A Generalised Coleman-Noll Procedure and the Balance Laws of Hyper-Anelasticity

Souhayl Sadik*[1] and Arash Yavari[2,3]

[1] *Department of Mechanical and Production Engineering, Aarhus University, 8000 Aarhus C, Denmark*
[2] *School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*
[3] *The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

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Abstract

It is known that the balance laws of hyperelasticity (Green elasticity), i.e., conservation of mass and balance of linear and angular momenta, can be derived using the first law of thermodynamics by postulating its invariance under superposed rigid body motions of the Euclidean ambient space—the Green-Naghdi-Rivlin theorem. In the case of a non-Euclidean ambient space, covariance of the energy balance—its invariance under arbitrary diffeomorphisms of the ambient space—gives all the balance laws and the Doyle-Ericksen formula—the Marsden-Hughes theorem. It is also known that, by assuming the balance laws, and positing the first and second laws of thermodynamics, the Doyle-Ericksen formula can be derived—the Coleman-Noll procedure. In this paper, we propose a generelization of the Coleman-Noll procedure: we show that the Doyle-Ericksen formula as well as the balance laws for both hyperelasticity and hyper-anelasticity can be derived using the first and second laws of thermodynamics without assuming any (observer) invariance.

Keywords: Nonlinear elasticity, hyperelasticity, Green elasticity, balance laws, laws of thermodynamics.

Mathematics Subject Classification 74A15 · 74B20

1 Introduction

There are several approaches to derive the balance laws in field theories, particularly in elasticity. One approach is the Lagrangian field theory (variational approach) of elasticity. In this method, Hamilton's principle is written for an elastic body, leading to the balance of linear momentum as the Euler-Lagrange equations of the variational principle. The balance of angular momentum is then derived using Noether's theorem, which relates the invariance of the Lagrangian density under ambient space rotations to angular momentum conservation. The balance of energy corresponds to the invariance of the Lagrangian density under time shifts. The second approach is the Hamiltonian mechanics formulation. In this framework, nonlinear elasticity is represented in phase space, where Hamilton's equations give the balance laws Simo et al. [1988]. More specifically, the governing equations of nonlinear elasticity can be derived using the canonical Hamilton equations $\dot{f} = \{f, H\}$, where $\{\}$ is the Poisson bracket of nonlinear elasticity, H is the Hamiltonian of nonlinear elasticity and f is an arbitrary scalar function.

Another approach to derive the balance laws of nonlinear hyperelasticity is to start from an energy balance (the first law of thermodynamics) and postulate its invariance under superposed rigid body motions of the ambient space (observer invariance). This idea is due to Green and Rivlin [1964] in the context of Euclidean ambient spaces (Green-Naghdi-Rivlin theorem). More specifically, Green and Rivlin [1964] postulated the balance of energy and its invariance under superposed translational and rotational motions of the Euclidean ambient space. A different version of this theorem is due to Noll [1963] who thought of the superposed motions

*Corresponding author, e-mail: sosa@mpe.au.dk passively as time-dependent coordinate charts for the Euclidean ambient space.[1] Effectively, Green and Rivlin [1964] viewed superimposed motions actively, whereas Noll [1963] viewed them passively. The invariance idea was subsequently extended to hyperelasticity (Green elasticity) with Riemannian ambient space manifolds by Hughes and Marsden [1977]; they postulated the invariance of the balance of energy under arbitrary diffeomorphisms of the ambient space—*covariance* of the energy balance. Hughes and Marsden [1977] showed that covariance of the energy balance gives all the balance laws of hyperelasticity and the Doyle-Ericksen formula Doyle and Ericksen [1956][2] (see also Marsden and Hughes [1983]; Yavari et al. [2006]). It is worth noting that the relationship between the covariance of the energy balance and the Lagrangian field theory of elasticity was explored in detail in Yavari and Marsden [2012].[3]

The second law of thermodynamics imposes constraints on the constitutive equations. In the classical Coleman and Noll [1963] procedure, it is assumed that the balances of linear and angular momenta are already satisfied. Subsequently, the second law of thermodynamics, in the form of the Clausius-Duhem inequality, places restrictions on the form of the constitutive equations.

In this paper, instead of using the first law of thermodynamics and its covariance, and as an extension to the Coleman-Noll procedure Coleman and Noll [1963], we show that, one can use the first and the second laws of thermodynamics to derive not only the Doyle-Ericksen formula but also all the balance laws of hyperelasticity. We also show that this generalised procedure holds for hyper-anelasticity.

This paper is organised as follows: In §2, we introduce some notational elements and discuss the kinematics framework, in particular we discuss the material geometry for finite elasticity and anelasticity including the Bilby-Kröner-Lee decomposition in the anelastic case. In §3 we discuss thernodynamics of hyperelastic solids, and without any assumptions of invariance, we prove that one is able to find not only the hyperelastic constitutive equations (classical Coleman-Noll procedure), but also the balance laws of hyperelasticity—hence the generalised Coleman-Noll procedure. In §4, we extend our generalised Coleman-Noll procedure to hyper-anelasticity. We further discuss the heat equation (in case of an isothermal process) and the kinetic equations governing the evolution of anelastic distortions. We conclude the paper with some final remarks in §5.

[1] Recall that a matrix can be regarded as either a linear transformation (active) or representing a change of basis (passive).

[2] It is worth emphasizing that both in Green and Rivlin [1964] and Hughes and Marsden [1977], it was assumed that the body is made of a material that has an underlying energy function, i.e., they restricted themselves to hyperelasticity. Similar invariance arguments can be used to derive the balance laws of anelasticity, provided that there exists an underlying energy function, e.g., Yavari [2010].

[3] A widely used approach for deriving the balance laws, particularly in computational mechanics, is the principle of virtual work (or virtual power) Antman and Osborn [1979]; Maugin [1980]. It can be shown that the principle of virtual work can be derived from the balance of energy and its covariance Marsden and Hughes [1983].

2 Kinematics

2.1 Kinematic and Mathematical Preliminaries

Consider a solid body B represented by an embedded 3-submanifold $\mathcal{B}$ within the ambient space $\mathcal{S}$.[4] Motion of the body B is represented by a time-parametrized family of maps $\varphi_t : \mathcal{B} \to \mathcal{C}_t \subset \mathcal{S}$, mapping the reference (material) configuration $\mathcal{B}$ of the body to its current (spatial) configuration $\mathcal{C}_t = \varphi_t(\mathcal{B})$. We adopt the following standard convention: objects and indices are denoted by uppercase characters in the material manifold $\mathcal{B}$ (e.g., $X \in \mathcal{B}$), and by lowercase characters in the spatial manifold $\varphi_t(\mathcal{B})$ (e.g., $x = \varphi_t(X) \in \varphi_t(\mathcal{B})$). We consider local coordinate charts on $\mathcal{B}$ and $\mathcal{S}$ and denote them by $\{X^A\}$ and $\{x^a\}$, respectively. The corresponding local coordinate bases are denoted by $\{\partial_A = \frac{\partial}{\partial X^A}\}$ and $\{\partial_a = \frac{\partial}{\partial x^a}\}$, and their respective dual bases are $\{dX^A\}$ and $\{dx^a\}$. We adopt Einstein's repeated index summation convention, e.g., $u^i v_i \coloneqq \sum_i u^i v_i$.

In the ambient space $\mathcal{S}$, given a vector $\mathbf{u} \in T_x\mathcal{S}$ and a 1-form $\boldsymbol{\omega} \in T_x^*\mathcal{S}$, their natural pairing is denoted by $\langle \boldsymbol{\omega}, \mathbf{u} \rangle = \boldsymbol{\omega}(\mathbf{u}) = \omega_a \, \mathrm{u}^a$. Similarly, in the reference manifold $\mathcal{B}$, given a vector $\mathbf{U} \in T_X\mathcal{B}$ and a 1-form $\boldsymbol{\Omega} \in T_X^*\mathcal{B}$, their natural pairing is also denoted by $\langle \boldsymbol{\Omega}, \mathbf{U} \rangle = \boldsymbol{\Omega}(\mathbf{U}) = \Omega_A \mathrm{U}^A$. Note that the natural pairing operation does not require any metric structure.

As a measure of strain in elastic solids, we typically use the derivative of the deformation mapping—known as the deformation gradient—denoted by $\mathbf{F}(X,t) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{C}_t$; in components it reads as $\mathrm{F}^a{}_A = \frac{\partial \varphi^a}{\partial X^A}$. The adjoint $\mathbf{F}^\star$ of $\mathbf{F}$ is defined as $\mathbf{F}^\star(X,t) : T_{\varphi_t(X)}\mathcal{C}_t \to T_X\mathcal{B}$, $\langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{U} \rangle = \langle \mathbf{F}^\star \boldsymbol{\alpha}, \mathbf{U} \rangle$, $\forall\, \mathbf{U} \in T_X\mathcal{B}$, $\forall\, \boldsymbol{\alpha} \in T^*_{\varphi(X)}\mathcal{S}$; it has components $(\mathrm{F}^\star)_A{}^a = \mathrm{F}^a{}_A$. Note that the definition of the adjoint, much like the natural pairing, does not require any metric structure either.

The ambient space has a background Euclidean metric $\mathbf{g} = \mathrm{g}_{ab}\, dx^a \otimes dx^b$. Given vectors $\mathbf{u}, \mathbf{w} \in T_x\mathcal{S}$, their dot product is denoted by $\langle\!\langle \mathbf{u}, \mathbf{w} \rangle\!\rangle_{\mathbf{g}} = \mathrm{u}^a \, \mathrm{w}^b \, \mathrm{g}_{ab}$. The spatial volume form is $dv = \sqrt{\det \mathbf{g}}\; dx^1 \wedge dx^2 \wedge dx^3$. The Levi-Civita connection of $(\mathcal{S}, \mathbf{g})$ is denoted by $\bar{\nabla}$, with Christoffel symbols $\gamma^a{}_{bc}$.

Remark 2.1. The same notation $\mathbf{g}$ is used for both the background metric of the ambient space $\mathcal{S} = \mathbb{R}^3$ and the spatial metric of the deformed configuration $\varphi_t(\mathcal{B})$, but they have a subtle yet important difference. The background metric is fixed and time-independent, describing the intrinsic geometry of the ambient space. In contrast, the spatial metric evolves over time due to its dependence on the deformation mapping φ_t, as its domain of definition is the deformed configuration $\varphi_t(\mathcal{B})$. Formally, the spatial metric $\mathbf{g}_t$ is the pull-back of the background metric $\mathbf{g}$ by the time-dependent inclusion map $\iota_t : \varphi_t(\mathcal{B}) \to \mathcal{S}$, i.e., $\mathbf{g}_t = \iota_t^* \mathbf{g}$. This distinction is essential: the background metric provides a constant geometric reference for measuring local distances, whereas the spatial metric reflects the dynamic geometry of the deforming body. For simplicity, we use the notation $\mathbf{g}$ for the spatial metric in the rest of the paper.

The material velocity of the motion is defined as $\mathbf{V} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{V}(X,t) \coloneqq \frac{\partial \varphi(X,t)}{\partial t}$; it has components $\mathrm{V}^a = \frac{\partial \varphi^a}{\partial t}$. The spatial velocity is defined as $\mathbf{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{v}(x,t) \coloneqq \mathbf{V}(\varphi_t^{-1}(x), t)$. The material acceleration of the motion is defined as $\mathbf{A} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{A}(X,t) \coloneqq D_t^{\mathbf{g}} \mathbf{V}(X,t)$, where $D_t^{\mathbf{g}}$ denotes the covariant derivative along $\varphi_X : t \mapsto \varphi(X,t)$; it reads in components as $\mathrm{A}^a = \frac{\partial \mathrm{V}^a}{\partial t} + \gamma^a{}_{bc} \, \mathrm{V}^b \, \mathrm{V}^c$. The spatial acceleration of the motion is defined as $\mathbf{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{a}(x,t) \coloneqq \mathbf{A}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S}$; it has components $\mathrm{a}^a = \frac{\partial \mathrm{v}^a}{\partial t} + \frac{\partial \mathrm{v}^a}{\partial x^b} \, \mathrm{v}^b + \gamma^a{}_{bc} \, \mathrm{v}^b \, \mathrm{v}^c$.

2.2 The Material Configuration in Finite Elasticity

In the context of finite elasticity, the body's natural reference configuration, also referred to as its material configuration, is stress-free and at rest within the Euclidean ambient space. Specifically, $\mathcal{B} \subset \mathcal{S} = \mathbb{R}^3$ and naturally inherits its geometry from the ambient space, meaning its metric is essentially a copy of the background metric $\mathbf{g}$. Denoting the material metric in finite elasticity by $\mathring{\mathbf{G}}$, it is formally defined as the pullback of the background metric under the inclusion embedding $\iota : \mathcal{B} \to \mathcal{S}$, such that $\mathring{\mathbf{G}} = \iota^* \mathbf{g}$.

[4] For most applications, the ambient space is the three-dimensional Euclidean space, i.e., $\mathcal{S} = \mathbb{R}^3$. However, in general, the ambient space may be curved, e.g., in modeling the dynamics of fluid membranes Arroyo and DeSimone [2009]. See Yavari et al. [2016] for a general framework on elasticity in evolving ambient spaces.

The material (natural stress-free) configuration for elasticity is $(\mathcal{B}, \mathring{\mathbf{G}})$ where distances and angles are measured using the material metric $\mathring{\mathbf{G}}$, which serves as the baseline against which deformations are quantified.[5] Given two referential vectors $\mathbf{U}, \mathbf{W} \in T_X\mathcal{B}$, their dot product in the material configuration $(\mathcal{B}, \mathring{\mathbf{G}})$ is expressed as $\langle\!\langle \mathbf{U}, \mathbf{W} \rangle\!\rangle_{\mathring{\mathbf{G}}} = \mathrm{U}^A \mathrm{W}^B \mathring{\mathrm{G}}_{AB}$. The material volume form given by $d\mathring{V} = \sqrt{\det \mathring{\mathbf{G}}}\, dX^1 \wedge dX^2 \wedge dX^3$. The Levi-Civita connection associated with $(\mathcal{B}, \mathring{\mathbf{G}})$, denoted by $\mathring{\nabla}$, has Christoffel symbols $\mathring{\Gamma}^A{}_{BC}$. The motion $\varphi : \mathcal{B} \to \mathcal{S}$ maps each material point in the reference configuration to its spatial position in the deformed configuration, and its Jacobian $\mathring{J}$ relates the material and spatial volume elements as $\varphi^* dv = \mathring{J} d\mathring{V}$. It can be shown that $\mathring{J} = \sqrt{\det \mathbf{g} / \det \mathring{\mathbf{G}}}\, \det \mathbf{F}$.

2.3 The Material Configuration in Finite Anelasticity

Anelasticity describes a class of material behaviours where intrinsic distortions, referred to as eigenstrains—derived from the German term *Eigenspannungsquellen* (sources of inherent stresses) introduced by Reissner Reissner [1931], contribute to the overall deformation of an otherwise elastic body. These eigenstrains, arising from non-elastic deformations or microstructural changes such as plasticity, growth, swelling, or thermal expansion, lead to a residually stressed state when the body is relaxed in a Euclidean ambient space. Indeed, due to the presence of eigenstrains, the natural (stress-free) configuration of the body generally cannot be isometrically embedded in the Euclidean ambient space, meaning it cannot exist as a stress-free body in physical space. The resulting residual stresses are, therefore, an expression of the non-Euclidean nature of the natural configuration. Consequently, the manifold $\mathcal{B}$ must be endowed with a material metric $\mathbf{G}$ that is generally distinct from the Euclidean metric $\mathring{\mathbf{G}}$.

To model the interplay between intrinsic, often irreversible and dissipative, distortions (eigenstrains) and the recoverable , nondissipative elastic deformations in anelastic materials, we introduce the multiplicative decomposition of the total deformation gradient: $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$—known as the Bilby-Köner-Lee decomposition Lubarda [2002]; Sadik and Yavari [2017a]. Here, $\overset{a}{\mathbf{F}}$ maps the Euclidean reference configuration $(\mathcal{B}, \mathring{\mathbf{G}})$ to the stress-free, but generally non-Euclidean, material configuration for anelasticity $(\mathcal{B}, \mathbf{G})$, while $\overset{e}{\mathbf{F}}$ represents the recoverable elastic deformation from the material (stress-free) configuration to the current configuration. Note, however, that while the total deformation gradient $\mathbf{F}$ is compatible—meaning it can be expressed as the derivative of a smooth mapping φ, such that $\mathbf{F} = T\varphi$—the individual distortions $\overset{e}{\mathbf{F}}$ and $\overset{a}{\mathbf{F}}$ are generally incompatible. This incompatibility arises because $\overset{e}{\mathbf{F}}$ and $\overset{a}{\mathbf{F}}$ are not derivatives of smooth mappings themselves but instead represent local, often non-homogeneous, distortions. As a matter of fact, these residual incompatibilities in $\overset{e}{\mathbf{F}}$ and $\overset{a}{\mathbf{F}}$ are a direct manifestation of the failure of the relaxed configuration to be embedded in the physical Euclidean space, and effectively justify the non-Euclidean character of the material configuration preventing the fully relaxed state from being realised as a smooth, globally compatible embedding. The material metric $\mathbf{G}$ associated with the stress-free configuration is related to the Euclidean reference metric $\mathring{\mathbf{G}}$ through the relationship $\mathbf{G} = \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{G}}$.[6]

Note that multiple sources of eigenstrains can coexist within a solid, including temperature changes, growth, remodeling, swelling, and plastic deformation. Each eigenstrain distribution contributes a distinct distortion field, leading to a combined total anelastic distortion. This total distortion can be expressed as a product of individual contributions: $\overset{a}{\mathbf{F}} = \prod_{j=1}^{N} \overset{j}{\mathbf{F}} = \overset{1}{\mathbf{F}} \dots \overset{N}{\mathbf{F}}$, where $\overset{j}{\mathbf{F}}$ represents the distortion field associated with the j-th anelastic process.

Remark 2.2 (Thermal Distortion in Anelasticity)**.** In the case of a non-isothermal process, characterised by an evolving temperature field $\Theta = \Theta(X, t)$, thermal distortion emerges as a distinct source of anelastic distortion. This distortion captures the local, generally incompatible, stretches induced by temperature variations Stojanovic et al. [1964]; Sadik and Yavari [2017b]. The total anelastic distortion $\overset{a}{\mathbf{F}}$ can hence be decomposed as

[5] Refering to Remark 2.1, note that, at time $t = t_0$, before the deformation begins, the deformation mapping reduces to the identity map on $\mathcal{B}$, i.e., $\varphi_{t_0} = \mathrm{id}_{\mathcal{B}}$, and the spatial metric at $t = t_0$ reduces to the Euclidean material metric, i.e., $\mathbf{g}_{t_0} = \mathring{\mathbf{G}}$.

[6] To see this, consider a curve on the reference configuration, $\gamma : \mathfrak{I} \to \mathcal{B}$, where $\mathfrak{I}$ is an open interval in $\mathbb{R}$. Due to the presence of eigenstrains, the curve γ is generally stressed in the Euclidean reference configuration $(\mathcal{B}, \mathring{\mathbf{G}})$. However, its push-forward under the anelastic distortion, $\overset{a}{\mathbf{F}}_* \gamma$, is locally stress-free. The squared arc-length element in this natural (stress-free) configuration is given by $\langle\!\langle \overset{a}{\mathbf{F}}_* \gamma'(t), \overset{a}{\mathbf{F}}_* \gamma'(t) \rangle\!\rangle_{\mathring{\mathbf{G}}} = \langle\!\langle \overset{a}{\mathbf{F}} \gamma'(t), \overset{a}{\mathbf{F}} \gamma'(t) \rangle\!\rangle_{\mathring{\mathbf{G}}} = \langle\!\langle \gamma'(t), \gamma'(t) \rangle\!\rangle_{\overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{G}}}$—see Yavari [2021]. Hence, the material metric $\mathbf{G}$, which measures natural distances in the stress-free configuration, is the pull-back of the Euclidean metric $\mathring{\mathbf{G}}$ under the anelastic distortion $\overset{a}{\mathbf{F}}$.

$\overset{a}{\mathbf{F}} = \overset{n}{\mathbf{F}}\overset{\theta}{\mathbf{F}}$, where $\overset{\theta}{\mathbf{F}}$ accounts for thermal distortion, while $\overset{n}{\mathbf{F}}$ encompasses all other anelastic mechanisms coexisting in the solid. What distinguishes thermal distortion is that its evolution is governed by the heat equation, making it directly dependent on the temperature field $\Theta = \Theta(X, t)$. We show in Appendix A that

$$\boldsymbol{\alpha} = \frac{\partial \overset{\theta}{\mathbf{F}}}{\partial \Theta} \overset{\theta}{\mathbf{F}}^{-1} = \overset{\theta}{\mathbf{F}}^{-1} \frac{\partial \overset{\theta}{\mathbf{F}}}{\partial \Theta} , \tag{2.1}$$

where $\boldsymbol{\alpha}$ is the linear thermal expansion coefficient tensor of the material.

For an anelastic solid, the material (natural stress-free) configuration is hence given by the abstract manifold $(\mathcal{B}, \mathbf{G})$ where distances and angles are measured using the Riemannian metric $\mathbf{G}$. Given two vectors $\mathbf{U}, \mathbf{W} \in T_X\mathcal{B}$, their dot product is expressed as: $\langle\!\langle \mathbf{U}, \mathbf{W} \rangle\!\rangle_{\mathbf{G}} = \mathrm{U}^A \mathrm{W}^B \mathrm{G}_{AB}$. The volume form in the anelastic material configuration is given by $dV = \sqrt{\det \mathbf{G}}\, dX^1 \wedge dX^2 \wedge dX^3$. The Levi-Civita connection associated with $(\mathcal{B}, \mathbf{G})$ is denoted ∇ with Christoffel symbols $\Gamma^A{}_{BC}$. In anelasticity, the elastic Jacobian $\overset{e}{J}$ of the motion relates the stress-free material and spatial volume elements as $\varphi^* dv = \overset{e}{J} dV$, and it can be shown that $\overset{e}{J} = \sqrt{\det \mathbf{g} / \det \mathbf{G}} \det \mathbf{F} = \sqrt{\det \mathbf{g} / \det \mathring{\mathbf{G}}} \det \overset{e}{\mathbf{F}}$. We may also introduce the athermal anelastic Jacobian related to the athermal anelastic distortion $\overset{n}{\mathbf{F}}$ as $\overset{n}{J} = \det \overset{n}{\mathbf{F}}$.

3 Thermodynamics and the Balance Laws of Hyperelasticity

In this section, we present the results of this work in the context of finite hyperelasticity—a subclass of elasticity in which the stress is derived from a scalar strain energy function as discussed below in Remark 3.1. Recall that in finite elasticity, the material metric $\mathring{\mathbf{G}}$ is a copy of the background Euclidean metric $\mathbf{g}$, i.e., $\mathring{\mathbf{G}} = \iota^* \mathbf{g}$, and serves as the geometric foundation for defining distances and angles in the stress-free reference configuration—see §2.2. In what follws, we begin by briefly reviewing the first and second laws of thermodynamics in the setting of nonlinear hyperelasticity. We then demonstrate how all the balance laws and the constitutive equations of hyperelasticity can be derived directly from these thermodynamic principles, without invoking (observer) invariance.[7]

Remark 3.1 (Hyperelasticity)**.** As formalised by Noll Noll [1958], a *simple elastic solid* is that for which the stress at a given point depends only on the local current deformation state at that point (e.g., via the Finger tensor $\mathbf{b}$), discarding any dependence on the material's deformation history or nonlocal effects such as higher-order spatial gradients. A particular subclass of simple elastic materials is known as *Cauchy elastic* materials, where the stress at any point is explicitly expressed as a function of the strain at that point, such as $\boldsymbol{\sigma} = \mathsf{f}(\mathbf{b})$ Cauchy [1828]; Truesdell [1952]. This subclass assumes a direct and explicit dependence of stress on strain, further refining the constitutive description. Within Cauchy elasticity, another important subset consists of materials for which the stress is derivable from a scalar strain energy function. These are referred to as *Green elastic* Green [1838, 1839]; Spencer [2015] or *hyperelastic* Truesdell [1952]. However, not all Cauchy elastic materials possess such an energy function, as Green elasticity imposes additional constraints on the constitutive relation. It is worth noting that not all simple elastic solids are Cauchy elastic. There are those whose constitutive relations are defined implicitly, taking the form $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{b}) = \mathbf{0}$, where the stress and deformation measures are related implicitly Morgan [1966]; Rajagopal [2003, 2007, 2011]. These implicit models encompass both Cauchy elastic materials and other, more complex elastic behaviours. Thus, Cauchy elastic solids form a proper subset of simple elastic materials, and Green elastic materials are a further subset of Cauchy elastic solids. This hierarchy reflects the progressively restrictive assumptions underpinning these material models.

3.1 First Law of Thermodynamics

The first law of thermodynamics posits the existence of an internal energy $\mathscr{E}$ as a state function, which satisfies the following balance equation as an expression of the conservation of energy principle Truesdell [1952]; Gurtin

[7] As discussed earlier in §1, the Green-Naghdi-Rivlin theorem (and its subsequent extensions) gives the balance of linear and angular momenta as a consequence of the invariance of the energy balance under superposed isometries of the ambient space. Here, the proposed derivation does not rely on invariance; instead, it generalizes the Colemon-Noll procedure Coleman and Noll [1963] to recover not only the constitutive equations, but also the balance laws.

[1974]; Marsden and Hughes [1983]

$$\frac{d}{dt}\int_{\mathcal{U}} \mathring{\rho}\left(\mathscr{E}+\frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^{2}\right) d\mathring{V}=\int_{\mathcal{U}} \mathring{\rho}\left(\langle\!\langle \mathbf{B},\mathbf{V}\rangle\!\rangle_{\mathbf{g}}+R\right) d\mathring{V}+\int_{\partial\mathcal{U}}\left(\langle\!\langle \mathbf{T},\mathbf{V}\rangle\!\rangle_{\mathbf{g}}-\langle\!\langle \mathbf{Q},\mathbf{N}\rangle\!\rangle_{\mathring{\mathbf{G}}}\right) d\mathring{A}\,, \tag{3.1}$$

for any open set $\mathcal{U}\subset\mathcal{B}$, where $\mathscr{E}$ stands for the specific internal energy, $\mathring{\rho}$ is the material mass density, $\mathbf{B}$ is the specific body force, $R=R(X,t)$ is the specific heat supply, $\mathbf{T}=\mathbf{T}(X,\mathbf{N})$ is the boundary traction vector field per unit material area, $\mathbf{N}$ is the $\mathring{\mathbf{G}}$-unit normal to the boundary $\partial\mathcal{U}$, and $\mathbf{Q}=\mathbf{Q}(X,\Theta,d\Theta,\mathbf{F},\mathring{\mathbf{G}},\mathbf{g})$ represents the external heat flux per unit material area, $d\Theta=\frac{\partial\Theta}{\partial X^A}dX^A$ is the exterior derivative of temperature Θ, and $d\mathring{A}$ is the material area element.

Using Marsden and Hughes's version of Cauchy's theorem [Marsden and Hughes, 1983, §2.1(1.9)],[8] it follows from (3.1) that there exists a unique material vector field $\mathbf{U}$ such that $\langle\!\langle \mathbf{U},\mathbf{N}\rangle\!\rangle_{\mathring{\mathbf{G}}}=\langle\!\langle \mathbf{T},\mathbf{V}\rangle\!\rangle_{\mathbf{g}}$. The linearity of $\langle\!\langle \mathbf{T},\mathbf{V}\rangle\!\rangle_{\mathbf{g}}$ with respect to $\mathbf{V}$ implies that $\mathbf{U}$ is also linear in $\mathbf{V}$, i.e., there exists a second-order two-point tensor field $\mathbf{M}$ such that $\mathbf{U}=\mathbf{M}\mathbf{V}$, which indeed is unique following the uniqueness of $\mathbf{U}$. We now have that $\langle\!\langle \mathbf{M}\mathbf{V},\mathbf{N}\rangle\!\rangle_{\mathring{\mathbf{G}}}=\langle\!\langle \mathbf{T},\mathbf{V}\rangle\!\rangle_{\mathbf{g}}$, which may be recast into $\langle\!\langle \mathbf{M}^{\mathring{\mathsf{T}}}\mathbf{N},\mathbf{V}\rangle\!\rangle_{\mathbf{g}}=\langle\!\langle \mathbf{T},\mathbf{V}\rangle\!\rangle_{\mathbf{g}}$. By virtue of the existence and uniqueness of $\mathbf{M}$, we may define the two-point tensor $\mathbf{P}:=\mathbf{M}^{\mathring{\mathsf{T}}}\mathbf{g}^{\sharp}$, and it follows by arbitrariness of $\mathbf{V}$ that $\mathbf{T}=\mathbf{P}\mathbf{N}^{\mathring{\flat}}$, where $(.)^{\mathring{\flat}}$ and $(.)^{\sharp}$ denote the musical isomorphisms for lowering and raising indices, respectively, with respect to $\mathring{\mathbf{G}}$ and $\mathbf{g}$. Now, we proceed to write the energy balance (3.1) in local form:

$$\mathring{\rho}\,\dot{\mathscr{E}}=\mathbf{P}\!:\!(\mathbf{g}\bar{\nabla}\mathbf{V})-\mathring{\mathrm{D}}\mathrm{iv}\,\mathbf{Q}+\mathring{\rho}R+\langle\!\langle \mathring{\mathrm{D}}\mathrm{iv}\,\mathbf{P}+\rho(\mathbf{B}-\mathbf{A}),\mathbf{V}\rangle\!\rangle_{\mathbf{g}}-\dot{\rho}_0\left(\mathscr{E}+\frac{1}{2}\|\mathbf{V}\|_{\mathbf{g}}^{2}\right)\,, \tag{3.2}$$

where a dotted quantity denotes its total time derivative, $\mathring{\mathrm{D}}\mathrm{iv}$ denotes the two-point divergence operator with respect to the connections $\mathring{\nabla}$ and $\bar{\nabla}$, and a colon denotes the double contraction product, i.e., $\mathbf{P}:(\mathbf{g}\bar{\nabla}\mathbf{V})=\mathrm{P}^{aA}\mathrm{V}_{a|A}$—the vertical stroke here denoting covariant differentiation with respect to $\mathbf{g}$ in components, i.e., $\mathrm{V}_{a|A}=\mathrm{V}_{a,A}-\mathrm{V}_b\gamma^b_{ac}\mathrm{F}^c{}_A$. Note that[9]

$$\mathbf{P}\!:\!(\mathbf{g}\bar{\nabla}\mathbf{V})=\mathbf{P}\!:\!(\mathbf{g}\bar{\mathrm{D}}_t\mathbf{F})=(\mathbf{F}^{-1}\mathbf{P})\!:\!(\overset{s}{\mathbf{D}}+\overset{a}{\mathbf{D}})\,, \tag{3.3}$$

where $\bar{\mathrm{D}}_t(.)\coloneqq\bar{\nabla}_{\mathbf{v}}(.)$ denotes the covariant time derivative,[10] $\overset{s}{\mathbf{D}}$ is the symmetric part of $\mathbf{F}^{\star}\mathbf{g}\dot{\mathbf{F}}$, and $\overset{a}{\mathbf{D}}$ is its anti-symmetric part:

$$\overset{s}{\mathbf{D}}=\frac{1}{2}\left[\mathbf{F}^{\star}\mathbf{g}\bar{\mathrm{D}}_t\mathbf{F}+\bar{\mathrm{D}}_t\mathbf{F}^{\star}\mathbf{g}\mathbf{F}\right]\,,\qquad \overset{a}{\mathbf{D}}=\frac{1}{2}\left[\mathbf{F}^{\star}\mathbf{g}\bar{\mathrm{D}}_t\mathbf{F}-\bar{\mathrm{D}}_t\mathbf{F}^{\star}\mathbf{g}\mathbf{F}\right]\,. \tag{3.4}$$

3.2 Second Law of Thermodynamics

The second law of thermodynamics posits the existence of specific entropy $\mathscr{N}$ as a state function, which satisfies the following inequality—known as the Clausius-Duhem inequality—as an expression of the principle of entropy production,[11] which steadily increases or remains constant within a closed system over time Truesdell [1952]; Gurtin [1974]; Marsden and Hughes [1983]

$$\frac{d}{dt}\int_{\mathcal{U}}\mathring{\rho}\mathscr{N}d\mathring{V}\geq\int_{\mathcal{U}}\mathring{\rho}\frac{R}{\Theta}d\mathring{V}+\int_{\partial\mathcal{U}}\frac{H}{\Theta}d\mathring{A}\,, \tag{3.5}$$

for any open set $\mathcal{U}\subset\mathcal{B}$. In localized form, the material Clausius-Duhem inequality (3.5) is written as

$$\dot{\eta}=\mathring{\rho}\Theta\dot{\mathscr{N}}+\dot{\mathring{\rho}}\Theta\mathscr{N}+\mathring{\mathrm{D}}\mathrm{iv}\,\mathbf{Q}-\mathring{\rho}R-\frac{1}{\Theta}\langle d\Theta,\mathbf{Q}\rangle\geq 0\,, \tag{3.6}$$

where $\dot{\eta}$ denotes the material rate of energy dissipation density.

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[8] Note that this version of Cauchy's theorem does not assume the balance of linear momentum and does not introduce the idea of stress—and neither do we, at this point in the paper.

[9] We thank Sanjay Govindjee for bringing this to our attention.

[10] Note that by using the symmetry lemma Nishikawa [2002], one may write $V^a{}_{|A}=\left(\frac{\partial\varphi^a}{\partial t}\right)_{|A}=\bar{\mathrm{D}}_t\left(\frac{\partial\varphi^a}{\partial X^A}\right)$, which implies that $\bar{\nabla}\mathbf{V}=\bar{\mathrm{D}}_t\mathbf{F}$. This in components reads $\bar{\mathrm{D}}_t\mathrm{F}^a{}_A=\frac{\partial}{\partial t}(\mathrm{F}^a{}_A)+\mathrm{F}^b{}_A\gamma^a{}_{bc}\mathrm{v}^c$.

[11] The entropy production for an open subset $\mathcal{U}$ in the body is written as

$$\Gamma=\frac{d}{dt}\int_{\mathcal{U}}\mathring{\rho}\mathscr{N}d\mathring{V}-\int_{\mathcal{U}}\mathring{\rho}\frac{R}{\Theta}d\mathring{V}-\int_{\partial\mathcal{U}}\frac{H}{\Theta}d\mathring{A}\,.$$

Remark 3.2. At a material point $X \in \mathcal{B}$, Coleman and Noll [1963] introduced the set[12]

$$\Big\{\varphi_t(X), \mathbf{P}(X,t), \mathbf{B}(X,t), \mathscr{E}(X,t), \mathbf{Q}(X,t), R(X,t), \mathscr{N}(X,t), \Theta(X,t)\Big\}, \tag{3.7}$$

and called it a *thermodynamic process*, provided all its eight fields satisfy the first law of thermodynamics and the balance of linear and angular momenta. A given hyperelastic material is specified by its constitutive assumptions, e.g., $\mathscr{E} = \mathscr{E}(X, \mathscr{N}, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$ and $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$. A thermodynamic process is *admissible* if the constitutive assumptions hold everywhere in the body and at all times. Coleman and Noll [1963] showed that requiring the Clausius-Duhem inequality (3.6) to hold for all admissible thermodynamic processes puts certain constraints on the constitutive assumptions, e.g., the Doyle-Ericksen formula.

In this work, we define an *extended thermodynamic process* at a material point $X \in \mathcal{B}$ as the set (3.7), provided all its eight fields satisfy the first law of thermodynamics—without requiring that $\mathbf{P}$ satisfies the balance of linear or angular momenta. An extended thermodynamic process is *admissible* if the constitutive assumptions hold everywhere in the body and at all times. We require that the Clausius-Duhem inequality (3.6) holds for all admissible extended thermodynamic processes.

3.3 Constitutive Equations and Balance Laws in Hyperelasticity

Let us first consider the case of a hyperelastic material. Hyperelasticity implies the existence an energy function that depends explicitly at every material point $X \in \mathcal{B}$ on the strain at that same point. We may hence write the specific free energy as $\Psi = \Psi(X, \Theta, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$ Truesdell [1952], which in fact is the Legendre transform of the specific internal energy $\mathscr{E}$ with respect to the conjugate variables temperature Θ and specific entropy $\mathscr{N}$:

$$\Psi = \mathscr{E} - \Theta \mathscr{N}. \tag{3.8}$$

Hence, we have that $\mathscr{E} = \mathscr{E}(X, \mathscr{N}, \mathbf{F}, \mathring{\mathbf{G}}, \mathbf{g})$, and consequently have[13]

$$\mathscr{N} = -\frac{\partial \Psi}{\partial \Theta}. \tag{3.9}$$

Proposition 3.1. *For a hyperelastic body, the first and second laws of thermodynamics* (3.2) *and* (3.6) *imply that*

$$\begin{cases} \dot{\mathring{\rho}}_0 = 0, & (3.10a) \\ \mathbf{P} = \mathring{\rho}\mathbf{g}^{\sharp}\dfrac{\partial \Psi}{\partial \mathbf{F}}, & (3.10b) \\ \mathring{\mathrm{Div}}\,\mathbf{P} + \mathring{\rho}\mathbf{B} = \mathring{\rho}\mathbf{A}, & (3.10c) \\ \mathbf{F}\mathbf{P}^{\star} = \mathbf{P}\mathbf{F}^{\star}, & (3.10d) \\ \dot{\eta} = -\dfrac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0. & (3.10e) \end{cases}$$

In other words, the first and second laws of thermodynamics imply the conservation of mass (3.10a)*, the Doyle-Ericksen formula* (3.10b)*, the balance of linear momentum* (3.10c)*, the balance of angular momentum* (3.10d)*, and the dissipation inequality* (3.10e)*. Note that* (3.10b) *effectively shows that the two-point tensor* $\mathbf{P}$ *is indeed the Piola-Kirchhoff stress tensor.*

Proof. From (3.8), one writes: $\mathring{\rho}\Theta\dot{\mathscr{N}} = \mathring{\rho}\dot{\mathscr{E}} - \mathring{\rho}\dot{\Psi} - \mathring{\rho}\dot{\Theta}\mathscr{N}$. Substituting this relation and (3.2) (while using (3.3)) into (3.6), one obtains

$$\begin{aligned} \dot{\eta} =& (\mathbf{F}^{-1}\mathbf{P}) : (\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}}) - \mathring{\rho}\dot{\Psi} - \mathring{\rho}\dot{\Theta}\mathscr{N} + \dot{\rho}_0 \Theta \mathscr{N} \\ &+ \langle\!\langle \mathring{\mathrm{Div}}\,\mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0 \left[\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0. \end{aligned} \tag{3.11}$$

[12] $\mathbf{P}$, in their treatment, is introduced, a priori, as the first Piola-Kirchhoff stress tensor (in fact, they use the Cauchy stress tensor) satisfying the balance of linear and angular momenta.

[13] Note that the Legendre transform (3.8) of $\mathscr{E}$ to Ψ with respect to the conjugate variables $\mathscr{N}$ and Θ is essentially a change of variable satisfying (3.9). See Arnold [1989]; Goldstein et al. [2002] for further details on Legendre transform in the context of Lagrangian mechanics and thermodynamics.

Applying Leibniz rule to $\dot{\Psi}$, one writes[14]

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \Theta}\dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{F}}:\bar{\mathrm{D}}_t\mathbf{F} + \frac{\partial \Psi}{\partial \mathbf{g}}:\bar{\mathrm{D}}_t\mathbf{g} = -\mathscr{N}\dot{\Theta} + \left(\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}}\right):(\overset{a}{\mathbf{D}} + \overset{s}{\mathbf{D}})\,. \tag{3.12}$$

It can be shown that[15]

$$\left(\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}}\right)^{\star} = \mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}}\,. \tag{3.13}$$

Hence $\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}}$ is a symmetric tensor, and since $\overset{a}{\mathbf{D}}$ is antisymmetric, it follows that

$$\left(\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}}\right):\overset{a}{\mathbf{D}} = 0\,. \tag{3.14}$$

Using the above identity, and substituting (3.12) into (3.11), the rate of dissipation is simplified to read

$$\begin{aligned}\dot{\eta} &= \left[\mathbf{F}^{-1}\left(\mathbf{P} - \mathring{\rho}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}}\right)\right]:\overset{s}{\mathbf{D}} + \mathbf{F}^{-1}\mathbf{P}:\overset{a}{\mathbf{D}} \\ &\quad + \langle\!\langle \mathring{\mathrm{D}}\mathrm{iv}\,\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left[\Psi + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,.\end{aligned} \tag{3.15}$$

This inequality must hold for all motions, i.e., all extended thermodynamic processes. As $\overset{s}{\mathbf{D}}$ (a symmetric tensor) and $\overset{a}{\mathbf{D}}$ (an antisymmetric tensor) can be chosen independently of all the other fields, one concludes that[16]

$$\mathbf{P} = \mathring{\rho}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}}\,, \qquad (\mathbf{F}^{-1}\mathbf{P})^{\star} = \mathbf{F}^{-1}\mathbf{P}\,. \tag{3.16}$$

Now, the rate of dissipation (3.15) is simplified to read

$$\dot{\eta} = \langle\!\langle \mathring{\mathrm{D}}\mathrm{iv}\,\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left[\Psi + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,. \tag{3.17}$$

One can choose the velocity vector arbitrarily while its norm $\|\mathbf{V}\|_{\mathbf{g}}$ is fixed. This implies that the inequality (3.17) can hold only if

$$\mathring{\mathrm{D}}\mathrm{iv}\,\mathbf{P} + \mathring{\rho}\mathbf{B} = \mathring{\rho}\mathbf{A}\,. \tag{3.18}$$

Now the rate of dissipation takes the following form

$$\dot{\eta} = -\dot{\rho}_0\left[\Psi + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,. \tag{3.19}$$

Next, one can choose the velocity vector norm $\|\mathbf{V}\|_{\mathbf{g}}$ arbitrarily while keeping the other fields fixed. For all these extended thermodynamics processes the above inequality must hold. This implies that $\dot{\rho}_0 = 0$ and $\dot{\eta} = -\frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0$. $\square$

[14] Note that since the connection is Levi-Civita, which is metric compatible, it follows that $\bar{\mathrm{D}}_t\mathbf{g} = \bar{\nabla}_{\mathbf{V}}\mathbf{g} = \mathbf{0}$.

[15] Using Leibniz rule, we write in components

$$\frac{\partial \Psi}{\partial \mathrm{F}^a{}_A} = \frac{\partial \Psi}{\partial(\mathrm{F}^i{}_K\,\mathrm{F}^j{}_L\,\mathrm{g}_{ij})}\frac{\partial(\mathrm{F}^k{}_K\,\mathrm{F}^l{}_L\,\mathrm{g}_{kl})}{\partial \mathrm{F}^a{}_A} = \frac{\partial \Psi}{\partial(\mathrm{F}^i{}_K\,\mathrm{F}^j{}_L\,\mathrm{g}_{ij})}\left(\delta^k{}_a\delta^A{}_K\,\mathrm{F}^l{}_L\,\mathrm{g}_{kl} + \mathrm{F}^k{}_K\,\delta^l{}_a\delta^A{}_L\,\mathrm{g}_{kl}\right)\,.$$

Using straightforward index manipulations, it follows that

$$\frac{\partial \Psi}{\partial \mathrm{F}^a{}_A} = 2\mathrm{g}_{al}\,\mathrm{F}^l{}_K\,\frac{\partial \Psi}{\partial(\mathrm{F}^i{}_K\,\mathrm{F}^j{}_A\,\mathrm{g}_{ij})}\,, \quad \text{and} \qquad \frac{\partial \Psi}{\partial \mathrm{F}^a{}_A} = 2\frac{\partial \Psi}{\partial(\mathrm{F}^i{}_A\mathrm{F}^j{}_L\mathrm{g}_{ij})}\,\mathrm{F}^k{}_L\,\mathrm{g}_{ka}\,,$$

which one may rewrite as

$$\frac{\partial \Psi}{\partial \mathbf{F}} = 2\mathbf{g}\mathbf{F}\frac{\partial \Psi}{\partial(\mathbf{F}^{\star}\mathbf{g}\mathbf{F})}\,, \quad \text{and} \qquad \left(\frac{\partial \Psi}{\partial \mathbf{F}}\right)^{\star} = 2\frac{\partial \Psi}{\partial(\mathbf{F}^{\star}\mathbf{g}\mathbf{F})}\mathbf{F}^{\star}\mathbf{g}\,.$$

Therefore, one finds that

$$\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}} = \left(\frac{\partial \Psi}{\partial \mathbf{F}}\right)^{\star}\mathring{\mathbf{g}}^{\sharp}\mathbf{F}^{-\star}\,,$$

which directly leads to (3.13).

[16] There are infinitely may extended thermodynamic processes for which everything except for $\overset{s}{\mathbf{D}}$ and $\overset{a}{\mathbf{D}}$ are the same. For the second law to hold for all such processes one must have (3.16).

Remark 3.3. Coleman and Noll [1963] showed that requiring (3.6) to hold for all admissible thermodynamic processes gives the Doyle-Ericksen formula (3.10b). We have shown that requiring (3.6) to hold for all admissible extended thermodynamic processes gives all the balance laws and the Doyle-Ericksen formula (3.10b). It is known that in hyperelaticity, balance of angular momentum implies objectivity Kadić [1980]; Yavari and Goriely [2024]. Therefore, we have shown that the first and second laws of thermodynamics imply objectivity as well.

Remark 3.4. For an incompressible hyperelastic solid, the Legendre transform (3.8) is modified to take into account the constraint of volume preservation $J = 1$ on motions as follows

$$\Psi - p(J-1) = \mathscr{E} - \Theta \mathscr{N} \,, \tag{3.20}$$

where $p(X,t)$ is the Lagrange multiplier associated with the incompressibility constraint. Computing $\dot{J}$, we find[17]

$$\dot{J} = J\mathbf{F}^{-\star}\!:\!\dot{\mathbf{F}} = \left(\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\mathbf{F}^{-\star}\right)\!:\!(\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}}) = \left(\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\mathbf{F}^{-\star}\right)\!:\!\overset{s}{\mathbf{D}} \,. \tag{3.21}$$

Revisiting the proof for Proposition 3.1, the results remain unchanged except for the Doyle-Ericksen formula (3.10b) which is modified to read

$$\mathbf{P} = \mathring{\rho}\mathring{\mathbf{g}}^{\sharp}\frac{\partial \Psi}{\partial \mathbf{F}} - p\,\mathring{\mathbf{g}}^{\sharp}\mathbf{F}^{-\star} \,. \tag{3.22}$$

Remark 3.5. Note that it is possible to recast the balance and constitutive equations (4.9) in their spatial (current) form as follows:

$$\begin{cases} \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0 \,, & (3.23a) \\ \boldsymbol{\sigma} = \rho \mathring{\mathbf{g}}^{\sharp}\dfrac{\partial \Psi}{\partial \mathbf{F}}\mathbf{F}^{\star} = 2\rho\dfrac{\partial \Psi}{\partial \mathbf{g}} \,, & (3.23b) \\ \operatorname{div} \boldsymbol{\sigma} + \rho\, \mathbf{b} = \rho\, \mathbf{a} \,, & (3.23c) \\ \boldsymbol{\sigma}^{\star} = \boldsymbol{\sigma} \,, & (3.23d) \\ \dot{\eta} = -\dfrac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0 \,, & (3.23e) \end{cases}$$

where ρ is the spatial mass density, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{b} = \mathbf{B} \circ \varphi_t^{-1}$, and div denotes the spatial divergence operator, i.e., in the manifold $(\mathcal{S}, \mathbf{g})$. The spatial balance of mass (3.23a) is derived from the material (reference) balance and the relation $\rho = \mathring{\rho}/\mathring{J}$, which follows from the definition of the Jacobian, $\varphi^* dv = \mathring{J} d\mathring{V}$. In (3.23b), the Cauchy stress tensor $\boldsymbol{\sigma}$ is the Piola transform of the first Piola-Kirchhoff stress tensor $\mathbf{P}$; consistently with the spatial forms of the balance of linear momentum (3.23c) and angular momentum (3.23d) obtained by pushing forward the respective material equations into the current configuration using the Piola transformation Marsden and Hughes [1983].

4 Thermodynamics and the Balance Laws of Hyper-Anelasticity

In this section, we extend our generalised Coleman-Noll procedure to the setting of hyper-anelasticity. As discussed in §2.3, the Bilby-Köner-Lee (BKL) decomposition, $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$, provides a framework for separating the recoverable elastic deformation, $\overset{e}{\mathbf{F}}$, from the intrinsic anelastic distortion, $\overset{a}{\mathbf{F}}$. As a consequence of this decomposition, the material metric $\mathbf{G} = \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{G}}$ emerges to describe the evolving, non-Euclidean geometry of the stress-free material configuration, reflecting the intrinsic distortions arising from eigenstrains. Building on this foundation, we generalise the first and second laws of thermodynamics to incorporate the interplay between elastic and anelastic distortions within the context of the evolving material metric. This generalised framework allows for the derivation of constitutive equations, balance laws, and thermodynamic constraints governing hyper-anelastic materials, all without invoking any assumption of invariance.

[17] We use the identity $\frac{d}{dt}[\det \mathbf{F}] = \left[\mathbf{F}^{-\star}\!:\!\bar{\mathrm{D}}_t\mathbf{F}\right] \det \mathbf{F}$, perform a computation similar to (3.3), and observe that $\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\mathbf{F}^{-\star}$ is a symmetric tensor.

4.1 The first law of thermodynamics

For an anelastic solid, we write energy balance as Epstein and Maugin [2000]; Lubarda and Hoger [2002][18]

$$\frac{d}{dt}\int_{\mathcal{U}} \rho \left(\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right) dV = \int_{\mathcal{U}} \rho \left(\langle\!\langle \mathbf{B}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + R\right) dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} - \langle\!\langle \mathbf{Q}, \mathbf{N} \rangle\!\rangle_{\mathbf{G}}\right) dA \\ + \int_{\mathcal{U}} S_m \left(\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right) dV\,, \tag{4.1}$$

for any open set $\mathcal{U} \subset \mathcal{B}$, where $\mathscr{E}$ is the specific internal energy, ρ is the material mass density, $S_m = S_m(X,t)$ is the material rate of change of mass per unit (stress-free) volume—it is identically equal to zero in the absence of bulk growth or shrinkage, $\mathbf{B}$ is the specific body force, $\mathbf{T} = \mathbf{T}(X, \mathbf{N})$ is the boundary traction vector field per unit (stress-free) material area, $\mathbf{N}$ is the $\mathbf{G}$-unit normal to the boundary $\partial\mathcal{U}$, $R = R(X,t)$ is the external specific heat supply, $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{C}, \mathbf{G})$ is the external heat flux per unit material (stress-free) area and, dA is the material area element.

By the same argument developed in §3.1, using Marsden and Hughes's version of Cauchy's theorem [Marsden and Hughes, 1983, §2.1(1.9)], we find $\mathbf{T} = \mathbf{P}\mathbf{N}^\flat$, where $\mathbf{P}$ is a second-order two-point tensor, and $(.)^\flat$ denotes the musical isomorphisms for lowering with respect to $\mathbf{G}$ and $\mathbf{g}$. Hence, the energy balance (3.1) in local form reads:[19]

$$\rho\, \dot{\mathscr{E}} = (\mathbf{F}^{-1}\mathbf{P}) \!:\! \left(\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}}\right) + \rho R - \operatorname{Div}\mathbf{Q} + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \\ + \left(S_m - \dot{\rho} - \frac{1}{2}\rho\, \dot{\mathbf{G}} \!:\! \mathbf{G}^\sharp\right)\left(\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right)\,, \tag{4.2}$$

where Div denotes the two-point divergence operator with respect to the connections ∇ and $\bar{\nabla}$, $(.)^\sharp$ denotes the musical isomorphisms for raising indices with respect to $\mathbf{G}$ and $\mathbf{g}$, and $\overset{s}{\mathbf{D}}$ and $\overset{a}{\mathbf{D}}$ are the symmetric and anti-symmetric material rate of deformation tensors, respectively, as defined in (3.4).

4.2 The second law of thermodynamics

For an anelastic solid, we write the Clausius-Duhem in the anelastic material manifold as Epstein and Maugin [2000]; Lubarda and Hoger [2002][20]

$$\frac{d}{dt}\int_{\mathcal{U}} \rho \mathscr{N} dV \geq \int_{\mathcal{U}} \rho \frac{R}{\Theta} dV + \int_{\partial\mathcal{U}} \frac{H}{\Theta} dA + \int_{\mathcal{U}} S_m \mathscr{N} dV\,, \tag{4.3}$$

for any open set $\mathcal{U} \subset \mathcal{B}$, where $\mathscr{N}$ is the specific entropy (i.e., entropy per unit mass). In localized form, the material Clausius-Duhem inequality (3.5) is written as

$$\dot{\eta} = \rho\, \dot{\mathscr{N}}\Theta + \Theta \operatorname{Div}\left(\frac{\mathbf{Q}}{\Theta}\right) - \rho R - \left(S_m - \dot{\rho} - \frac{1}{2}\rho\, \dot{\mathbf{G}} \!:\! \mathbf{G}^\sharp\right)\Theta \mathscr{N} \geq 0\,. \tag{4.4}$$

4.3 Constitutive Equations and Balance Laws in Hyper-Anelasticity

Let us now consider the case of a hyper-anelastic material. As discussed in §2.3, intrinsic distortions (eigen-strains) give rise to a non-Euclidean stress-free configuration, represented by the incompatible distortion $\overset{a}{\mathbf{F}}$. The recoverable elastic distortion $\overset{e}{\mathbf{F}}$ then maps this stress-free configuration to the current spatial configuration.

[18] Note that the last term in (4.1) is added to account for the change in the internal and kinetic energies of the system due to bulk growth or shrinkage with a material rate of change of mass S_m.

[19] Recall that $\mathbf{G} = \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{G}}$ and $dV = \sqrt{\det \mathbf{G}}\, dX^1 \wedge dX^2 \wedge dX^3$. Hence, we ought to consider the implicit time dependance of dV through $\det\mathbf{G}$ in the left-hand-side of (4.1) and we use the identity $\frac{d}{dt}\left[\det\mathbf{G}\right] = \left[\dot{\mathbf{G}} \!:\! \mathbf{G}^{-1}\right]\det\mathbf{G}$.

[20] Note that the last term in (3.5) is added to account for the change in the entropy of the system due to bulk growth or shrinkage with a material rate of change of mass S_m.

Consequently, the free energy for a hyper-anelastic material depends on $\overset{e}{\mathbf{F}}$ rather than the full deformation gradient $\mathbf{F}$, i.e., $\Psi = \Psi(X, \Theta, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g})$. As the Legendre transform of the specific internal energy $\mathscr{E}$ with respect to the conjugate variables temperature Θ and specific entropy $\mathscr{N}$, the specific free energy function reads

$$\Psi = \mathscr{E} - \Theta \mathscr{N} . \tag{4.5}$$

Hence, we find that $\mathscr{E} = \mathscr{E}(X, \mathscr{N}, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g})$ and

$$\mathscr{N} = -\frac{\partial \Psi}{\partial \Theta} . \tag{4.6}$$

Remark 4.1. The material symmetry group $\mathring{\mathcal{G}}_X$ of a hyper-anelastic material (constituvely defined by a free energy Ψ) at a point $X \in \mathcal{B}$ with respect to the Euclidean reference configuration $(\mathcal{B}, \mathring{\mathbf{G}})$ is the set of $\mathring{\mathbf{K}} \in \mathrm{Orth}(\mathring{\mathbf{G}}) = \{\mathbf{Q} : T_X\mathcal{B} \to T_X\mathcal{B} \mid \mathbf{Q}^\star \mathring{\mathbf{G}} \mathbf{Q} = \mathring{\mathbf{G}}\}$ such that

$$\mathring{\mathbf{K}}_* \Psi(X, \Theta, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \mathring{\mathbf{K}}^* \overset{e}{\mathbf{F}}, \mathring{\mathbf{K}}^* \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g}) , \tag{4.7}$$

for any elastic distortion $\overset{e}{\mathbf{F}}$. Introducing a set of structural tensors $\mathring{\mathbf{\Lambda}}$ for $\mathring{\mathcal{G}}_X$[21] as additional independent variables of the free energy, it follows that $\Psi = \Psi(X, \Theta, \overset{e}{\mathbf{F}}, \mathring{\mathbf{\Lambda}}, \mathring{\mathbf{G}}, \mathbf{g})$ is materially covariant, i.e., for any linear transformation $\mathbf{T} : T_X\mathcal{B} \to T_X\mathcal{B}$, $\Psi(X, \Theta, \overset{e}{\mathbf{F}}, \mathring{\mathbf{\Lambda}}, \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \mathbf{T}^* \overset{e}{\mathbf{F}}, \mathbf{T}^* \mathring{\mathbf{\Lambda}}, \mathbf{T}^* \mathring{\mathbf{G}}, \mathbf{g})$. If we take $\mathbf{T} = \overset{a}{\mathbf{F}}$, and recalling that $\overset{a}{\mathbf{F}}^* \overset{e}{\mathbf{F}} = \overset{e}{\mathbf{F}} \overset{a}{\mathbf{F}} = \mathbf{F}$ and $\mathbf{G} = \overset{a}{\mathbf{F}}^* \mathring{\mathbf{G}} = \overset{a}{\mathbf{F}}^\star \mathring{\mathbf{G}} \overset{a}{\mathbf{F}}$, it follows that

$$\Psi = \Psi(X, \Theta, \overset{e}{\mathbf{F}}, \mathring{\mathbf{\Lambda}}, \mathring{\mathbf{G}}, \mathbf{g}) = \Psi(X, \Theta, \mathbf{F}, \mathbf{\Lambda}, \mathbf{G}, \mathbf{g}) , \tag{4.8}$$

where $\mathbf{\Lambda} = \overset{a}{\mathbf{F}}^* \mathring{\mathbf{\Lambda}}$ is the set of structural tensors in $(\mathcal{B}, \mathbf{G})$. We also see in (4.8) how the material metric $\mathbf{G} = \overset{a}{\mathbf{F}}^* \mathring{\mathbf{G}}$ naturally emerges following the hyper-anelastic constitutive assumption.

Proposition 4.1. *For a hyper-anelastic body, the first and second laws of thermodynamics* (3.2) *and* (3.6) *imply that*

$$\begin{cases} \dot{\rho} + \dfrac{1}{2} \rho \, \dot{\mathbf{G}} \!:\! \mathbf{G}^{-1} = S_m , & \text{(4.9a)} \\ \mathbf{P} = \rho \mathbf{g}^\sharp \dfrac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{a}{\mathbf{F}}^{-\star} , & \text{(4.9b)} \\ \operatorname{Div} \mathbf{P} + \rho \mathbf{B} = \rho \mathbf{A} , & \text{(4.9c)} \\ \mathbf{F} \mathbf{P}^\star = \mathbf{P} \mathbf{F}^\star , & \text{(4.9d)} \\ \dot{\eta} = \rho \overset{e}{\mathbf{F}}^\star \dfrac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \!:\! \dot{\overset{n}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-1} + \rho \overset{n}{\mathbf{F}}^\star \overset{e}{\mathbf{F}}^\star \dfrac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-\star} \!:\! \boldsymbol{\alpha} \dot{\Theta} - \dfrac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0 . & \text{(4.9e)} \end{cases}$$

In other words, the first and second laws of thermodynamics imply the conservation of mass (4.9a)*, the Doyle-Ericksen formula* (4.9b)*, the balance of linear momentum* (4.9c)*, the balance of angular momentum* (4.9d)*, and the dissipation inequality* (4.9e)*. Note that* (4.9b) *effectively shows that the two-point tensor* $\mathbf{P}$ *is indeed the Piola-Kirchhoff stress tensor.*

Proof. Using (4.2), (4.4), and (4.5), one finds

$$\begin{aligned} \dot{\eta} = (\mathbf{F}^{-1} \mathbf{P}) \!:\! \big(\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}} \big) - \rho \dot{\Psi} - \rho \dot{\Theta} \mathscr{N} + \langle\!\langle \operatorname{Div} \mathbf{P} + \rho (\mathbf{B} - \mathbf{A}), \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \\ + \Big(S_m - \dot{\rho} - \frac{1}{2} \rho \, \dot{\mathbf{G}} \!:\! \mathbf{G}^\sharp \Big) \Big(\Psi + \frac{1}{2} \|\mathbf{V}\|_{\mathbf{g}}^2 \Big) - \frac{1}{\Theta} \langle d\Theta, \mathbf{Q} \rangle \geq 0 . \end{aligned} \tag{4.10}$$

[21]The subgroup $\mathring{\mathcal{G}}_X \leqslant \mathrm{Orth}(\mathring{\mathbf{G}})$ can be characterized by a finite collection of *structural tensors* $\mathring{\mathbf{\Lambda}}_i$, $i = 1, \ldots, N$, which is a basis for the space of $\mathring{\mathcal{G}}_X$-invariant tensors Liu [1982]; Boehler [1987]; Zheng and Spencer [1993]; Zheng [1994]; Lu and Papadopoulos [2000]; Mazzucato and Rachele [2006].

Expanding $\dot{\Psi}$ by Leibniz rule and using the product rule $\bar{\mathrm{D}}_t\overset{a}{\mathbf{F}} = (\bar{\mathrm{D}}_t\mathbf{F})\overset{a}{\mathbf{F}}^{-1} - \overset{e}{\mathbf{F}}\dot{\overset{a}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-1}$,[22] one obtains

$$
\begin{aligned}
\dot{\Psi} &= \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\bar{\mathrm{D}}_t\overset{e}{\mathbf{F}} + \frac{\partial\Psi}{\partial\mathbf{g}}:\bar{\mathrm{D}}_t\mathbf{g} \\
&= \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\Big[(\bar{\mathrm{D}}_t\mathbf{F})\overset{a}{\mathbf{F}}^{-1} - \overset{e}{\mathbf{F}}\dot{\overset{a}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-1}\Big] \\
&= \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}:\bar{\mathrm{D}}_t\mathbf{F} - \overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{a}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-1} \\
&= -\mathscr{N}\dot{\Theta} + \mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}:(\overset{a}{\mathbf{D}} + \overset{s}{\mathbf{D}}) - \overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{a}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-1}\,,
\end{aligned}
\tag{4.11}
$$

where we use metric compatibility to write $\bar{\mathrm{D}}_t\mathbf{g} = \mathbf{0}$, recall that $\frac{\partial\Psi}{\partial\Theta} = -\mathscr{N}$, and similarly to (3.3), we use the following identity:

$$
\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}:\bar{\mathrm{D}}_t\mathbf{F} = \mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}:(\overset{a}{\mathbf{D}} + \overset{s}{\mathbf{D}})\,.
\tag{4.12}
$$

Similarly to (3.13), it can be shown that $\mathbf{F}^{-1}\mathbf{g}^{\sharp}(\partial\Psi/\partial\overset{e}{\mathbf{F}})\overset{a}{\mathbf{F}}^{-\star}$ is a symmetric tensor, which then implies that

$$
\mathbf{F}^{-1}\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}:\overset{a}{\mathbf{D}} = 0\,.
\tag{4.13}
$$

Now substituting (4.11) into (4.10), one finds

$$
\begin{aligned}
\dot{\eta} = {} & \left[\mathbf{F}^{-1}\left(\mathbf{P} - \rho\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}\right)\right]:\overset{s}{\mathbf{D}} + (\mathbf{F}^{-1}\mathbf{P}):\overset{a}{\mathbf{D}} + \rho\overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{a}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-1} - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \\
& + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} + \left(S_m - \dot{\rho} - \frac{1}{2}\rho\,\dot{\mathbf{G}}:\mathbf{G}^{\sharp}\right)\left(\Psi + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right) \geq 0\,.
\end{aligned}
\tag{4.14}
$$

This inequality must hold for all motions, i.e., all extended thermodynamic processes. Similarly to the proof of Proposition 3.1, it follows from (4.14), by arbitrariness and independence of $\overset{s}{\mathbf{D}}$, $\overset{a}{\mathbf{D}}$, $\mathbf{V}/\|\mathbf{V}\|^2_{\mathbf{g}}$, and $\|\mathbf{V}\|^2_{\mathbf{g}}$, that

$$
\mathbf{P} = \rho\mathbf{g}^{\sharp}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}\,,
\tag{4.15a}
$$

$$
(\mathbf{F}^{-1}\mathbf{P})^{\star} = \mathbf{F}^{-1}\mathbf{P}\,,
\tag{4.15b}
$$

$$
\operatorname{Div}\mathbf{P} + \rho\mathbf{B} = \rho\mathbf{A}\,,
\tag{4.15c}
$$

$$
\dot{\rho} + \frac{1}{2}\rho\,\dot{\mathbf{G}}:\mathbf{G}^{\sharp} = S_m\,,
\tag{4.15d}
$$

respectively. Consequently, (4.14) simplifies to

$$
\dot{\eta} = \rho\overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{a}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-1} - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,.
\tag{4.16}
$$

Recalling the decomposition $\overset{a}{\mathbf{F}} = \overset{n}{\mathbf{F}}\overset{\Theta}{\mathbf{F}}$, we may rewrite the first term in (4.16) as

$$
\begin{aligned}
\overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{a}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-1} &= \overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} + \overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\overset{n}{\mathbf{F}}\dot{\overset{\Theta}{\mathbf{F}}}\overset{\Theta}{\mathbf{F}}^{-1}\overset{n}{\mathbf{F}}^{-1} \\
&= \overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} + \overset{n}{\mathbf{F}}^{\star}\overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}:\dot{\overset{\Theta}{\mathbf{F}}}\overset{\Theta}{\mathbf{F}}^{-1} \\
&= \overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}:\dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} + \overset{n}{\mathbf{F}}^{\star}\overset{e}{\mathbf{F}}^{\star}\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}:\boldsymbol{\alpha}\dot{\Theta}\,,
\end{aligned}
\tag{4.17}
$$

[22] Unlike the two-point tensors $\mathbf{F}$ and $\overset{e}{\mathbf{F}}$, which are sections of the product of the tangent bundle of the material manifold $\mathcal{B}$ and the cotangent bundle of the spatial manifold $\varphi_t(\mathcal{B})$, the referential tensor $\overset{a}{\mathbf{F}}$ is a section of the product of the tangent and cotangent bundles of the fixed material manifold $\mathcal{B}$. As a result, the total time derivatives of $\mathbf{F}$ and $\overset{e}{\mathbf{F}}$ are not well-defined due to the time-dependent nature of the spatial manifold $\varphi_t(\mathcal{B})$, and their material rates must instead be expressed using the spatial covariant derivative $\bar{\mathrm{D}}_t(.)$ to account for the evolving geometry of the spatial configuration. In contrast, the material rate of $\overset{a}{\mathbf{F}}$ is given by its well-defined total time derivative, $\dot{\overset{a}{\mathbf{F}}} \coloneqq d\overset{a}{\mathbf{F}}/dt$, since it is defined entirely on the fixed material manifold $\mathcal{B}$.

where following (2.1) we use $\dot{\overset{\Theta}{\mathbf{F}}}\overset{\Theta}{\mathbf{F}}^{-1} = \boldsymbol{\alpha}\dot{\Theta}$. Which then allows us to isolate the anelastic rate of dissipation in (4.16) into an athermal anelastic contribution and a purely thermal contribution as

$$\dot{\eta} = \rho \overset{e}{\mathbf{F}}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} : \dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} + \rho \overset{n}{\mathbf{F}}^{\star} \overset{e}{\mathbf{F}}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-\star} : \boldsymbol{\alpha}\dot{\Theta} - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0 \,. \tag{4.18}$$

□

Remark 4.2. For an incompressible hyper-anelastic solid, the Legendre transform (3.8) is modified to take into account the constraints of elastic incompressibility ($\overset{e}{J} = 1$) and/or athermal anelastic incompressibility ($\overset{n}{J} = 1$) as[23]

$$\rho\Psi - \overset{e}{p}(\overset{e}{J} - 1) - \overset{n}{p}(\overset{n}{J} - 1) = \rho\mathscr{E} - \Theta\rho\mathscr{N} \,, \tag{4.19}$$

where $\overset{e}{p}(X,t)$ and $\overset{n}{p}(X,t)$ are the Lagrange multipliers associated with the respective incompressibility constraints. Computing $\dot{\overset{e}{J}}$ and $\dot{\overset{n}{J}}$, we find

$$\dot{\overset{e}{J}} = \overset{e}{J}\left(\mathbf{F}^{-1}\mathring{\mathbf{g}}^{\sharp}\mathbf{F}^{-\star}\right) : \overset{s}{\mathbf{D}} - \overset{e}{J}\overset{n}{\mathbf{F}}^{-\star} : \dot{\overset{n}{\mathbf{F}}} - \overset{e}{J}(\mathbf{I} : \boldsymbol{\alpha})\dot{\Theta} \,, \qquad \dot{\overset{n}{J}} = \overset{n}{J}\,\overset{n}{\mathbf{F}}^{-\star} : \dot{\overset{n}{\mathbf{F}}} \,. \tag{4.20}$$

Revisiting the proof for Proposition 4.1, the results remain unchanged except for the Doyle-Ericksen formula (4.9b) which is modified to read

$$\mathbf{P} = \rho \mathbf{g}^{\sharp} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{a}{\mathbf{F}}^{-\star} - \overset{e}{p}\, \mathbf{g}^{\sharp}\mathbf{F}^{-\star} \,, \tag{4.21}$$

and the dissipation inequality (4.9e), which is modified to read

$$\dot{\eta} = \left(\rho \overset{e}{\mathbf{F}}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} - (\overset{e}{p} - \overset{n}{p})\mathbf{I}\right) : \dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} + \left(\rho \overset{n}{\mathbf{F}}^{\star} \overset{e}{\mathbf{F}}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-\star} - \overset{e}{p}\mathbf{I}\right) : \boldsymbol{\alpha}\dot{\Theta} - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0 \,. \tag{4.22}$$

4.4 Heat and Kinetic Equations

Heat Equation. Let us first derive the heat equation to characterise the evolution of the temperature field in the case of a non-isothermal process. Using the balance laws derived in the previous subsection, and using the Legendre transform, the localised energy balance (4.2) reads

$$\rho\, \dot{\Psi} + \rho\, \dot{\Theta}\mathscr{N} + \rho\, \Theta\dot{\mathscr{N}} = (\mathbf{F}^{-1}\mathbf{P}) : \overset{s}{\mathbf{D}} + \rho R - \operatorname{Div} \mathbf{Q} \,. \tag{4.23}$$

Recalling (4.11) and (4.17), we may write

$$\begin{aligned}
\dot{\Psi} &= -\mathscr{N}\dot{\Theta} + \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{a}{\mathbf{F}}^{-\star} : \bar{\mathrm{D}}_t\mathbf{F} - \overset{e}{\mathbf{F}}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} : \dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} - \overset{n}{\mathbf{F}}^{\star}\overset{e}{\mathbf{F}}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}^{-\star} : \boldsymbol{\alpha}\dot{\Theta} \,, && \text{(4.24a)} \\
\dot{\mathscr{N}} &= -\frac{d}{dt}\frac{\partial \hat{\Psi}}{\partial \Theta} = -\frac{\partial^2 \Psi}{\partial \Theta^2}\dot{\Theta} - \frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta} \overset{a}{\mathbf{F}}^{-\star} : \bar{\mathrm{D}}_t\mathbf{F} + \overset{e}{\mathbf{F}}^{\star} \frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta} : \dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} \\
&\quad + \overset{n}{\mathbf{F}}^{\star}\overset{e}{\mathbf{F}}^{\star} \frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta} \overset{n}{\mathbf{F}}^{-\star} : \boldsymbol{\alpha}\dot{\Theta} \,. && \text{(4.24b)}
\end{aligned}$$

Substituting (4.24) into (4.23) and using the Doyle-Ericksen formula (4.9b), one finds

$$\begin{aligned}
-\rho\left[\Theta\frac{\partial^2 \Psi}{\partial \Theta^2} + \overset{n}{\mathbf{F}}^{\star}\overset{e}{\mathbf{F}}^{\star}\left(\frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} - \Theta\frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta}\right)\overset{n}{\mathbf{F}}^{-\star} : \boldsymbol{\alpha}\right]\dot{\Theta} - \rho\overset{e}{\mathbf{F}}^{\star}\left[\frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} - \Theta\frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta}\right] : \dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} \\
-\rho\,\Theta\frac{\partial^2 \Psi}{\partial \overset{e}{\mathbf{F}} \partial \Theta}\overset{a}{\mathbf{F}}^{-\star} : \bar{\mathrm{D}}_t\mathbf{F} = \rho R - \operatorname{Div} \mathbf{Q} \,.
\end{aligned} \tag{4.25}$$

[23] Thermal effects can indeed lead to volumetric changes; however, the evolution of temperature is governed by the heat equation, which inherently accounts for these effects. Thus, thermal incompressibility is not defined separately, as the temperature field already dictates the thermal response, including any volumetric changes.

We introduce the specific heat capacity at constant strain, and denote it by c_E, as the quantity of heat required to produce a unit temperature increase in a unit mass of material at constant strain. It is given by the following equation

$$\operatorname{Div}\mathbf{Q} = -\rho c_E\dot{\Theta}\,. \tag{4.26}$$

It follows by identification of (4.26) with (4.25) at constant strain, i.e., $\bar{\mathrm{D}}_t\mathbf{F} = \dot{\overset{n}{\mathbf{F}}} = \mathbf{0}$, and in the absence of external specific heat supply, i.e., $R = 0$, that

$$c_E = -\Theta\frac{\partial^2\Psi}{\partial\Theta^2} - \overset{n}{\mathbf{F}}^\star\overset{e}{\mathbf{F}}^\star\left(\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}} - \Theta\frac{\partial^2\Psi}{\partial\overset{e}{\mathbf{F}}\partial\Theta}\right)\overset{n}{\mathbf{F}}^{-\star}:\boldsymbol{\alpha}\,. \tag{4.27}$$

Therefore, we find the heat equation for a hyper-anelastic solid as follows

$$\rho c_E\dot{\Theta} = -\operatorname{Div}\mathbf{Q} + \rho\,\Theta\frac{\partial^2\Psi}{\partial\overset{e}{\mathbf{F}}\partial\Theta}\overset{a}{\mathbf{F}}^{-\star}:\bar{\mathrm{D}}_t\mathbf{F} + \rho\overset{e}{\mathbf{F}}^\star\left[\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}} - \Theta\frac{\partial^2\Psi}{\partial\overset{e}{\mathbf{F}}\partial\Theta}\right]:\dot{\overset{n}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-1} + \rho R\,. \tag{4.28}$$

Configurational Forces. In a hyper-anelastic solid, the rate of anelastic energy dissipation may be written as the sum of athermal and thermal contributions Le Tallec et al. [1993]; Maugin [2010]

$$\dot{\eta} = -\overset{n}{\mathbf{B}}:\dot{\overset{n}{\mathbf{F}}} - \overset{\Theta}{\mathrm{B}}\dot{\Theta} - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle\,, \tag{4.29}$$

where $\overset{n}{\mathbf{B}}$ and $\overset{\Theta}{\mathrm{B}}$ are the generalised configurational forces governing the athermal and thermal anelastic distortions, respectively. Noting the arbitrariness of the independent variables $\dot{\overset{n}{\mathbf{F}}}$ and $\dot{\Theta}$, it follows by identification of (4.9e) and (4.29) that

$$\overset{n}{\mathbf{B}} = -\rho\overset{e}{\mathbf{F}}^\star\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}\,, \tag{4.30a}$$

$$\overset{\Theta}{\mathrm{B}} = -\rho\overset{n}{\mathbf{F}}^\star\overset{e}{\mathbf{F}}^\star\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}:\boldsymbol{\alpha}\,. \tag{4.30b}$$

For an incompressible solid, these are modified to read

$$\overset{n}{\mathbf{B}} = -\rho\overset{e}{\mathbf{F}}^\star\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star} + (\overset{e}{p} - \overset{n}{p})\overset{n}{\mathbf{F}}^{-\star}\,, \tag{4.31a}$$

$$\overset{\Theta}{\mathrm{B}} = -\rho\overset{n}{\mathbf{F}}^\star\overset{e}{\mathbf{F}}^\star\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}:\boldsymbol{\alpha} + \overset{e}{p}\operatorname{tr}\boldsymbol{\alpha}\,. \tag{4.31b}$$

Remark 4.3. It can be seen from the conservation of mass (4.9a) that $\rho = \rho(X, \mathbf{G}, t)$, which then allows one to write $\partial\rho/\partial\Theta = 0$.[24] Hence, we may write that

$$\begin{aligned}\rho\frac{\partial^2\Psi}{\partial\overset{e}{\mathbf{F}}\partial\Theta}\overset{a}{\mathbf{F}}^{-\star} &= \frac{\partial}{\partial\Theta}\left[\rho\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\right]\overset{a}{\mathbf{F}}^{-\star} = \frac{\partial}{\partial\Theta}\left[\rho\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{a}{\mathbf{F}}^{-\star}\right] - \rho\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}\frac{\partial\overset{\Theta}{\mathbf{F}}^{-\star}}{\partial\Theta} \\ &= \mathbf{g}\frac{\partial\mathbf{P}}{\partial\Theta} - \rho\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}\overset{\Theta}{\mathbf{F}}^{-\star}\boldsymbol{\alpha}^{-\star} = \mathbf{g}\frac{\partial\mathbf{P}}{\partial\Theta} - \mathbf{g}\mathbf{P}\boldsymbol{\alpha}^{-\star}\,,\end{aligned} \tag{4.33a}$$

$$\rho\overset{e}{\mathbf{F}}^\star\frac{\partial^2\Psi}{\partial\overset{e}{\mathbf{F}}\partial\Theta}\overset{n}{\mathbf{F}}^{-\star} = \frac{\partial}{\partial\Theta}\left[\rho\overset{e}{\mathbf{F}}^\star\frac{\partial\Psi}{\partial\overset{e}{\mathbf{F}}}\overset{n}{\mathbf{F}}^{-\star}\right] = -\frac{\partial\overset{n}{\mathbf{B}}}{\partial\Theta}\,. \tag{4.33b}$$

Therefore, the heat equation (4.28) may be recast as

$$\rho c_E\dot{\Theta} = -\operatorname{Div}\mathbf{Q} + \Theta\mathbf{g}\left[\frac{\partial\mathbf{P}}{\partial\Theta} - \mathbf{P}\boldsymbol{\alpha}^{-\star}\right]:\bar{\mathrm{D}}_t\mathbf{F} + \left[\Theta\frac{\partial\overset{n}{\mathbf{B}}}{\partial\Theta} - \overset{n}{\mathbf{B}}\right]:\dot{\overset{n}{\mathbf{F}}} + \rho R\,. \tag{4.34}$$

[24]Note, however, that its total derivative does not necessarily vanish, since it does implicitly depend on temperature through it metric dependance. Its total derivative it computed as follows

$$\frac{d\rho}{d\Theta} = \frac{d\rho}{d\mathbf{G}}:\frac{d\mathbf{G}}{d\Theta} = \frac{d\rho}{d\mathbf{G}}:\frac{\partial(\overset{a}{\mathbf{F}}^\star\mathring{\mathbf{G}}\overset{a}{\mathbf{F}})}{\partial\Theta} = \frac{d\rho}{d\mathbf{G}}:\left(\boldsymbol{\alpha}^\star\overset{a}{\mathbf{F}}^\star\mathring{\mathbf{G}}\overset{a}{\mathbf{F}} + \overset{a}{\mathbf{F}}^\star\mathring{\mathbf{G}}\overset{a}{\mathbf{F}}\boldsymbol{\alpha}\right) = 2\frac{d\rho}{d\mathbf{G}}:\mathbf{G}\boldsymbol{\alpha} = 2\frac{d\rho}{d\mathbf{G}}:\boldsymbol{\alpha}^\flat\,, \tag{4.32}$$

where recalling (2.1), we use $\partial\overset{\Theta}{\mathbf{F}}/\partial\Theta = \overset{\Theta}{\mathbf{F}}\boldsymbol{\alpha}$.

Kinetic Equations. While the evolution of temperature—and consequently that of the thermal distortion $\overset{\theta}{\mathbf{F}}$—is known to be governed by the heat equation (4.28), one has yet to prescribe a constitutive model for the generalised configurational force $\overset{n}{\mathbf{B}}$ in order to complete the set of governing equations for hyper-anelasticity. Such a constitutive model may be achieved by assuming the existence of a dissipation potential density (a Rayleigh function) $\phi = \phi(X, \mathbf{F}, \overset{n}{\mathbf{F}}, \dot{\overset{n}{\mathbf{F}}}, \mathring{\mathbf{G}}, \mathbf{g})$, and the generalized configurational forces are given by

$$\overset{n}{\mathbf{B}} = -\frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}}\,. \tag{4.35}$$

It is assumed that the dissipation potential is convex with respect to $\dot{\overset{n}{\mathbf{F}}}$ Ziegler [1958]; Ziegler and Wehrli [1987]; Germain et al. [1983]; Goldstein et al. [2002]; Kumar and Lopez-Pamies [2016]. This is equivalent to

$$\left(\frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}_2} - \frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}_1}\right) : \left(\dot{\overset{n}{\mathbf{F}}}_2 - \dot{\overset{n}{\mathbf{F}}}_1\right) \geq 0\,, \tag{4.36}$$

for any $\dot{\overset{n}{\mathbf{F}}}_1$ and $\dot{\overset{n}{\mathbf{F}}}_2$. From (4.31a) and (4.35), we find a set of athermal kinetic equations in tensorial form as follows

$$\frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}} - \rho \overset{e}{\mathbf{F}}{}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}{}^{-\star} = 0\,. \tag{4.37}$$

For an incompressible solid, it is modified to read

$$\frac{\partial \phi}{\partial \dot{\overset{n}{\mathbf{F}}}} - \rho \overset{e}{\mathbf{F}}{}^{\star} \frac{\partial \Psi}{\partial \overset{e}{\mathbf{F}}} \overset{n}{\mathbf{F}}{}^{-\star} + (\overset{e}{p} - \overset{n}{p}) \overset{n}{\mathbf{F}}{}^{-\star} = 0\,. \tag{4.38}$$

5 Concluding Remarks

In this paper, we present a generalization of the Coleman-Noll procedure Coleman and Noll [1963] to derive the balance laws of nonlinear hyperelasticity and hyper-anelasticity, including conservation of mass, balance of linear and angular momenta, and the Doyle-Ericksen formula, directly from the first and second laws of thermodynamics. Notably, this is achieved without invoking assumptions of observer invariance or pre-supposing the balance laws themselves—as it is customary done in the classical Coleman-Noll procedure. By reframing the Clausius-Duhem inequality in terms of extended thermodynamic processes—herein defined, we demonstrate how the constitutive equations and balance laws emerge naturally within this framework.

Unlike the Green-Naghdi-Rivlin theorem Green and Rivlin [1964] and its variations Noll [1963]; Hughes and Marsden [1977], which derives the balance laws by postulating the invariance of the energy balance under superposed rigid body motions of the Euclidean ambient space, the procedure presented here relies solely on the first and second laws of thermodynamics. By reframing the Clausius-Duhem inequality in terms of extended thermodynamic processes, this approach eliminates the need for explicit symmetry assumptions or pre-supposed invariance principles. Notably, the Green-Naghdi-Rivlin theorem shares a conceptual foundation with Noether's theorem, as both link symmetries to conserved quantities. In the Green-Naghdi-Rivlin framework, the invariance under rigid body motions corresponds to the translational and rotational symmetries central to Noether's theorem Noether [1918]; Günther [1962]; Knowles et al. [1972], and the derived conservation laws—linear momentum, angular momentum, and energy—are direct outcomes of these symmetries. In contrast, the approach presented herein fundamentally differs from these ideas by deriving the constitutive equations and balance laws without reliance on symmetry or invariance.

The proposed approach not only provides a new perspective on the foundational principles of continuum mechanics but also extends the applicability of classical methods to scenarios where invariance principles are not readily defined, such as in non-Euclidean or evolving ambient spaces. This generalisation connects thermodynamics, geometry, and mechanics in a unified framework, paving the way for new insights in hyperelastic and hyper-anelastic materials.

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Appendix

A Thermal distortion

In the presence of a nonuniform temperature field $\Theta = \Theta(X,t)$, thermal distortion is generally incompatible Stojanovic et al. [1964]; Sadik and Yavari [2017b]. We denote the thermal distortion by $\overset{\theta}{\mathbf{F}}$; we let all the other anelastic effects be combined together in a distortion $\overset{n}{\mathbf{F}}$ such that the total anelastic distortion is written as $\overset{a}{\mathbf{F}} = \overset{n}{\mathbf{F}}\overset{\theta}{\mathbf{F}}$. At any given point X of the body, thermal expansion may be described by three stretch ratios $\{\zeta_1(X,\Theta), \zeta_2(X,\Theta), \zeta_3(X,\Theta)\}$ along three mutually independent directions forming a basis of unit vectors $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$.[25] The thermal distortion may be expressed as Lubarda [2002]; Ozakin and Yavari [2010]

$$\overset{\theta}{\mathbf{F}} = \sum_{K=1}^{3} \zeta_K \mathbf{E}_K \otimes \mathbf{E}^K \,, \tag{A.1}$$

[25]Note that $\{\mathbf{E}_K\}$ is not necessarily a coordinate basis and is not necessarily orthogonal.

where $\{\mathbf{E}^K\}$ is the dual basis to $\{\mathbf{E}_K\}$. The vector $\overset{\Theta}{\mathbf{E}}_K = \overset{\Theta}{\mathbf{F}}.\mathbf{E}_K = \zeta_K \mathbf{E}_K$ (with no summation over K) represents the thermal distortion along the unit basis vector $\mathbf{E}_K$. Hence, the linear thermal expansion coefficient of the material in the direction $\mathbf{E}_K$ is given by

$$\alpha_K(X,\Theta) = \frac{1}{\|\overset{\Theta}{\mathbf{E}}_K\|_{\mathring{\mathbf{G}}}} \frac{\partial}{\partial \Theta}\left[\|\overset{\Theta}{\mathbf{E}}_K\|_{\mathring{\mathbf{G}}}\right] = \frac{1}{\zeta_K}\frac{\partial \zeta_K}{\partial \Theta}\,. \tag{A.2}$$

It follows that $\zeta_K = \exp\left[\int_{\Theta_0}^{\Theta} \alpha_K(X,\Xi)\, d\Xi\right]$, where $\Theta_0 = \Theta_0(X)$ is the temperature distribution such that $\zeta_K(\Theta_0) = 1$, i.e., the temperature distribution for which the body does not experience any thermal distortion. Therefore, we may let $\zeta_K = e^{\omega_K}$, and the thermal distortion may be written as

$$\overset{\Theta}{\mathbf{F}} = \sum_{K=1}^{3} e^{\omega_K} \mathbf{E}_K \otimes \mathbf{E}^K\,, \tag{A.3}$$

where $\omega_K = \int_{\Theta_0}^{\Theta} \alpha_K(X,\Xi)\, d\Xi$. We construct the tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ such that their representations with respect to the basis $\{\mathbf{E}_K\}$ and its dual are given by

$$\boldsymbol{\alpha} = \sum_{K=1}^{3} \alpha_K \mathbf{E}_K \otimes \mathbf{E}^K\,, \qquad \boldsymbol{\omega} = \sum_{K=1}^{3} \omega_K \mathbf{E}_K \otimes \mathbf{E}^K\,. \tag{A.4}$$

One may hence write

$$\frac{\partial \overset{\Theta}{\mathbf{F}}}{\partial \Theta} = \sum_{K=1}^{3} \frac{\partial e^{\omega_K}}{\partial \Theta} \mathbf{E}_K \otimes \mathbf{E}^K = \sum_{K=1}^{3} \alpha_K\, e^{\omega_K}\, \mathbf{E}_K \otimes \mathbf{E}^K = \boldsymbol{\alpha}\overset{\Theta}{\mathbf{F}} = \overset{\Theta}{\mathbf{F}}\boldsymbol{\alpha}\,, \tag{A.5}$$

which yields

$$\boldsymbol{\alpha} = \frac{\partial \overset{\Theta}{\mathbf{F}}}{\partial \Theta}\overset{\Theta}{\mathbf{F}}^{-1} = \overset{\Theta}{\mathbf{F}}^{-1}\frac{\partial \overset{\Theta}{\mathbf{F}}}{\partial \Theta}\,. \tag{A.6}$$

Let us denote by $[\mathrm{A}^I{}_J]$ the transformation matrix between the coordinate basis $\{\frac{\partial}{\partial X^K}\}$ and the basis $\{\mathbf{E}_K\}$, i.e., $\mathbf{E}_K = \mathrm{A}^I{}_K\, \partial/\partial X^I$ and $\mathbf{E}^K = \mathrm{A}^{-K}{}_J\, dX^J$, it follows that one may write

$$\sum_{K=1}^{3} T_K\, \mathbf{E}_K \otimes \mathbf{E}^K = \sum_{K=1}^{3} \mathrm{A}^I{}_K\, T_K\, \mathrm{A}^{-K}{}_J\, \frac{\partial}{\partial X^I} \otimes dX^J\,, \tag{A.7}$$

for any triplet of numbers $\{T_1, T_2, T_3\}$. In particular, the coordinate representations of $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ in $\{X^K\}$ are hence respectively given by

$$\boldsymbol{\omega} = \omega^I{}_J \frac{\partial}{\partial X^I} \otimes dX^J\,, \qquad \boldsymbol{\alpha} = \alpha^I{}_J \frac{\partial}{\partial X^I} \otimes dX^J\,, \tag{A.8}$$

where $\omega^I{}_J = \sum_{K=1}^{3} \mathrm{A}^I{}_K\, \omega_K\, \mathrm{A}^{-K}{}_J$ and $\alpha^I{}_J = \sum_{K=1}^{3} \mathrm{A}^I{}_K\, \alpha_K\, \mathrm{A}^{-K}{}_J$. Note that it still holds in tensorial form that $\boldsymbol{\omega}(X,\Theta) = \int_{\Theta_0}^{\Theta} \boldsymbol{\alpha}(X,\Xi)\, d\Xi$. One may write

$$\begin{aligned}
\overset{\Theta}{\mathbf{F}}(X,\Theta) &= \sum_{K=1}^{3} \zeta_K\, \mathbf{E}_K \otimes \mathbf{E}^K = \sum_{K=1}^{3} \mathrm{A}^I{}_K\, \zeta_K\, \mathrm{A}^{-K}{}_J\, \frac{\partial}{\partial X^I} \otimes dX^J \\
&= \sum_{K=1}^{3} \mathrm{A}^I{}_K\, e^{\omega_K} \mathrm{A}^{-K}{}_J\, \frac{\partial}{\partial X^I} \otimes dX^J = \sum_{K=1}^{3} \exp\left[\mathrm{A}^I{}_K\, \omega_K\, \mathrm{A}^{-K}{}_J\right] \frac{\partial}{\partial X^I} \otimes dX^J \\
&= \sum_{K=1}^{3} \exp\left[\omega^I{}_J\right] \frac{\partial}{\partial X^I} \otimes dX^J = \exp\left[\boldsymbol{\omega}(X,\Theta)\right]\,.
\end{aligned} \tag{A.9}$$

Therefore

$$\overset{\Theta}{\mathbf{F}}(X,\Theta) = \exp\left[\boldsymbol{\omega}(X,\Theta)\right] = \exp\left[\int_{\Theta_0}^{\Theta} \boldsymbol{\alpha}(X,\Xi)\, d\Xi\right]\,. \tag{A.10}$$

Note that for an isothermal process, one has $\Theta(X,t) = \Theta_0(X)\,, \forall X \in \mathcal{B}\,, \forall t$, and consequently, by using (A.10), $\overset{\Theta}{\mathbf{F}} = \mathbf{I}$, the identity tensor.