

# On the Balance Laws of Hyperelasticity

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## Abstract

It is known that the balance laws of hyperelasticity (Green elasticity), i.e., conservation of mass and balance of linear and angular momenta, can be derived using the first law of thermodynamics by postulating its invariance under superposed rigid body motions of the Euclidean ambient space—the Green-Naghdi-Rivlin theorem. In the case of a non-Euclidean ambient space, covariance of the energy balance—its invariance under arbitrary diffeomorphisms of the ambient space—gives all the balance laws and the Doyle-Ericksen formula—the Marsden-Hughes theorem. It is also known that, by assuming the balance laws, and positing the first and second laws of thermodynamics, the Doyle-Ericksen formula can be derived—the Coleman-Noll procedure. In this paper, we propose a generalization of the Coleman-Noll procedure: we show that the Doyle-Ericksen formula as well as the balance laws of hyperelasticity can be derived using the first and second laws of thermodynamics without assuming any (observer) invariance.

**Keywords:** Nonlinear elasticity, hyperelasticity, Green elasticity, balance laws, laws of thermodynamics.

**Mathematics Subject Classification** 74A15 · 74B20

## 1 Introduction

In nonlinear hyperelasticity, one can derive all the balance laws by starting from an energy balance (the first law of thermodynamics) and postulating its invariance under superposed rigid body motions of the ambient space (observer invariance). This idea is due to [Green and Rivlin \[1964\]](#) in the context of Euclidean ambient spaces (Green-Naghdi-Rivlin theorem). More specifically, [Green and Rivlin \[1964\]](#) postulated the balance of energy and its invariance under superposed translational and rotational motions of the Euclidean ambient space. A different version of this theorem is due to [Noll \[1963\]](#) who thought of the superposed motions passively as time-dependent coordinate charts for the Euclidean ambient space.<sup>1</sup> Effectively, [Green and Rivlin \[1964\]](#) viewed superimposed motions actively, whereas [Noll \[1963\]](#) viewed them passively. The invariance idea was subsequently extended to hyperelasticity (Green elasticity) with Riemannian ambient space manifolds by [Hughes and Marsden \[1977\]](#); they postulated the invariance of the balance of energy under arbitrary diffeomorphisms of the ambient space—*covariance* of the energy balance. [Hughes and Marsden \[1977\]](#) showed that covariance of the energy balance gives all the balance laws of hyperelasticity and the Doyle-Ericksen formula [[Doyle and Ericksen, 1956](#)]<sup>2</sup> (see also [[Marsden and Hughes, 1983](#); [Yavari et al., 2006](#)]). In this paper, instead of using the first law of thermodynamics and its covariance, and as an extension to the classical [Coleman and Noll \[1963\]](#) procedure, we show that, one can use the first and the second laws of thermodynamics to derive not only the Doyle-Ericksen formula but also all the balance laws of hyperelasticity.

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<sup>1</sup>Recall that a matrix can be regarded as either a linear transformation (active) or representing a change of basis (passive).

<sup>2</sup>It is worth emphasizing that in both [[Green and Rivlin, 1964](#)] and [[Hughes and Marsden, 1977](#)] it was assumed that the body is made of a material that has an underlying energy function, i.e., they restricted themselves to hyperelasticity. Similar invariance arguments can be used to derive the balance laws of anelasticity, provided that there exists an underlying energy function, e.g., [[Yavari, 2010](#)].

## 2 Kinematics of Finite Deformations

Consider a hyperelastic solid body  $\mathcal{B}$  represented by an embedded 3-submanifold  $\mathcal{B}$  within the ambient space  $\mathcal{S}$ .<sup>3</sup> Motion of the body  $\mathcal{B}$  is represented by a time-parametrized family of maps  $\varphi_t : \mathcal{B} \rightarrow \mathcal{C}_t \subset \mathcal{S}$ , mapping the reference (material) configuration  $\mathcal{B}$  of the body to its current (spatial) configuration  $\mathcal{C}_t = \varphi_t(\mathcal{B})$ . We adopt the following standard convention: objects and indices are denoted by uppercase characters in the material manifold  $\mathcal{B}$  (e.g.,  $X \in \mathcal{B}$ ), and by lowercase characters in the spatial manifold  $\varphi_t(\mathcal{B})$  (e.g.,  $x = \varphi_t(X) \in \varphi_t(\mathcal{B})$ ). We consider local coordinate charts on  $\mathcal{B}$  and  $\mathcal{S}$  denoted by  $\{X^A\}$  and  $\{x^a\}$ , respectively. The corresponding local coordinate bases are denoted by  $\{\partial_A = \frac{\partial}{\partial X^A}\}$  and  $\{\partial_a = \frac{\partial}{\partial x^a}\}$ , and their respective dual bases are  $\{dX^A\}$  and  $\{dx^a\}$ . We also adopt Einstein's repeated index summation convention, e.g.,  $u^i v_i := \sum_i u^i v_i$ .

The ambient space has a background metric  $\mathbf{g} = g_{ab} dx^a \otimes dx^b$ . Given vectors  $\mathbf{u}, \mathbf{w} \in T_x \mathcal{S}$ , their dot product is denoted by  $\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbf{g}} = u^a w^b g_{ab}$ . Given a vector  $\mathbf{u} \in T_x \mathcal{S}$  and a 1-form  $\boldsymbol{\omega} \in T_x^* \mathcal{S}$ , their natural pairing is denoted by  $\langle \boldsymbol{\omega}, \mathbf{u} \rangle = \boldsymbol{\omega}(\mathbf{u}) = \omega_a u^a$ . The spatial volume form is  $dv = \sqrt{\det \mathbf{g}} dx^1 \wedge dx^2 \wedge dx^3$ . The Levi-Civita connection of  $(\mathcal{S}, \mathbf{g})$  is denoted by  $\bar{\nabla}$ , with Christoffel symbols  $\gamma^a_{bc}$ . The metric  $\mathbf{g}$  of  $\mathcal{S}$  induces the metric  $\mathbf{G}$  on  $\mathcal{B}$  by which the natural distances in the body before deformation are calculated. Given vectors  $\mathbf{U}, \mathbf{W} \in T_X \mathcal{B}$ , their dot product is denoted by  $\langle \mathbf{U}, \mathbf{W} \rangle_{\mathbf{G}} = U^A W^B G_{AB}$ . Given a vector  $\mathbf{U} \in T_X \mathcal{B}$  and a 1-form  $\boldsymbol{\Omega} \in T_X^* \mathcal{B}$ , their natural pairing is denoted by  $\langle \boldsymbol{\Omega}, \mathbf{U} \rangle = \boldsymbol{\Omega}(\mathbf{U}) = \Omega_A U^A$ . The material volume form is  $dV = \sqrt{\det \mathbf{G}} dX^1 \wedge dX^2 \wedge dX^3$ . The Levi-Civita connection of  $(\mathcal{B}, \mathbf{G})$  is denoted by  $\nabla$ , with Christoffel symbols  $\Gamma^A_{BC}$ .

As a measure of strain in elastic solids, we typically use the derivative of the deformation mapping—known as the deformation gradient—denoted by  $\mathbf{F}(X, t) = T_{\varphi_t(X)} : T_X \mathcal{B} \rightarrow T_{\varphi_t(X)} \mathcal{C}_t$ ; in components it reads as  $F^a_A = \frac{\partial \varphi^a}{\partial X^A}$ . The dual  $\mathbf{F}^*$  of  $\mathbf{F}$  is defined as  $\mathbf{F}^*(X, t) : T_{\varphi_t(X)} \mathcal{C}_t \rightarrow T_X \mathcal{B}$ ,  $\langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{U} \rangle = \langle \mathbf{F}^* \boldsymbol{\alpha}, \mathbf{U} \rangle$ ,  $\forall \mathbf{U} \in T_X \mathcal{B}$ ,  $\forall \boldsymbol{\alpha} \in T_{\varphi_t(X)}^* \mathcal{C}_t$ ; it has components  $(\mathbf{F}^*)_A^a = F^a_A$ . The transpose  $\mathbf{F}^T$  of  $\mathbf{F}$  is defined as  $\mathbf{F}^T(X, t) : T_{\varphi_t(X)} \mathcal{C}_t \rightarrow T_X \mathcal{B}$ ,  $\langle \mathbf{F}\mathbf{U}, \mathbf{u} \rangle_{\mathbf{g}} = \langle \mathbf{U}, \mathbf{F}^T \mathbf{u} \rangle_{\mathbf{G}}$ ,  $\forall \mathbf{U} \in T_X \mathcal{B}$ ,  $\forall \mathbf{u} \in T_{\varphi_t(X)} \mathcal{C}_t$ ; in components it reads as  $(\mathbf{F}^T)^A_a = G^{AB} F^b_B g_{ba}$ . Note that  $\mathbf{F}^T = \mathbf{G}^\sharp \mathbf{F}^* \mathbf{g}$ , where  $(\cdot)^\sharp$  denotes the musical isomorphism for raising indices. The right Cauchy–Green deformation tensor is defined as  $\mathbf{C} := \mathbf{F}^T \mathbf{F}$ . Denoting by  $(\cdot)^b$  the musical isomorphism for lowering indices, one finds that  $\mathbf{C}^b$  corresponds to the pull-back of the spatial metric  $\mathbf{g}$  by  $\varphi$ , i.e.,  $\mathbf{C}^b = \varphi^* \mathbf{g} = \mathbf{F}^* \mathbf{g} \mathbf{F}$ . The Jacobian of the motion relates the material and spatial volume elements as  $\varphi^* dv = J dV$ , and it can be shown that  $J = \sqrt{\det \mathbf{g} / \det \mathbf{G}} \det \mathbf{F}$ .

The material velocity of the motion is defined as  $\mathbf{V} : \mathcal{B} \times \mathbb{R}^+ \rightarrow T\mathcal{S}$ ,  $\mathbf{V}(X, t) := \frac{\partial \varphi(X, t)}{\partial t}$ ; it has components  $V^a = \frac{\partial \varphi^a}{\partial t}$ . The spatial velocity is defined as  $\mathbf{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \rightarrow T\mathcal{S}$ ,  $\mathbf{v}(x, t) := \mathbf{V}(\varphi_t^{-1}(x), t)$ . The material acceleration of the motion is defined as  $\mathbf{A} : \mathcal{B} \times \mathbb{R}^+ \rightarrow T\mathcal{S}$ ,  $\mathbf{A}(X, t) := D_t^{\mathbf{g}} \mathbf{V}(X, t)$ , where  $D_t^{\mathbf{g}}$  denotes the co-variant derivative along  $\varphi_X : t \mapsto \varphi(X, t)$ ; it reads in components as  $A^a = \frac{\partial V^a}{\partial t} + \gamma^a_{bc} V^b V^c$ . The spatial acceleration of the motion is defined as  $\mathbf{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \rightarrow T\mathcal{S}$ ,  $\mathbf{a}(x, t) := \mathbf{A}(\varphi_t^{-1}(x), t) \in T_x \mathcal{S}$ ; it has components  $a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b} v^b + \gamma^a_{bc} v^b v^c$ .

## 3 Thermodynamics and the Balance Laws of Hyperelasticity

In this section, we first briefly review the first and the second laws of thermodynamics in the setting of nonlinear hyperelasticity. We then show how all the balance laws and constitutive equations of hyperelasticity can be derived departing from the first and the second laws of thermodynamics without assuming (observer) invariance.<sup>4</sup>

**Remark 3.1** (Hyperelasticity). In Cauchy elasticity, the stress at any given point is a function of the strain at that same point [Cauchy, 1828; Truesdell, 1952] without any history dependence. However, it is important to note that not all Cauchy elastic solids possess an energy function. Those Cauchy elastic solids that do possess energy functions are often referred to as Green elastic [Green, 1838, 1839; Spencer, 2015] or hyperelastic

<sup>3</sup>For most applications, the ambient space is the three-dimensional Euclidean space, i.e.,  $\mathcal{S} = \mathbb{R}^3$ . However, in general, the ambient space may be curved, e.g., in modeling the dynamics of fluid membranes [Arroyo and DeSimone, 2009]. See [Yavari et al., 2016] for a general framework on elasticity in evolving ambient spaces.

<sup>4</sup>As discussed earlier in §1, the Green-Naghdi-Rivlin theorem (and its subsequent extensions) yields the balance of linear and angular momentum as a consequence of the invariance of the energy balance under superposed isometries of the ambient space. Here, the proposed derivation does not rely on invariance; instead, it generalizes the Coleman-Noll procedure to recover not only the constitutive equations, but also the balance laws.

[Truesdell, 1952]. It is worth noting that not all elastic solids are Cauchy elastic. There are indeed elastic solids whose constitutive equations are implicit, such as those expressed as  $\mathbf{f}(\boldsymbol{\sigma}, \mathbf{b}) = \mathbf{0}$  [Morgan, 1966; Rajagopal, 2003, 2007, 2011]. Cauchy elastic solids are a proper subset of this class of solids; and Green elastic solids are a proper subset of Cauchy elastic solids.

### 3.1 First Law of Thermodynamics

The first law of thermodynamics posits the existence of an internal energy  $\mathcal{E}$  as a state function, which satisfies the following balance equation as an expression of the conservation of energy principle [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0 \left( \mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right) dV = \int_{\mathcal{U}} \rho_0 \left( \langle \mathbf{B}, \mathbf{V} \rangle_{\mathbf{g}} + R \right) dV + \int_{\partial \mathcal{U}} \left( \langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}} - \langle \mathbf{Q}, \mathbf{N} \rangle_{\mathbf{G}} \right) dA, \quad (3.1)$$

for any open set  $\mathcal{U} \subset \mathcal{B}$ , where  $\mathcal{E}$  stands for the specific internal energy,  $\rho_0$  is the material mass density,  $\mathbf{B}$  is the specific body force,  $\mathbf{T} = \mathbf{T}(X, \mathbf{N})$  is the boundary traction vector field per unit material area,  $R = R(X, t)$  is the specific heat supply, and  $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{C}^b, \mathbf{G})$  represents the external heat flux per unit material area,  $d\Theta = \frac{\partial \Theta}{\partial X^A} dX^A$  is the exterior derivative of temperature  $\Theta$ , and  $\mathbf{N}$  is the  $\mathbf{G}$ -unit normal to the boundary  $\partial \mathcal{U}$ .

Following Marsden and Hughes [1983, §2.1(1.9)]'s version of Cauchy's theorem,<sup>5</sup> there exists a unique material vector field  $\mathbf{M}$  such that  $\langle \mathbf{M}, \mathbf{N} \rangle_{\mathbf{G}} = \langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$ . The linearity of  $\langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$  with respect to  $\mathbf{V}$  implies that  $\mathbf{M}$  is also linear in  $\mathbf{V}$ , i.e., there exists a second order two-point tensor field  $\mathbf{R}$  such that  $\mathbf{M} = \mathbf{R}\mathbf{V}$ , which indeed is unique following the uniqueness of  $\mathbf{M}$ . We now have that  $\langle \mathbf{R}\mathbf{V}, \mathbf{N} \rangle_{\mathbf{G}} = \langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$ , which may be recast into  $\langle \mathbf{R}^T \mathbf{N}, \mathbf{V} \rangle_{\mathbf{g}} = \langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$ , and it follows by arbitrariness of  $\mathbf{V}$  that  $\mathbf{T} = \mathbf{P}\mathbf{N}^b$ , where  $\mathbf{P} := \mathbf{R}^T \mathbf{g}^\#$  is a well-defined two-point tensor by virtue of the existence and uniqueness of  $\mathbf{R}$ . Now, we proceed to write the energy balance (3.1) in local form:

$$\rho_0 \dot{\mathcal{E}} = \mathbf{P} : \nabla \mathbf{V}^b - \text{Div } \mathbf{Q} + \rho_0 R + \langle \text{Div } \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} - \dot{\rho}_0 \left( \mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right), \quad (3.2)$$

where a dotted quantity denotes its total time derivative,  $\text{Div}$  denotes the referential divergence operator, i.e., in the manifold  $(\mathcal{B}, \mathbf{G})$ , and a colon denotes the double contraction product, i.e.,  $\mathbf{P} : \nabla \mathbf{V}^b = P^{aA} V_{a|A}$ —the vertical stroke denoting covariant differentiation. Note that<sup>6</sup>

$$\mathbf{P} : \nabla \mathbf{V}^b = \mathbf{P} : (\mathbf{g}\dot{\mathbf{F}}) = (\mathbf{F}^{-1}\mathbf{P}) : (\overset{\circ}{\mathbf{D}} + \overset{\circ}{\mathbf{D}}), \quad (3.3)$$

where  $\overset{\circ}{\mathbf{D}}$  is the symmetric part of  $\mathbf{F}^* \mathbf{g}\dot{\mathbf{F}}$  and  $\overset{\circ}{\mathbf{D}}$  is its anti-symmetric part:<sup>7</sup>

$$\overset{\circ}{\mathbf{D}} = \frac{1}{2} [\mathbf{F}^* \mathbf{g}\dot{\mathbf{F}} + \dot{\mathbf{F}}^* \mathbf{g}\mathbf{F}] , \quad \overset{\circ}{\mathbf{D}} = \frac{1}{2} [\mathbf{F}^* \mathbf{g}\dot{\mathbf{F}} - \dot{\mathbf{F}}^* \mathbf{g}\mathbf{F}] .$$

### 3.2 Second Law of Thermodynamics

The second law of thermodynamics posits the existence of specific entropy  $\mathcal{N}$  as a state function, which satisfies the following inequality—known as the Clausius-Duhem inequality—as an expression of the principle of entropy production,<sup>8</sup> which steadily increases or remains constant within a closed system over time [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0 \mathcal{N} dV \geq \int_{\mathcal{U}} \rho_0 \frac{R}{\Theta} dV + \int_{\partial \mathcal{U}} \frac{H}{\Theta} dA, \quad (3.4)$$

<sup>5</sup>Note that this version of Cauchy's theorem does not assume the balance of linear momentum and does not introduce the idea of stress—and neither do we, at this stage.

<sup>6</sup>We thank Sanjay Govindjee for bringing this to our attention.

<sup>7</sup>Note that  $\overset{\circ}{\mathbf{D}} = \frac{1}{2} \dot{\mathbf{C}}^b$  is the symmetric material rate of the Cauchy-Green deformation tensor.

<sup>8</sup>The entropy production for an open subset  $\mathcal{U}$  in the body reads as

$$\Gamma = \frac{d}{dt} \int_{\mathcal{U}} \rho_0 \mathcal{N} dV - \int_{\mathcal{U}} \rho_0 \frac{R}{\Theta} dV - \int_{\partial \mathcal{U}} \frac{H}{\Theta} dA.$$

for any open set  $\mathcal{U} \subset \mathcal{B}$ . In localized form, the material Clausius-Duhem inequality (3.4) yields

$$\dot{\eta} = \rho_0 \Theta \dot{\mathcal{N}} + \dot{\rho}_0 \Theta \mathcal{N} + \text{Div } \mathbf{Q} - \rho_0 R - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0, \quad (3.5)$$

where  $\dot{\eta}$  denotes the material rate of energy dissipation density.

**Remark 3.2.** At a material point  $X \in \mathcal{B}$ , Coleman and Noll [1963] introduce the set<sup>9</sup>

$$\left\{ \varphi_t(X), \mathbf{P}(X, t), \mathbf{B}(X, t), \mathcal{E}(X, t), \mathbf{Q}(X, t), R(X, t), \mathcal{N}(X, t), \Theta(X, t) \right\}, \quad (3.6)$$

and call it a *thermodynamic process*, provided all its eight fields satisfy the first law of thermodynamics and the balance of linear and angular momenta. A given hyperelastic material is specified by its constitutive assumptions, e.g.,  $\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \mathbf{F}, \mathbf{G}, \mathbf{g})$  and  $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{F}, \mathbf{G}, \mathbf{g})$ . A thermodynamic process is *admissible* if the constitutive assumptions hold everywhere in the body and at all times. Coleman and Noll [1963] showed that requiring the Clausius-Duhem inequality (3.5) to hold for all admissible thermodynamic processes puts certain constraints on the constitutive assumptions, e.g., the Doyle-Ericksen formula.

In this work, we define an *extended thermodynamic process* at a material point  $X \in \mathcal{B}$  as the set (3.6), provided all its eight fields satisfy the first law of thermodynamics—without requiring that  $\mathbf{P}$  satisfies the balance of linear and angular momenta. An extended thermodynamic process is *admissible* if the constitutive assumptions hold everywhere in the body and at all times. We require that the Clausius-Duhem inequality (3.5) holds for all admissible extended thermodynamic processes.

### 3.3 Balance Laws of Nonlinear Hyperelasticity

Let us consider a body  $\mathcal{B}$  made of a hyperelastic material at every material point  $X \in \mathcal{B}$  for which there exists an energy function  $\mathcal{E}$  that depends explicitly on strain at the same material point  $X \in \mathcal{B}$ , i.e.,  $\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \mathbf{F}, \mathbf{G}, \mathbf{g})$  [Truesdell, 1952]. The specific free energy may hence be defined as the Legendre transform of the specific internal energy  $\mathcal{E}$  with respect to the conjugate variables temperature  $\Theta$  and specific entropy  $\mathcal{N}$ , i.e.,  $\Psi = \Psi(X, \Theta, \mathbf{F}, \mathbf{G}, \mathbf{g})$  and

$$\Psi = \mathcal{E} - \Theta \mathcal{N}, \quad (3.7)$$

such that  $\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \mathbf{F}, \mathbf{G}, \mathbf{g})$ . Thus<sup>10</sup>

$$\mathcal{N} = -\frac{\partial \Psi}{\partial \Theta}. \quad (3.8)$$

**Proposition 3.1.** For a hyperelastic body,<sup>11</sup> the first and second laws of thermodynamics (3.2) and (3.5) imply that

$$\left\{ \begin{array}{l} \dot{\rho}_0 = 0, \end{array} \right. \quad (3.9a)$$

$$\left\{ \begin{array}{l} \mathbf{P} = \rho_0 \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \mathbf{F}}, \end{array} \right. \quad (3.9b)$$

$$\left\{ \begin{array}{l} \text{Div } \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}, \end{array} \right. \quad (3.9c)$$

$$\left\{ \begin{array}{l} \mathbf{F} \mathbf{P}^* = \mathbf{P} \mathbf{F}^*, \end{array} \right. \quad (3.9d)$$

$$\left\{ \begin{array}{l} \dot{\eta} = -\frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \end{array} \right. \quad (3.9e)$$

In other words, the first and second laws of thermodynamics imply the conservation of mass (3.9a), the Doyle-Ericksen formula (3.9b), the balance of linear momentum (3.9c), the balance of angular momentum (3.9d), and the dissipation inequality (3.9e). Note that (3.9b) effectively shows that the two-point tensor  $\mathbf{P}$  is indeed the Piola-Kirchhoff stress tensor.

<sup>9</sup> $\mathbf{P}$ , in their treatment, is introduced a priori as the first Piola-Kirchhoff stress tensor (in fact, they use the Cauchy stress tensor) satisfying the balance of linear and angular momenta.

<sup>10</sup>Note that the Legendre transform (3.7) of  $\mathcal{E}$  to  $\Psi$  with respect to the conjugate variables  $\mathcal{N}$  and  $\Theta$  is essentially a change of variable, from the former to the latter, satisfying (3.8). See [Arnold, 1989; Goldstein et al., 2002] for further details on Legendre transform in the context of Lagrangian mechanics and thermodynamics.

<sup>11</sup>It should be emphasized that this result can be easily extended to hyper-anelasticity, where there is an energy function that explicitly depends on elastic distortions.

*Proof.* From (3.7), one writes:  $\rho_0 \dot{\Theta} \mathcal{N} = \rho_0 \dot{\mathcal{E}} - \rho_0 \dot{\Psi} - \rho_0 \dot{\Theta} \mathcal{N}$ . Substituting this relation and (3.2) (while using (3.3)) into (3.5), one obtains

$$\begin{aligned} \dot{\eta} = & (\mathbf{F}^{-1} \mathbf{P}) : (\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}}) - \rho_0 \dot{\Psi} - \rho_0 \dot{\Theta} \mathcal{N} + \dot{\rho}_0 \Theta \mathcal{N} \\ & + \langle \text{Div } \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} - \dot{\rho}_0 \left[ \mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right] - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \end{aligned} \quad (3.10)$$

Applying Leibniz rule to  $\dot{\Psi}$ , one writes

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} = -\mathcal{N} \dot{\Theta} + \left( \mathbf{F}^{-1} \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}} \right) : (\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}}). \quad (3.11)$$

It can be shown that<sup>12</sup>

$$\left( \mathbf{F}^{-1} \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}} \right)^* = \mathbf{F}^{-1} \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}}. \quad (3.12)$$

Hence  $\mathbf{F}^{-1} \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}}$  is a symmetric tensor, and since  $\overset{a}{\mathbf{D}}$  is antisymmetric, it follows that

$$\left( \mathbf{F}^{-1} \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}} \right) : \overset{a}{\mathbf{D}} = 0.$$

Using the above identity, and substituting (3.11) into (3.10), the rate of dissipation is simplified to read

$$\dot{\eta} = \left[ \mathbf{F}^{-1} \left( \mathbf{P} - \rho_0 \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}} \right) \right] : \overset{s}{\mathbf{D}} + \mathbf{F}^{-1} \mathbf{P} : \overset{a}{\mathbf{D}} + \langle \text{Div } \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} - \dot{\rho}_0 \left[ \Psi + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right] - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \quad (3.13)$$

This inequality must hold for all motions, i.e., all extended thermodynamic processes. As  $\overset{s}{\mathbf{D}}$  (a symmetric tensor) and  $\overset{a}{\mathbf{D}}$  (an antisymmetric tensor) can be chosen independently of all the other fields, one concludes that<sup>13</sup>

$$\mathbf{P} = \rho_0 \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}}, \quad (\mathbf{F}^{-1} \mathbf{P})^* = \mathbf{F}^{-1} \mathbf{P}. \quad (3.14)$$

Now, the rate of dissipation (3.13) is simplified to read

$$\dot{\eta} = \langle \text{Div } \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} - \dot{\rho}_0 \left[ \Psi + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right] - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \quad (3.15)$$

One can choose the velocity vector arbitrarily while its norm  $\|\mathbf{V}\|_{\mathbf{g}}$  is fixed. This implies that the inequality (3.15) can hold only if

$$\text{Div } \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}. \quad (3.16)$$

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<sup>12</sup>Using Leibniz rule, we write in components

$$\frac{\partial \Psi}{\partial \mathbf{F}^a{}_A} = \frac{\partial \Psi}{\partial (\mathbf{F}^i{}_K \mathbf{F}^j{}_L \mathbf{g}_{ij})} \frac{\partial (\mathbf{F}^k{}_K \mathbf{F}^l{}_L \mathbf{g}_{kl})}{\partial \mathbf{F}^a{}_A} = \frac{\partial \Psi}{\partial (\mathbf{F}^i{}_K \mathbf{F}^j{}_L \mathbf{g}_{ij})} \left( \delta^k{}_a \delta^A{}_K \mathbf{F}^l{}_L \mathbf{g}_{kl} + \mathbf{F}^k{}_K \delta^l{}_a \delta^A{}_L \mathbf{g}_{kl} \right).$$

Using straightforward index manipulations, it follows that

$$\frac{\partial \Psi}{\partial \mathbf{F}^a{}_A} = 2 \mathbf{g}_{al} \mathbf{F}^l{}_K \frac{\partial \Psi}{\partial (\mathbf{F}^i{}_K \mathbf{F}^j{}_A \mathbf{g}_{ij})}, \quad \text{and} \quad \frac{\partial \Psi}{\partial \mathbf{F}^a{}_A} = 2 \frac{\partial \Psi}{\partial (\mathbf{F}^i{}_A \mathbf{F}^j{}_L \mathbf{g}_{ij})} \mathbf{F}^k{}_L \mathbf{g}_{ka},$$

which one may rewrite as

$$\frac{\partial \Psi}{\partial \mathbf{F}} = 2 \mathbf{g} \mathbf{F} \frac{\partial \Psi}{\partial (\mathbf{F}^* \mathbf{g} \mathbf{F})}, \quad \text{and} \quad \left( \frac{\partial \Psi}{\partial \mathbf{F}} \right)^* = 2 \frac{\partial \Psi}{\partial (\mathbf{F}^* \mathbf{g} \mathbf{F})} \mathbf{F}^* \mathbf{g}.$$

Therefore, one finds that

$$\mathbf{F}^{-1} \mathbf{g}^\# \frac{\partial \Psi}{\partial \mathbf{F}} = \left( \frac{\partial \Psi}{\partial \mathbf{F}} \right)^* \mathbf{g}^\# \mathbf{F}^{-*},$$

which directly leads to (3.12).

<sup>13</sup>There are infinitely many extended thermodynamic processes for which everything except for  $\overset{s}{\mathbf{D}}$  and  $\overset{a}{\mathbf{D}}$  are the same. For the second law to hold for all such processes one must have (3.14).

Now the rate of dissipation takes the following form

$$\dot{\eta} = -\dot{\rho}_0 \left[ \Psi + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right] - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \quad (3.17)$$

Next, one can choose the velocity vector norm  $\|\mathbf{V}\|_{\mathbf{g}}$  arbitrarily while keeping the other fields fixed. For all these extended thermodynamics processes the above inequality must hold. This implies that  $\dot{\rho}_0 = 0$  and  $\dot{\eta} = -\frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0$ .  $\square$

**Remark 3.3.** Coleman and Noll [1963] showed that requiring (3.5) to hold for all admissible thermodynamic processes gives the Doyle-Ericksen formula (3.9b). We have shown that requiring (3.5) to hold for all admissible extended thermodynamic processes gives all the balance laws and the Doyle-Ericksen formula (3.9b).

**Remark 3.4.** For an incompressible hyperelastic solid, the Legendre transform (3.7) is modified to take into account the constraint of volume preservation  $J = 1$  on motions as follows

$$\Psi - p(J - 1) = \mathcal{E} - \Theta \mathcal{N}, \quad (3.18)$$

where  $p(X, t)$  is the Lagrange multiplier associated with the incompressibility constraint. Consequently, since  $\dot{J} = \frac{1}{2} J \mathbf{C}^{-\sharp} : \dot{\mathbf{C}}^{\flat} = (\mathbf{F}^{-1} \mathbf{g}^{\sharp} \mathbf{F}^{-\star}) : \dot{\mathbf{D}}$ , the Doyle-Ericksen formula (3.9b) is modified to read

$$\mathbf{P} = \rho_0 \mathbf{g}^{\sharp} \frac{\partial \Psi}{\partial \mathbf{F}} - p \mathbf{g}^{\sharp} \mathbf{F}^{-\star}. \quad (3.19)$$

Note that it is possible to recast the balance and constitutive equations (3.9) in their spatial (current) form as follows:

$$\left\{ \begin{array}{l} \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \end{array} \right. \quad (3.20a)$$

$$\left\{ \begin{array}{l} \boldsymbol{\sigma} = \rho \mathbf{g}^{\sharp} \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{\star} = 2\rho \frac{\partial \Psi}{\partial \mathbf{g}}, \end{array} \right. \quad (3.20b)$$

$$\left\{ \begin{array}{l} \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}, \end{array} \right. \quad (3.20c)$$

$$\left\{ \begin{array}{l} \boldsymbol{\sigma}^{\star} = \boldsymbol{\sigma}, \end{array} \right. \quad (3.20d)$$

$$\left\{ \begin{array}{l} \dot{\eta} = -\frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0, \end{array} \right. \quad (3.20e)$$

where  $\rho$  is the spatial density,  $\boldsymbol{\sigma}$  is the Cauchy stress tensor,  $\mathbf{b} = \mathbf{B} \circ \varphi_t^{-1}$ , and  $\operatorname{div}$  denotes the spatial divergence operator, i.e., in the manifold  $(\mathcal{S}, \mathbf{g})$ . The spatial balance of mass (3.20a) is derived from the material (reference) balance and the relation  $\rho = \rho_0/J$ , which follows from the definition of the Jacobian,  $\varphi^* dv = J dV$ . In (3.20b), the Cauchy stress tensor  $\boldsymbol{\sigma}$  is the Piola transform of the first Piola-Kirchhoff stress tensor  $\mathbf{P}$ ; consistently with the spatial forms of the balance of linear momentum (3.20c) and angular momentum (3.20d) obtained by pushing forward the respective material equations into the current configuration using the Piola transformation [Marsden and Hughes, 1983].

## 4 Concluding remarks

In this Letter, we present a generalization of the Coleman-Noll procedure to derive the balance laws of nonlinear hyperelasticity, including conservation of mass, balance of linear and angular momenta, and the Doyle-Ericksen formula, directly from the first and second laws of thermodynamics. Notably, this is achieved without invoking assumptions of observer invariance or pre-supposing the balance laws themselves—as it customary done in the classical Coleman-Noll procedure. By reframing the Clausius-Duhem inequality in terms of extended thermodynamic processes—herein defined, we demonstrate how the constitutive and balance equations emerge naturally within this framework.

The proposed approach not only provides a new perspective on the foundational principles of continuum mechanics but also extends the applicability of classical methods to scenarios where invariance principles are not readily defined, such as in non-Euclidean or evolving ambient spaces. This generalization connects thermodynamics, geometry, and mechanics in a unified framework, paving the way for new insights in hyperelastic

and hyper-anelastic materials. While the scope of this work is focused on hyperelasticity, extending the framework to address the complexities of coupled multiphysics problems appears to be straightforward. Indeed, the authors' treatment of viscoelasticity [Sadik and Yavari, 2024] initially inspired this work, and ongoing efforts towards formulating a more general framework of visco-anelasticity [Sadik and Yavari, 2025] extends the approach presented herein.

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