A Note on the Balance Laws of Nonlinear Hyperelasticity

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Abstract

It is known that the balance laws of hyperelasticity (Green elasticity), i.e., conservation of mass and balance of linear and angular momenta, can be derived using the first law of thermodynamics by postulating its invariance under rigid body motions of the Euclidean ambient space—the Green-Naghdi-Rivlin theorem. In the case of a non-Euclidean ambient space, covariance of the energy balance—its invariance under arbitrary diffeomorphisms of the ambient space—gives all the balance laws and the Doyle-Ericksen formula—the Marsden-Hughes theorem. In this note, we show that the constitutive equations as well as the balance laws of hyperelasticity can be derived using the first and second laws of thermodynamics without assuming any invariance.

Keywords: Nonlinear elasticity, hyperelasticity, balance laws, covariance, laws of thermodynamics.

Mathematics Subject Classification 74A15 · 74B20

1 Introduction

In nonlinear hyperelasticity, one can derive all the balance laws by starting from an energy balance (the first law of thermodynamics) and postulating its invariance under rigid body motions of the ambient space. This idea is due to Green and Rivlin [1964][1] in the context of Euclidean ambient spaces (Green-Naghdi-Rivlin theorem) and was subsequently expanded to hyperelasticity with Riemannian ambient space manifolds by Hughes and Marsden [1977] who postulated the invariance of the balance of energy under arbitrary diffeomorphisms of the ambient space—covariance of the energy balance. Hughes and Marsden [1977] showed that covariance of the energy balance gives all the balance laws of hyperelasticity and the Doyle-Ericksen formula [Doyle and Ericksen, 1956]. See also [Marsden and Hughes, 1983; Simo and Marsden, 1984; Yavari et al., 2006; Yavari and Ozakin, 2008; Yavari, 2008; Yavari and Marsden, 2009a,b; Yavari, 2010; Yavari and Golgoon, 2019]. In this note, as an extension to the classical Coleman and Noll [1963] procedure, we show that instead of using the first law of thermodynamics and its covariance, one can use the first and the second laws of thermodynamics to derive not only the hyperelastic constitutive equations but also all the balance laws of hyperelasticity—effectively, the second law (in the form of the Clausius-Duhem inequality) places restrictions on the first law (balance of energy) and yields the constitutive equations and the balance laws of nonlinear hyperelasticity.

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[1] A different version of this theorem is due to Noll [1963]. See [Marsden and Hughes, 1983] for more details.

2 Kinematics of Finite Deformations

Consider a hyperelastic solid body B represented by an embedded 3-submanifold $\mathcal{B}$ within the ambient space $\mathcal{S}$.[2] Motion of the body B is represented by a time-parametrized family of maps $\varphi_t : \mathcal{B} \to \mathcal{C}_t \subset \mathcal{S}$, mapping the reference (material) configuration $\mathcal{B}$ of the body to its current (spatial) configuration $\mathcal{C}_t = \varphi_t(\mathcal{B})$. We adopt the following standard convention: objects and indices are denoted by uppercase characters in the material manifold $\mathcal{B}$ (e.g., $X \in \mathcal{B}$), and by lowercase characters in the spatial manifold $\mathcal{S}$ (e.g., $x = \varphi_t(X) \in \varphi_t(\mathcal{B})$). We consider local coordinate charts on $\mathcal{B}$ and $\mathcal{S}$ denoted by $\{X^A\}$ and $\{x^a\}$, respectively. The corresponding local coordinate bases are denoted by $\{\partial_A = \frac{\partial}{\partial X^A}\}$ and $\{\partial_a = \frac{\partial}{\partial x^a}\}$, and their respective dual bases are $\{dX^A\}$ and $\{dx^a\}$. We also adopt Einstein's repeated index summation convention, e.g., $u^i v_i \coloneqq \sum_i u^i v_i$.

The ambient space has a background metric $\mathbf{g} = \mathrm{g}_{ab}\, dx^a \otimes dx^b$. Given vectors $\mathbf{u}, \mathbf{w} \in T_x\mathcal{S}$, their dot product is denoted by $\langle\!\langle \mathbf{u}, \mathbf{w} \rangle\!\rangle_{\mathbf{g}} = \mathrm{u}^a\, \mathrm{w}^b\, \mathrm{g}_{ab}$. Given a vector $\mathbf{u} \in T_x\mathcal{S}$ and a 1-form $\boldsymbol{\omega} \in T_x^*\mathcal{S}$, their natural pairing is denoted by $\langle \boldsymbol{\omega}, \mathbf{u} \rangle = \boldsymbol{\omega}(\mathbf{u}) = \omega_a\, \mathrm{u}^a$. The spatial volume form is $dv = \sqrt{\det \mathbf{g}}\; dx^1 \wedge dx^2 \wedge dx^3$. The Levi-Civita connection of $(\mathcal{S}, \mathbf{g})$ is denoted by $\nabla^{\mathbf{g}}$, with Christoffel symbols $\gamma^a{}_{bc}$. The metric $\mathbf{g}$ of $\mathcal{S}$ induces the metric $\mathbf{G}$ on $\mathcal{B}$ by which the natural distances in the body before deformation are calculated. Given vectors $\mathbf{U}, \mathbf{W} \in T_X\mathcal{B}$, their dot product is denoted by $\langle\!\langle \mathbf{U}, \mathbf{W} \rangle\!\rangle_{\mathbf{G}} = \mathrm{U}^A\, \mathrm{W}^B\, \mathrm{G}_{AB}$. Given a vector $\mathbf{U} \in T_X\mathcal{B}$ and a 1-form $\mathbf{\Omega} \in T_X^*\mathcal{B}$, their natural pairing is denoted by $\langle \mathbf{\Omega}, \mathbf{U} \rangle = \mathbf{\Omega}(\mathbf{U}) = \Omega_A\, \mathrm{U}^A$. The material volume form is $dV = \sqrt{\det \mathbf{G}}\; dX^1 \wedge dX^2 \wedge dX^3$. The Levi-Civita connection of $(\mathcal{B}, \mathbf{G})$ is denoted by $\nabla^{\mathbf{G}}$, with Christoffel symbols $\Gamma^A{}_{BC}$.

As a measure of strain in elastic solids, we typically use the derivative of the deformation mapping—known as the deformation gradient—denoted by $\mathbf{F}(X,t) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{C}_t$; in components it reads as $\mathrm{F}^a{}_A = \frac{\partial \varphi^a}{\partial X^A}$. The dual $\mathbf{F}^\star$ of $\mathbf{F}$ is defined as $\mathbf{F}^\star(X,t) : T_{\varphi_t(X)}\mathcal{C}_t \to T_X\mathcal{B}$, $\langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{U} \rangle = \langle \mathbf{F}^\star \boldsymbol{\alpha}, \mathbf{U} \rangle$, $\forall\, \mathbf{U} \in T_X\mathcal{B}$, $\forall\, \boldsymbol{\alpha} \in T^*_{\varphi(X)}\mathcal{S}$; it has components $(\mathrm{F}^\star)_A{}^a = \mathrm{F}^a{}_A$. The transpose $\mathbf{F}^\mathsf{T}$ of $\mathbf{F}$ is defined as $\mathbf{F}^\mathsf{T}(X,t) : T_{\varphi_t(X)}\mathcal{C}_t \to T_X\mathcal{B}$, $\langle\!\langle \mathbf{F}\mathbf{U}, \mathbf{u} \rangle\!\rangle_{\mathbf{g}} = \langle\!\langle \mathbf{U}, \mathbf{F}^\mathsf{T}\mathbf{u} \rangle\!\rangle_{\mathbf{G}}$, $\forall\, \mathbf{U} \in T_X\mathcal{B}$, $\forall\, \mathbf{u} \in T_{\varphi(X)}\mathcal{S}$; in components it reads as $\left(\mathrm{F}^\mathsf{T}\right)^A{}_a = \mathrm{G}^{AB}\, \mathrm{F}^b{}_B\, \mathrm{g}_{ba}$. Note that $\mathbf{F}^\mathsf{T} = \mathbf{G}^\sharp \mathbf{F}^\star \mathbf{g}$, where $(.)^\sharp$ denotes the musical isomorphism for raising indices. The right Cauchy–Green deformation tensor is defined as $\mathbf{C} \coloneqq \mathbf{F}^\mathsf{T}\mathbf{F}$. Denoting by $(.)^\flat$ the musical isomorphism for lowering indices, one finds that $\mathbf{C}^\flat$ corresponds to the pull-back of the spatial metric $\mathbf{g}$ by φ, i.e., $\mathbf{C}^\flat = \varphi^*\mathbf{g} = \mathbf{F}^\star \mathbf{g} \mathbf{F}$. The Jacobian of the motion relates the material and spatial volume elements as $\mathrm{d}v = J \mathrm{d}V$, and it can be shown that $J = \sqrt{\det \mathbf{C}} = \sqrt{\det \mathbf{g} / \det \mathbf{G}}\, \det \mathbf{F}$.

The material velocity of the motion is defined as $\mathbf{V} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{V}(X,t) \coloneqq \frac{\partial \varphi(X,t)}{\partial t}$; it has components $\mathrm{V}^a = \frac{\partial \varphi^a}{\partial t}$. The spatial velocity is defined as $\mathbf{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{v}(x,t) \coloneqq \mathbf{V}(\varphi_t^{-1}(x), t)$. The material acceleration of the motion is defined as $\mathbf{A} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{A}(X,t) \coloneqq D_t^{\mathbf{g}} \mathbf{V}(X,t)$, where $D_t^{\mathbf{g}}$ denotes the covariant derivative along $\varphi_X : t \mapsto \varphi(X,t)$; it reads in components as $\mathrm{A}^a = \frac{\partial \mathrm{V}^a}{\partial t} + \gamma^a{}_{bc}\, \mathrm{V}^b\, \mathrm{V}^c$. The spatial acceleration of the motion is defined as $\mathbf{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{a}(x,t) \coloneqq \mathbf{A}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S}$; it has components $\mathrm{a}^a = \frac{\partial \mathrm{v}^a}{\partial t} + \frac{\partial \mathrm{v}^a}{\partial x^b}\, \mathrm{v}^b + \gamma^a{}_{bc}\, \mathrm{v}^b\, \mathrm{v}^c$.

3 Thermodynamics and the Balance Laws of Hyperelasticity

In this section, we first briefly review the first and second laws of thermodynamics in the setting of nonlinear hyperelasticity. We then show how all the balance laws of hyperelasticity can be derived assuming the first and second laws of thermodynamics.

3.1 First Law of Thermodynamics

The first law of thermodynamics posits the existence of an internal energy $\mathscr{E}$ as a state function, which satisfies the following balance equation as an expression of the conservation of energy principle [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0 \left(\mathscr{E} + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + R \right) dV + \int_{\partial \mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + H \right) dA\,, \tag{3.1}$$

[2] For most applications the ambient space is the three-dimensional Euclidean space, i.e., $\mathcal{S} = \mathbb{R}^3$. However, in general, the ambient space may be curved, e.g., in modeling the dynamics of fluid membranes [Arroyo and DeSimone, 2009]. See [Yavari et al., 2016] for a general framework on elasticity in evolving ambient spaces.

for any open set $\mathcal{U} \subset \mathcal{B}$, where $\mathscr{E}$ stands for the specific internal energy, ρ_0 is the material mass density, $\mathbf{B}$ is the body force, $\mathbf{T}$ is the boundary traction vector field per unit (stress-free) material area, $R = R(X,t)$ is the specific heat supply, and $H = -\langle\!\langle \mathbf{Q}, \mathbf{N} \rangle\!\rangle_{\mathbf{G}}$ is the heat flux across a material surface where $\mathbf{Q} = \mathbf{Q}(X, T, dT, \mathbf{C}^\flat, \mathbf{G})$ represents the external heat flux per unit material (stress-free) area, T is temperature, and $\mathbf{N}$ is the $\mathbf{G}$-unit normal to the boundary $\partial\mathcal{B}$. Expressed in localized form, the energy balance (3.1) is written as

$$\rho_0\,\dot{\mathscr{E}} = \mathbf{S}\!:\!\mathbf{D} - \operatorname{Div}\mathbf{Q} + \rho_0 R + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0 \left(\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2 \right), \tag{3.2}$$

where a dotted quantity denotes its total time derivative, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor such that the traction vector is $\mathbf{T} = \mathbf{P}\mathbf{N}$, $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$ is the second Piola-Kirchhoff stress tensor, and $\mathbf{D} = \frac{1}{2}\dot{\mathbf{C}}^\flat$ is the material rate of deformation tensor.

3.2 Second Law of Thermodynamics

The second law of thermodynamics posits the existence of entropy $\mathscr{N}$ as a state function, which satisfies the following inequality—known as the Clausius-Duhem inequality—as an expression of the principle of entropy production,[3] which steadily increases or remains constant within a closed system over time [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \mathscr{N} dV \geq \int_{\mathcal{U}} \rho_0 \frac{R}{T} dV + \int_{\partial\mathcal{U}} \frac{H}{T} dA\,, \tag{3.3}$$

for any open set $\mathcal{U} \subset \mathcal{B}$, where $\mathscr{N}$ denotes the specific entropy. In localized form, the material Clausius-Duhem inequality (3.3) reads

$$\dot{\eta} = \rho_0 T \dot{\mathscr{N}} + \dot{\rho}_0 T \mathscr{N} + \operatorname{Div}\mathbf{Q} - \rho_0 R - \frac{1}{T}\langle dT, \mathbf{Q} \rangle \geq 0\,, \tag{3.4}$$

where $\dot{\eta}$ denotes the material rate of energy dissipation density.

3.3 Balance Laws of Nonlinear Hyperelasticity

The specific free energy $\Psi = \hat{\Psi}(X, T, \mathbf{C}^\flat, \mathbf{G})$ is the Legendre transform of the specific internal energy $\mathcal{E}$ with respect to the conjugate variables temperature T and specific entropy $\mathcal{N}$, i.e.,

$$\Psi = \mathscr{E} - T\mathscr{N}\,, \tag{3.5}$$

such that $\mathcal{E} = \hat{\mathcal{E}}(X, \mathcal{N}, \mathbf{C}^\flat, \mathbf{G})$. Thus

$$\mathscr{N} = -\frac{\partial \Psi}{\partial T}\,. \tag{3.6}$$

Proposition 3.1. *For a hyperelastic body, the first and second laws of thermodynamics* (3.2),(3.4) *imply that*

$$\begin{cases} \mathbf{P} = 2\rho_0 \mathbf{F} \dfrac{\partial \hat{\Psi}}{\partial \mathbf{C}^\flat}\,, & (3.7) \\ \operatorname{Div}\mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}\,, & (3.8) \\ \dot{\rho}_0 = 0\,, & (3.9) \\ \dot{\eta} = -\dfrac{1}{\Theta}\langle d\Theta, \mathbf{Q} \rangle \geq 0\,. & (3.10) \end{cases}$$

In other words, the first and second laws of thermodynamics imply the Doyle-Ericksen formula (3.7)*—and consequently the balance of angular momentum,*[4] *the balance of linear momentum* (3.8)*, and the conservation of mass* (3.9)*.*

[3]The entropy production for an open subset $\mathcal{U}$ in the body reads as

$$\Gamma = \frac{d}{dt}\int_{\mathcal{U}} \rho_0 \mathscr{N} dV - \int_{\mathcal{U}} \rho_0 \frac{R}{T} dV - \int_{\partial\mathcal{U}} \frac{H}{T}\, dA\,.$$

[4]Using the symmetry of the right Cauchy-Green deformation tensor (i.e., $\mathbf{C}^\star = \mathbf{C}$) in the Doyle-Ericksen formula yields the balance of linear momentum $\mathbf{P}^\star\mathbf{F}^{-\star} = \mathbf{F}^{-1}\mathbf{P}$, which is equivalent to $\mathbf{F}\mathbf{P}^\star = \mathbf{P}\mathbf{F}^\star$, i.e., symmetry of the Cauchy stress.

Proof. From (3.5), $\rho_0 T\dot{\mathscr{N}} = \rho_0\dot{\mathscr{E}} - \rho_0\dot{\Psi} - \rho_0\dot{T}\mathscr{N}$. Substituting this relation and (3.2) into (3.4) one obtains

$$\dot{\eta} = \mathbf{S}\!:\!\mathbf{D} - \rho_0\dot{\Psi} - \rho_0\dot{T}\mathscr{N} + \dot{\rho}_0 T\mathscr{N} + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left(\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2\right) - \frac{1}{T}\langle dT, \mathbf{Q}\rangle \geq 0\,. \tag{3.11}$$

Note that

$$\dot{\Psi} = \frac{\partial\Psi}{\partial T}\dot{T} + \frac{\partial\Psi}{\partial\mathbf{C}^\flat}\!:\!\dot{\mathbf{C}}^\flat = -\mathscr{N}\dot{T} + 2\frac{\partial\Psi}{\partial\mathbf{C}^\flat}\!:\!\dot{\mathbf{D}}\,. \tag{3.12}$$

Substituting this into (3.11), the rate of dissipation is simplified to read

$$\dot{\eta} = \left[\mathbf{S} - 2\rho_0\frac{\partial\Psi}{\partial\mathbf{C}^\flat}\right]\!:\!\mathbf{D} + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left(\Psi + \frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2\right) - \frac{1}{T}\langle dT, \mathbf{Q}\rangle \geq 0\,. \tag{3.13}$$

This inequality must hold for all motions. As $\mathbf{D}$ can be varied independently of all the other fields, one concludes that

$$\mathbf{S} = 2\rho_0\frac{\partial\hat{\Psi}}{\partial\mathbf{C}^\flat}\,, \tag{3.14}$$

and the rate of dissipation is simplified to read

$$\dot{\eta} = \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left(\Psi + \frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2\right) - \frac{1}{T}\langle dT, \mathbf{Q}\rangle \geq 0\,. \tag{3.15}$$

One can vary the velocity vector while its norm $\|\mathbf{V}\|_{\mathbf{g}}$ is fixed. This implies that the inequality (3.15) can hold only if

$$\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}\,. \tag{3.16}$$

Now the rate of dissipation takes the following form

$$\dot{\eta} = -\dot{\rho}_0\left(\Psi + \frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^2\right) - \frac{1}{T}\langle dT, \mathbf{Q}\rangle \geq 0\,. \tag{3.17}$$

Notice that adding an arbitrary constant to a free energy is thermodynamically inconsequential. This implies that the inequality (3.17) must be invariant under the transformations $\Psi \to \Psi + a$, $\forall\, a \in \mathbb{R}$. This implies that $\dot{\rho}_0 = 0$ and $\dot{\eta} = -\frac{1}{T}\langle dT, \mathbf{Q}\rangle \geq 0$. $\square$

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