NP-Completeness, part I

Christian Wulff-Nilsen
Advanced Algorithms and Data Structures
DIKU

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Overview for today

- Problems and decision problems

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- Polynomial-time solvable problems

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- The circuit-satisfiability problem

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- For `SHORTEST-PATH`, an instance is a triple $\langle G, s, t\rangle$.
- A solution is a sequence of vertices forming a shortest s-to-t path.

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- Instances with solution 1 are called *yes*-instances.
- Instances with solution 0 are called *no*-instances.
- Optimization problems (like $\mathtt{SHORTEST\text{-}PATH}$) can usually be turned into decision problems (like $\mathtt{PATH}$).

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- What is missing in this definition? Which encoding of the input is assumed?

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- These two ways of encoding k correspond to two different problems.

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- Encodings are always binary strings in our setting.

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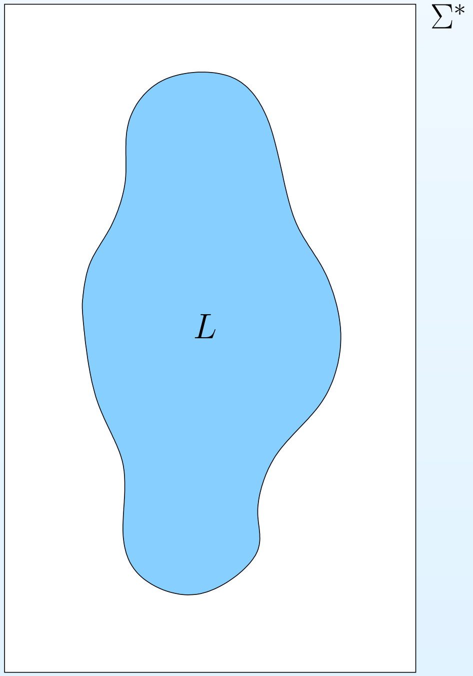
- Σ^* denotes the language of all strings (including ϵ).
- Any language L over Σ is a subset of Σ^*.



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- For instance, PATH is the language of binary strings $\langle G, u, v, k\rangle$ where G is a graph, u and v are vertices of G, and there is a u-to-v path in G with at most k edges.

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- Deciding a language is stronger than accepting it.

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- Example: `PATH` can both be accepted and decided in polynomial time.
- We can now define the complexity class P:

$$P = \{L \subseteq \{0,1\}^* | \text{there exists an algorithm } A \text{ that decides } L \text{ in polynomial time}\}.$$

P in terms of acceptance

- Lemma:

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- A' decides L and runs in polynomial time.

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- Consider an algorithm A taking two parameters, $x, c \in \Sigma^*$.
- Instead of trying to find a solution to x (which may take long time), A instead *verifies* that c is a solution to x.

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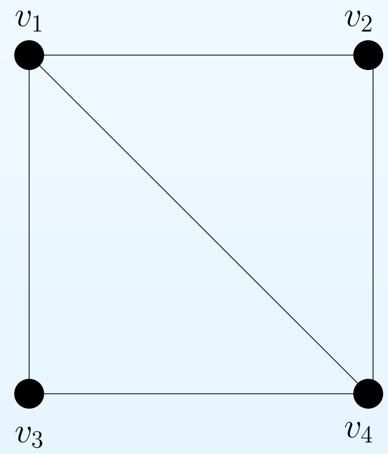

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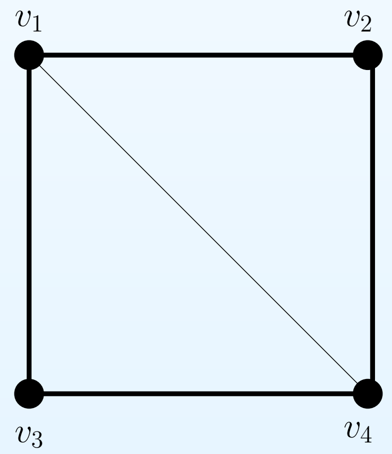

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- However, it is easy to show (next slide) that $\texttt{HAM-CYCLE}$ can be verified in polynomial time.

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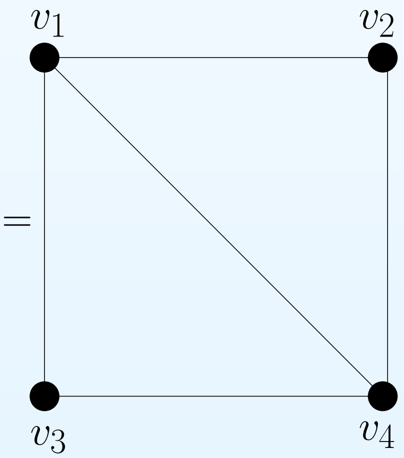
$G =$ 

$C = [v_1, v_2, v_3, v_4]$

- What is $A_{ham}(\langle G \rangle, \langle C \rangle)$?

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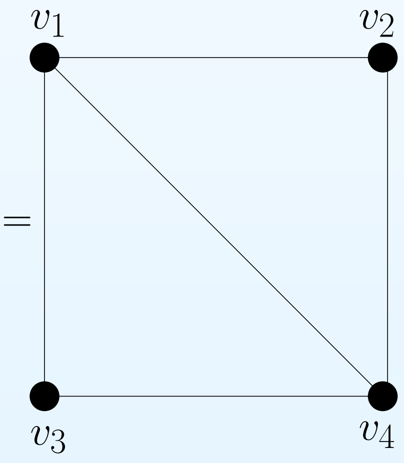
$G =$ 

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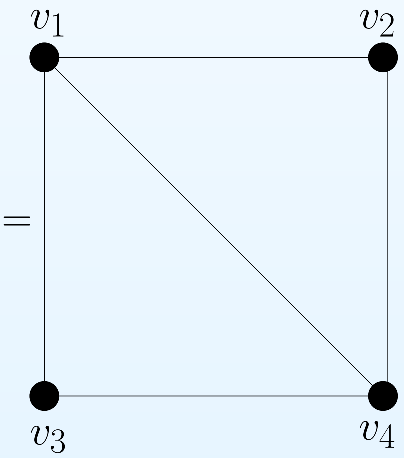

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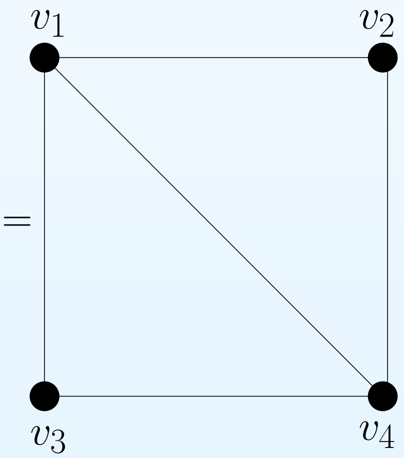
$G =$ 

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- Hence we can verify HAM-CYCLE in polynomial time.

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- Example:

$$\texttt{HAM-CYCLE} = \{x \in \{0,1\}^* | \text{there is a } y \in \{0,1\}^* \text{ such that } A_{ham}(x,y) = 1\}.$$

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- To define NP-completeness, we need to first define polynomial-time reducibility.

Polynomial-time reducibility

- Language L_1 is polynomial-time *reducible* to language L_2 if there is a polynomial-time computible function $f : \{0,1\}^* \to \{0,1\}^*$ such that for all $x \in \{0,1\}^*$,
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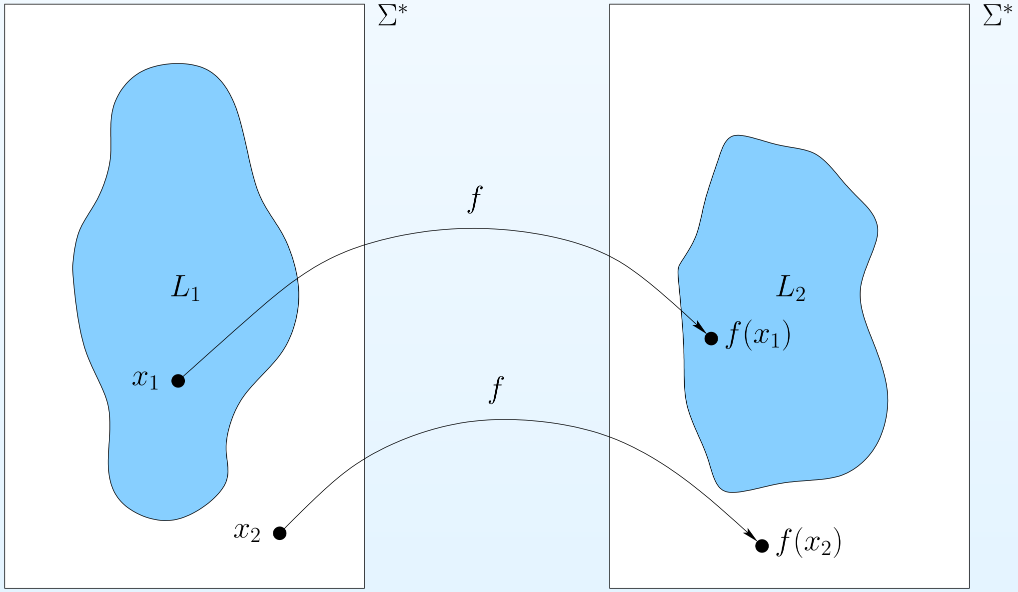

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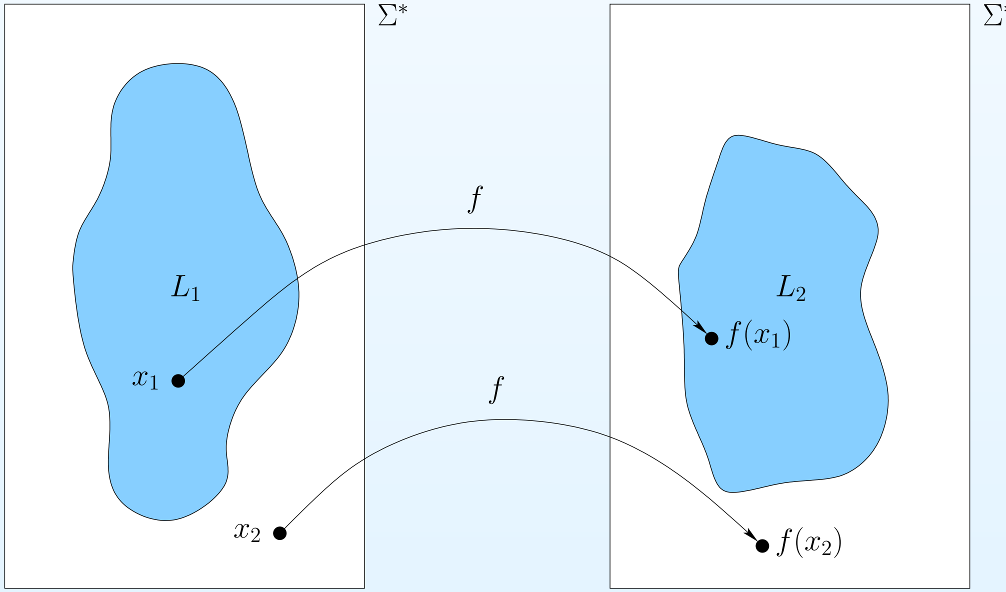

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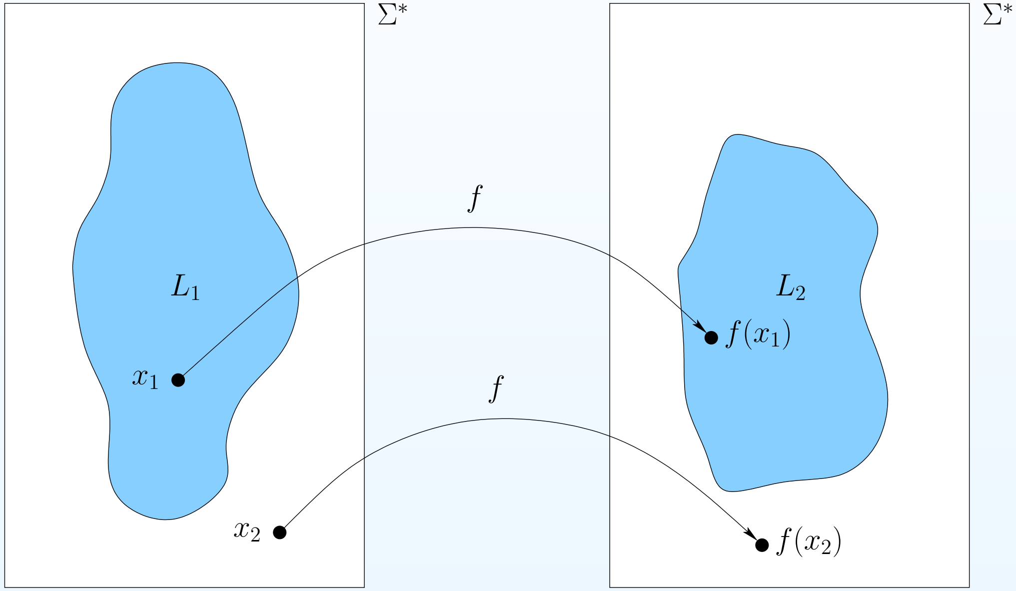

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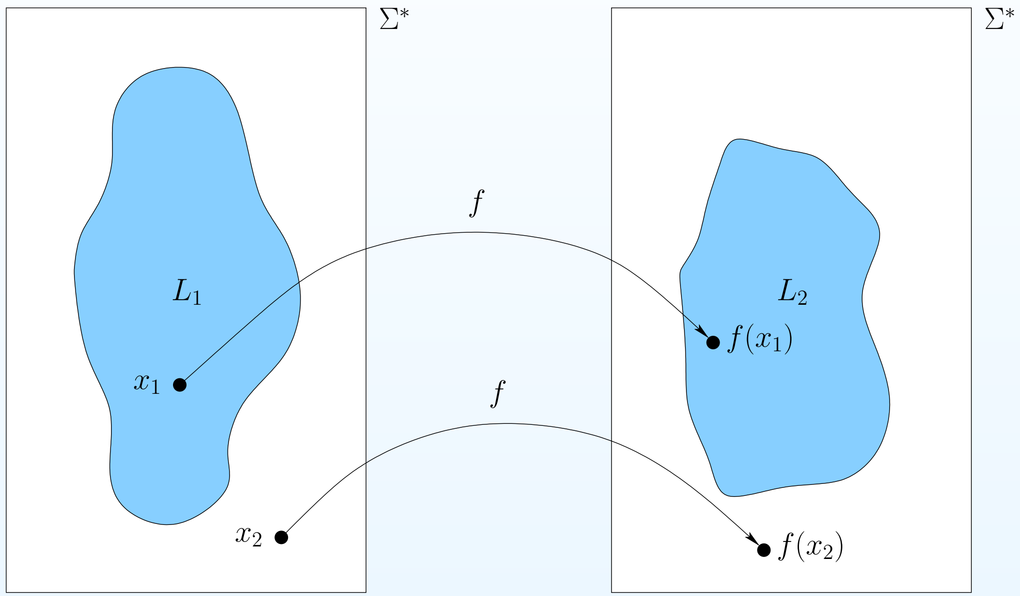

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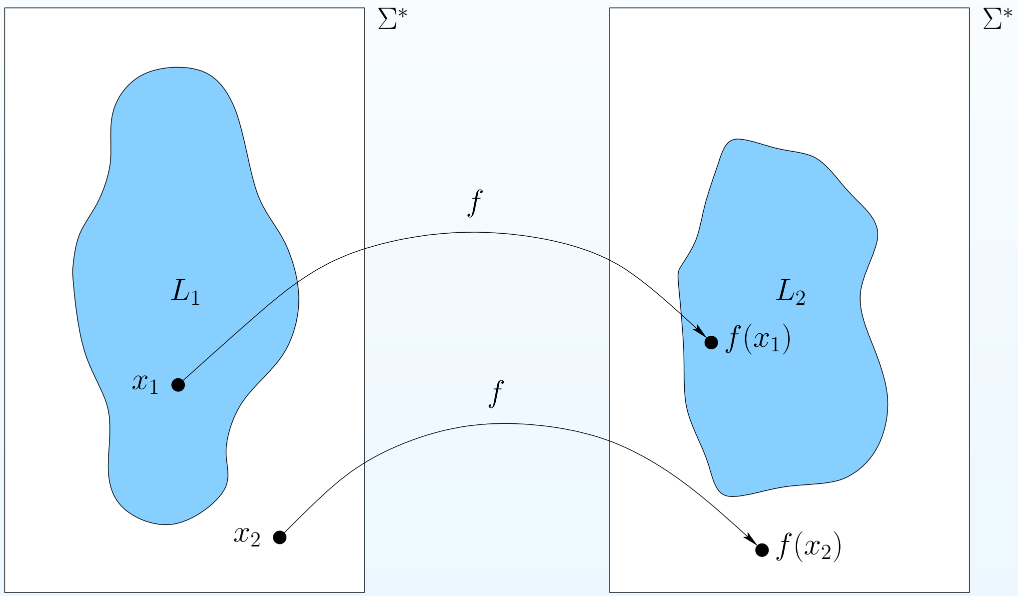

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- This follows since any instance x of L_1 can be solved by transforming it in polynomial time to an instance $y = f(x)$ of L_2 and then solving y with a polynomial-time algorithm for L_2.

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- We next show that the circuit satisfiability problem is NP-complete.

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- We can represent a circuit as an acyclic graph.

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- Thus, $\texttt{CIRCUIT-SAT} \in \text{NP}$.

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- Consider any language $L \in$ NP.
- We need to give a polynomial-time reduction from L to CIRCUIT-SAT.
- In other words, we need to find a polynomial-time algorithm A computing a function $f : \{0,1\}^* \to \{0,1\}^*$ such that

$$x \in L \Leftrightarrow f(x) \in \texttt{CIRCUIT-SAT}.$$

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- The size of the circuit is still polynomial in n, however.

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- Thus, `CIRCUIT-SAT` is NP-hard.
- Since also `CIRCUIT-SAT` $\in$ NP, it follows that `CIRCUIT-SAT` is NP-complete.

Plan for next lecture

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  - `(HAM-CYCLE)`
  - `TSP`
  - `SUBSET-SUM`

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- We also show that all these languages are in NP and hence they are NP-complete.