Mathematical Statistics

Axioms of Probability

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Overview

1. Sample Space and Events

Terminology

Definition # 1

In probability theory and statistics, the "Sample space" refers to the ***set*** *of all* ***possible*** *outcomes or results of an experiment, random process, or observation. It is denoted by the symbol "S".*

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Definition # 3 (Outcome)

A final result, observation, or measurement occurring from an experiment.

Definition # 4 (Sample Space)

A set that contains all possible outcomes of a given experiment.

Examples

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If the outcome of an experiment consists of the determination of the sex of a newborn child, then

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Example 2 (Horses)

If the outcome of an experiment is the order of finish in a race among the 7 horses having names A, B, C, D, E, F, and G, then

$$S = \{\text{all} \quad 7! \quad \text{permutations} \quad \text{of} \quad (\mathrm{A, B, C, D, E, F, G})\}$$

The outcome (B, C, A, F, E, D, G) means, for instance, that the horse B comes in first, then the horse C, then the horse A, and so on.

Examples

Example 3 (Coin)

If the experiment consists of flipping two coins, then the sample space consists of the following four points:

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Example 4

If the experiment consists of tossing two dice, then the sample space consists of the 36 points

$$S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$$

where the outcome (i, j) is said to occur if i appears on the leftmost die and j on the other die.

Example 5

If the experiment consists of measuring (in hours) the lifetime of a transistor, then the sample space consists of all nonnegative real numbers; that is, $S = \{x \mid 0 \leqslant x \leqslant \infty\}$

Event

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Definition # 7 (Occurrence of the Event)

An event is said to have occurred if the outcome of the experiment is an element of an event.

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If $E = \{$all outcomes in S starting with a C$\}$ then E is the event that horse C wins the race.

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Example 7 (Coin)

If $E = \{$(h, h), (h, t)$\}$, then E is the event that a head appears on the first coin.

Example 8

If $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, then E is the event that the sum of the dice equals 7.

New Events

Definition # 8 (Union)

For any two events E and F of a sample space S. Define the new event

$E \cup F$ *to consist of all outcomes that are* ***either*** *in E or in F or in both E and F.*

That is, the event $E \cup F$ will occur if either E or F occurs.

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That is, the event $E \cup F$ will occur if either E or F occurs.

Definition # 9 (Intersection)

For any two events E and F. Define the new event

$$EF \text{ to consist of all outcomes that are } \textbf{both} \text{ in } E \text{ and in } F.$$

$E \cap F$ is another notation for intersection of events.

Examples

Example 9 (Coin)

If $E = \{(\text{h, h}), (\text{h, t})\}$ is the event that the first coin lands heads, and $F = \{(\text{t, h}), (\text{h, h})\}$ is the event that the second coin lands heads, then

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is the event that at least one of the coins lands heads and thus will occur provided that both coins do not land tails.

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Example 10 (Coin)

If $E = \{(\text{h, h}), (\text{h, t}), (\text{t, h})\}$ is the event that at least 1 head occurs and $F = \{(\text{h, t}), (\text{t, h}), (\text{t, t})\}$ is the event that at least 1 tail occurs, then

$$EF = E \cap F = \{(h, t), (t, h)\}$$

is the event that exactly 1 head and 1 tail occur.

Definition # 10

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Empty Space

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Definition # 11 (Disjoint Events)

Events that are mutually exclusive, having no outcomes in common.
That is $EF = \varnothing$, then E and F are said to be mutually exclusive.

Countable Operations and Complement

Definition # 12 (Countable union & Intersection)

For events $E_1, E_2, \ldots$ define new events

$$\bigcup_{i=1}^{n} E_i$$

$$\bigcap_{i=1}^{n} E_i$$

- *The first one is the event that consists of all outcomes that are in E_i for at least one value of $i = 1, 2, \ldots$.*
- *The second is the event consisting of those outcomes that are in all of the events $E_i, i = 1, 2, \ldots$*

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For any event E, we define the new event E^c, the complement of E, to consist of all outcomes in the sample space S that are not in E.

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Statement: All of the outcomes in E are also in F.
Thus,if $E \subset F$, then the occurrence of E implies the occurrence of F

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If E is a subset of F then F is a superset of E.
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If E is a subset of F then F is a superset of E.
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If $E \subset F$ and $E \supset F$ then we say that they are equal and write $E = F$.

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Theorem 11 (DeMorgan's Law)

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c \tag{1}$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c \tag{2}$$