)} f(x,y)dy\right\} dx$

   where, $R = \{a \le x \le b$ and $g(x) \le y \le h(x)\}$

3. **When the limits of integration are constant with respect to 'y' and variable with respect to 'x'**

   Here, $\iint_R f(x,y)dxdy = \int_c^d \left\{\int_{x=g(y)}^{x=h(y)} f(x,y)dx\right\} dy$

   where, $R = \{g(y) \le x \le h(y)$ and $c \le y \le d\}$

Example28 Evaluate $\iint_R (x+y^2)dxdy$, where $R = \{0 \le x \le 2 \text{ and } 0 \le y \le 1\}$

Solution: $\iint_R (x+y^2)dxdy = \int_0^1 \left\{\int_0^2 (x+y^2)dx\right\} dy$

$$= \int_0^1 \left[\frac{x^2}{2} + y^2 x\right]_0^2 dy = \int_0^1 (2+2y^2)\, dy$$

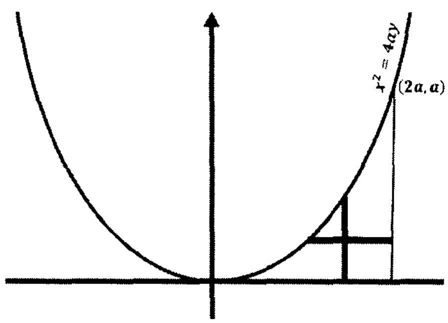
$$= \left[2y + \frac{2y^3}{3}\right]_0^1 = 2 + \frac{2}{3} = \frac{8}{3}$$

Note: $:\iint_R (x+y^2)dxdy = \int_0^1 \left\{\int_0^2 f(x+y^2)dx\right\} dy = \int_0^2 \left\{\int_0^1 f(x+y^2)dy\right\} dx = \frac{8}{3}$

Example29 Evaluate $\iint_R xy\, dxdy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Solution: Here $R = \left\{0 \le x \le 2a \text{ and } 0 \le y \le \frac{x^2}{4a}\right\}$

$$\therefore \iint_R xydxdy = \int_0^{2a} \left\{\int_{y=0}^{y=\frac{x^2}{4a}} xydy\right\} dx$$



$$= \int_0^{2a} \left[\frac{xy^2}{2}\right]_0^{\frac{x^2}{4a}} dx$$

$$= \frac{1}{2}\int_0^{2a} x\left(\frac{x^2}{4a}\right)^2 dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\, dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\left[\frac{64a^6}{6}\right] = \frac{a^4}{3}$$

12.1 Double Integral in Polar Coordinates

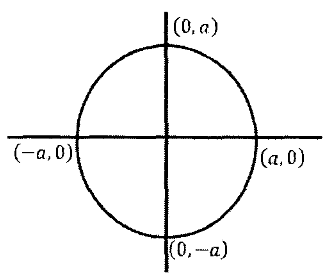
Sometimes a double integral may be found more easily by changing into polar coordinates.

Let $x = r\cos\theta$, $y = r\sin\theta$

Then, $\iint_R f(x,y)dxdy = \iint_R f(r\cos\theta, r\sin\theta)\, |J|\, drd\theta == \iint_R f(r\cos\theta, r\sin\theta)\, r\, drd\theta$

where $J = \frac{\partial(x,y)}{\partial(r,\theta)}$

Example30 Evaluate $\iint (x^2+y^2)^{\frac{3}{2}} dxdy$ over the circle $x^2 + y^2 = a^2$ by changing into polar coordinates.



Solution: Putting $x = r\cos\theta$, $y = r\sin\theta$

For the circle, $x^2 + y^2 = a^2$ in polar form,

θ varies from 0 to 2π and r varies from 0 to a

$$\therefore \iint (x^2+y^2)^{\frac{3}{2}} dxdy = \int_0^{2\pi}\int_0^a (r^2\cos^2\theta + r^2\cos^2\theta)^{\frac{3}{2}} rdrd\theta$$

$$= \int_0^{2\pi}\int_0^a r^4\, drd\theta = \int_0^{2\pi}\left[\frac{r^5}{5}\right]_0^a d\theta = \frac{a^5}{5}\int_0^{2\pi} d\theta = \frac{a^5}{5}[\theta]_0^{2\pi} = \frac{2\pi a^5}{5}$$

$\therefore \iint_R \left(\frac{xz}{\sqrt{16-x^2}} + x\right) dxdz = \int_0^5 \int_0^4 \left(\frac{xz}{\sqrt{16-x^2}} + x\right) dxdz$

$= \int_0^5 \left\{\int_0^4 \left(\frac{xz}{\sqrt{16-x^2}} + x\right) d\,x\right\} dz$

Now, $\int_0^4 \frac{xz}{\sqrt{16-x^2}} dx = -\frac{1}{2}\int_{16}^0 \frac{z}{\sqrt{t}} dt = \frac{1}{2}\int_0^{16} \frac{z}{\sqrt{t}} dt = \frac{1}{2}\left[\frac{z\sqrt{t}}{1/2}\right]_0^{16} = 4z$ putting $16 - x^2 = t$

$\therefore \iint_R \left(\frac{xz}{\sqrt{16-x^2}} + x\right) dxdz = \int_0^5 \{4z + 8\}\, dz = [2z^2 + 8z]_0^5 = 90$

Example33 Evaluate $\iint_S \vec{F}.\hat{n}\, dS$, where $\vec{F} = 2xz\hat{\imath} - y^3\hat{\jmath} + yz\hat{k}$ and S is surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

Solution: Here, S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0,$ $z = 1$, as shown by the $ABCDEFGH$ in the adjoining figure.

Let R_1 denote the front surface$ABCD$ of the cube

Then, $\hat{n} = \hat{\imath}$ $\because$ the surface $ABCD$ is parallel to yz plane

Also, $dS = \frac{dydz}{|\hat{n}.\hat{\imath}|} = \frac{dydz}{|\hat{\imath}.\hat{\imath}|} = dydz$

$\therefore \iint_{R_1} \vec{F}.\hat{n}\, dS = \iint_{R_1} (2xz\hat{\imath} - y^3\hat{\jmath} + yz\hat{k}).\hat{\imath}dydz = \iint_{R_1} 2xzdydz$

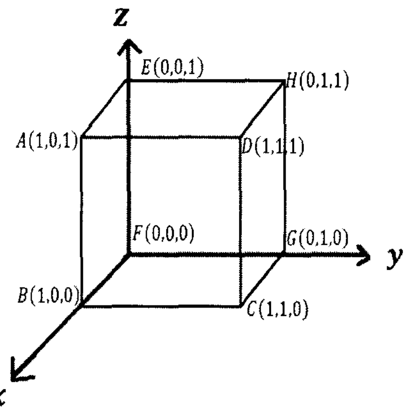
Now, on the front surface $ABCD$, $x = 1$, y varies from 0 to 1 and z varies from 0 to 1

$\therefore \iint_{R_1} 2xzdydz = \int_0^1 \int_0^1 2z\, dydz \because x = 1$

$= \int_0^1 \left\{\int_0^1 2z\, dy\right\} dz$

$= \int_0^1 [2yz]_0^1\, dz$

$= \int_0^1 2z\, dz = [z^2]_0^1 = 1$



Let R_2 denote the back surface $EFGH$of the cube

Then, $\hat{n} = -\hat{\imath} \because$ the surface $EFGH$is lies onyz plane

Also, $dS = \frac{dydz}{|\hat{n}.\hat{\imath}|} = \frac{dydz}{|-\hat{\imath}.\hat{\imath}|} = dydz$

$\therefore \iint_{R_2} \vec{F}.\hat{n}\, dS = \iint_{R_2} (2xz\hat{\imath} - y^3\hat{\jmath} + yz\hat{k}).(-\hat{\imath})dydz = -\iint_{R_2} 2xzdydz$

On the surface $EFGH$, $x = 0$, y varies from 0 to 1 and z varies from 0 to 1

$\therefore \iint_{R_2} 2xzdydz = -\int_0^1 \int_0^1 2(0)z\, dydz = 0 \because x = 0$

Let R_3 denote the surface $ABFE$ of the cube

Then, $\hat{n} = -\hat{\jmath} \because$ the surface $ABFE$ lies onxz plane

Also, $dS = \frac{dxdz}{|\hat{n}.\hat{\jmath}|} = \frac{dxdz}{|-\hat{\jmath}.\hat{\jmath}|} = dxdz$

$\therefore \iint_{R_3} \vec{F}.\hat{n}\, dS = \iint_{R_3} (2xz\hat{\imath} - y^3\hat{\jmath} + yz\hat{k}).(-\hat{\jmath})dxdz = \iint_{R_3} y^3 dxdz$

On the surface $ABFE$, $y = 0$, x varies from 0 to 1 and z varies from 0 to 1

$\therefore \iint_{R_3} y^3 dxdz = \int_0^1 \left\{\int_0^1 y^3\, dx\right\} dz = 0 \quad \because y = 0$

Let R_4 denote the surface $DCGH$ of the cube

Then, $\hat{n} = \hat{j}$ $\quad\because$ the surface $DCGH$ is parallel to xz plane

Also, $dS = \frac{dxdz}{|\hat{n}.\hat{j}|} = \frac{dxdz}{|\hat{j}.\hat{j}|} = dxdz$

$\therefore \iint_{R_4} \vec{F}.\hat{n}\, dS = \iint_{R_4} \left(2xz\hat{\imath} - y^3\hat{j} + yz\hat{k}\right).(\hat{j})dxdz = \iint_{R_4} -y^3 dxdz$

On the surface $DCGH$, $y = 1$, x varies from 0 to 1 and z varies from 0 to 1

$\therefore \iint_{R_4} -y^3 dxdz = -\int_0^1 \left\{\int_0^1 1\, dx\right\} dz \because y = 1$

$= -\int_0^1 [x]_0^1\, dz = -\int_0^1 1\, dz = -[z]_0^1 = -1$

Let R_5 denote the upper surface $ADHE$ of the cube

Then, $\hat{n} = \hat{k} \because$ the surface $ADHE$is parallel to xy plane

Also, $dS = \frac{dxdy}{|\hat{n}.\hat{k}|} = \frac{dxdy}{|\hat{k}.\hat{k}|} = dxdy$

$\therefore \iint_{R_5} \vec{F}.\hat{n}\, dS = \iint_{R_5} \left(2xz\hat{\imath} - y^3\hat{j} + yz\hat{k}\right).\left(\hat{k}\right)dxdy = \iint_{R_5} yzdxdy$

On the surface $ADHE$, $z = 1$, x varies from 0 to 1 and y varies from 0 to 1

$\therefore \iint_{R_5} yzdxdy = \int_0^1 \left\{\int_0^1 y\, dx\right\} dy \because z = 1$

$= \int_0^1 [yx]_0^1\, dy = \int_0^1 y\, dz = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$

Let R_6 denote the lower surface $FBCG$ of the cube

Then, $\hat{n} = -\hat{k} \because$ the surface $FBCG$lies on xy plane

Also, $dS = \frac{dxdy}{|\hat{n}.\hat{k}|} = \frac{dxdy}{|-\hat{k}.\hat{k}|} = dxdy$

$\therefore \iint_{R_6} \vec{F}.\hat{n}\, dS = \iint_{R_6} \left(2xz\hat{\imath} - y^3\hat{j} + yz\hat{k}\right).\left(-\hat{k}\right)dxdy = \iint_{R_6} -yzdxdy$

On the surface $FBCG$, $z = 0$, x varies from 0 to 1 and y varies from 0 to 1

$\therefore \iint_{R_6} yzdxdy = \int_0^1 \left\{\int_0^1 0\, dx\right\} dy = 0 \because z = 0$

Hence, $\iint_S \vec{F}.\hat{n}\, dS = \iint_{R_1} \vec{F}.\hat{n}\, dS + \iint_{R_2} \vec{F}.\hat{n}\, dS + \cdots + \iint_{R_6} \vec{F}.\hat{n}\, dS$

$= 1 + 0 + 0 - 1 + \frac{1}{2} + 0 = \frac{1}{2}$

14. Triple Integral

A triple integral is an extension of double integral for finding volume of a solid region and some other applications like moment of inertia, center of mass etc.

For a function $f(x, y, z)$, the triple integral over the region $R = \{a, b; c, d; e, f\}$, where $a \le x \le b, c \le y \le d,\ e \le z \le f$ is given by:

$\iiint_R f(x, y, z)dxdydz = \int_a^b \int_c^d \int_e^f f(x, y, z)dzdydx$

When the limits of integration are constant with respect to 'x' and variable with respect to 'y'and 'z', $\iiint_R f(x, y, z)dxdydz = \int_a^b \int_{y=g(x)}^{y=h(x)} \int_{z=\phi(x,y)}^{z=\psi(x,y)} f(x, y, z)dzdydx$

where, $R = \{a \le x \le b\,, g(x) \le y \le h(x), \phi(x,y) \le z \le \psi(x,y)\}$

Remark: Order of integration is immaterial if the limits in all the variables x, y and z are constants.

Example34 Evaluate $\iiint_R (x+y+z)dxdydz$, $R = \{0 \le x \le 2\,, 0 \le y \le 3, 0 \le z \le 1\}$

Solution: $\iiint_R (x+y+z)dxdydz = \int_0^2\int_0^3\int_0^1 (x+y+z)dzdydx$

$$= \int_0^2\int_0^3 \left[xz + yz + \frac{z^2}{2}\right]_0^1 dydx = \int_0^2\int_0^3 \left(x + y + \frac{1}{2}\right) dydx$$

$$= \int_0^2 \left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_0^3 dx = \int_0^2 \left(3x + \frac{9}{2} + \frac{3}{2}\right) dx$$

$$= 3\int_0^2 (x+2)\,dx = 3\left[\frac{x^2}{2} + 2x\right]_0^2 = 18$$

Example35 Evaluate $\int_{-1}^1\int_0^z\int_{x-z}^{x+z} (x+y+z)dxdydz$

Solution: Let $I = \int_{-1}^1\int_0^z\int_{x-z}^{x+z} (x+y+z)dxdydz = \int_{z=-1}^{z=1}\int_{x=0}^{x=z}\int_{y=x-z}^{y=x+z} (x+y+z)dydxdz$

$$\Rightarrow I = \int_{z=-1}^{z=1}\int_{x=0}^{x=z} \left[xy + \frac{y^2}{2} + zy\right]_{y=x-z}^{y=x+z} dxdz$$

$$= \int_{z=-1}^{z=1}\int_{x=0}^{x=z} \left[(x+z)[(x+z)-(x-z)] + \frac{1}{2}[(x+z)^2 - (x-z)^2]\right] dxdz$$

$$= \int_{z=-1}^{z=1}\int_{x=0}^{x=z} [2z(x+z) + 2xz]\, dxdz = 2\int_{z=-1}^{z=1}\int_{x=0}^{x=z} [2xz + z^2]\, dxdz$$

$$= 2\int_{z=-1}^{z=1} [x^2z + z^2x]_{x=0}^{x=z}\, dz = 4\int_{z=-1}^{z=1} z^3\, dz = [z^4]_{-1}^{1} = 0$$

Example36 Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}dxdydz$

Solution: Let $I = \int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}dxdydz = \int_{x=0}^{x=a}\int_{y=0}^{y=x}\int_{z=0}^{z=x+y} e^{x+y+z}dzdydx$

$$\Rightarrow I = \int_{x=0}^{x=a}\int_{y=0}^{y=x}\int_{z=0}^{z=x+y} e^{x+y}e^z dzdydx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=x} [e^{x+y}e^z]_{z=0}^{z=x+y}\, dydx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=x} [e^{x+y}(e^{x+y} - 1)]\, dydx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=x} [e^{2x}e^{2y} - e^x e^y]\, dydx$$

$$= \int_{x=0}^{x=a} \left[\frac{e^{2x}e^{2y}}{2} - e^x e^y\right]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=a} \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right] dx$$

$$= \int_{x=0}^{x=a} \left[\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^x\right] dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x\right]_{x=0}^{x=a} = \left[\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{1}{8} + \frac{3}{4} - 1\right]$$

$$= \left[\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}\right] = \frac{1}{8}[e^{4a} - 6e^{2a} + 8e^a - 3]$$

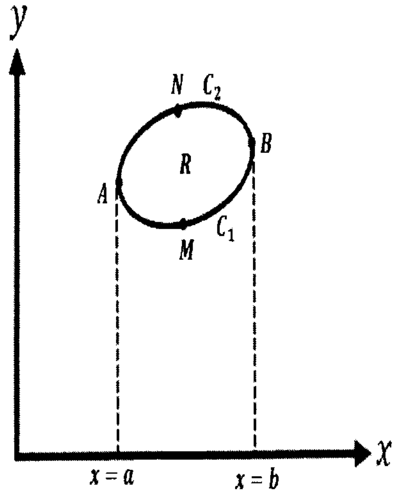
15. Green's Theorem

Green's theorem is useful for changing a line integral around a simple closed curve 'C' into a double integral over the region 'R' enclosed by 'C'.

Statement: If $P(x,y)$ and $Q(x,y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ in a region R of xy plane bounded by a simple closed curve C, then

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Proof: Let the curve C be divided into two curves C_1(AMB) and C_2(BNA), whose equations are given by $y = g_1(x)$ and $y = g_2(x)$ respectively.



Now, $\iint_R \frac{\partial P}{\partial y} dxdy = \int_{x=a}^{x=b} \left\{\int_{y=g_1(x)}^{y=g_2(x)} \frac{\partial P}{\partial y} dy\right\} dx$

$$= \int_a^b [P]_{g_1(x)}^{g_2(x)} dx$$

$$= \int_a^b [P(x,g_2) - P(x,g_1)]dx$$

$$= \int_a^b P(x,g_2)dx - \int_a^b P(x,g_1)dx$$

$$= -\int_b^a P(x,g_2)dx - \int_a^b P(x,g_1)dx$$

$$= -\left[\int_{C_1} P(x,g_1(x))dx + \int_{C_2} P(x,g_2(x))dx\right]$$

$$= -\left[\int_{C_1} P(x,y)dx + \int_{C_2} P(x,y)dx\right] = -\oint_C Pdx$$

$$\Rightarrow \oint_C Pdx = -\iint_R \frac{\partial P}{\partial y} dxdy \quad \text{...①}$$

Similarly, $\oint_C Qdy = \iint_R \frac{\partial Q}{\partial x} dxdy$ ...②

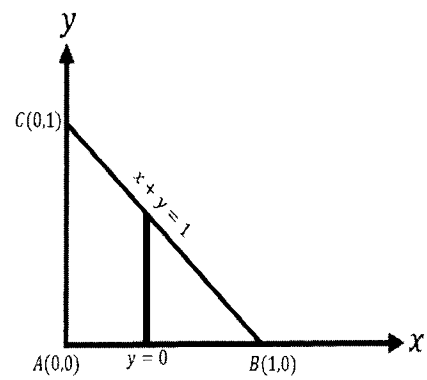
Adding ① and ②, we get $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

Example37 Verify Green's theorem in the plane for $\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$, where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$

Solution: Here, $P = 3x^2 - 8y^2, Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y \text{ and } \frac{\partial Q}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$$



Now, by Green's theorem

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\Rightarrow I = \oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$$

$$= \iint_R 10ydxdy$$

Now, the region R is defined by $x = 0, y = 0, x + y = 1$ as shown in adjoining figure.

$\therefore$ x varies from 0 to 1 and y varies from 0 to $1 - x$

$$\Rightarrow I = \int_0^1 \left\{\int_0^{1-x} 10ydy\right\} dx = 5\int_0^1 [y^2]_0^{1-x} dx = 5\int_0^1 (1 + x^2 - 2x)dx$$

$$= 5\left[x + \frac{x^3}{3} - x^2\right]_0^1 = \frac{5}{3}$$

Verification: $I = \oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$

Along AB, x varies from 0 to 1 and $y = 0$, $\therefore dy = 0$

$$\therefore I_1 = \int_0^1 3x^2 dx = 1$$

Along BC, x varies from 1 to 0 and $y = 1 - x$, $\therefore dy = -dx$

$$\therefore I_2 = \int_1^0 [(3x^2 - 8(1-x)^2)dx + \big(4(1-x) - 6x(1-x)\big)(-dx)]$$

$$= \int_1^0 (-11x^2 + 26x - 12)dx = \left[\frac{-11x^3}{3} + 13x^2 - 12x\right]_1^0$$

$$= \frac{11}{3} - 13 + 12 = \frac{8}{3}$$

Along CB, y varies from 1 to 0 and $x = 0$, $\therefore dx = 0$

$$\therefore I_3 = \int_1^0 4y dy = -2$$

Hence, $I = I_1 + I_2 + I_3 = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Example38 Verify Green's theorem in the plane for $\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$, where C is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$

Solution: Here, $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y \text{ and } \frac{\partial Q}{\partial x} = -6y$$

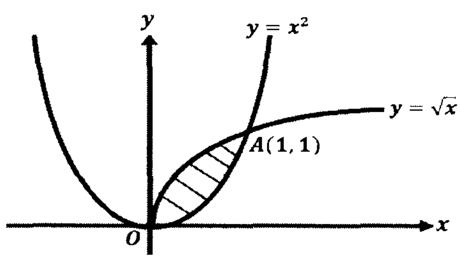
$$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$$

Now, by Green's theorem $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$\Rightarrow I = \oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy] = \iint_R 10y dxdy$

Now, the region R is defined by $y = \sqrt{x}$, $y = x^2$ as shown in adjoining figure.

$\therefore$ x varies from 0 to 1 and y varies from x^2 to $\sqrt{x}$



$$\Rightarrow I = \int_0^1 \left\{\int_{x^2}^{\sqrt{x}} 10y dy\right\} dx$$

$$= 5\int_0^1 [y^2]_0^{1-x} dx = 5\int_0^1 (x - x^4)dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{2}$$

Verification: $I = \oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$

Along OA, x varies from 0 to 1 and $y = x^2$, $\therefore dy = 2xdx$

$$\therefore I_1 = \int_0^1 [(3x^2 - 8x^4)dx + (4x^2 - 6x^3)2xdx]$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)dx$$

$$= [x^3 + 2x^4 - 4x^5]_0^1 = -1$$

Along AO, x varies from 1 to 0 and $y = \sqrt{x}$, $\therefore dy = \frac{1}{2\sqrt{x}} dx$

$$\therefore I_2 = \int_1^0 \left[(3x^2 - 8x)dx + \left(4\sqrt{x} - 6x^{\frac{3}{2}}\right)\frac{1}{2\sqrt{x}} dx\right]$$

$$= \int_1^0 (3x^2 - 11x + 2)dx = \left[x^3 - \frac{11x^2}{2} + 2x\right]_1^0$$

$$= -1 + \frac{11}{2} - 2 = \frac{5}{2}$$

Hence, $I = I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2}$

Example39 Evaluate $\int [\cos y\, \hat{\imath} + x\,(1 - \sin y)\hat{\jmath}].d\vec{r}$ on a closed curve given by $x^2 + y^2 = 1, z = 0$, using Green's theorem.

Solution: $\int [\cos y\, \hat{\imath} + x\,(1 - \sin y)\hat{\jmath}].d\vec{r} = \int [\cos y\, \hat{\imath} + x\,(1 - \sin y)\hat{\jmath}].[dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}]$

$$= \int [\cos y\, dx + x(1 - \sin y)dy]$$

Let $P = \cos y$, $Q = x(1 - \sin y)$

Then, by Green's theorem $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$$\Rightarrow \int [\cos y\, dx + x(1 - \sin y)dy] = \iint_R ((1 - \sin y) + \sin y)\, dxdy$$

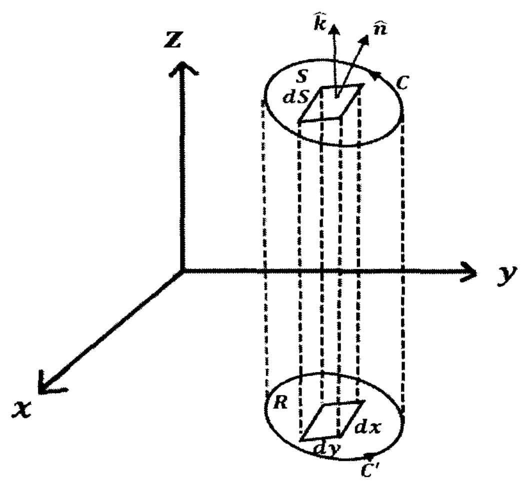
$$= \iint_R dxdy$$

Where, R is the unit circle $x^2 + y^2 = 1$ in xy plane

$\therefore \int [\cos y\, dx + x(1 - \sin y)dy] = \iint_R dxdy = \pi(1)^2 = \pi$

16. Stoke's Theorem

Stoke's theorem is useful for changing a line integral around a simple closed curve 'C' into a surface integral 'S' bounded by the curve 'C'.



Statement: If 'S' be an open surface bounded by a closed curve 'C' and $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be any differentiable vector point function, then $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS$, where $\hat{n}$ is unit outward drawn normal vector at any point of S.

Proof: Let $\hat{n}$ make angles α, β, γ with positive directions of x, y, z axes respectively.

Then $\hat{n} = \cos\alpha\hat{\imath} + \cos\beta\hat{\jmath} + \cos\gamma\, \hat{k}$

Also, $\hat{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, so that $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

Now, $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

$$\Rightarrow \text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$\therefore (\text{Curl}\vec{F}).\hat{n} = (\nabla \times \vec{F}).\hat{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma$

Also, $\vec{F}.d\vec{r} = (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}) = F_1dx + F_2dy + F_3dz$

Therefore, Stoke's theorem may be written as: $\oint_C (\boldsymbol{F_1dx + F_2dy + F_3dz})$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS \ldots (1)$$

First, we prove that $\int_C F_1dx = -\iint_S \left(\frac{\partial F_1}{\partial y}\cos\gamma - \frac{\partial F_1}{\partial z}\cos\beta\right) dS$

Let $z = \phi(x,y)$ be the equation of the surface S, whose projection on xy plane is R, and the projection of C on xy plane is the curve C', which bounds the region R.

Then, $\oint_C F_1(x,y,z)\,dx = \oint_C F_1(x,y,\phi)\,dx = \oint_C F_1(x,y,\phi)\,dx + 0dy \because z = \phi(x,y)$

Therefore, by Green's theorem in the plane for the region R

$$\oint_C F_1(x,y,\phi)\,dx + 0dy = -\iint_R \frac{\partial}{\partial y} F_1(x,y,\phi)dxdy$$

$$\Rightarrow \oint_C F_1(x,y,\phi)\,dx = -\iint_R \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}.\frac{\partial \phi}{\partial y}\right] dxdy \ldots (2) \because z = \phi(x,y)$$

Now, the direction ratios of the normal $\hat{n}$ to the surface $z = \phi(x,y)$ are $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, -1$

$$\Rightarrow \frac{\cos\alpha}{\frac{\partial\phi}{\partial x}} = \frac{\cos\beta}{\frac{\partial\phi}{\partial y}} = \frac{\cos\gamma}{-1} \Rightarrow \frac{\partial\phi}{\partial y} = -\frac{\cos\beta}{\cos\gamma} \ldots (3)$$

Also, $dxdy$ = projection of dS on xy plane= $\cos\gamma\, dS \ldots (4)$

Using (3) and (4) in (2), we get $\oint_C F_1(x,y,\phi)\,dx = -\iint_S \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\left(-\frac{\cos\beta}{\cos\gamma}\right)\right)\cos\gamma\, dS$

$$\Rightarrow \int_C F_1dy = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS \ldots (5)$$

Similarly, we can prove that: $\int_C F_2dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right) dS \ldots (6)$

And, $\int_C F_3dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) dS \ldots (7)$

Adding (5), (6), (7), we get the required result (1).

Example40 Evaluate $\oint_C \vec{F}.d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0,0,0), (1,0,0) and (1,1,0).

Solution: Using Stoke's theorem, we have: $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS$

Now, $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = (0-0)\hat{\imath} - (-1)\hat{\jmath} + (2x - 2y)\hat{k}$

$\therefore \text{Curl}\vec{F} = \nabla \times \vec{F} = \hat{\jmath} + 2(x-y)\hat{k}$

Now, since the z coordinate of each vertex of triangle is zero, C lies on xy plane,

hence $\hat{n} = \hat{k}$

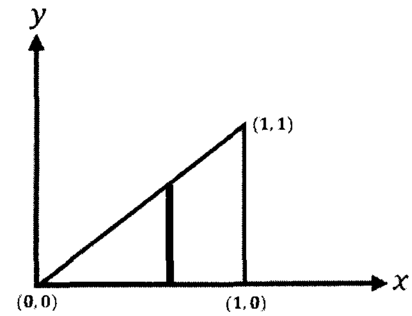
$\therefore\ (\text{Curl}\vec{F}).\hat{n} = (\hat{j} + 2(x-y)\hat{k}).\hat{k} = 2(x-y)$

Thus, $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS = \int_0^1 \int_0^x 2(x-y)dydx$

$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx$

$= 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right) dx$

$= \int_0^1 x^2 dx = \frac{1}{3}$



Example41 Evaluate by Stoke's theorem, the line integral for the function $\vec{F} = x^2\hat{\imath} - xy\hat{\jmath}$, integrated round the square in the plane $z = 0$ and bounded by the lines $x = 0, y = 0,$ $x = a, y = a.$

Solution: Using Stoke's theorem, we have: $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS$

Now, $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix} = -y\hat{k}$

$\therefore \text{Curl}\vec{F} = \nabla \times \vec{F} = -y\hat{k}$

Now, since the region bounded by the lines $x = 0, y = 0, x = a, y = a$lies on xy plane, hence $\hat{n} = \hat{k}$

$\therefore\ (\text{Curl}\vec{F}).\hat{n} = (-y\hat{k}).\hat{k} = -y$

Thus, $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS = \int_0^a \int_0^a -ydydx$

$= \int_0^a \left[-\frac{y^2}{2}\right]_0^a dx = \int_0^2 \left(-\frac{a^2}{2}\right) dx = -\frac{a^2}{2}[x]_0^a = -\frac{a^3}{2}$

Example42 Use Stoke's theorem to evaluate $\oint_C [(2x-y)dx - yz^2dy - y^2zdz]$, where C is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

Solution: Here, $\vec{F} = (2x-y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$

Using Stoke's theorem, we have: $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS$

Now, $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (-2yz + 2yz)\hat{\imath} - (0)\hat{\jmath} + (1)\hat{k}$

$\therefore \text{Curl}\vec{F} = \nabla \times \vec{F} = \hat{k}$

Now, since the circle $x^2 + y^2 = 1$lies on xy plane, hence $\hat{n} = \hat{k}$

$\therefore\ (\text{Curl}\vec{F}).\hat{n} = \hat{k}.\hat{k} = 1$

Thus, $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS = \iint_S dS =$ Area of the circle $x^2 + y^2 = 1$

$\Rightarrow \oint_C \vec{F}.d\vec{r} = \pi(1)^2 = \pi$

Example43 Use Stoke's theorem to evaluate $\oint_C [e^x dx + 2y dy - dz]$, where C is the curve $x^2 + y^2 = a^2$ and $z = 0$

Solution: Here, $\vec{F} = e^x\hat{\imath} + 2y\hat{\jmath} - \hat{k}$

Using Stoke's theorem, we have: $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS$

Now, $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (0-0)\hat{k}$

$\therefore \text{Curl}\vec{F} = \nabla \times \vec{F} = 0$

Hence, $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS = 0$

Example44 Use Stoke's theorem to evaluate $\oint_C [y^2 dx + xy dy + xz dz]$, where C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9, z > 0$ oriented in the positive direction.

Solution: Here, $\vec{F} = y^2\hat{\imath} + xy\hat{\jmath} + xz\hat{k}$

Using Stoke's theorem, we have: $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS$

Now, $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = (0-0)\hat{\imath} - (z-0)\hat{\jmath} + (y-2y)\hat{k}$

$\therefore \text{Curl}\vec{F} = \nabla \times \vec{F} = -z\hat{\jmath} - y\hat{k}$

Now, S is the hemisphere$x^2 + y^2 + z^2 = 9, z > 0$

Let the surface S be denoted by $\phi = x^2 + y^2 + z^2 - 9$

Then, a unit vector normal to ϕ is $\frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$= \frac{1}{3}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \qquad \because x^2 + y^2 + z^2 = 9$

$\therefore \hat{n} = \frac{1}{3}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$

Taking the projection of surface S on xy plane, $dS = \frac{dxdy}{|\hat{n}.\hat{k}|} = \frac{dxdy}{\left|\frac{z}{3}\right|} = \frac{3}{z}dxdy$

Now, $(\text{Curl}\vec{F}).\hat{n} = (-z\hat{\jmath} - y\hat{k}).\frac{1}{3}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = \frac{1}{3}(-yz - yz) = -\frac{2}{3}yz$

Thus, $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS = \iint_R -\frac{2}{3}yz\frac{3}{z}dxdy = \iint_R -2y dxdy$

Where R is the circle $x^2 + y^2 = 9$, as being projected on xy plane $z = 0$

Changing to polar form, i.e. putting $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$

$\oint_C \vec{F}.d\vec{r} = \int_0^{2\pi}\int_0^3 -2r\sin\theta\, rdrd\theta = -2\int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^3 \sin\theta\, d\theta$

$= -18\int_0^{2\pi}\sin\theta\, d\theta = 18[\cos\theta]_0^{2\pi} = 0$

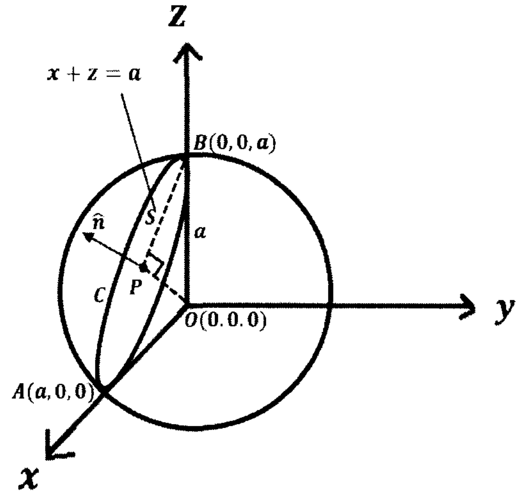
Example45 Apply Stoke's theorem to evaluate $\int_C [ydx + zdy + xdz]$, where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$

Solution: Here, $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$

Using Stoke's theorem, we have: $\int_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS$

Now, $\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (0-1)\hat{\imath} - (1-0)\hat{\jmath} + (0-1)\hat{k}$

$\therefore \text{Curl}\overrightarrow{F} = \nabla \times \vec{F} = -\hat{\imath} - \hat{\jmath} - \hat{k}$



Now, the surface S is formed by the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + z = a$, bounded by the circle C.

Let the surface S be denoted by $\phi = x + z - a$

Any vector normal to ϕ is given by:

$\nabla\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + z - a) = \hat{\imath} + \hat{k}$

Then, a unit vector normal to ϕ is given by:

$\frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{\imath}+\hat{k}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{k})$

$\therefore \hat{n} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{k})$

Now, $(\text{Curl}\vec{F}).\hat{n} = (-\hat{\imath} - \hat{\jmath} - \hat{k}).\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{k}) = \frac{-1-1}{\sqrt{2}} = -\sqrt{2}$

Thus, $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{Curl}\vec{F}).\hat{n}\, dS = -\sqrt{2}\iint_S dS$

where S is the surface bounded by the circle C

To find the radius PB of the circle C, $PB^2 = OB^2 - OP^2$ as shown in adjoining figure.

$OB = a$ and $OP =$ perpedicular distace from centre O to the plane $x + z - a$

$\Rightarrow OP = \left|\frac{0+0-a}{\sqrt{1+1}}\right| = \frac{a}{\sqrt{2}}$

$\therefore PB = OB^2 - OP^2 = \sqrt{a^2 - \frac{a^2}{2}} = \frac{a}{\sqrt{2}}$

Hence, the area of circle $C = \frac{\pi a^2}{2}$

Therefore, $\oint_C \vec{F}.d\vec{r} = -\sqrt{2}\iint_S dS = -(\sqrt{2})\frac{\pi a^2}{2} = -\frac{\pi a^2}{\sqrt{2}}$

17. Gauss Divergence Theorem

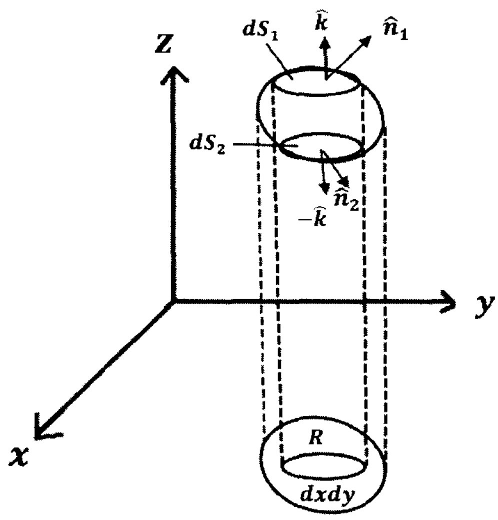
Gauss divergence theorem provides the relation between surface integral and volume integral.

Statement: If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a continuously differentiable vector point function, having continuous partial derivatives in the region 'R' bounded by a closed surface 'S', then

$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\vec{F}\, dV$,

where $\hat{n}$ is unit outward drawn normal vector at any point of S.

Proof: Let $\hat{n}$ make angles α, β, γ with positive directions of x, y, z axes respectively.



Then $\hat{n} = \cos\alpha\hat{\imath} + \cos\beta\hat{\jmath} + \cos\gamma\,\hat{k}$

$\therefore \iint_S \vec{F}.\hat{n}\, dS = \iint_S (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).(\cos\alpha\hat{\imath} + \cos\beta\hat{\jmath} + \cos\gamma\,\hat{k})dS$

$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_1\cos\gamma)dS$

Also, $\iiint_V div\vec{F}\, dV = \iiint_V \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})dV$

$\Rightarrow \iiint_V div\vec{F}\, dV = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$

Therefore, Gauss divergence theorem may be written as:

$\iint_S (F_1\cos\alpha + F_2\cos\beta + F_1\cos\gamma)dS = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$ ...①

Let 'S' be a closed surface and 'R' be the orthogonal projection of S on xy plane. Also let S_1 and S_2 be upper and lower portions of the surface S, with respective equations $Z = \phi_1(x,y)$ and $Z = \phi_2(x,y)$, such that $\phi_1(x,y) > \phi_2(x,y)$.

Consider the volume integral $\iiint_V \frac{\partial F_3}{\partial z} dV = \iiint_V \frac{\partial F_3}{\partial z} dxdydz = \iint_R \left\{\int_{\phi_2(x,y)}^{\phi_1(x,y)} \frac{\partial F_3}{\partial z} dz\right\} dxdy$

$\Rightarrow \iiint_V \frac{\partial F_3}{\partial z} dV = \iint_R [F_3(x,y,z)]_{\phi_2(x,y)}^{\phi_1(x,y)} dxdy = \iint_R [F_3(x,y,\phi_1) - F_3(x,y,\phi_2)]\, dxdy$ ...②

Let $\hat{n}_1$ be outward drawn normal to the upper surface S_1 making an acute angle γ_1 with $\hat{k}$

Then, the projection $dxdy$ of dS_1 on xy plane is given by $dxdy = dS_1\cos\gamma_1 = \hat{k}.\hat{n}_1 dS_1$

$\therefore \iint_R F_3(x,y,\phi_1)dxdy = \iint_{S_1} F_3(x,y,\phi_1)\hat{k}.\hat{n}_1\, dS_1$ ...③

Similarly, if $\hat{n}_2$ be outward drawn normal to the lower surface S_2 making an obtuse angle γ_2 with $\hat{k}$, $dxdy = dS_1\cos(\pi - \gamma_2) = -\hat{k}.\hat{n}_1 dS_2$

$\iint_R F_3(x,y,\phi_2)dxdy = -\iint_{S_2} F_3(x,y,\phi_2)\hat{k}.\hat{n}_2 dS_2$ ...④

From ②, ③ and ④, $\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_{S_1} F_3(x,y,\phi_1)\hat{k}.\hat{n}_1 dS_1 + \iint_{S_2} F_3(x,y,\phi_2)\hat{k}.\hat{n}_2 dS_2$

$\Rightarrow \iiint_V \frac{\partial F_3}{\partial z} dV = \iint_S F_3\hat{k}.\hat{n}\, dS$ ...⑤

Similarly, by taking projection of S on other coordinate planes, we have:

$\iiint_V \frac{\partial F_1}{\partial x} dV = \iint_S F_1 \hat{\imath}.\hat{n}\, dS \ldots$ ⑥ and $\iiint_V \frac{\partial F_2}{\partial x} dV = \iint_S F_2 \hat{\jmath}.\hat{n}\, dS \ldots$ ⑦

Adding ⑤, ⑥ and ⑦, we get

$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV = \iint_S (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).\hat{n}\, dS$

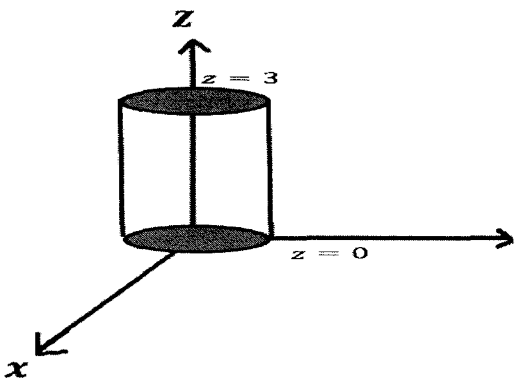
$\Rightarrow \iiint_V div\vec{F}\, dV = \iint_S \vec{F}\,.\hat{n}\, dS$

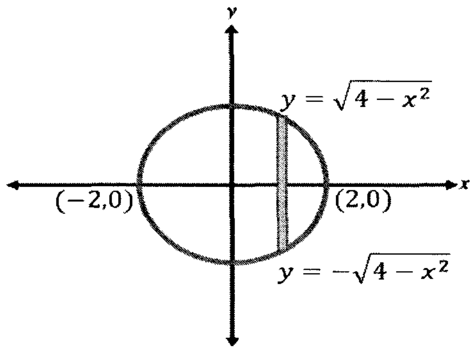
Example46 Evaluate $\iint_S \vec{F}.\hat{n}dS$, where $\vec{F} = 4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}$ and 'S' is the surface bounded by the region $x^2 + y^2 = 4$, $z = 0$and $z = 3$.

Solution: By Gauss divergence theorem $\iint_S \vec{F}\,.\hat{n}\, dS = \iiint_V div\vec{F}\, dV$

$\Rightarrow \iint_S \vec{F}\,.\hat{n}\, dS = \iiint_V \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}\right) dV$

$= \iiint_V (4 - 4y + 2z)\, dV$





$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)dzdydx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4z - 4yz + z^2]_0^3\, dydx$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dydx = \int_{-2}^{2} [21y - 6y^2]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\, dx$

$= \int_{-2}^{2}\left(42\sqrt{4 - x^2} - 6(4 - x^2) + 6(4 - x^2)\right) dx$

$\Rightarrow \iint_S \vec{F}\,.\hat{n}\, dS = 84\int_0^2 \sqrt{4 - x^2}\, dx = 84\left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 = 84\pi$

Example47 Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}$ taken over a rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$

Solution: Here, $\vec{F} = (x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}$

$\iint_S \vec{F}\,.\hat{n}\, dS = \iint_{S_1} \vec{F}\,.\hat{n}\, dS + \iint_{S_2} \vec{F}\,.\hat{n}\, dS + \cdots + \iint_{S_6} \vec{F}\,.\hat{n}\, dS \ldots$ ①

Where $S_1, S_2, \ldots, S_6$ are respective faces $FBCD$, $ADHE$, $FGHE$, $ABCD$, $FEAB$ and $DCGH$ of the parallelepiped, as shown in adjoining figure.

On the surfaceS_1, $z = 0$ and $\hat{n} = -\hat{k}$

$\therefore \vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} - xy\hat{k}$ and $\vec{F}.\hat{n} = xy$

$\Rightarrow \iint_{S_1} \vec{F}\,.\hat{n}\, dS = \int_0^a\int_0^b xydydx = \int_0^a \left[\frac{xy^2}{2}\right]_0^b dx = \frac{b^2}{2}\int_0^a xdx = \frac{a^2b^2}{4}$

On the surface S_2, $z = c$ and $\hat{n} = \hat{k}$

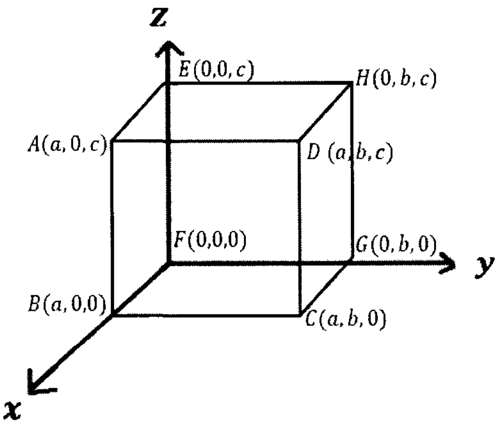
$\therefore \vec{F} = (x^2 - cy)\hat{\imath} + (y^2 - cx)\hat{\jmath} + (c^2 - xy)\hat{k}$

$\therefore \vec{F}.\hat{n} = c^2 - xy$

$\Rightarrow \iint_{S_2} \vec{F}.\hat{n}\, dS = \int_0^a \int_0^b (c^2 - xy)dydx\, dx$

$= \int_0^a \left[c^2 y - \frac{xy^2}{2}\right]_0^b$

$= \int_0^a \left(c^2 b - \frac{xb^2}{2}\right) dx = abc^2 - \frac{a^2b^2}{4}$



By symmetry,

$\iint_{S_3} \vec{F}.\hat{n}\, dS = \frac{b^2c^2}{4}$, $\iint_{S_4} \vec{F}.\hat{n}\, dS = bca^2 - \frac{b^2c^2}{4}$

$\iint_{S_5} \vec{F}.\hat{n}\, dS = \frac{c^2a^2}{4}$, $\iint_{S_4} \vec{F}.\hat{n}\, dS = cab^2 - \frac{c^2a^2}{4}$

$\therefore$ Using ① $\Rightarrow \iint_S \vec{F}.\hat{n}\, dS = abc^2 + bca^2 + cab^2 = abc(a + b + c) \ldots$ ②

Again, $div\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left((x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}\right)$

$= 2x + 2y + 2z = 2(x + y + z)$

$\iiint_V div\vec{F}\, dV = \int_0^a \int_0^b \int_0^c 2(x + y + z)dzdydx$

$= 2\int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2}\right]_0^c dydx = 2\int_0^a \int_0^b \left(cx + cy + \frac{c^2}{2}\right) dydx$

$= 2\int_0^a \left[cxy + \frac{cy^2}{2} + \frac{c^2y}{2}\right]_0^b dx = 2\int_0^a \left(bcx + \frac{cb^2}{2} + \frac{bc^2}{2}\right) dx$

$= 2\left[\frac{bcx^2}{2} + \frac{xcb^2}{2} + \frac{xbc^2}{2}\right]_0^a = 2\left[\frac{bca^2}{2} + \frac{acb^2}{2} + \frac{abc^2}{2}\right]$

$= bca^2 + acb^2 + abc^2 = abc(a + b + c) \ldots$ ③

Hence, from ② and ③, $\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\vec{F}\, dV$

Test Your Knowledge

1. Find $\vec{V}'(t)$ or $V'(t)$ in each of the following cases
   (a) $\vec{V}(t) = (\cos t + t^2)\left(t\hat{\imath} + \hat{\jmath} + 2\hat{k}\right)$
   (b) $\vec{V}(t) = \left(3t\hat{\imath} + 5t^2\hat{\jmath} + 6\hat{k}\right).\left(t^2\hat{\imath} - 2t\hat{\jmath} + t\hat{k}\right)$
   (c) $\vec{V}(t) = \left(t\hat{\imath} + e^t\hat{\jmath} - t^2\hat{k}\right) \times \left(t^2\hat{\imath} + \hat{\jmath} + t^3\hat{k}\right)$
2. Find arc length of the vector function $\vec{r}(t) = 2t\hat{\imath} + 3\sin 2t\,\hat{\jmath} + 3\cos 2t\,\hat{k}, 0 \le t \le 2\pi$
3. If a position vector $\vec{r}(t) = \langle t, 2\cos t, 2\sin t\rangle$, find the unit tangent, unit normal and binormal vectors.
4. Find the curvature and torsion of the curve $x = t, y = t^2\,,\; z = t^3$
5. Find the curvature and torsion of the curve $x = 3\sin t, y = 3\cos t\,,\; z = 4t$
6. If : $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, show that

(a) $grad\ r = \frac{\vec{r}}{r}$ (b) $grad\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

7. If $\nabla u = 2r^4\vec{r}$, find u
8. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1,2,-1)$
9. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. Also calculate the magnitude of the maximum directional derivative.
10. If $u = x^2 + y^2 + z^2$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then find $\text{div}(u\vec{r})$ in terms of u.
11. If $\vec{F} = (x^2 - y^2 + 2zx)\hat{\imath} + (xz - xy - yz)\hat{\jmath} + (z^2 + x^2)\hat{k}$ is a vector field, find $\text{Curl}\vec{F}$. Show that the vectors given by the $\text{Curl}\vec{F}$ at $P(1,2,-3)$ and $Q(2,3,-2)$ are orthogonal.
12. Show that the vector field $\vec{F} = 2x\hat{\imath} + 4y\hat{\jmath} + 8z\hat{k}$ be irrotational and find the scalar ϕ such that $\vec{F} = \nabla\phi$
13. Find the value of β for which the vector $\vec{F} = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x + \beta z)\hat{k}$ is solenoidal vector.
14. Show that Curl $(\vec{a} \times \vec{r}) = 2\vec{a}$ where $\vec{a}$ is a constant vector.
15. If $\vec{A} = x^2yz\hat{\imath} + xyz^2\hat{\jmath} + y^2z\hat{k}$, then determine $\text{Curl}(\text{Curl}\vec{A})$
16. If $\phi(x,y,z) = x^2yz$, $\vec{F} = x^2y\hat{\imath} + y^2x^3\hat{\jmath} - 3x^2z^2\hat{k}$ and $\vec{G} = 2xz^2\hat{\imath} - yz\hat{\jmath} + x^2y^3\hat{k}$, find the values of the following expressions at $P(1,2,1)$

    (a) $(\vec{F}.\nabla)\phi$ (c) $(\vec{F}.\nabla)\vec{G}$

    (b) $\nabla.(\phi F)$ (d) $(\vec{F} \times \nabla) \times \vec{G}$
17. Determine whether the line integral $\int_C (2xyz^2)dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$ is independent of path of integration. If so, then evaluate it from $(1,0,1)$ to $\left(0, \frac{\pi}{2}, 1\right)$
18. Prove that $\vec{F} = (y^2\cos x + z^3)\hat{\imath} + (2y\sin x - 4)\hat{\jmath} + (3xz^2 + 2)\hat{k}$ is

    (a) Conservative field

    (b) Find scalar potential of $\vec{F}$

    (c) Find work done in moving an object in this field from $A(0,1,-1)$ to $B\left(\frac{\pi}{2}, -1, 2\right)$
19. Evaluate $\int_0^1 \int_0^{1-x} x^2y\ dydx$
20. Evaluate $\iint_A xy\ dxdy$, where A is the domain bounded by x – axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$
21. Evaluate $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)dydx$, by changing to polar co-ordinates.
22. Find $\iint_S \vec{F}.\hat{n}\ dS$, where $\vec{F} = (x + y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$ and S is part of the plane $2x + y + 2z = 6$ in the first octant.
23. Evaluate

    (a) $\int_0^1 \int_0^2 \int_1^2 x^2yz\ dz\ dydx$

    (b) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dxdydz$

24. Verify Green's theorem in the plane for $\oint_C (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

25. Apply Green's theorem to evaluate $\oint_C (y - \sin x)dx + \cos x\, dy$ where C is the plane triangle enclosed by the lines $y = 0, x = \frac{\pi}{2}$ and $y = \frac{2}{\pi}x$

26. Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\hat{\imath} - 2xy\hat{\jmath}$ taken round the rectangle bounded by the lines $x = \pm a, y = 0, y = b$

27. Find $\iint_S \vec{F}.\hat{n}dS$ where $\vec{F} = (2x + 3z)\hat{\imath} - (xz + y)\hat{\jmath} + (y^2 + 2z)\hat{k}$ and S is the surface of the sphere having centre at $(3, -1, 2)$ and radius 3.

28. Evaluate $\iint_S \vec{F}.\hat{n}dS$ over the entire surface of the region above the xy − plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\vec{F} = 4xz\hat{\imath} + xyz^2\hat{\jmath} + 3z\hat{k}$

Answers

1. (a) $(\cos t - t\sin t + 3t^2)\,\hat{\imath} + (-\sin t + 2t)\hat{\jmath} + 2(-\sin t + 2t)\,\hat{k}$

   (b) $6 - 21t^2$

   (c) $[t^2e^t(t + 3) + 2t]\hat{\imath} - 8t^3\hat{\jmath} + [1 - te^t(t + 2)]\hat{k}$

2. $4\pi\sqrt{10}$

3. $\langle\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\sin t, \frac{2}{\sqrt{5}}\cos t,\rangle$, $\langle 0, -\cos t, -\sin t\rangle$, $\langle\frac{2}{\sqrt{5}}. \frac{1}{\sqrt{5}}\sin t, \frac{-1}{\sqrt{5}}\cos t\rangle$

4. $\frac{2\sqrt{9t^4+9t^2+1}}{(9t^4+9t^2+1)^{\frac{3}{2}}}, \frac{3}{9t^4+9t^2+1}$

5. $\frac{3}{25}, \frac{-4}{25}$

7. $u = \frac{1}{3}r^{3/2} + c$

8. $\frac{1}{\sqrt{14}}(-\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$

9. $\frac{28}{\sqrt{21}}, \sqrt{164}$

10. $5u$

11. $(y - x)\hat{\imath} + (y + z)\hat{k}$

12. $\phi = x^2 + 2y^2 + 4z^2 + c$

13. $\beta = -2$

15. $z^2\hat{\imath} + (2xz - 2xy + 2y)\hat{\jmath} + (2xy + 2xz - 2z)\hat{k}$

16. (a) 6 (b) 10 (c) $-8\hat{\imath} + 2\hat{\jmath} + 80\hat{k}$ (d) $-46\hat{\imath} - 44\hat{\jmath} + 3\hat{k}$

17. 1

18. (b) $y^2\sin x + xz^3 - 4y + 2z + c$ (c) $(4\pi + 15)$

19. $\frac{1}{60}$ 20. $\frac{a^4}{3}$ 21. $\frac{3}{4}\pi a^4$ 22. 81 23. (a) 1 (b) $\frac{1}{48}$

25. $-\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$

27. $108\,\pi$ 28. $320\,\pi$

MID TERM EXAMINATION , JANUARY, 2023

Paper Code: BS-111 Subject: Applied Mathematics-I

Time: 1 Hour 30 Mins Maximum Marks :30

Note: Attempt Q1 which is compulsory and any two more questions from remaining.

Q1. (a) Find the derivative with respect to x of the integral $\int_x^{x^2} \frac{\sin xt}{t}\, dt$

Solution: Given $I = \int_x^{x^2} \frac{\sin xt}{t}\, dt \ldots ①$

Differentiating ① with respect to parameter x

$$\frac{dI}{dx} = \int_x^{x^2} \frac{\partial}{\partial x}\left(\frac{sinxt}{t}\right) dt + f(x^2, x)\frac{dx^2}{dx} - f(x,x)\frac{dx}{dx} \quad \text{by Leibnitz's Rule}$$

$$\Rightarrow \frac{dI}{dx} = \int_x^{x^2} \frac{tcosxt}{t}\, dt + \frac{\sin x^3}{x^2}(2x) - \frac{\sin x^2}{x}(1)$$

$$= \left[\frac{\sin xt}{x}\right]_x^{x^2} + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x}$$

$$= \left(\frac{\sin x^3}{x} - \frac{\sin x^2}{x}\right) + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x} = \frac{1}{x}[3\sin x^3 - 2\sin x^2]$$

(b) Determine the differential $\left[\frac{x}{(x^2+y^2)}\right] dy - \left[\frac{y}{(x^2+y^2)}\right] dx$ is exact or not

Solution: Given differential is of the form $df = f_1(x,y)dx + f_2(x,y)dy$,

$f_1(x,y) = \frac{-y}{(x^2+y^2)}$, $f_2(x,y) = \frac{x}{(x^2+y^2)}$

We have $\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{(x^2+y^2)}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} \ldots ①$

Also, $\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} \ldots ②$

From ① and ② $\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial f_2(x,y)}{\partial x}$

$\therefore df = \left[\frac{x}{(x^2+y^2)}\right] dy - \left[\frac{y}{(x^2+y^2)}\right] dx$ is an exact differential.

(c) Express $5x^3 + x^2 - 2x + 1$in terms of Legendre polynomial

Solution: $1 = P_0(x)$, $x = P_1(x)$,

$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow x^2 = \frac{1}{3}(2P_2(x) + 1) = \frac{1}{3}\left(2P_2(x) + P_0(x)\right)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow x^3 = \frac{1}{5}(2P_3(x) + 3x) = \frac{1}{5}\left(2P_3(x) + 3P_1(x)\right)$

Let $F = (5x^3 + x^2 - 2x + 1)$

Substituting values of 1, x, x^2 and x^3in terms of Legendre's polynomials, we get

$$F = \left[\left(2P_3(x) + 3P_1(x)\right) + \left(\frac{1}{3}\left(2P_2(x) + P_0(x)\right)\right) - 2P_1(x) + P_0(x)\right]$$

$= 2P_3(x) + \frac{2}{3}P_2(x) + P_1(x) + \frac{4}{3}P_0(x)$

(d) Find the orthogonal trajectories of family of curves $y = cx^2$, c being a Parameter

Solution: Given family of circles is: $y = cx^2$ ... ①

Differentiating both sides of ① with respect to x

$\frac{dy}{dx} = 2cx = 2.\frac{y}{x^2}.x = 2\frac{y}{x} \Rightarrow \frac{dy}{dx} = 2\frac{y}{x} = m$,

where m is the slope of the family of curves$y = cx^2$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = -\frac{x}{2y}$

Hence the differential equation of orthogonal trajectories is given by

$\frac{dy}{dx} = -\frac{x}{2y}$ $\Rightarrow 2ydy = -xdx$

Integrating both sides, we have

$\int 2ydy = -\int xdx$

$\Rightarrow y^2 = \frac{-x^2}{2} + c$, c is an arbitrary constant

$\Rightarrow 2y^2 + x^2 = 2c \Rightarrow x^2 + 2y^2 = c'$

$\therefore x^2 + 2y^2 = c'$ is the required orthogonal trajectory.

Q2. (a) If $u = sin^{-1}(x - y), x = 3t$ and $y = 4t^3$, find $\frac{du}{dt}$

Solution: See example 8 (Unit-I)

(b) If$x = \rho cos\emptyset$ and $y = \rho sin\emptyset$, transform the equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ into one in ρ and $\emptyset$.

Solution: See example 36 (Unit-I)

Q3. (a) The temperature T at any point (x, y, z) of space is given by$T = 400xyz^2$. Find the highest temperature at the surface of the sphere $x^2 + y^2 + z^2 = 1$

Solution: Let $F(x, y, z, \lambda) = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1)$

$\Rightarrow F(x, y, z, \lambda) = 400xyz^2 + \lambda x^2 + \lambda y^2 + \lambda z^2 - \lambda$

Differentiating $F(x, y, z, \lambda)$ with respect to x, y, z and equating to zero

$\frac{\partial F}{\partial x} = 400yz^2 + 2\lambda x = 0 \Rightarrow \lambda = -\frac{200yz^2}{x}$ ... ①

$\frac{\partial F}{\partial y} = 400xz^2 + 2\lambda y = 0 \Rightarrow \lambda = -\frac{200xz^2}{y}$ ... ②

$\frac{\partial F}{\partial z} = 800xyz + 2\lambda z = 0 \Rightarrow \lambda = -400xy$ ... ③

From ①, ② and ③, we get

$-\frac{200yz^2}{x} = -\frac{200xz^2}{y} = -400xy$

From first two equations, we get

$\Rightarrow -200y^2z^2 = -200x^2z^2 \Rightarrow x^2 = y^2 \ldots (4)$

Similarly from last two equations, we get $z^2 = 2y^2 \ldots (5)$

Given $x^2 + y^2 + z^2 = 1 \ldots (6)$

Using (4) and (5) in (6), we get

$y^2 + y^2 + 2y^2 = 1 \Rightarrow 4y^2 = 1 \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

$\therefore x = \pm\frac{1}{2}, y = \pm\frac{1}{2}, z = \pm\frac{1}{\sqrt{2}}$

Hence, stationary points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$

At $\left(\frac{1}{2}, \frac{-1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{2}, \frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$, i.e, when x and y have opposite signs, $T = -50$

At $\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{2}, \frac{-1}{2}, \pm\frac{1}{\sqrt{2}}\right)$, i.e, when x and y have same signs, $T = 50$

Hence, highest temperature is 50 which is attained at $\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{2}, \frac{-1}{2}, \pm\frac{1}{\sqrt{2}}\right)$

(b) Prove that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - cosx\right)$

Solution: We know that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,\ J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

By recurrence formula, we have $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \ldots (1)$

Putting $n = \frac{1}{2}$ in (1)

$J_{1/2}(x) = x\left(J_{-1/2}(x) + J_{3/2}(x)\right)$

$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) \quad = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

Q4. (a) Solve by method of variation of parameters: $\frac{d^2y}{dx^2} + a^2y = sec\, ax$

Solution: $(D^2 + a^2)y = \sec x$

Auxiliary equation is: $(m^2 + a^2) = 0$

$\Rightarrow m = \pm ia$

C.F. $= c_1 \cos ax + c_2 \sin ax = c_1y_1 + c_2y_2$

$\therefore y_1 = \cos ax$ and $y_2 = \sin ax$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{\sin ax \sec ax}{a} dx = -\frac{1}{a}\int \tan ax \, dx = -\frac{log|\sec ax|}{a^2}$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{\cos ax \sec ax}{a} dx = \frac{1}{a}\int 1 \, dx = \frac{x}{a}$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$= -\frac{\cos ax \, log|\sec ax|}{a^2} + \frac{x}{a}\sin ax$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 \cos ax + c_2 \sin ax - \frac{\cos ax \, log|\cos ax|}{a^2} + \frac{x}{a}\sin ax$

(b) Solve $(1 + y^2)dx = (tan^{-1}y - x)dy$

Solution: The given equation may be written as:

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{tan^{-1}y}{1+y^2} \; \ldots\ldots \text{(1)}$$

This is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$

Where $P = \frac{1}{1+y^2}$ and $Q = \frac{tan^{-1}y}{1+y^2}$

IF $= e^{\int P\,dx} = e^{\int \frac{1}{1+y^2}dx} = e^{tan^{-1}y}$

$\therefore$ Solution of (1) is given by $x.\,e^{tan^{-1}y} = \int \frac{tan^{-1}y}{1+y^2}.\,e^{tan^{-1}y}dy + C$

$= \int te^t\,dt + C = te^t - \int e^t dt + C$ $\quad \left[\begin{array}{l} \text{put } tan^{-1}y = t \\ \frac{dy}{1+y^2} = dt \end{array}\right]$

$= te^t - e^t + C = (tan^{-1}y - 1)e^{tan^{-1}y} + C$

$\therefore x = tan^{-1}y - 1 + Ce^{tan^{-1}y}$

END TERM EXAMINATION, MARCH 2023

Paper Code: BS-111 **Subject:** Applied Mathematics-I

Time: 3 Hours **Maximum Marks :**75

Q1. Attempt **all** questions:

(a) If $\int_0^1 x^m dx = \frac{1}{m+1}$, then find the value of $\int_0^1 x^m (log x) dx$

Solution: Given $\int_0^1 x^m dx = \frac{1}{m+1} \dots (1)$

Differentiating (1) with respect to parameter m

$\int_0^1 x^m (log x) dx = \frac{-1}{(m+1)^2}$

(b) If $\vec{F} = (x + y + 1)\hat{\imath} + \hat{\jmath} - (x + y)\hat{k}$, then compute the value of $\vec{F}.Curl(\vec{F})$

Solution: $Curl(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$

$= (-1 - 0)\hat{\imath} - (-1 - 0)\hat{\jmath} + (0 - 1)\hat{k}$

$= -\hat{\imath} + \hat{\jmath} - \hat{k}$

$\vec{F}.Curl(\vec{F}) = [(x + y + 1)\hat{\imath} + \hat{\jmath} - (x + y)\hat{k}].[-\hat{\imath} + \hat{\jmath} - \hat{k}]$

$= (x + y + 1)(-1) + (1)(1) - (x + y)(-1)$

$= -x - y - 1 + 1 + x + y = 0$

(c) Find the particular integral for the linear differential equation

$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = sin3x$

Solution: Auxiliary equation is: $m^2 + 6m + 9 = 0$

$\Rightarrow (m + 3)(m + 3) = 0$

$\Rightarrow m = -3, -3$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin 3x = \frac{1}{D^2+6D+9} \sin 3x$

putting $D^2 = -3^2 = -9$

P.I $= \frac{1}{D^2+6D+9} \sin 3x$

$= \frac{1}{-9+6D+9} \sin 3x$, Putting $D^2 = -9$

$\therefore$ P. I. $= \frac{1}{6D} sin3x = \frac{1}{6} \int sin3x \, dx$

$= \frac{1}{6}\left[\frac{-cos3x}{3}\right] = \frac{-cos3x}{18}$

(d) Determine the rank of the matrix A $= \begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & 2 & 5 & 2 \\ 2 & 1 & 1 & 3 \end{pmatrix}$

Solution: Here, A is a matrix of order 3×4

$\rho(A) \leq min(3 \times 4)$

$\therefore\ \rho(A) \leq 3$

To find $\rho(A)$, let us check

$$|A|_{3\times3} = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{vmatrix} = 18 \neq 0 \therefore\ \rho(A) = 3$$

(e) Applying Gauss Divergence theorem, find the value of $\iint_S \vec{F}\,.\hat{n}ds$, for $\vec{F} = (x^2 - yz)\hat{\imath} + (y^2 - xz)\hat{\jmath} + (z^2 - yx)\hat{k}$ and S is the cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$

Solution: By Gauss Divergence theorem $\iint_S \vec{F}\,.\hat{n}\, dS = \iiint_V div\vec{F}\, dV$

Here, $\vec{F} = (x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}$

Now, $div\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left((x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}\right)$

$= 2x + 2y + 2z = 2(x + y + z)$

$\iiint_V div\vec{F}\, dV = \int_0^1\int_0^1\int_0^1 2(x + y + z)dzdydx$

$= 2\int_0^1\int_0^1 \left[xz + yz + \frac{z^2}{2}\right]_0^1 dydx$

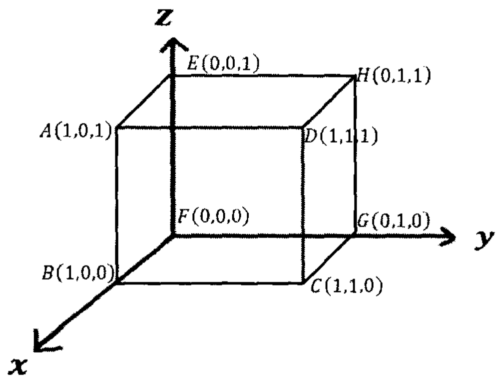
$= 2\int_0^1\int_0^1 \left(x + y + \frac{1}{2}\right) dydx$

$= 2\int_0^1 \left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_0^1 dx$

$= 2\int_0^1 \left(x + \frac{1}{2} + \frac{1}{2}\right) dx2$

$= \int_0^1 (x + 1)\, dx$

$= 2\left[\frac{x^2}{2} + x\right]_0^1 = 2\left[\frac{1}{2} + 1\right] = 3$



$\therefore \iint_S \vec{F}\,.\hat{n}\, dS = \iiint_V div\vec{F}\, dV = 3$

(f) Find the stationary values of the function $f(x,y) = x^3y^2(1 - x - y)$

Solution: Let $f(x,y) = x^3y^2(1 - x - y)$

For stationary points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\Rightarrow \frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \ldots (1)$

$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = 0 \ldots (2)$

Solving (1) and (2), we get (0,0) *and* $\left(\frac{1}{2}, \frac{1}{3}\right)$

For stationary values $[f]_{(0,0)} = 0$, $[f]_{\left(\frac{1}{2},\frac{1}{3}\right)} = \frac{1}{432}$

UNIT-I

Q2. (a) If $u = f(r)$ and $x = r\cos\theta, y = r\sin\theta$, then prove that

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

Solution: See example 3 (Unit-I)

(b) Evaluate $\int_0^a \frac{log(1+ax)}{1+x^2} dx$ and hence show that $\int_0^1 \frac{log(1+x)}{1+x^2} dx = \frac{\pi}{8} log2$

Solution: See example 28 (Unit-I)

Q3. (a) Find the shortest and longest distances from the point (1,2,-1) to the sphere $x^2 + y^2 + z^2 = 24$

Solution: See example 23 (Unit-I)

(b) If $u = x^2 - y^2, v = 2xy \text{ and } x = r\cos\theta, y = r\sin\theta$ then find the Jacobian $J = \frac{\partial(u,v)}{\partial(r,\theta)}$

Solution: Let $J = \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$ ...(1)

Now, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) = 4r^2$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$

Substituting the values in (1), we get

$J = \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 . r = 4r^3$

(c) If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution: See example 12 (Unit-I)

UNIT-II

Q4. (a) Solve the ordinary differential equation: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = logx\, sin(logx)$

Solution: This is a Euler–Cauchy Equation with variable coefficients.

Putting $x = e^t \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ (1) may be rewritten as $(D(D-1) + D + 1)y = tsint$

$\Rightarrow (D^2 + 1)y = tsint\ ,\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F. $= c_1\, cost + c_2 sint = c_1 \cos(logx) + c_2 \sin(logx)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} tsint =$ Imaginary part of $\frac{1}{D^2+1} te^{it}$

Now $\frac{1}{D^2+1} te^{it} = e^{it} \frac{1}{(D+i)^2+1} t = e^{it} \frac{1}{D^2+2iD} t$

$$= e^{it} \frac{1}{2iD\left(1+\frac{D}{2i}\right)} t = e^{it} \frac{1}{2iD\left(1-\frac{iD}{2}\right)} t$$

$$= \frac{1}{2i} e^{it} \frac{1}{D}\left(1-\frac{iD}{2}\right)^{-1} t$$

$$= \frac{1}{2i} e^{it} \frac{1}{D}\left(1+\frac{iD}{2}+\cdots \ldots.\right) t$$

$$= \frac{1}{2i} e^{it} \frac{1}{D}\left(t+\frac{i}{2}\right)$$

$$= \frac{1}{2i} e^{it} \int \left(t+\frac{i}{2}\right) d\, t$$

$$= -\frac{i}{2} e^{it} \left(\frac{t^2}{2}+\frac{i}{2}t\right)$$

$$= e^{it} \left(-\frac{i}{4}t^2+\frac{t}{4}\right)$$

$\therefore$ P.I. = Imaginary part of $\frac{1}{D^2+1} te^{it}$ =Imaginary part of $(cost + isint)\left(-\frac{i}{4}t^2+\frac{t}{4}\right)$

$$= -\frac{t^2}{4} cost + \frac{t}{4} sint = -\frac{1}{4}(logx)^2 cos(logx) + \frac{1}{4} logx\, sin(logx)$$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1\, cos(logx) + c_2 sin(logx) - \frac{1}{4}(logx)^2 cos(logx) + \frac{1}{4} logx\, sin(logx)$

(b) Prove that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, sinx$, where $J_n(x)$ represents the Bessel function of first kind.

Solution: See Article 16.1(3) (Unit-II)

Q5. (a) Let the electric equipotential lines (curves of constant potentials) between two concentric cylinders be given by $x^2 + y^2 = c$, where c is the constant. Find their orthogonal trajectories (known as curves of electric force).

Solution: Given family of circles is: $x^2 + y^2 = c$ ... ①

Differentiating both sides of ① with respect to x

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} = m\,,$$

where m is the slope of the family of circles $x^2 + y^2 = r^2$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = \frac{y}{x}$

Hence the differential equation of orthogonal trajectories is given by $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$

Integrating both sides, we have $\int \frac{dy}{y} = \int \frac{dx}{x}$

$\Rightarrow \log y = \log x + \log c$, c is an arbitrary constant

$\Rightarrow \log y = \log cx \Rightarrow y = cx$

$\therefore y = cx$ is the required orthogonal trajectory.

(b) Solve the ODE: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x\, logx$

Solution: Auxiliary equation is: $(m^2 - 2m + 1) = 0$

$\Rightarrow\ m = 1{,}1$

C.F. $= (c_1 + c_2\, x)e^x = c_1 e^x + c_2 x\, e^x = c_1 y_1 + c_2 y_2$

$\therefore\ y_1 = e^x$ and $y_2 = x\, e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x\, e^x \\ e^x & x\, e^x + e^x \end{vmatrix} = e^{2x}$$

$$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{xe^x e^x logx}{e^{2x}}\, dx = -\int xlogxdx = -\left(\frac{x^2}{2} logx - \frac{x^2}{4}\right)$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x logx}{e^{2x}}\, dx = \int logxdx = (\, xlogx - x)$$

$\therefore$P.I. $= u_1 y_1 + u_2 y_2$

$$= -\left(\frac{x^2}{2} logx - \frac{x^2}{4}\right) e^x + xe^x(\, xlogx - x) = \frac{1}{4} x^2 e^x (2logx - 3)$$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1 + c_2\, x)e^x + \frac{1}{4} x^2 e^x (2logx - 3)$

(c) Solve the ODE: $2ydx + x(2logx - y)dy = 0 \quad \cdots ①$

Solution: $M = 2y,\ N = 2x\log x - xy$

$$\frac{\partial M}{\partial y} = 2\,,\ \frac{\partial N}{\partial x} = 2(1 + \log x) - y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x)$ say

$\therefore$ IF $= e^{\int f(x)dx} = e^{\int \frac{-1}{x} dx} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$① may be rewritten after multiplying by IF as:

$\frac{2y}{x} dx + (2\log x - y)\, dy = 0 \ \ldots\ldots ②$

New $M = \frac{2y}{x}$, New $N = 2\log x - y$

$\frac{\partial M}{\partial y} = \frac{2}{x} = \quad \frac{\partial N}{\partial x} = \frac{2}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$② is an exact differential equation.

Solution of ②is given by:

$$\int_{y\ constant} \left(\frac{2y}{x}\right) dx + \int -ydy = C \Rightarrow 2y\log x - \frac{y^2}{2} = \mathrm{C}$$

UNIT-III

Q6. (a) Test the consistency and solve the system of equations

$3x + y + 2z = 3, 2x - 3y - z = -3, x + 2y + z = 4$

Solution: Let the system of equations be represented as $AX = B$

We can write the equations in the ascending order of coefficient of x

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -3 & -1 \\ 3 & 1 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$$

Forming the augmented matrix $C = [A:B]$

$$C = [A:B] = \begin{pmatrix} 1 & 2 & 1 & : 4 \\ 2 & -3 & -1 & :-3 \\ 3 & 1 & 2 & : 3 \end{pmatrix}$$

Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$C \sim \begin{pmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{pmatrix}$$

Applying $R_3 \to R_{3-}\frac{5}{7}R_2$

$$C \sim \begin{pmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & 0 & 8/7 & : & -8/7 \end{pmatrix} \ldots (1)$$

Matrix C is in echelon form

$\therefore$ Rank of matrix C is 3, i.e., $\rho(C) = 3$

By similar row transformations, Rank of matrix A is 3, i.e., $\rho(A) = 3$

$\rho(A) = \rho(C) = 3 = n$, where n is the number of variables in the system

$\therefore$ the given system of equations is having unique solution.

Using the augmented matrix given by (1)

$$x + 2y + z = 4 \ldots (2)$$

$$-7y - 3z = 11 \ldots (3)$$

$$\frac{8}{7} z = -\frac{8}{7} \ldots (4)$$

Solving (4), we get $z = -1$

Putting z in (3), we get

$$y = \frac{-11+3(-1)}{-7} = \frac{-14}{-7} = 2$$

Using values of y and z in (2), we get

$x = 4 - 2(2) - (-1) = 4 - 4 + 1 = 1$

$\therefore x = 1, y = 2, z = -1$

(b) Verify Cayley- Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \text{ and hence find } A^{-1}$$

Solution: The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley's Hamilton Theorem, $A^3 - 3A^2 - A + 9I = 0$

Verification

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & --2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$$

Now, $A^3 - 3A^2 - A + 9I$

$$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} - 3\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find Inverse of Matrix A

$A^3 - 3A^2 - A + 9I = 0$

On multiplying by A^{-1}, we get

$A^2 - 3A - I + 9A^{-1} = 0$

$A^{-1} = \frac{1}{9}(-A^2 + 3A + I)$

$$= \frac{1}{9}\left(\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

Q7. (a) Check whether the vectors $(2,1,1), (2,0,-1), (4,2,1)$ are Linearly dependent or independent?

Solution: Consider $C_1X_1 + C_2X_2 + C_3X_3 = 0$

$\Rightarrow C_1[2 \quad 1 \quad 1] + C_2[2 \quad 0 \quad -1] + C_3[4 \quad 2 \quad 1] = 0$

$\Rightarrow 2C_1 + 2C_2 + 4C_3 = 0$

$C_1 + 0C_2 + 2C_3 = 0$

$C_1 - C_2 + C_3 = 0$

$$\Rightarrow \begin{pmatrix} 2 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{1}{2}R_1, R_3 \to R_3 - \frac{1}{2}R_1$, we get

$$\begin{pmatrix} 2 & 2 & 4 \\ 0 & -1 & 0 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 2C_1 + 2C_2 + 4C_3 = 0 \ldots (1)$

$0C_1 - C_2 + 0C_3 = 0 \ldots (2)$

$0C_1 - 2C_2 - C_3 = 0 \quad \ldots (3)$

From(2) , we get $C_2 = 0$

Putting this value in (3) , we get $C_3 = 0$

Substituting the values of $C_2$and C_3in (1) , we get $C_1 = 0$

Hence the given vectors are linearly independent.

(b) Reduce the quadratic form $2xy + 2yz + 2zx$ into the canonical form and discuss the nature.

Solution: See example 31 (Unit-III)

UNIT-IV

Q8. (a) What is the directional derivative of $\emptyset = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $xlogz - y^2 = -4$ at $(-1,2,1)$.

Solution: See example 16 (Unit-IV)

(b) Apply Stoke's Theorem to evaluate $\int_C (ydx + zdy + xdz)$, where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$

Solution: See example 45 (Unit-IV)

Q9. (a) Find the curvature and torsion of the Helix $x = acost, y = asint, z = bt$

Solution: See example 6 (Unit-IV)

(b) Prove that $div(grad\ r^n) = \nabla^2(r^n) = n(n+1)\, r^{n-2}$, where $r = \sqrt{(x^2 + y^2 + z^2)}$

Solution: See example 24 (Unit-IV)