# All the shapes of spaces: a census of small 3-manifolds *

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**Abstract**

In this work we present a complete (no misses, no duplicates) catalogue for closed, connected, orientable and prime 3-manifolds induced by plane graphs with a bipartition of its edge set (blinks) up to 9 edges. Blinks form a universal encoding for such manifolds. In fact, each such a manifold is a subtle class of blinks, [19]. Blinks are in 1-1 correpondence with *blackboard framed links*, [10, 11] We hope that this census becomes as useful for the study of concrete examples of 3-manifolds as the tables of knots are in the study of knots and links.

## 1 Introduction

After presenting some instances of closed 3-manifolds, P. Alexandroff says in the English translation (1961) of his joint work with D. Hilbert [1], first published (1932) in German, [2]: *"These few examples will suffice. Let it be remarked here that, at present, in contrast with the two-dimensional case, the problem of enumerating the topological types of manifolds of three and more dimensions is in an apparently hopeless state. We are not only far removed from the solution, but even from the first step toward a solution, a plausible conjecture".*

John Hempel in his book (1976) *3-Manifolds* [8] writes at the opening of Section 15, entitled Open Problems: *"The ultimate goal of the theory would be in providing solutions to:* The homeomorphism problem*: provide an* effective *procedure for determining whether two* given *3-manifolds are homeomorphic, together with* The classification problem*:* effectively *generate a list containing exactly one 3-manifold from each homeomorphism class."*

### 1.1 A complete duplicate free census of 9-small 3-manifolds

In references [11], [16] and [18] we have defined and show how a *blink*, that is, a plane graph with an arbitrary bipartition of its edges (here presented as colors black and gray) induces a well defined closed oriented 3-manifold. Moreover each such a manifold is induced by a blink (in fact, by infinite blinks). An *n-small* is a closed, connected, orientable and prime 3-manifold is a manifold induced by a blink with at most $n$ edges. This paper concludes the proof of the following theorem:

**(1.1) Theorem (The first 489 closed orientable 3-manifolds).** *Let* $\mathbb{M}^3$ *be an 9-small 3-manifold. Then* $\mathbb{M}^3$ *is homeomorphic to exactly one of the 3-manifolds induced by the 489 blinks below. Morever all of these are pairwise non-homeomorphic. (However being redundant, we also present the corresponding census for the blackboard framed links. These census are enlarged in the Appendix.)*

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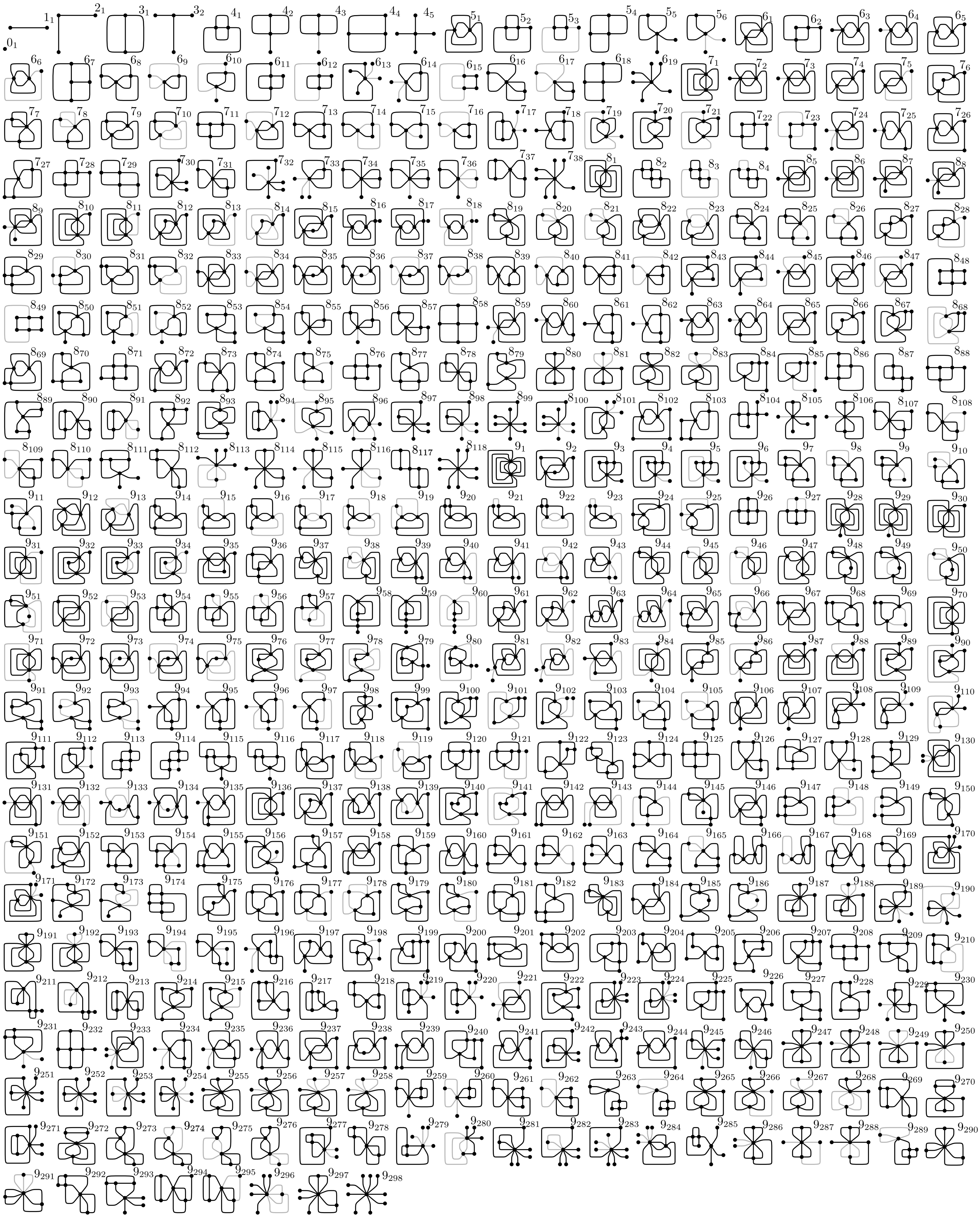

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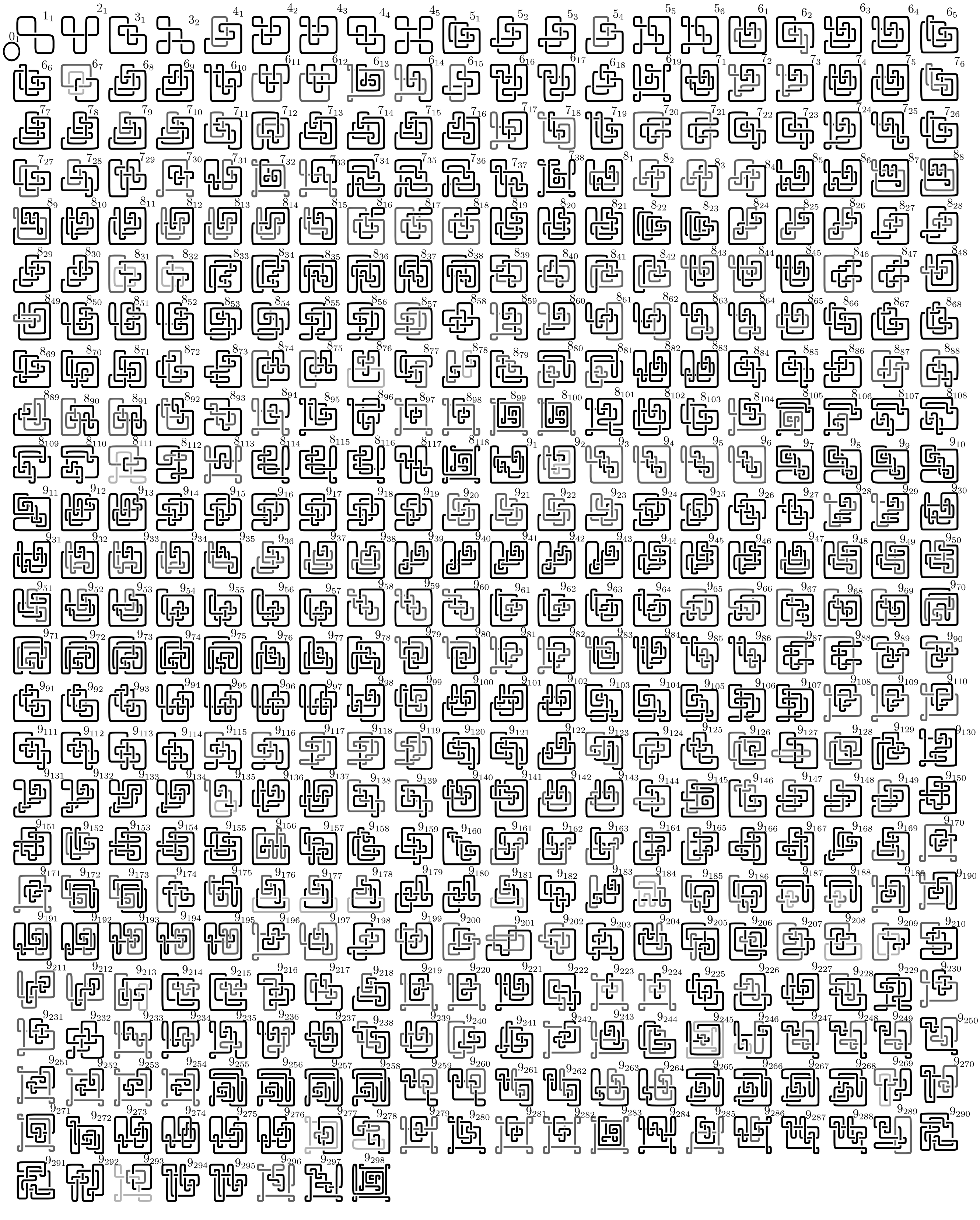


***Proof.*** The bulk of the proof follows from L. Lins' thesis under the supervision of S. Lins, [16]. In this work a theory for blink generation (missing no closed orientable 3-manifolds) is provided by pure combinatorics, lexicography and topological filtering of duplicates. This resulted in a set $U_9$ with 3437 blinks. This universal set of 9-small 3-manifold was partitioned by homology and $\text{WRT}_{12}$ into 487 classes, named $HG12QI$-classes. This is achieved by explicitly obtaining homeomorphisms between any two blinks in the same $HG12QI$-class. Each such homeomorphism is coded into a sequence of gems that is frozen in the data basis (Mysql) of BLINK forever, and, in principle can be reproduced at will. Exactly 75633 gems were used in the classifing sequences encoding homeomorphisms between the 3-manifolds of $U_9$. What remained to be done was to decide the status of the two $HG12QI$-classes $9_{126}$ and $9_{199}$ depicted in Fig. 7. After 6 years we posted these doubts as a Challenge in the arXiv, [21]. In a quick feedback, we got solutions for the two doubts: both pairs of 3-manifolds are non-homeomorphic. Among other people, M. Culler, N. Dunfield and C. Hodgson sent distinct proofs of this fact. Thus there are 489 classes of non-homeomorphic closed 9-small 3-manifolds. More details of the distinction are given in Section 3. □

Relative to page 109 of [16] the blinks of Theorem 1.1 have receive two additions, the representative blinks $U[1563]$ and $U[2165]$. Also the previous HG12QI-class $6_5$ became the homemorphism class $0_1$ corresponding to $\mathbb{S}^2 \times \mathbb{S}^1$. We have decreased by 1 the numbering of the $HG12QI$-classes $6_6, 6_7, \ldots, 6_{20}$ become the homeomorphisms classes $6_5, 6_7, \ldots, 6_{19}$. This is because the HG12QI-classes $9_{126}$ and $9_{199}$ of [16] split into two topological classes. An objective of the present work is to prove that the splitings indeed take place. We observe that the blinks are enlarged in the appendix, showing them together with the corresponding blackboard framed links. The notation $n_i$ attached to each blink below, is the name of its homeomorphism class, not merely its HG12QI-class, as in [16].

### 1.2 Brief historical overview

It is amazing how much the picture has changed in the 80 years since the book by Alexandroff and Hilbert. The progress initiated with the deep advances in the 1950's and 1960's, starting with the proof that 3-manifolds are triangulable by Moise (1952), [29]. Next the presentation of them by framed links by Lickorish (1962) [14]. Following that Kirby presented its calculus for framed links (1978)[12]. Starting in the early 1980's W. Thurston's provided gret breakthroughs, developing his conceptual theory on hyperbolic manifolds and of the geometrization conjecture. In the final 1980's early 1990's Witten [34] broke the psicological barrier that there were no good invariants for 3-manifolds. Following that a number of eastern European mathematicians like N. Reshetikhen, V. Turaev and O. Viro, [32, 30] using quantum groups were able to put in mathematical solid ground Witten's findings. One of us, S. Lins, was a witness of the excitement these developments caused. L. H. Kauffman and W. B. R. Lickorish discovered the relationship of the Temperley-Lieb algebra with the new invariants, [15]. Starting with a sabatical leave in to Chicago in 1990, S. Lins produced the joint monography with Kauffman [11], where blinks are first defined and extensive WRT-invariant computations were obtained from the theory developed from scratch, independently and simpler than that of quantum groups. In the early 2000's, G. Perelman revolutionized the field proving Poincaré's Conjecture and Thurston's Geometrization Conjecture. More recently in the 2010's, I. Agol is leading the field in this era post-Perelman. Of course, this is only a diagonal list of researchers. Many more have contributed and some are extremely active in this era pos-Perelman, [6]. Currently there is a great amount of important reserch issues going on and these are exciting times for 3-manifold theory. See the recent essay of E. Klarreich in the Simons Foundation, [13].

## 2 Blinks and gems

Unexplored simplicity. This was the reason for birth of this work many years ago. Repeating, a *blink* is a finite plane graph (that is, given embedded in the plane) together with an arbitrary bipartition of its edges into black and gray. For completeness we briefly recall the basic definitions of gem theory, leading to its definition, [18] and to its calculus bsed in dipole moves, [5, 17]. A *4-graph* $G$ is a finite bipartite 4-regular graph whose edges are partitioned into 4 colors, 0,1,2, and 3, so that at each vertex there is an edge of each color, a proper edge-coloration, [3]. For each $i \in \{0, 1, 2, 3\}$, let $E_i$ denote the set of $i$-colored edges of $G$. A $\{j, k\}$-residue in a 4-graph $G$ is a connected component of the subgraph induced by $E_j \cup E_k$. A 2-residue is a $\{j, k\}$-residue, for some distinct colors $j$ and $k$. A *gem* is a 4-graph $G$ such that for each color $i$, $G \backslash E_i$ can be embedded in the plane such that the boundary of each face is a 2-residue. From a gem there exists a straightforward algorithm to obtain a closed orientable 3-manifold, in two different, dual ways. Every such a manifold is obtainable in this way. An unecessary big gem is obtained from a triangulation $T$ for a manifold by taking the dual of the barycentric subdivision of $T$. Here the colors correspond to the dimensions. Given a pair of vertices $\{u, v\}$ of a gem $G$ with $k$ linking edges $k \in \{0, 1, 2, 3\}$, in $k$ distinct colors $K \subset \{0, 1, 2, 3\}$ is called a *$K$-dipole* if $u$ and $v$ are in distinct $(\{0, 1, 2, 3\})\backslash K$-residues of $G$. A *dipole cancellation* is the operation that remove $u$, $v$ and their $k$ linking edges leaving $4 - k$ pairs of pendant edges which are reunited along the same colors. The *dipole creation* is the inverse move. The dipole creations and cancellations do not change the induced 3-manifold. It is possible to simplify substantially the gem corresponding to the dual of the barycentric subdivison of a triangulation by cancelling dipoles. These simplifications in that gem completely destroys the correspondence of colors and dimensions. Two gems induce the same 3-manifold if and only if they are linked by a finite sequence of moves where each term is either a dipole cancellation or a diplole creation, [5, 17].

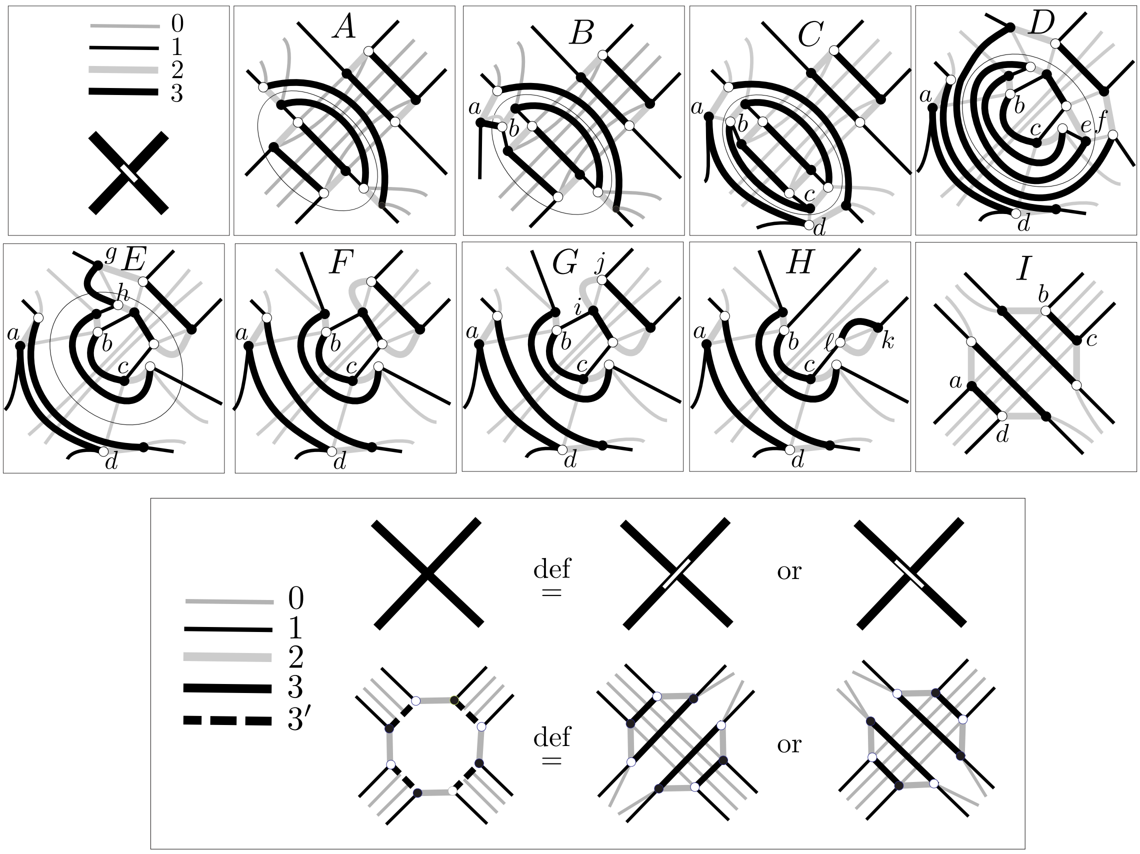



Figure 1: The $(12 \to 8)$-simplification and the BFL $\to$ gem algorithm.

In our context, a very important property of gems is that, given any blackboard framed link with $n$ crossings there is an algorithhm to present a gem having $8n$ vertices which induce the same 3-manifold.

This algorithm is displayed at the bottom part of Fig. 1. This was proved originally in Kauffman-Lins monography, page 175 of [11], which provide a gem with $12n$ vertices. By using an specific $TS$-move of [18] a gem with $8n$ vertices is obtained, page 81 of [16]. The $(12 \to 8)$-simplification by means of dipoles is depicted the upper part of Fig. 1.

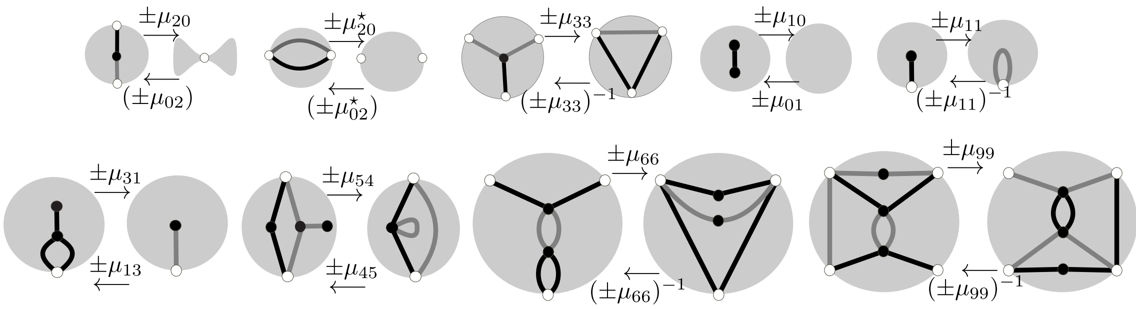



Figure 2: A (36-move,18-coin) reduced (sufficient) blink-coin calculus

### 2.1 A surprising new 1-1 correspondence and an $O(n^2)$-algorithm

Warning: the present work is completely independent of Fig. 2 and on the Theorems and on the Conjecture 2.1, 2.2, 2.3, 2.4 of this subsection, its reading can be skipped at no logical cost. But we strongly recomend the reader not to do so, because the results and the Conjecture are stated to situate our research in its appropriate context and timing.

Any closed, oriented 3-manifold is induced by some blink. Even though this object has been around since 1994 when it was introduced in the joint research monography of L. Kauffman and S. Lins, [11], the fact that they encode oriented closed 3-manifolds remains basically unkown. This is about to change because as a consequence, [19], of a recent result of B. Martelli, [26, 27], each such a 3-manifold becomes a subtle equivalence class of blinks. In [19], a new calculus, this time on blinks, named *the blink-coin calculus* with 8 types of local moves each applied to 8 types of related sub-blinks (named coins) are shown to capture the essence of homeomorphism between 3-manifolds, in the sense of Theorem 2.1. A sufficent reduced blink-coin calculus with 36 moves is shown in Fig. 2.

**(2.1) Theorem.** *Closed orientable 3-manifolds up to homeomorphism are blinks up to the blink-coin calculus.*

The meaning of the theorem is that there exists a bijection

$$\beta : \quad \frac{3\text{–}manifolds}{homeomorphisms} \quad \longrightarrow \quad \frac{blinks}{blink\text{–}coin\ calculus}.$$

If the manifold is given by a framed link, $\beta$ is obtained by a linear algorithm. However, if it is given by a gem, by a triangulation, by a special spine [28], or by a Heegaard diagram, then a polynomial algorithm to find $\beta$ seems to be an open problem. However, rescuing the situation there exists a recent work of S. Lins and his former student R. Machado, which is reported in a sequence of 3-papers posted in the arXiv proving that there exists an $O(n^2)$-algorithm to go from a *resoluble* gem to a blink inducing the same 3-manifold.

**(2.2) Theorem.** *There exists an $O(n^2)$-algorithm to go from a* resoluble *gem to a blink inducing the same 3-manifold.*

The parts of the work are available at [22, 23, 24]. Even though the work leading to the above theorem is still being polished, we feel that is important to mention it here because they imply that there is an $O(n^2)$-algorithm to find $\beta$ in the cases that the 3-manifold is given by a resoluble gem. The proper definition of resoluble gem is still unecessary complicated in the posted papers. Recently, in a still unpublished work, with the adequate definition of resolubility the same authors have proved the following theorem:

**(2.3) Theorem.** *If a 3-manifold is given by a framed link then there exists a linear algorithm to obtain a resoluble gem inducing the same manifold.*

Two gems induce the same 3-manifold if and only if they are linked by a finite sequence of *blobs moves* and *valid flip moves.* This result is proved for gems of arbitrary dimensions by S. Lins and M. Mulazzani in [17]. What is needed to close by polynomial algorithms this chain of objects is to establish the following Conjecture which we feel is impossible to be false:

**(2.4) Conjecture.** *A resoluble gem remains so under a blob move, or a valid flip move.*

If is true, as we are sure it is, the Conjecture means that there is an $O(n^2)$-algorithm to produce a blink inducing the same space as the one given by an arbitrary gem, an arbitrary triangulation, an arbitrary Heegaard diagram or an arbitrary special spine. Note that, as a bonus we gain the capability of computing the Witten-Reshetikhin-Turaev invariants [30, 11] from the latter 4 presentations, what is currently impossible, because the WRT-invariants are only computable from blinks.

We stress that there will be many interesting and deep consequences of Theorem 2.1 to the topology of 3-manifolds and to the combinatorics of plane graphs.

## 2.2 The complementary roles of blinks and gems

Blinks are in 1-1 correspondence with blackboard framed links which in turn encodes every closed oriented 3-manifold. If we consider these three encodings of the same 3-manifold given in Fig. 3, the blink is the one that has the smallest "perceptual complexity". The common manifold of this example is the binary tetrahedral space defined by $\mathbb{S}^3$ (as a topological continuos group) up to the action of the non-comutative binary tetrahedral group $\langle 3, 3, 2\rangle$, (see [25]) which has 24 elements. The 3-manifold $\mathbb{S}^3/\langle 3, 3, 2\rangle$ has an spherical geometry. Its attractor (definition given shortly) consists of the single gem depicted at the right of Fig. 3.

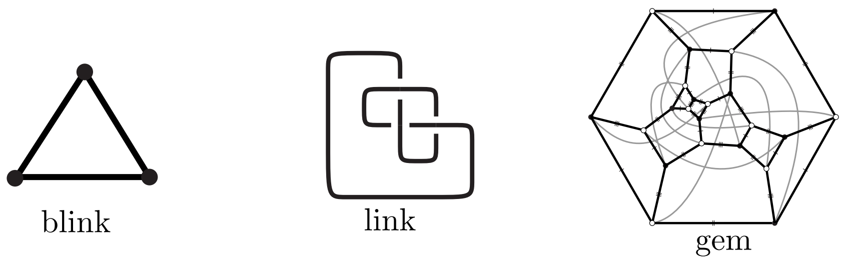



Figure 3: The minimum blink, minimum link and minimum gem inducing the binary tetrahedral space

Blinks are very easy to generate recursively. They have a rich simpifying theory which permits the generation of 3-manifold catalogues. Moreover, their isomorphism problem is computationally simple. Being blinks so good, why do we need gems? The answer is that to prove that blinks induce the same 3-manifold (when they do) remains (probably even with the new blink-coin calculus) a very difficult problem. It is straighforward to obtain a canonical gem from a blink, [11, 16]. Proving that two gems induce the same 3-manifold (when they do) is much easier because of the rich simplification theory of them (based on at least 4 intertwined planarities) leading to the *attractors of 3-manifolds*, [18]. These are defined as the set of gems inducing the manifold having the minimum set of vertices. The important computational point is that the attractor usually have few members, particularly if the manifold has a uniofolrm geometry euclidean, hyperbolic, spherical.

Blinks are very good at proving that two manifolds are distinct, because from them we can extract the WRT-invariants, which are very strong invariants, yet not a complete one. These invariants do not have a direct computation from triangulations, including gems. Thus, gems and blinks collaborate in a symbiotic dance to decide (at a computational level) whether two 3-manifolds are or are not homeomorphic. And many types of census become available!

We present here our modest contribution to the topic. It is placed in the confluency of two deep passions of the authors: the study of closed orientable 3-manifolds and the study of plane graphs. Here, we mean to provide a strategy for a segmented answer of Hempel's questions. The closed oriented 3-manifolds are partitioned by the number of edges in a minimum encoding of them by a blink.

Any plane drawing whatsoever of a graph with an arbitrary bipartition of its edge set, that is, a blink, corresponds to a unique closed oriented 3-manifold via the associated blackboard framed links. An important aspect about blinks is that each one possesses an easily obtainable *canonical form* inducing the same 3-manifold: it is named the *representative of the blink* and is obtained by lexicography from a small number of conventions, fixed in advance. This is explained, with a great amount of details, in L. Lins' thesis, [16].

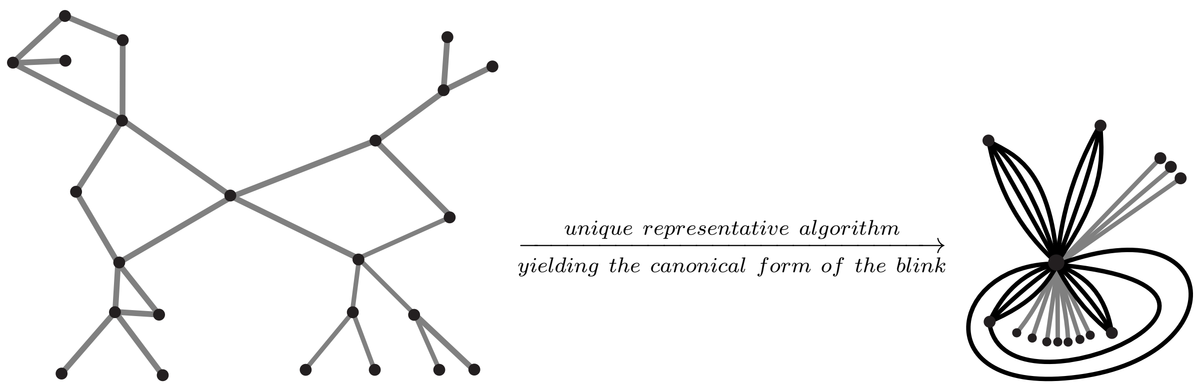
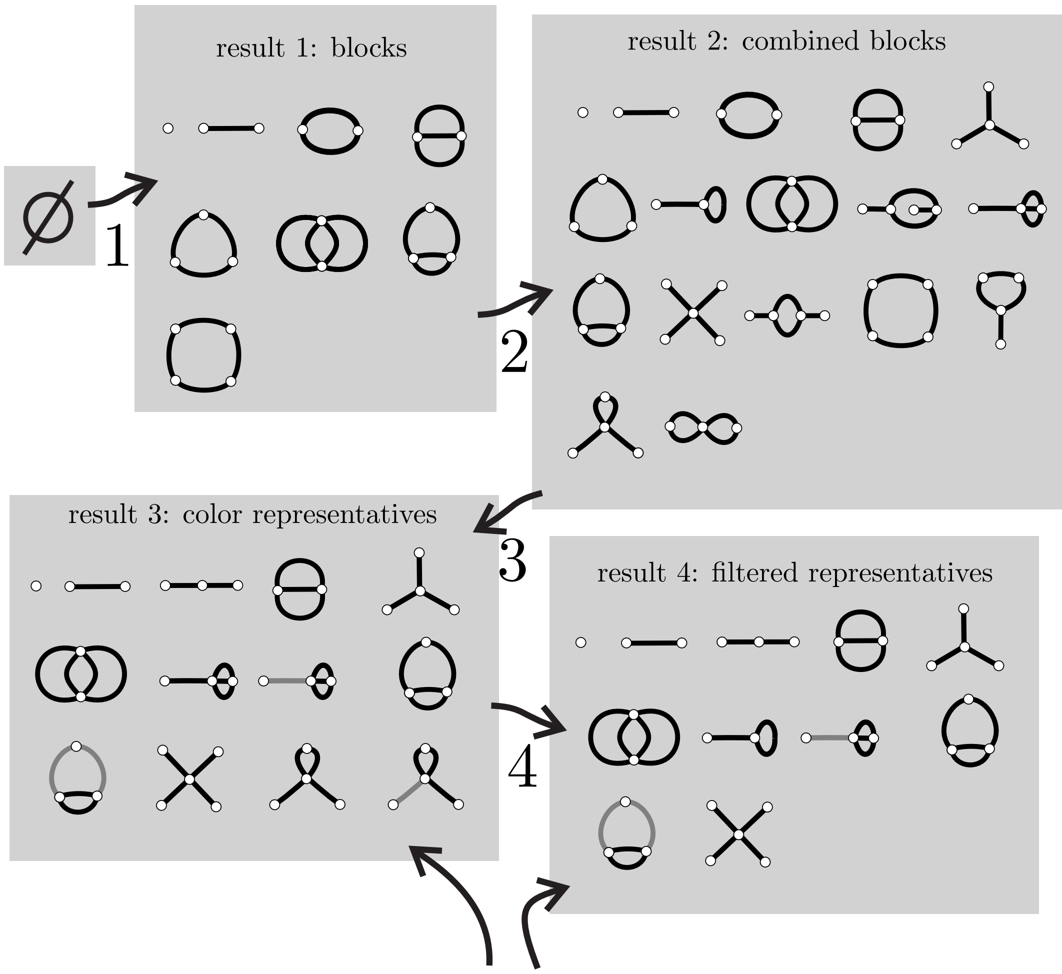



Figure 4: Blink representative algorithm: doglike blink with 27 edges and its representative with 25 edges.

Lexicography is used to define a representative unique plane graph (a canonical form) for each closed oriented 3-manifold. We explicitly solve the segmented problem up to 9 edges, see Theorem 1.1. This work provides an efficient algorithm to make available the canonical form of any closed orientable 3-manifold induced by plane graphs at the current level of the catalogue (currently 9 edges) and, theoretically, this could be extended 10, 11,. . . , $n$, for arbitrarily large $n$. Our theory gives a road to effectively name each 3-manifold classified by some set of invariants $INV$. We have a universal set of object, the blinks up to $n$ edges, which can be partitioned by these invariants. The $INV$-classes are then tried to be broken into homeomorphisms classes. New invariants are then discovered and added to $INV$ making them homeomorphisms classes of $n$-small manifolds. The difficult cases are going to appear naturally and they lead to enhancement of the theory. It is not at all impossible that this process stops and we get $INV$ so that the $INV$-classes be proved to be homeomorphisms classes for all $n$. The point we want to make is that good examples (hard to find, here exemplified by the $HG12QI$ classes $9_{126}$ and $9_{199}$) are important in obtaining progress in a general theory. A classifying $INV$ for the 9-small 3-manifolds is $INV = \{$ *homology*$, WRT_{12},$ *length of smallest geodesic* $\}$.

# 3 Universal set of blinks and the resolution of the doubts




Pipeline for obtaining the prime unavoidable class of blinks $U_4$

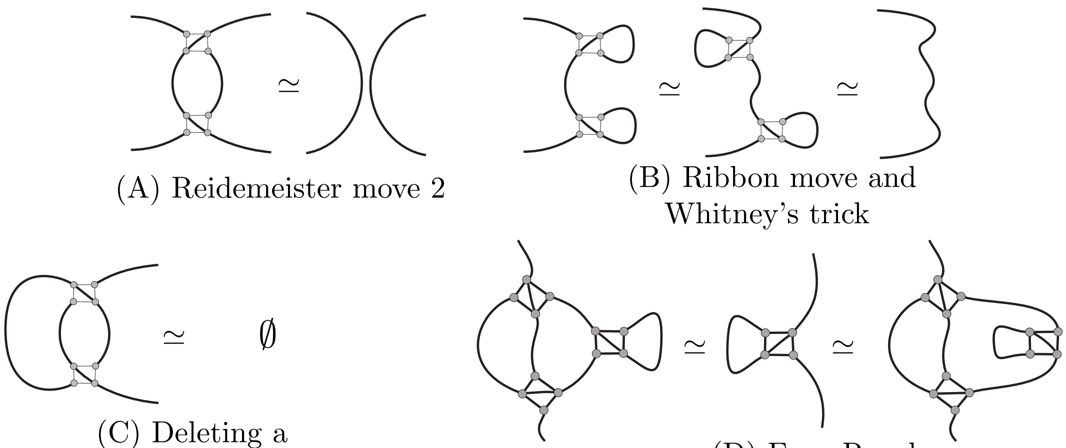
Figure 5: Pipeline for the generation of the set $U_k = U_4$

## 3.1 Topological filtering

## 3.2 Unveiling the mistery of the two doubts

The topological classification of the 9-small spaces was nearly completed in [16]. This work develops a theory for generating a distinguished set of blinks named $U_n$ and indexed lexicographically, $U_n[i]$ is the $i$-th such blink. The relevance of $U_n$ is that it misses no closed, orientable, prime and irreducible 3-manifold which is induced by a blink up to $n$ edges.

The 3-manifolds of [16] are classified by homology and the quantum WRT$_r$-invariants $r = 3, \ldots, u$, with 10 significant decimal digits forming $HGuQI$-classes of blinks. Our algorithm for computing the $WRT_r^u$-invariants are based on the theory developed in [11]. After 6 years we have put our doubts as a Challenge to topologists and group algebraists, [21]. They were quickly solved by some researchers among others M. Culler, N. Dunfield and C. Hodgson, at least in two different ways. They proved that the two pairs of manifolds which were left unresolved by BLINK are indeed non-homeomorphic. All the other pairs in the $HG12QI$-classes $9_{126}$ and $9_{199}$ have been checked to be homeomorphic, as BLINK proved 6 years ago. The new solutions were obtained using the software [4], which uses the



(A) Reidemeister move 2

(B) Ribbon move and Whitney's trick

(C) Deleting a circumcised pair Kauffman-Lins, pag 138

(D) Fenn-Rourke blow down of rank 1

Figure 6: Some topological filtering in the blink generation

kernel of [33]. They also use GAP ([7]) and Sage ([31]).

The first solution is by C. Hodgson using length spectra techniques, based in his joint paper with J. Weeks entitled *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds* ([9]). By using SnapPy Hodgson showed that even though the quantum WRT-invariants as well as the volumes of the hyperbolic $Z$-homology spheres induced by the blinks $U[1466]$ and $U[1563]$ are the same, the *length of the smallest geodesics of them are distinct*. As for the other pair of blinks, $U[2125]$ and $U[2165]$, the same facts apply. Here is a summary of Hodgson's findings extracted from the SnapPy session that he kindly sent us. As he explains: *"The output of the length spectrum command shows the* complex lengths *of closed geodesics — the real part is the actual length and the imaginary part is the rotation angle as you go once around the geodesic."*

Class $9_{126}$:

```
First geodesic of U[1466]: 1.0152103824828331+0.39992347315914334i.
First geodesic of U[1563]: 0.9359206605025168+2.333526236965665i.
Volume of both manifolds: 7.36429600733.
```

Class $9_{199}$:

```
First geodesic of U[2125]:  0.8939075859248593+0.761197185679321i.
First geodesic of U[2165]:  0.7978548001747316+2.9487425029345973i.
Volume of both manifolds: 7.12868652133.
```

# 4 Conclusion

A closed orientable 3-manifold is denoted *$n$-small* if it is induced by surgery on a blackboard framed link with at most $n$ crossings. We provide an instance of the general theory to produce a recursive indexation of $n$-small 3-manifolds up to homeomorphism. We solve this problem up to $n = 9$. Conceptually we could go on forever, finding in the way tougher and tougher examples to be distinguished

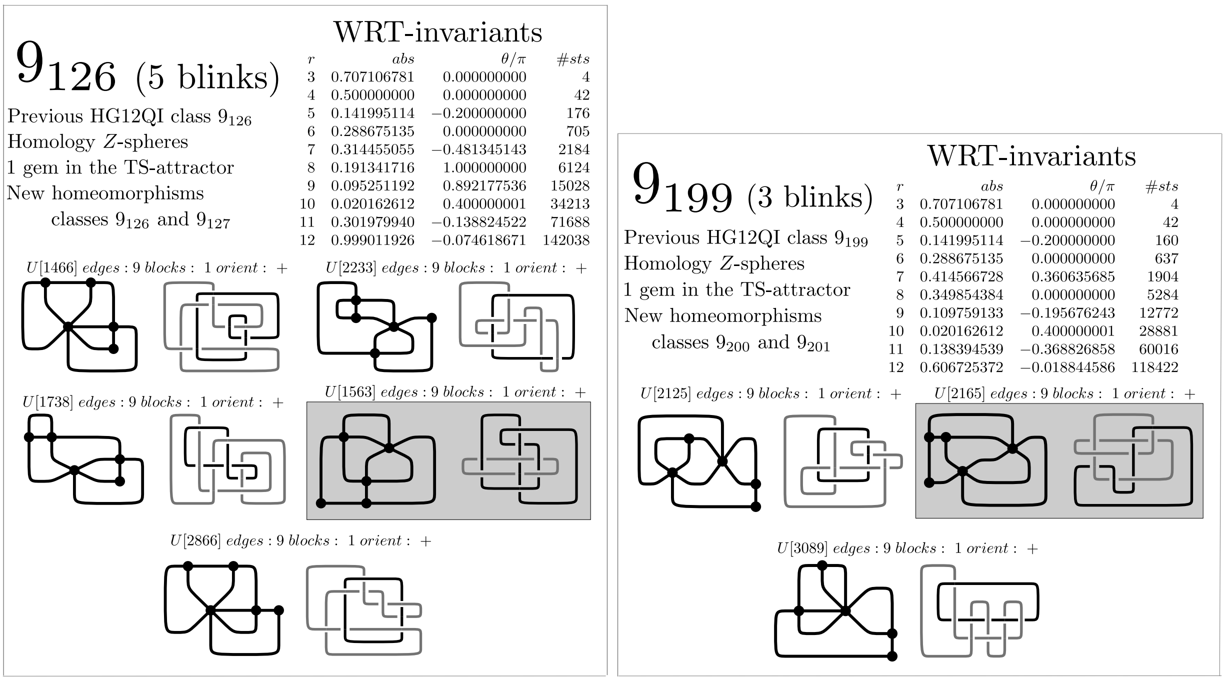



Figure 7: The two doubts left in L. Lins's thesis are solved In both cases the manifolds induced by the shaded blink-link pairs are proved very recently to be non-homeomorphic to the other in the same class, by geometric means. All the other pairs are proved to be homeomorphic by BLINK which keeps each such homeomorphism as a coded sequence which is stored in a data basis, and, in principle reproducible at will. This shows that BLINK finds all available homeomorphic pairs and, in conjunction with the length of the samllest geodesic (see next section), provides the topological classification of the 9-small 3-manifolds.

by yet to be found new invariants. The topological classification of the 9-small 3-manifolds involve three invariants:

$$INV = \{\ homology,\ WRT_{12},\ length\ of\ smallest\ geodesic\ \}.$$

The classification was nearly complete in [16], except for two doubts. Recently, after we posted a challenge in the arXiv, [21] these doubts were solved by M. Culler, N. Dunfield and C. Hodgson using SnapPy [4]. This made us add

$$length\ of\ smallest\ geodesic$$

which we define as 0, if the manifold is not hyperbolic, to our list of invariants. The 9-small 3-manifold classification maintains live the two Conjectures of page 15 of [18] based on two kinds of moves $TS$ and $U$: the $TS$- and $U$-moves yield an efficient algorithm to classify 3-manifolds by explicitly displaying homeomorphisms among them, whenever they exist. A recent finding by C. Hodgson concerning manifolds $T[71]$ and $T[79]$ forming the HG8QI-class $14^t_{24}$, in the notation of page 239 of [16] shows that the 3 invariants are not enough to decide the pair. This pair is the first one of 11 pairs that we display as some tougher challenges to 3-manifold topologists, [20]. Hodgson's finding is that the volume as well as the lengths of the smallest geodesics fail to distinguish $T[71]$ and $T[79]$. He proves them to be non-homeomorphic by more sofisticated techniques, involving drilling along the smallest geodesics to get non-isometric manifolds with toroidal boundary. Using SnapPy, GAP and Sage, N. Dunfield shows that $T[71]$ and $T[79]$ are also distinguished by the homology of its 5-covers. This solution depends on the existence of low index subgroups of the fundamental group of the manifold. What about if they do not exist? We feel that the majority of the *Tough Challenges* of [20] remains unresolved. Have we arived to our currently technological limit? Maybe so, but what we feel the difficulty of SnapPy and BLINK in dealing with the tough challenges teach us is that we need better subtle invariants computed quicky than the current ones. Invariants which are not so exponential as looking for small index coverings which depend on the fundamental group. The $HG8QI$-class $16_{42}$ is composed of 4 blinks $\{T[305],\ T[337],\ T[387],\ T[420]\}$, in the notation of [16, 20]. BLINK says that there is at most 2 homeomorphic classes. We believe this bound is attained. However SnapPy computed the homology of 4-covers of these spaces and they all agree. We conjecture that there is an invariant which break the $HG8QI$-class $16_{42}$ into two homeomorphic classes of two blinks each. We finish this paper with a focused challenge: prove or disprove this conjecture.

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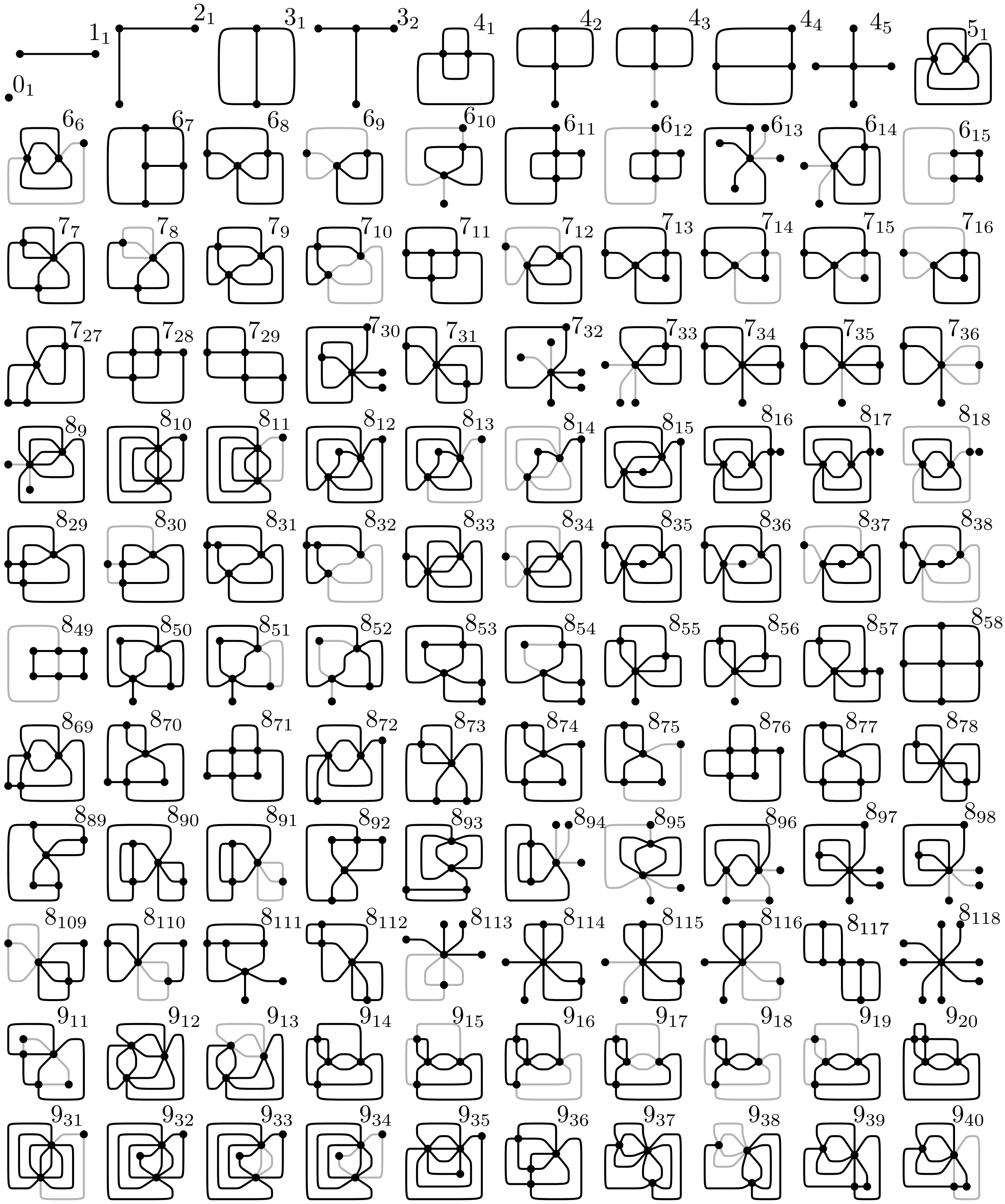
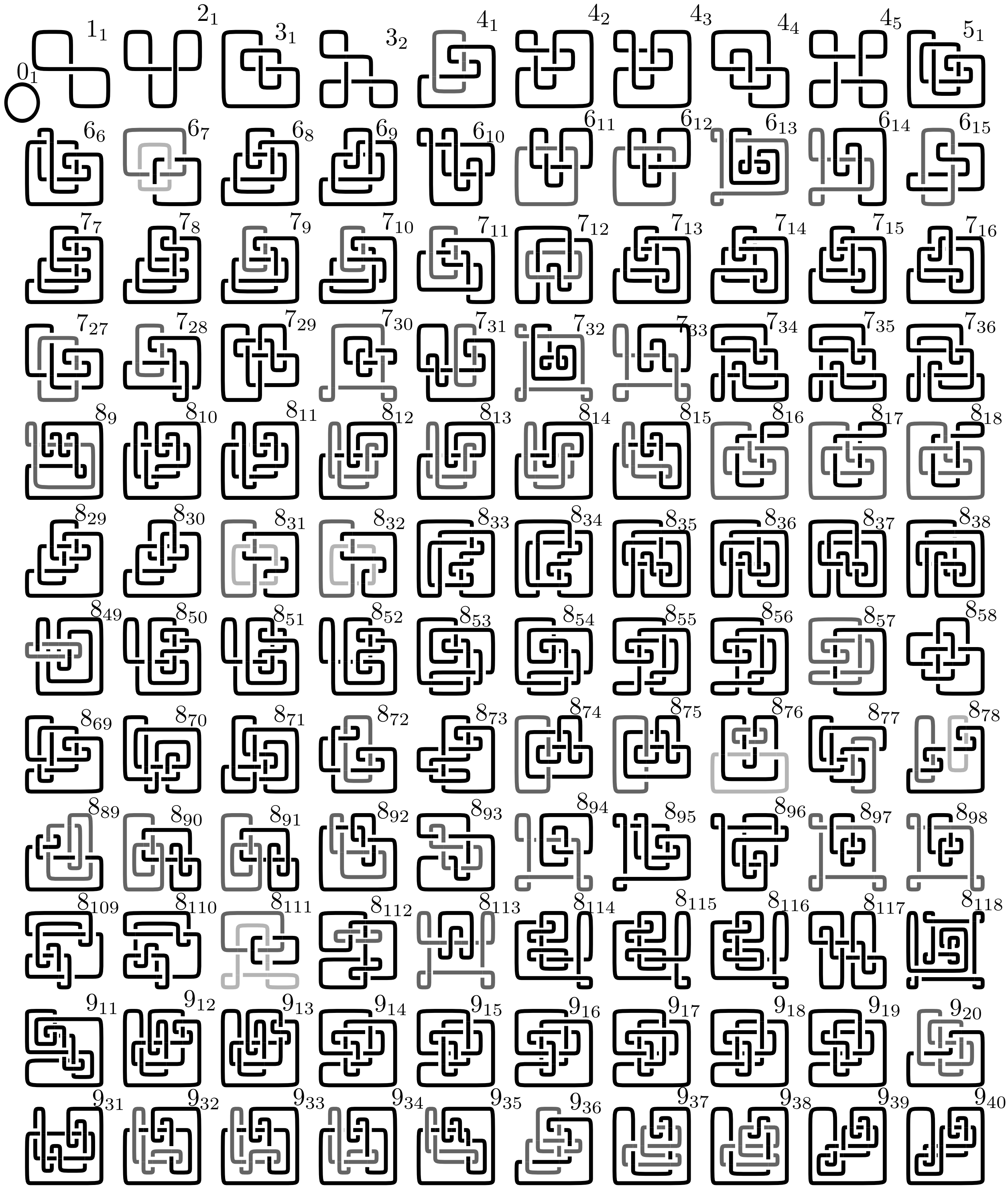
## 5 Appendix: census (no misses, no duplicates) of 9-small 3-manifolds

**Part 1/4 in terms of blinks:**



**Part 1/4 in terms of blackboard framed links:**



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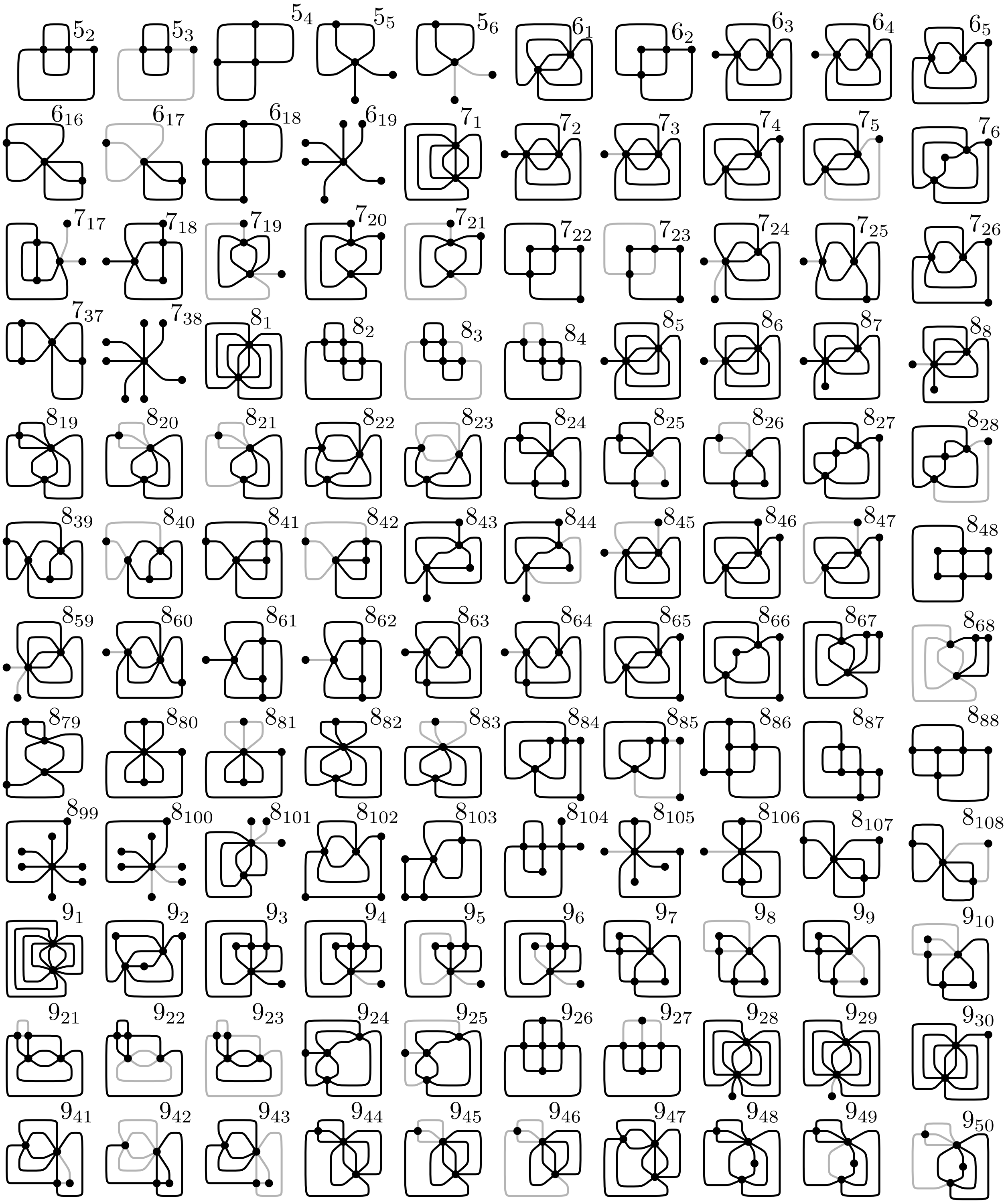

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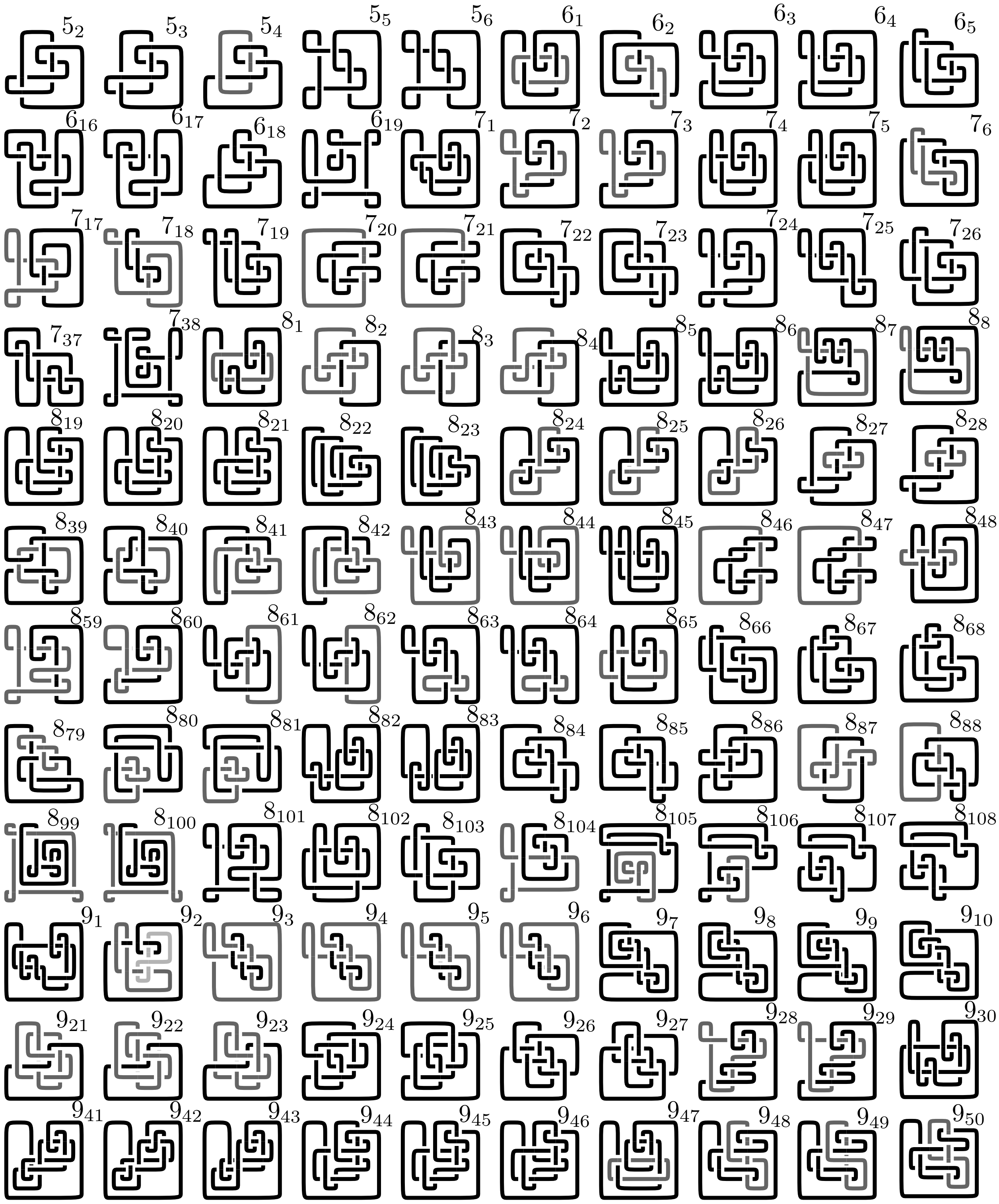
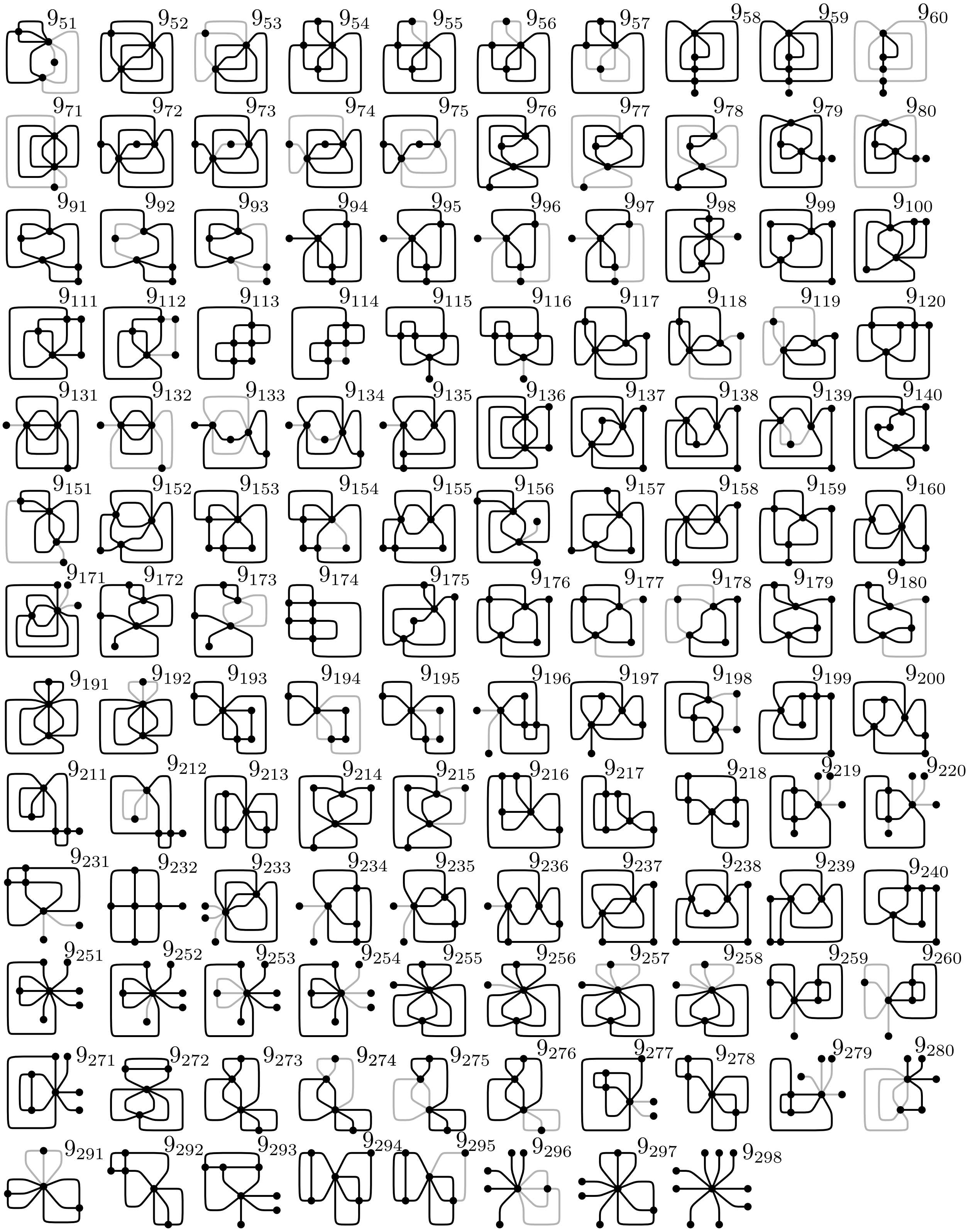


**Part 3/4 in terms of blinks:**



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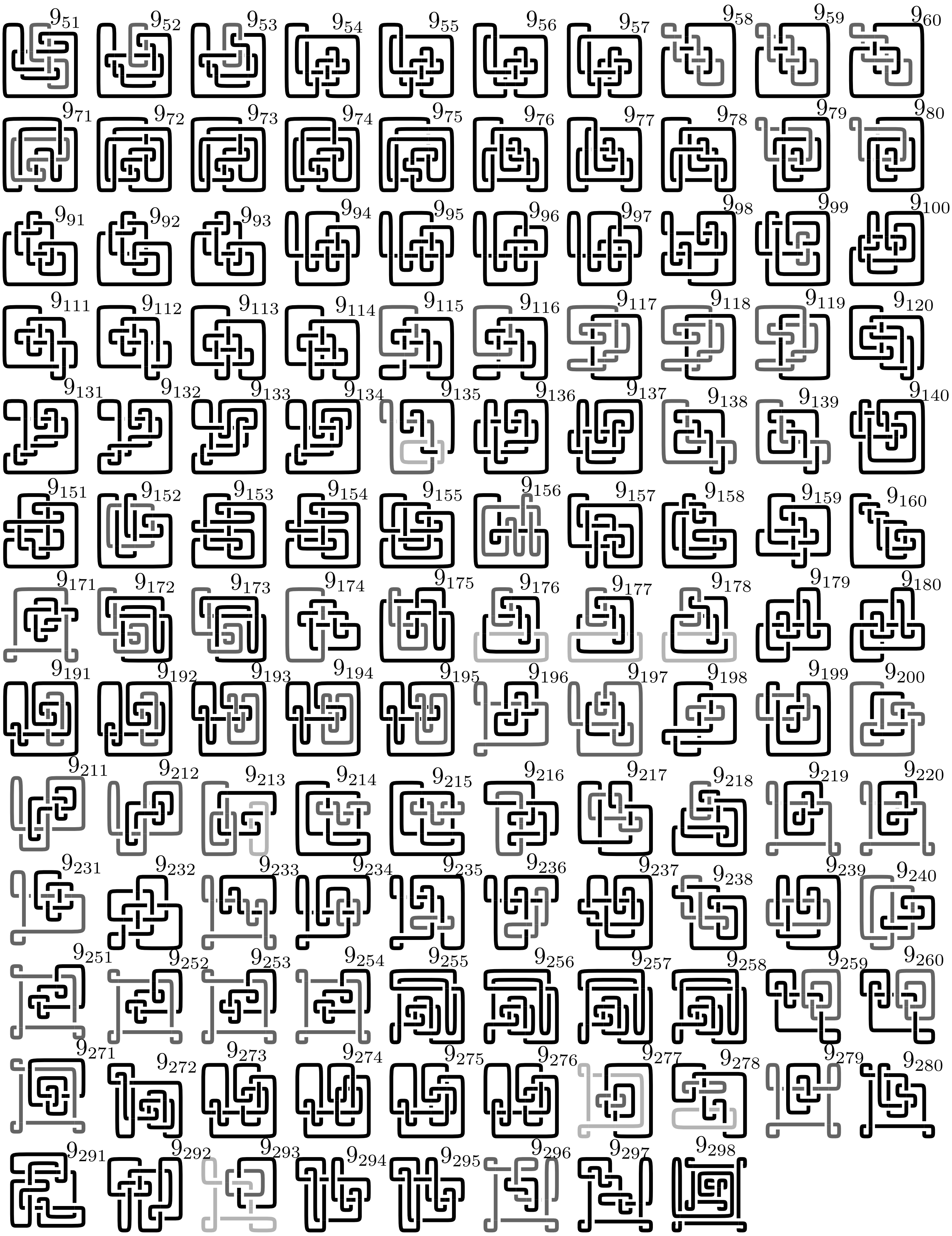
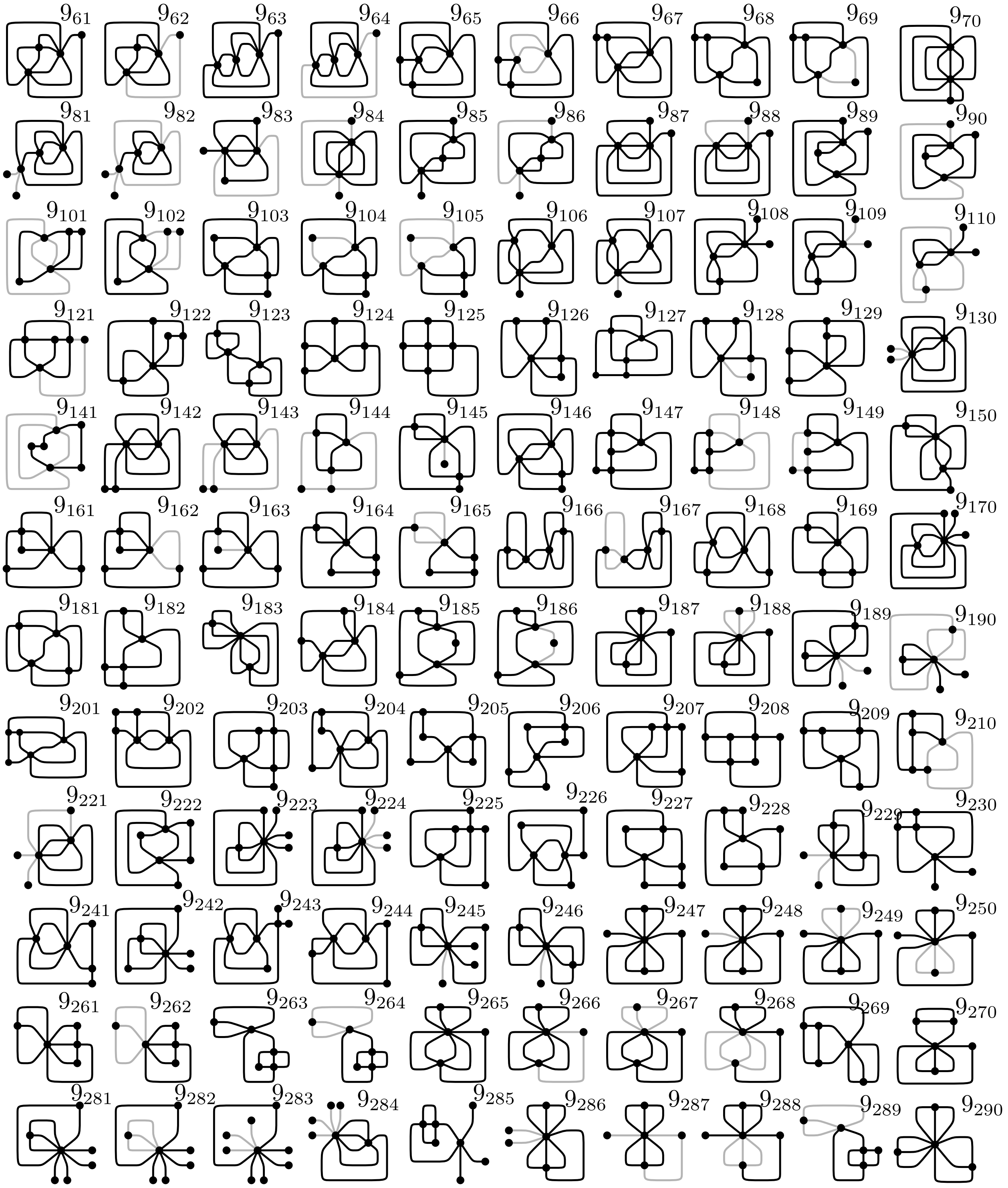


## Part 4/4 in terms of blinks:



**Part 4/4 in terms of blackboard framed links:**