PL-embedding the dual of J^2-gems into $\mathbb{S}^3$ by an $O(n^2)$-algorithm *

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Abstract

Let be given a *colored 3-pseudo-triangulation* $\mathcal{H}^\star$ with n tetrahedra. Colored means that each tetrahedron have vertices distinctively colored 0,1,2,3. In a *pseudo* 3-triangulation the intersection of simplices might be subsets of simplices of smaller dimensions, instead of singletons of such faces, as for true triangulations. If $\mathcal{H}^\star$ is the dual of a J^2-gem (shortly defined), then we show that $|\mathcal{H}^\star|$ is $\mathbb{S}^3$ and we make available an $O(n^2)$-algorithm to produce a PL-embedding ([10]) of $\mathcal{H}^\star$ into $\mathbb{S}^3$. This is rather surprising because such PL-embeddings are often of exponential size. This work is the first step towards obtaining, via an $O(n^2)$-algorithm, a framed link presentation inducing the same closed orientable 3-manifold as the one given by a colored pseudo-triangulation. Previous work on this topic appear in [5], [6] and [7]. However, the exposition and the new proofs of this paper are meant to be entirely self-contained.

1 Introduction

1.1 J^2-gems

A J^2*-gem* is a 4-regular, 4-edge-colored planar graph $\mathcal{H}$ obtained from the intersection pattern of two Jordan curves X and Y with $2n$ transversal crossings. These crossings define consecutive segments of X alternatively inside Y and outside Y. Color the first type 2 and the second type 3. The crossings also define consecutive segments of Y alternatively inside X and outside X. Color the first type 0 and the second type 1. This defines a 4-regular 4-edge-colored graph $\mathcal{H}$ where the vertices are the crossings and the edges are the colored colored segments. Let $\mathcal{H}^\star$ be the 3-dimensional abstract 3-complex formed by taking a set of vertex colored tetrahedra in 1–1 correspondence with the set of vertices of $\mathcal{H}$, $V(\mathcal{H})$, so that each tetrahedra has vertices of colors 0,1,2,3. This vertex coloring induces a face coloring of the triangular faces of the tetrahedron: color i the face opposite to the vertex colored i. For each i-colored edge of $\mathcal{H}$ with ends u and v paste the corresponding tetrahedra ∇_u and ∇_v so as to paste the two triangular faces that do not contain a vertex of color i in such a way as to to match vertices of the other three colors. We show that the topological space $|K|$ induced by $\mathcal{H}^\star$ is $\mathbb{S}^3$. Moreover we describe an $O(n^2)$-algorithm to make available a PL-embedding ([10]) of $\mathcal{H}^\star$ into $\mathbb{S}^3$. We get explicit coordinates in $\mathbb{S}^3$ for the 0-simplices and the p-simplices ($p \in \{1,2,3\}$) are linear simplices in the spherical geometry.

1.2 Gems and their duals

A *(3+1)-graph* $\mathcal{H}$ is a connected regular graph of degree 4 where to each vertex there are four incident differently colored edges in the color set $\{0,1,2,3\}$. For $I \subseteq \{0,1,2,3\}$, an I*-residue* is a component of the subgraph induced by the I-colored edges. Denote by $v(\mathcal{H})$ the number of 0-residues (vertices) of $\mathcal{H}$. For $0 \le i < j \le 3$, an $\{i,j\}$-residue is also called an ij*-gon* or an i- and j-colored *bigon* (it is an even polygon, where the edges are alternatively colored i and j). Denote by $b(\mathcal{H})$ the total number of ij-gons for $0 \le i < j \le 3$. Denote by $t(\mathcal{H})$ the total number of $\bar{i}$-residues for $0 \le i \le 3$, where $\bar{i}$ means complement of $\{i\}$ in $\{0,1,2,3\}$.

We briefly recall the definition of gems taken from [4]. A 3-*gem* is a $(3+1)$-graph $\mathcal{H}$ satisfying $v(\mathcal{H})+t(\mathcal{H}) = b(\mathcal{H})$. This relation is equivalent to having the vertices, edges and bigons restricted to any $\{i,j,k\}$-residue inducing a plane graph where the faces are bounded by the bigons. Therefore we can embed each such $\{i,j,k\}$-residue into a sphere $\mathbb{S}^2$. We consider the ball bounded this $\mathbb{S}^2$ as induced by the $\{i,j,k\}$-residue. For this reason an $\{i,j,k\}$-residue in a 3-gem, $i<j<k$, is also called a *triball*. An ij-gon appears once in the boundary

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary) of triball $\{i,j,k\}$ and once in the boundary of triball $\{i,j,h\}$. By pasting the triballs along disks bounded by all the pairs of ij-gons, $\{i,j\} \subset \{0,1,2,3\}$ of a gem $\mathcal{H}$, we obtain a closed 3-manifold denoted by $|\mathcal{H}|$. This general construction is dual to the one exemplified in the abstract and produces any closed 3-manifold. The manifold is orientable if and only if $\mathcal{H}$ is bipartite, [8]. A *crystallization* is a gem which remains connected after deleting all the edges of any given color, that is, it has one $\{i,j,k\}$-residue for each trio of colors $\{i,j,k\} \subset \{0,1,2,3\}$.

Let $\mathcal{H}^\star$ be the dual of a gem $\mathcal{H}$. An $\bar{i}$-residue of $\mathcal{H}$ corresponds in $\mathcal{H}^\star$ to a 0-simplex of $\mathcal{H}^\star$. Most 0-simplices of $\mathcal{H}^\star$ do not correspond to $\bar{i}$-residues of $\mathcal{H}$. An ij-gon of a gem $\mathcal{H}$ corresponds in $\mathcal{H}^\star$ to a *PL1-face* formed by a sequence of 1-simplices of $\mathcal{H}^\star$; this PL1-face is the intersection of two PL2-faces of colors i and j; their two bounding 0-simplices correspond to an $\bar{h}$- and to a $\bar{k}$-residue, where $\{h,i,j,k\} = \{0,1,2,3\}$. An i-colored edge of $\mathcal{H}$ corresponds to a *PL2-face* which is a 2-disk triangulated by a subset of i-colored 2-simplices of $\mathcal{H}^\star$. Finally to a vertex of $\mathcal{H}$, it corresponds a *PL3-face* of $\mathcal{H}^\star$ which is a 3-ball formed by a subset of 3-simplices of $\mathcal{H}^\star$.

(1.1) Proposition. *The* 3*-manifold induced by a* J^2*-gem* $\mathcal{H}$ *is* $\mathbb{S}^3$.

Proof. Removing from $\mathcal{H}$ all the edges of a given color still yields a connected graph which a plane graph and they come embedded so that the faces are the 2-residues. So $\mathcal{H}$ has four 3-residues, one of each type. Denote by b_{ij} the number of ij-gons of $\mathcal{H}$. Each one of these residues are planar graphs having $v = 2n$ vertices, $3v/2$ edges and $b_{12}+b_{13}+b_{23}$, $b_{02}+b_{03}+b_{23}$, $b_{01}+b_{13}+b_{03}$ and $b_{12}+b_{01}+b_{02}$ faces for, respectively, the $\bar{0}$-, $\bar{1}$-, $\bar{2}$-, $\bar{3}$-residue. Adding the four formulas for the Euler characteristic of the sphere imply that $v(\mathcal{H})+4 = b(\mathcal{H})$. Therefore, $\mathcal{H}$ is a crystallization having one $0i$-gon and one jk-gon. This implies that the fundamental group of the induced manifold is trivial: as proved in [3], the fundamental group of the space induced by a crystallization is generated by $b_{0i}-1$ generators, and in our case this number is 0. Since Poincaré Conjecture is now proved, we are done. However, we can avoid using this fact and, as a bonus, obtaining the validity of the next corollary, which is used in the sequel.

Assume that $\mathcal{H}$ is a J^2-gem which does not induce $\mathbb{S}^3$ and has the smallest possible number of vertices satisfying these assumptions. By planarity we must have a pair of edges of $\mathcal{H}$ having the same ends $\{p,q\}$. Consider the graph $\mathcal{H}fus\{p,q\}$ obtained from $\mathcal{H}$ by removing the vertices p, q and the 2 edges linking them as well as welding the 2 pairs of pendant edges along edges of the same color. In [2] S. Lins proves that if $\mathcal{H}$ is a gem, $\mathcal{H}' = \mathcal{H}fus\{p,q\}$ is also a gem and that two exclusive relations hold regarding $|\mathcal{H}|$ and $|\mathcal{H}'|$, their induced 3-manifolds: either $|\mathcal{H}| = |\mathcal{H}'|$ in the case that $\{p,q\}$ induces a 2-dipole or else $|\mathcal{H}| = |\mathcal{H}'|\#(\mathbb{S}^2 \times \mathbb{S}^1)$. Since $\mathcal{H}'$ is a J^2-gem, by our minimality hypothesis on $\mathcal{H}$ the valid alternative is the second. But this is a contradiction: the fundamental group of $|\mathcal{H}|$ would not be trivial, because of the summand $\mathbb{S}^2 \times \mathbb{S}^1$. $\square$

1.3 Dipoles, pillows and balloons

Suppose there are m edges linking vertices x and y of a gem, $m \in \{1,2,3\}$. We say that $\{u,v\}$ is an *m-dipole* if removing all edges in the colors of the ones linking x to y, these vertices are in distinct components of the graph induced by the edges in the complementary set of colors. To *cancel the dipole* means deleting the subgraph induced by $\{u,v\}$ and identify pairs of the hanging edges along the same remaining color. To *create the dipole* is the inverse operation. It is simple to prove that the manifold of a gem is invariant under dipole cancellation or creation. Even though is not relevant for the present work state the foundational result on gems: two 3-manifolds are homeomorphic if and only if any two gems inducing it are linked by a finite number of cancellations and creations of dipoles, [1, 9].

The dual of a 2-dipole $\{u,v\}$, with internal colors i,j is named a *pillow*. It consistes of two PL3-faces ∇_u and ∇_v sharing two PL2-faces colored i and j. The *thickening of a 2-dipole into a 3-dipole* is defined as follows. Let i,j be the two colors internal to 2-dipole $\{u,v\}$ and k a third color. Let a be the k-neighboor of x and b be the k-neighboor of y. Remove edges $[a,x]$ and $[b,y]$ and put back k-edges $[u,v]$ and $[r,s]$. This completes the thickening. It is simple to prove that the thickening in a gem produces a gem. We must be carefull because the inverse blind inverse operation *thinning a 3-dipole* not always produces a gem. The catch is that the result of the thinning perhaps is not a 2-dipole. In this sense the thinning move is not local: we must make sure that the result is a 2-dipole. To the data needed in thinning a 3-dipole $\{u,v\}$ with internal colors $\{i,j,k\}$ we must add the k-edge $[r,s]$. Note that the k-edges $[u,v]$ and $[r,s]$ are in the same hk-gon, where h is the fourth color. Denote by Δ_{rs} the dual of $[r,s]$. Let $\nabla_u \cup \nabla v \cup \Delta_{rs}$ be called a *balloon.* Note that it consists of 2 PL3-faces ∇_u and ∇_v sharing three PL2-faces in colors $\{i,j,k\}$ together with a k-colored PL2-face whose intersection with $\nabla_u \cup \nabla_v$ is a PL1-face corresponding to the dual of the hk-gon, where h is the fourth color. Let $\nabla_u \cup \nabla_v$ be the *balloon's head* and let Δ_{rs} be the *balloon's tail.*

1.4 The Strategy for obtaining the PL-embedding of the dual of a J^2-gem

We want to find a PL-embedding for the dual, $\mathcal{H}^\star$, of a J^2-gem $\mathcal{H}$ into $\mathbb{S}^3$. To this end we remove one PL3-face of $\mathcal{H}^\star$ find in $\mathbb{R}^3$ a PL-embedding of the complementary ball so that their union form PL-triangulated tetrahedron. After we use the inverse of a stereographic projection with center in the exterior of the triangulated tetrahedron. In this way we recover in $\mathbb{S}^3$ the missing PL3-face.

In this work we describe the PL-embedded PL3-faces of $\mathcal{H}^\star$ into $\mathbb{R}^3$ by making it geometrically clear that its boundary is a set of 4 PL2-faces, one of each color, forming an embedded $\mathbb{S}^2$ whose interior is disjoint from the interior of $\mathbb{S}^2$'s corresponding to others PL3-faces. Thus, for our purposes it will be only necessary to embed the 2-skeleton of $\mathcal{H}^\star$.

A direct approach to find the PL-embedding of the dual of a general J^2-gem with $2n$ vertices, seems very hard. We split the algorithm into 4 phases.

In the first phase we find a sequence of $n-1$ 2-dipole thickenings into 3-dipoles so that the final gem is simply a circular arrangement of n 3-dipoles with internal colors $0, 1, 2$. Such a canonical n-gem is named a *bloboid* and is denoted $\mathcal{B}_n$. This indexing decreasing sequence is easily obtainable from the primal objects, in the case, simplifying J^2B-gems:

$$\mathcal{H} = \mathcal{H}_n \xrightarrow[thick_1]{2dip} \mathcal{H}_{n-1} \xrightarrow[thick_2]{2dip} \cdots \xrightarrow[thick_{n-1}]{2dip} \mathcal{H}_1 = \mathcal{B}_n = \mathcal{B}.$$

In the second phase we first find (phase 2A) specific abstract PL-triangulations, for the PL2-faces for the index increasing sequence of abstract colored 2-dimensional PL-complexes. Each of these complexes, later, are going to be PL-embedded into $\mathbb{R}^3$ so that the PL2-faces are topologically 2-spheres with disjoint interior. Attaching 3-balls bounded by these spheres we get the dual of the J^2-gem $\mathcal{H}$ (with a vertex removed).

$$\mathcal{B}^\star = \mathcal{H}_1^\star \xrightarrow[move_1]{bp} \mathcal{H}_2^\star \xrightarrow[move_2]{bp} \cdots \xrightarrow[move_{n-1}]{bp} \mathcal{H}_n^\star = \mathcal{H}^\star.$$

In parallel to the construction of the sequence $\mathcal{H}_m^\star$'s we also construct (phase 2B) a sequence of *wings* $\mathcal{W}_m^\star$'s and their *nervures* $\mathcal{N}_m^\star$'s so that $\mathcal{W}_m^\star \cup \mathcal{N}_m^\star$ is an easy to define adequate planar graph:

$$\mathcal{W}_1^\star \cup \mathcal{N}_1^\star \xrightarrow[move_1]{wbp} \mathcal{W}_2^\star \cup \mathcal{N}_2^\star \xrightarrow[move_2]{wbp} \cdots \xrightarrow[move_{n-1}]{wbp} \mathcal{W}_n^\star \cup \mathcal{N}_n^\star = \mathcal{W}^\star \cup \mathcal{N}^\star.$$

Each wing $\mathcal{W}_m^\star$'s corresponds to a section of the previous sequence $\mathcal{H}_m^\star$'s by two adequated fixed semi-planes Π_ℓ and Π_r. The construction of the planar graphs $\mathcal{W}_m^\star \cup \mathcal{N}_m^\star$ is recursive. Initially $\mathcal{W}_1^\star$ is a pincel of lines and $\mathcal{N}_1^\star$ is $\varnothing$. Going from $\mathcal{W}_{m-1}^\star \cup \mathcal{N}_{m-1}^\star$ to $\mathcal{W}_m^\star \cup \mathcal{N}_m^\star$ is very simple: two new vertices and four new edges appear, so as to maintain planarity.

In the third phase we make the abstract final element $\mathcal{W}^\star \cup \mathcal{N}^\star$ of the second phase *rectilinearly* (that is each edge is a straihgt line segment) PL-embedded. By a cone construction we obtain from the rectilinearly PL-embedded $\mathcal{W}^\star \cup \mathcal{N}^\star$ a special PL-complex, named $\mathcal{H}_1^\diamond$. This complex does not correpond to a gem dual and it can be loosely explained as $\mathcal{H}_1^\star$ with all balloon's heads "opened".

The fourth phase, the *pillow filling phase* starts with $\mathcal{H}_1^\diamond$ and the uses the abstract sequence $\mathcal{H}_m^\star$'s to produce a pillow filling sequence

$$\mathcal{H}_1^\diamond \xrightarrow[filling_1]{pillow} \mathcal{H}_2^\diamond \xrightarrow[filling_2]{pillow} \cdots \xrightarrow[filling_{n-1}]{pillow} \mathcal{H}_n^\diamond = \mathcal{H}^\star.$$

In this phase everything is embedded into $\mathbb{R}^3$ and the last element is the PL-embedding that we seek.

The whole procedure can be implemented as a formal algorithm that takes $O(n^2)$-space and $O(n^2)$-time complexity, where $2n$ is the number of vertices of the original J^2-gem.

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2 Details of the whole construction

2.1 First phase: from a J^2-gem $\mathcal{H}$ to a bloboid $\mathcal{B}$

(2.1) Proposition. *Starting with a J^2-gem $\mathcal{J}^2$ with $2n$ vertices we can arrive to an n-bloboid $\mathcal{B}$ by means of $n-1$ operations which thickens a 2-dipole into a 3-dipole, producing a sequence of J^2B-gems each inducing $\mathbb{S}^3$,*

$$(\mathcal{J}^2 = \mathcal{H}_n, \mathcal{H}_{n-1}, \ldots, \mathcal{H}_2, \mathcal{H}_1 = \mathcal{B}).$$

Proof. The proof is by induction. For $\ell = n$ we have $\mathcal{H}_n = \mathcal{J}^2$ and so it is a J^2B-gem, establishing the basis of the induction. Assume that $\mathcal{H}_\ell$ is a J^2B-gem. For $\ell > 1$, let $\mathcal{H}'_\ell$ denote $\mathcal{H}_\ell$ after cancelling the blobs. Since $\mathcal{H}'_\ell$ is a J^2-gem by the Jordan curve theorem a 2-dipole is present in it. The same 2-dipole is also present in $\mathcal{H}_\ell$. Therefore it can be thickened, defining J^2B-gem $\mathcal{H}_{\ell-1}$, which establishes the inductive step. In face of Proposition 1.1 and from the fact that thickening dipoles on gems produce gems inducing the same manifold, every member of the sequence induces $\mathbb{S}^3$. $\square$

2.2 Second phase: colored abstract complexes, their wings and nervures

2.3 Third phase: a rectilinear PL-embedding for $\mathcal{W}^\star \cup \mathcal{N}^\star$ and its induced $\mathcal{H}_1^\diamond$

2.4 Fourth phase: filling the pillows to obtain the PL-embedding of $\mathcal{H}^\star$ we seek

3 Conclusion and further work

References

[1] M. Ferri and C. Gagliardi. Crystallisation moves. *Pacific J. Math*, 100(1):85–103, 1982.

[2] S. Lins. A simple proof of Gagliardi's handle recognition theorem. *Discrete mathematics*, 57(3):253–260, 1985.

[3] S. Lins. On the fundamental group of 3-gems and a planar class of 3-manifolds. *European Journal of Combinatorics*, 9(4):291–305, 1988.

[4] S. Lins. *Gems, Computers, and Attractors for 3-Manifolds*. World Scientific, 1995.

[5] S. Lins and R. Machado. Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, I: gems and their duals. *arXiv:1211.1953v2 [math.GT]*, 2012.

[6] S. Lins and R. Machado. Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, II: colored complexes and boundings in their complexity. *arXiv:1212.0826v2 [math.GT]*, 2012.

[7] S. Lins and R. Machado. Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, III: geometric complex $\mathcal{H}_n^\star$ embedded into $\mathbb{R}^3$. *arXiv:1212.0827v2 [math.GT]*, 2012.

[8] S. Lins and A. Mandel. Graph-encoded 3-manifolds. *Discrete Math.*, 57(3):261–284, 1985.

[9] S. Lins and M. Mulazzani. Blobs and flips on gems. *Journal of Knot Theory and its Ramifications*, 15(8):1001–1035, 2006.

[10] C.P. Rourke and B.J. Sanderson. *Introduction to piecewise-linear topology*, volume 69. Springer-Verlag, 1982.

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