PL-embedding the dual of J^2-gems into $\mathbb{S}^3$ by an $O(n^2)$-algorithm *

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May 6, 2013

Abstract

Let be given a *colored 3-pseudo-triangulation* K with n tetrahedra. Colored means that each tetrahedron have vertices distinctively colored 0,1,2,3. In a *pseudo* 3-triangulation the intersection of simplices might be subsets of simplices of smaller dimensions, instead of singletons of such faces, as for true triangulations. If K is the dual of a J^2-gem (shortly defined), then we show that $|K|$ is $\mathbb{S}^3$ and we make available an $O(n^2)$-algorithm to produce a PL-embedding ([6]) of K into $\mathbb{S}^3$. This is rather surprising because such PL-embeddings are often of exponential size. This work is the first step towards obtaining, via an $O(n^2)$-algorithm, a framed link presentation inducing the same closed orientable 3-manifold as the one given by a colored pseudo-triangulation. Previous work on this topic appear in [2], [3] and [4]. However, the exposition and new proofs of this paper are meant to be entirely self-contained.

1 Introduction

A J^2*-gem* is a 4-regular, 4-edge-colored planar graph G obtained from the intersection pattern of two Jordan curves X and Y with $2n$ transversal crossings. These crossings define consecutive segments of X alternatively inside Y and outside Y. Color the first type 2 and the second type 3. The crossings also define consecutive segments of Y alternatively inside X and outside X. Color the first type 0 and the second type 1. This defines a 4-regular 4-edge-colored graph G where the vertices are the crossings and the edges are the colored colored segments. Let K be the 3-dimensional abstract 3-complex formed by taking a set of vertex colored tetrahedra in 1–1 correspondence with $V(G)$ so that each tetrahedra has vertices of colors 0,1,2,3. For each i-colored edge of G with ends u and v paste the corresponding tetrahedra ∇_u and ∇_v so as to paste the two triangular faces that do not contain a vertex of color i in such a way as to to match vertices of the other three colors. We show that the topological space $|K|$ induced by K is $\mathbb{S}^3$. Moreover we describe an $O(n^2)$-algorithm to make available a PL-embedding ([6]) of $G^\star$ into $\mathbb{S}^3$. We get explicit coordinates in $\mathbb{S}^3$ for the 0-simplices and the p-simplices ($p \in \{1, 2, 3\}$) are linear simplices in the spherical geometry.

2 Gems and its duals

A *(3+1)-graph* G is a connected regular graph of degree 4 where to each vertex there are four incident differently colored edges in the color set $\{0, 1, 2, 3\}$. For $I \subseteq \{0, 1, 2, 3\}$, an I*-residue* is a component of the subgraph induced by the I-colored edges. Denote by $v(G)$ the number of 0-residues (vertices) of G. For $0 \le i < j \le 3$, an $\{i, j\}$-residue is also called an ij*-gon* or an i- and j-colored *bigon* (it is an even polygon, where the edges are alternatively colored i and j). Denote by $b(G)$ the total number of ij-gons for $0 \le i < j \le 3$. Denote by $t(G)$ the total number of $\bar{i}$-residues for $0 \le i \le 3$, where $\bar{i}$ means complement of $\{i\}$ in $\{0, 1, 2, 3\}$.

We briefly recall the definition of gems taken from [1]. A 3*-gem* is a $(3+1)$-graph G satisfying $v(G)+t(G) = b(G)$. This relation is equivalent to having the vertices, edges and bigons restricted to any $\{i, j, k\}$-residue inducing a plane graph where the faces are bounded by the bigons. Therefore we can embed each such $\{i, j, k\}$-residue into a sphere $\mathbb{S}^2$. We consider the ball bounded this $\mathbb{S}^2$ as induced by the $\{i, j, k\}$-residue. For this reason an $\{i, j, k\}$-residue in a 3-gem, $i < j < k$, is also called a *triball*. An ij-gon appears once in the boundary of triball $\{i, j, k\}$ and once in the boundary of triball $\{i, j, h\}$. By pasting the triballs along disks bounded by all the pairs of ij-gons, $\{i, j\} \subset \{0, 1, 2, 3\}$ of a gem G, we obtain a closed 3-manifold denoted by $|G|$. This general

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary) construction is dual to the one exemplified in the abstract and produces any closed 3-manifold. The manifold is orientable if and only if G is bipartite, [5].

Let K be the dual of a gem G. An $\bar{i}$-residue of G corresponds in K to a 0-simplex of K. Most 0-simplices of K do not correspond to $\bar{i}$-residues of G. An ij-gon of a gem G corresponds in K to a *PL1-face* formed by a sequence of 1-simplices of K; this PL1-face is the intersection of two PL2-faces of colors i and j. An i-colored edge of G corresponds to a *PL2-face* which is a 2-disk triangulated by a subset of i-colored 2-simplices of K. Finally to a vertex of G, it corresponds a *PL3-face* of K which is a 3-ball formed by a subset of 3-simplices of K.

In this work we describe the embedded $PL3$-faces of K by making it geometrically clear that its boundary is a set of 4 PL2-faces, one of each color forming an embedded $\mathbb{S}^2$ whose interior is disjoint from the interior of $\mathbb{S}^2$'s corresponding to others $PL3$-faces. Thus, for our purposes it will be only necessary to embed the 2-skeleton of K. Our strategy to embed K into $\mathbb{S}^3$ is removing one of its $PL3$-faces and obtain an embedding of the remaining ball into $\mathbb{R}^3$ after coming back to $\mathbb{S}^3$ by an inverse stereographic projection with its center in the exterior of the triangulated ball.

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