Mathematical Statistics

Sample Spaces Having Equally Likely Outcomes

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Overview

1. Sample Spaces Having Equally Likely Outcomes

2. Probability as a Continuous set Function

What if?

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Question # 1

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Example 1

That is, consider an experiment whose sample space S is a finite set, say, $S = \{1, 2, \dots, N\}$. Assume that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\}).$$

Observe that Axiom 2 and 3 implies then

$$P(\{i\}) = \frac{1}{N} \quad i = 1, 2, \dots, N$$

Equally Likely Outcomes

Proposition # 1

If we assume that all outcomes of an experiment are equally likely to occur, then

$$P(E) = \frac{\text{number of outcomes in E}}{\text{number of outcomes in S}}$$

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Example 4

Suppose 5 people are to be randomly selected from a group of 20 individuals consisting of 10 married couples, and we want to determine $P(N)$, the probability that the 5 chosen are all unrelated. (That is, no two are married to each other.)

Examples

Example 5 (HW)

A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

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An urn contains n balls, one of which is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

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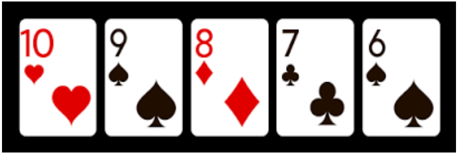
Example 7 (HW)

Suppose that $n + m$ balls, of which n are red and m are blue, are arranged in a linear order in such a way that all $(n + m)!$ possible orderings are equally likely. If we record the result of this experiment by listing only the colors of the successive balls, show that all the possible results remain equally likely.

Straight

Example 8

Poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?



but not

Full House

Example 9 (HW)

A 5-card poker hand is said to be a full house if it consists of 3 cards of the same denomination and 2 other cards of the same denomination (of course, different from the first denomination). Thus, a full house is three of a kind plus a pair. What is the probability that one is dealt a full house?

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Example 10 (HW)

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that

a. one of the players receives all 13 spades;
b. each player receives 1 ace?

Examples

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If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

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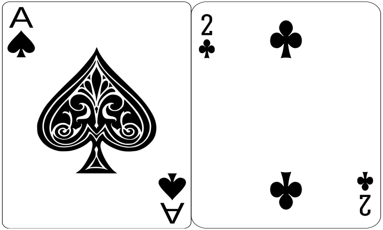
Example 12 (HW)

A deck of 52 playing cards is shuffled, and the cards are turned up one at a time until the first ace appears. Is the next card that is, the card following the first ace more likely to be the ace of spades or the two of clubs?



Football team

Football team

Example 13

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive-defensive roommate pairs? What is the probability that there are 2i offensive-defensive roommate pairs, $i = 1, 2, \ldots 10$?

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Definition # 1 (Stirling Approximation)

The Stirling approximation is given by:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

where n is a large positive integer. It's a useful approximation for estimating factorials of large numbers.

Examples

Example 14 (HW)

A total of 36 members of a club play tennis, 28 play squash, and 18 play badminton. Furthermore, 22 of the members play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton, and 4 play all three sports. How many members of this club play at least one of three sports?

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Example 15 (The matching problem)

Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. What is the probability that none of the men selects his own hat?

Examples

Example 16 (HW)

Compute the probability that if 10 married couples are seated at random at a round table, then no wife sits next to her husband.

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Example 17 (HW)

Suppose that a team has n wins and m losses. Assuming that all $\frac{(n+m)!}{(n!m!)} = \binom{n+m}{n}$ orderings are equally likely, let us determine the probability that there will be exactly r runs of wins.

Sequences

Definition # 2

A sequence of events $\{E_n, n \geqslant 1\}$ *is an* ***increasing*** *sequence if*

$$E_1 \subset E_2 \subset E_3 \subset \ldots \subset E_n \subset E_{n+1} \subset \ldots$$

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Example 18

Consider the event denoted as E_n, where the event consists of integer values that are divisible by the n^{th} power of 2.That is, $E_n = \{z \in \mathbb{Z} \mid 2^n \, \mathrm{divides\, z}\}$. Then $\{E_n, n \geqslant 1\}$ is a decreasing sequence of events.

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Definition # 4

If $\{E_n, n \geqslant 1\}$ *is an increasing sequence of events, then*

$$\lim_{n \to \infty} E_n := \bigcup_{i=1}^{\infty} E_i$$

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Definition # 5

If $\{E_n, n \geqslant 1\}$ *is a decreasing sequence of events, then*

$$\lim_{n\to\infty} E_n := \bigcap_{i=1}^{\infty} E_i$$

Limit and Probability

Proposition # 2

If $\{E_n, n \geqslant 1\}$ *is either an increasing or a decreasing sequence of events, then*

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$$

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Remark # 1

If $\{E_n, n \geqslant 1\}$ is a decreasing sequence, then $\{E_n^c, n \geqslant 1\}$ is an increasing sequence.

Probability and a "paradox"

Suppose that we possess an infinitely large box and an infinite collection of balls labeled ball number 1, number 2, number 3, and so on.

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Example 19

Let's say we start with an empty box. We put balls numbered from 1 to 10 into the box, and then we take out the ball number 10. After that, we add more balls from 11 to 20 into the box, and again we take out the ball number 20. This cycle keeps happening over and over without end. The question is, how many balls will be in the box when we repeat this process infinitely?

Examples

Example 20 (HW)

Assume we perform the same experiment and follow a procedure where we take out the ball numbered 1 initially, then the ball numbered 2 in the next step, and continue this pattern for all subsequent steps, and we keep repeating this process infinitely, how many balls will be inside the box at the end?

Example 20 (HW)

Assume we perform the same experiment and follow a procedure where we take out the ball numbered 1 initially, then the ball numbered 2 in the next step, and continue this pattern for all subsequent steps, and we keep repeating this process infinitely, how many balls will be inside the box at the end?

Example 21 (HW)

Let us now suppose that whenever a ball is to be withdrawn, that ball is randomly selected from among those present. That is, suppose that at 1^{st} step balls numbered 1 through 10 are placed in the box and a ball is randomly selected and withdrawn, and so on. In this case, how many balls are in the box at the end of experiment?