EXACT DIFF. EQUATIONS

LINEAR DIFFERENTIAL EQNs (LDE)

↳ with constant co-efficient

$$\underbrace{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y}_{\text{LHS}} = \underbrace{X}_{\text{RHS}}$$

$a_0, a_1, \ldots$ → constants

X → const or $f(x)$

SYMBOLIC OPERATOR

↳ Replace $D \rightarrow \frac{d}{dx}$

So $D^2 = \frac{d^2}{dx^2}$ $D^3 = \frac{d^3}{dx^3}$ .... $D^n = \frac{d^n}{dx^n}$

↳ LDE can be written as:

$$[f(D)]y = X$$

where y = Complete Solution = $\underbrace{\text{Complementary function (CF)} + \text{Particular Integral (PI)}}_{\text{Complete Solution (CS)}}$

AUXILIARY EQUATION

Basically, take the equation, set aside y and equate it to zero:

Aux Equation (AE) = $\boxed{f(D) = 0}$

$f(D)$ → function in D

FINDING CF (COMPLEMENTARY FUNCTION)

Consider $[f(D)]y = 0$ $\quad$ Roots of Eqn

$\hookrightarrow$ AE : $f(D) = 0 \quad \left\{ \begin{matrix} m_1 \\ m_2 \\ m_n \end{matrix} \right\}$

CASE 1: All roots are real and distinct

$$\boxed{CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}}$$

CASE 2: 2 roots are equal

$$\boxed{CF = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_2 x} + \dots}$$

3 roots equal
$\hookrightarrow (C_1 + C_2 x + C_3 x^2 \dots)$

Pair of equal roots

Distinct Roots (C1)

CASE 3: If 2 roots of auxiliary equation are imaginary

$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$

$$\hookrightarrow \boxed{CF = e^{\alpha x} \left(C_1 \cos(\beta x) + C_2 \sin(\beta x) \right)} \dots$$

CASE 4: If 2 pairs of imaginary roots are equal

$m_1 = m_2 = \alpha + i\beta \qquad m_3 = m_4 = \alpha - i\beta$

$$\boxed{CF = e^{\alpha x} \left[(C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x \right]}$$

Q CI1 : Solve $\frac{d^3y}{dx^3} - \frac{7dy}{dx} - 6y = 0$

Ans $\frac{d^3y}{dx^3} - \frac{7dy}{dx} - 6y = 0$

Writing the eqn in symbolic form :

$$(D^3 - 7D - 6)\,y = 0$$

AE : $D^3 - 7D - 6 = 0$

$\hookrightarrow (D+1) = 0$

$\Rightarrow (D+1)(D^2 - D - 6) = 0$

$\Rightarrow (D+1)(D-3)(D+2) = 0$

$$\begin{array}{r|l} & D^2 - D - 6 \\ \hline D+1 & D^3 - 7D - 6 \\ & \cancel{D^3} \pm D^2 \\ \hline & \mp D^2 - 7D - 6 \\ & \cancel{D^2} \mp D \\ \hline & -6D - 6 \\ & -6D - 6 \\ \hline & 0 \end{array}$$

$m_1 = -1$; $m_2 = -2$; $m_3 = 3$

$$y = CS = CF = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

Q CI2 : Solve $(D^3 - 4D^2 + 4D)\,y = 0$

Ans Aux Equation : $D^3 - 4D^2 + 4D = 0$

$D(D^2 - 4D + 4) = 0 \longrightarrow \boxed{m_1 = 0}$

$D(D-2)(D-2) = 0$ $\boxed{m_3 = 2}$ $\boxed{m_2 = 2}$

$$y = CS = CF = (C_1 + C_2 x)\,e^{2x} + C_3$$

QCI 3: Solve $\frac{d^4x}{dt^4} + 4x = 0$

Ans Let $D = \frac{d}{dt}$

$\hookrightarrow D^4 x + 4x = 0$

$\Rightarrow (D^4 + 4)x = 0$

Auxiliary equation : $D^4 + 4 = 0$

$\Rightarrow D^4 + 4 + 4D^2 - 4D^2 = 0$

$\Rightarrow (D^2+2)^2 - 4D^2 = 0$

$\Rightarrow (D^2+2-2D)(D^2+2+2D) = 0$

$D = \frac{+2 \pm \sqrt{4-8}}{2}$ $\quad D = \frac{-2 \pm \sqrt{4-8}}{2}$

$= 1 \pm i$ $\quad \Rightarrow -1 \pm i$

$x = CS = CF = e^{x}(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$

QCI4: Solve $\frac{d^2y}{dx^2} - \frac{4dy}{dx} + y = 0$

Ans let $\frac{d}{dx} = D$

$\Rightarrow D^2y - 4Dy + y = 0$

$\Rightarrow (D^2 - 4D + 1)y = 0$

Auxilary Equation : $D^2 - 4D + 1 = 0$

$D = \frac{+4 \pm \sqrt{16-4}}{2}$

$D = \frac{4 \pm \sqrt{12}}{2} \Rightarrow \frac{4 \pm 2\sqrt{3}}{2} \rightarrow 2+\sqrt{3}$ OR $2-\sqrt{3}$

$$y = CS = CF = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x}$$

PARTICULAR INTEGRAL

For differential equations $[f(D)y] = X \rightarrow$ function of x

$$\boxed{PI = \frac{1}{f(D)}(X)}$$

CASE (1) : when $X = e^{ax}$ (a is const.)

$$PI = \frac{1}{f(D)}(e^{ax})$$

Replace $[D = a]$

$$PI = \frac{1}{f(a)}(e^{ax})$$

$f(a)$ ↳ should $\neq 0$

CASE OF FAILURE

→ If $f(a) = 0$ then it is case of failure.

→ $$PI = \frac{x}{[f'(D)]_{x=a}} \cdot e^{ax}$$ { Multiply numr with x Differentiate denr }

→ If denominator still equal to zero, repeat the process.