

A Generalised Coleman-Noll Procedure and the Balance Laws of Hyper-Anelasticity

Souhayl Sadik^{*1} and Arash Yavari^{2,3}

¹*Department of Mechanical and Production Engineering, Aarhus University, 8000 Aarhus C, Denmark*

²*School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

³*The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

January 24, 2025

Abstract

It is known that the balance laws of hyperelasticity (Green elasticity), i.e., conservation of mass and balance of linear and angular momenta, can be derived using the first law of thermodynamics by postulating its invariance under superposed rigid body motions of the Euclidean ambient space—the Green-Naghdi-Rivlin theorem. In the case of a non-Euclidean ambient space, covariance of the energy balance—its invariance under arbitrary diffeomorphisms of the ambient space—gives all the balance laws and the Doyle-Ericksen formula—the Marsden-Hughes theorem. It is also known that, by assuming the balance laws, and positing the first and second laws of thermodynamics, the Doyle-Ericksen formula can be derived—the Coleman-Noll procedure. In this paper, we propose a generalization of the Coleman-Noll procedure: we show that the Doyle-Ericksen formula as well as the balance laws for both hyperelasticity and hyper-anelasticity can be derived using the first and second laws of thermodynamics without assuming any (observer) invariance.

Keywords: Nonlinear elasticity, hyperelasticity, Green elasticity, balance laws, laws of thermodynamics.

Mathematics Subject Classification 74A15 · 74B20

1 Introduction

There are several approaches to derive the balance laws in fields theories, particularly in elasticity. One approach is the Lagrangian field theory (variational approach) of elasticity. In this method, Hamilton’s principle is written for an elastic body, leading to the balance of linear momentum as the Euler-Lagrange equations of the variational principle. The balance of angular momentum is then derived using Noether’s theorem, which relates the invariance of the Lagrangian density under ambient space rotations to angular momentum conservation. The balance of energy corresponds to the invariance of the Lagrangian density under time shifts. The second approach is the Hamiltonian mechanics formulation. In this framework, nonlinear elasticity is represented in phase space, where Hamilton’s equations give the balance laws [Simo et al., 1988]. More specifically, the governing equations of nonlinear elasticity can be derived using the canonical Hamilton equations $\dot{f} = \{f, H\}$, where $\{\}$ is the Poisson bracket of nonlinear elasticity, H is the Hamiltonian of nonlinear elasticity and f is an arbitrary scalar function.

Another approach to derive the balance laws of nonlinear hyperelasticity is to start from an energy balance (the first law of thermodynamics) and postulate its invariance under superposed rigid body motions of the ambient space (observer invariance). This idea is due to Green and Rivlin [1964] in the context of Euclidean ambient spaces (Green-Naghdi-Rivlin theorem). More specifically, Green and Rivlin [1964] postulated the balance of energy and its invariance under superposed translational and rotational motions of the Euclidean ambient space. A different version of this theorem is due to Noll [1963] who thought of the superposed motions

^{*}Corresponding author, e-mail: sosa@mpe.au.dk

passively as time-dependent coordinate charts for the Euclidean ambient space.¹ Effectively, [Green and Rivlin \[1964\]](#) viewed superimposed motions actively, whereas [Noll \[1963\]](#) viewed them passively. The invariance idea was subsequently extended to hyperelasticity (Green elasticity) with Riemannian ambient space manifolds by [Hughes and Marsden \[1977\]](#); they postulated the invariance of the balance of energy under arbitrary diffeomorphisms of the ambient space—*covariance* of the energy balance. [Hughes and Marsden \[1977\]](#) showed that covariance of the energy balance gives all the balance laws of hyperelasticity and the Doyle-Ericksen formula [[Doyle and Ericksen, 1956](#)]² (see also [[Marsden and Hughes, 1983](#); [Yavari et al., 2006](#)]). It is worth noting that the relationship between the covariance of the energy balance and the Lagrangian field theory of elasticity was explored in detail in [[Yavari and Marsden, 2012](#)].³

The second law of thermodynamics imposes constraints on the constitutive equations. In the standard [Coleman and Noll \[1963\]](#) procedure, it is assumed that the balances of linear and angular momenta are already satisfied. Subsequently, the second law of thermodynamics, in the form of the Clausius-Duhem inequality, places restrictions on the form of the constitutive equations.

In this paper, instead of using the first law of thermodynamics and its covariance, and as an extension to the classical [Coleman and Noll \[1963\]](#) procedure, we show that, one can use the first and the second laws of thermodynamics to derive not only the Doyle-Ericksen formula but also all the balance laws of hyperelasticity. We also show that this generalised procedure holds for hyper-anelasticity.

This paper is organised as follows: In §2, we introduce some notational elements and discuss the kinematics framework, in particular we discuss the material geometry for finite elasticity and anelasticity including the Bilby-Kröner-Lee decomposition in the anelastic case. In §3 we discuss thermodynamics of hyperelastic solids, and without any assumptions of invariance, we prove that one is able to find not only the hyperelastic constitutive equations (classical Coleman-Noll procedure), but also the balance laws of hyperelasticity—hence the generalised Coleman-Noll procedure. In §4, we extend our generalised Coleman-Noll procedure to hyper-anelasticity. We further discuss the heat equation (in case of an isothermal transformation) and the kinetic equations governing the evolution of anelastic distortions. We finish the paper by some concluding remarks in §5.

2 Kinematics

2.1 Kinematic and Mathematical Preliminaries

Consider a solid body \mathbf{B} represented by an embedded 3-submanifold \mathcal{B} within the ambient space \mathcal{S} .⁴ Motion of the body \mathbf{B} is represented by a time-parametrized family of maps $\varphi_t : \mathcal{B} \rightarrow \mathcal{C}_t \subset \mathcal{S}$, mapping the reference (material) configuration \mathcal{B} of the body to its current (spatial) configuration $\mathcal{C}_t = \varphi_t(\mathcal{B})$. We adopt the following standard convention: objects and indices are denoted by uppercase characters in the material manifold \mathcal{B} (e.g., $X \in \mathcal{B}$), and by lowercase characters in the spatial manifold $\varphi_t(\mathcal{B})$ (e.g., $x = \varphi_t(X) \in \varphi_t(\mathcal{B})$). We consider local coordinate charts on \mathcal{B} and \mathcal{S} and denote them by $\{X^A\}$ and $\{x^a\}$, respectively. The corresponding local coordinate bases are denoted by $\{\partial_A = \frac{\partial}{\partial X^A}\}$ and $\{\partial_a = \frac{\partial}{\partial x^a}\}$, and their respective dual bases are $\{dX^A\}$ and $\{dx^a\}$. We adopt Einstein's repeated index summation convention, e.g., $u^i v_i := \sum_i u^i v_i$.

In the ambient space \mathcal{S} , given a vector $\mathbf{u} \in T_x \mathcal{S}$ and a 1-form $\boldsymbol{\omega} \in T_x^* \mathcal{S}$, their natural pairing is denoted by $\langle \boldsymbol{\omega}, \mathbf{u} \rangle = \boldsymbol{\omega}(\mathbf{u}) = \omega_a u^a$. Similarly, in the reference manifold \mathcal{B} , given a vector $\mathbf{U} \in T_X \mathcal{B}$ and a 1-form $\boldsymbol{\Omega} \in T_X^* \mathcal{B}$, their natural pairing is also denoted by $\langle \boldsymbol{\Omega}, \mathbf{U} \rangle = \boldsymbol{\Omega}(\mathbf{U}) = \Omega_A U^A$. Note that the natural pairing operation does not require any metric structure.

¹Recall that a matrix can be regarded as either a linear transformation (active) or representing a change of basis (passive).

²It is worth emphasizing that both in [[Green and Rivlin, 1964](#)] and [[Hughes and Marsden, 1977](#)], it was assumed that the body is made of a material that has an underlying energy function, i.e., they restricted themselves to hyperelasticity. Similar invariance arguments can be used to derive the balance laws of anelasticity, provided that there exists an underlying energy function, e.g., [[Yavari, 2010](#)].

³A widely used approach for deriving the balance laws, particularly in computational mechanics, is the principle of virtual work (or virtual power) [[Antman and Osborn, 1979](#); [Maugin, 1980](#)]. It can be shown that the principle of virtual work can be derived from the balance of energy [[Marsden and Hughes, 1983](#)].

⁴For most applications, the ambient space is the three-dimensional Euclidean space, i.e., $\mathcal{S} = \mathbb{R}^3$. However, in general, the ambient space may be curved, e.g., in modeling the dynamics of fluid membranes [[Arroyo and DeSimone, 2009](#)]. See [[Yavari et al., 2016](#)] for a general framework on elasticity in evolving ambient spaces.

As a measure of strain in elastic solids, we typically use the derivative of the deformation mapping—known as the deformation gradient—denoted by $\mathbf{F}(X, t) = T\varphi_t(X) : T_X\mathcal{B} \rightarrow T_{\varphi_t(X)}\mathcal{C}_t$; in components it reads as $F^a_A = \frac{\partial \varphi^a}{\partial X^A}$. The adjoint \mathbf{F}^* of \mathbf{F} is defined as $\mathbf{F}^*(X, t) : T_{\varphi_t(X)}\mathcal{C}_t \rightarrow T_X\mathcal{B}$, $\langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{U} \rangle = \langle \mathbf{F}^*\boldsymbol{\alpha}, \mathbf{U} \rangle$, $\forall \mathbf{U} \in T_X\mathcal{B}$, $\forall \boldsymbol{\alpha} \in T_{\varphi_t(X)}^*\mathcal{S}$; it has components $(\mathbf{F}^*)_A^a = F^a_A$. Note that definition of the adjoint, much like the natural pairing, does not require any metric structure either.

The ambient space has a background Euclidean metric $\mathbf{g} = g_{ab} dx^a \otimes dx^b$. Given vectors $\mathbf{u}, \mathbf{w} \in T_x\mathcal{S}$, their dot product is denoted by $\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbf{g}} = u^a w^b g_{ab}$. The spatial volume form is $dv = \sqrt{\det \mathbf{g}} dx^1 \wedge dx^2 \wedge dx^3$. The Levi-Civita connection of $(\mathcal{S}, \mathbf{g})$ is denoted by $\bar{\nabla}$, with Christoffel symbols γ^a_{bc} .

Remark 2.1. The same notation \mathbf{g} is used for both the background metric of the ambient space $\mathcal{S} = \mathbb{R}^3$ and the spatial metric of the deformed configuration $\varphi_t(\mathcal{B})$, but they have a subtle yet important difference. The background metric is fixed and time-independent, describing the intrinsic geometry of the ambient space. In contrast, the spatial metric evolves over time due to its dependence on the deformation mapping φ_t , as its domain of definition is the deformed configuration $\varphi_t(\mathcal{B})$. Formally, the spatial metric \mathbf{g}_t is the pull-back of the background metric \mathbf{g} by the time-dependent inclusion map $\iota_t : \varphi_t(\mathcal{B}) \rightarrow \mathcal{S}$, i.e., $\mathbf{g}_t = \iota_t^* \mathbf{g}$. This distinction is essential: the background metric provides a constant geometric reference for measuring local distances, whereas the spatial metric reflects the dynamic geometry of the deforming body. For simplicity, we use the notation \mathbf{g} for the spatial metric in the rest of the paper.

The material velocity of the motion is defined as $\mathbf{V} : \mathcal{B} \times \mathbb{R}^+ \rightarrow T\mathcal{S}$, $\mathbf{V}(X, t) := \frac{\partial \varphi(X, t)}{\partial t}$; it has components $V^a = \frac{\partial \varphi^a}{\partial t}$. The spatial velocity is defined as $\mathbf{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \rightarrow T\mathcal{S}$, $\mathbf{v}(x, t) := \mathbf{V}(\varphi_t^{-1}(x), t)$. The material acceleration of the motion is defined as $\mathbf{A} : \mathcal{B} \times \mathbb{R}^+ \rightarrow T\mathcal{S}$, $\mathbf{A}(X, t) := D_t^{\mathbf{g}} \mathbf{V}(X, t)$, where $D_t^{\mathbf{g}}$ denotes the covariant derivative along $\varphi_X : t \mapsto \varphi(X, t)$; it reads in components as $A^a = \frac{\partial V^a}{\partial t} + \gamma^a_{bc} V^b V^c$. The spatial acceleration of the motion is defined as $\mathbf{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \rightarrow T\mathcal{S}$, $\mathbf{a}(x, t) := \mathbf{A}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S}$; it has components $a^a = \frac{\partial v^a}{\partial t} + \gamma^a_{bc} v^b v^c$.

2.2 The Material Configuration in Finite Elasticity

In the context of finite elasticity, the body's natural reference configuration, also referred to as its material configuration, is stress-free and at rest within the Euclidean ambient space. Specifically, $\mathcal{B} \subset \mathcal{S} = \mathbb{R}^3$ and naturally inherits its geometry from the ambient space, meaning its metric is essentially a copy of the background metric \mathbf{g} . Denoting the material metric in finite elasticity by $\mathring{\mathbf{G}}$, it is formally defined as the pullback of the background metric under the inclusion embedding $\iota : \mathcal{B} \rightarrow \mathcal{S}$, such that $\mathring{\mathbf{G}} = \iota^* \mathbf{g}$.

The material (natural stress-free) configuration for elasticity is $(\mathcal{B}, \mathring{\mathbf{G}})$ where distances and angles are measured using the material metric $\mathring{\mathbf{G}}$, which serves as the baseline against which deformations are quantified.⁵ Given two referential vectors $\mathbf{U}, \mathbf{W} \in T_X\mathcal{B}$, their dot product in the material configuration $(\mathcal{B}, \mathring{\mathbf{G}})$ is expressed as $\langle \mathbf{U}, \mathbf{W} \rangle_{\mathring{\mathbf{G}}} = U^A W^B \mathring{G}_{AB}$. The material volume form given by $d\mathring{V} = (\det \mathring{\mathbf{G}})^{1/2} dX^1 \wedge dX^2 \wedge dX^3$. The Levi-Civita connection associated with $(\mathcal{B}, \mathring{\mathbf{G}})$, denoted by $\mathring{\nabla}$, has Christoffel symbols $\mathring{\Gamma}^A_{BC}$. The motion $\varphi : \mathcal{B} \rightarrow \mathcal{S}$ maps each material point in the reference configuration to its spatial position in the deformed configuration, and its Jacobian \mathring{J} relates the material and spatial volume elements as $\varphi^* dv = \mathring{J} d\mathring{V}$. It can be shown that $\mathring{J} = \sqrt{\det \mathbf{g} / \det \mathring{\mathbf{G}} \det \mathbf{F}}$.

2.3 The Material Configuration in Finite Anelasticity

Anelasticity describes a class of material behaviours where intrinsic distortions, referred to as eigenstrains—derived from the German term *Eigenspannungsquellen* (sources of inherent stresses) introduced by Reissner [1931], contribute to the overall deformation of an otherwise elastic body. These eigenstrains, arising from non-elastic deformations or microstructural changes such as plasticity, growth, swelling, or thermal expansion, lead to a residually stressed state when the body is relaxed in a Euclidean ambient space. Indeed, due to the presence of eigenstrains, the natural (stress-free) configuration of the body generally cannot be isometrically embedded in the Euclidean ambient space, meaning it cannot exist as a stress-free body in physical space. The resulting

⁵Referring to Remark 2.1, note that, at time $t = t_0$, before the deformation begins, the deformation mapping reduces to the identity map on \mathcal{B} , i.e., $\varphi_{t_0} = \text{id}_{\mathcal{B}}$, and the spatial metric at $t = t_0$ reduces to the Euclidean material metric, i.e., $\mathbf{g}_{t_0} = \mathring{\mathbf{G}}$.

residual stresses are, therefore, an expression of the non-Euclidean nature of the natural configuration. Consequently, the manifold \mathcal{B} must be endowed with a material metric \mathbf{G} that is generally distinct from the Euclidean metric $\hat{\mathbf{G}}$.

To model the interplay between intrinsic, often irreversible and dissipative, distortions (eigenstrains) and the recoverable, nondissipative elastic deformations in anelastic materials, we introduce the multiplicative decomposition of the total deformation gradient: $\mathbf{F} = \overset{\circ}{\mathbf{F}}\overset{\circ}{\mathbf{F}}$ —known as the Bilby-Köner-Lee decomposition [Lubarda, 2002; Sadik and Yavari, 2017a]. Here, $\overset{\circ}{\mathbf{F}}$ maps the Euclidean reference configuration $(\mathcal{B}, \hat{\mathbf{G}})$ to the stress-free, but generally non-Euclidean, material configuration for anelasticity $(\mathcal{B}, \mathbf{G})$, while $\overset{\circ}{\mathbf{F}}$ represents the recoverable elastic deformation from the material (stress-free) configuration to the current configuration. Note, however, that while the total deformation gradient \mathbf{F} is compatible—meaning it can be expressed as the derivative of a smooth mapping φ , such that $\mathbf{F} = T\varphi$ —the individual distortions $\overset{\circ}{\mathbf{F}}$ and $\overset{\circ}{\mathbf{F}}$ are generally incompatible. This incompatibility arises because $\overset{\circ}{\mathbf{F}}$ and $\overset{\circ}{\mathbf{F}}$ are not gradients of smooth mappings themselves but instead represent local, often non-homogeneous, distortions and deformations. Indeed, these residual incompatibilities in \mathbf{F}_e and $\overset{\circ}{\mathbf{F}}$ are a direct manifestation of the failure of the relaxed configuration to be embedded in the physical Euclidean space, and effectively justify the non-Euclidean character of the material configuration preventing the fully relaxed state from being realised as a smooth, globally compatible embedding. The material metric \mathbf{G} associated with the stress-free configuration is related to the Euclidean reference metric $\hat{\mathbf{G}}$ through the relationship $\mathbf{G} = \overset{\circ}{\mathbf{F}}^* \hat{\mathbf{G}}$.⁶

Note that multiple sources of eigenstrains can coexist within a solid, including temperature changes, growth, remodeling, swelling, and plastic deformation. Each eigenstrain distribution contributes a distinct distortion field, leading to a combined total anelastic distortion. This total distortion can be expressed as a product of individual contributions: $\overset{\circ}{\mathbf{F}} = \prod_{j=1}^N \overset{j}{\mathbf{F}} = \overset{1}{\mathbf{F}} \dots \overset{N}{\mathbf{F}}$, where $\overset{j}{\mathbf{F}}$ represents the distortion field associated with the j -th eigenstrain mechanism.

Remark 2.2 (Thermal Distortion in Anelasticity). In the case of a non-isothermal process, characterised by an evolving temperature field $\Theta = \Theta(X, t)$, thermal distortion emerges as a distinct source of anelastic distortion. This distortion captures the local, generally incompatible, stretches induced by temperature variations [Stojanovic et al., 1964; Sadik and Yavari, 2017b]. The total anelastic distortion $\overset{\circ}{\mathbf{F}}$ can hence be decomposed as $\overset{\circ}{\mathbf{F}} = \overset{\circ}{\mathbf{F}} \overset{\circ}{\mathbf{F}}$, where $\overset{\circ}{\mathbf{F}}$ accounts for thermal distortion, while $\overset{\circ}{\mathbf{F}}$ encompasses all other eigenstrain mechanisms coexisting in the solid. What distinguishes thermal distortion is that its evolution is governed by the heat equation, making it directly dependent on the temperature field $\Theta = \Theta(X, t)$. We show in Appendix A that

$$\boldsymbol{\alpha} = \frac{\partial \overset{\circ}{\mathbf{F}}}{\partial \Theta} \overset{\circ}{\mathbf{F}}^{-1} = \overset{\circ}{\mathbf{F}}^{-1} \frac{\partial \overset{\circ}{\mathbf{F}}}{\partial \Theta}, \quad (2.1)$$

where $\boldsymbol{\alpha}$ is the linear thermal expansion coefficient tensor of the material.

For an anelastic solid, the material (natural stress-free) configuration is hence given by the abstract manifold $(\mathcal{B}, \mathbf{G})$ where distances and angles are measured using the Riemannian metric \mathbf{G} . Given two vectors $\mathbf{U}, \mathbf{W} \in T_X \mathcal{B}$, their dot product is expressed as: $\langle \mathbf{U}, \mathbf{W} \rangle_{\mathbf{G}} = \mathbf{U}^A \mathbf{W}^B \mathbf{G}_{AB}$. The volume form in the anelastic material configuration is given by $dV = (\det \mathbf{G})^{1/2} dX^1 \wedge dX^2 \wedge dX^3$. The Levi-Civita connection associated with $(\mathcal{B}, \mathbf{G})$ is denoted ∇ with Christoffel symbols Γ^A_{BC} . In anelasticity, we define the elastic Jacobian $\overset{\circ}{J}$ of the motion relates the stress-free material and spatial volume elements as $\varphi^* dv = \overset{\circ}{J} dV$, and it can be shown that $\overset{\circ}{J} = \sqrt{\det \mathbf{g} / \det \mathbf{G}} \det \mathbf{F} = \sqrt{\det \mathbf{g} / \det \hat{\mathbf{G}}} \det \overset{\circ}{\mathbf{F}}$. We may also introduce the athermal anelastic Jacobian related to the athermal anelastic distortion $\overset{\circ}{\mathbf{F}}$: $\overset{\circ}{J} = \det \overset{\circ}{\mathbf{F}}$.

⁶To see this, consider a curve on the reference configuration, $\gamma : \mathcal{I} \rightarrow \mathcal{B}$, where \mathcal{I} is an open interval in \mathbb{R} . Due to the presence of eigenstrains, the curve γ is generally stressed in the Euclidean reference configuration $(\mathcal{B}, \hat{\mathbf{G}})$. However, its push-forward under the anelastic distortion, $\overset{\circ}{\mathbf{F}}_* \gamma$, is locally stress-free. The squared arc-length element in this natural (stress-free) configuration is given by $\langle \overset{\circ}{\mathbf{F}}_* \gamma'(t), \overset{\circ}{\mathbf{F}}_* \gamma'(t) \rangle_{\hat{\mathbf{G}}} = \langle \overset{\circ}{\mathbf{F}} \gamma'(t), \overset{\circ}{\mathbf{F}} \gamma'(t) \rangle_{\mathbf{G}} = \langle \gamma'(t), \gamma'(t) \rangle_{\hat{\mathbf{G}}}$ —see [Yavari, 2021]. Hence, the material metric \mathbf{G} , which measures natural distances in the stress-free configuration, is the pull-back of the Euclidean metric $\hat{\mathbf{G}}$ under the anelastic distortion $\overset{\circ}{\mathbf{F}}$.

3 Thermodynamics and the Balance Laws of Hyperelasticity

In this section, we present the results of this work in the context of finite hyperelasticity—a subclass of elasticity in which the stress is derived from a scalar strain energy function as discussed below in Remark 3.1. Recall that in finite elasticity, the material metric $\mathring{\mathbf{G}}$ is a copy of the background Euclidean metric \mathbf{g} , i.e., $\mathring{\mathbf{G}} = \iota^* \mathbf{g}$, and serves as the geometric foundation for defining distances and angles in the stress-free reference configuration—see §2.2. In what follows, we begin by briefly reviewing the first and second laws of thermodynamics in the setting of nonlinear hyperelasticity. We then demonstrate how all the balance laws and constitutive equations of hyperelasticity can be derived directly from these thermodynamic principles, without invoking (observer) invariance.⁷

Remark 3.1 (Hyperelasticity). As formalised by Noll [1958], a *simple elastic solid* is that for which the stress at a given point depends only on the local current deformation state at that point (e.g., via the deformation gradient \mathbf{F}), discarding any dependence on the material’s deformation history or nonlocal effects such as higher-order spatial gradients. A particular subclass of simple elastic materials is known as *Cauchy elastic* materials, where the stress at any point is explicitly expressed as a function of the strain at that point, such as $\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F})$ [Cauchy, 1828; Truesdell, 1952]. This subclass assumes a direct and explicit dependence of stress on strain, further refining the constitutive description. Within Cauchy elasticity, another important subset consists of materials for which the stress is derivable from a scalar strain energy function. These are referred to as *Green elastic* [Green, 1838, 1839; Spencer, 2015] or *hyperelastic* materials [Truesdell, 1952]. However, not all Cauchy elastic materials possess such an energy function, as Green elasticity imposes additional constraints on the constitutive relation. It is worth noting that not all simple elastic solids are Cauchy elastic. There are those whose constitutive relations are defined implicitly, taking the form $\mathbf{f}(\boldsymbol{\sigma}, \mathbf{F}) = \mathbf{0}$, where the stress and deformation measures are related implicitly [Morgan, 1966; Rajagopal, 2003, 2007, 2011]. These implicit models encompass both Cauchy elastic materials and other, more complex elastic behaviours. Thus, Cauchy elastic solids form a proper subset of simple elastic materials, and Green elastic materials are a further subset of Cauchy elastic solids. This hierarchy reflects the progressively restrictive assumptions underpinning these material models.

3.1 First Law of Thermodynamics

The first law of thermodynamics posits the existence of an internal energy \mathcal{E} as a state function, which satisfies the following balance equation as an expression of the conservation of energy principle [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt} \int_{\mathcal{U}} \dot{\rho} \left(\mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right) d\dot{V} = \int_{\mathcal{U}} \dot{\rho} \left(\langle \mathbf{B}, \mathbf{V} \rangle_{\mathbf{g}} + R \right) d\dot{V} + \int_{\partial \mathcal{U}} \left(\langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}} - \langle \mathbf{Q}, \mathbf{N} \rangle_{\mathring{\mathbf{G}}} \right) d\dot{A}, \quad (3.1)$$

for any open set $\mathcal{U} \subset \mathcal{B}$, where \mathcal{E} stands for the specific internal energy, $\dot{\rho}$ is the material mass density, \mathbf{B} is the specific body force, $R = R(X, t)$ is the specific heat supply, $\mathbf{T} = \mathbf{T}(X, \mathbf{N})$ is the boundary traction vector field per unit material area, \mathbf{N} is the $\mathring{\mathbf{G}}$ -unit normal to the boundary $\partial \mathcal{U}$, and $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$ represents the external heat flux per unit material area, $d\Theta = \frac{\partial \Theta}{\partial X^A} dX^A$ is the exterior derivative of temperature Θ , and $d\dot{A}$ is the material area element.

Using Marsden and Hughes [1983, §2.1(1.9)]’s version of Cauchy’s theorem,⁸ it follows from (3.1) that there exists a unique material vector field \mathbf{U} such that $\langle \mathbf{U}, \mathbf{N} \rangle_{\mathring{\mathbf{G}}} = \langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$. The linearity of $\langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$ with respect to \mathbf{V} implies that \mathbf{U} is also linear in \mathbf{V} , i.e., there exists a second order two-point tensor field \mathbf{M} such that $\mathbf{U} = \mathbf{M}\mathbf{V}$, which indeed is unique following the uniqueness of \mathbf{U} . We now have that $\langle \mathbf{M}\mathbf{V}, \mathbf{N} \rangle_{\mathring{\mathbf{G}}} = \langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$, which may be recast into $\langle \mathbf{M}^{\top} \mathbf{N}, \mathbf{V} \rangle_{\mathbf{g}} = \langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}}$. By virtue of the existence and uniqueness of \mathbf{M} , we may define the two-point tensor $\mathbf{P} := \mathbf{M}^{\top} \mathbf{g}^{\sharp}$, and it follows by arbitrariness of \mathbf{V} that $\mathbf{T} = \mathbf{P}\mathbf{N}^{\flat}$, where $(\cdot)^{\flat}$ and $(\cdot)^{\sharp}$

⁷As discussed earlier in §1, the Green-Naghdi-Rivlin theorem (and its subsequent extensions) yields the balance of linear and angular momentum as a consequence of the invariance of the energy balance under superposed isometries of the ambient space. Here, the proposed derivation does not rely on invariance; instead, it generalizes the Coleman-Noll procedure [Coleman and Noll, 1963] to recover not only the constitutive equations, but also the balance laws.

⁸Note that this version of Cauchy’s theorem does not assume the balance of linear momentum and does not introduce the idea of stress—and neither do we, at this stage of the paper.

denote the musical isomorphisms for lowering and raising indices, respectively, with respect to $\mathring{\mathbf{G}}$ and \mathbf{g} . Now, we proceed to write the energy balance (3.1) in local form:

$$\dot{\rho} \dot{\mathcal{E}} = \mathbf{P} : (\mathbf{g} \bar{\nabla} \mathbf{V}) - \text{Div } \mathbf{Q} + \dot{\rho} R + \langle \dot{\text{Div}} \mathbf{P} + \rho (\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} - \dot{\rho}_0 \left(\mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right), \quad (3.2)$$

where a dotted quantity denotes its total time derivative, $\dot{\text{Div}}$ denotes the two-point divergence operator with respect to the connections $\bar{\nabla}$ and $\bar{\nabla}$, and a colon denotes the double contraction product, i.e., $\mathbf{P} : (\mathbf{g} \bar{\nabla} \mathbf{V}) = \bar{\mathbf{P}}^{aA} V_{a|A}$ —the vertical stroke here denoting covariant differentiation with respect to \mathbf{g} in components, i.e., $V_{a|A} = V_{a,A} - V_b \gamma_{ac}^b F^c{}_A$. Note that⁹

$$\mathbf{P} : (\mathbf{g} \bar{\nabla} \mathbf{V}) = \mathbf{P} : (\mathbf{g} \bar{\text{D}}_t \mathbf{F}) = (\mathbf{F}^{-1} \mathbf{P}) : (\mathring{\mathbf{D}} + \mathring{\mathbf{D}}), \quad (3.3)$$

where $\bar{\text{D}}_t(\cdot) := \bar{\nabla}_{\mathbf{v}}(\cdot)$ denotes the covariant time derivative,¹⁰ $\mathring{\mathbf{D}}$ is the symmetric part of $\mathbf{F}^* \mathbf{g} \dot{\mathbf{F}}$, and $\mathring{\mathbf{D}}$ is its anti-symmetric part:

$$\mathring{\mathbf{D}} = \frac{1}{2} [\mathbf{F}^* \mathbf{g} \bar{\text{D}}_t \mathbf{F} + \bar{\text{D}}_t \mathbf{F}^* \mathbf{g} \mathbf{F}], \quad \mathring{\mathbf{D}} = \frac{1}{2} [\mathbf{F}^* \mathbf{g} \bar{\text{D}}_t \mathbf{F} - \bar{\text{D}}_t \mathbf{F}^* \mathbf{g} \mathbf{F}]. \quad (3.4)$$

3.2 Second Law of Thermodynamics

The second law of thermodynamics posits the existence of specific entropy \mathcal{N} as a state function, which satisfies the following inequality—known as the Clausius-Duhem inequality—as an expression of the principle of entropy production,¹¹ which steadily increases or remains constant within a closed system over time [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt} \int_{\mathcal{U}} \dot{\rho} \mathcal{N} d\dot{V} \geq \int_{\mathcal{U}} \dot{\rho} \frac{R}{\Theta} d\dot{V} + \int_{\partial \mathcal{U}} \frac{H}{\Theta} d\dot{A}, \quad (3.5)$$

for any open set $\mathcal{U} \subset \mathcal{B}$. In localized form, the material Clausius-Duhem inequality (3.5) yields

$$\dot{\eta} = \dot{\rho} \Theta \mathcal{N} + \dot{\rho} \dot{\Theta} \mathcal{N} + \dot{\text{Div}} \mathbf{Q} - \dot{\rho} R - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0, \quad (3.6)$$

where $\dot{\eta}$ denotes the material rate of energy dissipation density.

Remark 3.2. At a material point $X \in \mathcal{B}$, Coleman and Noll [1963] introduce the set¹²

$$\left\{ \varphi_t(X), \mathbf{P}(X, t), \mathbf{B}(X, t), \mathcal{E}(X, t), \mathbf{Q}(X, t), R(X, t), \mathcal{N}(X, t), \Theta(X, t) \right\}, \quad (3.7)$$

and call it a *thermodynamic process*, provided all its eight fields satisfy the first law of thermodynamics and the balance of linear and angular momenta. A given hyperelastic material is specified by its constitutive assumptions, e.g., $\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$ and $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$. A thermodynamic process is *admissible* if the constitutive assumptions hold everywhere in the body and at all times. Coleman and Noll [1963] showed that requiring the Clausius-Duhem inequality (3.6) to hold for all admissible thermodynamic processes puts certain constraints on the constitutive assumptions, e.g., the Doyle-Ericksen formula.

In this work, we define an *extended thermodynamic process* at a material point $X \in \mathcal{B}$ as the set (3.7), provided all its eight fields satisfy the first law of thermodynamics—without requiring that \mathbf{P} satisfies the balance of linear and angular momenta. An extended thermodynamic process is *admissible* if the constitutive assumptions hold everywhere in the body and at all times. We require that the Clausius-Duhem inequality (3.6) holds for all admissible extended thermodynamic processes.

⁹We thank Sanjay Govindjee for bringing this to our attention.

¹⁰Note that by using the symmetry lemma [Nishikawa, 2002], one may write $V^a|_A = \left(\frac{\partial \varphi^a}{\partial t} \right)|_A = \bar{\text{D}}_t \left(\frac{\partial \varphi^a}{\partial X^A} \right)$, which yields that $\bar{\nabla} \mathbf{V} = \bar{\text{D}}_t \mathbf{F}$, which, expanded in components, reads off as $\bar{\text{D}}_t F^a{}_A = \frac{\partial}{\partial t} (F^a{}_A) + F^b{}_A \gamma^a{}_{bc} v^c$.

¹¹The entropy production for an open subset \mathcal{U} in the body reads as

$$\Gamma = \frac{d}{dt} \int_{\mathcal{U}} \dot{\rho} \mathcal{N} d\dot{V} - \int_{\mathcal{U}} \dot{\rho} \frac{R}{\Theta} d\dot{V} - \int_{\partial \mathcal{U}} \frac{H}{\Theta} d\dot{A}.$$

¹² \mathbf{P} , in their treatment, is introduced, a priori, as the first Piola-Kirchhoff stress tensor (in fact, they use the Cauchy stress tensor) satisfying the balance of linear and angular momenta.

3.3 Constitutive and Balance Laws in Nonlinear Hyperelasticity

Let us first consider the case of a hyperelastic material. Hyperelasticity implies the existence an energy function that depends explicitly at every material point $X \in \mathcal{B}$ on the strain at that same point. We may hence write the specific free energy with the following dependence $\Psi = \Psi(X, \Theta, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$ [Truesdell, 1952], which in fact is the Legendre transform of the specific internal energy \mathcal{E} with respect to the conjugate variables temperature Θ and specific entropy \mathcal{N} :

$$\Psi = \mathcal{E} - \Theta \mathcal{N}. \quad (3.8)$$

Hence, we have that $\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$, and consequently find¹³

$$\mathcal{N} = -\frac{\partial \Psi}{\partial \Theta}. \quad (3.9)$$

Proposition 3.1. *For a hyperelastic body, the first and second laws of thermodynamics (3.2) and (3.6) imply that*

$$\begin{cases} \dot{\rho}_0 = 0, & (3.10a) \end{cases}$$

$$\begin{cases} \mathbf{P} = \dot{\rho} \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \mathbf{F}}, & (3.10b) \end{cases}$$

$$\begin{cases} \bar{\text{Div}} \mathbf{P} + \dot{\rho} \mathbf{B} = \dot{\rho} \mathbf{A}, & (3.10c) \end{cases}$$

$$\begin{cases} \mathbf{F} \mathbf{P}^* = \mathbf{P} \mathbf{F}^*, & (3.10d) \end{cases}$$

$$\begin{cases} \dot{\eta} = -\frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. & (3.10e) \end{cases}$$

In other words, the first and second laws of thermodynamics imply the conservation of mass (3.10a), the Doyle-Ericksen formula (3.10b), the balance of linear momentum (3.10c), the balance of angular momentum (3.10d), and the dissipation inequality (3.10e). Note that (3.10b) effectively shows that the two-point tensor \mathbf{P} is indeed the Piola-Kirchhoff stress tensor.

Proof. From (3.8), one writes: $\dot{\rho} \Theta \mathcal{N} = \dot{\rho} \mathcal{E} - \dot{\rho} \dot{\Psi} - \dot{\rho} \dot{\Theta} \mathcal{N}$. Substituting this relation and (3.2) (while using (3.3)) into (3.6), one obtains

$$\begin{aligned} \dot{\eta} = & (\mathbf{F}^{-1} \mathbf{P}) : (\bar{\mathbf{D}} + \mathring{\mathbf{D}}) - \dot{\rho} \dot{\Psi} - \dot{\rho} \dot{\Theta} \mathcal{N} + \dot{\rho}_0 \Theta \mathcal{N} \\ & + \langle \bar{\text{Div}} \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} - \dot{\rho}_0 \left[\mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right] - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \end{aligned} \quad (3.11)$$

Applying Leibniz rule to $\dot{\Psi}$, one writes¹⁴

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{F}} : \bar{\text{D}}_t \mathbf{F} + \frac{\partial \Psi}{\partial \mathbf{g}} : \bar{\text{D}}_t \mathbf{g} = -\mathcal{N} \dot{\Theta} + \left(\mathbf{F}^{-1} \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \mathbf{F}} \right) : (\bar{\mathbf{D}} + \mathring{\mathbf{D}}). \quad (3.12)$$

¹³Note that the Legendre transform (3.8) of \mathcal{E} to Ψ with respect to the conjugate variables \mathcal{N} and Θ is essentially a change of variable satisfying (3.9). See [Arnold, 1989; Goldstein et al., 2002] for further details on Legendre transform in the context of Lagrangian mechanics and thermodynamics.

¹⁴Note that since the connection is Levi-Civita, which is metric compatible, it follows that $\bar{\text{D}}_t \mathbf{g} = \bar{\nabla}_{\mathbf{v}} \mathbf{g} = \mathbf{0}$.

It can be shown that¹⁵

$$\left(\mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\mathbf{F}}\right)^{\star}=\mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\mathbf{F}}. \quad (3.13)$$

Hence $\mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\mathbf{F}}$ is a symmetric tensor, and since $\mathring{\mathbf{D}}$ is antisymmetric, it follows that

$$\left(\mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\mathbf{F}}\right):\mathring{\mathbf{D}}=0. \quad (3.14)$$

Using the above identity, and substituting (3.12) into (3.11), the rate of dissipation is simplified to read

$$\dot{\eta}=\left[\mathbf{F}^{-1}\left(\mathbf{P}-\dot{\rho}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\mathbf{F}}\right)\right]:\mathring{\mathbf{D}}+\mathbf{F}^{-1}\mathbf{P}:\mathring{\mathbf{D}}+\langle\mathring{\text{Div}}\mathbf{P}+\rho(\mathbf{B}-\mathbf{A}),\mathbf{V}\rangle_{\mathbf{g}}-\dot{\rho}_0\left[\Psi+\frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2\right]-\frac{1}{\Theta}\langle d\Theta,\mathbf{Q}\rangle\geq 0. \quad (3.15)$$

This inequality must hold for all motions, i.e., all extended thermodynamic processes. As $\mathring{\mathbf{D}}$ (a symmetric tensor) and $\mathring{\mathbf{D}}$ (an antisymmetric tensor) can be chosen independently of all the other fields, one concludes that¹⁶

$$\mathbf{P}=\dot{\rho}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\mathbf{F}}, \quad (\mathbf{F}^{-1}\mathbf{P})^{\star}=\mathbf{F}^{-1}\mathbf{P}. \quad (3.16)$$

Now, the rate of dissipation (3.15) is simplified to read

$$\dot{\eta}=\langle\mathring{\text{Div}}\mathbf{P}+\rho(\mathbf{B}-\mathbf{A}),\mathbf{V}\rangle_{\mathbf{g}}-\dot{\rho}_0\left[\Psi+\frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2\right]-\frac{1}{\Theta}\langle d\Theta,\mathbf{Q}\rangle\geq 0. \quad (3.17)$$

One can choose the velocity vector arbitrarily while its norm $\|\mathbf{V}\|_{\mathbf{g}}$ is fixed. This implies that the inequality (3.17) can hold only if

$$\mathring{\text{Div}}\mathbf{P}+\dot{\rho}\mathbf{B}=\dot{\rho}\mathbf{A}. \quad (3.18)$$

Now the rate of dissipation takes the following form

$$\dot{\eta}=-\dot{\rho}_0\left[\Psi+\frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2\right]-\frac{1}{\Theta}\langle d\Theta,\mathbf{Q}\rangle\geq 0. \quad (3.19)$$

Next, one can choose the velocity vector norm $\|\mathbf{V}\|_{\mathbf{g}}$ arbitrarily while keeping the other fields fixed. For all these extended thermodynamics processes the above inequality must hold. This implies that $\dot{\rho}_0=0$ and $\dot{\eta}=-\frac{1}{\Theta}\langle d\Theta,\mathbf{Q}\rangle\geq 0$. \square

Remark 3.3. Coleman and Noll [1963] showed that requiring (3.6) to hold for all admissible thermodynamic processes gives the Doyle-Ericksen formula (3.10b). We have shown that requiring (3.6) to hold for all admissible extended thermodynamic processes gives all the balance laws and the Doyle-Ericksen formula (3.10b). It is known that in hyperelasticity, balance of angular momentum implies objectivity [Kadić, 1980; Yavari and Goriely, 2024]. Therefore, we have shown that the first and second laws of thermodynamics imply objectivity as well.

¹⁵Using Leibniz rule, we write in components

$$\frac{\partial\Psi}{\partial\mathbf{F}^a{}_A}=\frac{\partial\Psi}{\partial(\mathbf{F}^i{}_K\mathbf{F}^j{}_L\mathbf{g}_{ij})}\frac{\partial(\mathbf{F}^k{}_K\mathbf{F}^l{}_L\mathbf{g}_{kl})}{\partial\mathbf{F}^a{}_A}=\frac{\partial\Psi}{\partial(\mathbf{F}^i{}_K\mathbf{F}^j{}_L\mathbf{g}_{ij})}\left(\delta^k{}_a\delta^A{}_K\mathbf{F}^l{}_L\mathbf{g}_{kl}+\mathbf{F}^k{}_K\delta^l{}_a\delta^A{}_L\mathbf{g}_{kl}\right).$$

Using straightforward index manipulations, it follows that

$$\frac{\partial\Psi}{\partial\mathbf{F}^a{}_A}=2\mathbf{g}_{al}\mathbf{F}^l{}_K\frac{\partial\Psi}{\partial(\mathbf{F}^i{}_K\mathbf{F}^j{}_A\mathbf{g}_{ij})}, \quad \text{and} \quad \frac{\partial\Psi}{\partial\mathbf{F}^a{}_A}=2\frac{\partial\Psi}{\partial(\mathbf{F}^i{}_A\mathbf{F}^j{}_L\mathbf{g}_{ij})}\mathbf{F}^k{}_L\mathbf{g}_{ka},$$

which one may rewrite as

$$\frac{\partial\Psi}{\partial\mathbf{F}}=2\mathbf{g}\mathbf{F}\frac{\partial\Psi}{\partial(\mathbf{F}^{\star}\mathbf{g}\mathbf{F})}, \quad \text{and} \quad \left(\frac{\partial\Psi}{\partial\mathbf{F}}\right)^{\star}=2\frac{\partial\Psi}{\partial(\mathbf{F}^{\star}\mathbf{g}\mathbf{F})}\mathbf{F}^{\star}\mathbf{g}.$$

Therefore, one finds that

$$\mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\mathbf{F}}=\left(\frac{\partial\Psi}{\partial\mathbf{F}}\right)^{\star}\mathbf{g}^{\sharp}\mathbf{F}^{-\star},$$

which directly leads to (3.13).

¹⁶There are infinitely many extended thermodynamic processes for which everything except for $\mathring{\mathbf{D}}$ and $\mathring{\mathbf{D}}$ are the same. For the second law to hold for all such processes one must have (3.16).

Remark 3.4. For an incompressible hyperelastic solid, the Legendre transform (3.8) is modified to take into account the constraint of volume preservation $J = 1$ on motions as follows

$$\Psi - p(J - 1) = \mathcal{E} - \Theta \mathcal{N}, \quad (3.20)$$

where $p(X, t)$ is the Lagrange multiplier associated with the incompressibility constraint. Computing \dot{J} , we find¹⁷

$$\dot{J} = J \mathbf{F}^{-\star} : \dot{\mathbf{F}} = (\mathbf{F}^{-1} \mathbf{g}^{\sharp} \mathbf{F}^{-\star}) : (\dot{\mathbf{D}} + \dot{\mathbf{D}}) = (\mathbf{F}^{-1} \mathbf{g}^{\sharp} \mathbf{F}^{-\star}) : \dot{\mathbf{D}}. \quad (3.21)$$

Revisiting the proof for Proposition 3.1, the results remain unchanged except for the Doyle-Ericksen formula (3.10b) which is modified to read

$$\mathbf{P} = \rho \mathbf{g}^{\sharp} \frac{\partial \Psi}{\partial \mathbf{F}} - p \mathbf{g}^{\sharp} \mathbf{F}^{-\star}. \quad (3.22)$$

Remark 3.5. Note that it is possible to recast the balance and constitutive equations (4.9) in their spatial (current) form as follows:

$$\begin{cases} \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, & (3.23a) \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma} = \rho \mathbf{g}^{\sharp} \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{\star} = 2\rho \frac{\partial \Psi}{\partial \mathbf{g}}, & (3.23b) \end{cases}$$

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}, & (3.23c) \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}^{\star} = \boldsymbol{\sigma}, & (3.23d) \end{cases}$$

$$\begin{cases} \dot{\eta} = -\frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0, & (3.23e) \end{cases}$$

where ρ is the spatial density, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{b} = \mathbf{B} \circ \varphi_t^{-1}$, and div denotes the spatial divergence operator, i.e., in the manifold $(\mathcal{S}, \mathbf{g})$. The spatial balance of mass (3.23a) is derived from the material (reference) balance and the relation $\rho = \dot{\rho}/\dot{J}$, which follows from the definition of the Jacobian, $\varphi^* dv = \dot{J} dV$. In (3.23b), the Cauchy stress tensor $\boldsymbol{\sigma}$ is the Piola transform of the first Piola-Kirchhoff stress tensor \mathbf{P} ; consistently with the spatial forms of the balance of linear momentum (3.23c) and angular momentum (3.23d) obtained by pushing forward the respective material equations into the current configuration using the Piola transformation [Marsden and Hughes, 1983].

4 Thermodynamics and the Balance Laws of Hyper-Anelasticity

In this section, we extend our generalised Coleman-Noll procedure to the setting of hyper-anelasticity. As discussed in §2.3, the Bilby-Köner-Lee (BKL) decomposition, $\mathbf{F} = \mathbf{\hat{F}} \mathbf{\check{F}}$, provides a framework for separating the recoverable elastic deformation, $\mathbf{\hat{F}}$, from the intrinsic anelastic distortion, $\mathbf{\check{F}}$. As a consequence of this decomposition, the material metric $\mathbf{G} = \mathbf{\hat{F}}^* \mathbf{\check{G}}$ emerges to describe the evolving, non-Euclidean geometry of the stress-free material configuration, reflecting the intrinsic distortions arising from eigenstrains. Building on this foundation, we generalise the first and second laws of thermodynamics to incorporate the interplay between elastic and anelastic deformations within the context of the evolving material metric. This generalised framework allows for the derivation of constitutive equations, balance laws, and thermodynamic constraints governing hyper-anelastic materials, all without invoking assumptions of invariance.

4.1 The first law of thermodynamics

For an anelastic solid, we write energy balance in the anelastic material manifold as follows [Epstein and Maugin, 2000; Lubarda and Hoger, 2002]¹⁸

¹⁷We use the identity $\frac{d}{dt} [\det \mathbf{F}] = [\mathbf{F}^{-\star} : \dot{\mathbf{D}}_t \mathbf{F}] \det \mathbf{F}$, perform a similar computation to (3.3), and observe that $\mathbf{F}^{-1} \mathbf{g}^{\sharp} \mathbf{F}^{-\star}$ is a symmetric tensor.

¹⁸Note that the last term in (4.1) is added to account for the change in the internal and kinetic energies of the system due to bulk growth or shrinkage with a material rate of change of mass S_m .

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{U}} \rho \left(\mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right) dV &= \int_{\mathcal{U}} \rho \left(\langle \mathbf{B}, \mathbf{V} \rangle_{\mathbf{g}} + R \right) dV + \int_{\partial \mathcal{U}} \left(\langle \mathbf{T}, \mathbf{V} \rangle_{\mathbf{g}} - \langle \mathbf{Q}, \mathbf{N} \rangle_{\mathbf{G}} \right) dA \\ &\quad + \int_{\mathcal{U}} S_m \left(\mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right) dV, \end{aligned} \quad (4.1)$$

for any open set $\mathcal{U} \subset \mathcal{B}$, where \mathcal{E} is the specific internal energy, ρ is the material mass density, $S_m = S_m(X, t)$ is the material rate of change of mass per unit (stress-free) volume—it is identically equal to zero in the absence of bulk growth or shrinkage, \mathbf{B} is the specific body force, $\mathbf{T} = \mathbf{T}(X, \mathbf{N})$ is the boundary traction vector field per unit (stress-free) material area, \mathbf{N} is the \mathbf{G} -unit normal to the boundary $\partial \mathcal{U}$, $R = R(X, t)$ is the external specific heat supply, $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{C}, \mathbf{G})$ is the external heat flux per unit material (stress-free) area and, dA is the material area element.

By the same argument developed in §3.1 using Marsden and Hughes [1983]’s version of Cauchy’s theorem, we find $\mathbf{T} = \mathbf{P}\mathbf{N}^{\flat}$, where \mathbf{P} is a second order two-point tensor, and $(\cdot)^{\flat}$ denotes the musical isomorphisms for lowering with respect to \mathbf{G} and \mathbf{g} . Hence, the energy balance (3.1) transforms to read in local form as follows:¹⁹

$$\rho \dot{\mathcal{E}} = (\mathbf{F}^{-1}\mathbf{P}) : (\overset{\circ}{\mathbf{D}} + \overset{\circ}{\mathbf{D}}) + \rho R - \text{Div } \mathbf{Q} + \langle \text{Div } \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} + \left(S_m - \dot{\rho} - \frac{1}{2} \rho \dot{\mathbf{G}} : \mathbf{G}^{\sharp} \right) \left(\mathcal{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right), \quad (4.2)$$

where Div denotes the two-point divergence operator with respect to the connections ∇ and $\bar{\nabla}$, $(\cdot)^{\sharp}$ denotes the musical isomorphisms for raising indices with respect to \mathbf{G} and \mathbf{g} , and $\overset{\circ}{\mathbf{D}}$ and $\overset{\circ}{\mathbf{D}}$ are the symmetric and anti-symmetric material rate of deformation tensors, respectively, as defined in (3.4).

4.2 The second law of thermodynamics

For an anelastic solid, we write the Clausius-Duhem in the anelastic material manifold as follows [Epstein and Maugin, 2000; Lubarda and Hoger, 2002]²⁰

$$\frac{d}{dt} \int_{\mathcal{U}} \rho \mathcal{N} dV \geq \int_{\mathcal{U}} \rho \frac{R}{\Theta} dV + \int_{\partial \mathcal{U}} \frac{H}{\Theta} dA + \int_{\mathcal{U}} S_m \mathcal{N} dV, \quad (4.3)$$

for any open set $\mathcal{U} \subset \mathcal{B}$, where \mathcal{N} is the specific entropy (i.e., entropy per unit mass). In localized form, the material Clausius-Duhem inequality (3.5) is written as

$$\dot{\eta} = \rho \dot{\mathcal{N}} \Theta + \Theta \text{Div} \left(\frac{\mathbf{Q}}{\Theta} \right) - \rho R - \left(S_m - \dot{\rho} - \frac{1}{2} \rho \dot{\mathbf{G}} : \mathbf{G}^{\sharp} \right) \Theta \mathcal{N} \geq 0. \quad (4.4)$$

4.3 Constitutive and Balance Laws in Nonlinear Hyper-Anelasticity

Let us now consider the case of a hyper-anelastic material. As discussed in §2.3, intrinsic distortions (eigenstrains) give rise to a non-Euclidean stress-free configuration, represented by the incompatible distortion $\overset{\circ}{\mathbf{F}}$. The recoverable elastic distortion $\overset{\circ}{\mathbf{F}}$ then maps this stress-free configuration to the current spatial configuration. Consequently, the free energy for a hyper-anelastic material depends on $\overset{\circ}{\mathbf{F}}$ rather than the full deformation gradient \mathbf{F} , i.e., $\Psi = \Psi(X, \Theta, \overset{\circ}{\mathbf{F}}, \overset{\circ}{\mathbf{G}}, \mathbf{g})$. As the Legendre transform of the specific internal energy \mathcal{E} with respect to the conjugate variables temperature Θ and specific entropy \mathcal{N} , the specific free energy function reads

$$\Psi = \mathcal{E} - \Theta \mathcal{N}. \quad (4.5)$$

Hence, we find that $\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \overset{\circ}{\mathbf{F}}, \overset{\circ}{\mathbf{G}}, \mathbf{g})$ and

$$\mathcal{N} = - \frac{\partial \Psi}{\partial \Theta}. \quad (4.6)$$

¹⁹Recall that $\mathbf{G} = \overset{\circ}{\mathbf{F}}^* \overset{\circ}{\mathbf{G}}$ and $dV = \sqrt{\det \mathbf{G}} dX^1 \wedge dX^2 \wedge dX^3$. Hence, we ought to consider the implicit time dependance of dV through $\det \mathbf{G}$ in the left-hand-side of (4.1) and we use the identity $\frac{d}{dt} [\det \mathbf{G}] = [\dot{\mathbf{G}} : \mathbf{G}^{-1}] \det \mathbf{G}$.

²⁰Note that the last term in (3.5) is added to account for the change in the entropy of the system due to bulk growth or shrinkage with a material rate of change of mass S_m .

Remark 4.1. The material symmetry group $\mathring{\mathcal{G}}_X$ of a hyper-anelastic material (constitutively defined by a free energy Ψ) at a point $X \in \mathcal{B}$ with respect to the Euclidean reference configuration $(\mathcal{B}, \mathring{\mathbf{G}})$ is the set of $\mathring{\mathbf{K}} \in \text{Orth}(\mathring{\mathbf{G}}) = \{\mathbf{Q} : T_X \mathcal{B} \rightarrow T_X \mathcal{B} \mid \mathbf{Q}^* \mathring{\mathbf{G}} \mathbf{Q} = \mathring{\mathbf{G}}\}$ such that

$$\mathring{\mathbf{K}}_* \Psi(X, \Theta, \mathring{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \mathring{\mathbf{K}}^* \mathring{\mathbf{F}}, \mathring{\mathbf{K}}^* \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \mathring{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g}), \quad (4.7)$$

for any elastic distortion $\mathring{\mathbf{F}}$. Introducing a set of structural tensors $\mathring{\mathbf{\Lambda}}$ for $\mathring{\mathcal{G}}_X$ ²¹ as additional independent variables of the free energy, it follows that $\Psi = \Psi(X, \Theta, \mathring{\mathbf{F}}, \mathring{\mathbf{\Lambda}}, \mathring{\mathbf{G}}, \mathbf{g})$ is materially covariant, i.e., for any linear transformation $\mathbf{T} : T_X \mathcal{B} \rightarrow T_X \mathcal{B}$, $\Psi(X, \Theta, \mathring{\mathbf{F}}, \mathring{\mathbf{\Lambda}}, \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \mathbf{T}^* \mathring{\mathbf{F}}, \mathbf{T}^* \mathring{\mathbf{\Lambda}}, \mathbf{T}^* \mathring{\mathbf{G}}, \mathbf{g})$. If we take $\mathbf{T} = \mathring{\mathbf{F}}$, and recalling that $\mathring{\mathbf{F}}^* \mathring{\mathbf{F}} = \mathring{\mathbf{F}} \mathring{\mathbf{F}} = \mathbf{F}$ and $\mathbf{G} = \mathring{\mathbf{F}}^* \mathring{\mathbf{G}} = \mathring{\mathbf{F}}^* \mathring{\mathbf{G}} \mathring{\mathbf{F}}$, it follows that

$$\Psi = \Psi(X, \Theta, \mathring{\mathbf{F}}, \mathring{\mathbf{\Lambda}}, \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \mathbf{F}, \mathbf{\Lambda}, \mathbf{G}, \mathbf{g}), \quad (4.8)$$

where $\mathbf{\Lambda} = \mathring{\mathbf{F}}^* \mathring{\mathbf{\Lambda}}$ is the set of structural tensors in $(\mathcal{B}, \mathbf{G})$. We also see in (4.8) how the material metric $\mathbf{G} = \mathring{\mathbf{F}}^* \mathring{\mathbf{G}}$ naturally emerges following the hyper-anelastic constitutive assumption.

Proposition 4.1. *For a hyper-anelastic body, the first and second laws of thermodynamics (3.2) and (3.6) imply that*

$$\begin{cases} \dot{\rho} + \frac{1}{2} \rho \mathring{\mathbf{G}} : \mathbf{G}^{-1} = S_m, & (4.9a) \\ \mathbf{P} = \rho \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} \mathring{\mathbf{F}}^{-*}, & (4.9b) \\ \text{Div } \mathbf{P} + \rho \mathbf{B} = \rho \mathbf{A}, & (4.9c) \\ \mathbf{F} \mathbf{P}^* = \mathbf{P} \mathbf{F}^*, & (4.9d) \\ \dot{\eta} = \rho \mathring{\mathbf{F}}^* \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} : \mathring{\mathbf{F}} \mathring{\mathbf{F}}^{-1} + \rho \mathring{\mathbf{F}}^* \mathring{\mathbf{F}}^* \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} \mathring{\mathbf{F}}^{-*} : \alpha \dot{\Theta} - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. & (4.9e) \end{cases}$$

In other words, the first and second laws of thermodynamics imply the conservation of mass (4.9a), the Doyle-Ericksen formula (4.9b), the balance of linear momentum (4.9c), the balance of angular momentum (4.9d), and the dissipation inequality (4.9e). Note that (4.9b) effectively shows that the two-point tensor \mathbf{P} is indeed the Piola-Kirchhoff stress tensor.

Proof. Using (4.2), (4.4), and (4.5), one finds

$$\begin{aligned} \dot{\eta} &= (\mathbf{F}^{-1} \mathbf{P}) : (\mathring{\mathbf{D}} + \mathring{\mathbf{D}}) - \rho \dot{\Psi} - \rho \dot{\Theta} \mathcal{N} + \langle \text{Div } \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} \\ &\quad + \left(S_m - \dot{\rho} - \frac{1}{2} \rho \mathring{\mathbf{G}} : \mathbf{G}^{-1} \right) \left(\Psi + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right) - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \end{aligned} \quad (4.10)$$

Expanding $\dot{\Psi}$ by Leibniz rule and using the product rule $\bar{\mathbf{D}}_t \mathring{\mathbf{F}} = (\bar{\mathbf{D}}_t \mathbf{F}) \mathring{\mathbf{F}}^{-1} - \mathring{\mathbf{F}} \mathring{\mathbf{F}}^{-1}$,²² one finds

$$\begin{aligned} \dot{\Psi} &= \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} : \bar{\mathbf{D}}_t \mathring{\mathbf{F}} + \frac{\partial \Psi}{\partial \mathbf{g}} : \bar{\mathbf{D}}_t \mathbf{g} \\ &= \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} : \left[(\bar{\mathbf{D}}_t \mathbf{F}) \mathring{\mathbf{F}}^{-1} - \mathring{\mathbf{F}} \mathring{\mathbf{F}}^{-1} \right] \\ &= \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} \mathring{\mathbf{F}}^{-*} : \bar{\mathbf{D}}_t \mathbf{F} - \mathring{\mathbf{F}}^* \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} : \mathring{\mathbf{F}} \mathring{\mathbf{F}}^{-1} \\ &= -\mathcal{N} \dot{\Theta} + \mathbf{F}^{-1} \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} \mathring{\mathbf{F}}^{-*} : (\mathring{\mathbf{D}} + \mathring{\mathbf{D}}) - \mathring{\mathbf{F}}^* \frac{\partial \Psi}{\partial \mathring{\mathbf{F}}} : \mathring{\mathbf{F}} \mathring{\mathbf{F}}^{-1}, \end{aligned} \quad (4.11)$$

²¹The subgroup $\mathring{\mathcal{G}}_X \leq \text{Orth}(\mathring{\mathbf{G}})$ can be characterized by a finite collection of *structural tensors* $\mathring{\mathbf{\Lambda}}_i$, $i = 1, \dots, N$, which is a basis for the space of $\mathring{\mathcal{G}}_X$ -invariant tensors [Liu, 1982; Boehler, 1987; Zheng and Spencer, 1993; Zheng, 1994; Lu and Papadopoulos, 2000; Mazzucato and Rachele, 2006].

²²Unlike the two-point tensors \mathbf{F} and $\mathring{\mathbf{F}}$, which are sections of the product of the tangent bundle of the material manifold \mathcal{B} and the cotangent bundle of the spatial manifold $\varphi_t(\mathcal{B})$, the referential tensor $\mathring{\mathbf{F}}$ is a section of the product of the tangent and cotangent bundles of the fixed material manifold \mathcal{B} . As a result, the total time derivatives of \mathbf{F} and $\mathring{\mathbf{F}}$ are not well-defined due to the time-dependent nature of the spatial manifold $\varphi_t(\mathcal{B})$, and their material rates must instead be expressed using the spatial covariant derivative $\bar{\mathbf{D}}_t(\cdot)$ to account for the evolving geometry of the spatial configuration. In contrast, the material rate of $\mathring{\mathbf{F}}$ is given by its well-defined total time derivative, $\dot{\mathring{\mathbf{F}}} := d\mathring{\mathbf{F}}/dt$, since it resides entirely on the fixed material manifold \mathcal{B} .

where we use metric compatibility to write $\bar{D}_t \mathbf{g} = \mathbf{0}$, recall that $\frac{\partial \Psi}{\partial \Theta} = -\mathcal{N}$, and similarly to (3.3), we use the following identity:

$$\frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{-\star} : \bar{D}_t \mathbf{F} = \mathbf{F}^{-1} \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{-\star} : (\bar{\mathbf{D}} + \dot{\bar{\mathbf{D}}}). \quad (4.12)$$

Similarly to (3.13), it can be shown that $\mathbf{F}^{-1} \mathbf{g}^\sharp (\partial \Psi / \partial \bar{\mathbf{F}}) \bar{\mathbf{F}}^{-\star}$ is a symmetric tensor, which then leads to

$$\mathbf{F}^{-1} \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{-\star} : \bar{\mathbf{D}} = 0. \quad (4.13)$$

Now substituting (4.11) in (4.10), one finds

$$\begin{aligned} \dot{\eta} = & \left[\mathbf{F}^{-1} \left(\mathbf{P} - \rho \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{-\star} \right) \right] : \bar{\mathbf{D}} + (\mathbf{F}^{-1} \mathbf{P}) : \bar{\mathbf{D}} + \rho \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \dot{\bar{\mathbf{F}}} \bar{\mathbf{F}}^{-1} - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \\ & + \langle \text{Div } \mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle_{\mathbf{g}} + \left(S_m - \dot{\rho} - \frac{1}{2} \rho \dot{\mathbf{G}} : \mathbf{G}^\sharp \right) \left(\Psi + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right) \geq 0. \end{aligned} \quad (4.14)$$

This inequality must hold for all motions, i.e., all extended thermodynamic processes. Similarly to the proof of Proposition 3.1, it follows from (4.14), by arbitrariness and independence of $\bar{\mathbf{D}}$, $\dot{\bar{\mathbf{D}}}$, $\mathbf{V} / \|\mathbf{V}\|_{\mathbf{g}}^2$, and $\|\mathbf{V}\|_{\mathbf{g}}^2$, that

$$\mathbf{P} = \rho \mathbf{g}^\sharp \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{-\star}, \quad (4.15a)$$

$$(\mathbf{F}^{-1} \mathbf{P})^\star = \mathbf{F}^{-1} \mathbf{P}, \quad (4.15b)$$

$$\text{Div } \mathbf{P} + \rho \mathbf{B} = \rho \mathbf{A}, \quad (4.15c)$$

$$\dot{\rho} + \frac{1}{2} \rho \dot{\mathbf{G}} : \mathbf{G}^\sharp = S_m, \quad (4.15d)$$

respectively. Consequently, (4.14) simplifies to

$$\dot{\eta} = \rho \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \dot{\bar{\mathbf{F}}} \bar{\mathbf{F}}^{-1} - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \quad (4.16)$$

Recalling the decomposition $\bar{\mathbf{F}} = \bar{\mathbf{F}}^{\bar{\mathbf{n}}} \bar{\mathbf{F}}^{\bar{\mathbf{e}}}$, we may rewrite the first term in (4.16) as

$$\begin{aligned} \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \dot{\bar{\mathbf{F}}} \bar{\mathbf{F}}^{-1} &= \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \dot{\bar{\mathbf{F}}}^{\bar{\mathbf{n}}} \bar{\mathbf{F}}^{\bar{\mathbf{e}}} \bar{\mathbf{F}}^{-1} + \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \bar{\mathbf{F}}^{\bar{\mathbf{n}}} \dot{\bar{\mathbf{F}}}^{\bar{\mathbf{e}}} \bar{\mathbf{F}}^{-1} \\ &= \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \dot{\bar{\mathbf{F}}}^{\bar{\mathbf{n}}} \bar{\mathbf{F}}^{\bar{\mathbf{e}}} \bar{\mathbf{F}}^{-1} + \bar{\mathbf{F}}^\star \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{\bar{\mathbf{n}}} : \dot{\bar{\mathbf{F}}}^{\bar{\mathbf{e}}} \bar{\mathbf{F}}^{-1} \\ &= \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \dot{\bar{\mathbf{F}}}^{\bar{\mathbf{n}}} \bar{\mathbf{F}}^{\bar{\mathbf{e}}} \bar{\mathbf{F}}^{-1} + \bar{\mathbf{F}}^\star \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{\bar{\mathbf{n}}} : \alpha \dot{\Theta}, \end{aligned} \quad (4.17)$$

where following (2.1) we use $\dot{\bar{\mathbf{F}}}^{\bar{\mathbf{e}}} \bar{\mathbf{F}}^{\bar{\mathbf{e}}} = \alpha \dot{\Theta}$. Which then allows us to isolate the anelastic rate of dissipation in (4.16) into an athermal anelastic contribution and a purely thermal contribution as follows

$$\dot{\eta} = \rho \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} : \dot{\bar{\mathbf{F}}}^{\bar{\mathbf{n}}} \bar{\mathbf{F}}^{\bar{\mathbf{e}}} \bar{\mathbf{F}}^{-1} + \rho \bar{\mathbf{F}}^\star \bar{\mathbf{F}}^\star \frac{\partial \Psi}{\partial \bar{\mathbf{F}}} \bar{\mathbf{F}}^{\bar{\mathbf{n}}} : \alpha \dot{\Theta} - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \quad (4.18)$$

□

Remark 4.2. For an incompressible hyper-anelastic solid, the Legendre transform (3.8) is modified to take into account the constraints of elastic incompressibility ($\bar{J} = 1$) and/or athermal anelastic incompressibility ($\bar{J}^{\bar{\mathbf{n}}} = 1$) as follows²³

$$\rho \Psi - \bar{p}(\bar{J}^{\bar{\mathbf{e}}} - 1) - \bar{p}(\bar{J}^{\bar{\mathbf{n}}} - 1) = \rho \mathcal{E} - \Theta \rho \mathcal{N}, \quad (4.19)$$

²³Thermal effects can indeed induce volumetric changes; however, the evolution of temperature is governed by the heat equation, which inherently accounts for these effects. Thus, thermal incompressibility is not defined separately, as the temperature field already dictates the thermal response, including any volumetric changes.

where $\hat{p}(X, t)$ and $\hat{p}^n(X, t)$ are the Lagrange multiplier associated with the respective incompressibility constraints. Computing $\dot{\hat{J}}$ and $\dot{\hat{J}}^n$, we find

$$\dot{\hat{J}} = \hat{J} \left(\mathbf{F}^{-1} \mathbf{g}^{\sharp} \mathbf{F}^{-\star} \right) : \dot{\mathbf{D}} - \hat{J} \mathbf{F}^{-\star} : \dot{\mathbf{F}} - \hat{J} (\mathbf{I} : \boldsymbol{\alpha}) \dot{\Theta}, \quad \dot{\hat{J}}^n = \hat{J}^n \mathbf{F}^{-\star} : \dot{\mathbf{F}}^n. \quad (4.20)$$

Revisiting the proof for Proposition 4.1, the results remain unchanged except for the Doyle-Ericksen formula (4.9b) which is modified to read

$$\mathbf{P} = \rho \mathbf{g}^{\sharp} \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{-\star} - \hat{p} \mathbf{g}^{\sharp} \mathbf{F}^{-\star}, \quad (4.21)$$

and the dissipation inequality (4.9e), which is modified to read

$$\dot{\eta} = \left(\rho \mathbf{F}^{\star} \frac{\partial \Psi}{\partial \mathbf{F}} - (\hat{p} - \hat{p}^n) \mathbf{I} \right) : \dot{\mathbf{F}} \mathbf{F}^{-1} + \left(\rho \mathbf{F}^{\star} \mathbf{F}^{\star} \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{-\star} - \hat{p} \mathbf{I} \right) : \boldsymbol{\alpha} \dot{\Theta} - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0. \quad (4.22)$$

4.4 Heat Equation and Kinetic Principles

Heat Equation Let us first derive the heat equation to characterise the evolution of the temperature field in the case of a non-isothermal transformation. Using the balance laws derived in the previous subsection, and using the Legendre transform, the localised energy balance (4.2) reduces to

$$\rho \dot{\Psi} + \rho \dot{\Theta} \mathcal{N} + \rho \Theta \dot{\mathcal{N}} = (\mathbf{F}^{-1} \mathbf{P}) : \dot{\mathbf{D}} + \rho R - \text{Div } \mathbf{Q}. \quad (4.23)$$

Recalling (4.11) and (4.17), we may write

$$\dot{\Psi} = -\mathcal{N} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{-\star} : \bar{\mathbf{D}}_t \mathbf{F} - \frac{\partial \Psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} \mathbf{F}^{-1} - \mathbf{F}^{\star} \mathbf{F}^{\star} \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{-\star} : \boldsymbol{\alpha} \dot{\Theta}, \quad (4.24a)$$

$$\dot{\mathcal{N}} = -\frac{d}{dt} \frac{\partial \hat{\Psi}}{\partial \Theta} = -\frac{\partial^2 \Psi}{\partial \Theta^2} \dot{\Theta} - \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \mathbf{F}^{-\star} : \bar{\mathbf{D}}_t \mathbf{F} + \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} : \dot{\mathbf{F}} \mathbf{F}^{-1} + \mathbf{F}^{\star} \mathbf{F}^{\star} \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \mathbf{F}^{-\star} : \boldsymbol{\alpha} \dot{\Theta}. \quad (4.24b)$$

Substituting (4.24) in (4.23) and using the Doyle-Ericksen formula (4.9b), one finds

$$\begin{aligned} -\rho \left[\Theta \frac{\partial^2 \Psi}{\partial \Theta^2} + \mathbf{F}^{\star} \mathbf{F}^{\star} \left(\frac{\partial \Psi}{\partial \mathbf{F}} - \Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \right) \mathbf{F}^{-\star} : \boldsymbol{\alpha} \right] \dot{\Theta} - \rho \mathbf{F}^{\star} \left[\frac{\partial \Psi}{\partial \mathbf{F}} - \Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \right] : \dot{\mathbf{F}} \mathbf{F}^{-1} \\ - \rho \Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \mathbf{F}^{-\star} : \bar{\mathbf{D}}_t \mathbf{F} = \rho R - \text{Div } \mathbf{Q}. \end{aligned} \quad (4.25)$$

We introduce the specific heat capacity at constant strain, and denote it c_E , as the quantity of heat required to produce a unit temperature increase in a unit mass of material at constant strain. It is given by the following equations

$$\text{Div } \mathbf{Q} = -\rho c_E \dot{\Theta}. \quad (4.26)$$

It follows by identification of (4.26), with (4.25) at constant strain, i.e., $\bar{\mathbf{D}}_t \mathbf{F} = \dot{\mathbf{F}} = \mathbf{0}$, and in the absence of external specific heat supply, i.e., $R = 0$, that

$$c_E = -\Theta \frac{\partial^2 \Psi}{\partial \Theta^2} - \mathbf{F}^{\star} \mathbf{F}^{\star} \left(\frac{\partial \Psi}{\partial \mathbf{F}} - \Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \right) \mathbf{F}^{-\star} : \boldsymbol{\alpha}. \quad (4.27)$$

Therefore, we find the heat equation for a hyper-anelastic solid as follows

$$\rho c_E \dot{\Theta} = -\text{Div } \mathbf{Q} + \rho \Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \mathbf{F}^{-\star} : \bar{\mathbf{D}}_t \mathbf{F} + \rho \mathbf{F}^{\star} \left[\frac{\partial \Psi}{\partial \mathbf{F}} - \Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} \right] : \dot{\mathbf{F}} \mathbf{F}^{-1} + \rho R. \quad (4.28)$$

Configurational Forces In a hyper-anelastic solid, the rate of anelastic energy dissipation may be written as the sum of athermal and thermal contributions [Le Tallec et al., 1993; Maugin, 2010]

$$\dot{\eta} = -\overset{n}{\mathbf{B}} : \dot{\overset{n}{\mathbf{F}}} - \overset{\circ}{\mathbf{B}} \dot{\Theta} - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle, \quad (4.29)$$

where $\overset{n}{\mathbf{B}}$ and $\overset{\circ}{\mathbf{B}}$ are the generalised configurational forces governing the athermal and thermal anelastic distortions, respectively. Noting the arbitrariness of the independent variables $\overset{n}{\mathbf{F}}$ and $\dot{\Theta}$, it follows by identification of (4.9e) and (4.29) that

$$\overset{n}{\mathbf{B}} = -\rho \overset{e}{\mathbf{F}}^* \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*}, \quad (4.30a)$$

$$\overset{\circ}{\mathbf{B}} = -\rho \overset{n}{\mathbf{F}}^* \overset{e}{\mathbf{F}}^* \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*} : \boldsymbol{\alpha}. \quad (4.30b)$$

For an incompressible solid, it is modified to read

$$\overset{n}{\mathbf{B}} = -\rho \overset{e}{\mathbf{F}}^* \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*} + (\bar{p} - \bar{p}) \overset{n}{\mathbf{F}}^{-*}, \quad (4.31a)$$

$$\overset{\circ}{\mathbf{B}} = -\rho \overset{n}{\mathbf{F}}^* \overset{e}{\mathbf{F}}^* \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*} : \boldsymbol{\alpha} + \bar{p} \operatorname{tr} \boldsymbol{\alpha}. \quad (4.31b)$$

Remark 4.3. It can be seen from the conservation of mass (4.9a) that $\rho = \rho(X, \mathbf{G}, t)$, which then allows us to write that $\partial \rho / \partial \Theta = 0$,²⁴ Hence, we may write that

$$\begin{aligned} \rho \frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta} \overset{n}{\mathbf{F}}^{-*} &= \frac{\partial}{\partial \Theta} \left[\rho \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \right] \overset{n}{\mathbf{F}}^{-*} = \frac{\partial}{\partial \Theta} \left[\rho \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*} \right] - \rho \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*} \frac{\partial \overset{n}{\mathbf{F}}^{-*}}{\partial \Theta} \\ &= \mathbf{g} \frac{\partial \mathbf{P}}{\partial \Theta} - \rho \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*} \overset{\circ}{\mathbf{F}}^{-*} \boldsymbol{\alpha}^{-*} = \mathbf{g} \frac{\partial \mathbf{P}}{\partial \Theta} - \mathbf{g} \mathbf{P} \boldsymbol{\alpha}^{-*}, \end{aligned} \quad (4.33a)$$

$$\rho \overset{e}{\mathbf{F}}^* \frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta} \overset{n}{\mathbf{F}}^{-*} = \frac{\partial}{\partial \Theta} \left[\rho \overset{e}{\mathbf{F}}^* \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-*} \right] = -\frac{\partial \overset{n}{\mathbf{B}}}{\partial \Theta}, \quad (4.33b)$$

Therefore, the heat equation (4.28) may be recast as follows

$$\rho c_E \dot{\Theta} = -\operatorname{Div} \mathbf{Q} + \Theta \mathbf{g} \left[\frac{\partial \mathbf{P}}{\partial \Theta} - \mathbf{P} \boldsymbol{\alpha}^{-*} \right] : \bar{\mathbf{D}}_t \mathbf{F} + \left[\Theta \frac{\partial \overset{n}{\mathbf{B}}}{\partial \Theta} - \overset{n}{\mathbf{B}} \right] : \dot{\overset{n}{\mathbf{F}}} + \rho R. \quad (4.34)$$

Kinetic Equations While the evolution of temperature—and consequently that of the thermal distortion $\overset{\circ}{\mathbf{F}}$ —is known to be governed by the heat equation (4.28), one has yet to prescribe a constitutive model for the generalised configurational force $\overset{n}{\mathbf{B}}$ in order to complete the set of governing equations for hyper-anelasticity. Such a constitutive model may be achieved by assuming the existence of a dissipation potential density (a Rayleigh function) $\phi = \phi(X, \mathbf{F}, \overset{n}{\mathbf{F}}, \dot{\overset{n}{\mathbf{F}}}, \overset{\circ}{\mathbf{F}}, \mathbf{g})$, and the generalized configurational forces are given by

$$\overset{n}{\mathbf{B}} = -\frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}}. \quad (4.35)$$

It is assumed that the dissipation potential is convex with respect to $\dot{\overset{n}{\mathbf{F}}}$ [Ziegler, 1958; Ziegler and Wehrli, 1987; Germain et al., 1983; Goldstein et al., 2002; Kumar and Lopez-Pamies, 2016]. This is equivalent to

$$\left(\frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}_2} - \frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}_1} \right) : (\dot{\overset{n}{\mathbf{F}}}_2 - \dot{\overset{n}{\mathbf{F}}}_1) \geq 0, \quad (4.36)$$

²⁴Note however that its total derivative does not necessarily vanish, since it does implicitly depend on temperature through its metric dependence. Its total derivative it computed as follows

$$\frac{d\rho}{d\Theta} = \frac{d\rho}{d\mathbf{G}} : \frac{d\mathbf{G}}{d\Theta} = \frac{d\rho}{d\mathbf{G}} : \frac{\partial(\overset{\circ}{\mathbf{F}}^* \overset{\circ}{\mathbf{G}} \overset{\circ}{\mathbf{F}})}{\partial \Theta} = \frac{d\rho}{d\mathbf{G}} : (\boldsymbol{\alpha}^* \overset{\circ}{\mathbf{F}}^* \overset{\circ}{\mathbf{G}} \overset{\circ}{\mathbf{F}} + \overset{\circ}{\mathbf{F}}^* \overset{\circ}{\mathbf{G}} \overset{\circ}{\mathbf{F}} \boldsymbol{\alpha}) = 2 \frac{d\rho}{d\mathbf{G}} : \mathbf{G} \boldsymbol{\alpha} = 2 \frac{d\rho}{d\mathbf{G}} : \boldsymbol{\alpha}^b, \quad (4.32)$$

where recalling (2.1), we use $\partial \overset{\circ}{\mathbf{F}} / \partial \Theta = \overset{\circ}{\mathbf{F}} \boldsymbol{\alpha}$.

for any $\dot{\mathbf{F}}_1$ and $\dot{\mathbf{F}}_2$. From (4.31a) and (4.35), we find a set of athermal kinetic equations in tensorial form as follows

$$\frac{\partial \phi}{\partial \dot{\mathbf{F}}} - \rho \mathbf{F}^* \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{-*} = 0. \quad (4.37)$$

For an incompressible solid, it is modified to read

$$\frac{\partial \phi}{\partial \dot{\mathbf{F}}} - \rho \mathbf{F}^* \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{-*} + (\bar{p} - \dot{p}) \mathbf{F}^{-*} = 0. \quad (4.38)$$

5 Concluding Remarks

In this paper, we present a generalization of the Coleman-Noll procedure [Coleman and Noll, 1963] to derive the balance laws of nonlinear hyperelasticity and hyper-anelasticity, including conservation of mass, balance of linear and angular momenta, and the Doyle-Ericksen formula, directly from the first and second laws of thermodynamics. Notably, this is achieved without invoking assumptions of observer invariance or pre-supposing the balance laws themselves—as it customary done in the classical Coleman-Noll procedure. By reframing the Clausius-Duhem inequality in terms of extended thermodynamic processes—herein defined, we demonstrate how the constitutive equations and balance laws emerge naturally within this framework.

Unlike the Green-Naghdi-Rivlin theorem [Green and Rivlin, 1964] and its variations [Noll, 1963; Hughes and Marsden, 1977], which derives the balance laws by postulating the invariance of the energy balance under superposed rigid body motions of the Euclidean ambient space, the procedure presented here relies solely on the first and second laws of thermodynamics. By reframing the Clausius-Duhem inequality in terms of extended thermodynamic processes, this approach eliminates the need for explicit symmetry assumptions or pre-supposed invariance principles. Notably, the Green-Naghdi-Rivlin theorem shares a conceptual foundation with Noether's theorem, as both link symmetries to conserved quantities. In the Green-Naghdi-Rivlin framework, the invariance under rigid body motions corresponds to the translational and rotational symmetries central to Noether's theorem [Noether, 1918; Günther, 1962; Knowles et al., 1972], and the derived conservation laws—linear momentum, angular momentum, and energy—are direct outcomes of these symmetries. In contrast, the approach presented herein fundamentally differs from these ideas by deriving the constitutive equations and balance laws without reliance on symmetry or invariance.

The proposed approach not only provides a new perspective on the foundational principles of continuum mechanics but also extends the applicability of classical methods to scenarios where invariance principles are not readily defined, such as in non-Euclidean or evolving ambient spaces. This generalisation connects thermodynamics, geometry, and mechanics in a unified framework, paving the way for new insights in hyperelastic and hyper-anelastic materials.

Acknowledgement

This work was partially supported by IFD – Eurostars III Grant No. 2103-00007B and NSF – Grant No. CMMI 1939901.

References

- S. S. Antman and J. E. Osborn. The principle of virtual work and integral laws of motion. *Archive for Rational Mechanics and Analysis*, 69:231–262, 1979.
- V. Arnold. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer New York, 2 edition, 1989.
- M. Arroyo and A. DeSimone. Relaxation dynamics of fluid membranes. *Physical Review E*, 79(3):031915, 2009.
- J.-P. Boehler. *Applications of Tensor Functions in Solid Mechanics*, volume 292. Springer, 1987.

- A.-L. Cauchy. Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique. *Exercices de Mathématiques*, 3:160–187, 1828.
- B. D. Coleman and W. Noll. The thermodynamics of elastic materials with heat conduction and viscosity. *Archive for Rational Mechanics and Analysis*, 13(1):167–178, 1963.
- T. C. Doyle and J. L. Ericksen. Nonlinear Elasticity. *Advances in Applied Mechanics*, 4:53–115, 1956.
- M. Epstein and G. A. Maugin. Thermomechanics of volumetric growth in uniform bodies. *International Journal of Plasticity*, 16(7-8):951–978, 2000.
- P. Germain, P. Suquet, and Q. S. Nguyen. Continuum thermodynamics. *ATJAM*, 50:1010–1020, 1983.
- H. Goldstein, C. Poole, and J. Safko. *Classical Mechanics*. Addison-Wesley, 3rd edition, 2002.
- A. E. Green and R. S. Rivlin. On Cauchy's equations of motion. *Zeitschrift für Angewandte Mathematik und Physik*, 15(3):290–292, 1964.
- G. Green. On the laws of the reflexion and refraction of light at the common surface of two non-crystallized media. *Transactions of the Cambridge Philosophical Society*, 7:1, 1838.
- G. Green. On the propagation of light in crystallized media. *Transactions of the Cambridge Philosophical Society*, 7:121, 1839.
- W. Günther. Über einige randintegrale der elastomechanik. *Abh. Braunschw. Wiss. Ges*, 14(53-72):176, 1962.
- M. E. Gurtin. Modern continuum thermodynamics. *Mechanics Today*, 1:168–213, 1974.
- T. J. R. Hughes and J. E. Marsden. Some applications of geometry in continuum mechanics. *Reports on Mathematical Physics*, 12(1):35–44, 1977.
- A. Kadić. Aspects of Cauchy elastic materials. *International Journal of Engineering Science*, 18(6):841–846, 1980.
- J. K. Knowles, E. Sternberg, et al. On a class of conservation laws in linearized and finite elastostatics. *Arch. Rat. Mech. Anal*, 44(3):187–211, 1972.
- A. Kumar and O. Lopez-Pamies. On the two-potential constitutive modeling of rubber viscoelastic materials. *Comptes Rendus Mecanique*, 344(2):102–112, 2016.
- P. Le Tallec, C. Rahier, and A. Kaiss. Three-dimensional incompressible viscoelasticity in large strains: formulation and numerical approximation. *Computer methods in applied mechanics and engineering*, 109(3-4):233–258, 1993.
- I. Liu. On representations of anisotropic invariants. *International Journal of Engineering Science*, 20(10):1099–1109, 1982.
- J. Lu and P. Papadopoulos. A covariant constitutive description of anisotropic non-linear elasticity. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 51(2):204–217, 2000.
- V. A. Lubarda. *Multiplicative decomposition of deformation gradient in continuum mechanics: thermoelectricity, elastoplasticity and biomechanics: Multiplikativna dekompozicija deformacionog gradijenta u mehanici kontinuuma: termoelastičnost, elastoplastičnost i biomehanika*. Crnogorska akademija nauka i umjetnosti, 2002.
- V. A. Lubarda and A. Hoger. On the mechanics of solids with a growing mass. *International Journal of Solids and Structures*, 39(18):4627–4664, 2002.
- J. E. Marsden and T. J. R. Hughes. *Mathematical Foundations of Elasticity*. Citeseer, 1983.
- G. Maugin. *Configurational Forces: Thermomechanics, Physics, Mathematics, and Numerics*. Chapman and Hall/CRC, 1st edition, 2010.

- G. A. Maugin. The method of virtual power in continuum mechanics: Application to coupled fields. *Acta Mechanica*, 35(1):1–70, 1980.
- A. L. Mazzucato and L. V. Rachele. Partial uniqueness and obstruction to uniqueness in inverse problems for anisotropic elastic media. *Journal of Elasticity*, 83(3):205–245, 2006.
- A. J. A. Morgan. Some properties of media defined by constitutive equations in implicit form. *International Journal of Engineering Science*, 4(2):155–178, 1966.
- S. Nishikawa. *Variational Problems in Geometry*. American Mathematical Society, 2002.
- E. Noether. Invariante variationsprobleme. *Nachr. Akad. Wiss. Göttingen, II*, pages 235–257, 1918.
- W. Noll. A mathematical theory of the mechanical behavior of continuous media. *Archive for Rational Mechanics and Analysis*, 2(1):197–226, 1958.
- W. Noll. La mecanique classique, basee sur un axiome d’objectivite. In *La Methode Axiomatique dans les Mecaniques Classiques et Nouvelles*, pages 47–56, Paris, 1963.
- A. Ozakin and A. Yavari. A geometric theory of thermal stresses. *Journal of Mathematical Physics*, 51(3), 2010.
- K. R. Rajagopal. On implicit constitutive theories. *Applications of Mathematics*, 48:279–319, 2003.
- K. R. Rajagopal. The elasticity of elasticity. *Zeitschrift für angewandte Mathematik und Physik*, 58:309–317, 2007.
- K. R. Rajagopal. Conspectus of concepts of elasticity. *Mathematics and Mechanics of Solids*, 16(5):536–562, 2011.
- H. Reissner. Eigenspannungen und eigenspannungsquellen. *Zamm-zeitschrift Fur Angewandte Mathematik Und Mechanik*, 11:1–8, 1931.
- S. Sadik and A. Yavari. On the origins of the idea of the multiplicative decomposition of the deformation gradient. *Mathematics and Mechanics of Solids*, 22(4):771–772, 2017a.
- S. Sadik and A. Yavari. Geometric nonlinear thermoelasticity and the time evolution of thermal stresses. *Mathematics and Mechanics of Solids*, 22(7):1546–1587, 2017b.
- S. Sadik and A. Yavari. Nonlinear anisotropic viscoelasticity. *Journal of the Mechanics and Physics of Solids*, 182:105461, 2024.
- S. Sadik and A. Yavari. Nonlinear anisotropic visco-anelasticity. *In preparation*, 2025.
- J. C. Simo, J. E. Marsden, and P. S. Krishnaprasad. The Hamiltonian structure of nonlinear elasticity: the material and convective representations of solids, rods, and plates. *Archive for Rational Mechanics and Analysis*, 104(2):125–183, 1988.
- A. Spencer. George Green and the foundations of the theory of elasticity. *Journal of Engineering Mathematics*, 95(1):5–6, 2015.
- R. Stojanovic, S. Djuric, and L. Vujosevic. On finite thermal deformations. *Archiwum Mechaniki Stosowanej*, 16:103–108, 1964.
- C. Truesdell. The mechanical foundations of elasticity and fluid dynamics. *Journal of Rational Mechanics and Analysis*, 1(1):125–300, 1952.
- A. Yavari. A geometric theory of growth mechanics. *Journal of Nonlinear Science*, 20(6):781–830, 2010.
- A. Yavari. On Eshelby’s inclusion problem in nonlinear anisotropic elasticity. *Journal of Micromechanics and Molecular Physics*, 6(01):2150002, 2021.
- A. Yavari and A. Gorieli. Nonlinear Cauchy elasticity. *arXiv preprint arXiv:2412.17090*, 2024.

- A. Yavari and J. E. Marsden. Covariantization of nonlinear elasticity. *Zeitschrift für Angewandte Mathematik und Physik*, 63(5):921–927, 2012.
- A. Yavari, J. E. Marsden, and M. Ortiz. On spatial and material covariant balance laws in elasticity. *Journal of Mathematical Physics*, 47(4):042903, 2006.
- A. Yavari, A. Ozakin, and S. Sadik. Nonlinear elasticity in a deforming ambient space. *Journal of Nonlinear Science*, pages 1–42, 2016.
- Q. S. Zheng. Theory of representations for tensor functions. *Applied Mechanics Reviews*, 47(11):545–587, 1994.
- Q.-S. Zheng and A. J. M. Spencer. Tensors which characterize anisotropies. *International Journal of Engineering Science*, 31(5):679–693, 1993.
- H. Ziegler. An attempt to generalize Onsager’s principle, and its significance for rheological problems. *Zeitschrift für angewandte Mathematik und Physik*, 9(5-6):748–763, 1958.
- H. Ziegler and C. Wehrli. The derivation of constitutive relations from the free energy and the dissipation function. In *Advances in applied mechanics*, volume 25, pages 183–238. Elsevier, 1987.

Appendix

A Thermal distortion

In the presence of a nonuniform temperature field $\Theta = \Theta(X, t)$, thermal distortion is generally incompatible [Stojanovic et al., 1964; Sadik and Yavari, 2017b]. We denote the thermal distortion by $\overset{\circ}{\mathbf{F}}$; we let all the other anelastic effects be combined together in a distortion $\overset{n}{\mathbf{F}}$ such that the total anelastic distortion is written as $\overset{a}{\mathbf{F}} = \overset{n}{\mathbf{F}} \overset{\circ}{\mathbf{F}}$. At any given point X of the body, thermal expansion may be described by three stretch ratios $\{\zeta_1(X, \Theta), \zeta_2(X, \Theta), \zeta_3(X, \Theta)\}$ along three mutually independent directions forming a basis of unit vectors $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$.²⁵ The thermal distortion may be expressed as [Lubarda, 2002; Ozakin and Yavari, 2010]

$$\overset{\circ}{\mathbf{F}} = \sum_{K=1}^3 \zeta_K \mathbf{E}_K \otimes \mathbf{E}^K, \quad (\text{A.1})$$

where $\{\mathbf{E}^K\}$ is the dual basis to $\{\mathbf{E}_K\}$. The vector $\overset{\circ}{\mathbf{E}}_K = \overset{\circ}{\mathbf{F}} \cdot \mathbf{E}_K = \zeta_K \mathbf{E}_K$ (with no summation over K) represents the thermal distortion along the unit basis vector \mathbf{E}_K . Hence, the linear thermal expansion coefficient of the material in the direction \mathbf{E}_K is given by

$$\alpha_K(X, \Theta) = \frac{1}{\|\overset{\circ}{\mathbf{E}}_K\|_{\overset{\circ}{\mathbf{G}}}} \frac{\partial}{\partial \Theta} [\|\overset{\circ}{\mathbf{E}}_K\|_{\overset{\circ}{\mathbf{G}}}] = \frac{1}{\zeta_K} \frac{\partial \zeta_K}{\partial \Theta}. \quad (\text{A.2})$$

It follows that $\zeta_K = \exp \left[\int_{\Theta_0}^{\Theta} \alpha_K(X, \Xi) d\Xi \right]$, where $\Theta_0 = \Theta_0(X)$ is the temperature distribution such that $\zeta_K(\Theta_0) = 1$, i.e., the temperature distribution for which the body does not experience any thermal distortion. Therefore, we may let $\zeta_K = e^{\omega_K}$, and the thermal distortion may be written as

$$\overset{\circ}{\mathbf{F}} = \sum_{K=1}^3 e^{\omega_K} \mathbf{E}_K \otimes \mathbf{E}^K, \quad (\text{A.3})$$

where $\omega_K = \int_{\Theta_0}^{\Theta} \alpha_K(X, \Xi) d\Xi$. We construct the tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ such that their representations with respect to the basis $\{\mathbf{E}_K\}$ and its dual are given by

$$\boldsymbol{\alpha} = \sum_{K=1}^3 \alpha_K \mathbf{E}_K \otimes \mathbf{E}^K, \quad \boldsymbol{\omega} = \sum_{K=1}^3 \omega_K \mathbf{E}_K \otimes \mathbf{E}^K. \quad (\text{A.4})$$

²⁵Note that $\{\mathbf{E}_K\}$ is not necessarily a coordinate basis and is not necessarily orthogonal.

One may hence write

$$\frac{\partial \overset{\circ}{\mathbf{F}}}{\partial \Theta} = \sum_{K=1}^3 \frac{\partial e^{\omega_K}}{\partial \Theta} \mathbf{E}_K \otimes \mathbf{E}^K = \sum_{K=1}^3 \alpha_K e^{\omega_K} \mathbf{E}_K \otimes \mathbf{E}^K = \boldsymbol{\alpha} \overset{\circ}{\mathbf{F}} = \overset{\circ}{\mathbf{F}} \boldsymbol{\alpha}, \quad (\text{A.5})$$

which yields

$$\boldsymbol{\alpha} = \frac{\partial \overset{\circ}{\mathbf{F}}}{\partial \Theta} \overset{\circ}{\mathbf{F}}^{-1} = \overset{\circ}{\mathbf{F}}^{-1} \frac{\partial \overset{\circ}{\mathbf{F}}}{\partial \Theta}. \quad (\text{A.6})$$

Let us denote by $[A^I_J]$ the transformation matrix between the coordinate basis $\{\frac{\partial}{\partial X^K}\}$ and the basis $\{\mathbf{E}_K\}$, i.e., $\mathbf{E}_K = A^I_K \partial/\partial X^I$ and $\mathbf{E}^K = A^{-K}_J dX^J$, it follows that one may write

$$\sum_{K=1}^3 T_K \mathbf{E}_K \otimes \mathbf{E}^K = \sum_{K=1}^3 A^I_K T_K A^{-K}_J \frac{\partial}{\partial X^I} \otimes dX^J, \quad (\text{A.7})$$

for any triplet of numbers $\{T_1, T_2, T_3\}$. In particular, the coordinate representations of $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ in $\{X^K\}$ are hence respectively given by

$$\boldsymbol{\omega} = \omega^I_J \frac{\partial}{\partial X^I} \otimes dX^J, \quad \boldsymbol{\alpha} = \alpha^I_J \frac{\partial}{\partial X^I} \otimes dX^J, \quad (\text{A.8})$$

where $\omega^I_J = \sum_{K=1}^3 A^I_K \omega_K A^{-K}_J$ and $\alpha^I_J = \sum_{K=1}^3 A^I_K \alpha_K A^{-K}_J$. Note that it still holds in tensorial form that $\boldsymbol{\omega}(X, \Theta) = \int_{\Theta_0}^{\Theta} \boldsymbol{\alpha}(X, \Xi) d\Xi$. One may write

$$\begin{aligned} \overset{\circ}{\mathbf{F}}(X, \Theta) &= \sum_{K=1}^3 \zeta_K \mathbf{E}_K \otimes \mathbf{E}^K = \sum_{K=1}^3 A^I_K \zeta_K A^{-K}_J \frac{\partial}{\partial X^I} \otimes dX^J = \sum_{K=1}^3 A^I_K e^{\omega_K} A^{-K}_J \frac{\partial}{\partial X^I} \otimes dX^J \\ &= \sum_{K=1}^3 \exp[A^I_K \omega_K A^{-K}_J] \frac{\partial}{\partial X^I} \otimes dX^J = \sum_{K=1}^3 \exp[\omega^I_J] \frac{\partial}{\partial X^I} \otimes dX^J = \exp[\boldsymbol{\omega}(X, \Theta)]. \end{aligned}$$

Therefore

$$\overset{\circ}{\mathbf{F}}(X, \Theta) = \exp[\boldsymbol{\omega}(X, \Theta)] = \exp\left[\int_{\Theta_0}^{\Theta} \boldsymbol{\alpha}(X, \Xi) d\Xi\right]. \quad (\text{A.9})$$

Note that for an isothermal process, one has $\Theta(X, t) = \Theta_0(X)$, $\forall X \in \mathcal{B}$, $\forall t$, and consequently, by using (A.9), that $\overset{\circ}{\mathbf{F}} = \mathbf{I}$, the identity tensor.