

# The $\kappa_r$ -version of the WRT $_r$ -invariants, monochromatic 3-connected blinks and evidence for a conjecture on their induced 3-manifolds \*

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## Abstract

A *blink* is a plane graph with a bicolouration of its edges into black and gray edges. Subtle equivalence class of blinks are in 1-1 correspondence with closed, oriented and connected 3-manifolds up to orientation preserving homeomorphisms, see <http://arxiv.org/abs/1305.4540>, [9]. Switching black and gray in a blink  $B$  reverses the orientation obtaining  $-B$ . The dual of the blink  $B$  in the sphere  $\mathbb{S}^2$  is denoted by  $B^*$ . Blinks  $B$  and  $-B^*$  induce the same 3-manifold. This paper concludes (up to a single uncertainty) the topological classification of  $T_{16}$ , the set of 381 monochromatic 3-connected blinks up to 16 edges. The classification justifies the Conjecture that if  $B' \notin \{B, -B^*\}$ , then  $B$  and  $B'$  induce distinct 3-manifolds. The isomorphism problem for blinks is solvable by an  $O(n^2 \log n)$ -algorithm.

## 1 Introduction

It is well known that a 3-connected graph has a unique pair of enantiomorphic embeddings in the 2-sphere [14]. The present paper suggests a manifestation of this fact in closed oriented connected 3-manifolds induced by 3-connected monochromatic blinks, restricting the 1-1 correspondence of <http://arxiv.org/abs/1305.4540>, [9] to that blinks.

### 1.1 The set of 381 blinks $T_{16}$

The set  $T_{16}$  with 381 blinks is explicitly generated and displayed in <http://arxiv.org/arXiv:math/0702057>, [6]. In the same work, the classification up to oriented homeomorphisms for the 3-manifolds induced by blinks in  $T_{16}$  was nearly performed by,  $\kappa_r$ ,  $r = 3, \dots, 8$ , the Kauffman-Lins version ([2]) of the WRT-invariants, leaving exactly 11 doubts. In section 2, for completeness, we give the full definition and an associated combinatorial algorithm to obtain these invariants using solely the blink as input. Ten of eleven doubts are solved in the present work. Here are the numbers of

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monochromatic 3-connected blinks up to 16 edges, indexed by a fixed number of edges:  $6 \mapsto 1, 8 \mapsto 1, 9 \mapsto 1, 10 \mapsto 2, 11 \mapsto 2, 12 \mapsto 9, 13 \mapsto 11, 14 \mapsto 36, 15 \mapsto 76, 16 \mapsto 242$ , so that  $T_{16}$  has a total of 381 blinks. Dual blinks are not filtered in [6]. This was on purpose to control the generation algorithms. When we perform the  $\kappa_r$ -classification, these dual blinks  $(\pm B, \mp B^*)$  must appear, as they do, together. As a matter of fact we only draw one of  $\pm B$ , the other is obtained by exchanging black and gray edges. The effect of taking the negative blink in  $\kappa_r(B)$  is to conjugate the complex number: for all blinks  $B$ ,  $\kappa_r(B) = \overline{\kappa_r(-B)}$  and  $\kappa_r(B) = \kappa_r(-B^*)$ .

The clarification of these doubts answers all but one of the challenges posted in <http://arxiv.org/abs/1305.2617>, [10]. From <http://arxiv.org/arXiv:math/0702057>, [6] we know that among all 3-connected monochromatic blinks up to 16 edges, forming the set  $T_{16}$ , there are only eleven that has a  $HG8QI$ -class with more than one pair  $(B, -B^*)$ . These are the following: (1)  $14_{24}^t$ , (2)  $15_{16}^t$ , (3)  $15_{19}^t$ , (4)  $15_{22}^t$ , (5)  $16_{42}^t$ , (6)  $16_{56}^t$ , (7)  $16_{140}^t$ , (8)  $16_{141}^t$ , (9)  $16_{142}^t$ , (10)  $16_{149}^t$ , (11)  $16_{233}^t$ . All the other pairs of blinks  $(B, -B^*)$  in  $T_{16}$  are complete graphical invariants for their induced 3-manifolds. We show that all but one (which remains unknown) of these classes break into two classes of homeomorphisms. The distinction in 9/10 cases is obtained by a combined action of the systems SnapPy, GAP and Sage. We find all the homeomorphisms of the fundamental group of the induced 3-manifold onto the symmetric group  $S_k$ , for small adequate  $k$ . Using them then we compute the homology of the cover 3-manifolds based in these homeomorphisms of the induced 3-manifolds. With this methodology we were able to distinguish all the  $HG8QI$ -classes, except  $16_{140}^t$ , which remains a big challenge. However, to conform with the other  $HG8QI$ -classes, we conjecture that it also breaks into two homeomorphism classes.

## 1.2 Acknowledgments

We are indebted to Marc Culler who kindly provided clues on how to install and how to use SnapPy and first found some of the distinguishing covers that follow. The first author thanks the financial support of CIn/UFPE and CNPq.

## 2 $\kappa_r$ : the Kauffman-Lins version of the WRT-invariants

To each blink  $B$  and each integer  $r \geq 3$  we will associate a complex number  $\kappa_r(B)$ . If  $B'$  is another blink obtained from  $B$  by moves in the coin calculus <http://arxiv.org/abs/1305.4540>, [9], then  $\kappa_r(B) = \kappa_r(B')$ . Therefore,  $\kappa_r$  is an *invariant of closed, orientable, connected 3-manifolds*. The invariant  $\kappa_r$  was obtained and justified in the Kauffman-Lins monography [2]. Its invariance relies in the properties of Kauffman's bracket and on the algebraic properties of the Temperley-Lieb algebra. Subsequently it was realized that this invariant is one of the manifestations of the Witten-Reshetikhin-Turaev invariants. Originally these invariants were found by Witten in the late 1980's using a physical formalism that was not perfectly mathematically satisfactory. Witten's result broke the prejudice that good invariants of 3-manifolds did not exist. Shortly after, some eastern european researchers as Turaev, Viro, Reshetikhin, Kirillov and others, produced full mathematical proofs of the invariance of Witten's results using quantum groups, [15, 11, 12, 4]. However, quantum groups is a rather complicated subject and Kauffman-Lins using some

ideas of Lickorish (invariance of second of Kirby's moves ([5]) via the Temperley-Lieb algebra and cubic graphs embedded into 3-space — see also page 144 of Kauffman-Lins monography, and Turaev (shadow world ([13]), which makes possible to combinatorialize the whole situation), provided the  $\kappa_r$ -invariant, which demands much less machinery to be understood. It is our intention here to provide a complete recipe for obtaining the  $\kappa_r$ -invariant in a way to be simply understood by a combinatorially inclined reader. Our idea is to make it available for the Combinatorics community these strong and mysterious quantum invariants: an infinite sequence of complex numbers which are deterministically possible to assign to blinks up to the coin calculus ([9]) and to (they are the same, there is a 1-1 correspondence) 3-manifolds up to orientable homeomorphisms. The invariance issue will remain untouched in this work; it is treated in detail in [2].

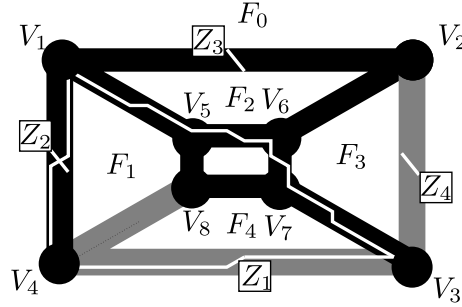


Figure 1: The zigzags of a blink  $B$  and the domain of an  $r$ -state  $\sigma$ : an  $r$ -state  $\sigma$  is a function from  $\mathbb{D}(B)$  to  $\mathcal{I}m(\sigma) = \mathcal{I} = \{3, \dots, r-2\}$ ; a zigzag in a plane graph is a closed path that alternates taking the leftmost turn and the rightmost turn at each vertex; zigzags in general surfaces (orientable and non-orientable) are studied in [7]. the bipartition of the edge colors of the blink plays no role in the definition of the zigzags; a zigzag traverses at most twice each edge, so that the total number of edges traversed by the set of zigzags is exactly  $2n$ , where  $n$  is the number of edges of the blink; The domain of  $\sigma$  is the union of its vertices, the faces and the zigzags of the blink  $B$ . In the example the domain is  $\mathcal{D}(\sigma) = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, F_0, F_1, F_2, F_3, F_4, F_5, Z_1, Z_2, Z_3, Z_4\}$ .

## 2.1 The ingredients for computing $\kappa_r$

This section is a reformulation of Chapter 7 of [8]. The invariance depends on the Jones polynomial and on the Temperley-Lieb algebra. The complete theory is developed from scratch in [2].

Let  $A = e^{\frac{\pi i}{2r}}$  be the “first” primitive  $4r$ -th root of unity,  $r \geq 3$ , and  $\mathcal{I} = \{0, \dots, r-2\}$ . In fact  $A$  could be any primitive root, but we need to be specific: the normalization used only works for the first root. It would have to be redone for other roots. For  $n$  in  $\mathcal{I}$ , let

$$\Delta_n = (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}, \quad [n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}} = (-1)^{n-1} \Delta_{n-1}.$$

Letting  $q = A^2$ , for reasons inherited from the physics we call  $[n]$  the  $q$ -deformed quantum integer and  $[n]! = \prod_{0 \leq m \leq n} [m]$  the  $q$ -deformed quantum factorial. Note that even though  $A$  is complex,  $\Delta_n$  and  $[n]$  assume only real values. Three numbers  $a, b, c \in \mathcal{I}$  form an

*r*-admissible triple if  $a + b + c \leq 2r - 4$  and  $a + b - c, b + c - a, c + a - b$  are non-negative and even. Let  $\theta : \mathcal{I}^3 \mapsto \mathbb{R}$ , defined on the *r*-admissible triples by means of

$$\theta(a, b, c) = \frac{(-1)^{m+n+p} [m+n+p+1]! [n]! [m]! [p]!}{[m+n]! [n+p]! [p+m]!}, \text{ where}$$

$m = (a + b - c)/2, n = (b + c - a)/2, p = (c + a - b)/2$ . We define  $\theta(a, b, c) = 0$  if  $(a, b, c)$  fails to be *r*-admissible.

Let  $\lambda : \mathcal{I}^3 \mapsto \mathbb{C}$ , be defined on the *r*-admissible triples by

$$\lambda(a, b, c) = \lambda_c^{ab} = (-1)^{(a+b-c)/2} A^{[a(a+2)+b(b+2)-c(c+2)]/2}$$

We let  $\lambda_c^{ab} = 0$  if  $(a, b, c)$  fails to be *r*-admissible. Finally, define  $Tet : \mathcal{I}^6 \mapsto \mathbb{R}$ , as follows. If  $(\alpha, \beta, \phi), (\alpha, \delta, \epsilon), (\gamma, \delta, \phi), (\beta, \gamma, \epsilon)$  are *r*-admissible, define

$$Tet(\alpha, \beta, \gamma, \delta, \epsilon, \phi) = Tet \begin{bmatrix} \alpha & \beta & \epsilon \\ \gamma & \delta & \phi \end{bmatrix} = \frac{Int!}{ext!} \sum_{m \leq s \leq M} \frac{(-1)^s [s+1]!}{\prod_i [s - a_i]! \prod_j [b_j - s]!}$$

where,

$$Int! = \prod_{i,j} [b_j - a_i]!$$

$$ext! = [\alpha]! [\beta]! [\gamma]! [\delta]! [\epsilon]! [\phi]!$$

$$a_1 = \frac{1}{2}(\alpha + \delta + \epsilon) \quad b_1 = \frac{1}{2}(\beta + \delta + \epsilon + \phi)$$

$$a_2 = \frac{1}{2}(\beta + \gamma + \epsilon) \quad b_2 = \frac{1}{2}(\alpha + \gamma + \epsilon + \phi)$$

$$a_3 = \frac{1}{2}(\alpha + \beta + \phi) \quad b_3 = \frac{1}{2}(\alpha + \beta + \gamma + \delta)$$

$$a_4 = \frac{1}{2}(\gamma + \delta + \phi) \quad M = \max\{a_i\} \quad m = \min\{b_j\}$$

If one of the four triples above fails to be *r*-admissible, define the value of *Tet* as null.

[9] These are all the algebraic ingredients we need to define a function  $\kappa_r$  on a *B*, where  $r \geq 3$  is an integer.

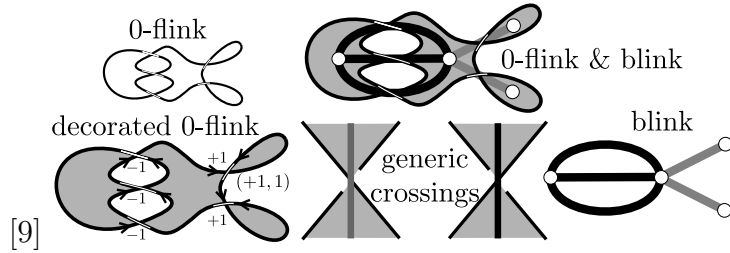


Figure 2: From 0-flink to blink and back: the projection of any 0-flink can be 2-face colorable into white and gray with the infinite face being white so that each sub-curve between two crossing have their incident faces receiving distinct colors. The above figure shows how to transform a 0-flink projection into a blink (with thicker edges than the curves representing the link projection). The vertices of the blink are distinguished fixed points represented by white disks in the interior of the gray faces. Each crossing of the link projection becomes an edge in the corresponding blink, An edge of the blink is gray if the upper strand that crosses it is from northwest to southeast, it is black if the the upper strand that crosses it is from northeast to southwest. The inverse procedure is clearly defined. In fact, the 0-flink is the so called *medial map* of the blink. Thus we have a 1-1 correspondence between 0-flinks and blinks.

We define a state sum  $\kappa'_r(B)$  which out to be an invariant under Kirby's second move, the *band move* or the *handle slide move*, [3], see also page 144 of [2] or <http://arxiv.org/abs/1305.4540>, [9]. Each  $r$ -state contributes with one monomial for  $\kappa'_r(B)$ . A face of a connected blink is a connected component of complement  $\mathbb{R}^2 \setminus B$ . There is a distinguished face (the infinite one) which is mapped to 0 by any admissible  $r$ -state  $\sigma$ . With this restriction,  $\kappa'_r$  is clearly multiplicative,  $\kappa'_r(B \cup B') = \kappa'_r(B) \times \kappa'_r(B')$  for disjoint blinks  $B$  and  $B'$ .

In obtaining the ingredients for the factors the bipartition of the edge set of the blink is disregarded, except for the  $\lambda(a, b, c)$ 's. They are also the only terms which are indeed complex:  $Tet(\alpha, \beta, \gamma, \delta, \epsilon, \phi)$ ,  $\theta(a, b, c)$  and  $\Delta_n$  are reals. A *state is  $r$ -admissible* if the triples of numbers associated to the incident trios (vertex, face, zigzag) is  $r$ -admissible. If a state is not admissible, its value is declared to be 0. From each  $r$ -state we produce a complex number and the value  $\kappa'_r(B)$  is the sum of all its  $r$ -admissible states. Having invariance under Kirby's band move, all that we need to have a 3-manifold invariant is to define

$$\kappa_r(B) = \kappa'_r(B) \left( \sin\left(\frac{\pi}{r}\right) \sqrt{\frac{2}{r}} \right)^{|F|+1} \left( (-1)^{r-2} e^{\frac{3i\pi(r-2)}{4r}} \right)^{-n(F)}.$$

Each blink  $B$  has associated with it an object called a 0-flink  $F$ , which is a decorated link projection as exemplified in Fig. 2. In fact, the flink is defined in ([9]) and generalizes Kauffman's notion of blackboard-framed link [1]. A 0-flink coincides with a blackboard-framed link. In the above formula,  $F$  is the 0-flink associated to  $B$ ,  $|F|$  is the number of components of  $F$ ,  $n(F)$  is the number of positive eigenvalues minus the number of negative eigenvalues of the *linking matrix* of  $F$ , which is easily obtained from the 0-flink  $F$ , see page 254 of [8]. The factor we introduce to go from  $\kappa'_r(B)$  to  $\kappa_r(B)$  compensates for the multiplicative change that happens in the state sum  $\kappa'_r(B)$  when we introduce an isolated gray or black edge in the blink: this is the blink manifestation of Kirby's move of type 1. With these definitions it is possible to prove (see pages 146-152 of [2]) that for all  $r \geq 3$ ,

$$\kappa_r(\circ) = 1 \text{ and } \kappa_r(\circ\text{---}\circ) = \sin\left(\frac{\pi}{r}\right) \sqrt{\frac{2}{r}},$$

where  $\circ$  denotes the isolated vertex blink, inducing  $\mathbb{S}^2 \times \mathbb{S}^1$  and  $\circ\text{---}\circ$  denotes the isolated single edge blink inducing  $\mathbb{S}^3$ . The computations of pages 146-152 only work for the first  $4r$ -th root of unity,  $A$ . See Fig. 3 to complement this material.

$$\sigma : V_B \cup F_B \cup Z_B \longrightarrow \mathcal{I} = \{0, 1, \dots, r-2\}; \quad \text{let } v, f, z, v_1, v_2, f_1, f_2, z_1, z_2 \in \text{Im}(\sigma) = \mathcal{I}.$$

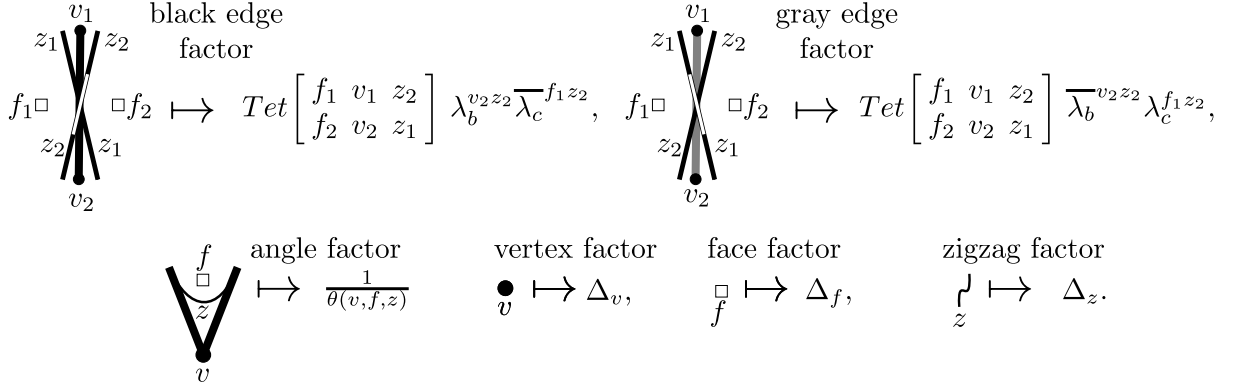


Figure 3: The *ingredients* and the *sites* for computing the  $\kappa_r$ -invariant from a blink  $B$ : black edge, gray edge, angle, vertex, face and zigzag are sites to have associated factors depending on a fixed  $r$ -state  $\sigma$  in the computation of the  $\kappa_r$ -invariant; it is obtained from blink  $B$  as follows: a state  $\sigma$  is a function from the union of the vertices, the faces and the zigags of  $B$  to an alphabet  $\mathcal{I} = \{0, \dots, r-2\}$ . The value of the  $r$ -state is the product of complex numbers in 1-1 correspondence with the factors for the black edges, the gray edges, the angles, the vertices, the faces and the zigzags (see Fig. 1); the value of  $\kappa'_r(B)$  is the sum of the  $r$ -admissible (as defined in Subsection 2.1) states of  $B$ ; in the above figure,  $v, f, z, v_1, v_2, f_1, f_2, z_1, z_2$  are the images under the specific  $r$ -state  $\sigma$  of, respectively,  $V, F, Z, V_1, V_2, F_1, F_2, Z_1, Z_2$ ; the evaluations of the ingredients for the factors,  $Tet(\alpha, \beta, \gamma, \delta, \epsilon, \phi)$ ,  $\lambda_{ab}^c$ ,  $\overline{\lambda_{ab}^c}$ ,  $\theta(a, b, c)$ ,  $\Delta_n$  are detailed in Subsection 2.1; there are symmetries in the situation which imply that the edge factors are invariant under a half turn rotation. There is an adequate factor which multiplied by  $\kappa'_r(B)$  produce a 3-manifold invariant  $\kappa_r(B)$ . See Subsection 2.1.

### 3 Solving 10/11 $HG8QI$ -classes as non-homeomorphic pairs

An  $HGnQI$ -class of blinks is the set of blinks inducing a 3-manifold with the same homology and the same  $\kappa_r$  invariants, for  $r = 3, \dots, n$ . We have used a combination of the softwares SnapPy/GAP/Sage to prove that that 10 out of the 11 pairs of blinks in these  $HG8QI$ -classes induce distinct 3-manifolds. The only exception was  $16_{140}$  which remains unconquered. We conjecture that it also breaks into two classes of homeomorphisms. In the figures of this section we present both the blink and its associated 0-flink (defined in see <http://arxiv.org/abs/1305.4540>, [9]). However, the 0-flinks and its pair of surgery coefficients attached to each component of the 0-flink (good for input in the SnapPy) are redundant, as they are implied by the small blink displayed at the south-west of each 0-flink. All the blinks are monochromatic, formed by black edges only. This corresponds to having only alternating 0-flinks. The running time of SnapPy/GAP/Sage for the 11 pairs varies a lot:  $14_{24}^t \mapsto 0m29.01s$ ,  $15_{16}^t \mapsto 2m17.75s$ ,  $15_{19}^t \mapsto 639m52.46s$ ,  $15_{22}^t \mapsto 125m51.74s$ ,  $16_{42}^t \mapsto 0m4.98s$ ,  $16_{56}^t \mapsto 0m2.63s$ ,  $16_{140}^t \mapsto \infty$ ,  $16_{141}^t \mapsto 0m1.18.90s$ ,  $16_{142}^t \mapsto 233m453.92s$ ,  $16_{149}^t \mapsto 34m7.37s$ ,  $16_{233}^t \mapsto 1m5.63s$ . In only one case,  $16_{140}^t$ , we

report an  $\infty$  because the case took more than three days, inconclusively. The status of  $16_{140}^t$  remains unknown.

### 3.1 Distinguishing the 2 members of the $HG8QI$ -class $14_{24}^t$

$HG8QI$ -class  $14_{24}^t$ : homology spheres  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	166
05	0.221839804	0.000000000	732
06	0.288675135	0.000000000	6153
07	0.569807123	0.000000000	26280
08	0.658347954	0.000000000	118404

$T[71]$  with 14 crossings

$T[79]$  with 14 crossings

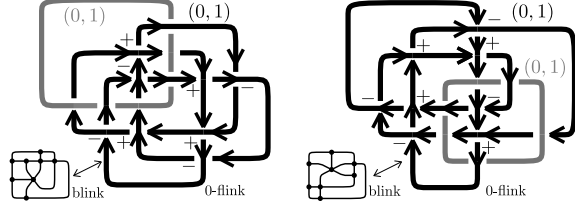


Figure 4: Distinction on the  $HG8QI$ -class  $14_{24}^t$ : We show the surgery coefficient pairs attached to each of the 0-flink components. These are determined by the blink. Covers of degree 5 perform the distinction. Homology of the three 5-covers of  $T[71]$ :  $[Z/3 + Z/3 + Z/3 + Z/3, Z/63 + Z/63, Z/132 + Z/132]$ . Homology of the three 5-covers of  $T[79]$ :  $[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]$ . Therefore  $T[71]$  and  $T[79]$  induce non-homeomorphic 3-manifolds. By the 1-1 correspondence of [9] we identify the blink and its induced 3-manifold. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[71]$  — 24.8073697340, volume of  $T[79]$  — 24.807369734. Sage is the front end of the system; it calls SnapPy and GAP. Sage Execution Time (CPU time 0m0.42s, Wall time 0m29.01s).

### 3.2 Distinguishing the 2 members of the $HG8QI$ -class $15_{16}^t$

$HG8QI$ -class  $15_{16}^t$ : homology spheres  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.626523527	-0.169924730	1684
06	0.288675135	0.000000000	19341
07	0.271887719	-0.143082082	111440
08	0.116351265	1.000000000	620624

$T[118]$  with 15 crossings

$T[119]$  with 15 crossings

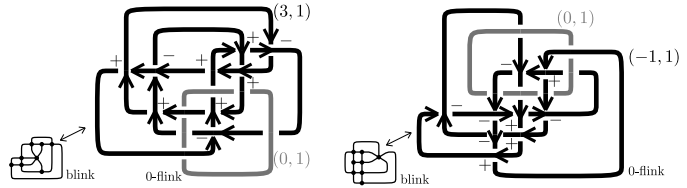


Figure 5: Distinction on the  $HG8QI$ -class  $15_{16}^t$ : Homology of the four 5-covers of  $T[118]$ :  $[Z/229773, Z/1110327, Z/3018207, Z/3699687]$ . Homology of the four 5-covers of  $T[119]$ :  $[Z/3 + Z/1299909, Z/126627, Z/1052067, Z/4117827]$ . Therefore  $T[118]$  and  $T[119]$  induce non-homeomorphic 3-manifolds. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[118]$  — 28.375305555, volume of  $T[119]$  — 28.3753055549. Sage Execution Time (CPU time 0m1.55s, Wall time 2m17.75s).

### 3.3 Distinguishing the 2 members of the $HG8QI$ -class $15_{19}^t$

The  $HG8QI$ -class  $15_{19}$  provided us with an example of the toughness of the computation involved to obtain the coverings. It took SnapPy/GAP/Sage more than 10 hours to obtain the (reported) 20 coverings.

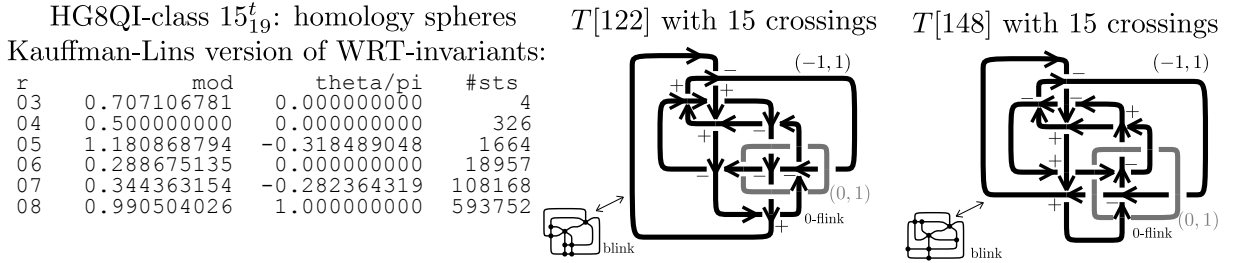


Figure 6: Distinction on the  $HG8QI$ -class  $15_{19}^t$ : Homology of the twenty 6-covers of  $T[122]$ :  $[Z/2 + Z/2 + Z/15486, Z/2 + Z/2 + Z/15486, Z/2 + Z/894, Z/2 + Z/894, Z/3 + Z/3 + Z/732, Z/3 + Z/3 + Z/732, Z/3 + Z/3 + Z/6552, Z/3 + Z/3 + Z/6552, Z/4 + Z/12, Z/4 + Z/12, Z/6 + Z/402, Z/6 + Z/402, Z/24 + Z, Z/24 + Z, Z/531, Z/531, Z/549, Z/549, Z/4716, Z/4716]$ . Homology of the twenty 6-covers of  $T[148]$ :  $[Z/2 + Z/2 + Z/114, Z/2 + Z/2 + Z/384, Z/2 + Z/966, Z/3 + Z/3 + Z/51, Z/3 + Z/3 + Z/1302, Z/3 + Z/24 + Z, Z/6 + Z/6 + Z/444, Z/6 + Z/138, Z/6 + Z/150, Z/6 + Z/162, Z/6 + Z/450, Z/6 + Z/4842, Z/8 + Z/24, Z/18 + Z/450, Z/1341, Z/4356, Z/4866, Z/11025, Z/66402, Z/71496]$ . Therefore  $T[122]$  and  $T[148]$  induce non-homeomorphic 3-manifolds. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[118]$  — 27.670218370, volume of  $T[119]$  — 27.670218370. Sage Execution Time (CPU time 2m8.60s, Wall time 639m52.46s).



### 3.4 Distinguishing the 2 members of the $HG8QI$ -class $15_{22}^t$

HG8QI-class  $15_{22}^t$ : homology spheres  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.943093903	-0.180036791	1680
06	0.288675135	0.000000000	20945
07	0.242490221	-0.482680140	111968
08	0.641616039	1.000000000	656072

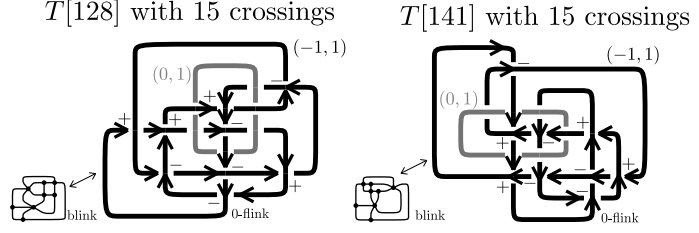


Figure 7: Distinction on the  $HG8QI$ -class  $15_{22}^t$ : Homology of the fifteen 6-covers of  $T[128]$ :  $[Z/2 + Z/162, Z/2 + Z/234, Z/2 + Z/414, Z/2 + Z/1086, Z/3 + Z/3 + Z, Z/3 + Z/6 + Z + Z, Z/3 + Z/6 + Z + Z, Z/3 + Z/6 + Z/18 + Z/720, Z/3 + Z/6 + Z/18 + Z/720, Z/6 + Z/6 + Z/1296, Z/6 + Z/6 + Z/10800, Z/9, Z/207, Z/5571, Z/5637]$ . Homology of the fifteen 6-covers of  $T[141]$ :  $[Z/2 + Z/2394, Z/3 + Z/3 + Z, Z/3 + Z/6 + Z + Z, Z/3 + Z/6 + Z + Z, Z/3 + Z/6 + Z/18 + Z/720, Z/3 + Z/6 + Z/18 + Z/720, Z/3 + Z/2439, Z/3 + Z/3921, Z/6 + Z/6 + Z/144, Z/6 + Z/6 + Z/5328, Z/6 + Z/1386, Z/6 + Z/1482, Z/9, Z/207, Z/2115]$ . Therefore  $T[128]$  and  $T[141]$  induce non-homeomorphic 3-manifolds. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[118]$  — 27.9322198834, volume of  $T[119]$  — 27.932219883. Sage Execution Time (CPU time 0m42.21s, Wall time 125m51.74s).

### 3.5 Distinguishing the 2 members of the $HG8QI$ -class $16_{42}^t$

HG8QI-class  $16_{42}^t$ : torsion, 2 2; Betti number, 0.  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.000000000	0.000000000	4
04	1.224744871	0.195913276	294
05	0.000000000	0.000000000	1900
06	0.577350269	-0.333333333	22883
07	0.000000000	0.000000000	159992
08	1.774989898	0.206598266	1001208

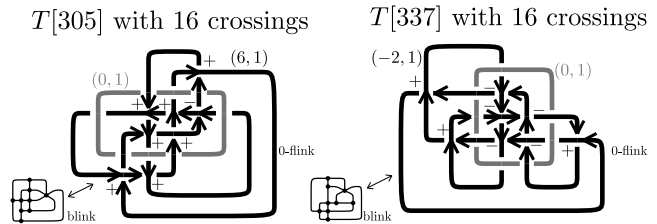


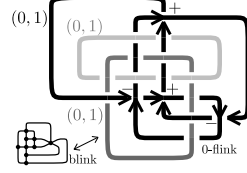
Figure 8: Distinction on the  $HG8QI$ -class  $16_{42}^t$ : Homology of the six 4-covers of  $T[305]$ :  $[Z/2 + Z/2 + Z/2 + Z/2 + Z/114, Z/2 + Z/29146, Z/2 + Z/2 + Z/2 + Z/2 + Z/6, Z/2 + Z/2 + Z/2 + Z/2 + Z/6, Z/2 + Z/1305434, Z/16169 + Z]$ . Homology of the six 4-covers of  $T[337]$ :  $[Z/2 + Z/2 + Z/2 + Z/6 + Z/6, Z/2 + Z/2 + Z/2 + Z/2 + Z/150, Z/2 + Z/2 + Z/2 + Z/2 + Z/150, Z/2 + Z/29146, Z/2 + Z/1305434, Z/16169 + Z]$ . Therefore  $T[305]$  and  $T[337]$  induce non-homeomorphic 3-manifolds. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[305]$  — 32.9085657755, volume of  $T[337]$  — 32.908565776. Sage Execution Time (CPU time 0m1.89s, Wall time 0m4.98s).

### 3.6 Distinguishing the 2 members of the $HG8QI$ -class $16_{56}^t$

HG8QI-class  $16_{56}^t$ : no torsion, 2 2; Betti number, 1.  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	1.000000000	0.000000000	8
04	1.000000000	0.000000000	420
05	1.249042228	0.093934688	4512
06	2.000000000	0.000000000	61009
07	1.748655511	-0.081646235	501032
08	0.801948466	0.000000000	3620288

$T[320]$  with 16 crossings



$T[357]$  with 16 crossings

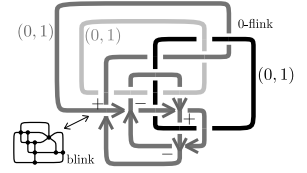


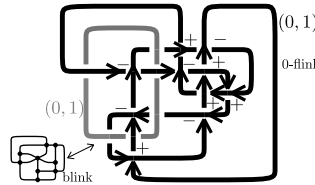
Figure 9: Distinction on the  $HG8QI$ -class  $16_{56}^t$ : Homology of the five 3-covers of  $T[320]$ :  $[Z, Z/2 + Z/34 + Z, Z/7 + Z/7 + Z, Z/37 + Z, Z/71 + Z]$ . Homology of the five 5-covers of  $T[357]$ :  $[Z, Z/2 + Z/2 + Z/2 + Z/4 + Z, Z/2 + Z/32 + Z, Z/2 + Z/40 + Z, Z/7 + Z/7 + Z]$ . Therefore  $T[320]$  and  $T[357]$  induce non-homeomorphic 3-manifolds. Note that the volumes also distinguish, on the contrary of the previous volumes: volume of  $T[320]$  — 29.4362635970, volume of  $T[357]$  — 29.460315997. Sage Execution Time (CPU time 0m1.40s, Wall time 0m2.63s).

### 3.7 The 2 members of the $HG8QI$ -class $16_{140}^t$ remain a big challenge

HG8QI-class  $16_{140}^t$ : homology spheres  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828810492	0.200682317	1736
06	0.288675135	0.000000000	19469
07	0.369735450	0.097853027	114084
08	0.499366312	1.000000000	624200

$T[423]$  with 16 crossings



$T[444]$  with 16 crossings

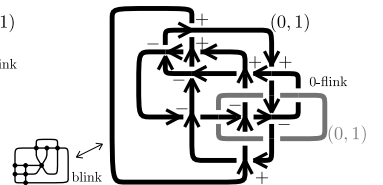


Figure 10: Trying to distinguish the two blinks in the  $HG8QI$ -class  $16_{140}^t$ : Inconclusive — there are no  $k$ -covers for  $k = 2, \dots, 6$  and the search for 7-covers took more than three days without ending. We did not try  $k$ -covers for  $k \geq 8$ . Both induced 3-manifolds have the same volume: 30.587901596. We conjecture that the induced manifolds are non-homeomorphic.

### 3.8 Distinguishing the 2 members of the $HG8QI$ -class $16_{141}^t$

$HG8QI$ -class  $16_{141}^t$ : homology spheres  
 Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828231334	0.000000000	1736
06	0.288675135	0.000000000	19469
07	0.591962818	0.000000000	114084
08	0.297863204	0.000000000	624200

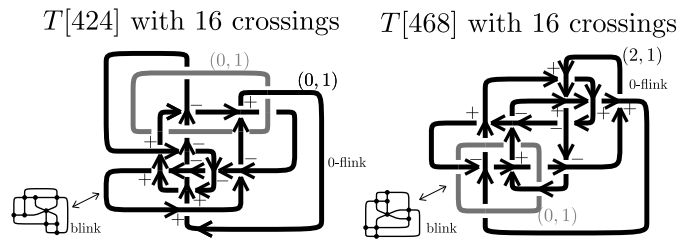


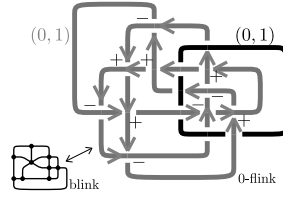
Figure 11: Distinction on the  $HG8QI$ -class  $16_{141}^t$ : Homology of the three 5-covers of  $T[320]$ :  $[Z/2 + Z/4285014, Z/2 + Z/4285014, Z/3 + Z/3 + Z/3 + Z/3]$ . Homology of the three 5-covers of  $T[357]$ :  $[Z/2 + Z/3357564, Z/2 + Z/3357564, Z/3 + Z/3 + Z/3 + Z/3]$ . Therefore  $T[424]$  and  $T[468]$  induce non-homeomorphic 3-manifolds. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[424]$  — 30.7074870213, volume of  $T[468]$  — 30.707487021. Sage Execution Time (CPU time 0m0.90s, Wall time 1m18.90s).

### 3.9 Distinguishing the 2 members of the $HG8QI$ -class $16_{142}^t$

HG8QI-class  $16_{142}^t$ : homology spheres  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1696
06	0.288675135	0.000000000	21113
07	1.546327427	0.000000000	113232
08	0.931913875	0.000000000	663132

$T[425]$  with 16 crossings



$T[435]$  with 16 crossings

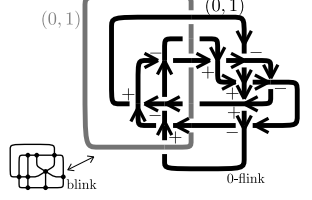


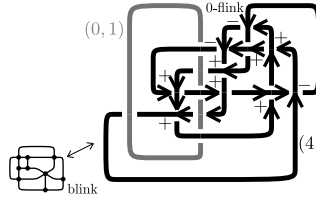
Figure 12: Distinction on the  $HG8QI$ -class  $16_{142}^t$ : Homology of the twenty nine 6-covers of  $T[425]$ :  $[Z/2 + Z/2 + Z/2 + Z/2 + Z/4, Z/2 + Z/2 + Z/2 + Z/2 + Z/4, Z/2 + Z/2 + Z/2 + Z/2 + Z/12 + Z/12 + Z, Z/2 + Z/2 + Z/168, Z/2 + Z/2 + Z/168, Z/2 + Z/2 + Z/12804, Z/2 + Z/2 + Z/12804, Z/2 + Z/2 + Z/42006, Z/2 + Z/2 + Z/42006, Z/3 + Z/3 + Z, Z/3 + Z/3 + Z, Z/3 + Z/3 + Z/162, Z/3 + Z/3 + Z/162, Z/3 + Z/3 + Z/1668, Z/3 + Z/3 + Z/1668, Z/3 + Z/84, Z/3 + Z/84, Z/3 + Z/990, Z/3 + Z/990, Z/3 + Z/13548, Z/3 + Z/13548, Z/3 + Z/26358, Z/3 + Z/26358, Z/4 + Z/4 + Z, Z/4 + Z/4 + Z, Z/6 + Z/6 + Z, Z/57 + Z/57 + Z, Z/7392, Z/7392]$ . Homology of the twenty nine 6-covers of  $T[435]$ :  $[Z/2 + Z/2 + Z/2 + Z/2 + Z/4, Z/2 + Z/2 + Z/2 + Z/2 + Z/4, Z/2 + Z/2 + Z/2 + Z/2 + Z/12 + Z/12 + Z, Z/3 + Z/3 + Z, Z/3 + Z/3 + Z, Z/3 + Z/468, Z/3 + Z/468, Z/3 + Z/1632, Z/3 + Z/1632, Z/3 + Z/3618, Z/3 + Z/3618, Z/3 + Z/47694, Z/3 + Z/47694, Z/4 + Z/4 + Z, Z/4 + Z/4 + Z, Z/6 + Z/6 + Z/6324, Z/6 + Z/6 + Z/6324, Z/6 + Z/6 + Z/61254, Z/6 + Z/6 + Z/61254, Z/9 + Z/9 + Z, Z/126 + Z/126 + Z, Z/4200, Z/4200, Z/10422, Z/10422, Z/13344, Z/13344, Z/14454, Z/14454]$ . Therefore  $T[425]$  and  $T[435]$  induce non-homeomorphic 3-manifolds. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[425]$  — 30.729338019, volume of  $T[435]$  — 30.7293380190. Sage Execution Time (CPU time 0m47.88s, Wall time 233m45.92s).

### 3.10 Distinguishing the 2 members of the $HG8QI$ -class $16_{149}^t$

HG8QI-class  $16_{149}^t$ : homology spheres  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1828
06	0.288675135	0.000000000	21553
07	0.745358881	0.081311555	126032
08	1.620357495	0.000000000	735988

$T[437]$  with 16 crossings



$T[464]$  with 16 crossings

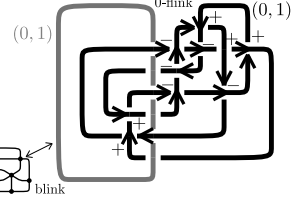


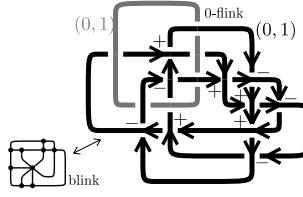
Figure 13: Distinction on the  $HG8QI$ -class  $16_{149}^t$ : Homology of the zero 5-covers of  $T[437]$ :  $[\ ]$ . Homology of the two 5-covers of  $T[464]$ :  $[Z/3 + Z/12 + Z, Z/3 + Z/12 + Z]$ . Note that the volumes are distinctly different, 33.464380115 for  $T[437]$  and 30.8673418910 for  $T[464]$ . Therefore  $T[437]$  and  $T[464]$  induce non-homeomorphic 3-manifolds. Sage Execution Time (CPU time 0m9.18s, Wall time 34m7.37s).

### 3.11 Distinguishing the 2 members of the $HG8QI$ -class $16_{233}^t$

HG8QI-class  $16_{233}^t$ : homology spheres  
Kauffman-Lins version of WRT-invariants:

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.706953028	0.000000000	1420
06	0.288675135	0.000000000	17019
07	0.472161455	0.000000000	72552
08	0.573693534	0.000000000	401400

$T[631]$  with 16 crossings



$T[663]$  with 16 crossings

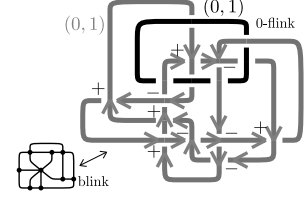


Figure 14: Distinction on the  $HG8QI$ -class  $16_{233}^t$ : Homology of the two 5-covers of  $T[631]$ :  $[Z/4 + Z/4 + Z/36 + Z/36, Z/21 + Z/21]$ . Homology of the two 5-covers of  $T[663]$ :  $[Z/2 + Z/2 + Z/258 + Z/258, Z/171 + Z/171]$ . Therefore  $T[631]$  and  $T[663]$  induce non-homeomorphic 3-manifolds. Note that the volumes (as the Kauffman-Lins version of the WRT-invariants) do not distinguish: volume of  $T[631]$  —29.624669407, volume of  $T[663]$  —29.6246694067. Sage Execution Time (CPU time 0m1.68s, Wall time 1m5.63s).

## 4 Conclusion

With the exception of the four blinks  $\{T[423], -(T[423])^*, T[444], -(T[444])^*\}$  in  $16_{140}^t$ , we have proved that if  $T_{16} \ni B' \notin \{B, -B^*\}$ , then  $B$  and  $B'$  induce non-homeomorphic 3-manifolds. We do not know whether  $T[423]$  and  $T[444]$  induce homeomorphic 3-manifolds or not. Our bet, to make 3-connected monochromatic blinks conform to their induced 3-manifolds, is that they are non-homeomorphic. We leave this doubt as a focused remaining challenge to the 3-manifold community. In the realm of the 381 members of  $T_{16}$ , this is the only pair that is unresolved from the 11 challenging  $HG8QI$ -classes presented in <http://arxiv.org/abs/1305.2617>, [10]. Maybe we have reach the current technological limit with this example. This strongly points to the research goal of finding still better invariants of 3-manifolds. We hope that <http://arxiv.org/abs/1305.4540>, [9] has something to say in this direction.

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