Divergence of a vector field

Let $(M, \boldsymbol{g})$ be an oriented smooth Riemannian manifold. We denote by "d" the (de-Rham) exterior derivative. Note that "d" does not require any connection to be defined, it only uses the smooth structure of the manifold. It is actually defined as a map

$$d : \Omega^k(M) \to \Omega^{k+1}(M)\,,$$

where $\Omega^k(M)$ is the set of differential k-forms on M, such that:

- $\forall\, \alpha, \beta \in \Omega^k(M)\,, \forall\, a, b \in \mathbb{R}\ :\ d(a\alpha + b\beta) = a(d\alpha) + b(d\beta)$
- $\forall\, \alpha \in \Omega^k(M)\,, \forall \beta \in \Omega^l(M)\ :\ d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^k \wedge (\beta)$
- $d \circ d = 0$
- If $f \in C^\infty(M) = \Omega^0(M)$, then $df = f_*$ where $f_* : TM \to T\mathbb{R}$ denotes the induced push-forward map of f.

For a vector field $\boldsymbol{v}$ and a k-form α on M, we denote by $\iota_{\boldsymbol{v}}\alpha$, the contraction of $\boldsymbol{v}$ with the first index of α. Let $d\mathrm{Vol}_g$ be the Riemannian volume form of $(M, \boldsymbol{g})$. In a local coordinates system $\{x^a\}_{a \in \{1,\ldots,n\}}$, it writes $d\mathrm{Vol}_g = \sqrt{\det \boldsymbol{g}}\, dx^1 \wedge \ldots \wedge x^n$. Finally, we can define the divergence of a vector field $\boldsymbol{v}$ based only on the smooth structure and the Riemannian metric of the manifold to be the unique function satisfying

$$d\left(\iota_{\boldsymbol{v}}(d\mathrm{Vol}_g)\right) = (\mathrm{div}(\boldsymbol{v}))d\mathrm{Vol}_g\,.$$

It turns out that this definition is actually equivalent to the definition of the divergence with respect to the **Levi-Civita** connection which in a local coordinate system $\{x^a\}_{a \in \{1,\ldots,n\}}$ writes

$$\mathrm{div}\, \boldsymbol{v} = v^a{}_{|a} = \frac{\partial v^a}{\partial x^a} + \gamma^a_{ab} v^b /, .$$

If we define the divergence based on any different connection, then it might not agree with the above.