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Chapter 1

Introduction to Thermodynamics

1.1 Introduction to Thermodynamics

Thermodynamics is the science that deals with the rules according to which bodies exchange energy. In the earlier times, thermodynamics was explained in terms of the relationship between mechanical and heat energy. However, with time thermodynamics got more developed by taking into account different branches of physics and chemistry showing inter-relationships between heat and all other forms of energy such as magnetic, chemical, electrical etc. We all know that all these forms of energy are inter-convertible thereby following the law of conservation of energy. The entire structure of thermodynamics rests on two well-known laws: the first and the second law of thermodynamics. The first law of thermodynamics is based on the law of conservation of energy i.e., energy can neither be created nor be destroyed but can be converted from one form to the other. The second law of thermodynamics expounds the idea that it is impossible to convert a given amount of heat fully into work. It explains the conditions under which conversion of heat into work or vice a versa can take place and it gives also an idea about the direction of heat transfer. The second law is also known as the law of increasing entropy.

1.2 The Continuum Model

A continuum model assumes that the substance of the object fills the space it occupies. Modelling objects in this way ignores the fact that matter is made of atoms, and so is not continuous; however, on length scales much greater than that of inter-atomic distances, such models are highly accurate. These models can be used to derive differential equations that describe the behaviour of such objects using physical laws, such as mass conservation, momentum conservation, and energy conservation, and some information about the material is provided by constitutive relationships. Continuum mechanics deals with the physical properties of solids and fluids which are independent of any particular coordinate system in which they are observed.

As a consequence of continuum approach fluid properties such as density, viscosity, thermal conductivity, temperature etc can be considered as continuous function of space and times.

1.3 Thermodynamic System: State Variables

In the thermodynamics, internal state of the system is our main concern and it is usual to describe it in terms of macroscopic quantities i.e., large scale or bulk properties of the system which depends on the internal state. These macroscopic quantities are called thermodynamic coordinates or state variables and the system is known as thermodynamic system. In case of gaseous system, the thermodynamic coordinates used are pressure(P), Volume(V), Temperature (T) and Entropy(S). Ex:

(a)a stretched bar has thermodynamic coordinates as length of the bar, tension to which it is subjected and the temperature(b) thin films like oil films on water or soap bubbles behave as stretched membranes and the thermodynamic coordinates are area of the film, surface tension and temperature. In these examples, we have taken pressure to be constant (atmospheric pressure) and we consider very small changes in volume.

An equation connecting the thermodynamic coordinates of the system (one of which becomes a dependent variable in the process) is called the Equation of state of the system and expresses the individual behavior of the system.

1.4 Zeroth Law of Thermodynamics: Concept of Temperature

This law was formulated after the first and the second law of thermodynamics and explains the concept of temperature of a system. Temperature can be defined as degree of hotness or coldness of a body.

It is seen that whenever a hot body is placed near to cold body, a change in temperature is observed. In other words, the body at higher temperature always gets cooler and the body with the lower temperature gets hotter. Thus, we can say that an energy or a heat exchange takes place between the two. As a result, the temperature of the bodies become equal and no more heat exchange takes place and we then can say that the two bodies are in thermal equilibrium. The thermal energy of each body now remains constant. If the two systems are not in thermal equilibrium, they will be at different temperatures. Thus, the temperature is not an absolute term rather it is a relative term and can now be defined as a quantity indicating the direction of heat exchange.

This concept of thermal equilibrium further led to the zeroth law of thermodynamics which states that "Two bodies A and B, each in thermal equilibrium with a third body C, are in thermal equilibrium with each other". Similarly, if we take a number of systems which are in thermal equilibrium with each other, the common property of the system can be represented by a single numerical value i.e., temperature. Zeroth law of thermodynamics thus indicates that the necessary and sufficient condition for thermal equilibrium is equality of temperature.

According to kinetic theory of gases, temperature is a measure of the average kinetic energy of the translational motion of the molecules of an ideal gas and signifies that the temperature of body is closely related to the energy of motion of its molecules. The greater the average kinetic energy per molecule of a body, the higher is its temperature. It means to change the temperature of a body, the average kinetic energy per molecule of the body is to be change. The best mechanism for this is, by heating or by cooling. For heating a body, heat energy must be supplied to it and to cool the body the heat energy must be taken away from it.

1.5 Heat

Julius Robert Mayer (1814-1878), a German doctor serving as a physician on a ship, conceived on 1842 the idea of the equivalence of heat and work. He hypothesized that heat is a form of energy. However, the relationship between heat and work was established in 1850 experimentally by James Joule. 1818-1989 the British physicist conducted a long series of experiments that conclusively demonstrated the equivalence of mechanical energy and heat energy.

Heat is basically the energy transferred or energy transferred from a hot body to the cold body. For example: If we consider a pan containing hot water (system), then after sometime the hot water starts getting colder as its temperature starts decreasing and tends to approach the room temperature. This is all because of exchange of energy between water (the system) and the surroundings. We now define

heat in a more general way. Heat is the energy that flows from one body or the system to another solely as a result of the temperature difference between them.
Thus, heat is not an intrinsic property of a body, a body containing certain amount of heat is meaningless. Also, if at any point we say the flow of heat stops, the word heat becomes meaningless. The term therefore can be used only and only when there is transfer of energy between two or more systems.

1.6 Heat and Work

The conversion of mechanical energy into heat and the reverse process of obtaining mechanical work at the expense of heat are the greatest interest in engineering. For example, in a thermal power station, thermal energy in the form of steam at high temperature and pressure drives the turbine and is converted into the mechanical energy. The turbine in turn drives the rotor of a generator which produces electricity.
Thermodynamics is the science that deals with work and heat, and those properties of substances related to heat and work. Many of the properties of the substance and many phenomena can be studied either with a microscopic or a macroscopic point of view. For instance, let us consider the familiar example of monoatomic gas confined in a small container. Quantities characteristics of the atoms such as an energy of an atom, its velocity, its mass, etc. can be used to describe the properties of the gas. These quantities can be computed from the theoretical analysis but cannot be measured. All such quantities are said to be microscopic. We may also describe the properties of the gas using quantities whose value can be measured. Such quantities are called macroscopic quantities. Temperature, pressure and volume are examples of macroscopic quantities. These are the quantities the gas as a whole has, but have no meaning when applied to individual atoms. Thus, we cannot speak of the pressure of temperature of the molecule. It is possible to develop the principles of thermodynamics from a microscopic point of view. Historically, the central concepts of thermodynamics were developed from a macroscopic view point without reference to microscopic models and details of the structure of matter. We study here the principles of thermodynamics from the macroscopic point of view.

1.7 Thermodynamic Concepts

There are certain terms in thermodynamics which we need to understand:

1.7.1 System

A well-defined system is a system in which the principles of thermodynamics are usually stated or are well applicable. A system is defined in thermodynamics as a quantity of matter of fixed mass and identity. Let us take an example of a thermodynamic system of a quantity of gas enclosed in a cylinder fitted with a movable piston The system and its environment are distinctly defined by drawing a boundary between them. The system can interact with its environment, mainly in two ways - by transfer of heat or by doing work.

1.7.2 Thermodynamic Equilibrium

In mechanics, equilibrium means state of rest. In thermodynamics, the concept is somewhat broader. A system is said to be in thermodynamic equilibrium if none of the thermodynamic variables determining its state changes with time. Thermodynamic equilibrium is easily understood in the case of a monoatomic gas confined in a cylinder. If the temperature of a gas in the cylinder is same at all points in the cylinder and the temperature of the walls of the cylinder is also the same, then the gas is said to be in thermal equilibrium with the cylinder. The heat of the gas does not flow from one part of the cylinder to the other. Further, when neither the pressure nor the chemical composition of the gas changes, it is said to be in thermodynamic equilibrium. Thus, a system will be in a state of thermodynamic equilibrium if it satisfies all the conditions mechanical, thermal and chemical equilibrium. This can further be explained as:

1. **Mechanical equilibrium:** For a system to be in mechanical equilibrium, the force exerted by the system must remain constant and uniform and must be balanced by the external force on the system. There should be no unbalanced forces acting on a part or the whole of the system.
2. **Thermal equilibrium:** For a system to be in thermal equilibrium there should be no difference in the temperatures between the parts of the system or between the system and the surroundings.
3. **Chemical equilibrium:** For a system to be in chemical equilibrium there should be no chemical reaction within the system or change in the internal structure because of it and no movement of any chemical constituent from one part of the system to the other on account of solution or diffusion etc.

Thus, we can clearly now define the state of thermodynamic equilibrium as the state when all the above discussed types of thermal equilibrium are satisfied to the fullest. When a system undergoes the thermodynamical equilibrium keeping surroundings unchanged, no motion will take place and no work will be done. However, if any one of the above-described equilibrium conditions are violated or unsatisfied the system will said to be in non-equilibrium state. E.g., if there is an unbalanced force either in the interior of the system or between the system and the surroundings, the mechanical equilibrium will be disturbed and as a result turbulence, waves and eddies would be set in the system or there can be an accelerated motion of the system itself. These things can then lead to non-uniformity in temperature within the system or a temperature difference between the system and the surroundings. The system may then pass-through non-equilibrium states, which then cannot be described in terms of the system as a whole.

A cartesian coordinate system is used to plot sets of values of P, V and T to indicate equilibrium states of a system. Different points on the graph correspond to different equilibrium states of the gas. An isolated system always reaches a state of thermodynamic equilibrium in course of time but can never depart from it spontaneously.

1.7.3 Process

Any thermodynamical state of a system can always be defined only with the help of thermodynamical coordinates or state variables of the system. By changing the thermodynamical coordinates one can change the states of system. This change of state by changing the thermodynamical coordinates is called a process. Let us consider two states of a system i.e., state A and state B, the change of state from A to B or vice a versa is a process.

Some of the typical processes are:

1.**Isothermal process:** If the change in pressure and volume of gaseous system or in other words if a system is perfectly conducting in such a way that its temperature remains the same throughout, it is called isothermal process. A graph between pressure and volume of gas at constant temperature is called an isothermal.

Let us suppose, a cylinder fitted with a piston contains gas which is maintained at room temperature and under atmospheric pressure. Work is done on the gas when the piston is pushed down such that the internal energy increases and so the temperature. If the temperature has to be maintained constant, then the extra heat must be conducted away to the surroundings. Similarly, if the compressed gas is allowed to expand and push the piston up a little i.e., some external work is done, the internal energy decreases and temperature decreases. Again, if the temperature has to be maintained constant, heat must be conducted to it from the surroundings.

Thus, in an isothermal process heat must be quickly conducted from the gas to the surroundings or vice a versa and this can be possible only when the containing vessel is a perfect conductor of heat.

If we want to ensure that the temperature of the system is constant, the withdrawal or supply of heat must keep pace with the rise and fall of temperature of the system. This means that the change in volume or pressure must be small as well as slow.

Thus, an ideal isothermal process must be infinitely slow and must consist of infinitely small steps in perfect thermal communication with the surroundings.

If the system is a perfect gas, the standard gas equation $PV = rT$ or $PV= RT$ will be applicable during an isothermal process where r and R are the gas constants per unit mass or per gram molecule of the gas respectively.

If we put it in the form $PV = rT = k$ or $PV = RT = k$, it gives the equation of the isothermal of a perfect gas at a temperature T.

2.**Adiabatic process:** During such type of process, the working substance is perfectly insulated from the surroundings. When work is done on the working substance, there is a rise in temperature because the external work done on the working substance increases its internal energy. When work is done by the working substance, it is done at the cost of its internal energy. As the system is perfectly insulated from the surroundings, there is fall in temperature.

Thus, during an adiabatic process, the working substance is perfectly insulated from the surroundings. All along the process, there is change in temperature. A curve between pressure and volume during an adiabatic process is called an adiabatic curve. e.g., Sudden bursting of a cycle tube is an adiabatic process. Also, in the example quoted in isothermal process above, if the cylinder and piston be of a perfectly insulating material and the gas in the cylinder be compressed it will rise in temperature since the heat developed cannot possibly escape out to the surroundings. If the gas is allowed to expand or to do external work, it will do at the expense of its own internal energy, since no heat can possibly enter the cylinder from outside and thus, the temperature of the gas will fall.

According to the first law of thermodynamics,

$$dQ = dU + dW$$

here since $dQ = 0$ so $dW = - dU$

This indicates that a decrease in internal energy takes place when for the system which is equal to the external work done by the system.

An adiabatic process is always accompanied by a change in the temperature of the system. Since no material is a perfect insulator of heat, some heat may escape from or enter into the system unless the process is quick and sudden. An adiabatic process is thus quick and sudden and takes place in thermal isolation from the surroundings.

3.**Isochoric process**: If the volume of the system is kept constant but pressure and temperature changing then the process is called an isochoric process. This is possible when the working substance is taken in a non-expanding chamber, the heat supplied will increase the pressure and temperature. The work done is zero here because there is no change in volume. The whole of the heat supplied increases the internal energy. The P-V or the indicator diagram of such a process will be a vertical line parallel to the pressure axis. As there is no change in the volume of the system, the work done is

$$\delta W = \int PdV = 0$$

Hence, if dQ be the heat supplied to the system, then

$$dQ = dU + dW$$

which gives $dQ = dU$

i.e., the whole of the heat supplied goes to increase the internal energy of the system

4. **Isobaric process**: If the working substance is taken in an expanding chamber kept at a constant pressure the process is called isobaric process. Here, both the temperature and volume change. The P-V or the indicator diagram of such a process will be a horizontal line parallel to the volume axis.

1.7.4 Quasistatic Process

We can remember from the previous discussion in section 1.6.2 about how a thermal equilibrium is achieved when the temperature of every part of the system is same with that to the surroundings. Therefore, there is a need in any process to describe the states of the system in terms of the same coordinates as those of the system itself. The external forces must not be varied to a very greater extent such that the unbalanced forces remain infinitesimal.

A quasi-static process is thus defined as the process in which the deviation from the thermodynamic equilibrium is infinitesimal and all the states through which the system passes during a quasi-static process can be considered as equilibrium states and may therefore be expressed in terms of thermodynamic coordinates of the system as a whole.

A quasi-static process is actually an ideal concept. Practically, only under rigorous conditions it can be satisfied. If somehow, we can slow down the process by proceeding step by step gradually, nearly quasi-static state can be achieved. e.g. if we decrease the pressure on a gas enclosed in a cylinder in small gradual steps by moving the piston up extremely slowly, the process of expansion may well be regarded as quasi-static. On the other hand, if we suddenly raise the piston up, the gas expands all at once. The process is therefore, far from quasi-static and at no stage in the process can the gas be regarded to be in an equilibrium state.

Work done in a quasistatic process – P-V or Indicator Diagrams

Work done in a quasi-static process such as for example changing the volume of a chemical or a gaseous system can be easily obtained with the help of what are called indicator diagrams or P-V diagrams. The changes in volume and pressure are due to the movement of piston in cylinder.

Let us discuss the cyclic and non-cyclic processes.

(i)**Non-Cyclic process**

Consider pressure along x axis, volume on y axis and let us suppose that the gas in the cylinder undergoes a change in pressure and volume at constant temperature. Let the pressure and volume changes gradually or quasi-statically along the curve AB in the P-V diagram or indicator diagram for the gas and represents the whole operation of expansion of the gas from its initial state at A to its final state at B.

Let the piston moves through an infinitesimal distance δx, an infinitesimal amount of work is done by the gas, given by

$\delta W = PA\delta x = P\delta V$

Where A is the cross-sectional area of the piston and δV the infinitesimal increase in volume.

This work is represented by the shaded area in Fig.1.0.

Thus, work done by the gas during the whole expansion from volume V_1 at A to volume V_2 at B is
$W = \int_A^B \delta W = \int_{V_1}^{V_2} P\delta V$ = the entire area GABH under the curve

If the gas were compressed along the same path from a volume V_2 at B to volume V_1 at A work done on the gas would be the same W numerically, but opposite in sign i.e., equal to -W. Conventionally, work done by the gas is positive and work done on it or absorbed by it as negative.

The area under the curve depends on different types of shapes with the same endpoints A and B. It is seen that work done by the gas depends not only on the initial and final states but also on the intermediate states it passes through i.e., on the path along which the change occurs. So work is a path function and not a point function like pressure or temperature.

Thus, $W = \int_A^B \delta W = \int_{V_1}^{V_2} P\delta V$ cannot be integrated to give $PV_2 - PV_1 = W_2 - W_1$ as δWor $P\delta V$ is an inexact differential.

(ii) **Cyclic process**

Let us suppose the gaseous system is subjected to a series of changes of pressure and volume such that its initial pressure and volume (represented by point A) change along the curve ACB and finally attain the values represented by point B. Let it now be subjected to a different series of pressures so that it comes back to its original state A by a different path BDA. The system is then said to perform a thermodynamic cycle of operations.

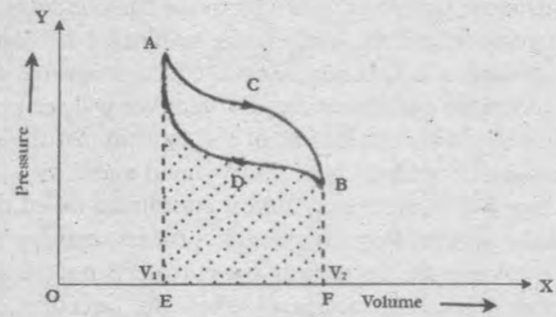

Fig. 1.0 Cyclic process

Work done by the system is given by the area ACBFE under curve ACB and shown shaded in Fig.1.0 The work done on the system is represented by area BDAEFB under the curve BDA and shown cross-shaded. So, the net external work(W) by the system is represented by the area of loop ACBDA and is equal to the difference of the areas ACBFEA and BDAEFB under the curves ACB and BDA respectively.

The area of loop is conventionally regarded as positive if the upperpart of the cycle is traversed towards the right i.e., expansion and negative, if it traverses towards left i.e., compression.

1.7.5 Reversible and Irreversible Processes

Reversible process

In thermodynamics, reversible process is the process in which for an infinitesimal small change in the external conditions can lead to changes taking place in the direct process but exactly repeated in the reverse order and in the opposite sense. The process should take place very slowly. There should be no loss of heat due to friction or radiation.

Let us consider a gas in a cylinder maintained at a certain temperature and pressure. This cylinder is attached with a frictionless piston. If the pressure is decreased, expansion of gas takes place very

slowly and constant temperature is maintained. The energy in this process is drawn continuously from the source(surroundings). If on the other hand pressure on the piston is increased, contraction of gas takes place slowly and temperature is maintained. The energy is liberated during compression and is given to sink(surroundings).
Hence, for a reversible process, there should be no loss of heat due to friction, radiation or conduction. If the changes take place very rapidly, the process will no more be reversible. Following are the conditions for a process to be reversible:
1.The pressure and the temperature of the working substance must not differ from those of the surroundings at any stage of the cycle of operation.
2.All the processes taking place in the cycle of operation must be infinitely slow.
3.There should be no heat losses.
Reversible process is only an ideal case. In reality there are always some losses of heat and the temperature and the pressure of the working substance changes from those to the surroundings.

Irreversible process
We know by this time that change of state from A to state B or vice a versa is a process. But the direction of the process is important and it always depends on a thermodynamical coordinate, called entropy. In practicality however all processes are actually not possible in the universe. e.g., Let us take two blocks 1 and 2 with different temperatures T_1 and T_2 such that $T_1 > T_2$ and they are in contact with each other but we insulate the system as whole from the surroundings. We will see that heat will be conducted between the two such that the temperature of block 1 falls and that of the temperature of block 2 rises and thermodynamical equilibrium is reached. Now reverse the direction i.e., the block 2 should transfer heat to 1 and initial conditions are restored. We will see that the reverse process will not be possible. In other words, the determination of the direction of the process cannot take place by knowing the thermodynamical coordinates in the two end states. So, in order to determine the direction, one has to introduce a new thermodynamical coordinate called the entropy of the system. Entropy is also the state of the system. For any possible process, entropy should increase or remain constant. The process which governs the decrease in the entropy is not possible.
Thus, a process is irreversible if entropy decreases when the direction of process is reversed. A process is said to be irreversible if it cannot be traced exactly back in the opposite direction. During an irreversible process, heat energy is always used to overcome friction. e.g.; all chemical reactions, all natural processes.

There are few more examples to understand the irreversibility of real thermal processes:
(i)**Free expansion of a gas:** Consider the example of a gas separated from vacuum by a membrane. If the membrane breaks, the gas fills the entire vessel. The system cannot be restored to the initial state unless the gas is compressed and the resulting heat is taken away. This, however, causes a change in the surroundings. Therefore, at the end of the process, the surroundings are not restored to their initial state. For this reason, the free expansion of a gas is an irreversible process.
(ii)**Diffusion:** Let us consider the case of two different gases, oxygen and hydrogen, separated by a membrane. If the membrane breaks, the diffusion of gases takes place spontaneously and a homogenous mixture of oxygen and hydrogen fill the entire volume. But the process will never reverse itself. The mixture of gases will never divide by itself in the initial components.
In practice, a mixture of gases can be divided into its initial components. But first, it involves application of energy. Secondly, the system will not pass again through the same intermediate states it

passes through the diffusion process. The system thus cannot be returned to its initial state without substantially altering the properties of the surroundings.

(iii) **Heat exchange:** Experience shows that the heat exchange like diffusion, is a one-way process. In heat exchange, energy is always transmitted from a body at the higher temperature to another body at lower temperature. Consequently, the heat exchange is always accompanied by an equalization of temperatures. The reverse process of transferring energy in the form of heat from cold bodies to hot bodies never occurs by itself. For instance, a hot cup of coffee cools through heat transfer to the surroundings. However, reverse will not happen.

1.7.6 Cycle

If a system in a given initial equilibrium state passes through a number of different changes of state and ultimately returns to its original equilibrium state, then the system is said to have gone through a thermodynamic cycle. The steps that constitute the cycle may be reversible or irreversible. If the system consists of a single homogenous substance and if all the steps are reversible, the cycle can be represented by a closed curve in a PV diagram as in Fig 1.1

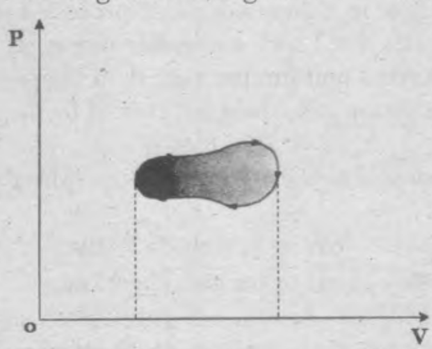

Fig.1.1 The PV diagram for an arbitrary cyclic process. The net work done in the process equals the area enclosed by the curve.

1.7.7 Heat Reservoir

A heat reservoir is a body from or to which heat can be transferred and it remains at constant temperature. For example: the atmosphere is a good heat reservoir. A heat reservoir from which heat is extracted by a body is called a high temperature heat reservoir (source) and heat reservoir to which heat is ejected from a body is called a low temperature reservoir(sink).

1.8 Work

The mechanical work performance by a force is most familiar form of work. In mechanics, work is said to have been performed when a force, acting on body displaces it through a distance x, the displacement being in the direction of force. That is:

$$W = F \times x \qquad (1)$$

The mechanical energy of a body changes when mechanical work is done on the body, Therefore, it serves as a measure of transfer of mechanical energy from one body to another. In general, we say work is a form of energy. Work is not stored in the body. Work is a form of transferring energy and is a measure of transferred energy.

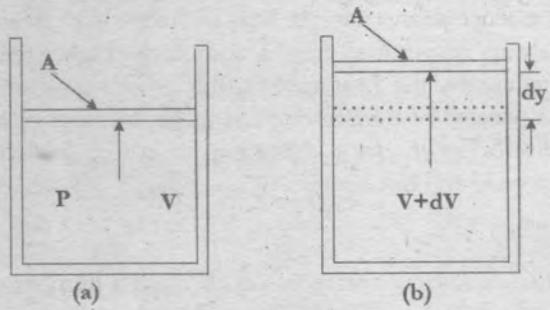

Fig. 1.2(a) Gas contained in a cylinder at a pressure P does work on the piston
(b) As the piston moves the system expands from a volume V to a volume V + dV

Now let us examine the work done in a thermodynamic process. Consider a thermodynamic system such as gas contained in a cylinder fitted with a movable piston, as in Fig 1.2(a). In equilibrium the gas occupies a volume V and exerts a uniform pressure P on the piston and the walls of the cylinder. If the cross-sectional area of the piston is A, the force exerted by the gas on the piston is

$$F = P \times A$$

When the gas expands quasistatically as the piston goes up a distance dy as in Fig.1.2(b), the work done by the gas on the piston is

$$\delta W = F \times dy = PAdy$$

But $Ady = dV$, the increase in the volume of the gas. Therefore,

$$\delta W = P\,dV \tag{2}$$

The above equation gives the work exclusively in terms of the thermodynamic variables of the system. The nature of the outside force and other characteristics of the surroundings are not reflected in this relation. The work $\delta W = P\,dV$ is often called thermodynamic work.

The convention is that work done by the system is regarded as positive and the work done on the system is negative. Thus, the work done by the gas expanding against a piston is positive and the work done by a piston compressing a gas is negative.

The total work done by the gas as its volume changes from V_1 to V_2 can be found by integrating equation (2). Thus,

$${}_1^2W = \int_1^2 \delta W = \int_1^2 P dV \tag{3}$$

The above integration can be performed only if we know the relationship between P and V during the process. In general, the pressure is not constant but depends on the volume and temperature. If the pressure and volume are known at each step of the process, the work done can be obtained graphically from the PV diagram as in Fig.1.3.

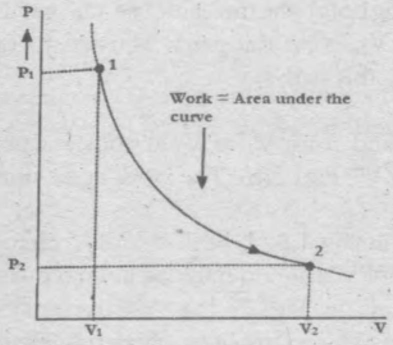

Fig. 1.3. A gas expands from an initial state 1 to final state 2. The work done by the gas equals the area under the PV curve

The work done during the process is given by the area V_1-1-2-V_2-V_1 under the curve 1-2 in Fig.1.3. Therefore, the total work done during the expansion of the gas from the initial state to the final state is the area under the curve in a PV diagram.

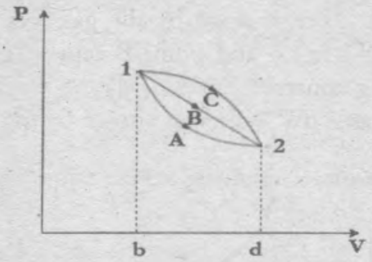

Fig 1.4. Three different paths connecting the same initial and final states. The work done by the gas is maximum for the path marked C which encloses the largest area.

As illustrated in Fig1.4. it is possible to go from the initial state 1 to the final state 2 along any path like A, B or C. The area under each curve symbolizes the work for each process. The amount of work done is a function of the end states of the process and on the path that is followed. To illustrate this important point we consider three different paths (Fig 1.5) connecting initial and final states.

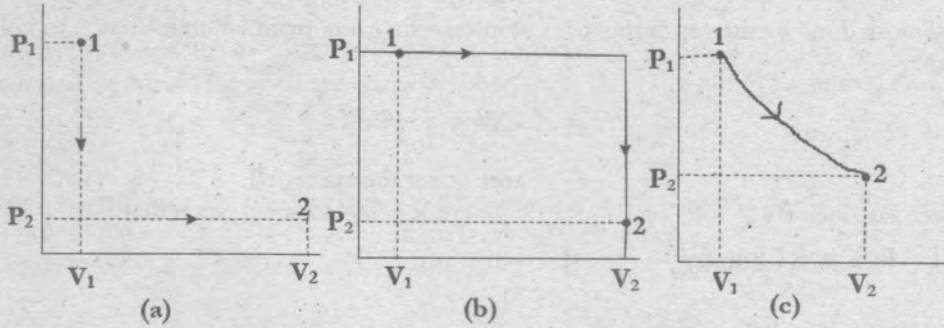

Fig. 1.5 The work done by an ideal gas as it is taken from an initial state 1 to final state 2 depends on the path between these states 1 and 2.

In the process represented in Fig1.5(a) the pressure of the gas is first reduced from P_1 to P_2 by cooling it at a constant volume V_1. Next the gas is allowed to expand from V_1 to V_2 at constant pressure P_2. The work done along this path is

$W_A = P_2(V_2 - V_1)$

The gas is first permitted to expand from V_1 to V_2 at constant pressure P_1 and then the pressure is reduced to P_2 at constant volume V_2, Fig1.5(b). The work done along this path is

$W_B = P_1(V_2 - V_1)$

Finally in the process described in Fig 1.5(c), both P and V change continuously. To complete the work in this case, the shape of the PV diagram must be known unambiguously.

It is clear from the PV diagrams in Fig1.5, that W_A is smaller than W_B and W_C has a value intermediate between W_A and W_B. This example amply demonstrates that the work done by a system depends on how the system goes from the initial to the final state. For this reason, thermodynamic work is not a point function but is a path function. dW cannot be therefore treated as exact differential in the mathematical sense.

1.8.1 Work done in Various Processes

(i)Isothermal Process

When a gas expands isothermally, work is done by the gas. Let P_1, V_1 be the initial pressure and volume represented by point A in Fig.1.6 and point B represents P_2, V_2 as the final pressure and volume, the temperature remaining constant. Suppose dV be the small increase in the volume of gas at pressure P. Work done by the gas is $dW = PdV$ as shown by the shaded area in Fig. 1.6

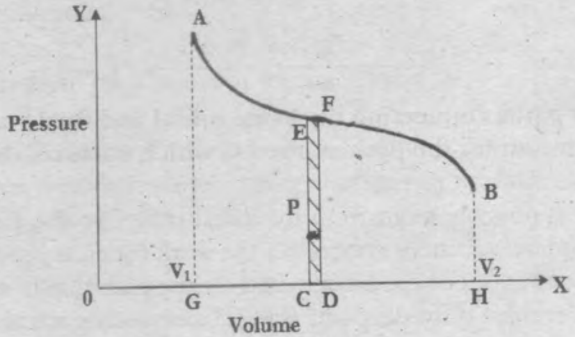

Fig. 1.6

Thus, total work done by the gas during the complete expansion from volume V_1 at A to volume V_2 at B such that

$$W = \int dW = \int_{V_1}^{V_2} PdV$$

= area under the curve AB.

For a perfect gas since $PV = RT$ or $P = RT/V$ where R is the constant, such that

$$W = RT\int_{V_1}^{V_2} PdV = RT\log_e(\frac{V_2}{V_1})$$

Or

$$W = 2.3026\ RT\log_{10}(\frac{V_2}{V_1}) \qquad (4)$$

As the temperature is constant, thus

$P_1V_1 = P_2V_2$ or $\frac{V_2}{V_1} = \frac{P_1}{P_2}$

From (4), we have

$$W = 2.3026\ RT\log_{10}(\frac{P_1}{P_2}) \tag{5}$$

As the temperature remains constant, the internal energy of the gas undergoes no change. The whole of the heat applied to the gas is thus converted into external work done by the gas.

(ii)**Adiabatic Process**

Let us consider an adiabatic expansion of a 1 gm mole of a perfect gas having P_1, V_1 as the initial pressure and volume and P_2, V_2 as the final pressure and volume.

Work done by the gas is: $W = \int dW = \int_{V_1}^{V_2} PdV$

During an adiabatic process,

$PV^{\Upsilon} = k$ a constant

Or $P = \frac{k}{V^{\Upsilon}}$

On substituting the value of P above we get,

$$W = \int_{V_1}^{V_2} \frac{k}{V^{\Upsilon}} dV = k\int_{V_1}^{V_2} V^{-\Upsilon} dV = k\,[\frac{V^{1-\Upsilon}}{1-\Upsilon}]_{V_1}^{V_2}$$
$$= \frac{k}{1-\Upsilon}\,[V_2^{1-\Upsilon} - V_1^{1-\Upsilon}]$$

Or

$$W = \frac{k}{\Upsilon-1}[\frac{1}{V_1^{\Upsilon-1}} - \frac{1}{V_2^{\Upsilon-1}}] \tag{6}$$

If V_2 is smaller than V_1, work will be done on the gas and the above expression (6) will then bear a negative sign.

Also, $P_1V_1{}^{\Upsilon} = k = P_2V_2{}^{\Upsilon}$, hence from (6) above

$$W = \frac{1}{\Upsilon-1}[\frac{P_1V_1{}^{\Upsilon}}{V_1^{\Upsilon-1}} - \frac{P_2V_2{}^{\Upsilon}}{V_2^{\Upsilon-1}}]$$

Or
$$W = \frac{1}{\Upsilon-1}[P_1V_1 - P_2V_2] \tag{7}$$

Again, since $P_1V_1{}^{\Upsilon} = k = P_2V_2{}^{\Upsilon}$, we have $V_1 = [\frac{k}{P_1}]^{1/\Upsilon}$ and $V_2 = [\frac{k}{P_2}]^{1/\Upsilon}$

Thus, $W = \frac{1}{\Upsilon-1}[P_1[\frac{k}{P_1}]^{\frac{1}{\Upsilon}} - P_2[\frac{k}{P_2}]^{1/\Upsilon}]$

Or
$$W = \frac{k^{1/\Upsilon}}{\Upsilon-1}[P_1{}^{(\Upsilon-1)/\Upsilon} - P_2{}^{(\Upsilon-1)/\Upsilon}] \tag{8}$$

If temperature of the gas changes from T_1 to T_2 we have $P_1V_1 = RT_1$ and $P_2V_2 = RT_2$ where R is the gas constant of gm-molecule of the gas. On substituting the values of P_1V_1 and P_2V_2 in (7) above we get,

$$W = (\frac{R}{\Upsilon-1})(T_1 - T_2) \tag{9}$$

Since in an adiabatic process, no heat is allowed to enter or leave the system, the external work W is done by the gas at the expense of its own internal energy, such that

Work done by the gas during an adiabatic process = decrease in internal energy of the gas

Thus, the work done by the gas during an adiabatic expansion from volume V_1 at pressure P_1 to volume V_2 at pressure P_2 is given by the area under the P-V curve for the gas i.e.

$$W = \int_{V_1}^{V_2} PdV = \frac{1}{\Upsilon-1}[P_1V_1 - P_2V_2] \doteq \text{area under the curve.}$$

1.9 Heat in Thermodynamics

A process of transfer of energy without work being done is called heat exchange.

Definition: Heat is a form of energy that is moved from a system at higher temperature to another system at a lower temperature.
So, heat is energy in transit. Heat transferred to a system is considered positive and from a system is negative.
Like work, heat is a path function. The heat transferred to or from a system depends on the process. We illustrate this point with the help of following examples:

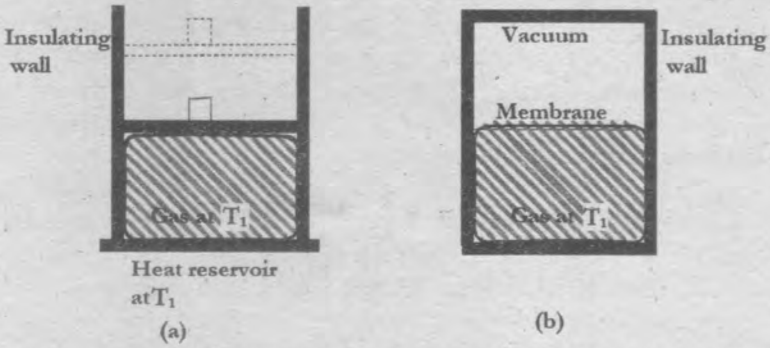

Fig.1.7(a) A gas at temperature T_1 expands slowly by absorbing heat from the reservoir at the same temperature (b)A gas expands rapidly into the evacuated region due to the rupture of the membrane.

Let us consider the case of a gas of volume V_1 which is in thermal contact with a heat reservoir Fig 1.7 (a) If the pressure of the gas is infinitesimally greater than atmospheric pressure the gas will expand and pushes up the piston. Ultimately the gas expands to a final volume V_2. The heat required to maintain the gas at a constant temperature T, during the process of expansion, is supplied from the reservoir to the gas. Work is done by the gas in this case.
In the second case, illustrated in Fig.1.7(b), a gas of the same volume V_1 is thermally insulated. It can neither receive nor give away heat. When the membrane is broken, the gas freely and rapidly expands into the vacuum until it occupies a volume V_2. No work is done in this case. Further, no heat is transferred through the walls. The initial and final states of both processes are identical but the path followed are different. We conclude that heat, like work, depends on how the system goes from its initial state to final state. Therefore, it is a path function. Like δW, δQ is an inexact differential.

1.10 Comparison of Heat and Work

In thermodynamics, work includes all forms of energy except heat. When a battery is charged, electrical work is performed and stored in chemical form. When a piece of steel is magnetized, magnetic work is performed.
Any kind of energy in the course of transformations may pass through many forms but invariably ends in the form of heat energy. In the process of mechanical motion, the kinetic energy of a body decreases owing to the action of frictional forces and gets transformed into heat. Similarly, the energy of an electric current and of chemical reactions transform into heat.
Both work and heat are transient phenomenon. Systems never contain work or heat. Heat and work exist only in a process of heat energy transfer and their numerical value depends on the kind of the

process. Either work or heat or both are transferred when a system undergoes a change of state. In real conditions, both ways of transferring energy to a system accompany each other. For instance, when a metal rod is heated, heat exchange takes place and at the same time thermal expansion of rod occurs. The latter implies that the work of expansion is done.
Heat and work are not equivalent forms of energy transfer from a qualitative viewpoint. The energy of ordered motion is transferred in the form of work. The energy of chaotic motion of particles constituting the body is increased when energy is transferred to a body in the form of heat.
Work can be fully converted into heat but heat cannot be entirely converted into work. As both heat and work depend on the path, neither quantity is individually conserved during a thermodynamic process.

1.11 Internal Energy

The total energy E_T of a system consists of (i)the kinetic energy of its macroscopic motion as a whole(ii)the potential energy due to the presence of external fields and (iii)internal energy, U. Thus,

$$E_T = K.E + P.E + U \tag{10}$$

The internal energy U of the system depends on the nature of the motion and interaction of the particles in the system. It consists of (a)the kinetic energy of random thermal motion of molecules (b)the potential energy of molecules due to the intermolecular interactions, (c)the kinetic and potential energies of atoms and electrons and (d) the nuclear energy. In thermodynamics, we do not concern ourselves with the form of internal energy.
Experiments have shown that the internal energy is determined by the thermodynamic state of the system and does not depend on how the system acquires the given state. Consequently, the internal energy is not related to the process of a change in the state of the temperature. All processes leading to particular pressure and temperature leave the gas with the same internal energy.
The internal energy is not of practical interest. The change in internal energy ΔU, when a system changes from one state to another is of actual interest. It is generally assumed that the internal energy of a system is zero at 0 K.

1.12 Law of Conservation of Energy

In a closed mechanical system, the sum of kinetic and potential energy is constant. The total energy can neither be created nor destroyed. It means that the energy is conserved. This is the law of conservation of mechanical energy. This law is applicable only in situation when there is no transformation of mechanical energy into heat energy. In real systems, the mechanical motion in general is accompanied with heating. When the engine of a running car is switched off, it gradually slows down and ultimately stops. Apparently, its kinetic energy has disappeared. In fact, its kinetic energy has transformed into heat energy by the frictional forces. Both, the tyre of the car and the ground are heated up. As a result, the random motion of particles constituting the interacting tyres and road acquires more velocity. To sum up, the mechanical energy (K. E of car) is transformed into the internal energy of the interacting bodies. If we take into account the internal energy, the law of conservation of energy can be extended to include thermodynamic systems also. The first law of thermodynamics generalizes the law of consideration of mechanical energy.

1.13 First Law of Thermodynamics

Thermodynamics involves the three general forms of energy i.e., heat, work and internal energy.

- Heat is energy transferred by virtue of temperature difference.

- Thermodynamic work is energy transferred between system and surroundings
- Internal energy is energy stored within a system

If the state of a system changes as a result of supplying a quantity of heat Q to it and as a consequence the system does the work W, then the law of conservation of energy states that the quantity of heat supplied to the system will be equal to the sum of the work performed by the system and the change in the internal energy of the system. That is,
Net heat transfer = Work + change in internal energy
Or mathematically, $Q = W + \Delta U$ (11)
Above equation (11) is the first law of thermodynamics and is a consequence of the conservation of energy. While the quantities Q and W are path dependent, the internal energy, does not depend on the path of the process.
Suppose a thermodynamic system undergoes a change from an initial state 1 to a final state 2 in which Q units of heat are absorbed (or removed) and W is the work done by (or on) the system. Expressing both Q and W in the same units (either thermal or mechanical) the difference (Q – W) can be calculated. If now we carry out this calculation for different paths between the same states 1 and 2, the quantity (Q – W) will be the same for all paths connecting the states 1 and 2. It follows that the internal energy change of a system is independent of the path. If U_1 is the internal energy in state 1 and U_2 is the internal energy in state 2, then the equation (11) can be written as
$U_2 - U_1 = \Delta U = Q - W$ (12)
When a thermodynamic process proceeds smoothly, it can be treated as a continuous sequence of small changes. Mathematically, we write equation (11) in the differential form as:
$dQ = dW + dU$ (13)
It is important to note that expressing heat and work as dQ and dW does not imply the existence of properties Q and W that measure the heat and work content of a system. dQ and dW denote small amounts of heat and work but they are not true differentials:
There is a series limitation on the first law of thermodynamics. The law tells us whether energy considerations permit a particular process to take a system from one equilibrium state to another. It does not tell us whether this process will actually occur or not. A certain process might be entirely consistent with the principle of energy conservation but still it will not take place.

1.14 Applications of the First Law
Let us study some special processes and the application of the first law to them
(i)Isolated system: An isolated system does not interact with its surroundings. Therefore, there is no heat flow and the work done is zero. That is $Q = 0$ and $W = 0$. It follows from equation (12) that
$\Delta U = 0$
$U_2 - U_1 = 0$
or $U_2 = U_1$ isolated system (14)
equation (14) means that the internal energy of an isolated system remains constant.
(ii)Cyclic process: In a cyclic process, the initial and final states of the system are the same. Thus,
$U_2 = U_1$
$\Delta U = 0$
It follows from equation (11) that

$$Q - W = 0$$

$Q = W$ cyclic process (15)

Above equation (15) means that the net work done by the system over a cycle equals the net heat absorbed over the cycle.

(iii)Adiabatic process: A process in which no heat is absorbed or ejected by the system is called an adiabatic process. Thus, $\Delta Q = 0$

$\Delta U = -W$ Adiabatic process (16)

Thus, the change in the internal energy of the system is equal in magnitude to the work done by the system. Heat flow into the system from the surroundings may be prevented in two ways- (i)by surrounding the system with a thick layer of insulating material or (ii)by performing the process quickly. The flow of heat requires finite time. So, any process performed quickly enough will be adiabatic.

(iv)Isochoric process: When a substance undergoes a process in which the volume remains unchanged, the process is called isochoric. If the volume of a system remains constant, it can do no work. Thus, W = 0 and the first law gives

$\Delta U = Q$ Isochoric process (17)

In this case, the heat that entered the system is stored as internal energy.

(v)Isothermal process: It is process happening at constant temperature and the quantities Q, W and ΔU are all non-zero. The first law does not have any special form for an isothermal process.

(vi)Isobaric process: A process taking place at constant pressure is called an isobaric process. As in the case of an isothermal process, Q, W and ΔU are all non-zero. The work done by a system that expands or contracts isobarically has a simple form. As the pressure is constant,

$$W = \int_1^2 P\,dV = P(V_2 - V_1) \quad \text{isobaric process} \qquad (18)$$

(vii)Isothermal expansion of an ideal gas: Let an ideal gas on expansion in quasistatic process at constant temperature by placing the gas in good thermal contact with a heat reservoir. The ideal gas equation PV = nRT for each point on the path

$$W = \int_{V_1}^{V_2} P\,dV = \int_{V_1}^{V_2} \frac{nRT}{V}\,dV$$

As T is constant in the process, we can write

$$W = nRT\int_{V_2}^{V_1} \frac{dV}{V}$$

$$W = nRT\ln\left(\frac{V_2}{V_1}\right) \qquad (19)$$

(viii)Adiabatic expansion of an ideal gas:

Let us find the relation between P and V for an adiabatic process carried out on an ideal gas.

According to equation (16), $\Delta U = -W$ in an adiabatic process

For an ideal gas $\Delta U = nC_v dT$ (at constant volume)

The work done during the process is given by $W = P\,dV$

$$P\,dV = -nC_v dT \qquad (20)$$

We can write the equation of the state of the gas in differential form as

$$d(PV) = d(nRT)$$

so

$$P\,dV + V\,dP = nR\,dT \qquad (21)$$

Using equation (20) into equation (21), we get

$$V\,dP = nC_v dT + nRdT\,(C_v + R)$$

But $C_v + R = C_P$

$$V\,dP = nC_P dT \tag{22}$$

Taking the ratio between equation (22) and equation (21), we get

$$\frac{V\,dP}{P\,dV} = -\frac{C_p}{C_v} = -\gamma$$

$$\frac{dP}{P} = -\gamma\frac{dV}{V}$$

Integrating on both the sides of the above equations, we get

$$\int_{P_1}^{P_2}\frac{dP}{P} = -\gamma\int_{V_1}^{V_2}\frac{dV}{V}$$

Or $\ln\frac{P_2}{P_1} = -\gamma\ln\frac{V_2}{V_1}$

Thus, $P_1V_1^{\gamma} = P_2V_2^{\gamma}$ (23)

Or $PV^{\gamma} = \text{constant}$ (24)

1.15 Heat Engine

A thermodynamic device that converts heat supplied to it into mechanical work is called heat engine. The automobile engines and steam turbines are examples of heat engines. Irrespective of design and features, for theoretical purposes, any heat engine can be conveniently represented by a diagram as shown in Fig.1.8

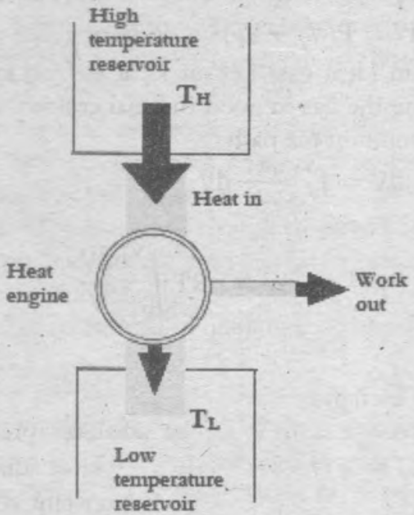

Fig. 1.8 Schematic representation of heat engine. The engine absorbs heat Q_1 from the high temperature reservoir, expels heat Q_2 from the high temperature reservoir and does work W.

Any heat engine consists of three main parts:

1.**Source:** There must be a source of heat or a hot body or a hot reservoir with infinite thermal capacity maintained at high temperature so that any amount of heat given or taken away does not affect its temperature.

2.**Sink:** There must be a condenser or sink or cold reservoir with infinite thermal capacity maintained at a lower constant temperature so that whenever any amount of heat is given or taken away does not affect its temperature.

3. **Working substance:** There must be a working medium or a substance through which heat can be absorbed from the source and rejected to sink. e.g. in case of steam engine, its steam and air in case of internal combustion engines of scooters, motor cars, aeroplanes etc.
The difference between the quantities of heat absorbed by the working substance from the source and rejected to the sink is converted into work (barring the heat lost in overcoming friction or by conduction etc). This means that the working substance falls in temperature during the process.

1.15.1 Thermal Efficiency

Practically no heat engine can convert whole of the heat supplied into work, a part of heat is usually rejected as unused heat. If the heat energy supplied is Q_1 and the heat energy rejected is Q_2 the amount of heat utilized in doing work is $Q_1 - Q_2$. Efficiency is thus defined as the ratio between its output of work and the input of heat and is denoted by η. It is a measure of how economical an engine is. We express cost efficiency in terms of fuel economy.

If the heat energy supplied is Q_1 and the heat energy rejected is Q_2 the amount of heat utilized in doing work and W is the work done by the engine, we have

Efficiency of the engine, $\eta = \dfrac{W}{Q_1}$

Since after every cycle, the working substance regains its original state the internal energy is not affected. Under ideal conditions there will be no dissipation of energy due to friction or conduction etc. We have,

$$W = Q_1 - Q_2 \tag{25}$$

$$\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

It is expressed as percentage.
For an ideal engine,

$$\eta = 1 - \frac{Q_2}{Q_1} \tag{26}$$

In practice, the useful work delivered by an engine is less than the work W owing to friction losses. Therefore, the overall efficiency is less than the thermal efficiency. Thus, for real engine

$$\eta \leq 1 - \frac{Q_2}{Q_1}$$

Where the less than sign refers to real engines and the equal sign to an ideal engine in which there are no losses.

1.16 The Carnot Cycle and Carnot Engine

In 1824 French Engineer, Sadi Carnot conceptualized an ideal engine and demonstrated that a heat engine operating in a reversible cycle between the two heat reservoirs would be the most efficient engine possible. This ideal engine is called Carnot engine.
The net work done by the working substance taken through the Carnot cycle is the largest possible for given amount of heat supplied to the substance.
The Carnot cycle is an idealization of the cycle of a real heat engine. It is assumed that there are no losses of energy by heat exchange with the environment and that there is no friction and other imperfections of actual engines. It must absorb all its heat at a constant high temperature and can reject whatever heat it has to at a constant low temperature and should work in a cycle of operations each of which is perfectly reversible.
He assumed the following things:

1.A cylinder of perfectly non-conducting walls with a perfectly conducting bottom fitted with a perfectly non-conducting and frictionless piston. The working substance in the piston can be assumed to be air which behaves like a perfect gas.
2.A hot reservoir or source of heat with infinite thermal capacity maintained at a high and constant temperature T_1.
3. A cold reservoir or sink with infinite thermal capacity maintained at a lower and constant temperature T_2.
4.A perfectly non-conducting stand on which when desired the cylinder can be moved to it without any friction.
The working substance has to undergo following quasi-static operations in order to obtain a continuous supply of work:

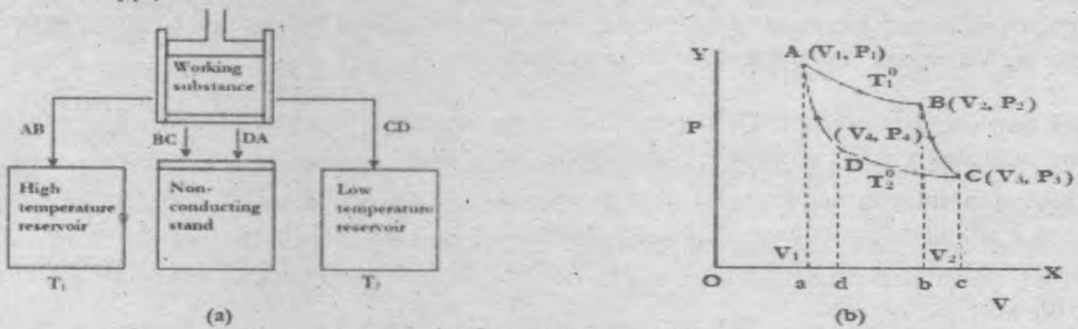

Fig. 1.9. The Carnot cycle

1.The cycle begins at the equilibrium state A Fig.1.9(b) with the cylinder in contact with heat source at a temperature T_2 Fig. 1.9(a). The working substance (the ideal gas) undergoes a slow quasistatic isothermal expansion to state B. As the gas expands it does external work against the pressure on the piston and its temperature tends to decrease. The fall in temperature is immediately made-up by-passing heat into the cylinder through its perfectly conducting base which is in contact with the source. Let the total amount of heat absorbed by the gas from the source during the process be Q_1. This heat energy input is converted directly to the work W_1 done in the moving piston. This operation is represented by the isothermal AB on the indicator diagram, shown in Fig 1.9(b) where A represents the initial volume V_1 and pressure P_1 of the gas and B, its final volume V_2 and pressure P_2 after absorbing Q_1 units of heat from the source and performance of W_1 units of external work.

From the first law of thermodynamics, the quantity of heat absorbed Q_1 must be equal to the external work W_1 done by the gas during isothermal expansion along AB from volume V_1, pressure P_1 to volume V_2, pressure P_2 at temperature T_2. This is represented by ABbaA. So,

$$Q_1 = W_1 \int_{V_1}^{V_2} PdV$$

$$= RT_1 \log_e \frac{V_2}{V_1} = \text{area ABbaA} \qquad (27)$$

2.With the system at B, the cylinder is removed from contact of source i.e., from the high temperature reservoir and placed on the non-conducting stand. This makes the gas thermally isolated from the surroundings. The gas slowly expands adiabatically along the path B to C to the state C. During this part of the cycle B to C no heat leaves or enters the system. However, work W_2 is done at the expense of reducing the internal energy. Consequently, the temperature falls from T_1 to T_2.

This operation is represented by adiabatic BC where C represents the volume V_3 and pressure P_3 of gas at temperature T_2. So that W_2 will be the work done by the gas it is represented by the area BCcbB on the indicator diagram. We thus have,

$$W_2 = \int_{V_1}^{V_2} PdV = \frac{R(T_1-T_2)}{\gamma-1} = \text{area BCcbB} \tag{28}$$

3.Next the cylinder is placed in contact with the low temperature reservoir i.e., sink at a temperature T_2. The gas is now slowly compressed (by an external agency here piston) so the work is done on the gas. During this part of cycle C to D, the temperature and hence the internal energy of the working substance are constant. The gas expels heat Q_1 to the reservoir and the work done on the gas by the external agent is W_3. This is shown here by the isothermal CD where D represents the volume V_4 and pressure P_4 of the gas when it has rejected Q_2 units of heat to the sink and W_3 units of work has been done upon it represented as by the area CcdDC. Then we have,

$$Q_2 = W_3 = \int_{V_3}^{V_4} PdV = RT_2 \log_e \frac{V_4}{V_3} = -RT_2 \log_e \frac{V_3}{V_4} \tag{29}$$

$$= \text{area CcdDC}$$

4.In the final part D to A, the cylinder is removed from the contact with the low temperature reservoir or sink and again placed on the non-conducting stand. The gas is compressed adiabatically and brought to the initial state A. The adiabatic compression is also the result of work W_4 done on the gas by the external energy. Consequently, to the adiabatic compression the system temperature increases from T_1 to T_2.

This step completes the Carnot cycle and returns the system to the initial condition.

The operation is represented by adiabatic DA and the work done on the gas is represented as DdaAD.

$$W_4 = \int_{V_4}^{V_1} PdV = -\frac{R(T_1-T_2)}{\gamma-1} = \text{area DdaAD} \tag{30}$$

Thus, it can be concluded that the Carnot cycle consist of two isothermal and two adiabatic strokes one each of compression and expansion, the two alternating with each other.

Work done by the engine per cycle:

The net work done in this reversible process is equal to the area enclosed by the path ABCDA of the PV diagram Fig 1.9(b). In a reversible cycle, the change in internal energy is zero. It follows from first law that the net work done in one cycle equals the net heat transferred into the system. For a system that undergoes a Carnot cycle, no heat is supplied to or rejected by the system during the adiabatic paths BC and DA as in Fig 1.9(b). An amount of heat Q_1 is supplied to the system during the isothermal expansion AB and an amount Q_2 is rejected during the isothermal compression CD. Thus, the first law can be written as:

$W = Q_1 - Q_2$ all heat engines

Above equation is applicable for a complete cycle

Of any heat engine the thermal efficiency is given by:

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

1.16.1 Efficiency

Let us now calculate Efficiency (η) in terms of the temperatures T_1 and T_2 of the source and sink respectively and in terms of the expansion ratio (ρ) of the gas.

I

n terms of the temperatures T_1 and T_2 of the source and sink

$$\frac{Q_1}{Q_2} = \frac{W_1}{W_2} = \frac{RT_1 \log_e \frac{V_2}{V_1}}{RT_2 \log_e \frac{V_3}{V_4}} = \frac{T_1 \log_e \frac{V_2}{V_1}}{T_2 \log_e \frac{V_3}{V_4}} \tag{31}$$

Since points B and C lie on the same adiabatic BC, we have

$$T_1 V_2^{\gamma-1} = T_2 V_3^{\gamma-1} \text{ or}$$

$$\frac{T_1}{T_2} = [\frac{V_3}{V_2}]^{\gamma-1} \tag{32}$$

Where γ is the ratio between Cp and Cv for the gas.

Similarly, points D and A lie on the same adiabatic DA, we have

$$T_1 V_1^{\gamma-1} = T_2 V_4^{\gamma-1} \text{ or}$$

$$\frac{T_1}{T_2} = [\frac{V_4}{V_1}]^{\gamma-1} \tag{33}$$

From equation (26) and equation (27) :

$$\frac{V_3}{V_2} = \frac{V_4}{V_1} \quad \text{or} \quad \frac{V_2}{V_1} = \frac{V_3}{V_4}$$

Thus,

$$\log_e \frac{V_2}{V_1} = \log_e \frac{V_3}{V_4} \tag{34}$$

Substituting in equation (31) we have,

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \tag{35}$$

Thus, efficiency,

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}$$

Efficiency always depends on the temperatures T_1 and T_2 of the source and sink respectively and that the greater the value of $(T_1 - T_2)$, greater the difference between the temperatures of the source and sink, higher will be the efficiency. Since, $(T_1 - T_2)$, must always be less than T_1, it is clear that the efficiency must be less than 1 or 100%

In terms of expansion ratio:

We have seen here that

$$\frac{V_3}{V_2} = \frac{V_4}{V_1} \tag{36}$$

Each one of the ratios is the adiabatic expansion ratio of the gas, and so is equal to ρ

From equation (32) and (33),

$$\rho = \frac{V_3}{V_2} = \frac{V_4}{V_1} = [\frac{T_1}{T_2}]^{1/\gamma-1} \text{ whence, } \frac{T_2}{T_1} = [\frac{1}{\rho}]^{\gamma-1} \tag{37}$$

Thus efficiency,

$$\eta = 1 - [\frac{1}{\rho}]^{\gamma-1} \tag{38}$$

1.16.2 Carnot's Theorem

This theorem discloses a fundamental restriction on the conversion of heat into work. It is the most efficient heat engine operating between the two heat reservoirs.

Carnot's Theorem Statement: No heat engine can be more efficient than a reversible Carnot engine operating between the same limits of temperature i.e., between the same source and sink and all reversible engines operating between the same limits of temperature have the same efficiency.

1.17 Heat Pump

A device that transfers heat from a sink to a high temperature reservoir i.e., source is known as heat pump. This is the reverse function of a heat engine. It requires work input, because the heat transfer from low temperature reservoir to high temperature reservoir will not take place spontaneously.
The household refrigerators and air conditioners are examples of the most common heat pumps. In an ordinary household refrigerator, the working substance is a liquid that circulates within the system. It takes heat from the low temperature reservoir, which is the cold chamber in which food article are stored and transfers it to outside air in the room where the unit is kept. The work is supplied by a compressor that uses electrical energy. The working substance is a fluid that readily undergoes a liquid to gas phase change at the operating temperature. The most common refrigerants are Freon-12 (CCl_2F_2) and ammonia.

1.18 Second Law of Thermodynamics

The first law of thermodynamics is called the law of equivalence and can only tell us about the inter-convertibility of heat and mechanical work. It tells us nothing about the conditions in which the conversion can occur neither it tells us the limit within which it is possible. Different workers have defined it in different ways. Here we can discuss two classical statements of the second law of thermodynamics: Kelvin-Planck statement and Clausius statement.

According to Kelvin-Planck: It is impossible to construct a heat engine that will operate in a cycle and which will receive a given amount of heat from a high temperature reservoir and does an equal amount of work. The only alternative is that some heat must be transferred from the working fluid to a low temperature reservoir. Therefore, work can be done by the transfer of heat only if there are two temperature levels involved.

According to Rudolf Clausius: It is impossible to construct a device that operates in a cycle and produce no effect other than the transfer of heat from a cold body to the hot body. Thus, it is impossible to construct a refrigerator that operates without an input of work.
The first law of thermodynamics is concerned with conservation of energy. As long as the energy is conserved in a process, the first law is satisfied. The second law underlines the fact that the processes proceed in a certain direction but not in the opposite direction. For example: a cup of tea cools by heat transfer to the surroundings but heat will not flow from the cooler surroundings to hot the tea cup. Thus, real processes progress only in one direction.
A cycle will occur only when both the first and second law of thermodynamics are satisfied.
The first law of thermodynamics is a common declaration of the conservation of energy. It makes no distinction between the different forms of energy. The second law of thermodynamics asserts that thermal energy is different from all other forms of energy. Various forms of energy can be converted into thermal energy spontaneously and completely; whereas the reverse transformation is never complete. The impossibility of converting heat completely into mechanical energy forms the basis of Kelvin-Planck statement of second law.
The basis of second law lies in the difference between the nature of mechanical energy and the nature of internal energy. Mechanical energy is the energy of ordered motion of a body. Internal energy is the energy of random motion of molecules within it as a whole in the direction of velocity of the body. The energy associated with this ordered motion of molecules is the internal energy. When a moving body comes to rest due to friction, the ordered portion of the kinetic energy becomes converted into energy of random molecular motion. It is impossible to reconvert the energy of

random motion completely to the energy of ordered motion, since we cannot control the motions of individual molecules. We can convert only a portion of it. That is what a heat engine does.

1.19 Entropy

The differential form of the first law of thermodynamics is written as

$$dQ = dU + dW \tag{39}$$

The work dW done depends on the path of process and therefore it is not a function of the state. The same is the case with dQ, the quantity of heat supplied or taken away. The work dW can be expressed in terms of thermodynamic variables and their changes. For instance, we have expressed dW in equation (39) as

$$dW = PdV \tag{40}$$

It is found that dQ can also be expressed in a similar fashion, in case of reversible processes. We write

$$dQ = T\,dS \tag{41}$$

where dS is called the change in entropy and T is the temperature. Now, we define the change in entropy as

$$dS = dQ/T \qquad \text{reversible process} \tag{42}$$

The change in entropy dS in the course of an infinitesimal change is equal to the quantity of heat dQ divided by the absolute temperature T, where dQ is the heat absorbed or rejected when the change is carried out in a reversible manner.

The total entropy change in a reversible process may be obtained by integrating equation (42) thus,

$$\Delta S = S_2 - S_1 = \int_1^2 \frac{dQ}{T} \tag{43}$$

Where S_1 and S_2 are the entropies of the initial and final states of the system.

The importance of the entropy S is that it is a function of state like the internal energy U. Both these parameters depend only on the initial and final states of the system and not on the path of the process that takes the system from the initial state to final state. Equation (43) assumes a simpler form when the process is an isothermal process. As T is constant in isothermal process equation (43) may be written as

$$\Delta S = \int_1^2 \frac{dQ}{T} = \frac{1}{T}\int_1^2 dQ = \frac{Q}{T} \tag{44}$$

Thus, $\Delta S = \frac{Q}{T}$ isothermal process

The units of entropy and entropy change are J/K

In practice, the value of entropy S is not of much interest. We have to know the change in the entropy when the system changes from one state to another.

1.19.1 Entropy, Disorder and Second Law

When processes occur, in general they are irreversible and the degree of disorder increases as a result of these processes. As an example, let us take the case of isothermal expansion of an isothermal gas. As the gas absorbs heat, it slowly expands. At the end of the process the gas occupies a greater volume than at the beginning. The gas molecules are more disordered now. The gas will not, by its own accord, give up its thermal energy and segregate itself to confine to the initial volume. We thus, observe that the flow of heat takes place in the direction that increases the amount of disorder. The same type of order to disorder change occurs when free expansion of gas occurs, when one gas diffuses into another, and in similar other spontaneous processes.

Rudolf Clausius the German physicist, introduced the quantity entropy which is regarded as a measure of disorder in a system. An increase in disorder is equivalent to an increase in entropy. Irreversible processes are processes for which entropy increases.
These considerations led Clausius to reformulate the second law of thermodynamics in terms of entropy. According to it, the entropy of an isolated system always tends to increase. Mathematically, it is expressed as

$$\Delta S_{\text{isolated system}} \geq 0$$

1.19.2 Points of Discussion

(i)The net change in entropy in any reversible cycle is zero
Let us take the case of Carnot cycle as an example. There is no change in entropy of the working substance during two adiabatic paths. Either during adiabatic expansion or compression Q = 0. Therefore, S = 0. However, there is an increase in entropy during isothermal expansion as heat Q_1 is added at a constant temperature T_1. The consequent increase in entropy $\Delta S_1 = \frac{Q_1}{T_1}$. There is a decrease in entropy during isothermal compression in which heat Q_2 is rejected at a temperature T_2. Thus, $\Delta S_2 = -\frac{Q_2}{T_2}$
The net change in entropy is given by

$$\Delta S = \Delta S_1 + \Delta S_2 = \frac{Q_1}{T_1} - \frac{Q_2}{T_2}$$

But $\quad \frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

Thus, ?? $\quad \Delta S = 0$ reversible cycle

(ii)Entropy increases in all reversible processes: It is proved that there is net increase in the entropy during irreversible processes. Since all real processes taking place in the universe are irreversible, there is a continuous increase in its entropy. For this reason, entropy is not conserved. In this respect entropy differs from energy.
We can illustrate this by taking example. Suppose a small quantity of heat dQ is radiated away from a hot body A, at a temperature T_H to a cold body B at a temperature T_C. Let dQ be so small that T_H and T_C are not altered appreciably due to the exchange of heat. However, the entropy of A decreases by $-dQ/T_H$ whereas that of B increases by dQ/T_C in this process.

$$\Delta S = \frac{dQ}{T_C} - \frac{dQ}{T_H}$$

As $T_H > T_C$, $\quad \Delta S > 0$

(iii)Entropy indicates the direction in which processes proceed in nature: all natural processes are irreversible. They proceed in the direction of increasing entropy.

(iv)Entropy represents the unavailability of energy: In the thermodynamic sense, entropy is a measure of the capability to do work or transfer heat. A system at a higher temperature will tend to do work on and/or transfer heat to its lower temperature surroundings. In the process, the entropy of the system increases and greater the entropy the less available is the energy.

Let us consider the example of Carnot engine. The efficiency of Carnot engine is given by

$$\eta = 1 - \frac{T_2}{T_1}$$

As Q_H is the heat input, heat converted into work = $Q_H(1 - \frac{T_2}{T_1})$
Thus, heat unavailable for work = $Q_1 = T_2$
But Q_1/T_1 represents the increase in entropy ΔS during isothermal expansion
Energy wasted = $T_2 \Delta S$
If T_2 is constant, the amount of energy wasted is proportional to the increase in entropy.

1.20 Third Law of Thermodynamics

With a decrease in temperature, a greater degree of order prevails in any system. If we could cool a system to 0K, the maximum conceivable order would be established in the system and the minimum entropy would correspond to this state. Now, suppose we apply a pressure on the system at 0K. What does happen to the entropy of the system? On the basis of experiments conducted at low temperatures, W. Nernst concluded that at 0K any change in the state of a system takes place without change in the entropy. This is called Nernst's theorem. It is called the third law of thermodynamics. Third law of thermodynamics is sometimes known as the principle of unattainability of absolute zero. It is stated as follows: It is impossible to attain a temperature of 0K.

******Solved examples******

Based on equivalence of heat and work

Ex.1 The height of Niagra falls is 50m. Calculate the difference between the temperature of water at the top and bottom of the fall, if J = 4.2J/cal/kg^{0}C, J = 4.2J/cal
Sol. Given: h = 50m, g = 9.8m/s^2, s = 1kcal
Let m kg of water falls in one second. The potential energy lost in one second is

$$W = mgh = m \times 9.8 \times 50J$$

The lost energy is converted into heat. If Q be the heat produced, then

$$Q = \frac{W}{J} = \frac{m \times 9.8 \times 50}{4.2} = 117\text{mcal} = 0.117\text{m kcal}$$

If this heat causes temperature rise ΔT in water, then
Q = mass × specific heat of water × temperature rise

$$0.117m = m \times 1 \times \Delta T$$

$\Delta T = 0.117^0$ C

Based on work done in different process

Ex.2 If in an isothermal expansion, the volume of 1g mole of gas at 27^{0}C is doubled, calculate the work done in the process (R = 8.3 Jmol^{-1}K^{-1})
Sol. Given: T = 27^{0}C= 300K, R = 8.3 Jmol^{-1}K^{-1}, $V_2 = 2V_1$
The work done in isothermal process is

$$W = 2.3026 \text{ RTlog}_{10}\frac{V_2}{V_1}$$
$$= 2.3026 \times 8.3 \times 300\log_{10}\frac{2V}{V}$$
$$= 2.3026 \times 8.3 \times 300 \times 0.3010 = 1725.8\text{ J}$$

Ex.3 A definite mass of a perfect gas is compressed adiabatically to half of its original volume. Determine the resultant pressure if its initial pressure was 1 atmosphere. [$\Upsilon = 1.4$ and $2^{1.4} = 2.64$]

Sol. Given: $P_1 = 1$ atmosphere, $V_2 = V_1/2$, $\gamma = 1.4$ and $2^{1.4} = 2.64$
For an adiabatic change

$$P_1V_1^{\gamma} = P_2V_2^{\gamma}$$

Thus, $P_2 = P_1[\frac{V_1}{V_2}]^{\gamma} = 1 \times [\frac{V_1}{V_1/2}]^{1.4} = 1 \times (2)^{1.4} = 2.64$ **atmosphere**

Ex.4 What is the highest possible theoretical efficiency of a heat engine operating with a heat reservoir of furnace gases at 2100⁰C, when cooling water available is at 15⁰C?
Sol. Given: Temperature of furnace, $T_1 = 2100 + 273 = 2373$ K and Temperature of cooling water, $T_2 = 15 + 273$ K $= 288$K
Now, $\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{288}{2373} = 0.878$ or 87.8%
Ex.5 The efficiency of Carnot cycle is 1/6. On reducing the temperature of the sink by 60^0C, the efficiency increases to 1/3. Find the initial and final temperatures which the cycle is working.
Sol. Let T_1 and T_2 be the initial Kelvin temperature of the source and sink respectively
Then the efficiency is given by: $\eta = 1 - \frac{T_2}{T_1} = 1/6$ (i)
When T_2 is decreased to $T_2 - 60^0$C (1^0C = 1K in size). The new efficiency is
$\eta' = 1 - \frac{T_2 - 60}{T_1} = 1/3$ (ii)
From equations (i) and (ii)

$$\frac{T_2}{T_1} = 1 - \frac{1}{6} = \frac{5}{6}$$

And $\frac{T_2 - 60}{T_1} = 1 - \frac{1}{3} = \frac{2}{3}$

Dividing (ii) by (i) we get

$$\frac{T_2}{T_2 - 60} = \frac{5}{6}$$

Solving we get $T_2 = 300\text{K} = 27^0\text{C}$
Putting the value of T_2 in (i), we get

$$\frac{300}{T_1} = \frac{1}{6}$$

This gives $T_1 = 360\text{K} = 87^0\text{C}$
The cycle is initially working between 87^0C and 27^0C. finally the temperature of sink is reduced 60^0C, so that the cycle works between 87^0C and -33^0C
Ex.6 A Carnot's refrigerator absorbs heat from water at 0⁰C and rejects it at the room temperature 37⁰C. Calculate the amount of work required to convert 10kg water at 0⁰C into ice at same temperature (latent heat of ice = 3.4×10^5 Jkg^{-1}. Also find the coefficient of performance of the refrigerator.
Sol. Given: m = 10kg, L = 3.4×10^5 Jkg^{-1}, $T_1 = 37^0\text{C} = 310\text{K}$, $T_2 = 0^0\text{C} = 273\text{K}$
We know that for an ideal Carnot's cycle

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

$T_2 \rightarrow$ lower Temp

$T_1 \rightarrow$ Higher Temp

Or $Q_1 = Q_2\frac{T_1}{T_2}$

Here, $Q_2 = mL = 10 \times 3.4 \times 10^6 = 3.4 \times 10^6$J
Then $Q_1 = 3.4 \times 10^6 \times \frac{310}{273} = 3.86 \times 10^6$J

Therefore, work done by the refrigerator

$$W = Q_1 - Q_2 = (3.86 - 3.4) \times 10^6 = 0.46 \times 10^6 = 4.6 \times 10^5 J$$

Coefficient of performance, $\beta = \frac{Q_2}{W} = \frac{3.4 \times 10^6}{4.6 \times 10^5} = \frac{34}{4.6} = 7.4$

Imp **Ex.7** A bullet moving with velocity 40m/s falls down after striking a target. If only half of the heat produced is absorbed by the bullet, find rise in its temperature. Specific heat of lead = 120 $Jkg^{-1}degC^{-1}$

Sol. Given: v = 40m/s, s = 120 $Jkg^{-1}degC^{-1}$

Let the mass of the bullet be m kg and its velocity be v.

Then before striking the target, the kinetic energy of the bullet $= \frac{1}{2}mv^2$

On striking the target this is converted into heat

$$Q = \frac{1}{2}mv^2$$

As the bullet absorbs only half of the heat produced in the impact, heat received by the bullet.

$$Q' = 0.5, \quad Q = \frac{1}{2} \times \frac{1}{2}mv^2$$

If the increase in the temperature of the bullet is ΔT^0C,

Then $Q' = ms\Delta T$ cal

Hence $ms\Delta T = \frac{1}{2} \times \frac{1}{2}mv^2$

Thus, $\Delta T = \frac{v^2}{4s} = \frac{40^2}{4 \times 120} = 3.33°C$

Ex.8 The volume of 1g mole of a gas filled in a container at standard pressure $(1 \times 10^5 \ N/m^2)$ and the temperature (0°C) is $22.4 \times 10^{-3}m^3$. The volume of the gas is reduced to half of its original value by increasing the pressure(i)isothermally (ii) adiabatically. In each case calculate the final pressure of the gas and the amount of work done ($R = 8.3\ Jmol^{-1}K^{-1}, \Upsilon = 1.4$)

Sol. Given , $P_1 = 1 \times 10^5 \ N/m^2$, $T_1 = 0°C = 273K$, $V_1 = 22.4 \times 10^{-3}m^3$, $V_2 = \frac{1}{2}V_1 = 11.2 \times 10^{-3}m^3$

(i) In an isothermal process

$$P_1V_1 = P_2V_2$$

Or $$P_2 = \frac{P_1V_1}{V_2} = 1 \times 10^3 = \frac{V_1}{V_1/2} = 2 \times 10^5 \ N/m^2$$

Work done in the process

$$W = 2.3026 \ RT\log_{10}\frac{V_2}{V_1} = 2.3026 \times 8.3 \times 273 \log_{10}\frac{V_1}{2V_1}$$

$$= 2.3026 \times 8.3 \times 273 \log_{10}\frac{1}{2_1} = 2.3026 \times 8.3 \times 273 \times (-0.3010)$$

$$= -1.57 \times 10^3 J$$

(ii) In an adiabatic process

$$P_1V_1^{\Upsilon} = P_2V_2^{\Upsilon}$$

or $$P_2 = P_1[\frac{V_1}{V_2}]^{\Upsilon}$$

Final temperature of the gas

$$T_2V_2^{\Upsilon-1} = T_1V_1^{\Upsilon-1}$$

Or $$T_2 = T_1[\frac{V_1}{V_2}]^{\Upsilon-1} = 273 \times [\frac{V_1}{V_1/2}]^{1.4-1} = 273 \times 2^{0.4} = 360.2K]$$

Work done $W = \frac{R}{\Upsilon-1}(T_1 - T_2) = \frac{8.73\times(273-360.2)}{1.4-1} = \frac{8.3\times(-87.2)}{0.4}$

$= -1.81 \times 10^3 J$

heat input   work done

Ex.9 A person consumes a diet of 10^4 J/day and spends total energy of 1.2×10^4 J/day. Determine the daily change in the internal energy. If the net energy spent comes from sucrose at the rate of ~~2×10^5J/kg~~, in how many days will the person reduce his mass by 1 kg?

1.6×10^4

Sol. Given $dQ = 10^4$ J/day, $dW = 1.2 \times 10^4$ J/day.

From the first law of thermodynamics

$$dU = dQ - dW = 1 \times 10^4 - 1.2 \times 10^4 = -2.0 \times 10^3 J/day$$

This decrease corresponds to the loss of sucrose. Therefore, sucrose lost per day

$$= \frac{2.0 \times 10^3}{1.6 \times 10^4} kg = 0.125 \text{ kg/day}$$

Thus, number of days required for the loss of 1kg

$$= \frac{1}{0.125} = 8 \text{ days}$$

Ex.10 At normal temperature (0^0C) and at normal pressure ($1.1.013 \times 10^6 N/m^2$) when one gm of water freezes its volume increases by 0.09 /cm³. Calculate the change in its internal energy (latent heat of melting ice = 80 cals/gm, 1cal = 4.2 J)

Sol. 1.P $= 1.013 \times 10^6 \frac{N}{m^2} = 1.013 \times 10^6 \frac{dynes}{cm^2}$, m = 1gm, V_1=1cm³, V_2=0.091 cm³

Work done $dW = PdV = -1.013 \times 10^6 \times 0.091 = -9.2 \times 10^4 erg$

$$dQ = 80cal = 80 \times 4.2 \times 10^7 erg = 336 \times 10^7 erg$$

According to the first law of thermodynamics

$$dQ = dU + dW$$
$$dU = dQ - dW$$
$$dU = 336 \times 10^7 + 9.2 \times 10^4 = (336 + 0.0092) \times 10^7 erg = 336 J$$

Ex.11 1 mole of a gas at 127^0C expands isothermally until its volume is doubled. Calculate the work done.

Sol. In isothermal process,

$\boxed{1 \text{ erg} = 10^{-7} \text{ Joule}}$

$W = RT \ln\frac{V_2}{V_1}$ $[V_2 = 2V_1]$

$R = 8.3 \times 10^7$, T = 127+273=400K $\boxed{2310.1 J}$

So $W = 22.9 \times 10^9 ergs$

Ex.12 A Carnot's engine whose low temperature reservoirs at 7^0C has an efficiency of 40%. To increase the efficiency to 50% by how many degree should the temperature of the source be increased?

Sol. The efficiency is given by : $\eta = 1 - \frac{T_2}{T_1}$

Or $\eta' = 1 - \frac{T_2}{T_1'}$, $T_2 = 280K$

T_1= 466.67K, $T_1' = 560K$

ΔT = 560 - 466.67 = 93.33K = -179.67^{0}C

Ex.13 A Carnot's engine working between temperatures 100^0C and 0^0C absorbs 10^4 calories of heat at higher temperature. What will be the work done when the cycle of operation is over.

Sol. $\eta = 1 - \frac{T_2}{T_1} = \frac{W}{\text{heat absorbed}}$

$W = 1.126 \times 10^4 J$

Ex.14 Calculate the change in entropy when 2g of ice melts into water at the temperature 0^0C. Latent heat of ice = 80cal/g.

Sol. $dQ = T\, dS$

$dS = dQ/T = 160/273 = 0.58 cal/K = 0.58\ cal/^0C$

*****Review Questions*****

Based on laws of thermodynamics

1.What are fundamental ideals of thermodynamics?
2.Discuss continuum model of thermodynamics
3.Define the following: (a) system (b) state (c) equilibrium (d) process.
4.Explain how first law of thermodynamics leads to concept of internal energy ?
5.Why does the temperature of a gas drops when it is subjected to adiabatic expansion? Explain
6.State First law of thermodynamics and prove that internal energy is a thermodynamical variable.
7.Write notes on following (a) Zeroth law of thermodynamics and concept of temperature (b) Zeroth law of thermodynamics (c) Concept of internal energy (d) First law of thermodynamics
8.Explain in short first law of thermodynamics.
9.What do you mean by indicator diagram? Draw indicator diagram for isochoric and isobaric processes.
10.What do you understand by internal energy of a system?
11.What is zeroth law of thermodynamics? State its importance?
12.State first law of thermodynamics. What are its limitations ?
13.State the zeroth law of thermodynamics. How is mercury in thermometer able to find temperature of a body using the zeroth law of thermodynamics ?
14.Explain the first law of thermodynamics for the closed system undergoing a cyclic change.
15.Write the zeroth law of thermodynamics. On this basis explain the concept of temperature. Explain the zeroth law of thermodynamics.
16.What is the concept of temperature? Define temperature.
17.Prove that heat and work both are the path function.
18.State first law of thermodynamics.
19.Show that for a cyclic process the heat supplied to a system is equal to the work done by the system. Define reversible and irreversible process with one example of each process.
20.State zeroth law of thermodynamics and explain the concept of temperature on the basis of this law.
21.State the first law of thermodynamics. Discuss its significance.
22.Calculate work done in an adiabatic expansion of a perfect gas.
23. State first law of thermodynamics. and show that heat and work are path functions but their difference is a point function.
24.Write down the zeroth law of thermodynamics. Explain how it is introduces the temperature of a system as its function of state.
25.State the first law of thermodynamics and use it to derive a relation between the volume and temperature of a perfect gas undergoing an adiabatic change.
26.Explain the first law of thermodynamics. Explain the latent heat on the basis of it.
27.State two statements of second law of thermodynamics and show their equivalence.
28. State second law of thermodynamics.

29.Show that work is a path function and not a property.
30.State the first law for a closed system undergoing a change of state.
31.What is thermodynamics? State the first, second and third laws of thermodynamics and discuss their significance
32.Why the second law is called a directional law of nature?
33.Give the Nernst statement of the third law of thermodynamics?
34.Give the Kelvin-Planck statement of the second law.
35.Give the Clausius statement of the second law.

Based on Carnot Cycle, Carnot Engine, Entropy

36.Derive the expression for the efficiency of a Carnot engine directly from a T-S diagram.
37.What are the reversible processes?
38.Write the Kelvin-Planck and Clausius statements of the second law of thermodynamics.
39.Between two given temperatures no ordinary engine can be more efficient than the Carnot engine and all Carnot's engines are equally efficient. Prove the statement.
40.State the essential conditions for a process to be reversible.
41.What do you understand by entropy? State the second law of thermodynamics in the entropy.
42.What is Carnot's theorem? Prove it.
43.Explain need of second law of thermodynamics. State it's both statements and show the equivalence.
44.Prove that the efficiency of a Carnot's engine depends only upon the two temperature between which it works.
45.Differentiate between reversible and irreversible processes.
46.What is a reversible process? A reversible process should not leave any evidence to show that the process had ever occurred. Explain.
47.All spontaneous processes are irreversible. Explain.
48.Distinguish between a reversible and an irreversible process. Illustrate your answer by some examples.
49.A Carnot's engine and refrigerator work between same temperatures T_1K and T_2K. Write expressions for the efficiency η of Carnot's engine and coefficient of performance β, the refrigerator and inter relationship between η and β.
50.Write the expression for efficiency of a Carnot's reversible engine.
51.Establish a relation between the efficiency η of an ideal Carnot engine and the coefficient of performance β of an ideal refrigerator working between temperatures T_1K and T_2K
52.Explain second law of thermodynamics.
53.What is significance of second law of thermodynamics?
54.What is physical significance of entropy?
55.The entropy of a substance is a unique function of its state. Explain.
56.Prove that the integral $\int_1^2 \frac{dQ}{T}$ does not depend on path for a reversible process.
57.Define and explain entropy. Explain why unavailable energy in the universe tends to increase?
58. Show that entropy remains constant in a reversible process whether it increases for an irreversible process.
59.Define entropy and explain its physical significance?
60.What is the absolute scale to temperature and how has it been derived? Explain clearly why the scale is called absolute and why the zero of this scale is considered to the lowest temperature possible?

61.State and Prove Carnot's theorem.
62.What is a cyclic heat engine?
63. Define the thermal efficiency of a heat engine cycle. Can this be 100%?
64. What is a Carnot Cycle? What are the four processes which constitute the cycle?
65.Describe the different operations involved in a Carnot's cycle. Derive the efficiency of a Carnot engine in terms of source and sink temperatures.
66.What is a reversed heat engine?
67.Show that the efficiency of a reversible engine operating between two given constant temperatures is the maximum.
68.How does the efficiency of a reversible engine vary as the source and sink temperatures is varied? When does the efficiency become 100%?
69.What do you understand .by the entropy principle?
70.Show that the transfer of heat through a finite temperature difference is irreversible?
71.Show that the adiabatic mixing of two fluids is irreversible.
72.What is the maximum work obtainable from two finite bodies at temperatures T_1 and T_2?

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Unit - II

Waves and Oscillations: Wave motion, simple harmonic motion, wave equation, superposition principle. Introduction to Electromagnetic Theory, Maxwell's equations, Work done by the electromagnetic field, Poynting's theorem, Momentum, Angular momentum in electromagnetic fields, Electromagnetic waves: the wave equation, plane electromagnetic waves, energy carried by electromagnetic waves

Chapter 2

Wave and Oscillations

Introduction

Before wave motion, we have to understand oscillatory motion which is a kind of periodic motion. Therefore, we need to understand first the periodic motion.

Periodic Motion: A motion that repeats itself after a certain period of time, is called periodic motion. Repetition of the motion may be in a circle or a back-and-forth motion about a fixed mean point.

Oscillatory Motion: When the motion repeats itself in a back-and-forth manner about a fixed point, it is termed as oscillatory motion. It may also be called vibratory motion if the frequency of repetitions is quite high.
From the above discussion, it is clear that oscillatory motion should have the following characteristics.

Frequency: The numbers of repetitions per time, done by the oscillatory body, is called the frequency (f), of the periodic motion.
Time Period: Time taken to complete, one oscillation, is called the time period (T).
Here, we can have $T = \frac{1}{f}$
Amplitude: The maximum distance travelled on either side of the fixed point, is called amplitude.

2.1 Simple Harmonic Oscillation

It is a kind of oscillatory motion in which the motion is in a straight path on both the sides of the fixed point. If we look at the system of S.H.M., as a whole, it can be said that a force, called restoring force is needed to perform the S.H.M and it always follows Hook's Law, i.e.

$$F = -kx,$$

Here, x is the displacement from the mean position and k is the force constant.

2.1.1 Equation of S. H. M. and its Solution

Consider a particle of mass 'm' executing S.H.M along x axis about its mean position such that the displacement from the mean position be x. Then, according to the condition to perform S.H.M, restoring force F will act on the particle to oppose its motion and is proportional to the displacement. Therefore;

$$F \propto -x \text{ or } F = -kx \qquad (1)$$

Here, k is the constant of proportionality and is known as force constant or force per unit displacement.
Now, according to the Newton's second law of motion, "**the acceleration produced, is defined as the force applied per unit mass of a moving object**" i.e.

$$\frac{d^2x}{dt^2} = \frac{F}{m}$$

Or we can write, Force = mass x acceleration i.e.

$$F = m\frac{d^2x}{dt^2} \tag{2}$$

From (1) and (2)

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Putting $\boxed{\frac{k}{m} = \omega^2}$ (3)

We have

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{4}$$

This is a second order differential equation and its solution can be written as;

$$x = A\sin(\omega t + \delta) \tag{5}$$

The equation (5), gives the instantaneous displacement of the particle, executing S.H.M at any instant of time. A is the maximum displacement of the particle and is called the displacement amplitude and $(\omega t + \delta)$ is called instantaneous phase angle.

2.2 Ideal Simple Harmonic Motion

The ideal S.H.M. can be defined with the help of a particle moving in a circle. "If a particle is moving on a circular path and a perpendicular is drawn on any diameter of the circle, then the motion of foot of the perpendicular about the centre of the circle is called ideal S.H.M" as shown in Fig. 2.1 The equation for instantaneous displacement can be obtained very easily using Fig. 2.1

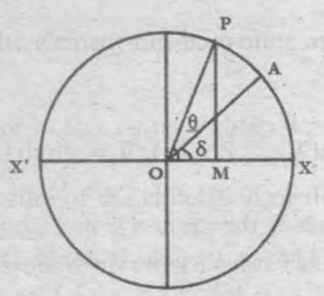

Fig. 2.1

Let a particle, start its motion on a circular track with the initial phase XOA as δ. Let us consider any arbitrary position P of the particle such that angle AOP is θ. Let PM is a perpendicular drawn on the diameter XX′. As the particle moves on the circular path and completes a circle, the point M i.e., the foot of the perpendicular, moves on the straight path XX′, about the mean point O and also complete one oscillation. The motion of this point M, is an oscillatory motion about the centre of the circle and it is in perfect straight path, termed as ideal simple harmonic motion. So "the motion of the foot of the perpendicular, drawn on any diameter from a particle, moving on a circular track is called ideal simple harmonic motion". The amplitude of the oscillation A is equal to the radius of the circle OX. Now from the diagram, we can write, θ

$$x = A\sin(\theta + \delta)$$

Where $\theta = \omega t$, and ω is the angular frequency, i.e., angular distance per unit time.

Or $\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi f$

From equation (3), we have the angular frequency $\omega = \sqrt{\frac{k}{m}}$, so the time period and the frequency of S.H.M will be written as;

$$T = 2\pi\sqrt{\frac{m}{k}}$$

And,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

2.2.1 Velocity and Acceleration in S.H.M

We can find the velocity of the particle executing S.H.M. from the expression of the displacement.

$$x = A\sin(\omega t + \delta) \tag{6}$$

Differentiating it w.r.t. time we get;

$$v = \frac{dx}{dt} = A\omega\cos(\omega t + \delta)$$

$$= A\omega\sqrt{1 - \sin^2}(\omega t + \delta) = \omega\sqrt{A^2 - A^2\sin^2(\omega t + \delta)}$$

$$\boxed{v = \omega\sqrt{A^2 - x^2}} \tag{7}$$

This is the expression for the velocity of the particle at any point which has a displacement x from its mean position. It is well known that particle has maximum velocity at the mean position and minimum at the extreme point, due to obvious reasons. So, the maximum velocity can be obtained by putting $x = 0$ and minimum at $x = A$. i.e.

$v_{max} = A\omega$ (at mean position)

$v_{min} = 0$ (at extreme position)

Again, differentiating equation (7) w.r.t time, we get acceleration as;

$$\alpha = \frac{dv}{dt} = -A\omega^2\sin(\omega t + \delta)$$

$$\alpha = -\omega^2 x \tag{8}$$

The above equation gives acceleration of the oscillating particle at any displacement. From equation (8), it can be seen that acceleration is proportional to the displacement and in opposite direction of the motion. This is a standard characteristic of a simple harmonic motion. It is clear from equation (8) that for the maximum acceleration x is equal to the amplitude A.

So, the maximum acceleration i.e., $\alpha_{max} = \omega^2 A$ (at the extreme position)

And the minimum acceleration i.e., $\alpha_{min} = 0$ (at the mean position)

2.3 Energy of a Simple Harmonic Oscillator

A simple harmonic oscillator possesses both the kinetic energy and the potential energy. Here it can also be seen that in ideal condition, the law of conservation of energy also holds good during the complete oscillation of the particle.

The potential energy is given by the amount of work done stored in displacing the particle from the position 0 to x by applying some force. Thus,

$$P.E. = \int \vec{F}.\,\vec{dx} = \int_0^x kx\,dx = \boxed{\frac{1}{2}kx^2} \tag{9}$$

Putting the values of k and x, the instantaneous potential energy at any position is;

$$\boxed{P.E. = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \delta)} \qquad (10)$$

And the corresponding kinetic energy of the harmonic oscillator, at any instance, is given as

$$K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\frac{dx}{dt}\right]^2$$
$$= \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \delta)$$

So, the total energy E of the oscillator at any displacement from the mean position can be obtained as

$$E = K.E + P.E$$
$$= \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \delta) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \delta)$$
$$E = \frac{1}{2}m\omega^2A^2 \qquad (11)$$

Hence, the total energy comes out to be constant and independent of the position of the particle. It is obvious that the maximum possible value of K.E and P.E would be same as $\frac{1}{2}m\omega^2A^2$. The K.E and P.E of the harmonic oscillator can also be calculated with the passage of time, in place of displacement. If the oscillation starts at t = 0 at its extreme position, then;

The average P.E of the simple harmonic oscillator over a cycle, is given as:

$$< P.E > = \frac{1}{T}\int_0^T \frac{1}{2}kx^2 dt$$
$$= \frac{\int_0^T \frac{1}{2}kA^2\sin^2(\omega t + \delta)dt}{T}$$
$$= \frac{1}{2}\frac{\int_0^T m\omega^2A^2\sin^2(\omega t + \delta)dt}{T}$$
$$= \frac{1}{2}m\omega^2A^2\frac{\int_0^T \sin^2(\omega t + \delta)dt}{T}$$
$$< P.E > = \frac{1}{4}m\omega^2A^2 \qquad \text{as } \left[\frac{\int_0^T \sin^2(\omega t + \delta)dt}{T} = 1/2\right] \quad (12)$$

And similarly, the average kinetic energy for one complete cycle is;

$$< K.E > = \frac{1}{T}\int_0^T \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 dt$$
$$= \frac{\int_0^T \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t+\delta)\, dt}{T}$$
$$= \frac{1}{2}m\omega^2A^2\frac{\int_0^T \cos^2(\omega t+\delta)\, dt}{T}$$
$$< K.E > = \frac{1}{4}m\omega^2A^2$$
$$\Rightarrow < K.E > = < P.E > = \frac{1}{4}m\omega^2A^2$$
$$= \frac{1}{4}m\omega^2A^2 \qquad (13)$$

From equations (12) and (13), it can be seen that the total energy is equally divided into kinetic as well as potential energy. From here, it is also clear that the total energy of a simple harmonic oscillator is always conserved.

2.4 Examples of S.H.M

2.4.1 Simple Pendulum

A point mass attached to a massless, inextensible string is called a simple pendulum as in Fig.2.2. Let us find out various physical parameters for a simple pendulum.

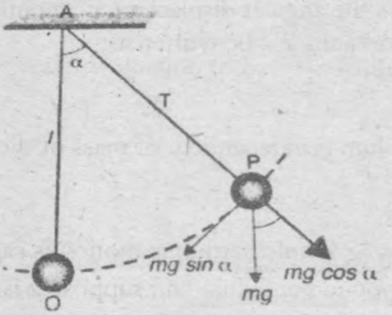

Fig. 2.2

Let a particle of mass m, at point P, is under the action of various forces:

(i) The weight mg of the bob acting vertically downward

(ii) Tension 'T', in the string along PA.

From the diagram, it can be seen that the tension is opposed by the radial component of weight, $mg\cos\alpha$, therefore, the force $T - mg\cos\alpha$ provides centripetal force for circular arc and the tangential component $mg\sin\alpha$ tends to bring the bob back to its initial position 'o'. Here, $mg\sin\alpha$, is known as restoring force and is responsible for performing oscillations of the particle. So,

$$F = -mg\sin\alpha$$

The negative sign indicates that the acceleration and the displacement are oppositely directed. If $\frac{d^2x}{dt^2}$ be the acceleration at any time t, in the direction of increasing x, then the force must be;

$$F = m\frac{d^2x}{dt^2}, \text{ so}$$

$$m\frac{d^2x}{dt^2} = -mg\sin\alpha$$

$$\frac{d^2x}{dt^2} = -g\sin\alpha$$

For the condition to perform S.H.M, the angle α should be small, so the distance x (arc) can be written as;

$x = l\alpha,$ (where, l is the radius of the circular path, i.e., length of the thread)

On differentiating two times, it gives;

$$\frac{d^2x}{dt^2} = l\frac{d^2\alpha}{dt^2}$$

$$l\frac{d^2\alpha}{dt^2} = -g\alpha \quad (\sin\alpha \approx \alpha)$$

$$\frac{d^2\alpha}{dt^2} = -\frac{g}{l}\alpha$$

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l}\alpha = 0$$

Considering, $\omega^2 = \frac{g}{l}$, we have;

$$\frac{d^2\alpha}{dt^2} + \omega^2\alpha = 0 \qquad (14)$$

The equation (14), is a second order differential equation and is the equation of the motion of simple pendulum, with solution as;

$$\boxed{\alpha = \alpha_0 \sin(\omega t + \delta)}$$

Here, δ is the initial phase and α_0 is the angular displacement amplitude.
The time period for the simple pendulum will be written as;

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \qquad (15)$$

The time period of a simple pendulum is independent of mass of the bob.

2.4.2 Compound Pendulum

An object of laminar shape, executing simple harmonic motion is called a compound pendulum. In Fig. 2.3, let 'm' is the mass of the compound pendulum and suppose it is making small oscillations under the influence of gravity, about an axis through a point O for considering the motion as S.H.M. Let G be the centre of gravity of the compound pendulum and l be the distance OG.

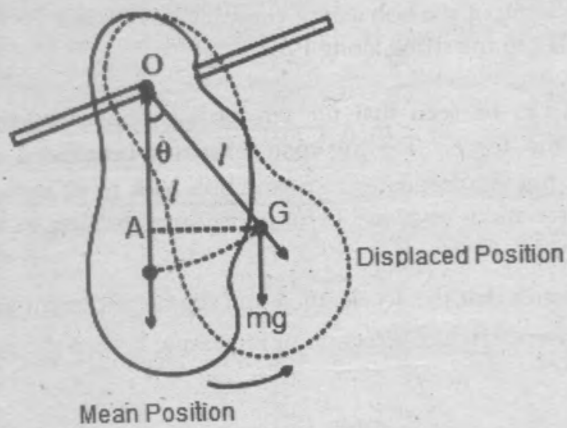

Fig. 2.3

The pendulum is displaced from its mean position so that OG makes a small angle θ with vertical line OA. In the displaced position, the weight mg of the pendulum, acting vertically downwards produces a torque which tends to bring the pendulum to its initial position and is given by;

$$\tau = mg \times GA = mgl\sin\theta$$ (force × length of arm)

As we know the torque can also be written as

$$\tau = -I\alpha$$

Where I is the moment of inertia of the pendulum about O and $\alpha = \frac{d^2\theta}{dt^2}$ is the angular acceleration.
So, we have;

$$\boxed{mgl\sin\theta = -I\left(\frac{d^2\theta}{dt^2}\right)}$$

$$I\frac{d^2\theta}{dt^2} + mgl\sin\theta = 0$$

Here θ is small for obvious reason, So, $\sin\theta = \theta$. Thus,

$$I\frac{d^2\theta}{dt^2} + mgl\,\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0 \qquad (16)$$

Considering, $\omega^2 = \frac{mgl}{I}$, equation (16) can be written as

$\frac{d^2\alpha}{dt^2} + \omega^2\alpha = 0$, which is again a standard equation of S.H.M., with the time period

$$T = \frac{2\pi}{\omega}$$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

If K is the radius of gyration of the pendulum about a parallel axis through G then using, parallel axis theorem, the moment of inertia is written as;

$$I = mK^2 + ml^2$$

Thus;

$$T = 2\pi\sqrt{\frac{K^2 + l^2}{gl}}$$

$$T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}} \qquad (17)$$

Now if we compare equations (15) and (17), we see that both the equations are similar, if we put $\frac{K^2}{l} + l$ = L, i.e. the formula for the time period for a compound pendulum, takes the form as;

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad (18)$$

Here, $L = \frac{K^2}{l} + l$ is the equivalent length of the compound pendulum. It means, if we are able to find the equivalent length 'L', of an oscillating compound pendulum, then the compound pendulum can be considered as a simple pendulum with length of the thread as 'L'. The equation (18), is a very important equation to find the earth's gravitational acceleration, g, at any place without using sophisticated instrumentation.

2.5 Applications of Simple Harmonic Oscillations

As discussed above, besides finding acceleration due to gravity, there are enormous practical examples that can be considered as the direct applications of simple harmonic motion, but due to limitations to our topics of concern. Here we will discuss a few ideal cases, where no damped or forced oscillations are involved.

2.5.1 Oscillations of L-C circuit

If a charged capacitor is attached to an inductor, the electrical energy stored in the capacitor, is converted into the magnetic energy of the inductor and again the magnetic energy converted back into the electrical energy of the capacitor due to the oscillations of the charge of the capacitor in the circuit. This conversion goes on for infinite time if there is no dissipation of energy during this process. Practically it is not possible, so this may be treated as an application of an ideal case of simple harmonic oscillation.

Suppose a charged capacitor C, is connected with an inductor L, Fig. 2.4. Let q_0 is the total charge stored in the capacitor of capacitance 'C', then the potential difference between the plates of the capacitor will be $\frac{q_0}{C}$.

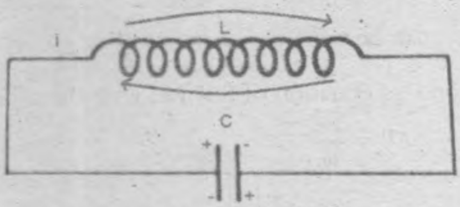

Fig. 2.4.

Now if the circuit is closed, the charge from positive plate of the capacitor starts flowing towards the negative plate of the capacitor, through the inductor 'L'. Let 'q' be the instantaneous charge and 'i' is the instantaneous current in the circuit, then the e.m.f. induced between the ends of the inductor will be $L\frac{di}{dt}$. As there is no external source, then in a close circuit, the total potential will be zero i.e.

$\frac{q}{C} + L\frac{di}{dt} = 0$, or

$\frac{q}{C} + L\frac{d^2q}{dt^2} = 0$, or

$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$, or

$$\frac{d^2q}{dt^2} + \omega^2 q = 0 \qquad (19)$$

The equation (19) is a standard differential equation of simple harmonic motion. Here, $\frac{1}{\sqrt{LC}}$ is the angular frequency of the oscillations and charge 'q', is the physical variable of the motion, having amplitude as q_0. The frequency is given as,

$\nu = \frac{\omega}{2\pi}$. i.e.

$$\nu = \frac{1}{2\pi\sqrt{LC}}$$

And the instantaneous charge is calculated by the solution of the equation (19), as;

$q = q_0 \sin(\omega t + \phi)$

And the instantaneous current will be;

$i = \frac{dq}{dt} = \omega q_0 \cos(\omega t + \phi)$

The total energy in the circuit either electrical or magnetic will be

$\frac{1}{2}CV^2 = \frac{1}{2}\frac{q_0^2}{C}$, (Purely electrical)

$\frac{1}{2}CV^2 = \frac{1}{2}Li_0^2$ (Purely magnetic)

Or

$\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$ (Instantaneous energy, electrical and magnetic)

2.6 Wave Motion

When a particle in a material medium executes S.H.M. then due to the inter molecular force the neighbouring particles of the medium, also start oscillating simple harmonically, either in the direction of motion of the particle (longitudinally) or in the perpendicular direction of the motion of the source particle (transversely) or in both the directions. It means the energy of the oscillating particle gets transferred to its neighbouring particle without the actual movement of the particle of the medium. This process of movement of the energy is termed as wave motion and this transfer/movement of energy/disturbance, in the medium is called a wave. It may be progressing through the medium, called progressive wave or redistributed among the particle in the still medium, called as standing wave. The origin of energy transfer in the medium is the inter-molecular force between the particles of the medium (in case of a mechanical wave). The mechanism of propagation of an electromagnetic wave is different from the mechanical wave.

2.7 Wave Equation

As the wave motion through the medium is associated with the oscillation of the particle at the source which is oscillating simple harmonically or otherwise, the wave equation can be derived with the help of the equation of simple harmonic motion of the source particle. As the motion of the particle at the source is governed by an oscillating external force, in the same manner the wave motion is governed by the oscillation of the particle at the source. It can be seen that the particle at the source oscillates with the frequency of governing force. Similarly, the frequency of the wave is also equal to the frequency of the particle at the source and again the wave motion is simple harmonic if the particle at the source is oscillating simple harmonically.

Now, let us write the equation of simple harmonic oscillations, performed by the particle as;

$$y = A\sin(\omega t + \delta)$$

Here y is the instantaneous displacement of the particle from the mean position, oscillating simple harmonically along y-axis. A is the displacement amplitude. ω and δ are the frequency and initial phase of the particle, respectively. Now starting the motion with initial phase as zero, the displacement of the particle at $x = 0$, will be;

$$y = A\sin\omega t$$

So, the displacement of the particle at position x from the mean position and at time 't', will be;

$$y = A\sin\omega\left(t - \frac{x}{v}\right) = A\sin\left(\omega t - \frac{\omega x}{v}\right) \qquad (20)$$

The equation (20) may be called as an equation of a wave which is associated with the displacement of the particle at the source. Here v is the velocity of the wave and x is the displacement of the wave from the origin and $\frac{\omega x}{v}$ is the phase w.r. t., the origin. From here it can also be seen that after a certain distance say X, the phase will repeat itself after one or more than one time periods. Then the equation (20) can be written as;

$$y = A\sin\omega\left(t - \frac{x}{v}\right) = A\sin\omega\left(t - \frac{x+X}{v}\right)$$

$$\frac{dy}{dt} = A\omega\cos\omega\left(t - \frac{x}{v}\right) = A\omega\cos\omega\left(t - \frac{x+X}{v}\right)$$

From here we can write

$$\omega\left(t - \frac{x}{v}\right) = \omega\left(t - \frac{x+X}{v}\right) + 2n\pi$$

Or

$$\boxed{X = \frac{v}{\omega}2n\pi}$$

For two consecutive point of same phase, $n = 1$, so

$$X = \frac{2\pi v}{\omega} = vT \qquad (21)$$

Equation (21) gives the distance travelled by the wave in a complete time period T. This length is called wavelength and denoted by λ. The equation can be re-written in its well-known form, i.e.

$$v = \nu\lambda \qquad (22)$$

The other forms of equations of wave are

$y = A\sin(\omega t - \omega \frac{x}{v}) \Rightarrow y = A\sin(\omega t - kx)$,

Where, $k = \frac{w}{v} = \frac{2\pi v}{v} = \frac{2\pi}{\lambda}$ is called wave number.

Also, another form of equation is;

$y = A\sin 2\pi\,(\frac{t}{T} - \frac{x}{\lambda})$,

2.8 Particle Velocity and Wave Velocity

The particle velocity of the particle, executing S.H.M at the source, is given by

$$\frac{dy}{dt} = A\omega\cos\omega\,(t - \frac{x}{v}) \tag{23}$$

Now, differentiating (23) w.r.t. x, we have

$$\frac{dy}{dx} = -\frac{A\omega}{v}\cos\omega(t - \frac{x}{v}) \tag{24}$$

From equations (23) and (24), the particle velocity u, can be written as

$$u = \frac{dy}{dt} = v\frac{dy}{dx} \quad \text{(Ignoring -ve sign)} \tag{25}$$

Equation (25) relates the particle velocity to the velocity of the wave associated with the particle, executing S.H.M. at the source.

2.9 General Form of Equation of Wave

Differentiating equations (23) w.r.t. t and (24) w.r.t. x, respectively, we get

$$\frac{d^2y}{dt^2} = A\omega^2\sin\omega(t - \frac{x}{v}) \tag{26}$$

$$\frac{d^2y}{dx^2} = \frac{A\omega^2}{v^2}\sin\omega(t - \frac{x}{v}) \tag{27}$$

From equation (26) and (27) we can write;

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}} \tag{28}$$

This is called general wave equation in differential form.

2.10 Superposition Principle for Waves

2.10.1 Waves in Same Direction (Progressive Wave)

"When two or more than two waves superimpose at a point in the medium, the instantaneous amplitude of the resultant waves is given by the vector sum of all the instantaneous amplitudes of all the waves". This is called superposition principle of waves. Hence, if there are 'n' number of waves having amplitudes $y_1, y_2, y_3, \ldots\ldots y_n$, then;

$y = y_1 + y_2 + y_3 + \ldots\ldots y_n$

Here, y is the instantaneous amplitude of resultant wave due to the superposition of the n waves.

For simplicity let us take the case of two waves only. Let

$y_1 = A_1\sin(\omega t + kx)$ and

$y_2 = A_2\sin(\omega t + kx + \emptyset)$,

According to the principle of superposition, the resultant will be;

$$y = A_1\sin(\omega t + kx) + A_2\sin(\omega t + kx + \emptyset) \tag{29}$$

On solving equation (29) we have;

$$y = A\sin(\omega t + kx + \delta) \tag{30}$$

Where A is the amplitude of the resultant wave and δ is the phase of the resultant wave, given as;

$y = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$

and

$$\tan\delta = \frac{A_2 \sin\phi}{A_1 + A_2\cos\phi}$$

From equation (30), it is clear that the resultant of the superposition of two simple harmonic waves travelling in the same direction will again be simple harmonic of same frequency as that of the source waves with some phase angle with the source.

2.10.2 Waves in Opposite Direction (Standing Waves)

Let us take two simple harmonic waves travelling in opposite direction to each other. i.e.

$$y_1 = A_1 \sin(\omega t + kx)$$
$$y_2 = A_2 \sin(\omega t - kx + \emptyset)$$

Using superposition principle, the resultant amplitude will be given as;

$$y = A\ \sin(\omega t + kx) + A\ \sin(\omega t - kx)$$

Here, phase difference $\emptyset$ is taken as zero, and the amplitude is taken same as a special case.
On simplification we have,

$$y = (2A\cos kx)\sin\omega t \qquad (31)$$

The equation (31) is the equation of standing wave, in which the amplitude i.e., $2A\cos kx$, of resultant waves, also varies sinusoidally. The points of maximum amplitude are called antinode and the points of minimum amplitude are termed as nodes.

→ sin graph.

2.11 Category of waves

Fundamentally, waves can be divided into two categories. One, which needs a material medium, called mechanical waves and the other which do not need any material medium for their propagation, are called electromagnetic waves.

2.12 Electromagnetic Field Theory

Introduction

A static charge produces static electric field around it and the theory is called electrostatic field theory. The theory behind uniformly moving charge is called magnetostatic field theory. But if the charge is accelerating or oscillating, the theory, which explain this phenomenon, is termed as electromagnetic field theory. An accelerating or oscillating charge produces oscillating electric field and the oscillating electric field produces, oscillating magnetic field. We have the equation of oscillating electric field as;

$$E_y = E_{y_0}\sin(\omega t + kx) \qquad (32)$$

And the equation of corresponding magnetic field will be

$$B_z = B_{z_0}\sin(\omega t + kx) \qquad (33)$$

Later in this chapter, it will be proved that the in case of an electromagnetic wave, the oscillations of electric and magnetic field vectors are mutually perpendicular to each other and also perpendicular to the propagation of the wave.

Maxwell derived and developed the fundamental equations of electromagnetic field theory and ultimately the theory of electromagnetic waves. He also predicted that light is an electromagnetic wave having electric and magnetic field vectors, oscillating perpendicular to each other and also perpendicular to the direction of propagation of electromagnetic energy.

2.13 Maxwell's Equations

A set of four equations of electromagnetism (with some modifications), are called the Maxwell's equations of electromagnetic field theory. Maxwell rigorously analysed four fundamental equations of electromagnetism, i.e., of electric field, magnetic field and electromagnetic induction and came to the conclusion that these equations are interlinked although, they are written separately for electrostatics, magnetostatics or electromagnetic induction. After doing a lot of mathematical operations on these equations, he also derived nearly all mathematical formulations for electromagnetic field theory. Later, this theory led to the theory of electromagnetic waves. Heinrich Hertz was the first scientist who had successfully generated the electromagnetic waves.

After doing mathematical operations on the four fundamental equations of electrostatics, magnetism and electromagnetism, Maxwell found very interesting physical interpretation of these equations in integral as well as in the differential form. He concluded that these equations are not ideal rather they are interlinked to each other and lead to the fundamental foundation of the electromagnetic wave theory.

These four equations of electromagnetism can be converted into two, using the concept of electromagnetic potential. In relativistic physics and using the concept of electromagnetic four-vector potential, there comes out to be only one equation of electromagnetism. Later, this will be explained in detail, in the chapter of "special theory of relativity", under the topic of 'invariance of Maxwell's equations under Lorentz's Transformation'.

2.13.1 Modifications of the Equations of Electromagnetism

Here are the four equations of electromagnetism as;

(i) $\oint E.\, ds = \frac{q}{\epsilon_0}$ Gauss's Law of Electrostatics

(ii) $\oint B.\, ds = 0$ Gauss's Law of Magnetostatics

(iii) $\xi = -\frac{\partial \phi_B}{\partial t}$ Faraday's Law of electromagnetic Induction

(iv) $\oint B.\, dl = \mu_0\, I$ Ampere's Circuital Law

Initially, Maxwell checked the consistency of each equation in every situation and eventually found that the Ampere's circuital law to be inconsistent with an AC circuit, if a capacitor is connected into an AC circuit. Then, he modified the equation, made it consistent and put all the four equations of electromagnetism, under one umbrella. All together they are called Maxwell's equations of electromagnetism.

2.14 Inconsistency of Ampere's Circuital Law — Maxwell's Displacement Current

According to the Ampere's circuital law, the line integral of the magnetic induction B around any closed path around a current carrying conductor is μ_0 times the current flowing through the conductor and written as;

$$\oint B.\, dl = \mu_0\, I \qquad (34)$$

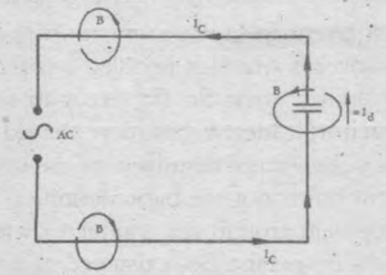

Fig.2.5

As there is no current flowing between the plates of the capacitor, so in the loop over the free space between the plates of the capacitor, the Ampere's law gives;

$$\oint B.\,dl = \mu_0\ I_C = 0 \qquad (35)$$

So, from (35), it is clear that Ampere's circuital law is not consistent for a circuit in which a capacitor is maintained with alternating electric field, as there is not current flowing in the free space between the plates of the capacitor, Fig. 2.5. To remove this inconsistency, the concept of Maxwell's displacement current was introduced. The expression for displacement current can be derived as follows;

Let an electric circuit contains a capacitor, connected to an AC source. For any instantaneous value of surface charge density σ on the plates, the electric field between the plates is given by;

$E = \frac{\sigma}{\epsilon_0}$, where $\sigma = \frac{q}{A}$,

q is the instantaneous charge and A is the area of the plates and ϵ_0 is the permittivity of the free space between the plates.

Thus, $E = \frac{q}{\epsilon_0 A}$

As the capacitor is connected to an AC source, the time variation of the electric field between the plates is given by;

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{\partial q}{\partial t},$$

Or

$$\frac{\partial q}{\partial t} = \epsilon_0 A\ \frac{\partial E}{\partial t} \qquad (36)$$

The equation (36), has Ampere as the unit on both the sides, so it may be taken as some sort of current, called Maxwell's displacement current.

On rearranging, equation (36) we have;

$$\frac{\partial q}{\partial t} = A\ \frac{\partial \epsilon_0 E}{\partial t} = A\frac{\partial D}{\partial t} = I_d \quad \text{, (where } D = \epsilon_0 E \text{ is electric displacement vector)} \qquad (37)$$

Here, I_d may be called as Maxwell's displacement current.

The equation (37) is the expression for Maxwell's Displacement current. As $D = \epsilon E$ is known as electric displacement vector and this current is proportional to time rate of electric displacement vector, that's why it is called Maxwell's displacement current.

For the consistency of Ampere's circuital law for direct as well as AC circuit, Maxwell modified, the equation (34) as;

$$\oint B.\,dl = \mu_0\,(I_C + I_d) \qquad (38)$$

Or

$$\oint H.\,dl = (I_C + I_d)$$

Or

$$\oint H.\,dl = I_C + I_d = \oint (J_C + J_d).\,ds \qquad (39)$$

The equation (38) or (39), is called modified Ampere's circuital law which is consistent for DC and AC circuits.

*Maxwell's displacement current is different from the conventional current as there is no flow of charge in this case. Now the question that arises is why this is called a current? The explanation to this is that the origin of any kind of magnetic field is current. So, if there is an alternating magnetic field in the free space between the plates of the capacitor, it means that there should be some sort of current. Actually, the flow of charge per unit time is not the proper definition of the current. This can be used to measure the amount of the conduction current but is not the basic definition of current. Whenever charge flow in the conductors, there is a magnetic field around the conductor which means, even in the case of the conduction current, magnetic field always persist. So, existence of alternating magnetic field in the free space between the two plates of conductor, there is always a Maxwell displacement current, whether it is a capacitor or an electric dipole. *

2.15 Maxwell's Equations of Electromagnetism – Foundation of Electromagnetic Waves

Four equations of electrostatics, magnetism, electromagnetic induction with modified Ampere's circuital law, are termed as the Maxwell's equations of electromagnetism.

(i) $\oint E.ds = \frac{q}{\epsilon_0}$	Gauss's Law of Electrostatics
(ii) $\oint B.ds = 0$	Gauss's Law of Magnetostatics
(iii $\xi = \oint E.dl = -\frac{\partial\phi_B}{\partial t}$	Faraday's Law of electromagnetic Induction
(iv) $\oint H.dl = I + I_d = \oint(J + J_d).ds$	Modified Ampere's Circuital Law

These equations are called Maxwell's equations of electromagnetic field in integral from.
Maxwell performed rigorous mathematical operations on these equations and converted these equations into differential form. This helped in interpreting the physical significance of the Maxwell's equations in a very different and an exhaustive manner. He reached on the conclusion that these equations cannot be treated as separated from each other but are very well connected to each other. In the combined form, the physical significance of these equations also gives the foundation of other important physical quantity called as electromagnetic waves.

To convert these equations in differential form, a mathematical operator called Del (∇) is very much needed. Without this operator it is impossible to convert integral form into differential form. So let us have a brief description and discussion about this operator after which it will be very easy to convert the integral form of four equations of electromagnetism into differential form. The physical interpretation of the equations will reveal the hidden concepts of electromagnetic waves in it and finally the foundation for the fundamental equations of electromagnetic waves could be understood.

2.16 Del Operator

The operator, del (or nabla) ' ∇ ' is a mathematical operator, defined as (in rectangular coordinates)

$$\nabla = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

Note: Sometimes the 'del' operator is represented with a vector sign, but it should be kept in mind that 'del' is not a vector quantity; it is an operator which operates in a way similar to a vector.
Some of the operations of $\nabla = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$ are defined as follows: -

2.16.1 Scalar Multiplication of ∇: Gradient of a Vector

The gradient of a scalar point function f(x, y, z) , is represented by ∇f or grad(f) and can be obtained by operating the del operator '∇' on the function f. Thus, the gradient of f is given by,

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) f = \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right) \qquad (9)$$

Therefore, gradient of a scalar function is a quantity whose x, y and z components are respectively the partial derivatives of f with respect to x, y and z.

2.16.2 Physical Interpretation of Gradient: If f(x, y, z) is a scalar function at a point (x, y, z)then on changing the x, y and z coordinates by dx, dy, dz the differential change in the scalar function 'f' can be written as,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \nabla f . d\vec{r}$$

Where, $d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

So, we can write,

$$df = |\nabla f||d\vec{r}|\cos\theta$$

Where θ is the angle between $d\vec{r}$ and ∇f.

From the above equations, we can see that for a fixed value of dr (magnitude of $d\vec{r}$), df will be maximum when $\cos\theta = 1$, i.e., when $d\vec{r}$ is in the direction of ∇f. Therefore, df is maximum in the direction ∇f, so we can say that, the gradient ∇f of a scalar function 'f', points in the direction of maximum increase of the function. Further if $(df)_{max}$ is the maximum change in the function for a fixed value of dr, we can write,

$$(df)_{max} = |\nabla f||d\vec{r}| = |\nabla f| dr$$

Thus, the magnitude of gradient of f is given by,

$$|\nabla f| = \frac{(df)_{max}}{dr} \qquad (40)$$

The gradient of a scalar field thus is a vector field which points in the direction of the greatest increase of the scalar field, and whose magnitude is equal to the greatest rate of change of the function with distance.

The gradient can also be used to measure how a scalar field changes in other direction, rather than just the direction of greatest change, by taking its dot product with the unit vector along that direction. Another important property of gradient of a function 'f' is that it is normal to the surface over which f is constant.

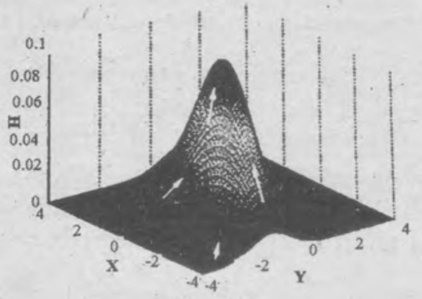

Fig. 2.6

Illustration: Consider that the temperature in a room at different points (x, y, z) of the room is given by (x, y, z). The gradient at each point in the room will show the direction in which the temperature changes most quickly. The magnitude of the gradient determines that how fast the temperature changes in that direction. Further consider a hill whose height at a point (x, y) is H(x, y). Then, the gradient of H at a point is a vector pointing in the direction of the steepest slope at that point (as shown in Fig. 2.6). The steepness of the slope at that point is given by the magnitude of the gradient of H.

2.16.3 Scalar Product of ∇ : Divergence

The divergence of a vector point function $\vec{f}(x,y,z)= f_x\hat{x} + f_y\hat{y} + f_z\hat{z}$ {where f_x, f_y and f_z are respectively the x, y and z components of the vector field $\vec{f}$ at a point having coordinates x, y, z is represented by div($\vec{f}$)or $\nabla.\vec{f}$ and can be obtained by operating the del operator (∇) on the vector field through dot product. Thus, the divergence of the vector field $\vec{f}$ is given by,

$$\nabla.\vec{f} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right).\vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

The divergence of a vector field thus is a scalar quantity.

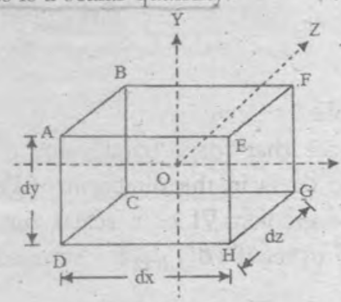

Fig. 2.7

2.16.4 Physical Interpretation of Divergence

Consider an infinitesimal volume with sides dx, dy and dz as shown in Fig. 2.7. If a vector field is $\vec{f}(x,y,z) = f_x\hat{x} + f_y\hat{y} + f_z\hat{z}$ in the middle of the volume at the point 'O', having coordinates (x, y, z), then the x-component of $\vec{f}$ at the middle of the face ABCD, can be of x-component of $\vec{f}$, same over the face thus, the flux of the field $\vec{f}$ through the face ABCD can be written as $\left(f_x - \frac{1}{2}\frac{\partial f_x}{\partial x}dx\right)dxdz$ {the y and z component of $\vec{f}$ do not contribute anything to the flux since these components are perpendicular to the normal to the surface }. Similarly, the x-component of $\vec{f}$ at the face EFGH can be written as $\vec{f}+\frac{1}{2}\frac{\partial f_x}{\partial x}dx$ and the flux through the face EFGH can be written as $\left(\vec{f} + \frac{1}{2}\frac{\partial f_x}{\partial x}dx\right)dydz$. Therefore, the net amount of flux of the field that is diverging from the face EFGH, can be written as , $\left(\vec{f} + \frac{1}{2}\frac{\partial f_x}{\partial x}dx\right)dydz - \left(f_x - \frac{1}{2}\frac{\partial f_x}{\partial x}dx\right)dxdz = \frac{\partial f_x}{\partial x}dxdydz$. Similarly, the flux diverging from the faces ABFE and BCGF, can be written as $\frac{\partial f_y}{\partial y}dxdydz$ and $\frac{\partial f_z}{\partial z}dxdydz$, respectively. Thus, the total flux diverging from the infinitesimal volume is given by, $\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right)dxdydz$. The amount of flux diverging through an infinitesimal volume per unit of its volume is given by $\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$, which, as we know, is equal to the divergence of the field. We can now define the divergence of a vector field as the net amount of flux of the field diverging through an infinitesimal volume per unit of its volume or we can say that the divergence basically represents the flux generation per unit volume at each point of the field. If 'S' is the surface that bounds a volume 'V' we can alternatively write the divergence of a vector field $\vec{f}$ as,

$$\nabla.\vec{f} = V\rightarrow 0\,\frac{\oint_s \vec{f}.d\vec{S}}{V} \tag{41}$$

The divergence of a vector field at a point tells that how much the vector field diverges or spreads out from that point. The point from where the field lines diverge, can be called a source of the field while the point where the field lines converge can be called a sink of the field, so a point of positive divergence is a source and a point of negative divergence is a sink. If the divergence of a vector field is non-zero, then there must be a source or sink of the field.

2.16.5 Solenoidal Vector Fields

The vector fields that have zero divergence everywhere are called solenoidal or divergence-less vector fields.

2.16.6 Gauss' Divergence Theorem

From the above discussion, we have clearly understood that the amount of flux diverging through an infinitesimal volume per unit of its volume is also called divergence of the field and if 'S' is the surface that bounds a volume 'V' we can also write the divergence of a vector field $\vec{A}$ as,

$$\nabla.\vec{A} = V \rightarrow 0\,\frac{\oint_s \vec{A}.d\vec{S}}{V}$$

Thus, we can write,

$$\oint_s \vec{A}.\,d\vec{S} = \int_V \left(\nabla.\vec{A}\right)dV \tag{42}$$

In other words, the total inward or outward flux of a vector point function through a closed surface is equal to the divergence of the vector point function, from the volume enclosed by that surface. The equation (42) is known as Gauss' divergence theorem of the vector fields. In simple way it is used to convert any surface integral into volume integral enclosed by the surface.

2.16.7 Vector Product of ∇: Curl

The curl of a vector point function $\vec{f}(x,y,z)=f_x\hat{x} + f_y\hat{y} + f_z\hat{z}$ (where f_x, f_y and f_z respectively are the x, y and z components of the vector field $\vec{f}$ at a point having coordinates x,y,z) is represented by $\nabla\times\vec{f}$ and can be obtained by operating the del operator (∇) on the vector field through cross product. Thus, the curl of $\vec{f}$ is given by,

$$\nabla\times\vec{f} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\times\vec{f}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \hat{x}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{dz}\right) + \hat{y}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{dx}\right) + \hat{z}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{dy}\right) \tag{43}$$

Therefore, the curl of a vector field is a vector quantity.

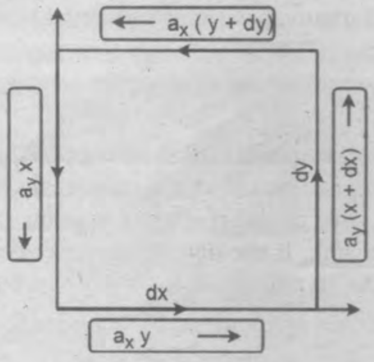

Fig. 2.8

Fig. 2.9

2.16.8 Physical Interpretation of Curl

To see, what the curl of a vector means, consider a rectangular element of length dx and width dy in the region of a vector field $\vec{a}$ as shown in Fig. 2.8. We can write the x-components of the field $\vec{a}$ at the bottom and at the top of the element as,

$a_x(y)$ and $a_x(y+dy)= a_x(y) + \frac{\partial a_x}{\partial y}dy$.

Similarly, we can write, the y-components of the field $\vec{a}$ at the left and right side of the element as,

$a_y(x)$ and $a_y(x+dx) = a_y(x)+ \frac{\partial a_y}{\partial x}dx$.

Now working round the clockwise sense, the circulation of the vector around the element can be written as $a_x(y)dx + a_y(x+dx) - a_x(y+dy)dx - a_y(x)dy$, where the minus signs in the last two terms arise because there the path is opposite to the direction of the field.

The circulation of the field $\vec{a}$ around the element can be written as, $a_x(y)dx+\left\{a_y(x) + \frac{\partial a_y}{\partial x}dx\right\}dy -$
$\left\{a_x(y) + \frac{\partial a_x}{\partial y}dy\right\}dx - a_y(x)dy$

Or, $\frac{\partial a_y}{\partial x}dxdy - \frac{\partial a_x}{\partial y}dydx= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)dxdy =(\nabla \times \vec{a}).d\vec{S}$ (Where $d\vec{S} = dx\,dy\,\hat{z}$)

The curl of a vector field is thus a measure of the circulation of the vector field per unit area, i.e. also called vertically of the field.

Now the circulation of the vector field around any closed curve can be written as

$\oint_l \vec{a}.d\vec{l} = \oint_l dC= \int_S(\nabla \times \vec{a}).d\vec{S}$

If the integral of a vector field around a closed loop is not zero, then it implies that there is some circulation of the vector field around the loop i.e., a non–zero curl implies that there is a circulation of the vector field. However, if the curl of the vector is zero everywhere, then there cannot be any circulation of the vector field, anywhere in space. Hence, the name 'curl' is given for $\nabla \times \vec{a}$.

The curl of a vector field tells us about the circulation of rotation per unit area the field has at any point.

The magnitude of the curl tells us how much rotation there is and its direction tells us, by the right – hand rule (four fingers of the right hand are curled in the direction of the vector field, then the thumb points in the direction of the rotation) that about which axis the field is rotating. That is why; curl of a vector field is also called rot (short for rotor).

Illustration: Consider a vector a vector field $\vec{f} = y\hat{x} - x\hat{y}$. This vector field is shown in Fig. 2.9.

From Figure, it can be seen that $\vec{f}$ is circulating around the point 'O'. Using the right – hand rule, we expect the curl to be into the page or in the negative z-direction.

The curl of $\vec{f}$ is given by

$$\nabla \times \vec{f} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \times \hat{f}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = \hat{z}\left(-\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right) = -2\hat{z}$$

It, indeed is in the negative z-direction, as expected. In this case, the curl is actually a constant, irrespective of position. The amount of rotation in the above vector field is the same at any point (x, y).

2.16.9 Irrotational Vector Fields: The vector fields which have zero curl everywhere are called irrotational or curl-less vector fields.

2.16.10 Stoke's Theorem (line to surface integral)

For any vector field $\vec{A}$, we can write as,

$$\int_S (\nabla \times \vec{A}).d\vec{S} = \oint_l \vec{A}.\,d\vec{l} \qquad (44)$$

This equation is called Stokes' theorem.

2.17 Equation of Continuity – Conservation of Charge

The flow of conduction current in a circuit can be written as

$$I = -\frac{\partial q}{\partial t} = -\frac{\partial}{\partial t}\int \underbrace{\rho}_{\text{volume charge density}}\, dv \qquad (45)$$

We also have;

$$I = \int J.\,ds \qquad (46)$$

From (45) and (46), we have;

$$\int J.\,ds = -\frac{\partial}{\partial t}\int \rho\, dv$$

$$\int J.\,ds = \int \frac{\partial \rho}{\partial t}\, dv$$

Using Gauss's Div. Theorem

$$\int \nabla.J\, dv = -\int \frac{\partial \rho}{\partial t}\, dv$$

$$\int (\nabla.J + \frac{\partial \rho}{\partial t})dv = 0$$

As the integral is arbitrary, so the integrand vanishes to zero. i.e.

$$\boxed{\nabla.J + \frac{\partial \rho}{\partial t} = 0} \qquad (47)$$

This equation is called the equation of continuity.

2.17.1 Physical Interpretation of Equation of Continuity

The equation (47) shows that if the divergence of conduction current density is zero, i.e., $\nabla.J = 0$ then volume charge density ρ is constant; it means charge is static or charge is conserved. If time rate of change of charge density is zero, then the conduction current density will not be originated from any source. It will flow in a loop like magnetic field lines of force. It means the equation of continuity tells whether the circuit is maintained by AC or DC.

2.18 Maxwell's Equations in Integral Form

The set of four equations of electromagnetism are known as Maxwell's equations in integral form.

(i) $\oint \mathbf{E}.\,d\mathbf{S} = \frac{q}{\epsilon_0}$ Gauss's Law of Electrostatics

(ii) $\oint \mathbf{B}.\,d\mathbf{S} = 0$ Gauss's Law of Magnetostatics

(iii) $\oint \mathbf{E}.\,d\mathbf{l} = -\frac{\partial \phi_B}{\partial t}$ Faraday's Law of EM Induction

(iv) $\oint \mathbf{H}.\,d\mathbf{l} = I + I_d = \oint (J + J_d).\,d\mathbf{S}$ Modified Ampere's Circuital Law

2.18.1 Physical Significance

The physical significance of these is given as;

(i) This equation is Gauss's law in electrostatics which states that, the total outward electric flux over any closed surface is equal to total charge enclosed within volume surrounded by the surface.

(ii) The total outward flux of magnetic flux B through any closed surface 'S' is equal to zero i.e., monopole does not exist.

(iii) If an electric circuit is placed in a magnetic field and the magnetic flux close to a circuit changes, an electromotive force (e.m.f) is induced in the circuit. The magnitude of which is proportional to the rate of change of flux and the direction of the induced e.m.f is given by Lenz's law which states that "the direction of the induced e.m.f. is such that the magnetic flux associated with the current generated by it opposes the original change of flux causing e.m.f".

(iv)This equation has been derived from ampere's law in circuital form for a magnetic field accompanying an electric current. This law states that the line integral of magnetic field around a closed path is equal to the total current crossing any surface bounded by the line integral path.

Most of these equations are more or less as it is, as given by several researchers and does not lead to any specific correlations between them. But when Maxwell converted these equations into differential form, a strong inter-connectivity of these equations come into picture. He also showed that these equations, not only are coupled to each other but are the basic equations of electromagnetic field theory. These equations can then be used further to derive the equations of electromagnetic waves.

2.18.2 Conversion of Maxwell's equations - Integral Form into Differential Form

1. Conversion of Gauss law of electrostatics

Here we have, $\oint \mathbf{E}.\,d\mathbf{S} = \frac{q}{\epsilon_0}$

If ρ is the charge density, then $\rho\, dV$ is the charge in volume element dV which contributes $\frac{1}{\epsilon_0}\rho\, dV$ to the surface integral. If the surface 's' encloses a volume V then

$$\oint \mathbf{E}.\,d\mathbf{S} = \frac{1}{\epsilon_0}\oint \rho\, dV$$

But if the surface is bounded volume V of a dielectric, then the total charge must include both the free and polarisation charges. Thus, the total charge density at a point in a small volume dV, should be $(\rho + \rho')$ where ρ' is polarisation charge density and ρ is free charge density. Thus, the above equation can be expressed as

$$\oint \mathbf{E}.\,d\mathbf{S} = \frac{1}{\epsilon_0}\oint (\rho + \rho')\, dV$$

If P is polarisation i.e., electric dipole moment per unit volume then div P or $\nabla \cdot P$ is the amount of polarised charge in a unit volume. As the polarised charge is reverse in nature with respect to real charge, thus

$$\rho' = -\nabla \cdot P = -\text{div } P$$

$$\oint \mathbf{E}.d\mathbf{S} = \frac{1}{\epsilon_0}\oint(\rho - \text{div } P)\, dV$$

$$\oint \mathbf{E}.d\mathbf{S} = \frac{1}{\epsilon_0}\oint \rho\, dV - \oint \text{div } P\, dV$$

Applying Gauss Divergence Theorem to change surface integral to volume integral, we get

$$\oint \epsilon_0\mathbf{E}.d\mathbf{S} = \oint \text{div}\,(\epsilon_0\mathbf{E})dV = \oint \rho dV - \oint \text{div } P\, dV$$

$$\oint \text{div}(\epsilon_0\mathbf{E} + \mathbf{P})dV = \oint \rho\, dV$$

The quantity, $\epsilon_0\mathbf{E} + \mathbf{P} = \mathbf{D}$, called electric displacement vector,

$$\oint \text{div } \mathbf{D}\, dV = \oint \rho\, dV$$

$$\oint \text{div}(\mathbf{D} - \rho)dV = 0$$

Since the equation is true for arbitrary volume, the integrand must vanish,

Thus

$$\text{div}(\mathbf{D} - \rho) = 0$$

$$\text{div } \mathbf{D} = \rho$$

$$\nabla.\mathbf{D} = \rho \qquad (48)$$

Now we have, $\mathbf{D} = \epsilon\mathbf{E}$ where ϵ is the permittivity of the dielectric medium.

In free space, $\mathbf{D} = \epsilon_0\mathbf{E}$, ϵ_0 is the permittivity of free space.

Thus,

$$\boxed{\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0}} \qquad (49)$$

Equations (48) or (49) is first equation in differential form.

2. Conversion of Gauss law of Magnetostatics

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any arbitrary close surface is exactly the same as leaving it. This means the total outward flux of magnetic induction B through any closed surface 's' is equal to zero i.e.

$$\oint \mathbf{B}.d\mathbf{S} = 0$$

Transforming the surface integral into volume integral, using Gauss divergence theorem, we get

$$\oint \nabla.\mathbf{B}\, dV = 0$$

The integrand should vanish for the surface boundary as the volume is arbitrary i.e.

$$\text{div } \mathbf{B} = 0$$

or

$$\boxed{\nabla.\mathbf{B} = 0} \qquad (50)$$

This is differential form of second equation.

3. Conversion of Faradays law of electromagnetic induction and Lenz law

We know according to Faraday's law of electromagnetic induction, the induced e.m.f is given by;

$\xi = -\frac{\partial \phi_B}{dt}$,

i.e., time rate of change of magnetic flux produces an induced e.m.f. and it is produced to oppose this change. Also, the induced e.m.f. is equal to the work done per unit charge and can be written as the line integral of the induced electric field E around the circuit i.e.

$e = \oint \mathbf{E}.\,\mathbf{dl}$

And the rate of change of magnetic flux through the circuit is equal to

$$\frac{\partial \phi_B}{dt} = \oint \frac{\partial(\mathbf{B}.\,\mathbf{dS})}{\partial t}$$

Where the integral is taken over any area 's' bounded by the circuit. Since the surface ds does not change its shape or position with time, we can write the above equation as:

$$\oint \mathbf{E}.\,\mathbf{dl} = -\oint \frac{\partial \mathbf{B}}{\partial t}.\,\mathbf{dS}$$

The total time derivative has been changed to partial derivative as we are only concerned with the changes in the field **B** with time at a fixed position of the elemental area. it signifies that "the electromotive force around a closed path is equal to the time derivative of the magnetic displacement through any surface bounded by the path.

Using Stokes theorem, the line integral can be transformed into the surface integral i.e.

$\oint \mathbf{E}.\,\mathbf{dl} = \oint \text{curl}\,\mathbf{E}.\,\mathbf{dS} = -\oint \frac{\partial \mathbf{B}}{\partial t}.\,\mathbf{dS}$

Or $\oint(\text{curl}\,\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}).\,\mathbf{dS} = 0$

This equation must hold for any arbitrary surface in the field, thus the integrand should vanish i.e.

$\text{curl}\,\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

Or

$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$ (51)

This is called differential form of Maxwell third equation.

4. Conversion of Ampere's Circuital Law

Ampere's law states that the line integral of magnetic field intensity around a closed path is equal to the total current crossing any surface bounded by the line integral path i.e.

$$\oint \mathbf{H}.\,\mathbf{dl} = \mathrm{I} = \oint \mathbf{J}.\,\mathbf{dS}$$

Now changing the line integral into the surface integral by the use of Stoke's theorem,

$\oint \mathbf{H}.\,\mathbf{dl} = \oint \text{curl}\,\mathbf{H}.\,\mathbf{dS} = \oint \mathbf{J}.\,\mathbf{dS}$ Or

$\oint(\nabla \times \mathbf{H} - \mathbf{J}).\,\mathbf{dS} = 0$

For an arbitrary surface, the integrand should vanish, thus

$\nabla \times \mathbf{H} - \mathbf{J} = 0$, Or

$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$, Or

$\text{curl}\,\mathbf{H} = \mathbf{J}$ (52)

This is the differential form of Ampere's law for steady current only.

2.18.2.1 Consistency of Ampere's Law for Time Varying Fields on the basis of Maxwell's equations

This is another way to check the consistency of Ampere's law for DC and AC fields. As we can see from the equation of continuity, the divergence of current density can reveal whether the field is DC or AC. So, taking the divergence of equation (23), we have;

$\nabla.(\nabla \times \mathbf{H}) = \nabla.\mathbf{J}$

$(\nabla.\nabla)\mathbf{H} - \nabla(\nabla.\mathbf{H}) = \nabla.\mathbf{J}$, Or

$0 = \nabla.\mathbf{J}$

It means the divergence of current density is zero.

According to the equation of continuity;

$$\nabla.\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \qquad (53)$$

If $\nabla.\mathbf{J} = 0$, then ρ is constant i.e., volume charge density is not function of time. This shows that the Ampere's circuital is true only for the circuits in which charge does not change with time i.e., it is true for direct fields only.

To make it consistent for a circuit having connected a capacitor or an electric dipole maintained by alternating electric filed, Maxwell added some quantity in the right-hand side of the equation (52) let it be $\mathbf{J}'$. So, the Ampere's equation is modified as;

$$\text{curl}\,\mathbf{H} = \mathbf{J} + \mathbf{J}' \qquad (54)$$

As, the addition is in current density so $\mathbf{J}'$is also some kind of current density. Now the divergence of equation (54) gives as;

$\nabla.(\text{curl}\,\mathbf{H}) = \nabla.(\mathbf{J} + \mathbf{J}')$

Or

$$\nabla.\mathbf{J} + \nabla.\mathbf{J}' = 0 \qquad (55)$$

Using equation (53) and (55);

$$\nabla.\mathbf{J}' = \frac{\partial\rho}{\partial t} \qquad (56)$$

From (48) and (56);

$\nabla.\mathbf{J}' = \frac{\partial(\nabla.\mathbf{D})}{\partial t}$

Or

$\nabla.\mathbf{J}' = \nabla.\frac{\partial\mathbf{D}}{\partial t}$

Or

$\mathbf{J}' = \frac{\partial\mathbf{D}}{\partial t}$ or

$$I' = A\frac{\partial\mathbf{D}}{\partial t} \qquad (57)$$

As the current given by the equation (57), is directly proportional to the time rate of change of electric displacement vector $\mathbf{D}$, this may be called as displacement current or Maxwell's displacement current.

In space, the magnitude of Maxwell's displacement current can be calculated as;

$I_d = A\epsilon_0\frac{\partial E}{\partial t}$

Where A is the area, ϵ_0 is the permittivity of free space where the electric field is changing with time.

So, the differential form of modified Ampere's law can be written as the Maxwell's fourth equation of electromagnetic wave in differential form. That is;

$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}' \qquad (58)$$

Here, it is very important to note that the equation (58) is true for a DC circuit as well as AC circuit having a capacitor like component or an electric dipole operated by an alternating electric field (Fig.2.5). In the portion of the circuit containing conducting wires only, i.e., in the portion of circuit containing conduction current, only the conduction current density comes into picture and the displacement current density will be zero. But in the portion of the circuit where there is free space, for example the space between the plates of a capacitor or space between the two poles of an electric dipole operated by alternating electric field, only the displacement current density will come into picture and at that place the conduction current density will be zero. From this, it may be concluded that in a circuit shown in Fig. 2.5, the magnitude of conduction current and Maxwell current should be equal as it may be considered that the current in whole of the circuit remains the same. In the wires, it is conduction current I_C but in the free space it is Maxwell displacement current I_d.

2.18.3 Physical Significance of Maxwell Equations: Differential Form

Maxwell equations in differential form are given as;

$$\left.\begin{aligned} &\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \text{ or } \nabla.\mathbf{D} = \rho && \text{(i)} \\ &\nabla.\mathbf{B} = 0 && \text{(ii)} \\ &\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \mu_0 \frac{\partial \mathbf{H}}{\partial t} && \text{(iii)} \\ &\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && \text{(iv)} \end{aligned}\right\} \qquad (59)$$

Physical Significance

$\Delta \cdot D$ ρ

1. The divergence of electric displacement vector gives the volume charge density. If it is positive than the position of charge is the origin of source of the electric field, if it is negative, the electric field is sinking at the position of charge and if it is zero then the electric field entering from one side is the same as coming out from the other side. But if the value of charge divergence is oscillating between a maximum and minimum value then the charge density is changing with time i.e., there is oscillating electric field at that point.
2. The divergence of magnetic field at any point is always zero. This verifies the concept that monopole does not exist.
3. This equation shows that curling effect of electric field give us the changing magnetic field with time and also perpendicular to the electric field. This also shows that space varying electric field gives time varying magnetic field.
4. This equation is one of the most important contributions of Maxwell. If the Ampere's law was not modified by Maxwell, the equation 59(iv) shows that space varying magnetic field gives conduction current density and there is no time varying electric field. After the introduction of displacement current, it is possible that space varying magnetic field can generate time varying electric field in the free space. And that is the basic concept of electromagnetic waves.

The physical significance of the equation 59(iii) and 59(iv) is the foundation for the development of the theory of electromagnetic waves. As 59(iii) and 59(iv) equations show vice a versa effects of varying electric and magnetic fields i.e., if magnetic field vary at any point in space, it propagates varying electric field in the space around it and if electric field vary with time at any point, the varying magnetic field propagates in the space around it. It means that propagation of varying electric and magnetic field is due to the time variation of magnetic and electric field respectively. As the propagation of electromagnetic wave in any medium is the propagation of varying electric and magnetic field in the medium, so we can conclude that these four equations of Maxwell are the fundamental equations which can be used for further derivation of equations of electromagnetic waves in different media. Now let us derive the equations of plane electromagnetic waves in various media.

2.19 Work done by the electromagnetic field

The propagation of electromagnetic waves i.e., oscillating electric and magnetic fields, in free space or in any medium is possible on account of the energy carried by with the wave. This is generated due to the work done by the electromagnetic field.

2.19.1 Energy of Electromagnetic Field

Like all other types of waves, electromagnetic waves also transport energy as they travel from one place to other. As, the energy, in an electromagnetic system is due to the oscillating electric and magnetic fields. So, the total energy associated with electromagnetic wave will be;

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

We have, $B = \frac{E}{C}$, so

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\frac{E^2}{C^2},$$

Using $C = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 E^2 = \epsilon_0 E^2$

This also proves that the electric field energy is equal to magnetic field energy in an electromagnetic wave.

2.20 Propagation of Electromagnetic Field Energy – Poynting Vector

Now we have to find out, how and in which direction the electromagnetic field energy propagates. Let a plane electromagnetic wave of cross-sectional area ds, travelling with velocity C. The volume, crossing in unit time will be C.d**S**. So, the energy density passing with this region per second is;

$$d\mathbf{U} = u\,C\,d\mathbf{S} = \epsilon_0 E^2 C\,d\mathbf{S}$$

So, the energy flowing per time per unit cross-sectional area will be;

$$\mathbf{S} = \frac{d\mathbf{U}}{d\mathbf{S}} = \epsilon_0 E^2 C = uC \qquad (60)$$

$$= \epsilon_0 E^2 . \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$= E^2 \sqrt{\frac{\epsilon_0}{\mu_0}}$$

$$= E^2 \frac{1}{\mu_0}\sqrt{\mu_0 \epsilon_0}$$

$$= \frac{EE}{C}\frac{1}{\mu_0} \qquad (\text{since } \frac{E}{B} = C)$$

So, $\boxed{S = \frac{1}{\mu_0}EB}$

As we know in an electromagnetic wave, electric and magnetic field vectors are perpendicular to each other, so the above expression, in vector form can be written as;

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \qquad (61)$$

Here **S** is a vector form of flow of electromagnetic energy per unit cross-sectional area per unit time, which is perpendicular to both the electric and magnetic field vectors, called Poynting vector, also can be written as;

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad (62) \quad H = \frac{B}{\mu_0}$$

2.21 Electromagnetic Energy Theorem - Poynting Theorem

Poynting theorem in electromagnetism is equivalent to mass - energy theorem in mechanics. It states that "rate of decrease of the energy in the electromagnetic fields in any volume minus flow of energy per second through the surface of that volume is equal to the rate of work done on the charges by the electromagnetic force or power transferred in the space". It can be written as

$$-\frac{\partial}{\partial t}\oint_V\left(\frac{1}{2}\frac{B^2}{\mu_0}+\frac{1}{2}\epsilon_0 E^2\right)dV-\frac{1}{\mu_0}\oint(\mathbf{E}\times\mathbf{B}).\,d\mathbf{S}=\oint_V(\mathbf{E}.\,\mathbf{J})dV$$

$$\text{Or } -\frac{\partial}{\partial t}\oint_V\left(\frac{1}{2}\mathbf{E}.\,\mathbf{D}+\frac{1}{2}\mathbf{H}.\,\mathbf{B}\right)-\oint(\mathbf{E}\times\mathbf{H}).\,d\mathbf{S}=\oint_V(\mathbf{E}.\,\mathbf{J})dV \quad (63)$$

Where

(i) $-\frac{\partial}{\partial t}\oint_V\{\frac{1}{2}\frac{B^2}{\mu_0}+\frac{1}{2}\epsilon_0 E^2\}dV$ = Rate of decrease of electromagnetic energy in the electromagnetic fields in the volume V

(ii) $-\frac{1}{\mu_0}\oint(\mathbf{E}\times\mathbf{B}).\,d\mathbf{S}$ = Rate at which the energy is propagated by the electromagnetic fields through the cross-sectional area that bounds the volume V (Energy flow / Area)

(iii) $\oint_V(\mathbf{E}.\,\mathbf{J})dV$ = Power transferred into the free space or rate of work done by the electromagnetic fields in the volume V (Power loss)

2.21.1 Derivation of Poynting Theorem

Writing Maxwell's equations in differential form

$$\left.\begin{array}{lll} \nabla.\,\mathbf{D}=\rho & \text{or}\quad \text{div}\,\mathbf{D}=\rho & \text{(i)}\\ \nabla.\,\mathbf{B}=0 & \text{or}\quad \text{div}\,\mathbf{B}=0 & \text{(ii)}\\ \nabla\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} & \text{or}\quad \text{curl}\,\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} & \text{(iii)}\\ \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t} & \text{or}\quad \text{curl}\,\mathbf{H}=\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t} & \text{(iv)} \end{array}\right\} \quad (64)$$

Taking scalar product of equation {64(iii)} with **H** and equation {64(iv)} with **E**, we get;

$$\mathbf{H}.\,(\nabla\times\mathbf{E})=-\mathbf{H}.\,\frac{\partial \mathbf{B}}{\partial t} \quad (65)$$

$$\mathbf{E}.\,(\nabla\times\mathbf{H})=\mathbf{E}.\,\mathbf{J}+\mathbf{E}.\,\frac{\partial \mathbf{D}}{\partial t} \quad (66)$$

Now subtracting equation (66) from equation (65), we get;

$$\mathbf{H}.\,(\nabla\times\mathbf{E})-\mathbf{E}.\,(\nabla\times\mathbf{H})=-\mathbf{H}.\,\frac{\partial \mathbf{B}}{\partial t}-\mathbf{E}.\,\frac{\partial \mathbf{D}}{\partial t}-\mathbf{E}.\,\mathbf{J}$$

$$\mathbf{H}.\,(\nabla\times\mathbf{E})-\mathbf{E}.\,(\nabla\times\mathbf{H})=-\left(\mathbf{H}.\,\frac{\partial \mathbf{B}}{\partial t}+\mathbf{E}.\,\frac{\partial \mathbf{D}}{\partial t}\right)-\mathbf{E}.\,\mathbf{J} \quad (67)$$

We have the vector identity;

$$\nabla.\,(\mathbf{E}\times\mathbf{H})=\mathbf{H}.\,\text{curl}\,\mathbf{E}-\mathbf{E}.\,\text{curl}\,\mathbf{H}$$

So, the equation (67) takes the form;

$$\nabla.\,(\mathbf{E}\times\mathbf{H})=-\left(\mathbf{H}.\,\frac{\partial \mathbf{B}}{\partial t}+\mathbf{E}.\,\frac{\partial \mathbf{D}}{\partial t}\right)-\mathbf{E}.\,\mathbf{J} \quad (68)$$

For a linear medium we can write;

$$\mathbf{B}=\mu\mathbf{H} \text{ and } \mathbf{D}=\epsilon\mathbf{E}$$

So, the equation (68) be written as;

$$\nabla.\,(\mathbf{E}\times\mathbf{H})=-\left[\mathbf{H}.\,\frac{\partial(\mu\mathbf{H})}{\partial t}+\mathbf{E}.\,\frac{\partial(\epsilon\mathbf{E})}{\partial t}\right]-\mathbf{E}.\,\mathbf{J} \quad (69)$$

Now rearranging $\mathbf{E}.\,\frac{\partial(\epsilon\mathbf{E})}{\partial t}$ and $\mathbf{H}.\,\frac{\partial(\mu\mathbf{H})}{\partial t}$, we have;

$$\mathbf{E}.\,\frac{\partial}{\partial t}(\epsilon\mathbf{E})=\frac{1}{2}\epsilon\frac{\partial}{\partial t}(E^2)=\frac{\partial}{\partial t}\left(\frac{1}{2}\,\epsilon E^2\right)=\frac{\partial}{\partial t}\left(\frac{1}{2}\,\mathbf{E}.\,\mathbf{D}\right) \quad (70)$$

$$\mathbf{H}.\frac{\partial}{\partial t}(\mu\mathbf{H}) = \frac{1}{2}\mu\frac{\partial}{\partial t}(H^2) = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\ \mathbf{H}.\mathbf{B}\right) \qquad (71)$$

Using expression (70) and (71), the equation (69) takes the form;

$$\nabla.(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}(\mathbf{E}.\mathbf{D}+\mathbf{H}.\mathbf{B})\right] - \mathbf{J}.\mathbf{E} \qquad (72)$$

On integration, the equation (52) over a volume V bounded by surface S, we have;

$$\int_V \nabla.(\mathbf{E}\times\mathbf{H})\,d\mathbf{V} = -\int_V\left\{\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E}.\mathbf{D}+\frac{1}{2}\mathbf{H}.\mathbf{B}\right)\right\}d\mathbf{V} - \int_V \mathbf{J}.\mathbf{E}\,d\mathbf{V}$$

Using Gauss divergence theorem to change volume integral on L.H.S. of the above equation into surface integral, we get

$$\int_V \mathbf{J}.\mathbf{E}\,d\mathbf{V} = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\mathbf{E}.\mathbf{D}+\frac{1}{2}\mathbf{H}.\mathbf{B}\right)d\mathbf{V} - \oint_s(\mathbf{E}\times\mathbf{H}).\,d\mathbf{S} \qquad (73)$$

This is the Poynting theorem.

2.21.2 Physical Significance of Poynting Theorem

Let us interpret each of the three terms individually.

(i) Physical Interpretation of $-\oint_s(\mathbf{E}\times\mathbf{H}).\,d\mathbf{S}$

$\mathbf{E}\times\mathbf{H} = \mathbf{S}$ is a quantity called poynting vector and is defined as the energy flow per unit time per unit area i.e., power flow per unit area and is perpendicular to $\mathbf{E}$ and $\mathbf{H}$ both. So $\oint_s(\mathbf{E}\times\mathbf{H}).\,d\mathbf{S}$ may be defined as the rate at with the energy of the electromagnetic fields is decreased through the cross-sectional area that bounds the volume V

(ii) Physical Interpretation of $-\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\mathbf{E}.\mathbf{D}+\frac{1}{2}\mathbf{H}.\mathbf{B}\right)dV$

Here, we have;

$\int_V \frac{1}{2}\mathbf{E}.\mathbf{D}\,d\mathbf{V} = U_e$, Electrostatic potential energy in volume V

$\int_V \frac{1}{2}\mathbf{H}.\mathbf{B}\,d\mathbf{V} = U_m$, Magnetic energy in volume V

$U = \int_V \frac{1}{2}(E.D + H.B)dV$, represents some sort of potential energy of electromagnetic field, as it exists due to static fields (electric and magnetic), known as electromagnetic field energy in volume V. A concept such as energy stored in the field itself rather than residing with the particles is a basic concept of electromagnetic theory.

Obviously $\frac{1}{2}(\mathbf{E}.\mathbf{D}+\mathbf{H}.\mathbf{B})$ represents energy density of electromagnetic field i.e.

$$U = \frac{1}{2}(\mathbf{E}.\mathbf{D}+\mathbf{H}.\mathbf{B})$$

Consequently, the term, $-\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\mathbf{E}.\mathbf{D}+\frac{1}{2}\mathbf{H}.\mathbf{B}\right)d\mathbf{V}$ represents the rate of decrease of stored electromagnetic energy in volume V.

(ii) Physical Interpretation of $\int_V \mathbf{J}.\mathbf{E}\,dV$

To understand the physical significance of this term, let us consider a charged particle q in the charge distribution acted upon by Lorentz force $\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$, displaced by an amount $\mathbf{dl}$. Where $\mathbf{v}$ may be considered as the drift velocity of the charges. So, the work done in this displacement will be given as;

$dW = \mathbf{F}.\,\mathbf{dl} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\,\mathbf{dl} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\,\mathbf{v}dt$

Or $dW = q\mathbf{E}.\,\mathbf{v}dt = \mathbf{F}.\,\mathbf{v}dt$

Or $\frac{\partial W}{\partial t} = \mathbf{F}.\mathbf{v}$

We can also get the above expression as for an electromagnetic force due to field vectors E and B acting on the charged particle, the magnetic force $\mathbf{q}(\mathbf{v} \times \mathbf{B})$ is always perpendicular to velocity. Hence, the magnetic field does no work. Therefore, for a single charge q the rate of doing work by electromagnetic field E and B is;

$$\frac{\partial W}{\partial t} = \mathbf{F}.\mathbf{v} = q\mathbf{E}.\mathbf{v} \qquad (74)$$

If an electromagnetic field consists of a group of charges moving with different velocities e.g. n_i charge carriers each with charge q_i, moving with velocity v_i (i-1,2,3...); then equation (54) can be written as;

$$\frac{\partial W}{\partial t} = -\Sigma\, n_i q_i \mathbf{v}_i.\mathbf{E}_i \qquad (75)$$

In this case, the total current density $\mathbf{J} = \Sigma\, \mathbf{J}_i = \Sigma\, n_i q_i \mathbf{v}_i$. So, equation (55) becomes

$$\frac{\partial W}{\partial t} = -\Sigma\, \mathbf{J}_i.\mathbf{E}_i = -\mathbf{J}.\mathbf{E} \qquad (76)$$

Therefore, the expression, $\int_V \mathrm{J}.\mathrm{E}\, dV$ represents rate of energy transferred into the electromagnetic field through the motion of free charge in volume V.

The physical significance of the equation of Poynting theorem is that the time rate of decrease of electromagnetic energy with a certain volume plus time rate of the energy flowing out through the boundary surface is equal to the power transferred into the electromagnetic field. This is also the statement of conservation of energy in electromagnetism which is known as Poynting theorem.

2.22 Momentum of Electromagnetic Field

As we know, anything containing energy, also have momentum. So, electromagnetic waves also must have momentum. The expression for momentum of electromagnetic waves can be derived with the analogy of momentum of a d' Broglie wave.

We have $P = \frac{h}{\lambda}$, here P is the momentum of the d' Broglie wave, λ is the wavelength and h is Planck's constant.

$u = h\nu = h\frac{C}{\lambda}$

Or $u = PC$

Or $P = \frac{u}{C}$ (77)

From equation (60), we have the energy density per unit area per unit time as;

S = uC (here u is the energy density)

So $\frac{u}{C} = \frac{S}{C^2}$ (78)

From (77) and (78), we can have the momentum density of electromagnetic waves as

$$\mathbf{P} = \frac{u}{C} = \frac{S}{C^2} \qquad (79)$$

Or

$$\mathbf{P} = \frac{1}{\mu_0 C^2}(\mathbf{E} \times \mathbf{B}) \qquad \text{as } \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \qquad (80)$$

Or

$$\boxed{\mathbf{P} = \frac{1}{C^2}(\mathbf{E} \times \mathbf{H})} \qquad (81)$$

Using equation (80) or (81), the momentum transferred by the electromagnetic waves into the medium, can be calculated, if we have electric and magnetic field vectors at that point of the medium.

2.23 Angular Momentum in Electromagnetic Fields

Angular momentum is a physical property of a rotating object. Now, the first question arises that what is the concept behind the angular momentum of electromagnetic waves? The propagation of a beam of an electromagnetic wave is considered due to the helical type rotation of electric as well as magnetic fields. So, if we look through the cross section of a light beam, the propagation of the beam of electromagnetic wave will be seen as, composed of helical as well as spin type motion of light beam. That's why electromagnetic wave should also possess angular momentum.

As according to the fundamental definition of angular momentum, "moment of linear momentum is called angular momentum of a rotating object". So, according to this definition, the angular momentum of beam of electromagnetic waves can be calculated as;

$$\mathbf{L} = \mathbf{r} \times \mathbf{P}$$

Here P is the linear momentum of electromagnetic field/wave and L is the moment of linear momentum about the axis of rotation having r as the distance from axis of rotation, called angular momentum. Now substituting the value of 'P' from (79) into the above equation, we have;

$$\mathbf{L} = \frac{1}{C^2}(\mathbf{r} \times \mathbf{S}) \text{, or}$$

$$\mathbf{L} = \frac{1}{\mu_0 C^2}(\mathbf{r} \times \mathbf{E} \times \mathbf{B}) \tag{82}$$

$$\mathbf{L} = \frac{1}{C^2}(\mathbf{r} \times \mathbf{E} \times \mathbf{H}) \tag{83}$$

Here, it should be noted that angular momentum calculated from (82) or (83), is the angular momentum density. The actual angular momentum will be found out using;

$$\mathbf{L} = \frac{1}{\mu_0 C^2}\int(\mathbf{r} \times \mathbf{E} \times \mathbf{B})\, dv \tag{84}$$

$$\mathbf{L} = \frac{1}{C^2}\int(\mathbf{r} \times \mathbf{E} \times \mathbf{H})\, dv \tag{85}$$

Obviously, the angular momentum of electromagnetic field, at any point can be calculated by taking moment of the cross product of electric and magnetic field vectors of an electromagnetic wave at that point. From equations (84) or (85), it can also be seen that angular momentum of an electromagnetic wave is always perpendicular to the direction of propagation of electromagnetic wave.

The propagation of a beam of an electromagnetic wave, as viewed through the cross section of the beam, is composed of two types of motion, one as the helical motion called orbital motion and the other is the spin on its own axis. The beam of electromagnetic waves can be actually considered as rotating around its own axis while propagating in helical path. So, the angular momentum of an electromagnetic wave should be composed of spin angular momentum (SAM) and orbital angular momentum (OAM) both. i.e.

$$\mathbf{L} = \mathbf{L_{SAM}} + \mathbf{L_{OAM}}$$

For a well collimated beam, the optical polarization also called circular polarization is exclusively due to Spin Angular Momentum while the Orbital Angular Momentum is related with the spatial field distribution, and in particular with the wave-front helical shape. However, for highly focused or diverging beam or otherwise in general, total angular momentum '**L**', may server the purpose. Spin angular momentum is widely being used in radar applications while orbital angular momentum is being employed in optical fibre transmission.

2.24 Idea of Electromagnetic Waves

Ampère's circuital law was written and well explained for a circuit in which a conduction current is flowing in the circuit. But if a circuit has a region where no conventional current is flowing, the Ampere circuital law will not be valid at that place for example, the Ampere's circuital law cannot be applied throughout the whole circuitry, if a capacitor is connected in a circuit powered by an alternating field

Fig.2.1. During the flow of current inside the circuit, the Ampere's circuital law is applicable throughout the whole circuit except in the space between the plates of the capacitor as there is no conventional current. So, the Ampere's circuital law i.e., $\oint \mathbf{B}.\mathbf{dl} = \mu_0 \mathbf{I} = 0$ and is not valid in the space between the plates of a capacitor. But the circuit is complete otherwise current will not flow in the connecting wires. It means two plates of the capacitor should be connected with each other internally so that that the current may flow in the external circuit. As, the whole circuit contains conduction current except the gap between the plates of the capacitor, Maxwell gave a hypothesis that there must be some kind of current in between the plates of the capacitor. Later, he termed it Maxwell displacement current which was due to the presence of variable electric and magnetic fields between the plates. It means some energy propagates from one plate to the other through the free space between the plates during the charging and discharging of the capacitor over the complete cycle of the alternating field. As this energy is propagating in the form of oscillating electric and magnetic field in the free space between the plates, this may be termed as electromagnetic waves.

2.24.1 Discovery of Electromagnetic Waves

The existence of electromagnetic waves was first investigated by Heinrich Hertz, who succeeded in generating and detecting radio waves. But he could not lay down the fundamental formulations for the electromagnetic waves. Maxwell is well known for his pioneer work in the area of electromagnetic waves. Using fundamental equations of electromagnetism, he not only developed the equations for electromagnetic waves but also proved that light is an electromagnetic wave and is transverse in nature. Today, we all know about complete electromagnetic wave spectrum which we receive from the Sun.

2.25 Electromagnetic Waves in Free Space

Although the four equations written in differential form are sufficient to understand the concept of electromagnetic waves, but there is a need to formulate the equation of electromagnetic waves which resembles the general wave equation so that we can treat and analyse that equation to find various physical parameters of a wave. By doing some mathematical operations on the Maxwell equation in differential form, we can obtain the required equations of electromagnetic waves.

General Maxwell equations in differential form are as follows;

$$\left.\begin{aligned}
&\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \text{ or } \nabla.\mathbf{D} = \rho && \text{(i)}\\
&\nabla.\mathbf{B} = 0 && \text{(ii)}\\
&\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} && \text{(iii)}\\
&\nabla\times\mathbf{H} = \mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t} && \text{(iv)}
\end{aligned}\right\} \quad (86)$$

For free space, the charge density ρ and conduction current density J are both zero. So, the Maxwell's equations reduce to:

$$\left.\begin{aligned}
&\nabla.\mathbf{E} = 0 && \text{(i)}\\
&\nabla\times\mathbf{B} = 0 && \text{(ii)}\\
&\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} && \text{(iii)}\\
&\nabla\times\mathbf{H} = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} && \text{(iv)}
\end{aligned}\right\} \quad (87)$$

Where μ_0 and ϵ_0 is the permeability and permittivity of free space.

Now taking curl of the equations {(87)(iii)}, we have

$$\nabla\times(\nabla\times\mathbf{E}) = -\mu_0\frac{\partial(\nabla\times\mathbf{H})}{\partial t} \quad (88)$$

Simplifying the vector triple product and putting the value of $\nabla \times \mathbf{H}$ from {(87)(iv)} into (88);

$$\nabla.(\nabla.\mathbf{E}) - \nabla^2\mathbf{E} = \mu_0 \frac{\partial}{\partial t}[\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}]$$

$$\nabla^2\mathbf{E} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{dt^2} = 0 \qquad (\text{As } \nabla.\mathbf{E} = 0 \text{ for free space}) \qquad (89)$$

Similarly, taking curl of the equations {(87)(iv)}, we have

$$\nabla \times (\nabla \times \mathbf{H}) = -\epsilon_0 \frac{\partial(\nabla \times \mathbf{E})}{\partial t} \qquad (90)$$

Simplifying the vector triple product and putting the value of $\nabla \times \mathbf{E}$ from {(87)(iii)} into (90);

$$\nabla.(\nabla.\mathbf{H}) - \nabla^2\mathbf{H} = -\epsilon_0\mu_0 \frac{\partial}{\partial t}[\frac{\partial \mathbf{H}}{\partial t}]$$

$$\nabla^2\mathbf{H} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{H}}{dt^2} = 0 \qquad (91)$$

The equations (89) and (91) resemble the general wave equation (28), generally written as;

$$\nabla^2\mathbf{u} - \frac{1}{v^2}\frac{\partial^2 \mathbf{u}}{dt^2} = 0 \qquad (92)$$

Where **u** is a wave variable and **v** is the velocity of the wave.

Now comparing the equations (89) and (91) with (92), we can say that **E** or **H** are the variables of the wave represented by these equations and the velocity of the waves is given by;

$$\frac{1}{v^2} = \mu_0\epsilon_0, \qquad \text{Or}$$

$$v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = C \qquad (93)$$

Substituting the values of μ_0 and ϵ_0, the velocity of electromagnetic waves in free space comes out to be 2.99×10^8 m/s, which is equal to the already calculated speed of light in vacuum.

As we have already shown that the variable electric and magnetic fields are coupled together. So, when we write equation of electromagnetic wave, either of the one is sufficient to consider for further treatment of the wave equation. One most important conclusion can be made here; the velocity of electromagnetic wave in free space comes out to be equation to the velocity of light in vacuum. So, in first impression, it may be said that light is an electromagnetic wave. We shall however find other physical parameters of light with these equations to finally conclude the electromagnetic nature of light. The following coupled equations having E and H, as wave parameters are called equations of electromagnetic waves in the free space.

$$\left.\begin{aligned} \nabla^2\mathbf{E} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{dt^2} = 0 \\ \nabla^2\mathbf{H} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{H}}{dt^2} = 0 \end{aligned}\right] \qquad (94)$$

And the velocity of electromagnetic waves in free space is calculated by;

$$C = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

For simplicity we can treat either of the two for finding various physical parameters of the electromagnetic waves.

2.26 Electromagnetic Waves in a Non-Conducting Isotropic Dielectric

Let us study the propagation of electromagnetic waves in a medium which is linear, non-conducting dielectric, isotropic and homogeneous. Let ϵ and μ are the permittivity and permeability respectively of the medium. Let the medium is source free, the electric field enters from outside the medium then the charge density ρ will be zero. Being non conducting, the conductivity $\sigma = 0$ and hence the conduction current density $\vec{J} = \sigma\vec{E}$, is also zero.

Writing, the Maxwell's equations in differential form;

$$
\left.\begin{array}{lll}
\nabla.\mathbf{D} = \rho & \text{or} \quad \text{div}\,\mathbf{D} = \rho & \text{(i)} \\
\nabla.\mathbf{B} = 0 & \text{or} \quad \text{div}\,\mathbf{B} = 0 & \text{(ii)} \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \text{or} \quad \text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \text{(iii)} \\
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \text{or} \quad \text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \text{(iv)}
\end{array}\right] \quad (95)
$$

Now putting, $\mathbf{D} = \epsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}, \mathbf{J} = \sigma\mathbf{E} = 0$ and $\rho = 0$, in (95), we have;

$$
\left.\begin{array}{ll}
\nabla.\mathbf{E} = 0 & \text{(i)} \\
\nabla.\mathbf{H} = 0 & \text{(ii)} \\
\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t} & \text{(iii)} \\
\nabla \times \mathbf{H} = \epsilon\frac{\partial \mathbf{E}}{\partial t} & \text{(iv)}
\end{array}\right] \quad (96)
$$

Taking curl of equation {(96)(iii)}, we get;

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\frac{\partial(\nabla \times \mathbf{H})}{\partial t}$$

Simplifying the vector triple product and putting the value of $\nabla \times \mathbf{H}$ from {(96)(iv)}, we have;

$$\nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E} = -\mu\frac{\partial}{\partial t}[\epsilon\frac{\partial \mathbf{E}}{\partial t}]$$

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{dt^2} = 0 \quad (97)$$

Similarly, we can have following equation for magnetic field intensity, i.e.

$$\nabla^2\mathbf{H} - \mu\epsilon\frac{\partial^2\mathbf{H}}{dt^2} = 0 \quad (98)$$

So, in coupled form, the equations (97) and (98) are the equations of electromagnetic waves in a linear, non-conducting dielectric, isotropic and homogeneous medium. Here, **E** and **H**, are the wave variables. Comparing (97) and (98), with the general wave equation, the speed of electromagnetic wave in isotropic, homogeneous, dielectric medium will be;

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

$$= \frac{1}{(K_m\mu_0K_e\epsilon_0)} \quad (99)$$

Where K_m and K_e are the relative permeability and permittivity of the medium respectively.

As we have $C = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ is the speed of electromagnetic waves in free space. Hence, the velocity of electromagnetic waves will be;

$$\therefore v = \frac{C}{\sqrt{K_mK_e}} \quad (100)$$

As $K_m > 1$ and $K_e > 1$: therefore we can say the speed of electromagnetic waves in an isotropic dielectric is less than the speed of electromagnetic waves in free space.

The ratio of velocity of electromagnetic waves in vacuum to the medium gives another physical parameter which is widely being used for light as a relative parameter between two media called refractive index of any medium i.e.

Refractive index $= \frac{C}{v} = \sqrt{K_mK_e} = n$(say)

For a non-magnetic material $K_m = 1$; therefore, $n = \sqrt{K_e}$, i.e. $n^2 = K_e$

2.27 Transverse Nature of Electromagnetic Waves

To show the transverse nature of electromagnetic waves in a linear, non-conducting dielectric, isotropic and homogeneous medium, let us have the wave equations as derived above;

$$\nabla^2\mathbf{E} - \frac{1}{v^2}.\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \qquad (101)$$

$$\nabla^2\mathbf{H} - \frac{1}{v^2}.\frac{\partial^2\mathbf{H}}{\partial t^2} = 0 \qquad (102)$$

Mathematically these are the second order differential equations. So, the plane-wave solutions of equations (101) and (102), may be written as;

$$\mathbf{E}(r,t) = E_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} \qquad (103)$$

$$\mathbf{H}(r,t) = H_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} \qquad (104)$$

Where E_0 and H_0 are complex amplitudes which are constant in space and time, while k is wave propagation vector given by

$$\mathbf{K} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{\omega}{v}\hat{n} \qquad (105)$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation vector.

Now, as $\mathbf{E}(r,t)$ and $\mathbf{H}(r,t)$ are the solutions of the Maxwell waves equations so these should satisfy all the four Maxwell equations, i.e.

$$\nabla.\mathbf{E}(r,t) = 0 \qquad (106)$$

$$\nabla.\mathbf{H}(r,t) = 0 \qquad (107)$$

$$\nabla\times\mathbf{E}(r,t) = -\mu\frac{\partial\mathbf{H}(r,t)}{\partial t} \qquad (108)$$

$$\nabla\times\mathbf{H}(r,t) = \epsilon\frac{\partial\mathbf{E}(r,t)}{\partial t} \qquad (109)$$

Substituting $\mathbf{E}(r,t)$ and $\mathbf{H}(r,t)$ from equations (65) and (66) respectively we have;

$$\nabla.E_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} = 0 \qquad (110)$$

$$\nabla.H_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} = 0 \qquad (111)$$

Let us simplify equation (110), i.e., taking the dot product of Del operator and electric field vector we have;

$$\nabla.\mathbf{E} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{dz}\right).E_0 e^{i\mathbf{k}.\mathbf{r}-i\omega t}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{dz}\right)\left[\left(\hat{\imath}E_{0x} + \hat{\jmath}E_{0y} + \hat{k}E_{0z}\right)e^{i(k_x x+k_y y+k_z z)-i\omega t}\right]$$

$$\left[\text{since } \mathbf{k}.\mathbf{r} = \left(\hat{\imath}k_x + \hat{\jmath}k_y + \hat{k}k_z\right).(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)\right]$$

$$= \left[k_x x + k_y y + k_z z\right]$$

$$\therefore\ \nabla.\mathbf{E} = (E_{0x}\hat{\imath}k_x + E_{0y}\hat{\imath}k_y + E_{0z}\hat{\imath}k_z)\, e^{i\mathbf{K}.\mathbf{r}-i\omega t}$$

$$= \hat{\imath}(E_{0x}k_x + E_{0y}k_y + E_{0z}k_z)\, e^{i\mathbf{K}.\mathbf{r}-i\omega t}$$

$$= \hat{\imath}\left(\hat{\imath}k_x + \hat{\jmath}k_y + \hat{k}k_z\right).(\hat{\imath}E_{0x} + \hat{\jmath}E_{0y} + \hat{k}E_{0z})\, e^{i\mathbf{K}.\mathbf{r}-i\omega t}$$

$$= \hat{\imath}\mathbf{K}.E_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} \quad = \hat{\imath}\mathbf{K}.\mathbf{E} = 0$$

Thus $\nabla.\mathbf{E} = 0$ implies that

$$\mathbf{K}.\mathbf{E} = 0 \qquad (112)$$

Now, $\mathbf{K}.\mathbf{E} = 0$, implies the wave propagation vector **K** and the electric field vector **E** of the electromagnetic wave are perpendicular to each other.

On doing similar operation on equation (111), we have;

$$\nabla.\mathbf{H} = \hat{\imath}\,\mathbf{K}.\mathbf{H}$$

Or

$$\mathbf{K}.\mathbf{H} = 0 \qquad (113)$$

Again, we can infer the propagation vector **K** and the oscillating magnetic field vector **H** are also perpendicular to each other. The above mathematical operations, shows that both the wave variables **E** and **H** oscillate perpendicular to the propagation of wave. This proves the transverse character of electromagnetic waves.

2.28 Mutually Perpendicularity of Electric and Magnetic Field Vectors

On satisfying the equations (101) and (102) with the solutions given by (103) and (104), we have;

$$\nabla \times \mathbf{E}(r,t) = -\mu \frac{\partial \mathbf{H}(r,t)}{\partial t} \tag{114}$$

$$\nabla \times \mathbf{H}(r,t) = \epsilon \frac{\partial \mathbf{E}(r,t)}{\partial t} \tag{115}$$

$$\nabla \times E_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} = -\mu \frac{\partial H_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t}}{\partial t} \tag{116}$$

$$\nabla \times H_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} = \epsilon \frac{\partial E_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t}}{\partial t} \tag{117}$$

On simplifying the mathematical operations of the equations (116) and (117), we find;

$$i\mathbf{K} \times \mathbf{E} = i\mu\omega\mathbf{H} \quad \text{i.e.}\ \mathbf{K} \times \mathbf{E} = \mu\omega\mathbf{H} \tag{118}$$

$$\mathbf{K} \times \mathbf{H} = -\epsilon\omega\mathbf{E} \tag{119}$$

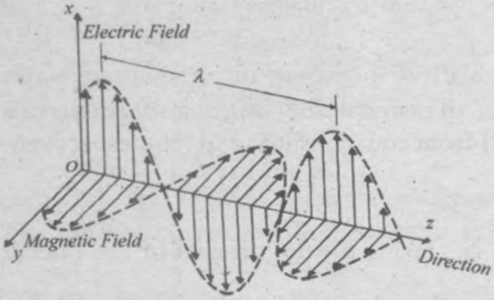

Fig. 2.10 Transverse Character of Electromagnetic wave

The equations (118) and (119) shows that the wave propagation vector **k**, electric field vector **E** and Magnetic field vector **H** are mutually perpendicular to each other. Simply we can say the electric and magnetic fields are oscillating perpendicular to each other.

2.29 Some Other Physical Parameters of Electromagnetic Wave

By doing other mathematical manipulations on the equations derived above we are able to define a lot of physical quantities related to the electromagnetic waves.

2.29.1 Wave Impedance

On rearranging the equation (118), we have;

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{K} \times \mathbf{E}) = \frac{k}{\mu\omega}(\hat{n} \times \mathbf{E})$$

Or

$$\left|\frac{E}{H}\right| = \frac{\mu\omega}{K} = \mu v, \qquad \left(\text{since } \mathbf{K} = \frac{\omega}{v}\right)$$

Or

$$\left|\frac{E}{H}\right| = \frac{\mu\omega}{K} = \frac{\mu}{\sqrt{\mu\epsilon}} \qquad \left(\text{since } v = \frac{1}{\sqrt{\mu\epsilon}}\right)$$

Or

$$\left|\frac{E}{H}\right| = \sqrt{\frac{\mu}{\epsilon}} \tag{120}$$

Now let us see what this quantity is showing. Putting the units of each term involved in it, we have;

$$\frac{\text{volt/m}}{\text{amp-turn/m}} = \frac{\text{volt}}{\text{amp}} = \text{ohm}$$

This implies that the magnitude of $\left|\frac{E}{H}\right| = \sqrt{\frac{\mu}{\epsilon}}$ is giving some physical quantity which has the unit as ohm. This unit represents impedance. Here, this is called wave impedance offered by medium when electromagnetic waves propagate through it and is given by the modulus of the ratio of electric field vector and magnetic field vector. This can also be calculated by the square root of the ratio of permeability and permittivity of the medium. So, the wave impedance is given as;

$$Z = \left|\frac{E}{H}\right| = \sqrt{\frac{\mu}{\epsilon}}$$

As we have the values of the permeability and permittivity of the free space as μ_0 and ϵ_0 , the impedance offered by the vacuum when electromagnetic waves propagate through it is given by;

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

Substituting the values of μ_0 and ϵ_0, the value of Z_0 comes out to be as;

$$Z_0 = 376.66 \text{ ohm}$$

2.29.2 Phase between Electric Field and Magnetic Field Vector

A simple way to find whether the two vector quantities are in phase or out of phase is that if the modulus of the ratio of the two quantities is real then those two vectors are said to be in phase but if the modulus of the ratio, is a complex quantity then the two vector quantities are said to be out of phase. As we already have the modulus of the ratio of E and H in free space as well as in isotropic dielectric, i.e.

$$Z_0 = \left|\frac{E_0}{H_0}\right| = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.6 \text{ ohm, and } Z = \left|\frac{E}{H}\right| = \sqrt{\frac{\mu}{\epsilon}} = \left(\frac{K_m \mu_0}{K_e\, \epsilon_0}\right)$$

Both of these are real. So, we can say when electromagnetic waves propagate in free space and isotropic dielectric, the oscillations of electric field and magnetic field are in phase with each other.

2.29.3 Poynting Vector for a Plane Electromagnetic Wave in an Isotropic Dielectric

Let us find out the flow of electromagnetic energy when electromagnetic waves propagate through an isotropic dielectric. The well-defined way of doing this is to find out the poynting vector of electromagnetic waves in corresponding medium.

The poynting vector is calculated as;

$$\begin{aligned}
\mathbf{S} &= \mathbf{E} \times \mathbf{H} \\
&= \mathbf{E} \times \left(\frac{K}{\mu\omega}(\hat{n} \times \mathbf{E})\right) \\
&= \mathbf{E} \times (\hat{n} \times \mathbf{E}).\frac{1}{\mu v} \\
&= \mathbf{E} \times (\hat{n} \times \mathbf{E}).\sqrt{\frac{\epsilon}{\mu}} \\
&= \frac{\mathbf{E} \times (\hat{n} \times \mathbf{E})}{Z} \qquad \left(\text{since } Z = \sqrt{\frac{\mu}{\epsilon}}\right) \\
&= \frac{(\mathbf{E}.\mathbf{E})\hat{n} - (\mathbf{E}.\hat{n})\mathbf{E}^2}{Z} \\
&= \frac{E^2}{Z}\hat{n} \qquad (\text{since } \mathrm{E}.\hat{n} = 0, \text{ as E and } \hat{n} \text{ are perpendicular}) \qquad (121)
\end{aligned}$$

As **E** and **H** are oscillating between maximum and minimum values of these physical quantities, so to find the flow of energy of electromagnetic waves, we have to take time average of poynting vector over a complete cycle of electric of magnetic field vector i.e.;

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \langle \frac{E^2}{Z} \hat{n} \rangle$$

$$= \frac{1}{Z} \langle \left(E_0 e^{i\mathbf{K}.\mathbf{r} - i\omega t} \right)^2_{real} \rangle \hat{n}$$

Since for finding actual physical fields, we often take real parts of complex exponentials. So, we have;

$$\langle \mathbf{S} \rangle = \frac{1}{Z} E_0{}^2 \langle \cos^2(\omega t - \mathbf{K}.\mathbf{r}) \rangle \hat{n}$$

$$= \frac{1}{Z} E_0{}^2 . \frac{1}{2} \hat{n} = \frac{E_0{}^2}{2Z} \hat{n}$$

$$= \frac{E_{rms}^2}{Z} \hat{n} \qquad \left(\text{since } E_{rms} = \frac{E_0}{\sqrt{2}}\right) \qquad (122)$$

The equation (122) gives the average energy flow of electromagnetic waves, per unit area per time i.e., energy flux. It also shows that the flow of energy is along the direction of propagation of electromagnetic wave.

Similarly for propagation in free space we have the time average poynting vector as;

$$\langle S \rangle_{\text{free space}} = \frac{E_{rms}^2}{Z_0} \hat{n} \qquad (123)$$

2.29.4 Power Flow and Energy Density

Let us find the ratio of electrostatic and magnetostatic energy densities in an electromagnetic wave field i.e.

$$\frac{u_e}{u_m} = \frac{\frac{1}{2} \epsilon E^2}{\frac{1}{2} \mu H^2} = \frac{\epsilon}{\mu} \frac{E^2}{H^2} = \frac{\epsilon}{\mu} Z^2 = \frac{\epsilon}{\mu} . \frac{\mu}{\epsilon} = 1 \qquad (124)$$

This implies that for the case of electromagnetic waves in an isotropic dielectric the electrostatic energy density (u_e) is equal to the magnetostatic energy density (u_m).

Therefore, total electromagnetic energy density

$$u = u_e + u_m = 2u_e \text{ (since } u_e = u_m)$$

$$= 2 . \frac{1}{2} \epsilon E^2 = \epsilon E^2$$

And the time average of energy density

$$\langle u \rangle = \epsilon \langle E^2 \rangle = \epsilon \langle \left(E_0 e^{i\mathbf{K}.\mathbf{r} - i\omega t} \right)^2_{real} \rangle$$

$$= \epsilon E^2 \langle \cos^2(\omega t - \mathbf{K}.\mathbf{r}) \rangle$$

$$= \frac{1}{2} \epsilon E_0^2$$

i.e., the total electromagnetic energy density $\langle u \rangle = \frac{1}{2} \epsilon E_0^2 = \epsilon E_{rms}^2$ (125)

2.29.5 Relation between Energy Flux and Energy Density of Electromagnetic Waves

Dividing equation (123) and (125), we have;

$$\frac{\langle S \rangle}{\langle u \rangle} = \frac{E_{rms}^2 \hat{n}/Z}{\epsilon E_{rms}^2} = \frac{1}{\epsilon Z} \hat{n} = \frac{1}{\sqrt{\frac{\mu}{\epsilon}} \epsilon} \hat{n} \qquad \left(\text{since } Z = \sqrt{\frac{\mu}{\epsilon}}\right)$$

$$= \frac{1}{\sqrt{\mu \epsilon}} \hat{n} = v\hat{n} \qquad \text{As } v = \frac{1}{\sqrt{\mu \epsilon}}$$

$$\frac{\langle S \rangle}{\langle u \rangle} = v\hat{n}$$

$$\langle S \rangle = \langle u \rangle v\hat{n} \qquad (126)$$

Energy Flux = Energy Density × velocity of em waves in the medium

This equation implies that the energy density associated with an electromagnetic wave in the dielectrics, flows with the same speed of the wave in the dielectrics and it is in the direction of propagation of the wave.

Following points can be summarised for the electromagnetic waves in isotropic dielectric:

- The electromagnetic waves travel with a speed less than the speed of light in case of an isotropic dielectric.
- The electromagnetic waves are transverse in nature as the field vectors E and H are mutually perpendicular and also perpendicular to the direction of propagation the electromagnetic wave.
- The phase of field vectors E and H is same.
- The direction of flow of electromagnetic energy and the direction of wave propagation is same and the energy flowing per second is represented as $\langle S\rangle \frac{E_{rms}^2 \hat{n}}{Z} = \langle u\rangle v\hat{n}$
- The electrostatic energy density and the magnetostatic energy density are equal and the total energy density is given as $\langle u\rangle = \epsilon E_{rms}^2$
- The energy density associated with an electromagnetic wave in dielectrics flows with the same speed as of the wave in dielectric and in the direction of propagation of the wave.

2.30 Plane Electromagnetic Waves in a Conducting Medium

Maxwell equations in differential form are;

$$\text{div}\,\mathbf{D} = \nabla.\mathbf{D} = \rho$$

$$\text{div}\,\mathbf{B} = \nabla.\mathbf{B} = 0$$

$$\text{curl}\,\mathbf{E} = \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\text{curl}\,\mathbf{H} = \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Let us assume that medium is linear and isotropic and is characterised by permittivity ϵ permeability μ and conductivity σ, but not any charge or any current other than that determined by Ohm's law. Then the parameters of the medium are $\mathbf{D} = \epsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}, \mathbf{J} = \sigma\mathbf{E}$ and $\rho = 0$

So that Maxwell's equations for a linear, isotropic, homogeneous and conducting medium will be;

$$\left.\begin{aligned} &\nabla.\mathbf{E} = 0 && \text{(i)} \\ &\nabla.\mathbf{H} = 0 && \text{(ii)} \\ &\nabla\times\mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t} && \text{(iii)} \\ &\nabla\times\mathbf{H} = \sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t} && \text{(iv)} \end{aligned}\right\} \qquad (127)$$

Taking curl of equation {(127),(iii)}, we get;

$$\nabla\times(\nabla\times\mathbf{E}) = -\mu\frac{\partial}{\partial t}(\nabla\times\mathbf{H}) \qquad (128)$$

Substituting $(\nabla\times\mathbf{H})$ from equation{(127) (iv)} into (128), we get;

$$\nabla\times(\nabla\times\mathbf{E}) = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla\times(\nabla\times\mathbf{E}) = -\sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (129)$$

As we know, $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E}$ (130)

Using identity (130) and putting $\nabla.\mathbf{E} = 0$ for the medium, the equation (129) takes the form;

$$\nabla^2\mathbf{E} = -\sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (131)$$

Similarly doing similar operation for magnetic field, we get;

$$\nabla^2 \mathbf{H} = -\sigma\mu\frac{\partial \mathbf{H}}{\partial t} - \epsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} \quad (132)$$

The equations (131) and (132) represent the equations of electromagnetic wave in a linear, homogeneous, isotropic, conducting medium of conductivity σ.

The general solutions of these second order differential equation (131) and (132)), is written as;

$$\mathbf{E} = E_0 e^{i\mathbf{K}.\mathbf{r} - i\omega t} \quad (133)$$

$$\mathbf{H} = H_0 e^{i\mathbf{K}.\mathbf{r} - i\omega t} \quad (134)$$

Where the wave vector, **K**, may be complex, while E_0 and H_0 are complex amplitudes which are constant in space and time. Satisfying the solutions from (133) and (134) with the corresponding differential equation (131) and (132) respectively, we get;

$$(-k^2 + i\sigma\mu\omega + \mu\epsilon\omega^2)\mathbf{E} = 0$$

$$(-k^2 + i\sigma\mu\omega + \mu\epsilon\omega^2)\mathbf{H} = 0$$

As the fields **E** or **H** are arbitrary, therefore this equation holds only if

$$(-k^2 + i\sigma\mu\omega + \mu\epsilon\omega^2) = 0$$

$$k^2 = \mu\epsilon\omega^2\left(1 + \frac{i\sigma}{\omega\epsilon}\right) = \mu\epsilon\omega^2 + i\,\sigma\mu\omega \quad (135)$$

From above equation, it seems that the wave propagation vector **K** is a complex quantity in the case, when the electromagnetic waves propagate in a linear, isotropic, homogeneous and conducting medium.

2.30.1 Other Physical Parameters of the Wave

2.30.1.1 Propagation Vector

Let us further simplify the equation (135) to find out some more physical parameters of electromagnetic waves in a conducting medium.

Let $\mathbf{K} = \alpha + i\beta$

So $\mathbf{K}^2 = \alpha^2 - \beta^2 + 2i\alpha\beta \quad (136)$

Comparing equation (97) and (98), we have

$$\alpha^2 - \beta^2 = \mu\epsilon\omega^2 \quad (137)$$

And

$$2\alpha\beta = \mu\omega\sigma \quad (138)$$

Or

$$\beta = \frac{\mu\omega\sigma}{2\alpha},$$

So, we have;

$$\alpha^2 - \left(\frac{\mu\omega\sigma}{2\alpha}\right)^2 = \mu\epsilon\omega^2$$

$$\alpha^4 - \mu\epsilon\omega^2\alpha^2 - \frac{\mu^2\omega^2\sigma^2}{4} = 0$$

$$(\alpha^2)^2 - \mu\epsilon\omega^2\alpha^2 - \frac{\mu^2\omega^2\sigma^2}{4} = 0$$

Comparing the above equation with the standard quadratic equation, we have the coefficients, a, b and c as;

$$a = 1, b = -\mu\epsilon\omega^2, \quad c = -\frac{\mu^2\omega^2\sigma^2}{4}$$

$$\alpha^2 = \mu\epsilon\omega^2 \pm \frac{\sqrt{\mu^2\epsilon^2\omega^4 + 4\times 1\times\frac{\mu^2\omega^2\sigma^2}{4}}}{2}$$

$$= \frac{\mu\epsilon\omega^2 \pm \sqrt{\mu^2\epsilon^2\omega^4 + \mu^2\omega^2\sigma^2}}{2}$$

$$= \frac{\mu\epsilon\omega^2 \pm \sqrt{\mu^2\epsilon^2\omega^4\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]}}{2}$$

$$\alpha = \sqrt{\mu\epsilon}\,.\,\omega \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}} + 1}{2}\right]^{1/2} \tag{139}$$

Similarly, the value of β comes out to be;

$$\beta = \sqrt{\mu\epsilon}\,.\,\omega \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}} - 1}{2}\right]^{1/2} \tag{140}$$

So, the wave propagation vector K of an electromagnetic wave travelling in a conducting medium having conductivity σ , μ permeability, ϵ permittivity and ω as the frequency of em waves, can be calculated as;

$$\mathbf{K} = \sqrt{\mu\epsilon}\,.\,\omega \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}} + 1}{2}\right]^{1/2} + i\sqrt{\mu\epsilon}\,.\,\omega \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}} - 1}{2}\right]^{1/2} \tag{141}$$

Which is a complex quantity.

Case 1. Propagation vector for Very Good Conductor

For a good conductor $\frac{\sigma}{\omega\epsilon} \gg 1$, so 1 can be neglected in comparison of $\frac{\sigma}{\omega\epsilon}$ so that α and β are approximately equal i.e.

$$\alpha \approx \beta = \sqrt{\mu\epsilon}.\,\omega\sqrt{\frac{\frac{\sigma}{\omega\epsilon}}{2}} = \sqrt{\frac{\mu\sigma\omega}{2}}$$

So, the propagation constant comes out to be

$$\mathbf{K} \approx \alpha + i\beta = \sqrt{\frac{\mu\sigma\omega}{2}} + i\sqrt{\frac{\mu\sigma\omega}{2}}$$

Case 2. Propagation vector for Very Poor Conductor

For very poor conductor, $\frac{\sigma}{\omega\epsilon} \ll 1$, and be neglected then

$\alpha = \sqrt{\mu\epsilon}.\,\omega$ and β may neglected or can be taken as zero.

$$\therefore\ \mathbf{K} = \alpha + i\beta = \sqrt{\mu\epsilon}.\,\omega \tag{142}$$

Or $\mathbf{K} = \frac{\omega}{v}$, since $v = \frac{1}{\sqrt{\mu\epsilon}}$

This comes out to be as in the case of non-conductor and also shows negligible or zero attenuation of electromagnetic waves in a very poor conductor, i.e., may be treated as a homogeneous isotropic non-conducting medium.

2.30.1.2 Attenuation of Electromagnetic Waves in a Conducting Medium

Now putting the value of $\mathbf{K} = \alpha + i\beta$, in the equations (95) and (96) we have:

$$E = E_0 e^{i(\alpha+i\beta)\,\mathbf{n.r}-i\omega t} = E_0 e^{-\beta \mathbf{n.r}} e^{i\,(\alpha \mathbf{n.r}-\omega t)} \quad (143)$$

$$H = H_0 e^{i(\alpha+i\beta)\,\mathbf{n.r}-i\omega t} = H_0 e^{-\beta \mathbf{n.r}} e^{i\,(\alpha \mathbf{n.r}-\omega t)} \quad (144)$$

From equations (133) and (134), it can be seen that field amplitudes of the electric and magnetic oscillations i.e., $E_0 e^{-\beta \mathbf{n.r}}$ and $H_0 e^{-\beta \mathbf{n.r}}$ respectively, are attenuated with distance due to the term $e^{-\beta \mathbf{n.r}}$. We can say the amplitude of electric and magnetic fields vectors decay exponentially. The quantity β is a characteristic of the medium and is known as decay constant or absorption coefficient. So, the attenuation coefficient of any electromagnetic wave in a conducting medium is given by;

$$\beta = \sqrt{\mu\epsilon}\,.\,\omega \left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}}-1}{2}\right]^{1/2} \quad (145)$$

2.30.1.3 Penetration of Electromagnetic Wave into a Medium - skin Depth or Penetration Depth

Now, we have seen that as the electromagnetic waves travel through a conducting medium, the amplitude of the field vectors, attenuate with distance. The distance/depth from the surface of the medium, at which the amplitude becomes $\frac{1}{e} = 0.369$, of its initial amplitude at the surface, i.e., $E = \frac{1}{e} E_0$, is called skin depth or penetration depth.

For this, the distance r should be $\frac{1}{\beta}$ and is denoted by δ.

So, the penetration of skin depth is given by;

$$\delta = r = \frac{1}{\beta} = \frac{1}{\sqrt{\mu\epsilon}\,.\omega}\left[\frac{2}{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}}-1}\right]^{1/2} \quad (146)$$

For good conductor, the penetration depth is given by

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\sigma\omega}} \quad (147)$$

This expression may be useful to find the thickness of a conducting enclosure for protection from electromagnetic radiation. This also shows that very high frequency electromagnetic waves travel through the surface of the conductors.

Let us take a few examples:

(i) For copper at 60 cycles δ is 0.86 cm, but at 1 megacycle, it has dropped to 0.0067. That is why in high frequency circuits current flows only on the surface of the conductors. The major importance of the skin depth is that it measures the depth to which an electromagnetic wave can penetrate a conducting medium. Therefore, the conducting sheets which are used as electromagnetic shields must be thicker than the skin depth.

(ii)) For silver $\sigma \approx 10^7 \text{mho/m}$ at a typical microwave frequency$\approx 10^8 \frac{c}{s}$, the skin depth $\approx 10^{-4}$ cm. Thus at microwave frequencies the skin depth in silver is very small and consequently performance of a pure silver component and a silver-plated brass component would be expected to be indistinguishable.

(iii) For sea water $\sigma \approx 4.3\ \mathrm{mho/m}$ at a frequency of 60 kc/s ; so that $\delta \approx 1\text{meter}$. That is why radiocommunication with submerged submarine becomes increasing difficult at several skin depths.

2.31 Transverse Character of Electromagnetic Waves in Conducting Medium

Now satisfying the general solutions of differential equations of electromagnetic waves, we get;

$$i\mathbf{K}.\mathrm{E} = 0 \quad \text{or} \quad \mathbf{K}.\mathrm{E} = 0 \tag{148}$$

$$i\mathbf{K}.\mathrm{H} = 0 \quad \text{or} \quad \mathbf{K}.\mathrm{H} = 0 \tag{149}$$

These equations imply that field vectors **E** and **H** are both perpendicular to the direction of propagation vector **K**. This implies that electromagnetic waves in a conducting medium are transverse in nature.

On satisfying third and fourth equations we get

$$\nabla \times \mathbf{E}(\mathrm{r}, \mathrm{t}) = -\mu \frac{\partial \mathbf{H}(\mathrm{r,t})}{\partial \mathrm{t}} \tag{150}$$

$$\nabla \times \mathbf{H}(\mathrm{r}, \mathrm{t}) = \epsilon \frac{\partial \mathbf{E}(\mathrm{r,t})}{\partial \mathrm{t}} \tag{151}$$

$$\nabla \times \mathrm{E}_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} = -\mu \frac{\partial \mathrm{H}_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t}}{\partial t} \tag{152}$$

$$\nabla \times \mathrm{H}_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t} = \epsilon \frac{\partial \mathrm{E}_0 e^{i\mathbf{K}.\mathbf{r}-i\omega t}}{\partial t} \tag{153}$$

On simplification (152) and (153), we get;

$$i\mathbf{K} \times \mathbf{E} = i\mu\omega\mathbf{H} \text{ i.e. } \mathbf{K} \times \mathbf{E} = \mu\omega\, \mathrm{H} \tag{154}$$

and

$$i\mathbf{K} \times \mathbf{H} = (\sigma - i\epsilon\omega)\mathbf{E} \text{ ie. } \mathbf{K} \times \mathbf{H} = (\epsilon\omega + i\sigma)\mathbf{E} \tag{155}$$

The equations (154) and (155), imply that the electromagnetic field vectors **E** and **H** are mutually perpendicular and also they are perpendicular to the direction of propagation vector **K**, in a conducting medium also.

2.32 Relative Phase of E and H

Using equations (154) and (155), we have;

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{K} \times \mathbf{E}) = \frac{1}{\mu\omega} k(\mathrm{n} \times \mathbf{E})$$

$$= \frac{(\alpha+i\beta)}{\mu\omega}(\mathrm{n} \times \mathbf{E}) \tag{156}$$

This implies that $\left|\frac{\mathrm{H}}{\mathrm{E}}\right| = \frac{\mathrm{H}_0}{\mathrm{E}_0} = \frac{\alpha+i\beta}{\mu\omega} = \text{complex quantity}$ (157)

This implies that the field vectors H and E are out of phase in a conductor. The magnitude and phase of complex wave vector is written as $= |k|e^{i\phi}$. and can be calculated as;

$$|\mathbf{K}| = |\alpha + i\beta| = \sqrt{(\alpha^2 + \beta^2)} = \sqrt{\mu\epsilon}\omega\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4} \tag{158}$$

$$\text{and } \phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right) \tag{159}$$

So equation (156) may be expressed as

$$\mathrm{H} = \frac{1}{\mu\omega}\sqrt{\mu\epsilon}\omega\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}} e^{i\phi}\,(\mathrm{n} \times \mathrm{E})$$

$$= \frac{\epsilon}{\mu}\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}} e^{i(\phi)}\,(\mathrm{n} \times \mathrm{E}) \tag{160}$$

This equation shows that H lags behind E by the phase angle ϕ given by equation (159) and the relative magnitude of magnetic and electric field is;

$$\left|\frac{H}{E}\right| = \frac{H_0}{E_0} = \sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}} = \frac{1}{Z},$$ Where Z is the wave impedence in the conducting medium. (161)

2.33 Poynting Vector or Energy Flow of Electromagnetic Waves in Conducting Medium

The Poynting vector is given by:

$\mathbf{S = (E \times H)}$

And the time average of poynting vector may expressed as

$$S_{av} = \frac{1}{2}\mathrm{Re}\left[E \times \left\{\frac{\epsilon}{\mu}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} e^{i(-\phi)}\,(n \times E)\right\}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}} \mathrm{Re}\{E \times (n \times E^*)e^{i(-\phi)}\}$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}} \mathrm{Re}\left[\{(E.E^*)n) - (E.n)E^*\}e^{i(-\phi)}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}} E_0^2\, e^{-2\beta n.r} n \cos\emptyset \tag{162}$$

$\left[\text{Since } (E.E^*) = E_0^2\, e^{-2\beta n.r} \text{ and } \mathrm{Re}\,(e^{i(-\phi)} = \cos\emptyset\right]$

For good conductor $\sigma/\epsilon\omega \gg 1$ so that $\emptyset = \pi/4$ and also $E_{rms} = \frac{E_0}{\sqrt{2}}$

Hence $S_{av} = \left\{\left(\frac{\sigma}{2\mu\omega}\right)\right\} E_{rms}^2\; e^{-2\beta n.r}$

Energy density: The total energy density of electromagnetic field is given by

$$u = u_e + u_m$$

where electrostatic energy $u_e = \frac{1}{2}\,\mathrm{Re}\frac{1}{2}(E.D^*)$

$$= \frac{1}{2}\epsilon E_0^2 e^{-2\beta n.r} \tag{163}$$

$$= \frac{1}{2}\epsilon E_{rms}^2\; e^{-2\beta n.r} \tag{164}$$

and magnetic energy density $u_m = \frac{1}{2}\;\mathrm{Re}\frac{1}{2}(H.B^*)$

$$= \frac{1}{4}\; u\,\mathrm{Re}(H.H^*)$$

$$= \frac{1}{4}\;\mu\, H_0^2\; e^{-2\beta n.r}$$

$$= \frac{1}{4}\;\mu\frac{\epsilon}{\mu}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} E_0^2\; e^{-2\beta n.r} \quad \text{using (161)}$$

$$= \frac{1}{4}\epsilon\left\{\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} E_0^2\; e^{-2\beta n.r}\right\}$$

$$= \left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} u_e \tag{165}$$

So, the total energy density $u = u_e + u_m = u_e + \left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} u_e$

$$= \left[1+\left\{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}\right]^{1/2} u_e$$

$$= \left[1 + \left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\}\right]^{1/2} \times \frac{1}{2}\epsilon E_{rms}^2 \; e^{-2\beta n.r} \tag{166}$$

From equations (162) and (166) it is clear that the energy flux and energy density are damped as the electromagnetic wave propagates in a conducting medium. This energy lose is due to Joule heating of the medium.

Following points can be summarised for the electromagnetic waves in conducting medium

- The electromagnetic waves are transverse in nature in which the electric field vector E and magnetic field vector H are mutually perpendicular and perpendicular to the direction of propagation of electromagnetic wave.
- The amplitudes of electric and magnetic field vectors E and H respectively are damped exponentially as the wave propagates deeper in the conductors.
- The electric and magnetic field vectors E and H of the electromagnetic wave are not in the same phase as H lags behind E by angle Ø given by
$$\emptyset = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\epsilon\omega}\right)$$
- And the magnitude of H is much greater that than of E.
- The energy flow is along the direction of propagation of electromagnetic wave and is damped exponentially as the wave propagates in the conducting medium.
- The magnetic energy density is much greater than electric energy density and both are damped exponentially as the wave propagates in the conducting medium.

******Solved examples******

Based on S.H.M

Ex.1 If a particle moves in a potential energy field $U = U_0 - ax + bx^2$ where a, and b are positive constants, obtain an expression for the force acting on it as a function of position. At what point does the force vanish? Is this point of stable equilibrium? Calculate the force constant, time period and frequency of the particle.

Sol. Force acting on the particle is given as

(i) $F = -\frac{dU}{dx} = -\frac{d}{dx}(U_0 - ax + bx^2) = a - 2bx$

(ii) The force vanishes at the point where $\frac{dU}{dx} = 0$ i.e.

$$a - 2bx = 0 \quad \text{or} \quad x = a/2b$$

Which gives the position of point where the force vanishes

(iii) $\frac{d^2U}{dx^2} = 2b$ such that b is positive, the point x = a/2b represents the point of minimum potential energy on the energy curve of the particle. It is point of stable equilibrium

(iv) From the expression of force F in (i)above, the linear restoring force and force constant k is equal to 2b.

Ex.2 What is the frequeney of a simple pendulum 2.0 meters long?

Sol. The time period of a simple pendulum is given by:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Frequency,

$$n = 1/T = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

Here $l = 2m$, $g = 9.8m/s^2$

Frequency

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{9.8}{2}} = \frac{1}{2\pi}\sqrt{4.9} = 0.3524/\text{sec}$$

Based on gradient, divergence, curl

Ex.1 A scalar field u is given by $u = x^3y - xz^2 + yz$. Find grad u and its value at the point (0, 2,-1).

Sol. We have $u = x^3y - xz^2 + yz$

$\therefore$ grad $u = \nabla u = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})[x^3y - xz^2 + yz]$

$= \mathbf{i}(3x^2y - z^2) + \mathbf{j}(x^2+z) + \mathbf{k}(+2x+y)$

This is grad u. It's value at (0, 2, -1)is given as

$(\text{grad } u)_{(0,2,-1)} = \mathbf{i}(-1) + \mathbf{j}(-1) + \mathbf{k}(2)$

$= -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Ex.2 Find grad $\mathbf{r}^n$ where r is a position vector.

Sol. We have $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ and $\mathbf{r} = |r| = \sqrt{x^2 + y^2 + z^2}$

$\therefore r^n = (x^2 + y^2 + z^2)^{n/2}$

$\therefore r^n = \nabla r^n = [\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}](x^2 + y^2 + z^2)^{n/2}$

$= \mathbf{i}\frac{n}{2}(x^2 + y^2 + z^2)^{(n/2)-1}.2x + \mathbf{j}.\frac{n}{2}(x^2 + y^2 + z^2)^{(n/2)-1}\,2y + \mathbf{k}.\frac{n}{2}(x^2 + y^2 + z^2)^{(n/2)-1}.2z$

$= (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)[n(x^2 + y^2 + z^2)^{(n/2)-1}]$

$= \cancel{(\mathbf{r}.(nr^2)^{(n/2)-1}} = r.n\,r^{n-2} = r.n(r^{2(\frac{n}{2}-1)})$

Thus, grad $r^2 = nr^{n-2} = nr^{n-1}\hat{r}$

In particular, if $n \mp -1$ or, respectively, we have grad $r = \hat{r}$

And grad $\frac{1}{r} = -\frac{\hat{r}}{r^2}$. We can obtain these results by direct calculation also.

Ex.3 Show that grad log $\mathbf{r} = \frac{\hat{r}}{r}$

Sol. We have

grad log $r = \nabla \log r = (\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z})\log(\sqrt{x^2 + y^2 + z^2})$

$= \mathbf{i}\frac{1}{\sqrt{x^2+y^2+z^2}}.\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}.2x + \mathbf{j}.\frac{1}{\sqrt{x^2+y^2+z^2}}.\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}.2y + \mathbf{k}.$

$\frac{1}{\sqrt{x^2+y^2+z^2}}.\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}.2z$

$= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2} = r\frac{\hat{r}}{r} = \frac{\hat{r}}{r}$

Thus, grad log r = $\frac{\hat{r}}{r}$

Ex.4 A spherical equipotential surface is given by

$V(x, y, z) = a(x^2 + y^2 + z^2)$

Where a is a constant. Show that the force field is radial.

Sol. We know

$$F = -\text{ grad } V = [\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}]$$

$$\frac{\partial V}{\partial x} = 2ax, \frac{\partial V}{\partial y} = 2ay \text{ and } \frac{\partial V}{\partial z} = 2az$$

$$F = -2a\,[x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}] = -2ar$$

Where $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ is the position vector of point (x, y, z). since $F \propto r$, it is a radial force field.

Ex.5 If **r** is the position vector ($\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$), evaluate div r and div $\hat{\mathbf{r}}$.

Sol. We have

$$\text{div } r = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z} = \frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z} = 1+1+1 = 3$$

$$\text{div}\,\hat{\mathbf{r}} = \text{div}\left(\frac{\mathbf{r}}{r}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y} + \left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right) + \text{two similar terms}$$

$$= \frac{r.1 - x.\frac{1}{2}.\frac{1}{(x^2+y^2+z^2)^{1/2}}.2x}{(x^2+y^2+z^2)} + \text{two similar terms}$$

$$= \frac{r - \frac{x^2}{r}}{r^2} + \text{two similar terms} = \left(\frac{1}{r} + \frac{x^2}{r^3}\right) + \text{two similar terms}$$

$$= \left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{z^2}{r^3}\right) = \frac{3}{r} - \frac{1}{r}\left(\frac{x^2+y^2+z^2}{r^2}\right) = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

Thus div $\hat{\mathbf{r}} = \frac{2}{\mathbf{r}}$

Ex.6 Show that div $\left(\frac{\hat{r}}{r^2}\right)$is zero

Sol. We have

$$\text{div}\left(\frac{\hat{r}}{r^2}\right) = \left(\frac{r}{r^3}\right) = \text{div}\left[\frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \text{two similar terms}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{3}{2}}.1 - x.\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}.2x}{(x^2+y^2+z^2)^3} + \text{two similar terms}$$

$$= \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{3/2}}\right] + \text{two similar terms}$$

$$= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \text{two similar terms}$$

$$= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)$$

$$= \frac{3}{r^3} - \frac{3}{r^5}(x^2+y^2+z^2)\,\frac{3}{r^3} - \frac{3}{r^5}(r^2)$$

$$= \frac{3}{r^3} - \frac{3}{r^3} = 0$$

Ex.7 What must be the value of p so that the vector field **p** represented by $\mathbf{p} = \mathbf{i}(3x + py - 2z) + \mathbf{j}(2x + y - 2pz) + \mathbf{k}(4px - 3pz)$ is a solenoidal field ?

Sol. We want div p = 0

Now $\quad \text{div } p = \frac{\partial}{\partial x} p_x + \frac{\partial}{\partial y} p_y + \frac{\partial}{\partial z} p$

$= \frac{\partial}{\partial x}(3x + py - 2z) + \frac{\partial}{\partial y}(2x + y - 2pz)$

$= 3+1-3 = 4 - 3p$

$\therefore \quad \text{div } p = 0 \text{ if } \quad 4 - 3p = 0$

Or $\quad = 4/3$

Ex.8 Evaluate grad div B and div (grad u)

Sol. We have

$$\text{Grad (div B)} = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)$$

$$= \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\mathbf{k}\right]\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right]$$

$$= \mathbf{i}\left(\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_y}{\partial x\,\partial y} + \frac{\partial^2 B_z}{\partial x\,\partial z}\right) + \mathbf{j}\left(\frac{\partial^2 B_x}{\partial y\,\partial x} + \frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_z}{\partial y\,\partial z}\right) + \mathbf{k}\left(\frac{\partial^2 B_x}{\partial z\,\partial x} + \frac{\partial^2 B_y}{\partial z\,\partial y} + \frac{\partial^2 B_z}{\partial z^2}\right)$$

Again,

$$\text{Div (grad u)} = \text{div}\left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial z}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u$$

Ex.9 A central force field **F** is $\mathbf{F} = k\, r^n \mathbf{r}$. The field **F** is solenoidal such that div $\mathbf{F} = 0$.

Sol. Since $k \neq 0$; for field to be solenoidal, so

$$\text{div}(r^n r) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\hat{k}\right).r^n\,(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial x}(r^n z)$$

Let us evaluate $\frac{\partial}{\partial x}(r^n x)$,

$$\frac{\partial}{\partial x}(r^n x) = \frac{\partial}{\partial x}\left[(x^2 + y^2 + z^2)^{\frac{n}{2}}.x\right]$$

$$= (x^2 + y^2 + z^2)^{\frac{n}{2}}.1 + \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \times 2x$$

$$= r^n + nx^2(r^2)^{\frac{n}{2}-1} = r^n + nx^2 r^{n-2}$$

Similarly, $\frac{\partial}{\partial x}(r^n y) = r^n + ny^2 r^{n-2}$

and $\quad \frac{\partial}{\partial x}(r^n z) = r^n + nz^2 r^{n-2}$

$$\text{div}(r^n r) = 3r^n + (x^2 + y^2 + z^2)r^{n-2}$$

$$= 3r^n + nr^n = (3 + n)r^n$$

In order that div F = 0 ; 3 + n = 0

Or $\quad n = -3$

$F(r) = k/r^3$

Ex.10 Show that the position vector $\mathbf{r} = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$ is irrotational

Sol. We have to show that curl $\mathbf{r} = 0$

$$\text{Now curl } \mathbf{r} = \nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \text{two similar terms}$$

Now as z does not depend on x and y and so on, we have each of $\frac{\partial z}{\partial y}$ and $\frac{\partial y}{\partial z}$ equal to zero.

Hence $\quad$ curl $\mathbf{r} = \mathbf{i}(0) + \mathbf{j}(0) + \mathbf{k}(0) = 0$

Since curl **r** is zero, **r** is an irrotational vector.

Ex.11 Given that the vector **E** defined by.

$\mathbf{E} = (2x - 5y + pz)\mathbf{i} + (qx + 3y)\mathbf{j} + (ry + 6z + 3x)\mathbf{k}$ is irrotational, find p, q, r.

Sol. We have

$$(\text{curl } E)_x = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) \text{and so on.}$$

Hence
$$(\text{curl } E)_x = (r - 0) = 0$$
$$(\text{curl } E)_y = (p - 3) = 0$$
$$(\text{curl } E)_z = (q - 5) = 0$$

Thus, $\quad p = 0, q = 5, r = 0$

Ex.12 If $\mathbf{B} = yz^2\mathbf{i} - xyz\mathbf{j} + 3x^2z^3\,\mathbf{k}$, find curl **B** at the point (0, 1, 1).

Sol. We have

$$(\text{curl } B)_x = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = \frac{\partial}{\partial y}(3x^2z^3) - \frac{\partial}{\partial y}(-xyz) = 0 + xy = xy$$

Similarly, $$(\text{curl } B)_y = \frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial y}(3x^2z^3) = 2yz + 6xz^3$$

$$(\text{curl } B)_z = \frac{\partial}{\partial y}(-xyz) - \frac{\partial}{\partial y}(yz^2) = -yz - z^2$$

$\therefore$ curl B = $\mathbf{i}(xy) + \mathbf{j}(2yz + 6xz^3) - \mathbf{k}(yz + z^2)$

Thus $(\text{curl } \mathbf{B})_{0,1,1} = \mathbf{i}(0 \times 1) + \mathbf{j}(2 \times 1 \times 1 + 6 \times 0 \times 1^3) - \mathbf{k}(1 \times 1 \times 1^2)$

$$= 2(\mathbf{j} - \mathbf{k})$$

Ex.13 If $\mathbf{F} = (xy)\mathbf{i} - (yz)\mathbf{j} + (zx)\mathbf{k}$, find curl **F** and curl(curl **F**). Evaluate it at (0, 0, -1).

Sol. We have

$$\text{Curl } \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial x}{\partial y}\right)$$
$$= \mathbf{i}\left[\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(-yz)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(xy)\right]$$
$$= \mathbf{i}(+y) + \mathbf{j}(-z) + \mathbf{k}(-x)$$

$$\therefore \text{Curl (curl } \mathbf{F}) = \mathbf{i}\left[\frac{\partial}{\partial y}(-x) - \mathbf{j}\frac{\partial}{\partial z}(-z)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(x)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(+y)\right]$$
$$= \mathbf{i}(+1) + \mathbf{j}(+1) + \mathbf{k}(-1)$$

$$\text{Curl(curl } \mathbf{F}) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

It has the same value at all points and value at (0, 0, −1) is also $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Based on Equation of continuity, Displacement current

Ex.14 A parallel plate capacitor having circular plates of radius 5 cm is being charged. If the electric field between the plates during charging is changing at the rate of 10^{12} V/ms, find the displacement current between the plates.

Sol. The displacement current is

$$I_d = \varepsilon_0 \frac{d\phi_E}{dt}$$
$$= \frac{d}{dt}(\pi R^2 E)$$
$$= \varepsilon_0 \pi R^2 \frac{dE}{dt}$$

$$k = \alpha = {}^{2\pi}/_{\lambda} = \frac{\omega}{c}$$
$$= \frac{10^8}{3\times10^8} = \frac{1}{3}\,m^{-1}$$

The frequency of the wave is

$$\nu = \frac{\omega}{2\pi} = \frac{10^8}{2\times3.14} = 1.67\times10^7\,Hz$$

The magnetic field is similar in form to the electric field:

$\mathbf{B} = B_0\exp[-\mathbf{i}\,(10^8\,t + \frac{1}{3}z)]$

The magnitude of B_0 is given by

$$B = \frac{E_0}{c} = \frac{60}{c}$$
$$= \frac{60}{3\times10^8} = 2\times10^{-7}\,\text{tesla}$$

Ex.21 A plane electromagnetic wave is travelling in the $-\hat{x}$ direction. Its frequency is 100 MHz and the electric field is perpendicular to $\hat{z}$ direction. Write down the expressions for the **E** and **B** fields that specify the wave.

Sol. Since the wave is travelling in the $-\hat{x}$ direction, the **E** field is normal to $\hat{x}$. Also, **E** is perpendicular to $\hat{z}$ direction. Therefore, **E** is in the $\hat{y}$ direction. The expression for the **E** field can be written as

$$\mathbf{E} = E_0\hat{y}\cos(kx + \omega t)$$

Where $\omega = 2\pi\nu = 2\pi\times10^8$ Hz

And $k = \frac{\omega}{c} = \frac{2\pi}{3}\,m^{-1}$

Thus,

$$\mathbf{E} = E_0\hat{y}\cos[2\pi(\frac{x}{3}x + 10^8t)$$

The corresponding **B** field is given by

$$B = \frac{1}{\omega}(k\times E) = \frac{k}{\omega}(\hat{k}\times E)$$
$$= \frac{1}{c}\hat{k}\times E$$
$$= \frac{1}{c}(-\hat{x}\times\hat{y})E_0\cos[2\pi\left(\frac{x}{3}x + 10^8t\right)]$$
$$= -\frac{E_0}{c}\hat{z}\cos[2\pi\left(\frac{x}{3}x + 10^8t\right)]$$

Ex.22 The electric field of an electromagnetic wave in free space is given by

$E_x = 0,\ E_y = 50\sin(2\pi\times10^8t - \frac{2\pi}{5}x),\ E_z = 0$

Where all the quantities are in SI units. Determine a) the wavelength of the wave b) the direction of propagation of the wave and c) the direction of the magnetic field.

Sol. The problem can be solved by comparing the given expression for E_y with the standard expression

$\mathbf{E_y} = E_0\sin(\omega t - \mathbf{k}x)$,

$\omega = 2\pi\times10^8$ rad/s, $\mathbf{k} = \frac{2\pi}{5}m^{-1}$

a) Wavelength $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\frac{2\pi/5}{5}} = 5m$

b) The direction of propagation is the +x direction.

c) Since **E** is the +y direction and **k** is along the +x direction, the **B** field is along the + z direction.

Based on electromagnetic wave travelling in a dielectric medium

Ex.23 A plane electromagnetic wave travelling in positive z-direction in an unbounded lossless dielectric medium with relative permeability $\mu_r = 1$ and relative permittivity $\varepsilon_r = 3$ has a peak electric field intensity $E_0 = 6\ V/m$. Find

i) The speed of the wave

ii) the intrinsic impedance of the medium

iii) The peak magnetic field intensity (H_0), and

iv) The peak Poynting vector S (z, t).

Sol. $E_0 = \sqrt{\left(E^2{}_{ox} + E^2{}_{oy}\right)} = 6V/m,\ \varepsilon_r = 3,\ \mu_r = 1.$

i) The speed of electromagnetic wave

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{(\mu_r\mu_0\varepsilon_r\varepsilon_0)}}$$

$$= \frac{1}{\sqrt{(\mu_0\varepsilon_0)}} \cdot \frac{1}{\sqrt{(\mu_r\varepsilon_r)}} = \frac{c}{\sqrt{(\mu_r\varepsilon_r)}}$$

$$= \frac{3\times10^8}{\sqrt{(1\times3)}} = 1\cdot73\ \times\ 10^8 m/s.$$

ii) Impedance of medium

$$z = \sqrt{\left(\frac{\mu}{\varepsilon}\right)} = \sqrt{\left(\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}\right)} = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)} \cdot \sqrt{\left(\frac{\mu_r}{\varepsilon_r}\right)}$$

$$= \sqrt{\left(\frac{4\pi\times10^{-7}}{8\cdot86\times10^{-12}}\right)} \cdot \sqrt{\left(\frac{1}{3}\right)} = \frac{376\cdot6}{\sqrt{3}}$$

$$= 217\cdot6\ \Omega.$$

iii) Peak value of magnetic field

$$H_0 = \frac{E_0}{Z} = \frac{6}{217\cdot6} = 2\cdot76\ \times\ 10^{-2} A/m.$$

iv) Poynting vector $\qquad S = E \times H$

Peak Poynting vector $= E_0H_0 = \frac{{E_0}^2}{Z}$

$$= \frac{6^2}{217\cdot6} = 0\cdot165\ W/m^2$$

Ex.24 In a homogenous non-conducting medium the electric and magnetic fields of electromagnetic wave are given by:

$E = 30\pi\hat{z}\exp[\mathbf{i}\left(\omega t - \frac{4}{3}y\right)]$ V/m and $H = 1\hat{x}\exp[\mathbf{i}\left(\omega t - \frac{4}{3}y\right)]$ A/m

It is given that for the medium, μ_r=1. Calculate a) ε_r b) the velocity of light in this medium c) ω

Sol. a) From the equation:

$$Z = \frac{E}{H} = 377\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$30\pi = 377\sqrt{\frac{1}{\varepsilon_r}}$$

Or $\varepsilon_r = (\frac{377}{30\pi})^2 = 16$

b) The velocity of light in the medium is

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$$
$$= (\frac{1}{\sqrt{\mu_0\varepsilon_0}})(\frac{1}{\sqrt{\mu_r\varepsilon_r}})$$
$$= \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{1 \times 16}}$$
$$= \frac{c}{4} = \frac{3 \times 10^8}{4}$$
$$= 7.5 \times 10^7 \text{m/s}$$

c) $\omega = vk = \left(\frac{c}{4}\right)\left(\frac{4}{3}\right) = \frac{c}{3} = 10^8 \text{rad/s}$

Based on skin depth

Ex.25. Calculate the skin depth for 3 MHz electromagnetic wave through copper. (Given conductivity $\sigma = 6 \times 10^7 \text{mho/m}$, $\mu = 4\pi \times 10^{-7} \text{Henry/m}$.

Sol. $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2}{\mu\sigma(2\pi v)}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 6 \times 10^7(2\pi \times 3 \times 10^6)}} = 37.5\mu\text{m}$

Ex.26 The constitution parameter of aluminium is given by $\mu_r = 1$, $\varepsilon_r = 1$ and $\sigma = 3.54 \times 10^7 \text{mho/m}$. Find the frequency for which the skin depth/penetration depth of aluminium is 0.01mm.

Sol. $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{1}{\mu\sigma(\pi v)}}$

Or $v = \sqrt{\frac{1}{\pi\delta^2\mu\sigma}}$

$$= \frac{1}{3.14 \times (0.01 \times 10^{-3})^2 \times 1 \times 3.54 \times 10^7}$$
$$= \frac{1000}{3.14 \times 3.54} = 89.96 \text{ Hz}$$

Ex.27 For silver, $\sigma = 5.0\text{MS/m}$. At what frequency will the depth of penetration δ be 1mm?

Sol. $v = \sqrt{\frac{1}{\pi\delta^2\mu\sigma}} = 84.4\text{kHz}$

Ex.28 Show that for a good conductor the magnetic field lags the electric field by 45^0. Determine the ratio of their amplitudes.

Sol. $\alpha + i\beta = (1 + i)\sqrt{\frac{\mu\sigma\omega}{2}}$

$\alpha = \sqrt{\frac{\mu\sigma\omega}{2}}$ and $\beta = \sqrt{\frac{\mu\sigma\omega}{2}}$.

$\emptyset = \tan^{-1}\frac{\beta}{\alpha} = 45^0$

*****Review Questions*****

Based on Simple harmonic motion

1.What is meant by a harmonic oscillator? Obtain expressions for (1) its displacement and velocity at a given instant (2) time period and frequency

2.Show that for a simple harmonic oscillator, mechanical energy remains conserved and that its energy is on an average, half kinetic and half potential in form. At what particular displacement is this exactly so? What is the ratio between its kinetic and potential energies at a displacement equal to half of its amplitude?

3. Solve the differential equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$ to obtain the expression $x = A\sin(\omega t + \delta)$ for the displacement of a particle executing S.H.M.

4.Show that for a S.H.M may be expressed as either a sine or a cosine wave function, there being only a difference of initial phase in the two cases.

5. Is it really possible to construct a truly simple pendulum? What are the drawbacks of simple pendulum?

6.How does a compound pendulum differ from a simple pendulum? Obtain an expression for its time period and mention its points of superiority over simple pendulum.

7. Define centres of suspension and oscillation of a compound pendulum and show that they are interchangeable. What length of the pendulum has its minimum time period?

Based on Equation of continuity, displacement current

8.State the equation of continuity.

9.What is the physical significance of equation of continuity?

10.What is meant by displacement current?

11.In what way the displacement current is different from the conventional current?

12.What is the value of displacement current when a capacitor becomes fully charged? Give explicit reasoning for your answer.

13.Give the equation of continuity of electromagnetic theory. Explain the inconsistency of Ampere's law for transient currents. How was the law modified in its generalized form to overcome the inconsistency?

14.Discuss in brief the inconsistencies in Ampere's law and describe how Maxwell fixed up this. Further discuss in brief the characteristics of displacement current.

15.Illustrate mathematically how and under what conditions does Ampere's circuital law fail. How did Maxwell modify Ampere's law to make it consistent under all conditions? Give the mathematical justification to prove this consistency

Based on Maxwell's equation

16.Write Maxwell's equation differential form

17.Write Maxwell's equation in integral form.

18.One of the Maxwell's equations is $\nabla \cdot B = 0$. What is its physical implication?

19.What was the most important consequence of Maxwell's equations.

20.Explain how Maxwell's equations were developed. Write them in integral and differential forms.

21.Write Maxwell's equation in integral form. Discuss the physical meaning of each of these.

22.Enumerate Maxwell's equations and show how they predict the existence of electromagnetic waves.

Based on electromagnetic waves in free space

23.What is the velocity of electromagnetic waves in free space?

24.Show that the electromagnetic waves are transverse waves.

25.Show that in an electromagnetic wave the E and B fields are perpendicular to each other and also perpendicular to the direction of propagation.

26.What is wave impedance? What is its value for free space?
27.Write the expression for the refractive index of a medium in terms of electric and magnetic quantities.
28.State Maxwell's equations for the electromagnetic fields and obtain the wave equations for E and B in free space.
29.Discuss the propagation of plane electromagnetic waves through free space. Establish the transverse nature of these waves. What is the expression for wave impedance?

Based on Poynting theorem and Poynting vector
30.State the Poynting theorem. Explain the term Poynting vector.
31.What is Poynting vector and what does it represent?
32.State Poynting theorem.
33.What is Poynting vector? How is the Poynting theorem derived from Maxwell's curl equations? Explain Poynting theorem.

Based on em waves in isotropic dielectric
34.Write down Maxwell's equations for electromagnetic fields in a homogeneous isotropic dielectric. Solve these equations to get the velocity of propagation of electromagnetic waves. Why do we regard these waves as transverse? Show that the wave energy is equally shared between the electric and magnetic fields.
35.Discuss the propagation of plane electromagnetic waves in a non-conducting medium.
36.Obtain the wave equation for a plane E M wave in an isotropic dielectric medium and show that its velocity of propagation is less than the speed of light.
37.(a) Write Maxwell's equations in differential form and give their significance.
(b) Derive the wave equation for an isotropic dielectric medium. Prove the orthogonality of E, H and K vectors. Find the wave impedance of the medium.

Based on Wave in conducting medium
38.What do you understand by skin depth?
39.(a) Discuss the propagation of monochromatic plane electromagnetic waves in a conducting medium. (b) Show that in a good conductor the magnetic field lags the electric field by 45°. Determine the ratio of their amplitudes.
40.Define skin depth. Show that in case of good conductor, the skin depth is given by $\delta = (\frac{2}{\omega\sigma\mu})^{1/2}$
Show that inside the conducting medium electromagnetic wave is damped and obtain an expression for skin depth.
41.Obtain the equation of plane em wave in a conducting medium. Prove the orthogonality of electric vector E, intensity of magnetic field H and propagation wave vector $\hat{k}$.

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