№3

$$\xi \sim p(x) = \frac{e^{-\frac{x}{\theta}}}{\theta}\{[0,+\infty)\} \quad \theta>0, \quad n=3$$

а) 1) $\tilde{\theta}_1 = \bar{x}$ ; $M[\tilde{\theta}_1] = \frac{1}{n}\sum_{i=1}^{n} M[x_i] = \frac{1}{n}\cdot nM\xi = \theta$ несмещ.

$$\bullet\ M[x_i] = M\xi = \int_{-\infty}^{\infty} x p(x)dx = \int_0^{\infty} x\frac{e^{-\frac{x}{\theta}}}{\theta}dx = \int_0^{\infty} x e^{-\frac{x}{\theta}} d\left(\frac{x}{\theta}\right) =$$

$$= \cancel{\theta} \int_0^{\infty} x de^{-\frac{x}{\theta}} = \cancel{-xe^{-\frac{x}{\theta}}\Big|_0^{\infty}} + \int_0^{\infty} e^{-\frac{x}{\theta}}dx = \theta\int_0^{\infty} e^{-\frac{x}{\theta}}d\frac{x}{\theta} =$$

$$= -\theta e^{-\frac{x}{\theta}}\Big|_0^{\infty} = \theta(e^0 - e^{-\infty}) = \theta$$

2) $\tilde{\theta}_2 = x_{(2)} \sim h(t) = p(t)\cdot nC_{n-1}^{k-1}(F(t))^{k-1}(1-F(t))^{n-k}$

$$F(x) = \int_{-\infty}^{x} p(t)dt = \int_0^{x}\frac{e^{-\frac{t}{\theta}}}{\theta}dt = -\int_0^{x} de^{-\frac{t}{\theta}} = -e^{-\frac{t}{\theta}}\Big|_0^{x} = e^0 - e^{-\frac{x}{\theta}} = 1-e^{-\frac{x}{\theta}}$$

$$h(t) = 3\cdot\frac{e^{-\frac{t}{\theta}}}{\theta}\cdot C_2^1\cdot(1-e^{-\frac{t}{\theta}})e^{-\frac{t}{\theta}}\{[0,+\infty)\} = 6\cdot\frac{e^{-\frac{2t}{\theta}}}{\theta}(1-e^{-\frac{t}{\theta}})\{[0,+\infty)\}$$

$$\bullet\ M[\tilde{\theta}_2] = \int_{-\infty}^{\infty} x h(x)dx = \int_0^{\infty} x\frac{6}{\theta}(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}})dx =$$

$$= 6\cdot\frac{1}{2}\int_0^{\infty} xe^{-\frac{2x}{\theta}}d\frac{2x}{\theta} - 2\int_0^{\infty} xe^{-\frac{3x}{\theta}}d\frac{3x}{\theta} = -3\int_0^{\infty} xde^{-\frac{2x}{\theta}} +$$

$$+2\int_0^{\infty} xde^{-\frac{3x}{\theta}} = -3\left(\cancel{xe^{-\frac{2x}{\theta}}\Big|_0^{\infty}} - \int_0^{\infty} e^{-\frac{2x}{\theta}}dx\right) + 2\left(\cancel{xe^{-\frac{3x}{\theta}}\Big|_0^{\infty}} -\right.$$

$$\left.-\int_0^{\infty} e^{-\frac{3x}{\theta}}dx\right) = 3\frac{\theta}{2}\int_0^{\infty} e^{-\frac{2x}{\theta}}d\frac{2x}{\theta} - 2\frac{\theta}{3}\int_0^{\infty} e^{-\frac{3x}{\theta}}d\frac{3x}{\theta} = -3\frac{\theta}{2}e^{-\frac{2x}{\theta}}\Big|_0^{\infty} +$$

$$+2\frac{\theta}{3}e^{-\frac{3x}{\theta}}\Big|_0^{\infty} = \frac{3\theta}{2}\cdot 1 + \frac{2\theta}{3}(-1) = \frac{5}{6}\theta \ \text{— смещ.}$$

$\tilde{\theta}_2' = \frac{6}{5}\tilde{\theta}_2$ , $M[\tilde{\theta}_2'] = \theta$ — несмещ.

$$\bullet\ D\xi = M\xi^2 - (M\xi)^2 = 2\theta^2 - \theta^2 = \theta^2$$

$$б) M\xi^2 = \int_{-\infty}^{\infty} x^2 p(x)dx = \int_0^{\infty} x^2\frac{e^{-\frac{x}{\theta}}}{\theta}dx = -\int_0^{\infty} x^2 de^{-\frac{x}{\theta}} = \cancel{-x^2e^{-\frac{x}{\theta}}\Big|_0^{\infty}} +$$

$+ 2\int\limits_0^\infty e^{-\frac{x}{\theta}} x dx = 2\theta^2$

$\cdot D[\tilde{\theta}_1] = \frac{1}{n^2}\sum\limits_{i=1}^n D\xi = \frac{1}{n} D\xi = \frac{\theta^2}{3}$

$\cdot M[\tilde{\theta}_2^2] - M[\tilde{\theta}_2]^2 = \frac{19}{18}\theta^2 - \frac{25}{36}\theta^2 = \frac{13}{36}\theta^2$

$\int\limits_{-\infty}^\infty x^2 h(x) dx = \int\limits_0^\infty x^2 \frac{6}{\theta}(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}) dx = -3\int\limits_0^\infty x^2 d e^{-\frac{2x}{\theta}} +$

$+ 2\int\limits_0^\infty x^2 d e^{-\frac{3x}{\theta}} = \cancel{-3x^2 e^{-\frac{2x}{\theta}}\Big|_0^\infty} + 3\cdot 2\int\limits_0^\infty x e^{-\frac{2x}{\theta}} dx + \cancel{2x^2 e^{-\frac{3x}{\theta}}\Big|_0^\infty} -$

$- 2\cdot 2\int\limits_0^\infty e^{-\frac{3x}{\theta}} x dx = -3\theta\int\limits_0^\infty x d e^{-\frac{2x}{\theta}} + 4\frac{\theta}{3}\int\limits_0^\infty x d e^{-\frac{3x}{\theta}} =$

$= \cancel{-3\theta x e^{-\frac{2x}{\theta}}\Big|_0^\infty} + 3\theta\int\limits_0^\infty e^{-\frac{2x}{\theta}} dx + \cancel{4\frac{\theta}{3} x e^{-\frac{3x}{\theta}}\Big|_0^\infty} - 4\frac{\theta}{3}\int\limits_0^\infty e^{-\frac{3x}{\theta}} dx =$

$= 3\frac{\theta^2}{2}\int\limits_0^\infty e^{-\frac{2x}{\theta}} d\frac{2x}{\theta} - 4\frac{\theta^2}{9}\int\limits_0^\infty e^{-\frac{3x}{\theta}} d\frac{3x}{\theta} = \frac{3\theta^2}{2} - \frac{4\theta^2}{9} = \frac{19}{18}\theta^2$

$\cdot D[\tilde{\theta}_2] = \frac{36}{25}\cdot\frac{13}{36}\theta^2 = \frac{13}{25}\theta^2 > \frac{\theta^2}{3} \quad \forall\theta > 0 \Rightarrow \tilde{\theta}_1$ эф. $\tilde{\theta}_2$

c) 1) $\cdot I(\theta) = \left[M\left(\frac{\partial \ln p(x)}{\partial\theta}\right)^2\right] = M\left[\left(\frac{\partial \ln \frac{e^{-\frac{x}{\theta}}}{\theta}}{\partial\theta}\right)^2\right] =$

$= M\left[\left(\frac{\theta}{e^{-\frac{x}{\theta}}}\cdot\frac{\theta\cdot x\frac{1}{\theta^2}e^{-\frac{x}{\theta}} - e^{-\frac{x}{\theta}}}{\theta^2}\right)^2\right] = M\left[\left(\frac{x-\theta}{\theta^2}\right)^2\right]$

$\circ \int\limits_{-\infty}^\infty \frac{\partial}{\partial\theta} p(x) dx \overset{\text{беск. дифф.}\ \theta>0}{=} \int\limits_{-\infty}^\infty \frac{\partial}{\partial\theta}\frac{e^{-\frac{x}{\theta}}}{\theta} dx = \int\limits_0^\infty\left[\frac{\theta x\frac{1}{\theta^2}e^{-\frac{x}{\theta}} - e^{-\frac{x}{\theta}}}{\theta^2}\right] dx =$

$= \int\limits_0^\infty \frac{\frac{1}{\theta}x - 1}{\theta^2} e^{-\frac{x}{\theta}} dx = -\int\limits_0^\infty \frac{x-\theta}{\theta^2} d e^{-\frac{x}{\theta}} = -\frac{x-\theta}{\theta^2} e^{-\frac{x}{\theta}}\Big|_0^\infty +$

$+ \frac{1}{\theta}\int\limits_0^\infty e^{-\frac{x}{\theta}} dx = -\frac{1}{\theta} - \frac{1}{\theta}\int\limits_0^\infty d e^{-\frac{x}{\theta}} = -\frac{1}{\theta} - \frac{1}{\theta} e^{-\frac{x}{\theta}}\Big|_0^\infty = \underline{\underline{0}}$

регулярность:

$\circ \int\limits_{-\infty}^\infty \frac{\partial^2}{\partial\theta^2} p(x) dx = \int\limits_{-\infty}^\infty \frac{\partial^2}{\partial\theta^2}\frac{e^{-\frac{x}{\theta}}}{\theta} dx = \int\limits_0^\infty \frac{\partial}{\partial\theta}\left(\frac{x-\theta}{\theta^3} e^{-\frac{x}{\theta}}\right) dx =$

$= \int\limits_0^\infty \frac{\partial}{\partial\theta}\left(\left(\frac{x}{\theta^3} - \frac{1}{\theta^2}\right) e^{-\frac{x}{\theta}}\right) dx = \int\limits_0^\infty \left(\left(-\frac{3x}{\theta^4} + \frac{2}{\theta^3}\right) + \left(\frac{x}{\theta^3} - \frac{1}{\theta^2}\right)\cdot\frac{1}{\theta^2}\cdot\right.$

$\left.\cdot e^{-\frac{x}{\theta}} dx\right) = \int\limits_0^\infty\left[-\frac{3x}{\theta^4} + \frac{2}{\theta^3} + \frac{x^2}{\theta^5} - \frac{x}{\theta^4}\right] e^{-\frac{x}{\theta}} dx = -\left[\frac{x^2}{\theta^4} - \frac{4x}{\theta^3} + \frac{2}{\theta^2}\right]\cdot$

$\cdot e^{-\frac{x}{\theta}}\Big|_0^\infty + \int\limits_0^\infty\left(\frac{2x}{\theta^4} - \frac{4}{\theta^3}\right) e^{-\frac{x}{\theta}} dx = \frac{2}{\theta^2} - \int\limits_0^\infty\left(\frac{2x}{\theta^3} - \frac{4}{\theta^2}\right) d e^{-\frac{x}{\theta}} = \frac{2}{\theta^2} - \left(\frac{2x}{\theta^3} - \frac{4}{\theta^2}\right)\cdot$

$$e^{-\frac{x}{\theta}}\Big|_0^\infty + \int_0^\infty e^{-\frac{x}{\theta}} \cdot \frac{2}{\theta^3}dx = -\frac{2}{\theta^2} + \frac{2}{\theta^2}\left(-\int_0^\infty d e^{-\frac{x}{\theta}}\right) = \underline{0}$$

$$\Rightarrow I(\theta) = -M\left[\frac{\partial^2 \ln p(x)}{\partial \theta^2}\right] = -M\left[\frac{\partial}{\partial\theta}\left(\frac{x-\theta}{\theta^2}\right)\right] =$$

$$= -M\left[\frac{\partial}{\partial\theta}\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)\right] = -M\left[-\frac{2x}{\theta^3} + \frac{1}{\theta^2}\right] = M\left[\frac{2x}{\theta^3} - \frac{1}{\theta^2}\right] =$$

$$= \int_{-\infty}^{\infty}\left(\frac{2x}{\theta^3} - \frac{1}{\theta^2}\right)\frac{e^{-\frac{x}{\theta}}}{\theta}dx = -\left(\frac{2x}{\theta^3} - \frac{1}{\theta^2}\right)e^{-\frac{x}{\theta}}\Big|_0^\infty + \int_0^\infty e^{-\frac{x}{\theta}}\frac{2}{\theta^3}dx =$$

$$= -\frac{1}{\theta^2} + \frac{2}{\theta^2}\int_0^\infty e^{-\frac{x}{\theta}} d\frac{x}{\theta} = -\frac{1}{\theta^2} - \frac{2}{\theta^2}e^{-\frac{x}{\theta}}\Big|_0^\infty = \frac{1}{\theta^2} > 0$$

модель <u>регулярна</u>

• $D[\tilde{\theta}_1] = \frac{\theta^2}{3}$ – огр., $\tilde{\theta}_1$ – несмещ. и модель регул. $\Rightarrow \tilde{\theta}_1$ – регул. по дост. усл.

$$D[\tilde{\theta}] \geq \frac{1}{nI(\theta)} = \frac{1}{3 \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{3} = D[\tilde{\theta}_1] = \inf D[\tilde{\theta}] \Rightarrow \tilde{\theta}_1$$

$\tilde{\theta}_1$ – <u>эф.</u>

2) $\tilde{\theta}_2$ – смещ. → нерегул. → <u>не</u> эф.

$\tilde{\theta}_2'$ – регулярна

• $D[\tilde{\theta}_2'] = \frac{13}{25}\theta^2$ – огр., $\tilde{\theta}_2'$ несмещ. и модель регул. $\Rightarrow \tilde{\theta}_2'$ – <u>регул.</u>

$$D[\tilde{\theta}] \geq \frac{1}{nI(\theta)} = \frac{\theta^2}{3} = D[\tilde{\theta}_1] < D[\tilde{\theta}_2'] = \frac{13}{25}\theta^2 \Rightarrow \tilde{\theta}_2' \text{ – } \underline{\text{не}} \text{ эф.}$$