№T1

$\xi \sim R(0,\theta)$ $\quad\theta>0$ $\quad$ вер. модель

$\vec{X}_n$ - выборка

$\tilde\theta_1 = 2\bar{x} = 2\frac{1}{n}\sum_{i=1}^n x_i$; $\tilde\theta_2 = X_{min}$; $\tilde\theta_3 = X_{max}$; $\tilde\theta_4 = x_1 + \frac{\sum_{k=2}^n x_k}{(n-1)}$

1) $\tilde\theta_1$: несмещ. $M[\tilde\theta_1]=\theta$

$x_i \sim R(0,\theta)$

• $M[2\frac{1}{n}\sum x_i] = \frac{2}{n} M[\sum x_i] = \frac{2}{n} \sum M[X_i] = 2M\xi = \theta$ — несмещ.

• $M[\xi] = \int_0^\theta x\frac{1}{\theta} dx = \frac{\theta}{2}$, $\quad p(x)=\frac{1}{\theta} \{(0,\theta)\}$

• $D[\tilde\theta_1] = D[\frac{2}{n}\sum x_i] = \frac{4}{n^2}D[\sum x_i] = \frac{4}{n^2}\sum Dx_i = \frac{4}{n} D\xi = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$

незав. сл. велич.

• $D\xi = M[\xi^2]-M^2[\xi] = \frac{\theta^2}{12}$

• $M[\xi^2]=\frac{\theta^2}{3}$; $M^2[\xi]=\frac{\theta^2}{4}$ $\quad$ по (T) - сост.

2) $\tilde\theta_2 = X_{min} = X_{(1)}$

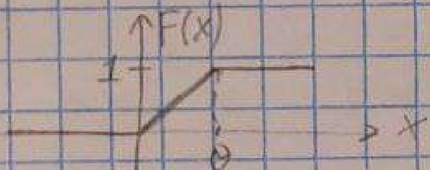
• $M[\tilde\theta_2] = \int_{-\infty}^{\infty} y\, q(y)dy = \int_0^\theta y n(1-\frac{y}{\theta})\frac{1}{\theta} dy = \left\{\begin{matrix} t = 1-\frac{y}{\theta} \\ y=(1-t)\theta\end{matrix}\right\} =$

$X_i\sim R(0,\theta)$

$X_i \sim F(y)$

$X_1 \sim 1-(1-F(y))^n$

$\varphi(y)$



$q(y) = n(1-F(y))^{n-1}F'(y) = n(1-\frac{y}{\theta})^{n-1}\frac{1}{\theta}\{(0,\theta)\}$

$= -\int_1^0 (1-t)\theta n t^{n-1}\frac{1}{\theta} \theta dt = n\theta(\int_0^1 t^{n-1}dt - \int_0^1 t^n dt) =$

$= n\theta\frac{1}{n} - n\theta\frac{1}{n+1} = \theta\frac{1}{n+1}$ - смещ.

$\tilde\theta_2' = (n+1)\tilde\theta_2 = (n-1)X_{min}$; $M[\tilde\theta_2']=\theta$ - несмещ.

• $D[\tilde\theta_2] = M[\tilde\theta_2^2] - M^2[\tilde\theta_2]$

• $M[\tilde\theta_2^2] = \int_{-\infty}^{\infty} y^2 q(y)dy = \int_0^\theta y^2 n(1-\frac{y}{\theta})^{n-1}\frac{1}{\theta} dy = \{t=1-\frac{y}{\theta}\} =$

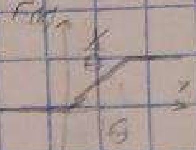
$= -\int_1^0 \theta^2(1-t)^2 n t^{n-1}\frac{1}{\theta}\theta dt = n\theta^2(\int_0^1 t^{n-1}dt - 2\int_0^1 t^n dt +$

$+\int_0^1 t^{n+1}dt) = n\theta^2\left(\frac{1}{n} - 2\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{n\theta^2}{(n+1)^2(n+2)} \xrightarrow{n\to\infty} 0$

смещ. дост. ус. не раб

- $D[\tilde{\theta}_2'] = D[(n+1)\tilde{\theta}_2] = \frac{\theta^2 n}{n+2} \underset{n\to\infty}{\not\to} 0$
- По опр. состоятельности: $\tilde{\theta}_2 \xrightarrow{P} 0 \ \forall\theta>0$

$\forall\varepsilon>0 \quad P(|\tilde{\theta}_2-\theta|\ge\varepsilon) \xrightarrow{n\to\infty} 0$



$\tilde{\theta}_2 \le \theta-\varepsilon \ / \ \theta+\varepsilon\le\tilde{\theta}_2$ $\qquad X_i \sim R(0,\theta)$

$P(X_{min} \ge \theta+\varepsilon) = 0$ $\qquad \varphi(y) = 1-(1-F(y))^n$

в силу непр

$P(X_{min} \le \theta-\varepsilon) = P(X_{min} < \theta-\varepsilon) = \varphi(\theta-\varepsilon) =$

$= 1-(1-F(\theta-\varepsilon))^n = 1-\left(1-\frac{\theta-\varepsilon}{\theta}\right)^n = 1-\left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow{n\to\infty} 1$ не сост.

- $\tilde{\theta}_2' \xrightarrow{P} \theta \ \forall\theta>0$

$\forall\varepsilon>0 \quad P(|\tilde{\theta}_2'-\theta|\ge\varepsilon) \xrightarrow{n\to\infty} 0$

$P(|(n+1)X_{min}-\theta|\ge\varepsilon) \ge P(X_{min}(n+1)\ge\theta+\varepsilon) = P\left(X_{min} \ge \frac{\theta+\varepsilon}{n+1}\right)$

$= 1-P\left(X_{min} < \frac{\theta+\varepsilon}{n+1}\right) = 1-\varphi\left(\frac{\theta+\varepsilon}{n+1}\right) = \left(1-F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n \underset{n>>}{=} \left(1-\frac{\theta+\varepsilon}{\theta(n+1)}\right)^n$

$\to e^{-\frac{\theta+\varepsilon}{\theta}} > 0$ — не сост.

3) $\tilde{\theta}_3 = X_{max}$

- $M[\tilde{\theta}_3] = \int_{-\infty}^{\infty} y\, q(y)\,dy = \int_0^\theta y^n \frac{n}{\theta^n}\,dy = \frac{n}{\theta^n}\frac{\theta^{n+1}}{n+1} = \frac{\theta n}{n+1}$ смещ.

$\Psi(y) = (F(y))^n$, $q(y) = \Psi'(y) = n(F(y))^{n-1}F'(y) = n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}\{(0,\theta)\}$

$\tilde{\theta}_3' = \frac{n+1}{n}X_{max}$, $M[\tilde{\theta}_3'] = \theta$

- $D[\tilde{\theta}_3] = M[\tilde{\theta}_3^2] - M^2[\tilde{\theta}_3] = \theta^2\frac{n}{n+2} - \theta^2\frac{n^2}{(n+1)^2} = \theta^2\frac{n}{(n+2)(n+1)^2}$

— несмещ.

- $M[\tilde{\theta}_3^2] = \int_0^\theta y^2 q(y)\,dy = \int_0^\theta n\frac{1}{\theta^n}\cdot y^{n+1}\,dy = \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \theta^2\frac{n}{n+2}$

- $D[\tilde{\theta}_3'] = D\left[\frac{n+1}{n}X_{max}\right] = \frac{\cancel{(n+1)^2}}{n^2}\theta^2\frac{\cancel{n}}{(n+2)\cancel{(n+1)^2}} \xrightarrow{n\to 0} 0$ смещ.

По опр: $\tilde{\theta}_3 \xrightarrow{P} \theta \quad \forall \theta > 0$

$\forall \varepsilon > 0 \quad P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) \xrightarrow{n \to \infty} 0$

$\tilde{\theta}_3 = X_{max} \qquad \theta + \varepsilon \leq X_{max} \leq \theta - \varepsilon$

$P(X_{max} \geq \theta + \varepsilon) = 0$

$P(X_{max}) \leq \theta - \varepsilon) = P(X_{max} < \theta - \varepsilon) = \Psi(\theta - \varepsilon) = \begin{cases} \left(\dfrac{\theta - \varepsilon}{\theta}\right)^n, & 0 < \varepsilon < \theta \\ 0, & \varepsilon \geq \theta \end{cases}$

$= \begin{cases} \left(1 - \dfrac{\varepsilon}{\theta}\right)^n, & 0 < \varepsilon < \theta \\ 0, & \varepsilon \geq \theta \end{cases} \xrightarrow{n \to \infty} 0$ — <u>сост.</u>

4) $\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$

- $M[\tilde{\theta}_4] = M[x_1 + \frac{1}{n-1}\sum x_i] = M[x_1] + \frac{1}{n-1}\sum M[x_i] =$

$= M\xi + M\xi = 2M\xi = \theta$ не смещ.

- $D[\tilde{\theta}_4] = D[x_1] = \frac{1}{(n-1)^2}\sum_{i=2}^{n} Dx_i = \frac{\theta^2}{12} + \frac{1}{n-1}\frac{\theta^2}{12} \xrightarrow{n\to\infty} \frac{\theta^2}{12}$ не раб. дост. усл.

По опр. $\tilde{\theta}_4 \xrightarrow{P} \theta$

$\xi_n \xrightarrow{P} \xi$, $\eta_n \xrightarrow{P} \eta$, тогда $\xi_n + \eta_n \xrightarrow{P} \xi + \eta$

ЗБЧ Хинчина: ξ_i незав. и одинак. распр. и $M\xi_i < \infty$, тогда $\frac{1}{n}\sum_{i=1}^{n}\xi_i \xrightarrow{P} M\xi_i$

$\tilde{\theta}_4 = \underbrace{x_1}_{\xi_n} + \underbrace{\frac{1}{n-1}\sum_{i=2}^{n} x_i}_{\eta_n} \xrightarrow{P} \xi + \frac{\theta}{2}$ – не сост.

$\xi_n \xrightarrow{P} x_1$, $\eta_n \xrightarrow{P} \overset{Mx_i}{M\xi} = \frac{\theta}{2}$

) Сравнение оценок

$D\tilde{\theta}_1 = \frac{\theta^2}{3n}$ $\quad \frac{1}{3n} > \frac{1}{(n+2)n} \to n+2 > 3 \to n > 1 \Rightarrow \tilde{\theta}_3' \text{ эф. } \tilde{\theta}_1$

$D\tilde{\theta}_3' = \frac{\theta^2}{(n+2)n}$

при $n=1$: $\hat{\theta} = \sqrt{2}x_1$ оптим.