PL-embedding the dual of J^2-gems into $\mathbb{S}^3$ by an $O(n^2)$-algorithm *

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Abstract

Let be given a *colored 3-pseudo-triangulation* $\mathcal{H}^\star$ with n tetrahedra. Colored means that each tetrahedron have vertices distinctively colored 0,1,2,3. In a *pseudo* 3-triangulation the intersection of simplices might be subsets of simplices of smaller dimensions, instead of singletons of such faces, as for true triangulations. If $\mathcal{H}^\star$ is the dual of a J^2-gem (shortly defined), then we show that $|\mathcal{H}^\star|$ is $\mathbb{S}^3$ and we make available an $O(n^2)$-algorithm to produce a PL-embedding ([10]) of $\mathcal{H}^\star$ into $\mathbb{S}^3$. This is rather surprising because such PL-embeddings are often of exponential size. This work is the first step towards obtaining, via an $O(n^2)$-algorithm, a framed link presentation inducing the same closed orientable 3-manifold as the one given by a colored pseudo-triangulation. Previous work on this topic appear in [5], [6] and [7]. However, the exposition and new proofs of this paper are meant to be entirely self-contained.

1 Introduction

1.1 J^2-gems

A J^2*-gem* is a 4-regular, 4-edge-colored planar graph $\mathcal{H}$ obtained from the intersection pattern of two Jordan curves X and Y with $2n$ transversal crossings. These crossings define consecutive segments of X alternatively inside Y and outside Y. Color the first type 2 and the second type 3. The crossings also define consecutive segments of Y alternatively inside X and outside X. Color the first type 0 and the second type 1. This defines a 4-regular 4-edge-colored graph $\mathcal{H}$ where the vertices are the crossings and the edges are the colored colored segments. Let $\mathcal{H}^\star$ be the 3-dimensional abstract 3-complex formed by taking a set of vertex colored tetrahedra in 1–1 correspondence with the set of vertices of $\mathcal{H}$, $V(\mathcal{H})$, so that each tetrahedra has vertices of colors 0,1,2,3. This vertex coloring induces a face coloring of the triangular faces of the tetrahedron: color i the face opposite to the vertex colored i. For each i-colored edge of $\mathcal{H}$ with ends u and v paste the corresponding tetrahedra ∇_u and ∇_v so as to paste the two triangular faces that do not contain a vertex of color i in such a way as to to match vertices of the other three colors. We show that the topological space $|K|$ induced by $\mathcal{H}^\star$ is $\mathbb{S}^3$. Moreover we describe an $O(n^2)$-algorithm to make available a PL-embedding ([10]) of $\mathcal{H}^\star$ into $\mathbb{S}^3$. We get explicit coordinates in $\mathbb{S}^3$ for the 0-simplices and the p-simplices ($p \in \{1,2,3\}$) are linear simplices in the spherical geometry.

1.2 Gems and their duals

A *(3+1)-graph* $\mathcal{H}$ is a connected regular graph of degree 4 where to each vertex there are four incident differently colored edges in the color set $\{0,1,2,3\}$. For $I \subseteq \{0,1,2,3\}$, an I*-residue* is a component of the subgraph induced by the I-colored edges. Denote by $v(\mathcal{H})$ the number of 0-residues (vertices) of $\mathcal{H}$. For $0 \le i < j \le 3$, an $\{i,j\}$-residue is also called an ij*-gon* or an i- and j-colored *bigon* (it is an even polygon, where the edges are alternatively colored i and j). Denote by $b(\mathcal{H})$ the total number of ij-gons for $0 \le i < j \le 3$. Denote by $t(\mathcal{H})$ the total number of $\bar{i}$-residues for $0 \le i \le 3$, where $\bar{i}$ means complement of $\{i\}$ in $\{0,1,2,3\}$.

We briefly recall the definition of gems taken from [4]. A 3-*gem* is a $(3+1)$-graph $\mathcal{H}$ satisfying $v(\mathcal{H}) + t(\mathcal{H}) = b(\mathcal{H})$. This relation is equivalent to having the vertices, edges and bigons restricted to any $\{i,j,k\}$-residue inducing a plane graph where the faces are bounded by the bigons. Therefore we can embed each such $\{i,j,k\}$-residue into a sphere $\mathbb{S}^2$. We consider the ball bounded this $\mathbb{S}^2$ as induced by the $\{i,j,k\}$-residue. For this reason an $\{i,j,k\}$-residue in a 3-gem, $i<j<k$, is also called a *triball*. An ij-gon appears once in the boundary

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of triball $\{i, j, k\}$ and once in the boundary of triball $\{i, j, h\}$. By pasting the triballs along disks bounded by all the pairs of ij-gons, $\{i, j\} \subset \{0, 1, 2, 3\}$ of a gem $\mathcal{H}$, we obtain a closed 3-manifold denoted by $|\mathcal{H}|$. This general construction is dual to the one exemplified in the abstract and produces any closed 3-manifold. The manifold is orientable if and only if $\mathcal{H}$ is bipartite, [8]. A *crystallization* is a gem which remains connected after deleting all the edges of any given color, that is, it has one $\{i, j, k\}$-residue for each trio of colors $\{i, j, k\} \subset \{0, 1, 2, 3\}$.

Let $\mathcal{H}^\star$ be the dual of a gem $\mathcal{H}$. An $\bar{i}$-residue of $\mathcal{H}$ corresponds in $\mathcal{H}^\star$ to a 0-simplex of $\mathcal{H}^\star$. Most 0-simplices of $\mathcal{H}^\star$ do not correspond to $\bar{i}$-residues of $\mathcal{H}$. An ij-gon of a gem $\mathcal{H}$ corresponds in $\mathcal{H}^\star$ to a *PL1-face* formed by a sequence of 1-simplices of $\mathcal{H}^\star$; this PL1-face is the intersection of two PL2-faces of colors i and j; their two bounding 0-simplices correspond to an $\bar{h}$- and to a $\bar{k}$-residue, where $\{h, i, j, k\} = \{0, 1, 2, 3\}$. An i-colored edge of $\mathcal{H}$ corresponds to a *PL2-face* which is a 2-disk triangulated by a subset of i-colored 2-simplices of $\mathcal{H}^\star$. Finally to a vertex of $\mathcal{H}$, it corresponds a *PL3-face* of $\mathcal{H}^\star$ which is a 3-ball formed by a subset of 3-simplices of $\mathcal{H}^\star$.

(1.1) Proposition. *The* 3*-manifold induced by a* J^2*-gem* $\mathcal{H}$ *is* $\mathbb{S}^3$.

Proof. Removing from $\mathcal{H}$ all the edges of a given color still yields a connected graph which a plane graph and they come embedded so that the faces are the 2-residues. So $\mathcal{H}$ has four 3-residues, one of each type. Denote by b_{ij} the number of ij-gons of $\mathcal{H}$. Each one of these residues are planar graphs having $v = 2n$ vertices, $3v/2$ edges and $b_{12} + b_{13} + b_{23}$, $b_{02} + b_{03} + b_{23}$, $b_{01} + b_{13} + b_{03}$ and $b_{12} + b_{01} + b_{02}$ faces for, respectively, the $\bar{0}$-, $\bar{1}$-, $\bar{2}$-, $\bar{3}$-residue. Adding the four formulas for the Euler characteristic of the sphere imply that $v(\mathcal{H}) + 4 = b(\mathcal{H})$. Therefore, $\mathcal{H}$ is a crystallization having one $0i$-gon and one jk-gon. This implies that the fundamental group of the induced manifold is trivial: as proved in [3], the fundamental group of the space induced by a crystallization is generated by $b_{0i} - 1$ generators, and in our case this number is 0. Since Poincaré Conjecture is now proved, we are done. However, we can avoid using this fact and, as a bonus, obtaining the validity of the next corollary, which is used in the sequel.

Assume that $\mathcal{H}$ is a J^2-gem which does not induce $\mathbb{S}^3$ and has the smallest possible number of vertices satisfying these assumptions. By planarity we must have a pair of edges of $\mathcal{H}$ having the same ends $\{p, q\}$. Consider the graph $\mathcal{H}fus\{p, q\}$ obtained from $\mathcal{H}$ by removing the vertices p, q and the 2 edges linking them as well as welding the 2 pairs of pendant edges along edges of the same color. In [2] S. Lins proves that if $\mathcal{H}$ is a gem, $\mathcal{H}' = \mathcal{H}fus\{p, q\}$ is also a gem and that two exclusive relations hold regarding $|\mathcal{H}|$ and $|\mathcal{H}'|$, their induced 3-manifolds: either $|\mathcal{H}| = |\mathcal{H}'|$ in the case that $\{p, q\}$ induces a 2-dipole or else $|\mathcal{H}| = |\mathcal{H}'|\#(\mathbb{S}^2 \times \mathbb{S}^1)$. Since $\mathcal{H}'$ is a J^2-gem, by our minimality hypothesis on $\mathcal{H}$ the valid alternative is the second. But this is a contradiction: the fundamental group of $|\mathcal{H}|$ would not be trivial, because of the summand $\mathbb{S}^2 \times \mathbb{S}^1$. □

1.3 Dipoles, pillows and balloons

Suppose there are m edges linking vertices x and y of a gem, $m \in \{1, 2, 3\}$. We say that $\{u, v\}$ is an *m-dipole* if removing all edges in the colors of the ones linking x to y, these vertices are in distinct components of the graph induced by the edges in the complementary set of colors. To *cancel the dipole* means deleting the subgraph induced by $\{u, v\}$ and identify pairs of the hanging edges along the same remaining color. To *create the dipole* is the inverse operation. It is simple to prove that the manifold of a gem is invariant under dipole cancellation or creation. Even though is not relevant for the present work state the foundational result on gems: two 3-manifolds are homeomorphic if and only if any two gems inducing it are linked by a finite number of cancellations and creations of dipoles, [1, 9].

The dual of a 2-dipole $\{u, v\}$, with internal colors i, j is named a *pillow*. It consistes of two PL3-faces ∇_u and ∇_v sharing two PL2-faces colored i and j. The *thickening of a 2-dipole into a 3-dipole* is defined as follows. Let i, j be the two colors internal to 2-dipole $\{u, v\}$ and k a third color. Let a be the k-neighboor of x and b be the k-neighboor of y. Remove edges $[a, x]$ and $[b, y]$ and put back k-edges $[u, v]$ and $[r, s]$. This completes the thickening. It is simple to prove that the thickening in a gem produces a gem. We must be carefull because the inverse blind inverse operation *thinning a 3-dipole* not always produces a gem. The catch is that the result of the thinning perhaps is not a 2-dipole. In this sense the thinning move is not local: we must make sure that the result is a 2-dipole. To the data needed in thinning a 3-dipole $\{u, v\}$ with internal colors $\{i, j, k\}$ we must add the k-edge $[r, s]$. Note that the k-edges $[u, v]$ and $[r, s]$ are in the same hk-gon, where h is the fourth color. Denote by Δ_{rs} the dual of $[r, s]$. Let $\nabla_u \cup \nabla v \cup \Delta_{rs}$ be called a *balloon.* Note that it consists of 2 PL3-faces ∇_u and ∇_v sharing three PL2-faces in colors $\{i, j, k\}$ together with a k-colored PL2-face whose intersection with $\nabla_u \cup \nabla_v$ is a PL1-face corresponding to the dual of the hk-gon, where h is the fourth color. Let $\nabla_u \cup \nabla_v$ be the *balloon's head* and let Δ_{rs} be the *balloon's tail.*

1.4 The Strategy for obtaining the PL-embedding of the dual of a J^2-gem

We want to find a PL-embedding for the dual, $\mathcal{H}^\star$, of a J^2-gem $\mathcal{H}$ into $\mathbb{S}^3$. To this end we remove one PL3-face of $\mathcal{H}^\star$ find in $\mathbb{R}^3$ a PL-embedding of the complementary ball so that their union form triangulated tetrahedron. After we use the inverse of a stereographic projection with center in further in the exterior of the triangulated tetrahedron. In this way we recover in $\mathbb{S}^3$ the missing PL3-face.

In this work we describe the PL-embedded PL3-faces of $\mathcal{H}^\star$ into $\mathbb{R}^3$ by making it geometrically clear that its boundary is a set of 4 PL2-faces, one of each color, forming an embedded $\mathbb{S}^2$ whose interior is disjoint from the interior of $\mathbb{S}^2$'s corresponding to others PL3-faces. Thus, for our purposes it will be only necessary to embed the 2-skeleton of $\mathcal{H}^\star$.

A direct approach to find the PL-embedding of the dual of a general J^2-gem with $2n$ vertices, seems very hard. We split the algorithm into 3 phases. First we find a sequence of $n-1$ 2-dipole thickenings into 3-dipoles so that the final gem is simply a circular arrangement of n 3-dipoles with internal colors $0, 1, 2$. Such a canonical n-gem is named a *bloboid* and is denoted $\mathcal{B}_n$. This indexing decreasing sequence is easily obtainable from the primal objects, the gems:

$$\mathcal{H} = \mathcal{H}_n \xrightarrow[thick_1]{2dip} \mathcal{H}_{n-1} \xrightarrow[thick_2]{2dip} \dots \xrightarrow[thick_{n-1}]{2dip} \mathcal{H}_1 = \mathcal{B}_n = \mathcal{B}.$$

In the second phase we find specific abstract PL-triangulations of the PL2-faces for the index increasing sequence of abstract colored 2-dimensional PL-complexes:

$$\mathcal{B}^\star = \mathcal{H}_1^\star \xrightarrow[move_1]{bp} \mathcal{H}_2^\star \xrightarrow[move_2]{bp} \dots \xrightarrow[move_{n-1}]{bp} \mathcal{H}_n^\star = \mathcal{H}^\star.$$

At the same time that we get we get also a sequence of *wings*

$$\mathcal{W}_1^\star \xrightarrow[move_1]{wbp} \mathcal{W}_2^\star \xrightarrow[move_2]{wbp} \dots \xrightarrow[move_{n-1}]{wbp} \mathcal{W}_n^\star = \mathcal{W}^\star.$$

Each wing $\mathcal{W}_m$'s corresponds to a section of the previous sequence $\mathcal{H}_m$'s by two adequated fixed semi-planes Π_ℓ and Π_r. Each wing $\mathcal{W}_m$ has an associated *nervure*, denoted $\mathcal{N}_m$. Each nervure is an auxilary spanning tree which helps in finding canonical embeddings for the planar graph $\mathcal{U}_m = \mathcal{W}_m \cup \mathcal{N}_m$ $(m = n-1, \dots, 1)$, in Π_ℓ and Π_r. The third phase uses the final wing and the sequence of $\mathcal{H}_m^\star$'s to produce a pillow filling sequence

$$\mathcal{H}_1^\diamond \xrightarrow[filling_1]{pillow} \mathcal{H}_2^\diamond \xrightarrow[filling_2]{pillow} \dots \xrightarrow[filling_{n-1}]{pillow} \mathcal{H}_n^\diamond = \mathcal{H}^\star.$$

In this phase everything is embedded into $\mathbb{R}^3$ and the last element is the PL-embedding that we seek. The whole procedure can be implemented as a formal algorithm that takes $O(n^2)$-space and $O(n^2)$-time complexity, where $2n$ is the number of vertices of the original J^2-gem.

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2 Abstract 2-dimensional PL-complexes, their wings and nervures

3 Diamond complexes

4 Further work

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