All the shapes of spaces: a census of small 3-manifolds *

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Abstract

In this work we present a complete (no misses, no duplicates) catalogue for closed, orientable and prime 3-manifolds induced by plane graphs with a bipartition of its edge set (blinks) up to 9 edges. Blinks form a universal encoding for such manifolds. We hope that this census becomes as useful for the study of concrete examples of 3-manifolds as the tables of knots are in the study of knots and links. Along the years we have made an issue in our computational work that it must be reproducible and independently checked by other researchers. Our software BLINK is available, but currently it lacks yet a good documentation and help is welcome to change this. An Wiki open source project is starting.

1 Introduction

After presenting some instances of closed 3-manifolds, P. Alexandroff says in the English translation (1961) of his joint work with D. Hilbert [1], first published (1932) in German, [2]: *"These few examples will suffice. Let it be remarked here that, at present, in contrast with the two-dimensional case, the problem of enumerating the topological types of manifolds of three and more dimensions is in an apparently hopeless state. We are not only far removed from the solution, but even from the first step toward a solution, a plausible conjecture"*.

John Hempel in his book (1976) *3-Manifolds* [8] writes at the opening of Section 15, entitled Open Problems: *"The ultimate goal of the theory would be in providing solutions to:* The homeomorphism problem*: provide an* effective *procedure for determining whether two* given *3-manifolds are homeomorphic, together with* The classification problem*:* effectively *generate a list containing exactly one 3-manifold from each homeomorphism class."*

It is amazing how much the picture has changed in the 80 years since Alexandroff's-Hilbert book. The progress was due to the deep advances in the 1950's and 1960's, starting with the proof that 3-manifolds are triangulable by Moise (1952), [21]. Next the presentation of them by framed links by Lickorish (1962) [14]. Following that Kirby presented its calculus for framed links (1978)[12]. starting in the early 1980's W. Thurston's breakthroughs, developing his conceptual theory on hyperbolic manifolds and of the geometrization conjecture. In the final 1980's early 1990's Witten [27] broke the psicological barrier that there were no good invariants for 3-manifolds. Following that a number of eastern European mathematicians like N. Reshetikhen, V. Turaev and O. Viro, [25, 22] using quantum groups were able to put in mathematical solid ground Witten's

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary) findings. One of us, S. Lins, was a witness of the excitement these developments caused. L. H. Kauffman and W. B. R. Lickorish discovered the relationship of the Temperley-Lieb algebra with the new invariants, [15]. Starting with a sabatical leave in to Chicago in 1990, S. Lins produced the joint monography with Kauffman [11], where blinks are first defined and extensive WRT-invariant computations were obtained from the theory developed from scratch, independently and simpler that that of quantum groups. In the early 2000's, G. Perelman revolutionized the field proving Poincaré's Conjecture and Thurston's Geometrization Conjecture. More recently in the 2010's, I. Agol is leading the field in this era post-Perelman. Of course, this is only a diagonal list of researchers. Many more have contributed and some are extremely active in this era Pós-Perelman, [6]. Currently there is a great amount of important reserch issues going on and these are exciting times for 3-manifold theory. See the recent essay of E. Klarreich in the Simons Foundation, [13].

We present here our modest contribution to the topic. It is placed in the confluency of two deep research passions of the authors, apparently very far apart: The study of closed orientable 3-manifolds and the study of plane graphs. We mean to provide a strategy for a segmented answer of Hempel's questions. The closed oriented 3-manifolds are partitioned by the number of edges in a minimum encoding of them by a certain class of plane graphs, shortly to be defined. Lexicography is used to define a representative unique plane graph (a canonical form) for each closed oriented 3-manifold. We explicitly solve the segmented problem up to 9 edges, see Theorem 4.1. This work provides an efficient algorithm to make available the canonical form of any closed orientable 3-manifold induced by plane graphs at the current level of the catalogue (currently 9 edges) and, theoretically, this could be extended 10, 11,..., n, for arbitrarily large n. Our theory gives a road to effectively name each 3-manifold classified by some set of invariants INV. We have a universal set of object, the blinks up to n edges, which can be partitioned by these invariants. The INV-classes are then tried to be broken into homeomorphisms classes. New invariants are then discovered and added to INV making them homeomorphisms classes of n-small manifolds. The difficult cases are going to appear naturally and they lead to enhancement of the theory. It is not at all impossible that this process stops and we get INV so that the INV-classes be proved to be homeomorphisms classes for all n. The point we want to make is that good examples (hard to find) are important in obtaining progress in a general theory. The classifying INV for the 9-small 3-manifolds is

$$INV = \{\ homology, WRT_{12},\ length\ of\ smallest\ geodesic\ \}.$$

2 Encoding closed orientable 3-manifolds by plane graphs

Unexplored simplicity. This was the reason for birth of this work. A *blink* is a plane graph with an arbitrary bipartition of its edges in gray and black. Even though this object is around since 1994 when it was introduced in the joint research monography of L. Kauffman and S. Lins, [11], the fact that they encode oriented closed 3-manifolds remains basically unkown.

Blinks are in 1-1 correspondence with blackboard framed links which in turn encodes in a very simple way an specific 3-gem inducing the 3-manifold. If we consider these three encodings of a 3-manifold, the blink is the one that has the smallest "perceptual complexity", see Fig. 1. Also, blinks are very easy to generate recursively. They have a rich simpifying theory which permits the generation of 3-manifold catalogues. Yet, their isomorphism problem is computationally simple. Being blinks so good, why do we need gems? The answer is that to prove that blinks induce the same 3-manifold (when they do) remains very difficult. It is straighforward to obtain a

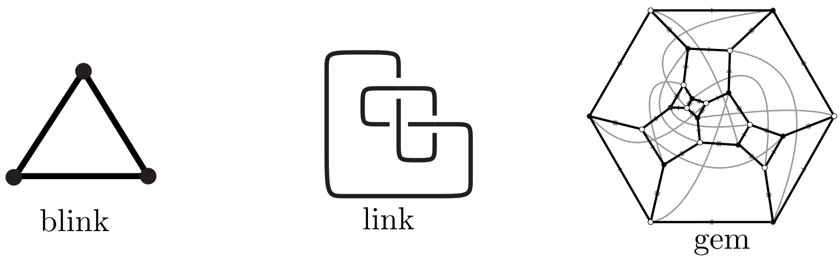


Figure 1: The minimum blink, minimum link and minimum gem inducing the binary tetrahedral space

canonical gem from a blink. Proving that two gems induce the same 3-manifold (when they do) is much easier because of the rich simplification theory of them (based on at least 4 intertwined planarities) leading to the attractors of 3-manifolds, [17]. Blinks are very good at proving that two manifolds are distinct, because from them we can extract the WRT-invariants, which are very strong invariants, yet not a complete one. These invariants do not have a direct computation from triangulations, including gems. Thus, gems and blinks collaborate in a symbiotic dance to decide (at a computational level) whether two 3-manifolds are or are not homeomorphic. And census become available!

We make here explicit for the first time that each class of homeomorphic closed oriented 3-manifolds is a subtle class of blinks where two membres of each class is linked by means of a small number of local simple moves. Our moves (see Fig. 7) are a slight reformulation for blinks of the one for framed links on Kirby's calculus ([12]) using Fenn and Rourke moves for the calculus, [5] recently published by Martelli, [20]. Reformulation of the original Kirby's moves and of the Fenn-Rourke move for blackboard framed links are worked by Kauffman in [10]. I got influenced by Kauffman's insistence in considering blackboard framed links [10] to use the plane to remove the number attached to the components, thus obtaining, esssentially, the blinks. No doubt that blinks are simple mathematical objects. Nevertheless, as a consequence of Martelli's reformulation of Kirby's calculus as complemented by Fenn-Rourke and Kauffman, I will prove shortly that they hold, in their gist, the mystery of 3-manifolds.

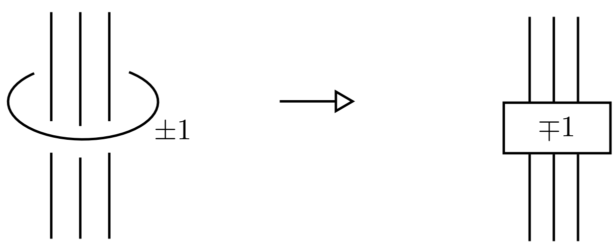


Figure 2: The Fenn-Rourke single n-parametrized move: a topological blow-down in the number of components. The number $n \geq 0$ of vertical strands crossing the unknot is arbitrary (here $n = 3$). The box marked with $+1$ (-1) indicates one full counterclockwise (clockwise) twist.

In a grounding breaking work, Lickorish in 1962, [14], proved that each closed orientable 3-manifold $\mathbb{M}^3$ can be encoded by a link in $\mathbb{S}^3$ where each one of its k components is endowed with an irreducible fraction (the framing) $\frac{\pm p}{q}$ where q could be 0, and in the case the fraction becomes $\pm\infty$. To construct the 3-manifold $\mathbb{M}^3$ from the framed link we act as follows: after removing an

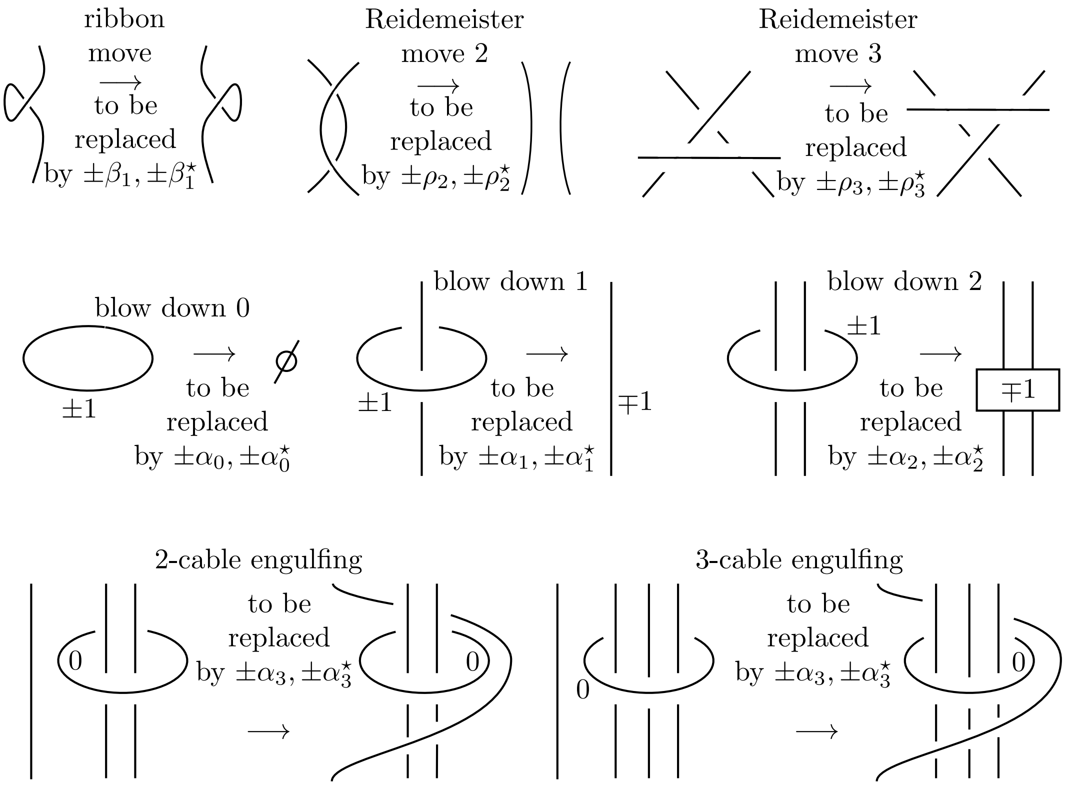


Figure 3: Martelli's finite set of 14 local moves for framed links. The ribbon move and the Reidemeister moves are redundant in the calculus, since the configurations are in $\mathbb{R}^3$. It is included to help the translation to blinks, which is a strictly planar calculus.

ϵ-neighborhood of each link component we are left with $\mathbb{M}^3\backslash(\mathbb{S}^1 \times \mathbb{D}^2)_k = \mathbb{S}^3\backslash(\mathbb{S}^1 \times \mathbb{D}^2)_k$. The fraction specifies, in the toroidal boundary inside $\mathbb{S}^3$, the homology type $(\pm p, q)$ of the curve that is contractible in the solid torus inside $\mathbb{M}^3$. For each component, we then identify the simple curve given by the homological base pair with the meridian of a canonical copy of a solid torus in $\mathbb{R}^3$ so as to completely specify the pasting of the solid torus closing the toroidal hole. Lickorish's breakthrough was to prove that any $\mathbb{M}^3$ has inside it a finite number k of disjoint solid tori so that $\mathbb{M}^3\backslash(\mathbb{S}^1 \times \mathbb{D}^2)_k = \mathbb{S}^3\backslash(\mathbb{S}^1 \times \mathbb{D}^2)_k$.

(2.1) Proposition. *Given any fractional framed link it is possible to obtain, by an effective algorithm, an integer framed link inducing the same 3-manifold.*

Proof. See [23] for an algorithmic proof and a simple complete example in page 79 of [17], which illustrate the general algorithm. This example relates to the closed hyperbolic orientable 3-manifold with smallest volume known, [9]. □

A *blackboard framed link* is an integer framed link, given as a fixed projection in the plane (with upper and lower strands recorded) where the integer framing of a component is the algebraic sum of the signs of its self-crossings.

(2.2) Proposition. *Given any integer framed link it is possible to obtain a blackboard framed link inducing the same 3-manifold.*

Proof. Just introduce an adequate set of positive or negative curls to the components to have their self-writhes agreeing with the required integer framing. □

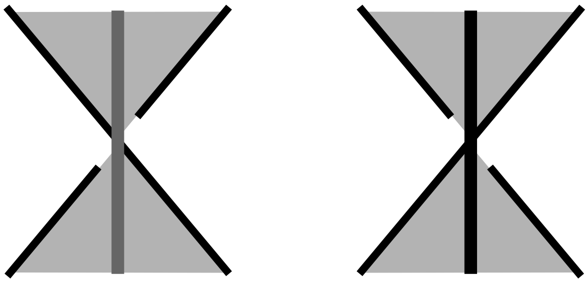


Figure 4: Obtaining the blink from the link. The inverse procedure is straighforward.

3 A formal calculus on blinks

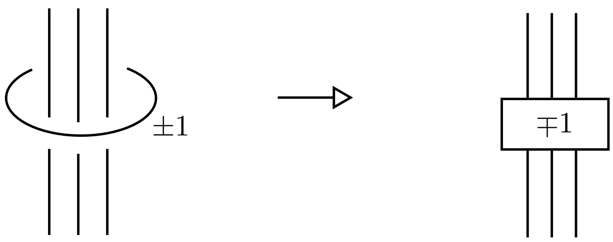


Figure 5: The Fenn-Rourke single n-parametrized move: a topological blow-down in the number of components. The number $n \geq 0$ of vertical strands crossing the unknot is arbitrary (here $n = 3$). The box marked with $+1$ (-1) indicates one full counterclockwise (clockwise) twist.

(3.1) Theorem. *Two blinks represent the same oriented closed 3-manifold if and only if they are linked by a finite sequence where each term is in the subset*

$$\{\pm\rho_2, \pm\rho_2^\star, \pm\rho_3, \pm\alpha_0, \pm\beta_1, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4\}$$

of Fig. 7 or their inverses.

Proof. The moves are basically a reformulation for blinks of the calculus in [20] restricted to blackboard framed links. The only difference are in moves $\pm\alpha_3, \pm\alpha_3^\star$ and $\pm\alpha_4, \pm\alpha_4^\star$ which can be easily checked to produce an equivalent set of moves for blackboard framed links as those of Martelli's. □

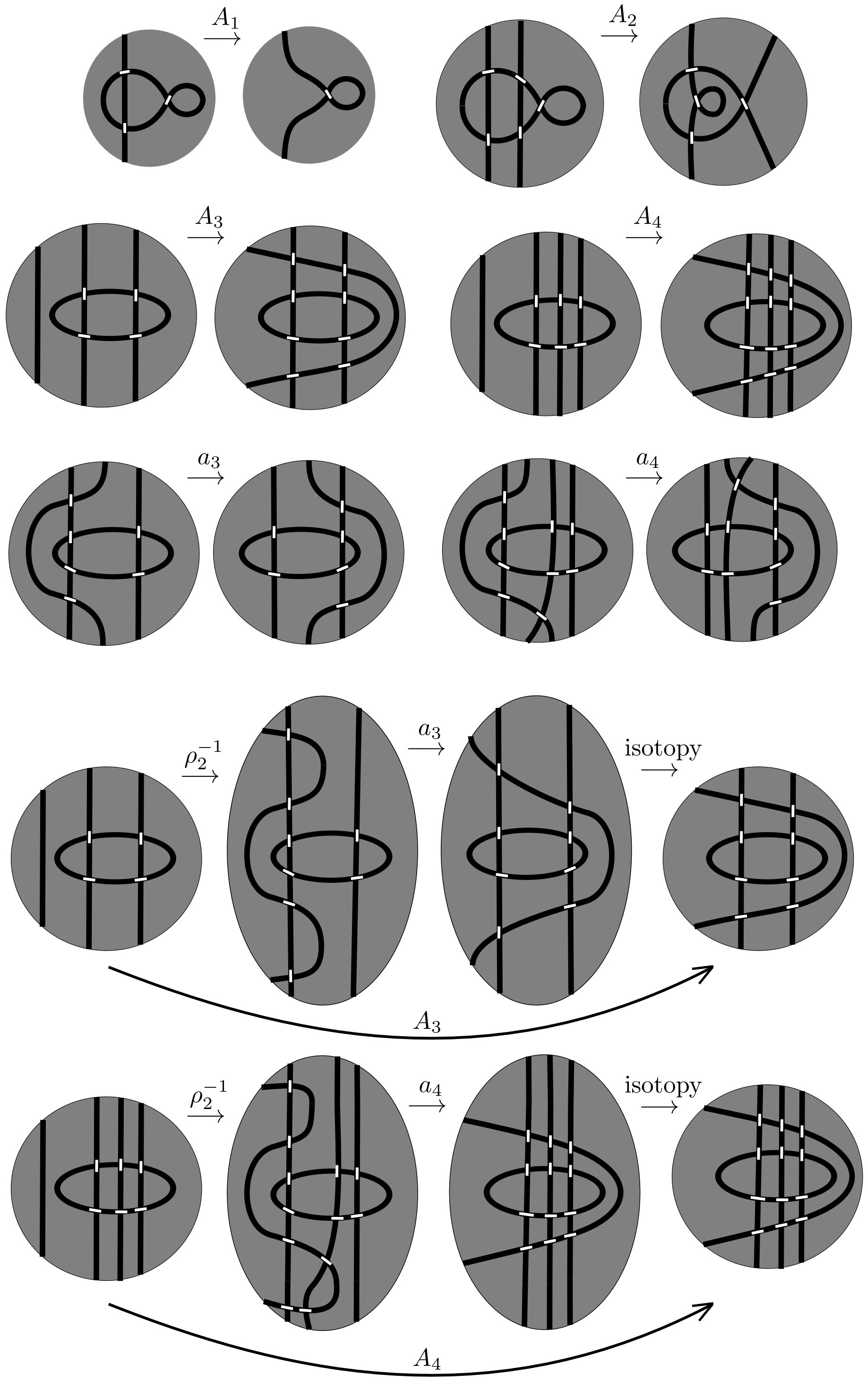


Figure 6: Reformulation of moves A_3 and A_4 into a_3 and a_4

Any plane drawing whatsoever of a graph with an arbitrary bipartition of its edge set, that is, a blink, corresponds to a unique closed oriented 3-manifold via the associated blackboard framed links. An important aspect about blinks is that each one possesses an easily obtainable *canonical form* inducing the same 3-manifold: it is named the *representative of the blink* and is obtained

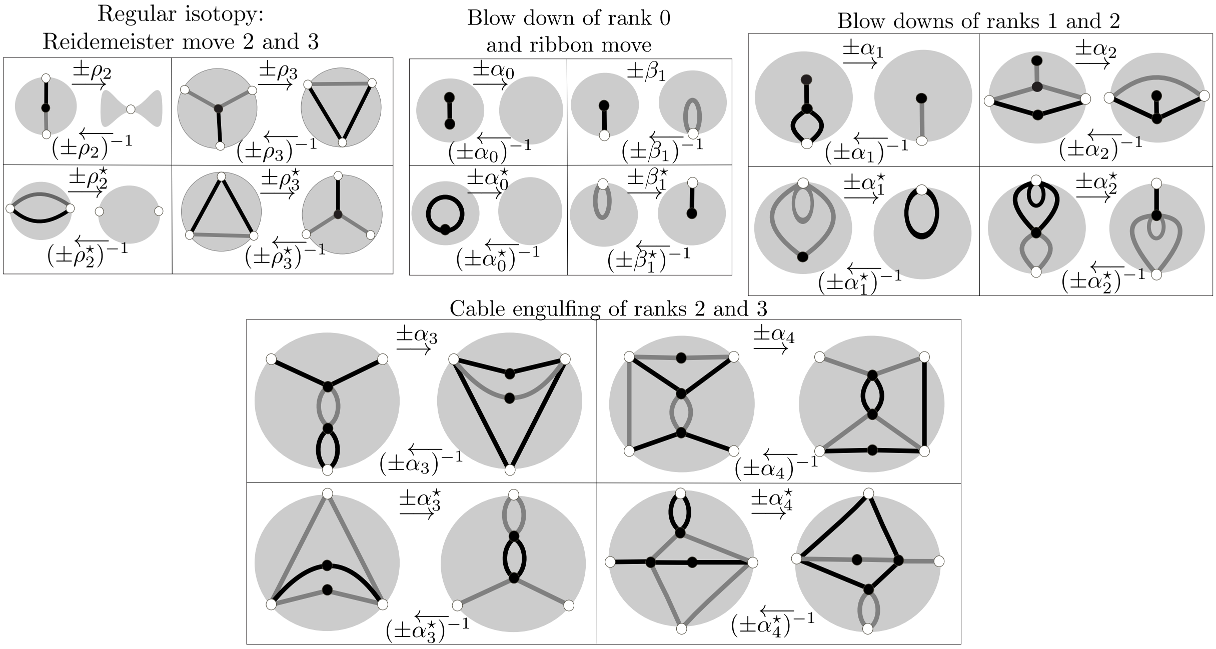


Figure 7: **Blink-coin reformulation of Martelli's calculus**: there are 8 types of moves, each having four moves, a total of 32 local inverse pairs of moves. The four moves of the same type are obtained either by changing sign which means to interchange the gray and black colors of the edges (changing the parity) or by plane duality (represented by the starred moves) followed by the interchanging of gray and black (which produces the same associated blackboard framed link). There are a grand total of 64 moves, because each move goes from left to right (which names the moves) or from right to left (its inverse move, which must have a -1 exponente) in each one of the 32 pairs. For each move a local disk (named *coin*) with an internal specific configuration is replaced by another coin with another internal specific configuration. The complement of the disk contains a completely arbitrary blink whose intersection with the the internal blink is formed by some isolated attachment vertices represented by small white circle in the boundary of the coin. The number of such attachment vertices is the first index of the move. In all but moves $\pm\rho_2$ corresponding to Reidemeister move 2 in the blackboard framed link, the replacing coin is a disk; in the four exceptional moves one disk becomes a pinched disk (or vice-versa), but this causes no harm. We made no effort in minimizing the number of moves. For instance moves $\pm\rho_3$, $\pm\rho_3^\star$ and $\pm\alpha_3$, $\pm\alpha_3^\star$ are equivalent because they are self-dual. As a matter of fact, beyound the self-dual moves we can show that every $\alpha_{ij}^\star$ move is redundant by global considerations. The last 4 types of moves, on the contrary of the previous ones, are not direct translations from Martelli' s calculus restricted to blackboard framed links. However it is easy to show that our whole set of moves imply and are implied by the direct translation of Martelli' s. Our last four types of moves seem more convenient as compared with those of Martelli's, because they maintain the number of edges and are either self-dual, $\{\pm\alpha_3, \pm\alpha_3^\star\}$ involving 6 edges, or clearly involutions, the biggest ones involving 9 edges, $\{\pm\alpha_4, \pm\alpha_4^\star\}$.

by lexicography from a small number of conventions, fixed in advance. This is explained, with a great amount of details, in L. Lins' thesis, [16].

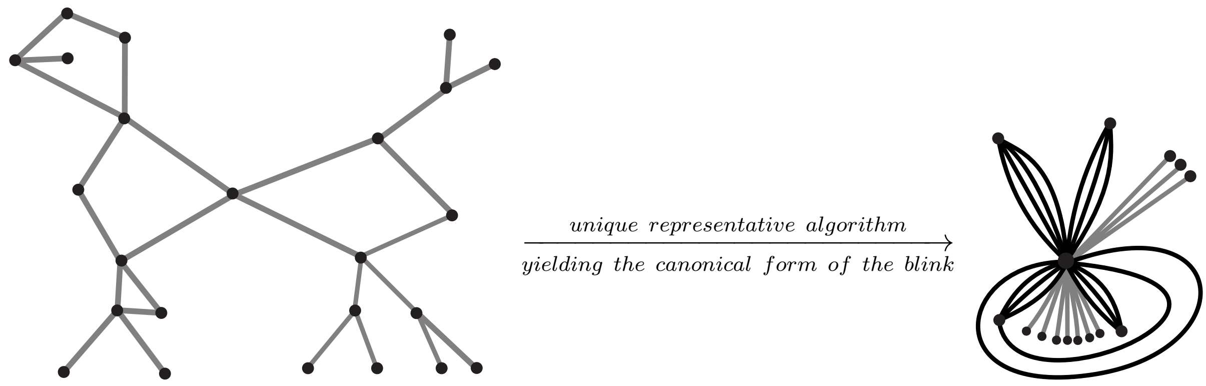


Figure 8: Blink representative algorithm: doglike blink with 27 edges and its representative with 25 edges.

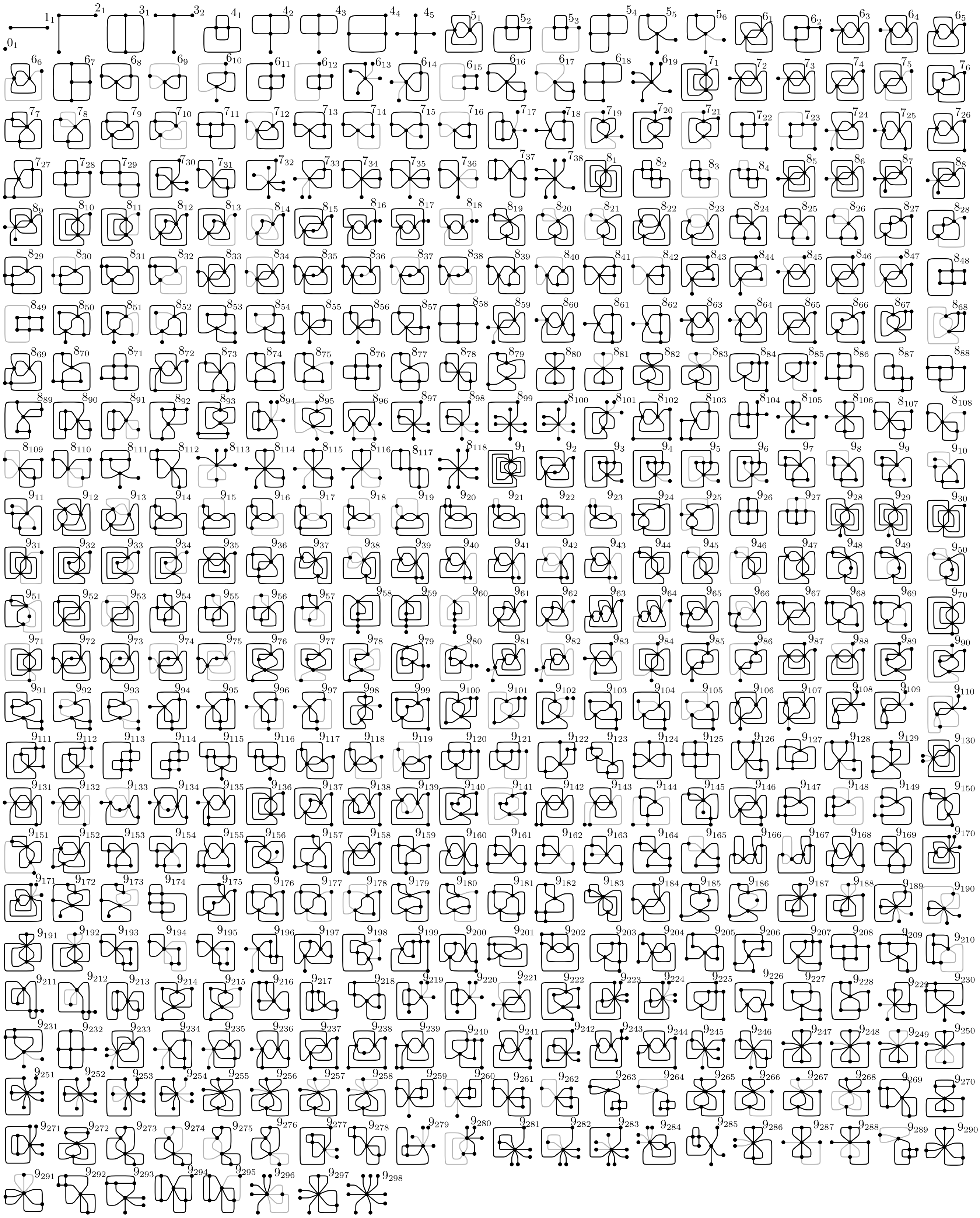
4 A complete duplicate free census of 9-small 3-manifolds

In references [11], [16] and [17] we have defined and show how a *blink*, that is, a plane graph with an arbitrary partition of its edges (here presented as colors black and gray) induces a well defined closed oriented 3-manifold. Moreover each such a manifold is induced by a blink (in fact, by infinite blinks). An *n-small* is a closed, orientable, and prime 3-manifold is a manifold induced by a blink with at most n edges. Relative to [16] the blinks of next theorem have receive two additions, the representative blinks $U[1563]$ and $U[2165]$. Also the previous HG12QI-class 6_5 became the homemorphism class 0_1 corresponding to $\mathbb{S}^2 \times \mathbb{S}^1$. We have decreased by 1 the numbering of the $HG12QI$-classes $6_6, 6_7, \ldots, 6_{20}$ become the homeomorphisms classes $6_5, 6_7, \ldots, 6_{19}$. This is because the HG12QI-classes 9_{126} and 9_{199} of [16] split into two topological classes. An objective of the present work is to prove that the splitings indeed take place.

We observe that the blinks are enlarged in the appendix, showing them together with the corresponding blackboard framed links. The notation n_i attached to each blink below, is the name of its homeomorphism class, not merely its HG12QI-class, as in [16].

This paper concludes the proof of the following theorem:

(4.1) Theorem (The first 489 closed orientable 3-manifolds). *Let $\mathbb{M}^3$ be a closed, oriented and prime 3-manifold induced by a blink with at most 9 edges. Then $\mathbb{M}^3$ is homeomorphic to exactly one of the 3-manifolds induced by the 489 blinks below. Morever all of these are pairwise non-homeomorphic. (However being redundant, we also present the corresponding census for the blackboard framed links. These census are enlarged in the Appendix)*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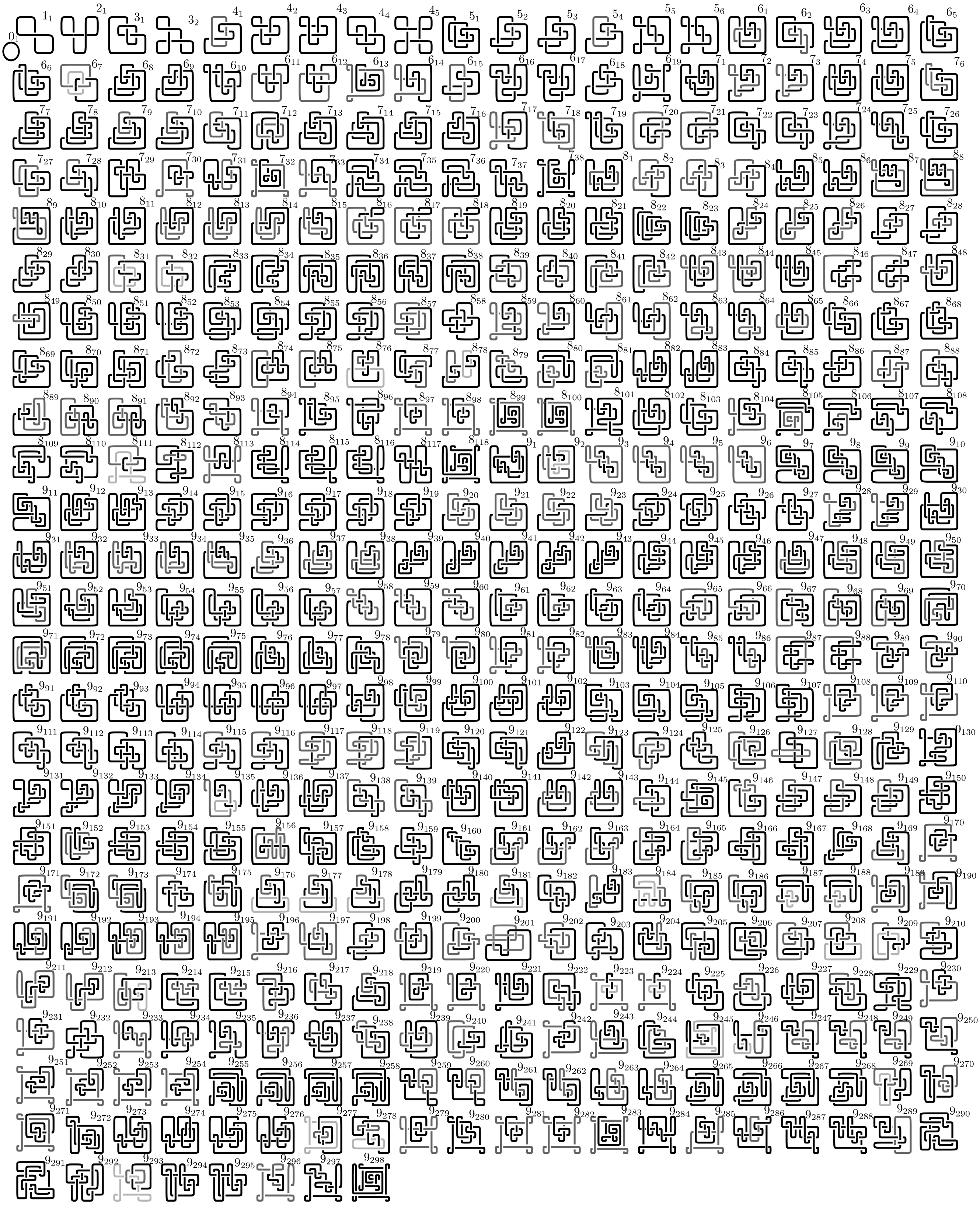

Proof. The proof follows from L.Lins' thesis and from the discussion about the lengths of the smallest geodesics of the classes 9_{126} and 9_{199} $\square$

5 Definition of Gem

For completeness we briefly recall the basic definitions of gem theory, leading to its definition, [17]. A *4-graph* G is a finite bipartite 4-regular graph whose edges are partitioned into 4 colors, 0,1,2, and 3, so that at each vertex there is an edge of each color, a proper edge-coloration, [3]. For each $i \in \{0, 1, 2, 3\}$, let E_i denote the set of i-colored edges of G. A $\{j, k\}$-residue in a 4-graph G is a connected component of the subgraph induced by $E_j \cup E_k$. A 2-residue is a $\{j, k\}$-residue, for some distinct colors j and k. A *gem* is a 4-graph G such that for each color i, $G\backslash E_i$ can be embedded in the plane such that the boundary of each face is a 2-residue. From a gem there exists a straightforward algorithm to obtain a closed orientable 3-manifold, in two different, dual ways. Every such a manifold is obtainable in this way. An unecessary big gem is obtained from a triangulation T for a manifold by taking the dual of the barycentric subdivision of T. Here the colors corresponds to the dimensions. Doing simplifications in the gem completely destroys this correspondence.

6 The resolution of the doubts left in L. Lins' thesis

The topological classification of the 9-small spaces was nearly completed in [16]. This work develops a theory for generating a distinguished set of blinks named U_n and indexed lexicographically, $U_n[i]$ is the i-th such blink. The relevance of U_n is that it misses no closed, orientable, prime and irreducible 3-manifold which is induced by a blink up to n edges.

The 3-manifolds of [16] are classified by homology and the quantum WRT$_r$-invariants $r = 3, \ldots, u$, with 10 significant decimal digits forming $HGuQI$-classes of blinks. Our algorithm for computing the WRT_r^u-invariants are based on the theory developed in [11]. After 6 years we have put our doubts as a Challenge to topologists and group algebraists, [19]. They were quickly solved by some researchers among others M. Culler, N. Dunfield and C. Hodgson, at least in two different ways. They proved that the two pairs of manifolds which were left unresolved by BLINK are indeed non-homeomorphic. All the other pairs in the $HG12QI$-classes 9_{126} and 9_{199} have been checked to be homeomorphic, as BLINK proved 6 years ago. The new solutions were obtained using the software [4], which uses the kernel of [26]. They also use GAP ([7]) and Sage ([24]).

The first solution that we got, and that still blows our mind, was by Craig Hodgson using length spectra techniques, based in his joint paper with J. Weeks entitled *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds* ([9]). By using SnapPy Craig showed that even though the quantum WRT-invariants as well as the volumes of the hyperbolic Z-homology spheres induced by the blinks $U[1466]$ and $U[1563]$ are the same, the *length of the smallest geodesics of them are distinct*. As for the other pair of blinks, $U[2125]$ and $U[2165]$, the same facts apply. Here is a summary of Craig's findings extracted from the SnapPy session that he kindly sent me. As Craig writes: *"The output of the length spectrum command shows the* complex lengths *of closed geodesics — the real part is the actual length and the imaginary part is the rotation angle as you go once around the geodesic."*

Class 9_{126}:

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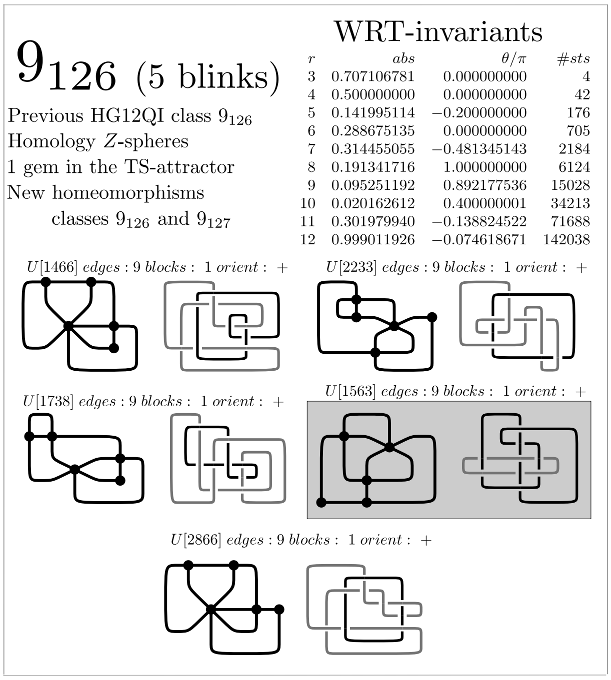
First geodesic of U[1466]: 1.0152103824828331+0.39992347315914334i.
First geodesic of U[1563]: 0.9359206605025168+2.333526236965665i.
Volume of both manifolds: 7.36429600733.
```



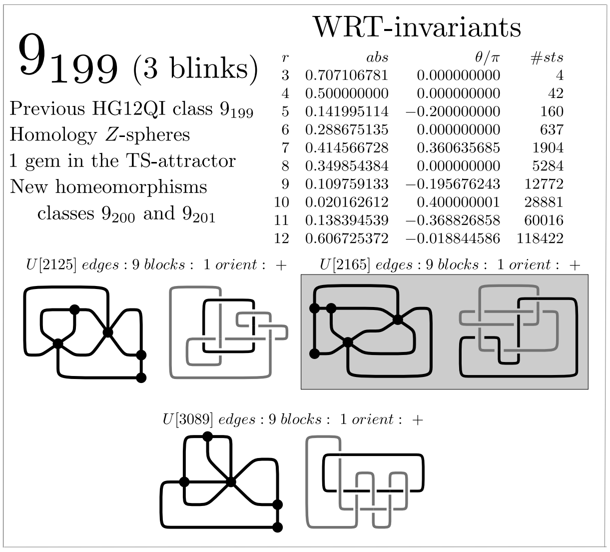


Figure 9: The two wo doubts left in L. Lins's thesis are solved showing that BLINK alone provides the topological classification of the 9-small 3-manifolds. In both cases the manifolds induced by the shaded blink-link pairs are not homeomorphic to the others in the class. These are already known to be homeomorphic by BLINK.

Class 9_{199}:

```
First geodesic of U[2125]:  0.8939075859248593+0.761197185679321i.
First geodesic of U[2165]:  0.7978548001747316+2.9487425029345973i.
Volume of both manifolds: 7.12868652133.
```

7 Conclusion

A closed orientable 3-manifold is denoted *n-small* if it is induced by surgery on a blackboard framed link with at most n crossings. We provide an instance of the general theory to produce a recursive indexation of n-small 3-manifolds up to homeomorphism. We solve this problem up to $n = 9$. Conceptually we could go on forever, finding in the way tougher and tougher examples to be distinguished by yet to be found new invariants. The topological classification of the 9-small 3-manifolds involve three invariants:

$$INV = \{\ \textit{homology},\ WRT_{12},\ \textit{length of smallest geodesic}\ \}.$$

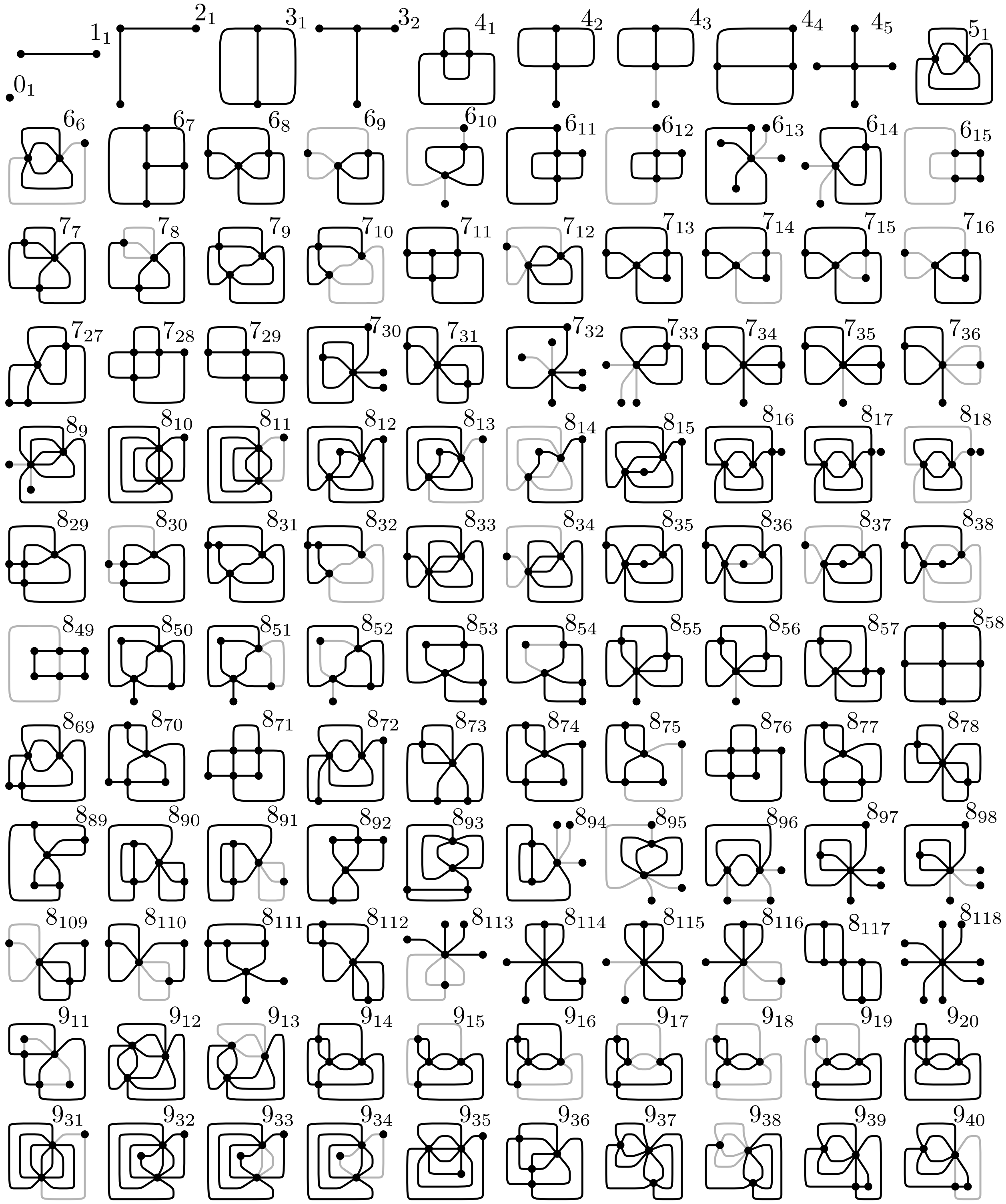
The classification was nearly complete in [16], except for two doubts. Recently, after we posted a challenge in the arXiv, [19] these doubts were solved by M. Culler, N. Dunfield and C. Hodgson using SnapPy [4]. This made us add

$$\textit{length of smallest geodesic}$$

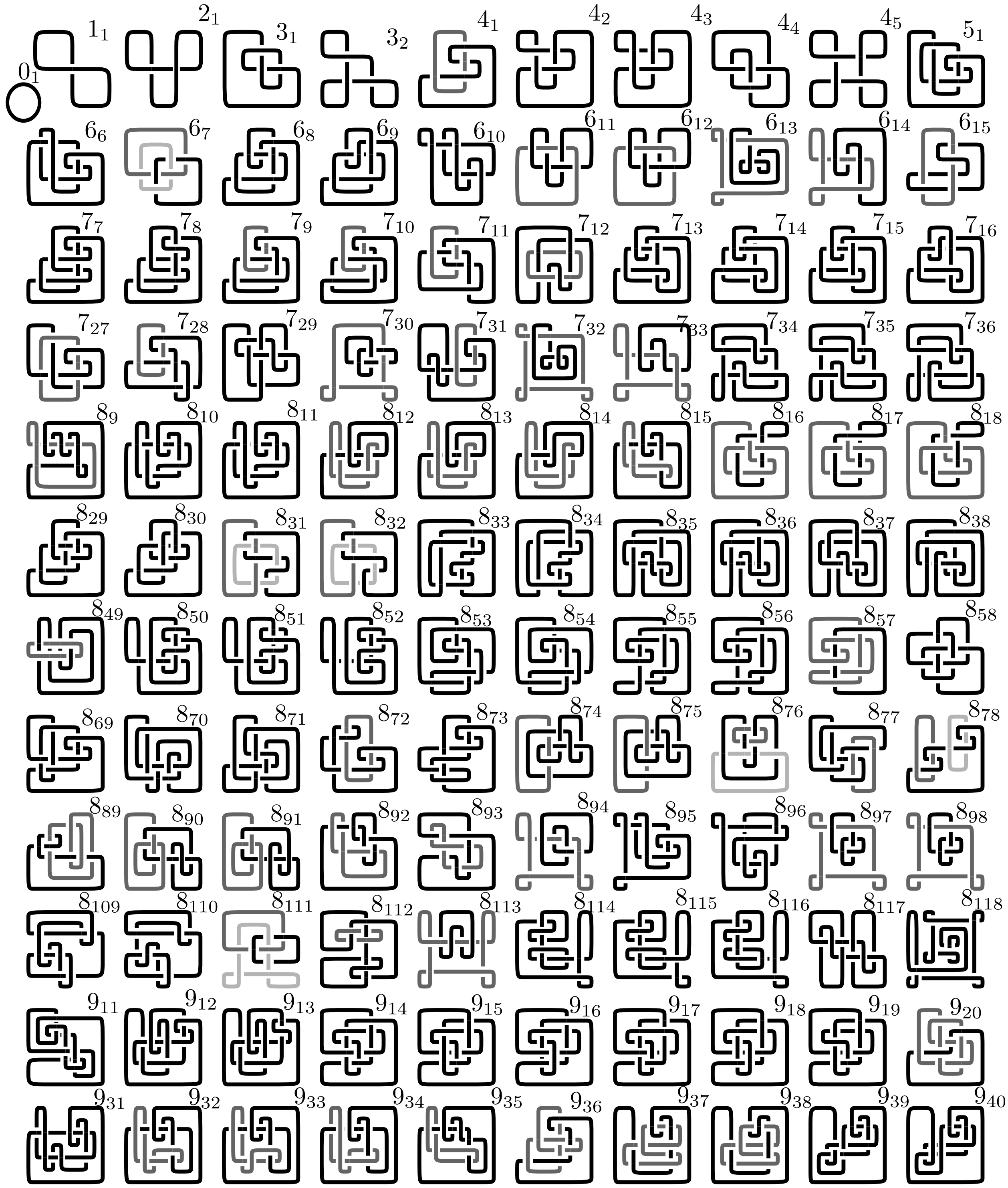
which we define as 0, if the manifold is not hyperbolic, to our list of invariants, The 9-small 3-manifold classification maintains live the two Conjectures of page 15 of [17]: the TS- and u^k-moves yield an efficient algorithm to classify n-small 3-manifolds by explicitly displaying homeomorphisms among them, whenever they exist. In the second of these conjectures $k = 1$. A recent finding by C. Hodgson concerning manifolds $T[71]$ and $T[79]$ forming the HG8QI-class 14^t_{24}, in the notation of page 239 of [16] shows that the 3 invariants are not enough to decide the pair. This pair is the first one of 11 pairs that we display as some tougher challenges to 3-manifold topologists, [18]. Craig's finding is that the volume as well as the lengths of the smallest geodesics fail to distinguish $T[71]$ and $T[79]$. He proves them to be non-homeomorphic by more sofisticated techniques, involving drilling along the smallest geodesics to get non-isometric manifolds with toroidal boundary. Using SnapPy, GAP and Sage, N. Dunfield shows that $T[71]$ and $T[79]$ are distinguished by its 5-covers.

8 Appendix: census (no misses, no duplicates) of 9-small 3-manifolds

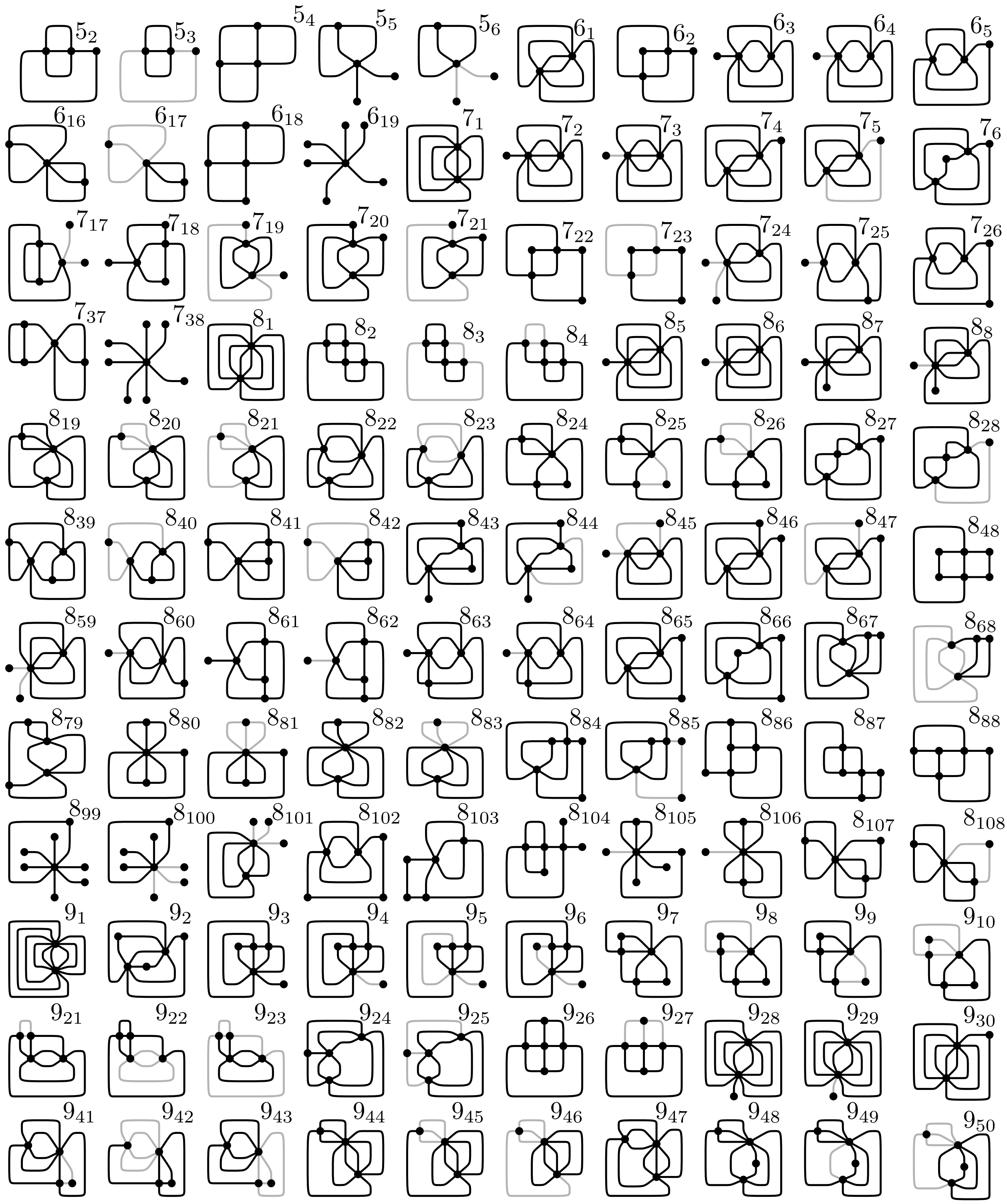
Part 1/4 in terms of blinks:



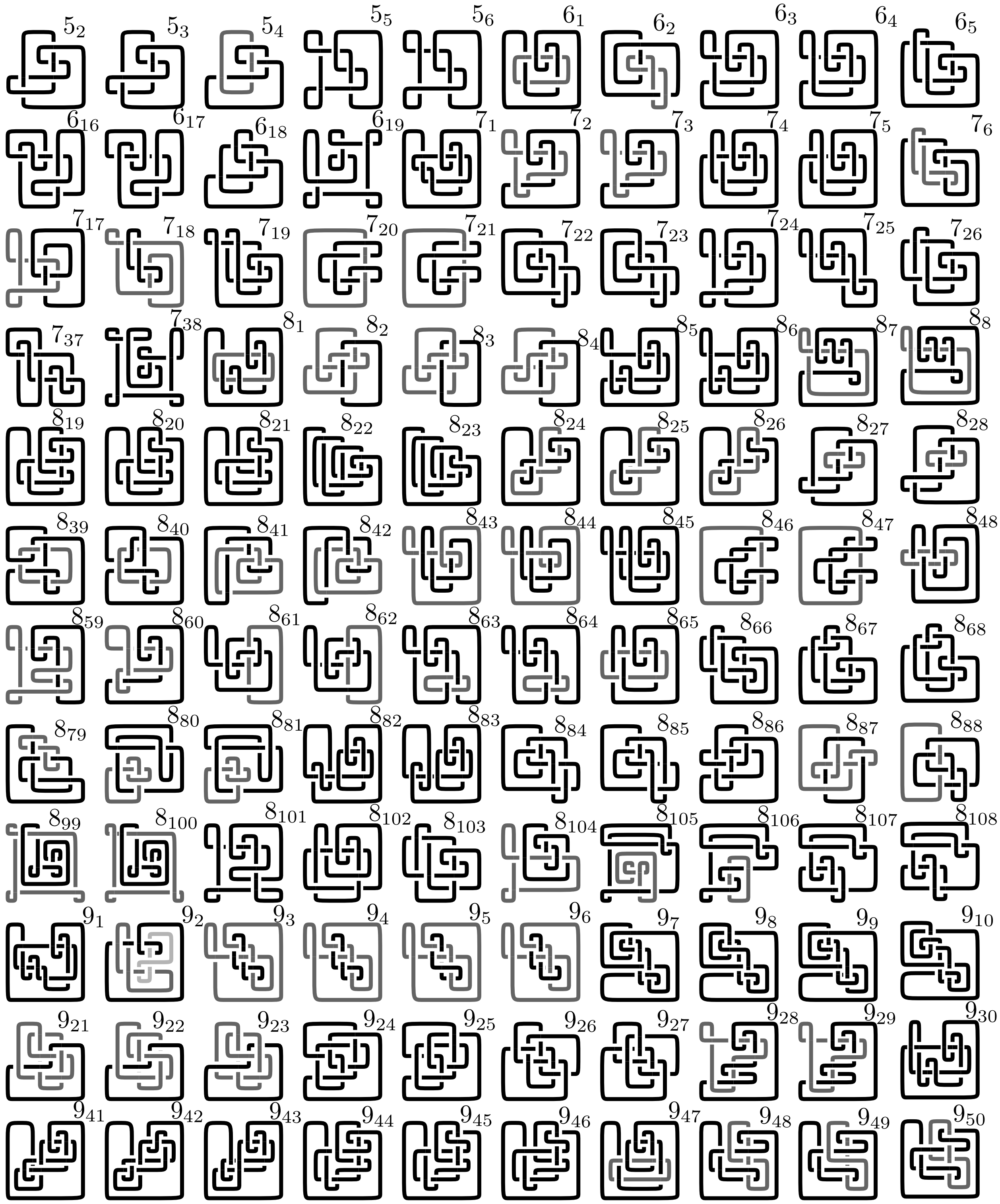
Part 1/4 in terms of blackboard framed links:



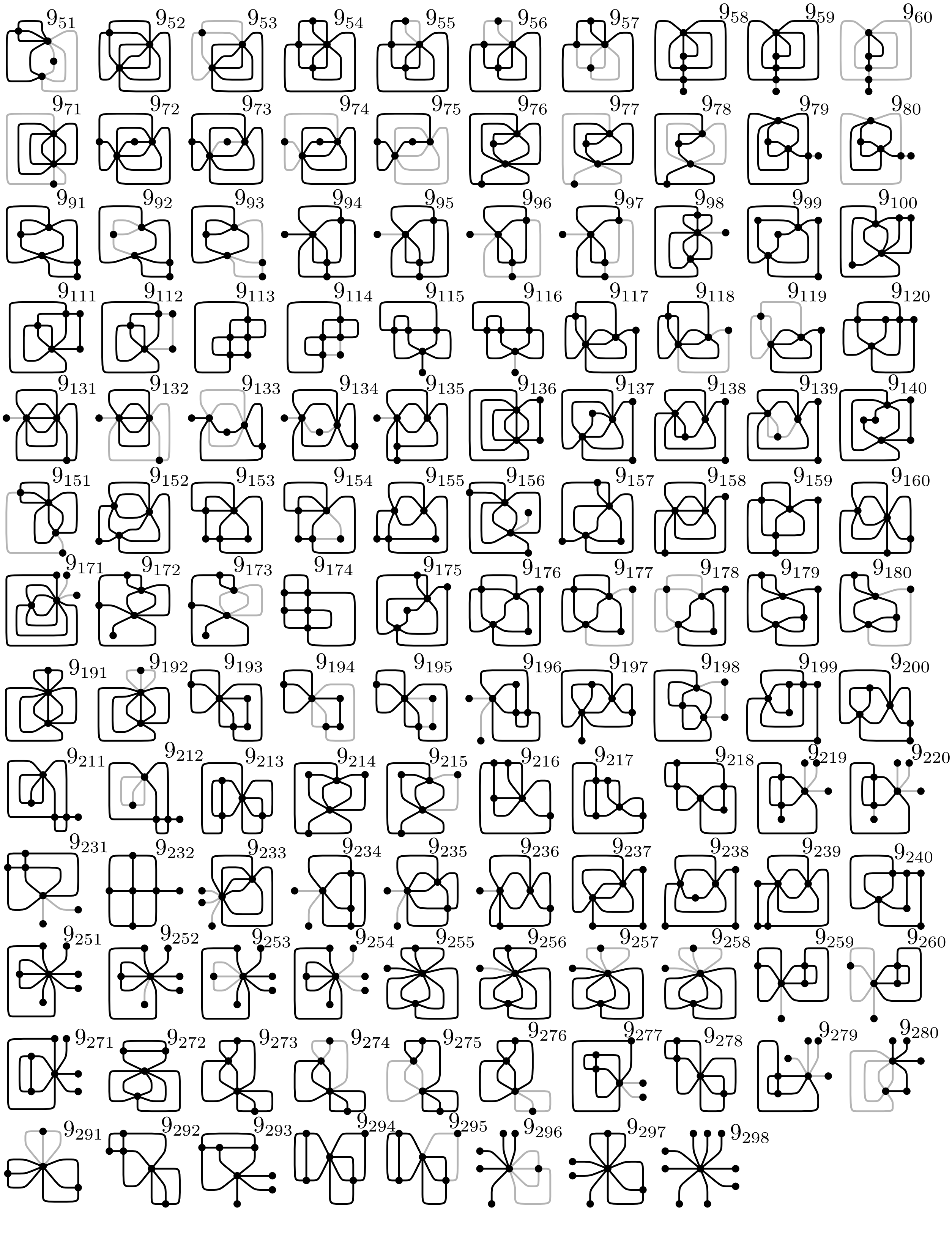
Part 2/4 in terms of blinks:



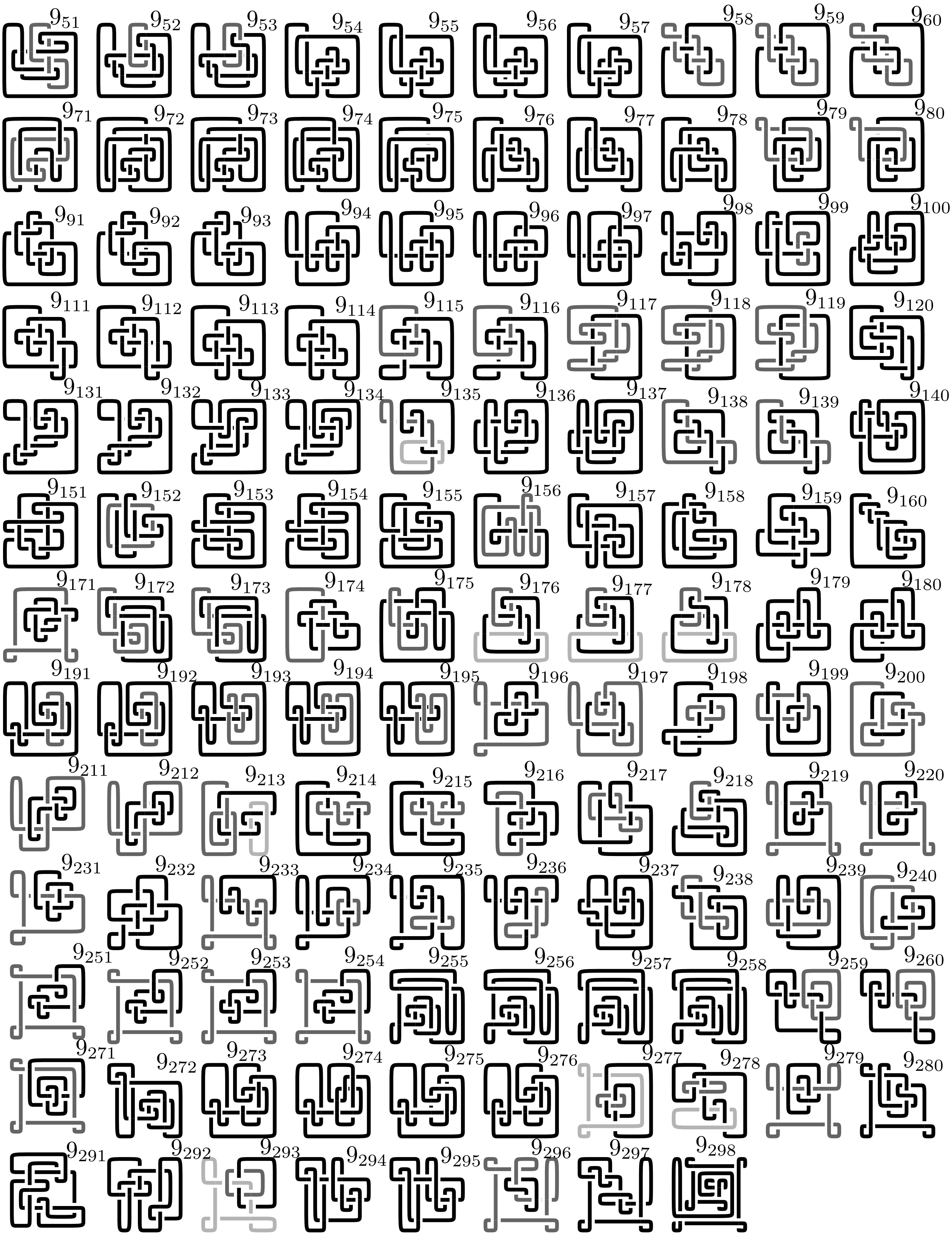
Part 2/4 in terms of blackboard framed links:



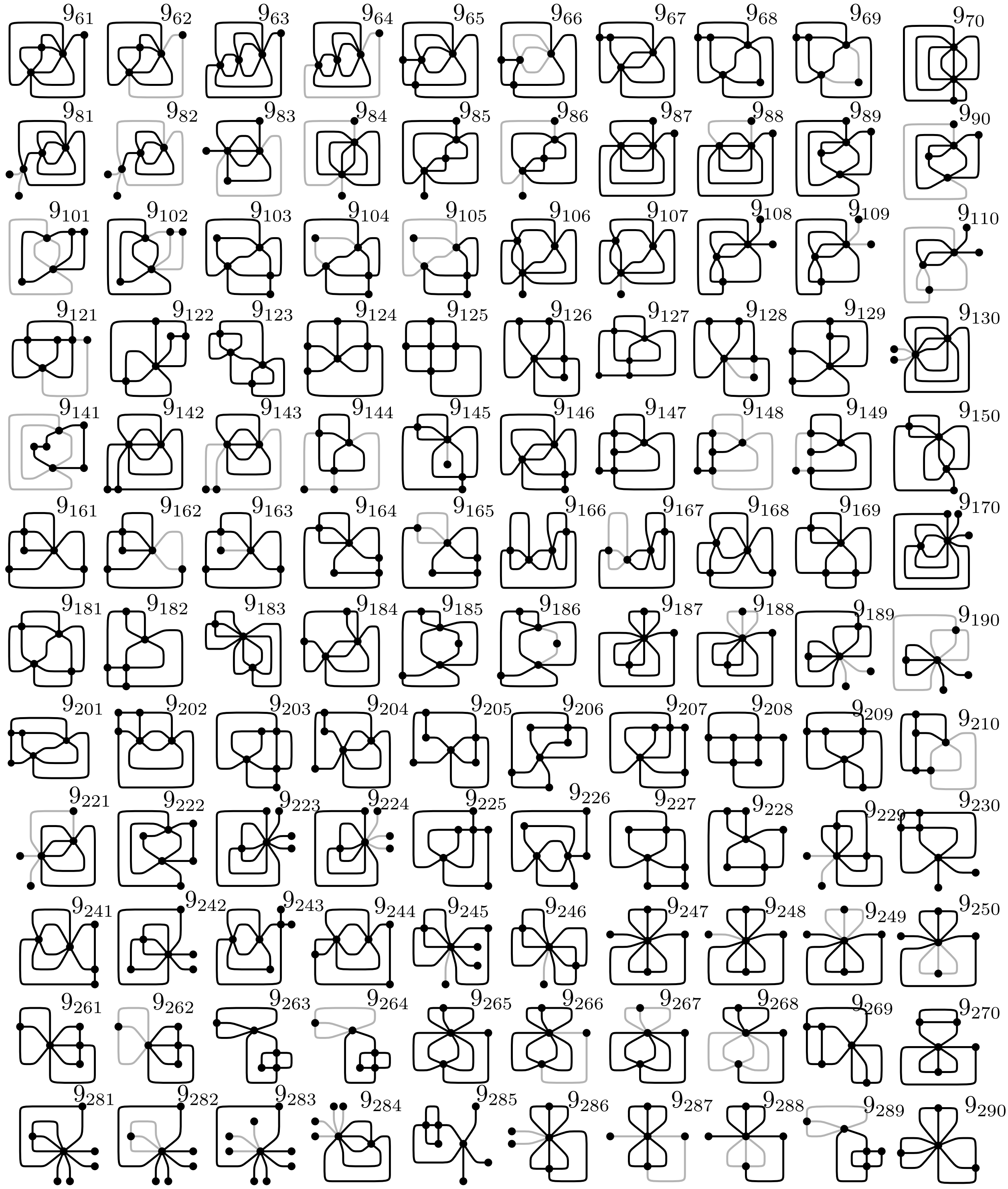
Part 3/4 in terms of blinks:



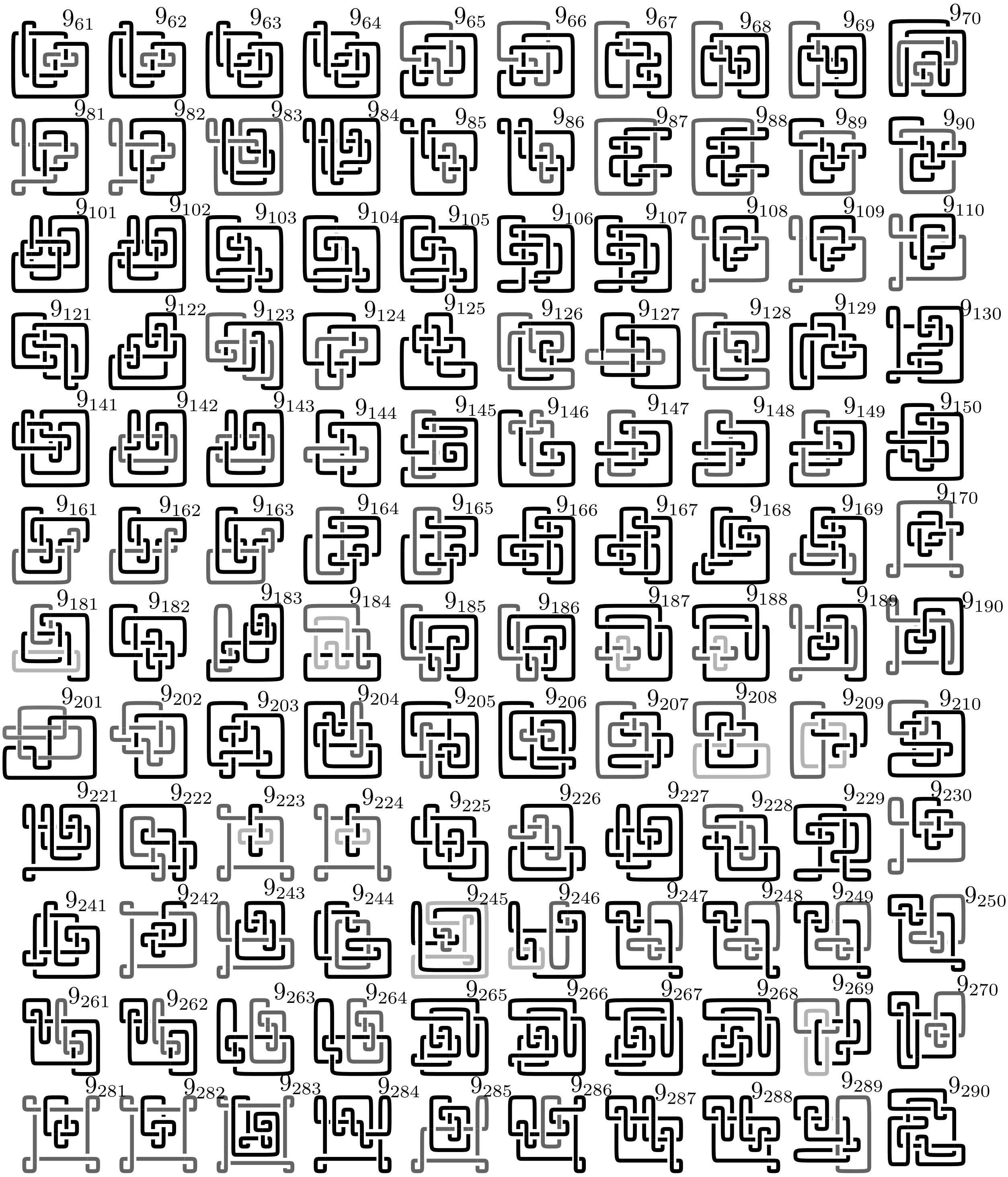
Part 3/4 in terms of blackboard framed links:



Part 4/4 in terms of blinks:



Part 4/4 in terms of blackboard framed links:



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