

PL-embedding the dual of J^2 -gems into \mathbb{S}^3 by an $O(n^2)$ -algorithm *

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Abstract

Let be given a *colored 3-pseudo-triangulation* \mathcal{H}^* with n tetrahedra. Colored means that each tetrahedron have vertices distinctively colored 0,1,2,3. In a *pseudo* 3-triangulation the intersection of simplices might be subsets of simplices of smaller dimensions, instead of singletons of such faces, as for true triangulations. If \mathcal{H}^* is the dual of a J^2 -gem (shortly defined), then we show that $|\mathcal{H}^*|$ is \mathbb{S}^3 and we make available an $O(n^2)$ -algorithm to produce a PL-embedding ([10]) of \mathcal{H}^* into \mathbb{S}^3 . This is rather surprising because such PL-embeddings are often of exponential size. This work is the first step towards obtaining, via an $O(n^2)$ -algorithm, a framed link presentation inducing the same closed orientable 3-manifold as the one given by a colored pseudo-triangulation. Previous work on this topic appear in [5], [6] and [7]. However, the exposition and new proofs of this paper are meant to be entirely self-contained.

1 Introduction

1.1 J^2 -gems

A J^2 -gem is a 4-regular, 4-edge-colored planar graph \mathcal{H} obtained from the intersection pattern of two Jordan curves X and Y with $2n$ transversal crossings. These crossings define consecutive segments of X alternatively inside Y and outside Y . Color the first type 2 and the second type 3. The crossings also define consecutive segments of Y alternatively inside X and outside X . Color the first type 0 and the second type 1. This defines a 4-regular 4-edge-colored graph \mathcal{H} where the vertices are the crossings and the edges are the colored segments. Let \mathcal{H}^* be the 3-dimensional abstract 3-complex formed by taking a set of vertex colored tetrahedra in 1–1 correspondence with the set of vertices of \mathcal{H} , $V(\mathcal{H})$, so that each tetrahedra has vertices of colors 0,1,2,3. This vertex coloring induces a face coloring of the triangular faces of the tetrahedron: color i the face opposite to the vertex colored i . For each i -colored edge of \mathcal{H} with ends u and v paste the corresponding tetrahedra ∇_u and ∇_v so as to paste the two triangular faces that do not contain a vertex of color i in such a way as to match vertices of the other three colors. We show that the topological space $|K|$ induced by \mathcal{H}^* is \mathbb{S}^3 . Moreover we describe an $O(n^2)$ -algorithm to make available a PL-embedding ([10]) of \mathcal{H}^* into \mathbb{S}^3 . We get explicit coordinates in \mathbb{S}^3 for the 0-simplices and the p -simplices ($p \in \{1, 2, 3\}$) are linear simplices in the spherical geometry.

1.2 Gems and their duals

A $(3+1)$ -graph \mathcal{H} is a connected regular graph of degree 4 where to each vertex there are four incident differently colored edges in the color set $\{0, 1, 2, 3\}$. For $I \subseteq \{0, 1, 2, 3\}$, an I -residue is a component of the subgraph induced by the I -colored edges. Denote by $v(\mathcal{H})$ the number of 0-residues (vertices) of \mathcal{H} . For $0 \leq i < j \leq 3$, an $\{i, j\}$ -residue is also called an ij -gon or an i - and j -colored *bigon* (it is an even polygon, where the edges are alternatively colored i and j). Denote by $b(\mathcal{H})$ the total number of ij -gons for $0 \leq i < j \leq 3$. Denote by $t(\mathcal{H})$ the total number of \bar{i} -residues for $0 \leq i \leq 3$, where \bar{i} means complement of $\{i\}$ in $\{0, 1, 2, 3\}$.

We briefly recall the definition of gems taken from [4]. A 3-gem is a $(3+1)$ -graph \mathcal{H} satisfying $v(\mathcal{H}) + t(\mathcal{H}) = b(\mathcal{H})$. This relation is equivalent to having the vertices, edges and bigons restricted to any $\{i, j, k\}$ -residue inducing a plane graph where the faces are bounded by the bigons. Therefore we can embed each such $\{i, j, k\}$ -residue into a sphere \mathbb{S}^2 . We consider the ball bounded this \mathbb{S}^2 as induced by the $\{i, j, k\}$ -residue. For this reason an $\{i, j, k\}$ -residue in a 3-gem, $i < j < k$, is also called a *triball*. An ij -gon appears once in the boundary

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of triball $\{i, j, k\}$ and once in the boundary of triball $\{i, j, h\}$. By pasting the triballs along disks bounded by all the pairs of ij -gons, $\{i, j\} \subset \{0, 1, 2, 3\}$ of a gem \mathcal{H} , we obtain a closed 3-manifold denoted by $|\mathcal{H}|$. This general construction is dual to the one exemplified in the abstract and produces any closed 3-manifold. The manifold is orientable if and only if \mathcal{H} is bipartite, [8]. A *crystallization* is a gem which remains connected after deleting all the edges of any given color, that is, it has one $\{i, j, k\}$ -residue for each trio of colors $\{i, j, k\} \subset \{0, 1, 2, 3\}$.

Let \mathcal{H}^* be the dual of a gem \mathcal{H} . An \bar{i} -residue of \mathcal{H} corresponds in \mathcal{H}^* to a 0-simplex of \mathcal{H}^* . Most 0-simplices of \mathcal{H}^* do not correspond to \bar{i} -residues of \mathcal{H} . An ij -gon of a gem \mathcal{H} corresponds in \mathcal{H}^* to a *PL1-face* formed by a sequence of 1-simplices of \mathcal{H}^* ; this PL1-face is the intersection of two PL2-faces of colors i and j ; their two bounding 0-simplices correspond to an \bar{h} - and to a \bar{k} -residue, where $\{h, i, j, k\} = \{0, 1, 2, 3\}$. An i -colored edge of \mathcal{H} corresponds to a *PL2-face* which is a 2-disk triangulated by a subset of i -colored 2-simplices of \mathcal{H}^* . Finally to a vertex of \mathcal{H} , it corresponds a *PL3-face* of \mathcal{H}^* which is a 3-ball formed by a subset of 3-simplices of \mathcal{H}^* .

(1.1) Proposition. *The 3-manifold induced by a J^2 -gem \mathcal{H} is \mathbb{S}^3 .*

Proof. Removing from \mathcal{H} all the edges of a given color still yields a connected graph which a plane graph and they come embedded so that the faces are the 2-residues. So \mathcal{H} has four 3-residues, one of each type. Denote by b_{ij} the number of ij -gons of \mathcal{H} . Each one of these residues are planar graphs having $v = 2n$ vertices, $3v/2$ edges and $b_{12} + b_{13} + b_{23}$, $b_{02} + b_{03} + b_{23}$, $b_{01} + b_{13} + b_{03}$ and $b_{12} + b_{01} + b_{02}$ faces for, respectively, the $\bar{0}$ -, $\bar{1}$ -, $\bar{2}$ -, $\bar{3}$ -residue. Adding the four formulas for the Euler characteristic of the sphere imply that $v(\mathcal{H}) + 4 = b(\mathcal{H})$. Therefore, \mathcal{H} is a crystallization having one $0i$ -gon and one jk -gon. This implies that the fundamental group of the induced manifold is trivial: as proved in [3], the fundamental group of the space induced by a crystallization is generated by $b_{0i} - 1$ generators, and in our case this number is 0. Since Poincaré Conjecture is now proved, we are done. However, we can avoid using this fact and, as a bonus, obtaining the validity of the next corollary, which is used in the sequel.

Assume that \mathcal{H} is a J^2 -gem which does not induce \mathbb{S}^3 and has the smallest possible number of vertices satisfying these assumptions. By planarity we must have a pair of edges of \mathcal{H} having the same ends $\{p, q\}$. Consider the graph $\mathcal{H}fus\{p, q\}$ obtained from \mathcal{H} by removing the vertices p, q and the 2 edges linking them as well as welding the 2 pairs of pendant edges along edges of the same color. In [2] S. Lins proves that if \mathcal{H} is a gem, $\mathcal{H}' = \mathcal{H}fus\{p, q\}$ is also a gem and that two exclusive relations hold regarding $|\mathcal{H}|$ and $|\mathcal{H}'|$, their induced 3-manifolds: either $|\mathcal{H}| = |\mathcal{H}'|$ in the case that $\{p, q\}$ induces a 2-dipole or else $|\mathcal{H}| = |\mathcal{H}'| \# (\mathbb{S}^2 \times \mathbb{S}^1)$. Since \mathcal{H}' is a J^2 -gem, by our minimality hypothesis on \mathcal{H} the valid alternative is the second. But this is a contradiction: the fundamental group of $|\mathcal{H}|$ would not be trivial, because of the summand $\mathbb{S}^2 \times \mathbb{S}^1$. \square

1.3 Dipoles, pillows and balloons

Suppose there are m edges linking vertices x and y of a gem, $m \in \{1, 2, 3\}$. We say that $\{u, v\}$ is an m -*dipole* if removing all edges in the colors of the ones linking x to y , these vertices are in distinct components of the graph induced by the edges in the complementary set of colors. To *cancel the dipole* means deleting the subgraph induced by $\{u, v\}$ and identify pairs of the hanging edges along the same remaining color. To *create the dipole* is the inverse operation. It is simple to prove that the manifold of a gem is invariant under dipole cancellation or creation. Even though is not relevant for the present work state the foundational result on gems: two 3-manifolds are homeomorphic if and only if any two gems inducing it are linked by a finite number of cancellations and creations of dipoles, [1, 9].

The dual of a 2-dipole $\{u, v\}$, with internal colors i, j is named a *pillow*. It consists of two PL3-faces ∇_u and ∇_v sharing two PL2-faces colored i and j . The *thickening of a 2-dipole into a 3-dipole* is defined as follows. Let i, j be the two colors internal to 2-dipole $\{u, v\}$ and k a third color. Let a be the k -neighbor of x and b be the k -neighbor of y . Remove edges $[a, x]$ and $[b, y]$ and put back k -edges $[u, v]$ and $[r, s]$. This completes the thickening. It is simple to prove that the thickening in a gem produces a gem. We must be carefull because the inverse blind inverse operation *thinning a 3-dipole* not always produces a gem. The catch is that the result of the thinning perhaps is not a 2-dipole. In this sense the thinning move is not local: we must make sure that the result is a 2-dipole. To the data needed in thinning a 3-dipole $\{u, v\}$ with internal colors $\{i, j, k\}$ we must add the k -edge $[r, s]$. Note that the k -edges $[u, v]$ and $[r, s]$ are in the same hk -gon, where h is the fourth color. Denote by Δ_{rs} the dual of $[r, s]$. Let $\nabla_u \cup \nabla_v \cup \Delta_{rs}$ be called a *balloon*. Note that it consists of 2 PL3-faces ∇_u and ∇_v sharing three PL2-faces in colors $\{i, j, k\}$ together with a k -colored PL2-face whose intersection with $\nabla_u \cup \nabla_v$ is a PL1-face corresponding to the dual of the hk -gon, where h is the fourth color. Let $\nabla_u \cup \nabla_v$ be the *balloon's head* and let Δ_{rs} be the *balloon's tail*.

1.4 The Strategy for obtaining the PL-embedding we seek

We want to find a PL-embedding for the dual, \mathcal{H}^* , of a J^2 -gem \mathcal{H} into \mathbb{S}^3 . To this end we remove one PL3-face of \mathcal{H}^* find in \mathbb{R}^3 a PL-embedding of the complementary ball so that their union form triangulated tetrahedron. After we use the inverse of a stereographic projection with center in further in the exterior of the triangulated tetrahedron. In this way we recover in \mathbb{S}^3 the missing PL3-face.

In this work we describe the PL-embedded PL3-faces of \mathcal{H}^* into \mathbb{R}^3 by making it geometrically clear that its boundary is a set of 4 PL2-faces, one of each color, forming an embedded \mathbb{S}^2 whose interior is disjoint from the interior of \mathbb{S}^2 's corresponding to others PL3-faces. Thus, for our purposes it will be only necessary to embed the 2-skeleton of \mathcal{H}^* .

A direct approach to find the PL-embedding of the dual of a general J^2 -gem with $2n$ vertices, seems very hard. We split the algorithm into 3 phases. First we find a sequence of $n - 1$ 2-dipole thickenings into 3-dipoles so that the final gem is simply a circular arrangement of n 3-dipoles with internal colors 0, 1, 2. Such a canonical n -gem is named a *bloboïd* and is denoted \mathcal{B}_n . This sequence is easily obtainable from the primal objects, the gems: $\mathcal{H} = \mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{n-1} = \mathcal{B}_n$. In the second phase we find specific abstract PL-triangulations of the PL2-faces for the inverse sequence of colored 2-dimensional PL-complexes: $\mathcal{B}_n^* = \mathcal{H}_{n-1}^*, \dots, \mathcal{H}_2^*, \mathcal{H}_1^* = \mathcal{H}^*$. At the same time that we get we get also another sequence $\mathcal{W}_{n-1}^*, \dots, \mathcal{W}_2^*, \mathcal{W}_1^* = \mathcal{W}^*$. Each term of this \mathcal{W} -sequence is called a *wing* and it is a section of the previous \mathcal{H} -sequence by two adequate semi-planes. Each wing \mathcal{W}_m has an associated *nervure*, denoted \mathcal{N}_m . Each nervure is an auxiliary spanning tree which helps in finding canonical embeddings for the graph $\mathcal{U}_m = \mathcal{W}_m \cup \mathcal{N}_m$ ($m = n - 1, \dots, 1$), in two fixed semi-planes: Π_ℓ and Π_r . The third phase uses the wings and the abstract 2-dimensional PL-complexes to produce a new sequence $\mathcal{H}_1^\diamond, \mathcal{H}_2^\diamond, \dots, \mathcal{H}_n^\diamond = \mathcal{H}^*$. In this phase everything is embedded into \mathbb{R}^3 and the last element is the PL-embedding that we seek.

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2 Abstract 2-dimensional PL-complexes, their wings and nervures

3 Diamond complexes

4 Further work

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