All the shapes of spaces: a census of small 3-manifolds *

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Abstract

In this work we present a complete (no misses, no duplicates) catalogue for closed, connected, orientable and prime 3-manifolds induced by plane graphs with a bipartition of its edge set (blinks) up to $k = 9$ edges. Blinks form a universal encoding for such manifolds. In fact, each such a manifold is a subtle class of blinks, [20]. Blinks are in 1-1 correpondence with *blackboard framed links*, [11, 12] We hope that this census becomes as useful for the study of concrete examples of 3-manifolds as the tables of knots are in the study of knots and links. We also present in this work a complete theory capable of going on obtaining the similar census of 3-manifolds homeomorphism classes encoded by blinks up to $k = 10, 11, 12, \ldots$ edges. The limit is the technological state of the art. The algorithms are fully parallelizable and this limit seems far away. This work brings to light a new technique for obtaining drawings of links with curls and blinks with loops and pendant edges. These decorative objects in the usual theory of links become fundamental in this work. Links are drawn so as to deterministically minimize the number of right angles turns in a grid. This comes as an application of network flow optimization theory). Another contribution of this work is in finding good examples to test the splendid general theory of (we are focused at closed oriented) 3-manifold which is flourishing in this era post-Perelman. We have done this by recently posting in the arXiv two papers with 2+11=13 classes of 3-manifolds left open in L. Lins' Thesis under the supervision of S. Lins, [17]. The first challenge, containing 2 classes was solved quickly by various researchers in the 3-manifold community. However the second with tougher challenges having 11 classes of 3-manifolds mysteriously got no feedback. In the final section of this paper we explain why these challenges should be taken seriously.

1 Introduction

After presenting some instances of closed 3-manifolds, P. Alexandroff says in the English translation (1961) of his joint work with D. Hilbert [2], first published (1932) in German, [3]: *"These few examples will suffice. Let it be remarked here that, at present, in contrast with the two-dimensional case, the problem of enumerating the topological types of manifolds of three and more dimensions is in an apparently hopeless state. We are not only far removed from the solution, but even from the first step toward a solution, a plausible conjecture".*

John Hempel in his book (1976) *3-Manifolds* [9] writes at the opening of Section 15, entitled Open Problems: *"The ultimate goal of the theory would be in providing solutions to:* The homeomorphism problem*: provide an* effective *procedure for determining whether two* given *3-manifolds are homeomorphic, together with* The classification problem*:* effectively *generate a list containing exactly one 3-manifold from each homeomorphism class."*

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary)

1.1 A surprising new 1-1 correspondence: $\frac{3-manifolds}{homeomorphisms} = \frac{blinks}{coin\ \ calculus}$

A *blink* is a finite plane graph with an edge bipartition. Any closed, connected oriented 3-manifold is induced by some blink. Even though this object has been around since 1994 when it was introduced in the joint research monography of L. Kauffman and S. Lins, [12], the fact that they encode oriented closed 3-manifolds remains basically unkown. This is about to change because as a consequence, [20], of a recent result of B. Martelli, [28, 29], each such a 3-manifold becomes a subtle equivalence class of blinks. In [20], a new calculus, this time on blinks, named *the coin calculus* with 8 types of local moves each applied to 8 types of related sub-blinks (named coins) are shown to capture the essence of homeomorphism between 3-manifolds, in the sense of Theorem 1.1. A sufficent reduced coin calculus with 36 moves and 36 blinks is shown in Fig. 1.

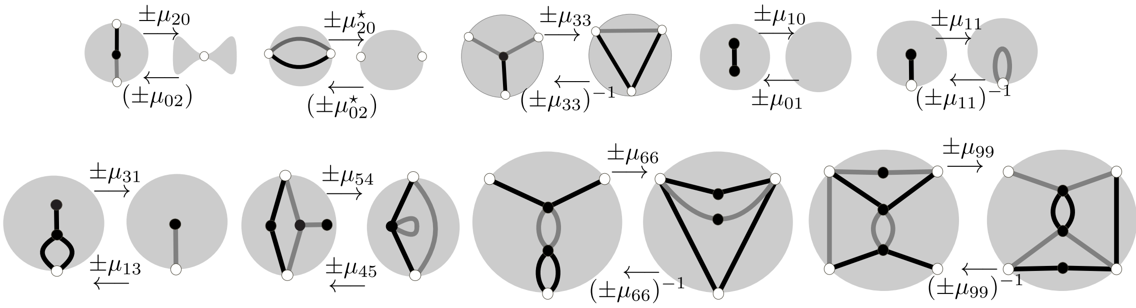


Figure 1: A 36-move, 36-coins (not all distinct) reduced (sufficient) coin calculus: a coin is a sub-blink which lies in a disk (or in a pinched disk, in the case of the right coin at the right of the moves $\pm\mu_{20}$). The complementary sub-blink is completely arbitrary, provided the intersection with the coins are vertices contained in the set of attachment vertices of the coins, each represented by a hollow small circle in the boundary of the coin, as shown in Fig. 1.

Throughout this work *3-manifold* means a closed, oriented and connected 3-manifold.

(1.1) Theorem. *Two blinks induce the same 3-manifolds if and only if they are linked by a finite sequence of moves where each move is one of the thirty six in the coin calculus described in Fig. 1.*

The meaning of the theorem is that there exists a bijection

$$\beta: \quad \frac{3-manifolds}{homeomorphisms} \quad \longrightarrow \quad \frac{blinks}{coin\ calculus}.$$

If the manifold is given by a framed link, β is obtained by a linear algorithm. However, if it is given by a gem, by a triangulation, by a special spine [30], or by a Heegaard diagram, then a polynomial algorithm to find β seems to be an open problem. However, rescuing the situation there exists a recent work of S. Lins and his former student R. Machado, which is reported in a sequence of 3-papers posted in the arXiv proving that there exists an $O(n^2)$-algorithm to go from a *resoluble* gem to a blink inducing the same 3-manifold. The definition of *resoluble gems* and the status of this theory is the subject of Subsection 2.1. We antecipated that it solves the general problem up to a conjecture which is true, with high probability. From the practical point of view we have an $O(n^2)$-algorithm to obtain a framed link presentation from an arbitrary triangulation.

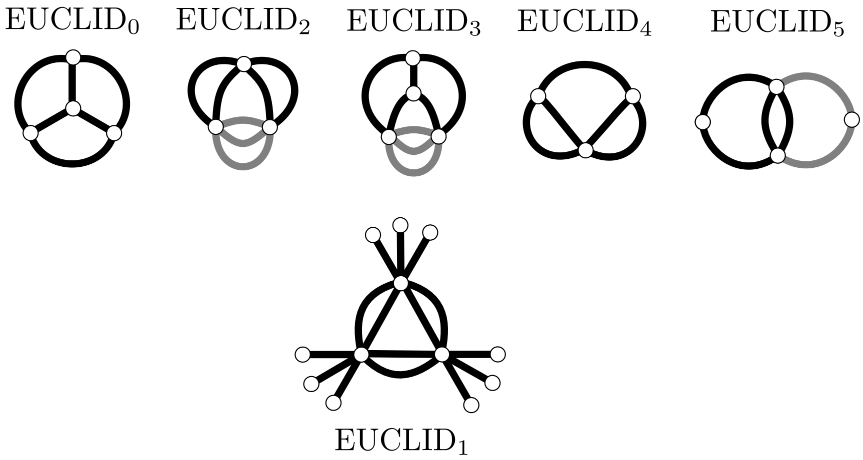
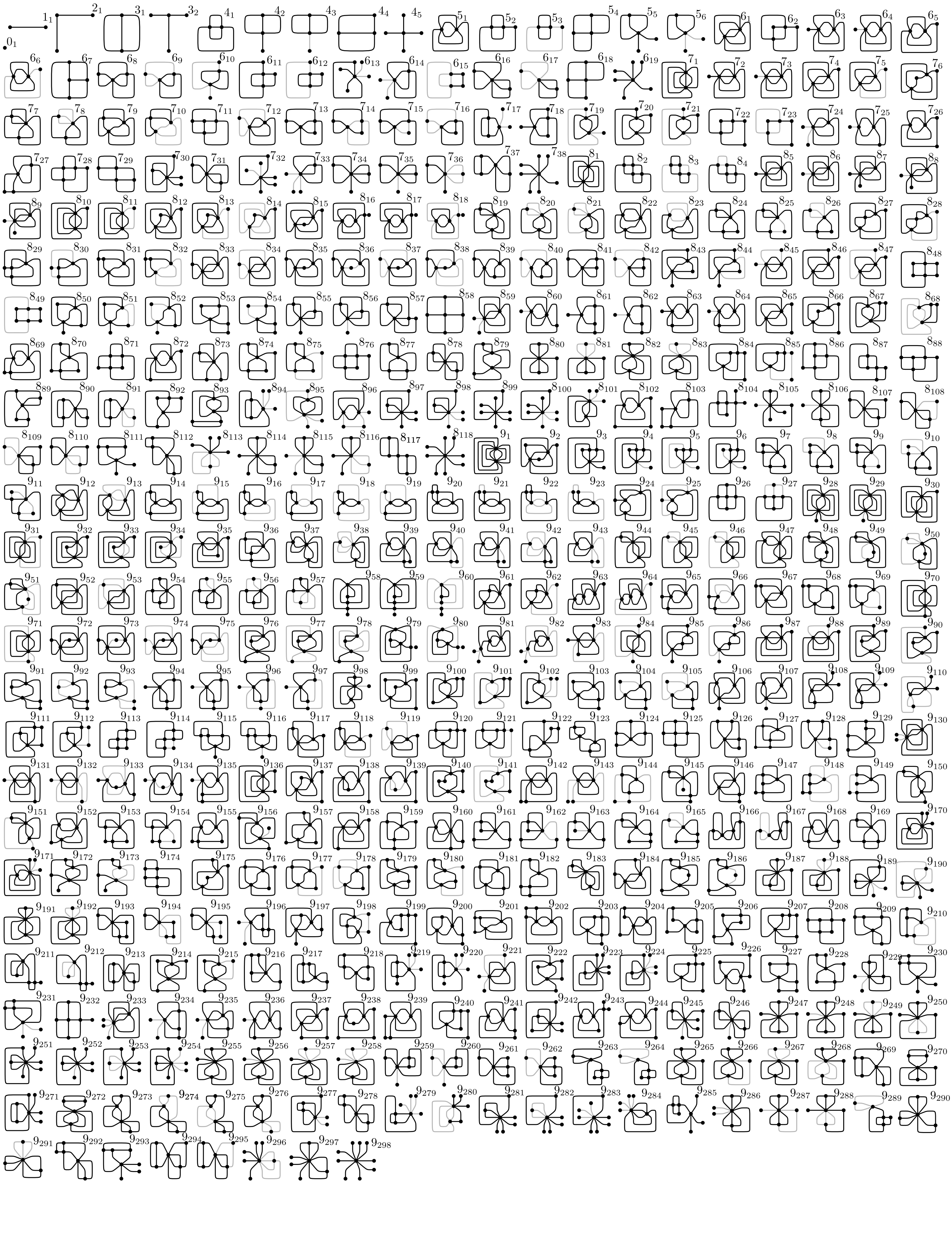


Figure 2: The discovery of a blink for EUCLID$_1$, solving an open problem left in page 117 of [17]: of the six euclidean 3-manifolds only EUCLID$_1$ did not have a blink presentation. It was necessary to develop and understanding the deep geometric properties of gems, to find this blink by the theory developed in [23, 24, 25]. The new blink has more than twice the number of edges of the other blinks in the euclidean family. No wonder it was so difficult to find. The new blink is correct: R. Machado applied to its associated blackboard framed link the linear algorithm (given in Fig. 4) to produce the canonical gem G inducing the same 3-manifold. The simplifying procedure of BLINK was used to go from G to the *superattractor*, (the unique minimum gem with 24 vertices inducing $|\mathrm{EUCLID}_1|$), ([19]) of EUCLID$_1$.

1.2 A complete duplicate free census of 9-small 3-manifolds

This paper concludes the proof of the following theorem, where 3-manifold means a closed orientable one and a *k-small* 3-manifold is a closed, connected, orientable and prime induced by a (necessarily connected) blink with at most k edges.

(1.2) Theorem. *Let $\mathbb{M}^3$ be an 9-small 3-manifold. Then $\mathbb{M}^3$ is homeomorphic to exactly one of the 3-manifolds induced by the 489 blinks below. Morever all of these are pairwise non-homeomorphic. (However being redundant, we also present the corresponding census for the blackboard framed links. These census are enlarged in the Appendix.)*



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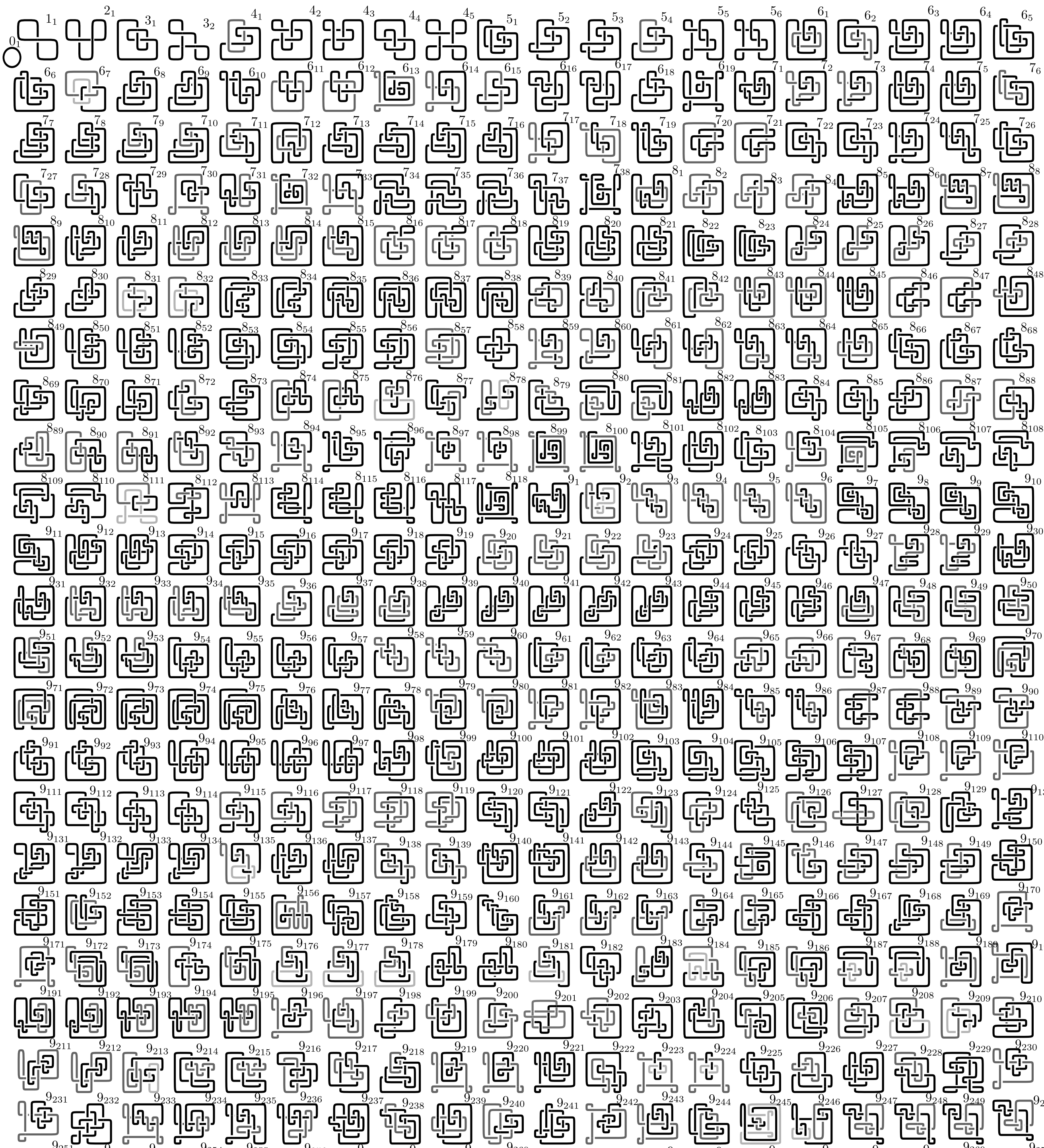

In references [12], [17] and [19] we have defined and show how a *blink*, that is, a plane graph with an arbitrary bipartition of its edges (here presented as colors black and gray) induces a well defined closed oriented 3-manifold. Moreover each such a manifold is induced by a blink (in fact, by infinite blinks).

Proof. **(Of the Theorem above)** The bulk of the proof follows from L. Lins' thesis under the supervision of S. Lins, [17]. In this work a theory for blink generation (missing no closed orientable 3-manifolds) is provided by pure combinatorics, lexicography and topological filtering of duplicates. This resulted in a set U_9 with 3437 blinks. This universal set of 9-small 3-manifold was partitioned by homology and WRT_{12} (the Witten-Reshetikhin-Turaev invariant computed, as in [12] for $r = 3, 4, \ldots, 12$) into 487 classes, named $HG12QI$-classes. This is achieved by explicitly obtaining homeomorphisms between any two blinks in the same $HG12QI$-class. Each such homeomorphism is coded into a sequence of gems (defined in the next section) that is frozen in the data basis (Mysql) of BLINK forever, and, in principle can be reproduced at will. Exactly 75633 gems were used in the classifing sequences encoding homeomorphisms between the 3-manifolds of U_9. What remained to be done was to decide the status of the two $HG12QI$-classes 9_{126} and 9_{199} depicted in Fig. 10. After 6 years we posted these doubts as a Challenge in the arXiv, [22]. In a quick feedback, many Mathematicians got interested in the Challenge (in the order of their comments in the blog)

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Comments were by H. Wilton, N. Dunfield, S. Friedl, M. Culler, D. Huberman, J. Berge, C. Doria. Also, N. Dunfield M. Culler and C. Hodgson contacted one of us (S.Lins) via e-mail. And we got solutions for the two 6 year old doubts: to the benefit of BLINK both pairs of 3-manifolds are non-homeomorphic. M. Culler, N. Dunfield and C. Hodgson sent distinct proofs of this fact. Thus there are 489 classes of non-homeomorphic closed 9-small 3-manifolds. More details of the distinction are given in Section 4. □

Relative to page 109 of [17] the blinks of Theorem 1.2 have receive two additions, the representative blinks $U[1563]$ and $U[2165]$. Also the previous HG12QI-class 6_5 became the homemorphism class 0_1 corresponding to $\mathbb{S}^2 \times \mathbb{S}^1$. We have decreased by 1 the numbering of the $HG12QI$-classes $6_6, 6_7, \ldots, 6_{20}$ which become the homeomorphisms classes $6_5, 6_7, \ldots, 6_{19}$: the HG12QI-classes 9_{126} and 9_{199} of [17] split into two homeomorphism classes. We observe that the blinks are enlarged in the appendix, showing them together with the corresponding blackboard framed links. The notation n_i attached to each blink below, is the name of its homeomorphism class, not merely its HG12QI-class, as in [17].

In Fig. 3 we depict the function which associates a blink to a given blackboard framed link via a checkerboard bicoloration of the faces of the link diagram. The function has clearly an inverse and associated linear algorithms to go from BFL to link and from link to BFL are defined.

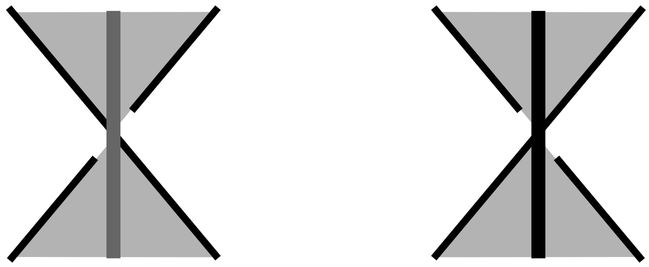


Figure 3: The linear algorithm to get a blink from a blackboard framed link. The inverse algorithm is also linear.

1.3 A word about the drawings in this work

As important as the Mathematical content of this paper is the possibility of easy visualization of the objects. The elegant drawings of blinks and blackboard framed links produced by the software BLINK are possible due the groundbreaking algorithm of R. Tamasia [35]. We could implement the drawings very fast because we had at hand the implementation of network flow algorithms we had done for a project to solve *practical timetable (!) problems*. This is an example of the unicity in Mathematics, advocated by L. Lovasz in his famous essay [26]. To get the drawings one has to apply three times the full strength of network flow theory, [1]. The drawings BLINK present are in an integer grid and deterministically minimize the number of $\pi/2$-bents in the blackboarded framed links. In particular, it permit us to deal with the unavoidable curls in the BFL's and the corresponding loops and pendant edges which adjust the integer framings to self-writhes in the best possible way: we do not care about them. The drawings for the companion blinks require a slight modification: it replaces each p-valent vertex $p > 4$, by a p-polygon inducing 3-valent ones. The final result is massaged a bit to produce aesthetically pleasing and unambiguous figures in a homogeneous grid.

1.4 Brief historical overview

It is amazing how much the picture has changed in the 80 years since the book by Alexandroff and Hilbert. The progress initiated with the deep advances in the 1950's and 1960's, starting with the proof that 3-manifolds are triangulable by Moise (1952), [31]. Next the presentation of them by framed links by Lickorish (1962) [15]. Following that Kirby presented its calculus for framed links (1978)[13]. Starting in the early 1980's W. Thurston's provided gret breakthroughs, developing his conceptual theory on hyperbolic manifolds and of the geometrization conjecture. In the final 1980's early 1990's Witten [38] broke the psicological barrier that there were no good invariants for 3-manifolds. Following that a number of eastern European mathematicians like N. Reshetikhen, V. Turaev and O. Viro, [36, 32] using quantum groups were able to put in mathematical solid ground Witten's findings. One of us, S. Lins, was a witness of the excitement these developments caused. L. H. Kauffman and W. B. R. Lickorish discovered the relationship of the Temperley-Lieb algebra with the new invariants, [16]. Starting with a sabatical leave in to Chicago in 1990, S. Lins produced the joint monography with Kauffman [12], where blinks are first defined and extensive WRT-invariant computations were obtained from the theory developed from scratch, independently and simpler than that of quantum groups. In the early 2000's, G. Perelman revolutionized the field proving Poincaré's Conjecture and Thurston's Geometrization Conjecture. More recently in the 2010's, I. Agol is leading the field in this era post-Perelman. Of course, this is only a diagonal list of researchers. Many more have contributed and some are extremely active in this era post-Perelman, [7]. Currently there is a great amount of important reserch issues going on and these are exciting times for 3-manifold theory. See the recent essay of E. Klarreich in the Simons Foundation, [14].

2 Blinks and gems

Unexplored simplicity. This was the reason for birth of this work many years ago. Repeating, a *blink* is a finite plane graph (that is, given embedded in the plane) together with an arbitrary bipartition of its edges into black and gray. For completeness we briefly recall the basic definitions of gem theory, leading to its definition, [19] and to its calculus based in dipole moves, [6, 18]. A *4-graph* G is a finite bipartite 4-regular graph whose edges are partitioned into 4 colors, 0,1,2, and 3, so that at each vertex there is an edge of each color, a proper edge-coloration, [4]. For each $i \in \{0, 1, 2, 3\}$, let E_i denote the set of i-colored edges of G. A $\{j, k\}$-residue in a 4-graph G is a connected component of the subgraph induced by $E_j \cup E_k$. A 2-residue is a $\{j, k\}$-residue, for some distinct colors j and k. A *gem* is a 4-graph G such that for each color i, $G\backslash E_i$ can be embedded in the plane such that the boundary of each face is a 2-residue. From a gem there exists a straightforward algorithm to obtain a closed orientable 3-manifold, in two different, dual ways. Every such a manifold is obtainable in this way. An unecessary big gem is obtained from a triangulation T for a manifold by taking the dual of the barycentric subdivision of T. Here the colors correspond to the dimensions. Given a pair of vertices $\{u, v\}$ of a gem G with k linking edges $k \in \{0, 1, 2, 3\}$, in k distinct colors $K \subset \{0, 1, 2, 3\}$ is called a K*-dipole* if u and v are in distinct $(\{0, 1, 2, 3\}\backslash K)$-residues of G. A *dipole cancellation* is the operation that remove u, v and their k linking edges leaving $4-k$ pairs of pendant edges which are reunited along the same colors. The *dipole creation* is the inverse move. The dipole creations and cancellations do not change the induced 3-manifold. It is possible to simplify substantially the gem corresponding to the dual of the barycentric subdivison of a triangulation by cancelling dipoles. These simplifications in that gem completely destroys the correspondence of colors and dimensions. Two gems induce the same 3-manifold if and only if they are linked by a finite sequence of moves where each term is either a dipole cancellation or a diplole creation, [6, 18].

In our context, a very important property of gems is that, given any blackboard framed link with n crossings there is a linear algorithhm to present a gem having $8n$ vertices which induce the same 3-manifold. This algorithm is displayed at the bottom part of Fig. 4. This was proved originally in Kauffman-Lins monography, page 175 of [12], which provide a gem with $12n$ vertices. By using an specific TS-move of [19] a gem with $8n$ vertices is obtained, page 81 of [17]. The $(12 \to 8)$-simplification by means of dipoles is depicted the upper part of Fig. 4.

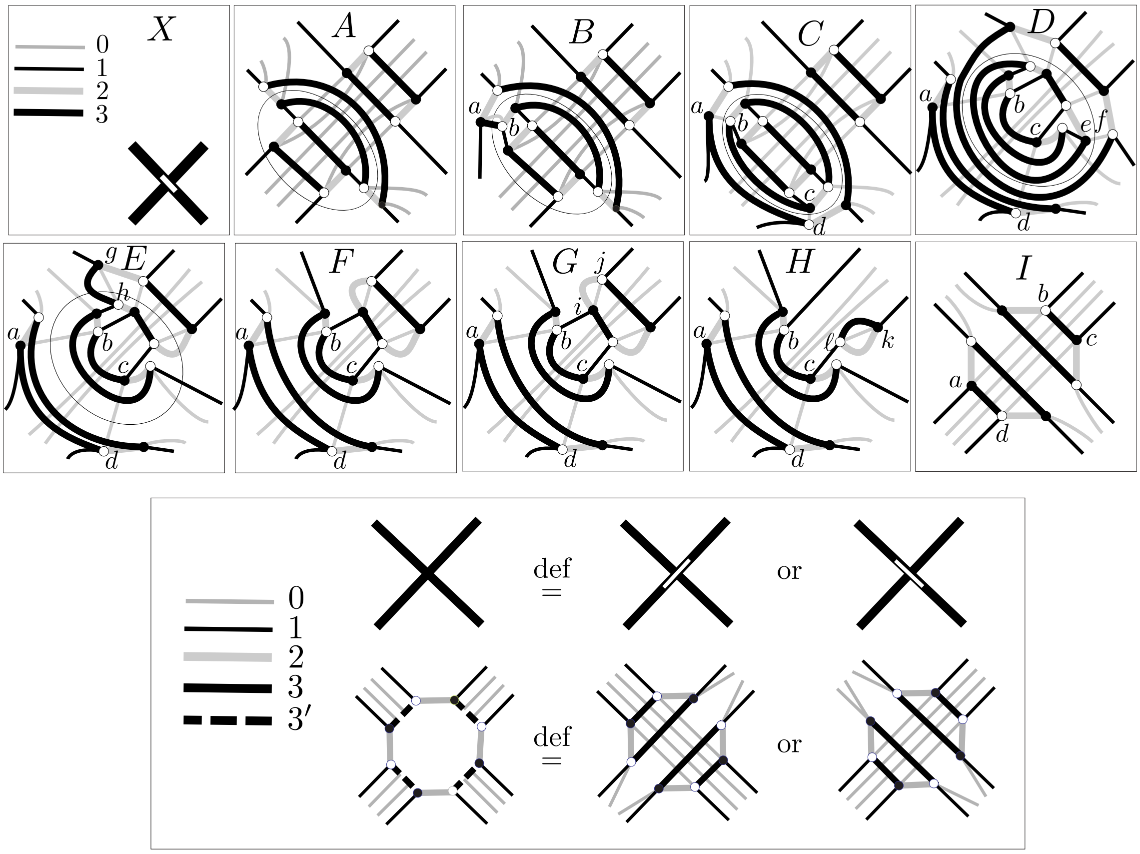


Figure 4: The passage $X \to A$ corresponds to the $12X$-algorithm to go from a BFL to a gem, is defined and proved to be correct in [12]. The $(12 \to 8)$-simplification by dipole moves (passages $A \to B \to \ldots \to I$) first appears in [17]. The final $8X$-algorithm to go from a BFL to a gem is depicted in the lower part of the figure.

2.1 A new $O(n^2)$-algorithm: from a resoluble gem to a blink

The present work is independent of Fig. 1, on Theorem 1.1 of Subsection, on Theorems 2.1, 2.2 and on the Conjecture 2.3, of this Subsection, which can be skipped at no logical cost. But we recommend the reader not to do so, because the Theorems and the Conjecture are stated to give a glympse and to clarify our research in a broader appropriate context and timing. Consider the following 9-element set of universal presentations of 3-manifolds:

UP3M = { *(1) triangulation (E. Moise), (2) Hegaard diagram, (3) gem (Italian School of crystallizations &S. Lins), (4) special spine (Matveev), (5) integer framed link (Lickorish & Kirby), (6) blackboard framed link (L. Kauffman), (7) blink (S. Lins), (8)* $\pm\infty$*-fractional framed link (D. Rolfsen), (9) homology based framed link (SnapPy)* }.

The first 4 are *triangulation based types of presentations*. The last 5 are the *framed link type based presentations*. Two triangulation based presentations are obtainable, one from the other, by inverse linear algorithms. If the integers are bounded in a presentation (5), then (5,6,7) are also linked by such algorithms. Presentations (8,9) are more general than those of (5,6,7): the denominator of the fraction are 1 and the second number of the canonical homology basis are 1. Thus going from (5,6,7) to (8,9), nothing needs to be done. The inverse algorithm, is easily to be shown to be finite, although the complexity analysis is more difficult, [33]. Fortunately, these algorithms are not needed because we start with the blink, and so (5,6,7) are equivalent up to linear algorithms.

It is straighforward to go from a blink to a gem by a linear algorithm: we take the associated blackboard framed link and obtain the canonical gem by the linear algorithm of Fig. 4. However, to go from a triangulated type presentation to a framed link type presentation by a polynomial algorithm is, at present an open problem.

In a sequence of papers which S. Lins and R. Machado posed in the arXiv, [23, 24, 25] an $O(n^2)$-algorithm is developed to produce a blink from a special kind of gems, named *resoluble.* The importance of resoluble gems stems from the following theorem:

(2.1) Theorem. *There exists an $O(n^2)$-algorithm to go from a* resoluble *gem to a blink inducing the same 3-manifold.*

The parts of the proof of this theorem are available at [23, 24, 25]. Even though the work leading to the above result is still being polished, we feel that is important to mention it here because it implies that there is an $O(n^2)$-algorithm to find the image of β in Subsection 2.1. in the case that the 3-manifold is given by a resoluble gem. The proper definition of resoluble gem is still unecessary complicated in the posted papers. Here is the *currently adequate (we are still trying to improving it)* definition of resoluble gem. Let G be a gem and u, v distinct vertices. For $j \in \{2, 3\}$ we say that $\{u, v\}$ is a *j-double meeting, or a dm_j* if u, v are in the same $\{0, j\}$-residue, in the same $\{1, k\}$-residue and in distinct $\{2, 3\}$-residues. Note that $\{0, 1, j, k\} = \{0, 1, 2, 3\}$. Let D_2 be a set of new edges (in color 4) linking u to v, whenever $\{u, v\}$ is a dm_2 in G. Let D_3 be a set of new edges (in color 5) linking u to v, whenever $\{u, v\}$ is a dm_3 in G. Let $G_1 = G$, G_2 be G with its colors permuted under the 3-cycle $(1, 2, 3)$ and G_3 be G with its edges permuted under the 3-cycle $(1, 3, 2)$. We say that G is a *resoluble gem* if $G_i \backslash \{E_0 \cup E_1\} \cup (D_2 \cup D_3)$ is connected, for some $i \in \{1, 2, 3\}$. Recently, in a still unpublished work, with the above adequate definition of resolubility the same authors have proved the following theorem:

(2.2) Theorem. *If a 3-manifold is given by a framed link then there exists a linear algorithm to obtain a resoluble gem inducing the same manifold.*

Two gems induce the same 3-manifold if and only if they are linked by a finite sequence of *blob moves* and *valid flip moves.* This result is proved for gems of arbitrary dimensions by S. Lins and M. Mulazzani in [18]. A *blob* $\{u, v\}$ is an $\{i, j, k\}$-dipole for 3 distinct colors i, j, k. A *flip in a gem* is the recoupling of two equally colored edges so as to maintain the bipartiteness. Flips that transform the I-dipole $\{u, v\}$ into the the $I \cup \{j\}$-dipole ($j \notin I \subset \{0, 1, 2, 3\}$) and their inverses are named *valid flips.* What is needed to link by or $O(n)$ or $O(n^2)$ algorithms any two of the following set of universal presentations of the same 3-manifold ($UP3M$), is the following Conjecture:

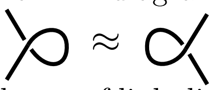
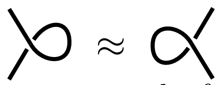
(2.3) Conjecture. *A resoluble gem remains so under a blob move, or a valid flip move and straightforward simplifications using $TS\rho$-moves, [19].*

This Conjecture is one of those annoying results that we know to be true (there is too much freedom in the situation) but which stubbornly refuses to comport. We still did not put enough effort in trying to prove it, mainly because of lack of time. For the present status of this whole theory what imports is that simplifying a gem by TS-moves we invariably obtain a resoluble gem. For our computational purposes there is an $O(n^2)$-algorithm to produce a blink inducing the same space as the one given by an arbitrary member of $UP3M$. At the level of the catalogue, simplifying the gems under TS_ρ produce resoluble gems. Note that, as a bonus we gain the capability of computing the Witten-Reshetikhin-Turaev invariants [32, 12] from all the members of $UP3M$, what is currently impossible, because the WRT-invariants are only computable from framed link type presentions (6,7).

We stress our belief that many interesting and deep consequences to be investigated of Theorem 1.1 to the topology of 3-manifolds and to the combinatorics of plane graphs ought to exist. Here is an example: what does it means, in terms of blinks a closed orientable 3-manifold having a homogeneous Riemannian geometry?

2.2 Blackboard framed links

The pictures and some definitions of this subsection are reproduced from Section 12.1 of [12]. They are included here for completeness since the concept is central in this paper. Instead of working with the link in $\mathbb{R}^3$ it is to our advantage to work with a *general position decorated projection* of the link into $\mathbb{R}^2$. Such projections are named *link diagrams*. General position means that the pre-image of any point of $\mathbb{R}^2$ in the link is at most two points. A point that has two pre-image is a transversal crossing of two strands in the projection. Decorated means that we keep the information of which strand is the lower and which is the upper, usually by removing a small segment of the lower strand centered at the crossing.

If L is a link in $\mathbb{R}^3$ a *framed link based at* L is an embedding $f : \mathbb{S}^1 \times [0,1] \longrightarrow \mathbb{R}^3$ so that $L = f(\mathbb{S}^1 \times \{0\})$. *Regular isotopy* is the class of link diagrams up to Reidemeister moves 2 and 3. Framed links are represented by regular isotopy classes of link diagrams, with the extra move, the *ribbon equivalence*, which is generated by the relations  and .

With these extra relations, the ribbon equivalence classes of link diagrams represent the framed links via the *blackboard framing.* This framing is obtained by placing a vector field normal to each link component with all the normals in the plane indicated by the diagram. We can indicate this this framing by sketching the tips of the normals as a second component, so that each component is indicated by an immersed band, see Fig. 5.

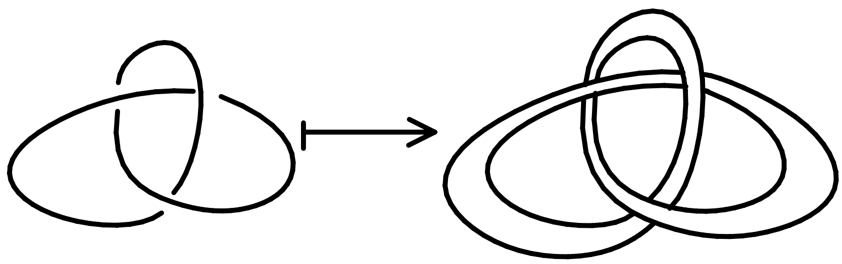
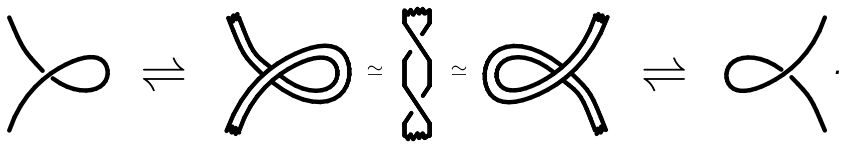
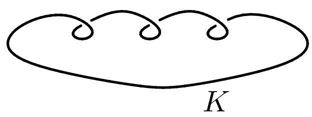


Figure 5: How the tips of the normal vector field draws a second parallel component

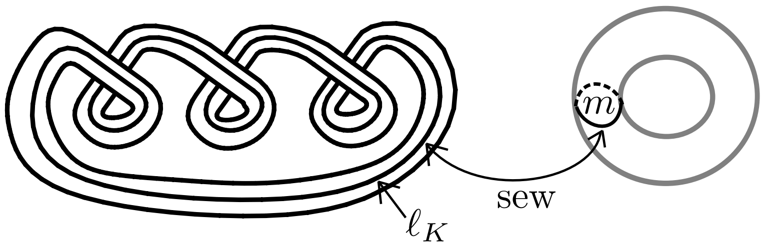
The reason for the ribbon equivalence is manifest in that the framings corresponding to these two curls (with distinct winding number in the plane) are ambient isotopic:



We finish this subsection with a representative example of how to construct the 3-manifold from the blackboard framed link. Suppose that the link diagram K has just one component and 3 curls corresponding to framing number 3 in $\mathbb{R}^3$:



Then $\mathbb{M}^3(K)$ is obtained by sewing $m \subset \mathbb{D}^2 \times \mathbb{S}^1$ to $\ell_K \subset \mathbb{S}^3 \backslash (\mathbb{S}^1 \times \mathbb{D}^2)^o$, where X^o means the interior of X. The sewing, forming the lens space $L_{3,1}$, is depicted below:



Note that the longitudinal curve ℓ_K in the boundary of the toroidal hole (which is identified with m to close the toroidal hole) never touches the boundary of the projection of the band formed by K and its copy.

2.3 The complementary roles of blinks and gems

Blinks are in 1-1 correspondence with blackboard framed links which in turn encodes every closed oriented 3-manifold. If we consider these three encodings of the same 3-manifold given in Fig. 6, the blink is the one that has the smallest "perceptual complexity". The common manifold of this example is the binary tetrahedral space defined by $\mathbb{S}^3$ (as a topological continuos group) up to the action of the non-comutative binary tetrahedral group $\langle 3, 3, 2\rangle$, (see [27]) which has 24 elements. The 3-manifold $\mathbb{S}^3/\langle 3, 3, 2\rangle$ has an spherical geometry. Its attractor (definition given shortly) consists of the single gem depicted at the right of Fig. 6.

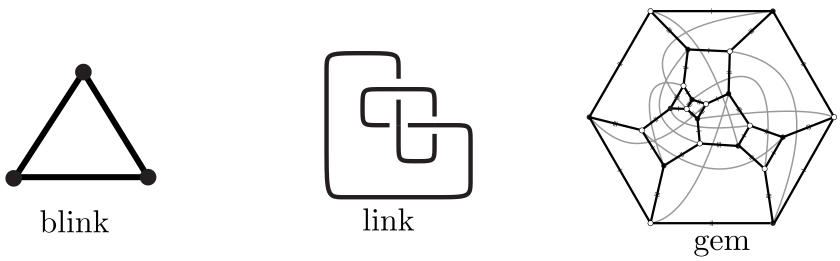


Figure 6: The minimum blink, minimum link and minimum gem inducing the binary tetrahedral space

Blinks are very easy to generate recursively. They have a rich simpifying theory which permits the generation of 3-manifold catalogues. Moreover, their isomorphism problem is computationally simple. Being blinks so good, why do we need gems? The answer is that to prove that blinks induce the same 3-manifold (when they do) remains (probably even with the new coin calculus) a very difficult problem. It is straighforward to obtain a canonical gem from a blink, [12, 17]. Proving that two gems induce the same 3-manifold (when they do) is much easier because of the rich simplification theory of them (based on at least 4 intertwined planarities) leading to the *attractors of 3-manifolds*, [19]. These are defined as the set of gems inducing the manifold having the minimum set of vertices. The important computational point is that the attractor usually have few members, particularly if the manifold has a uniform geometry euclidean, spherical, hyperbolic.

Blinks are very good at proving that two manifolds are distinct, because from them we can extract the WRT-invariant, which is a very strong invariant, yet not a complete one. These invariants currently do not have a direct computation neither from triangulations, neither from gems, neither from Heegaard diagrams, neither from special spines of Matveev. In this respect see subsection 2.1, where we report progress on this issue.

Gems and blinks collaborate in a symbiotic dance to decide (at a computational level) whether two 3-manifolds are or are not homeomorphic. And many types of census become available! We present here our contribution to the topic. It is placed in the confluency of two deep passions of the authors: the study of closed orientable 3-manifolds and the study of plane graphs. Here, we mean to provide a strategy for a segmented answer of Hempel's questions posed at the introduction. The closed oriented 3-manifolds are partitioned by the number of edges in a minimum encoding of them by a blink.

Any plane drawing whatsoever (see Fig. 7) of a graph with an arbitrary bipartition of its edge set, that is, a blink, corresponds to a unique closed oriented 3-manifold via the associated blackboard framed links. An important aspect about blinks is that each one possesses an easily obtainable *canonical form* inducing the same 3-manifold: it is named the *representative of the blink* and is obtained by lexicography from a small number of conventions, fixed in advance. This is explained, with a great amount of details, in L. Lins' thesis, [17].

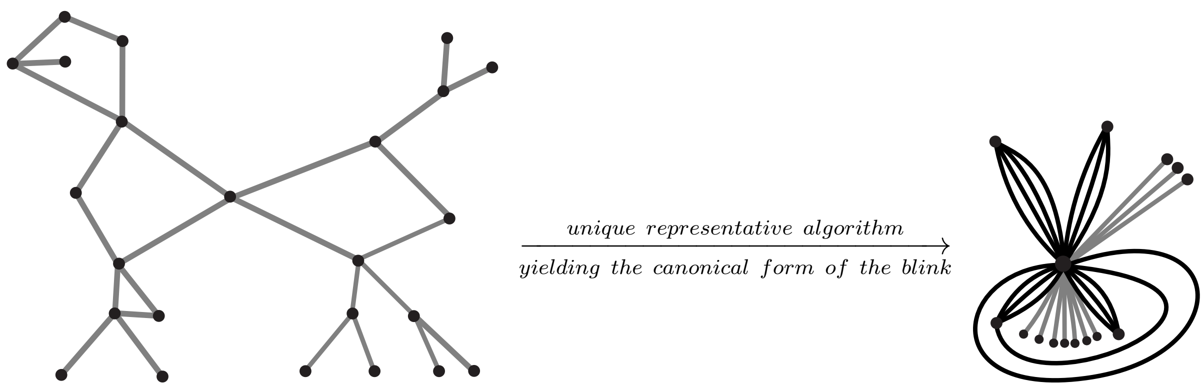


Figure 7: Blink unique representative obtained by an $O(n \log n)$-algorithm implemented by BLINK: monochromatic doglike blink with 27 edges and its representative with 25 edges.

Lexicography is used to define a representative unique plane graph (a canonical form) for each closed oriented 3-manifold. We explicitly solve the segmented problem up to 9 edges, see Theorem 1.2. This work provides an efficient algorithm to make available the canonical form of any closed orientable 3-manifold induced by plane graphs at the current level of the catalogue (currently 9 edges) and, theoretically, this could be extended 10, 11,. . . , n, for arbitrarily large n. Our theory gives a road to effectively name each 3-manifold classified by some set of invariants INV. We have a universal set of object, the blinks up to n edges, which can be partitioned by these invariants. The INV-classes are then tried to be broken into homeomorphisms classes. New invariants are then discovered and added to INV making them homeomorphisms classes of n-small manifolds. The difficult cases are going to appear naturally and they lead to enhancement of the theory. It is not at all impossible that this process stops and we get INV so that the INV-classes be proved to be homeomorphisms classes for all n. The point we want to make is that good examples (hard to find, here exemplified by the $HG12QI$ classes 9_{126} and 9_{199}) are important in obtaining progress in a general theory. A classifying INV for the 9-small 3-manifolds is $INV = \{$ *homology*, WRT_{12}, *length of smallest geodesic* $\}$.

3 Generating the k-universal set of blinks U_k

A set of blinks is said to be k-universal if its members induce all k-small prime 3-manifolds. Note that the components of a disconnected blink correspond to the summands in the connected sum of the 3-manifolds induced by the components of the blink. Therefore, our minimal sets of k-universal blinks are composed of connected blinks. It is not difficult to implement an algorithm that produces a specific k-universal set U_k. We start by (1) constructing all the 2-connected graphs having up to k edges, forming a set A. This is a standard procedure in graph theory. Then we take (2) all unions of blocks having a common vertex so that their edges set has at most k edges. Regular isotopy shows that we can restrict to having all blocks with a common vertex. And, by lexicography we do this in a unique way. Denote by B the set of these combined blocks. Next (3) we consider all bicolorations of the edges so that the number of black edges is not smaller than the number of gray edges. We put the blink in the third set only if it coincides with its representative. This forms the set C. Finally we use some topological filtering to decrease the size of C. The surviving blinks forms the set D, which is our U_k. A final comment we make about the generation is that the connected blinks are implemented as embedded in $\mathbb{S}^2$. The external face of a blink is well defined: namely is the face indexed with number 1 in the numbering which produces its code. At any rate, changing the external face is easily done in the coin calculus: it involve regular isotopy moves, one ribbon move and one Whitney trick simplification.

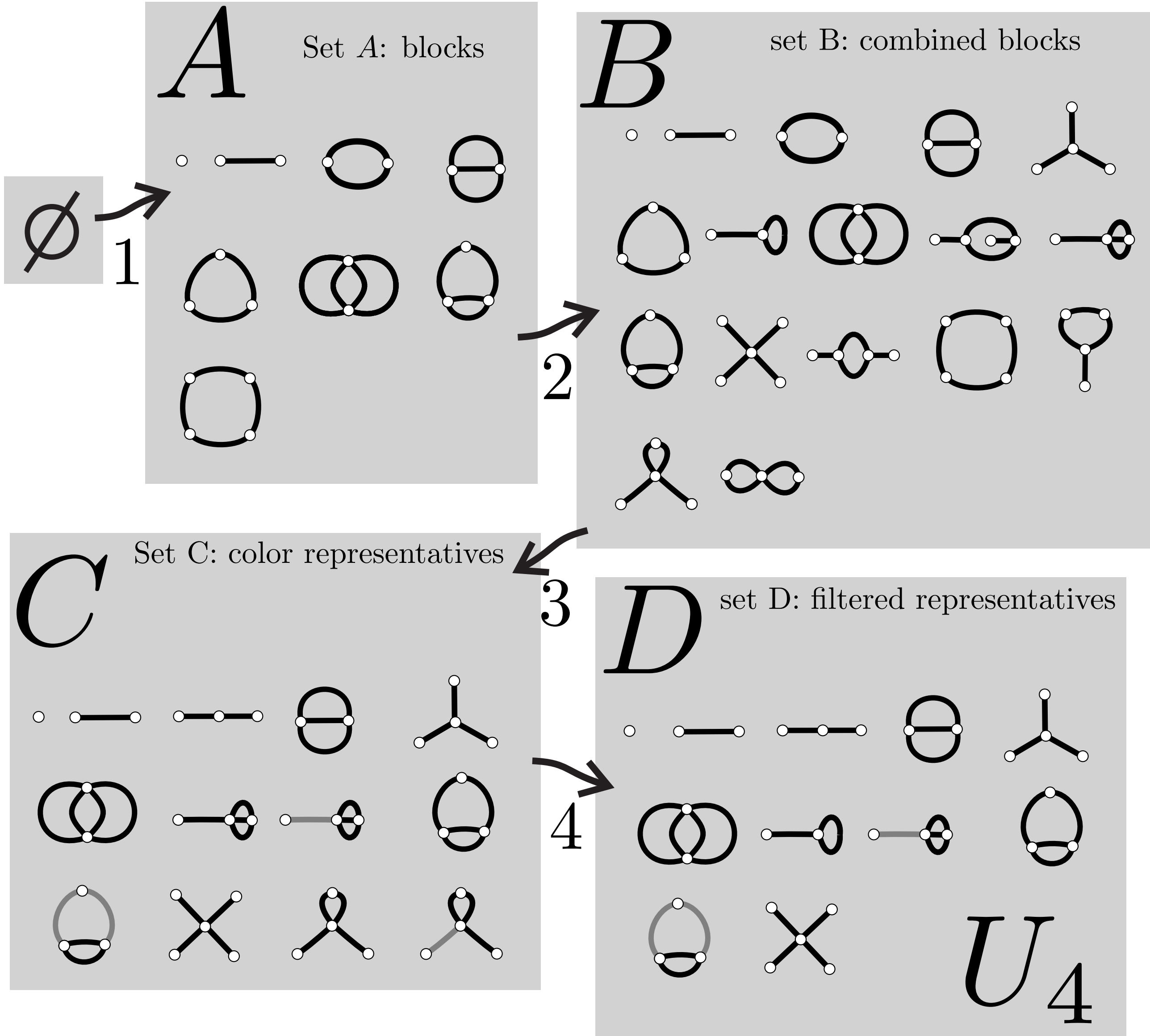
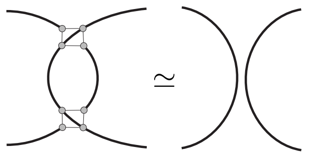
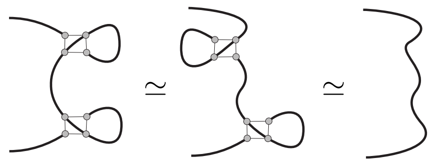


Figure 8: Pipeline for obtaining, in 4 steps, the set U_k exemplified at $k = 4$. Note that, starting with the empty set, the input of a phase is the output of the previous phase, that it, we have a pipeline. We must include the single vertex, as it corresponds to the unknot, inducing $\mathbb{S}^2 \times \mathbb{S}^1$. The cardinality of U_4 is only 11. BLINK produced, in its data base, the sets U_9 and U_{10} with respectively 3437 and 17948 blinks.

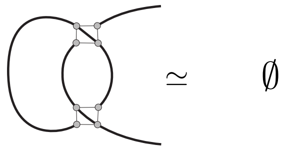
The topological filters that we use are depicted in Fig. 9.



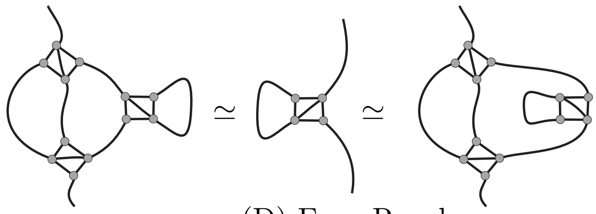
(A) Reidemeister move 2



(B) Ribbon move and Whitney's trick



(C) Deleting a circumcised pair Kauffman-Lins, pag 138

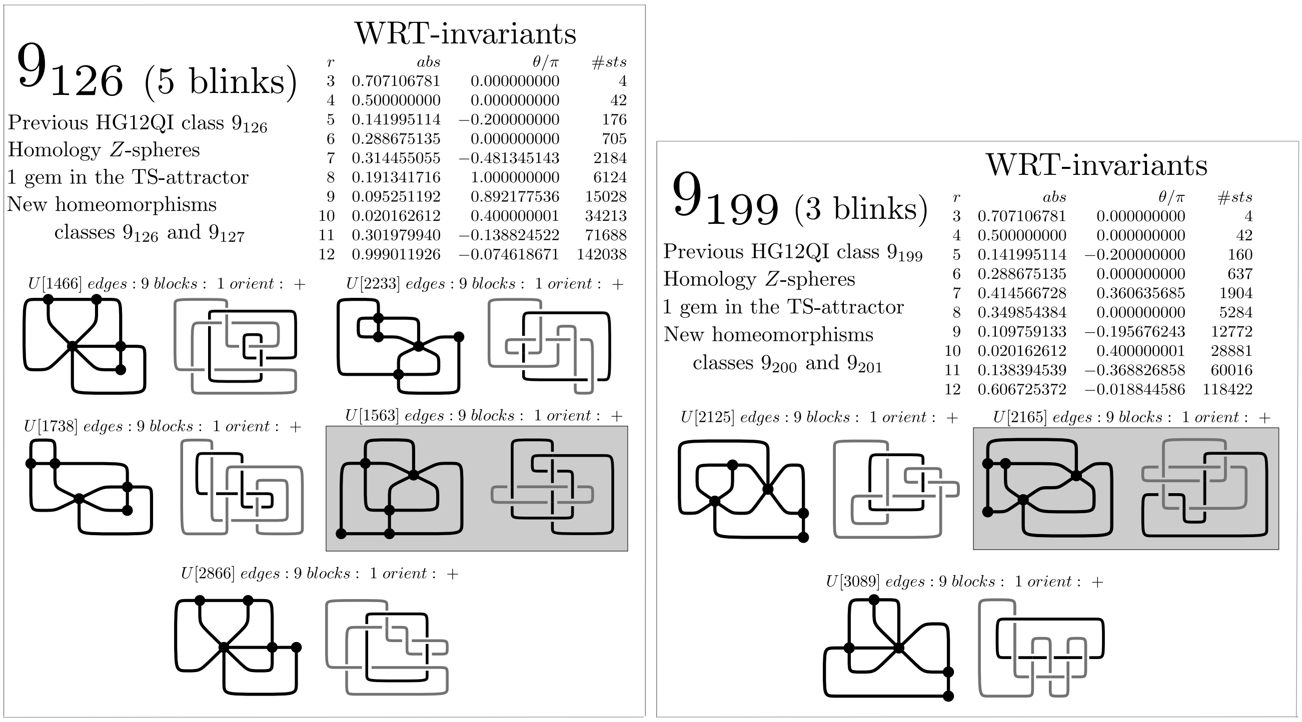


(D) Fenn-Rourke blow down of rank 1

Figure 9: The topological filtering (at the level of blackborad framed links) in the generation of U_k. A final ingredient that we use in the topological filtering is to drop blinks that produce non-prime 3-manifolds. They can be detected easily by finding a separating handle in the associated gems: just a coboundary with 4 differently colored edges. There are only 14/3437 composite $HG12QI$-classes of blinks in U_9.

4 Unveiling the mystery of the two doubts in L. Lins thesis

The topological classification of the 9-small spaces was nearly completed in [17]. This work develops a theory for generating a distinguished set of blinks named U_n and indexed lexicographically, $U_n[i]$ is the i-th such blink. This has been revewed The relevance of U_n is that it misses no closed, orientable, prime and irreducible 3-manifold which is induced by a blink up to n edges.



r	abs	θ/π	$\#sts$
3	0.707106781	0.000000000	4
4	0.500000000	0.000000000	42
5	0.141995114	−0.200000000	176
6	0.288675135	0.000000000	705
7	0.314455055	−0.481345143	2184
8	0.191341716	1.000000000	6124
9	0.095251192	0.892177536	15028
10	0.020162612	0.400000001	34213
11	0.301979940	−0.138824522	71688
12	0.999011926	−0.074618671	142038

r	abs	θ/π	$\#sts$
3	0.707106781	0.000000000	4
4	0.500000000	0.000000000	42
5	0.141995114	−0.200000000	160
6	0.288675135	0.000000000	637
7	0.414566728	0.360635685	1904
8	0.349854384	0.000000000	5284
9	0.109759133	−0.195676243	12772
10	0.020162612	0.400000001	28881
11	0.138394539	−0.368826858	60016
12	0.606725372	−0.018844586	118422

Figure 10: The two doubts left in L. Lins's thesis are solved In both cases the manifolds induced by the shaded blink-link pairs are proved very recently to be non-homeomorphic to the other in the same class, by geometric means. All the other pairs are proved to be homeomorphic by BLINK which keeps each such homeomorphism as a coded sequence which is stored in a data basis, and, in principle reproducible at will. This shows that BLINK finds all available homeomorphic pairs and, in conjunction with the length of the smallest geodesic (see the end of this section), provides the topological classification of the 9-small 3-manifolds.

The 3-manifolds of [17] are classified by homology and the quantum WRT$_r$-invariants $r = 3, \dots, u$, with 10 significant decimal digits forming $HGuQI$-classes of blinks. Our algorithm for computing the WRT^u_r-invariants are based on the theory developed in [12]. After 6 years we have put our doubts as a Challenge to topologists and group algebraists, [22]. They were quickly solved by some researchers among others M. Culler, N. Dunfield and C. Hodgson, at least in two different ways. They proved that the two pairs of manifolds which were left unresolved by BLINK are indeed non-homeomorphic. All the other pairs in the $HG12QI$-classes 9_{126} and 9_{199} have been checked to be homeomorphic, as BLINK proved 6 years ago. The new solutions were obtained using the software [5], which uses the kernel of [37]. They also use GAP ([8]) and Sage ([34]).

The first solution we obtained was by C. Hodgson using length spectra techniques, based in his joint paper with J. Weeks entitled *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds* ([10]). By using SnapPy Hodgson showed that even though the quantum WRT-invariants as well as the volumes of the hyperbolic Z-homology spheres induced by the blinks $U[1466]$ and $U[1563]$ are the same, the *length of the smallest geodesics of them are distinct.* As for the other pair of blinks, $U[2125]$ and $U[2165]$, the same facts apply. Here is a summary of Hodgson's findings extracted from the SnapPy session that he kindly sent us. As he explains: *"The output of the length spectrum command shows the* complex lengths *of closed geodesics — the real part is the actual length and the imaginary part is the rotation angle as you go once around the geodesic."*

Class 9_{126}:

```
First geodesic of U[1466]: 1.0152103824828331+0.39992347315914334i.
First geodesic of U[1563]: 0.9359206605025168+2.333526236965665i.
Volume of both manifolds: 7.36429600733.
```

Class 9_{199}:

```
First geodesic of U[2125]:  0.8939075859248593+0.761197185679321i.
First geodesic of U[2165]:  0.7978548001747316+2.9487425029345973i.
Volume of both manifolds: 7.12868652133.
```

Both N. Dunfield and M. Culler also classified topologically all the 3-manifolds in the two $HG12QI$-classes of 3-manifolds, by k-coverings techniques, an independent check of Hodgson's proof. We thank to the these 3 researchers: M. Culler, N. Dunfield and C. Hodgson for unveiling our 6-year old mystery: as we believed, both $HG12QI$-classes breaks into two classes of homeomorphisms. The solution of the mystery was the incentive to produce this paper. It reveals scientific collaboration at its best.

5 Conclusion

A closed orientable 3-manifold is denoted *n-small* if it is induced by surgery on a blackboard framed link with at most n crossings. We provide an instance of the general theory to produce a recursive indexation of n-small 3-manifolds up to homeomorphism. We solve this problem up to $n = 9$. Conceptually we could go on forever, finding in the way tougher and tougher examples to be distinguished by yet to be found new invariants. This is particularly reachable, if we consider that the census generating algorithms are fully parallellizable, and this parallelization was not used. The topological classification of the 9-small 3-manifolds involve three invariants:

$$INV = \{ \; homology, \; WRT_{12}, \; length\ of\ smallest\ geodesic \; \}.$$

The classification was nearly complete in [17], except for two doubts. Recently, after we posted a challenge in the arXiv, [22] these doubts were solved by M. Culler, N. Dunfield and C. Hodgson using SnapPy [5]. This made us add

$$length\ of\ smallest\ geodesic$$

which we define as 0, if the manifold is not hyperbolic, to our list of invariants. The 9-small 3-manifold classification maintains live the two Conjectures of page 15 of [19] based on two kinds of moves TS and U: the TS- and U-moves yield an efficient algorithm to classify 3-manifolds by explicitly displaying homeomorphisms among them, whenever they exist. A recent finding by C. Hodgson concerning manifolds $T[71]$ and $T[79]$ forming the HG8QI-class 14^t_{24}, in the notation of page 239 of [17] shows that the 3 invariants are not enough to decide the pair. This pair is the first one of 11 pairs that we display as some tougher challenges to 3-manifold topologists, [21]. Hodgson's finding is that the volume as well as the lengths of the smallest geodesics fail to distinguish $T[71]$ and $T[79]$. He proves them to be non-homeomorphic by more sofisticated techniques, involving drilling along the smallest geodesics to get non-isometric manifolds with toroidal boundary. Using SnapPy, GAP and Sage, N. Dunfield shows that $T[71]$ and $T[79]$ are also distinguished by the homology of its 5-covers. This solution depends on the existence of low index subgroups of the fundamental group of the manifold. What about if they do not exist? We feel that the majority of the *Tough Challenges* of [21] remains unresolved. Have we arived to our currently technological limit? Maybe so, but what we feel the difficulty of SnapPy and BLINK in dealing with the tough challenges teach us is that we need to find better subtle invariants which performs quicker than the current ones. Invariants which are not so exponential as looking for small index coverings which depend on the unpredictability idiosyncrasies of the fundamental group.

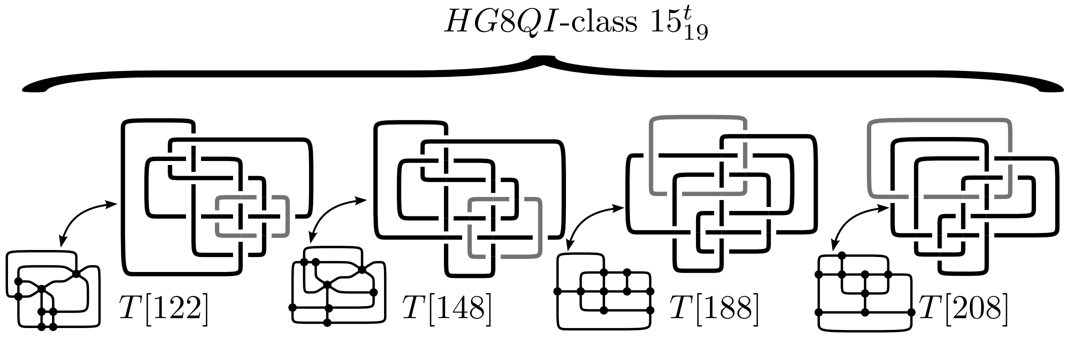


Figure 11: A focused challenge: the $HG8QI$-class 15^t_{19}. We conjecture that there are exactly two homeomorphism classes induced by these 4 blackboard framed links in 15^t_{19}, and that each class is induced by 2 of the BFL's above. BLINK tell us that there is at most 2 homeomorphism classes, (each induced by two blinks). Using SnapPy/Sage/GAP we could not find the Dirichlet domain, due to numerical inaccuracies, and could not compute the volumes. Also low index k-covering methods, take long and fail for $k = 3, 4, 5, 6, 7, 8$. We do not know whether SnapPy can get isomophic triangulations for the 4 blinks. We bet not, BLINK would have found them. Asssuming that there are two homeomorphism classes, which computable invariant will distinguish them? Have we reached our current technological limit with such a small example? If so, we need badly to discover new invariants. The eleven examples (of which $HG8QI$-class $1t^t_{19}$ is the third) forming the tougher challenges in [21] must be taken seriously. To us most of them remain open. It seems that if they can be solved at all, it will be by luck, involving very expensive computations. We need something better. After all, our input is a very small combinatorial object: all the information is stored in a 16-edge plane graph (the blinks in the T-census are monochromatic). It is our hope that the close connection between 3-manifolds and plane graphs of [20] may help us in the future to find some new subtle invariant which does not take so long to execute while having some guaranty of success whithin a bounding time pre-established. It does not have to be a polynomial algorithm (if it is, the better), but a finite algorithm that we know, for sure it will take at most a reasonable time limit that we know in advance. Currently, the better example of such an algorithm is the WRT-invariant (within a segment $r \in \{3, 4, \ldots, u\}$) computed as in [12]

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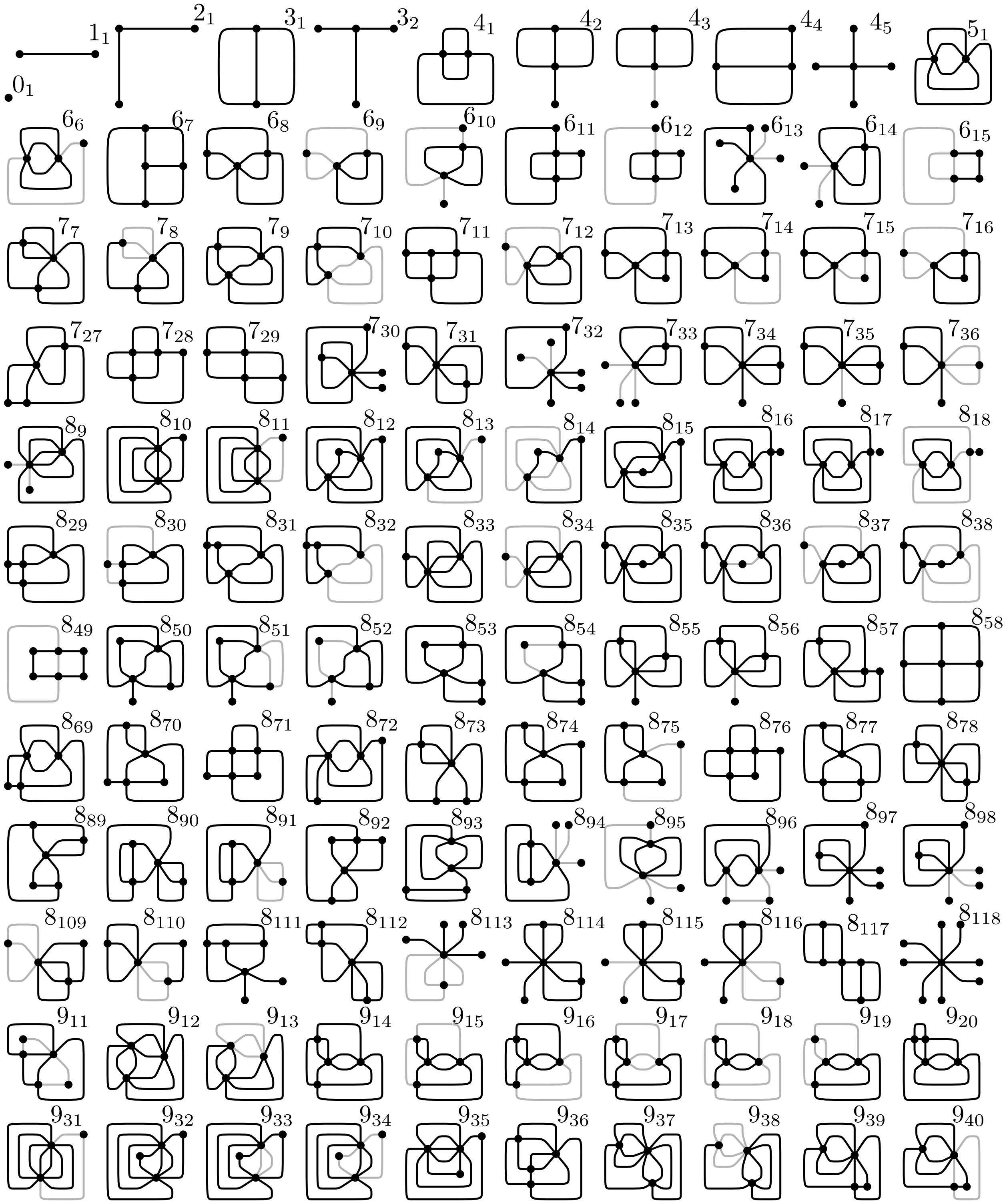
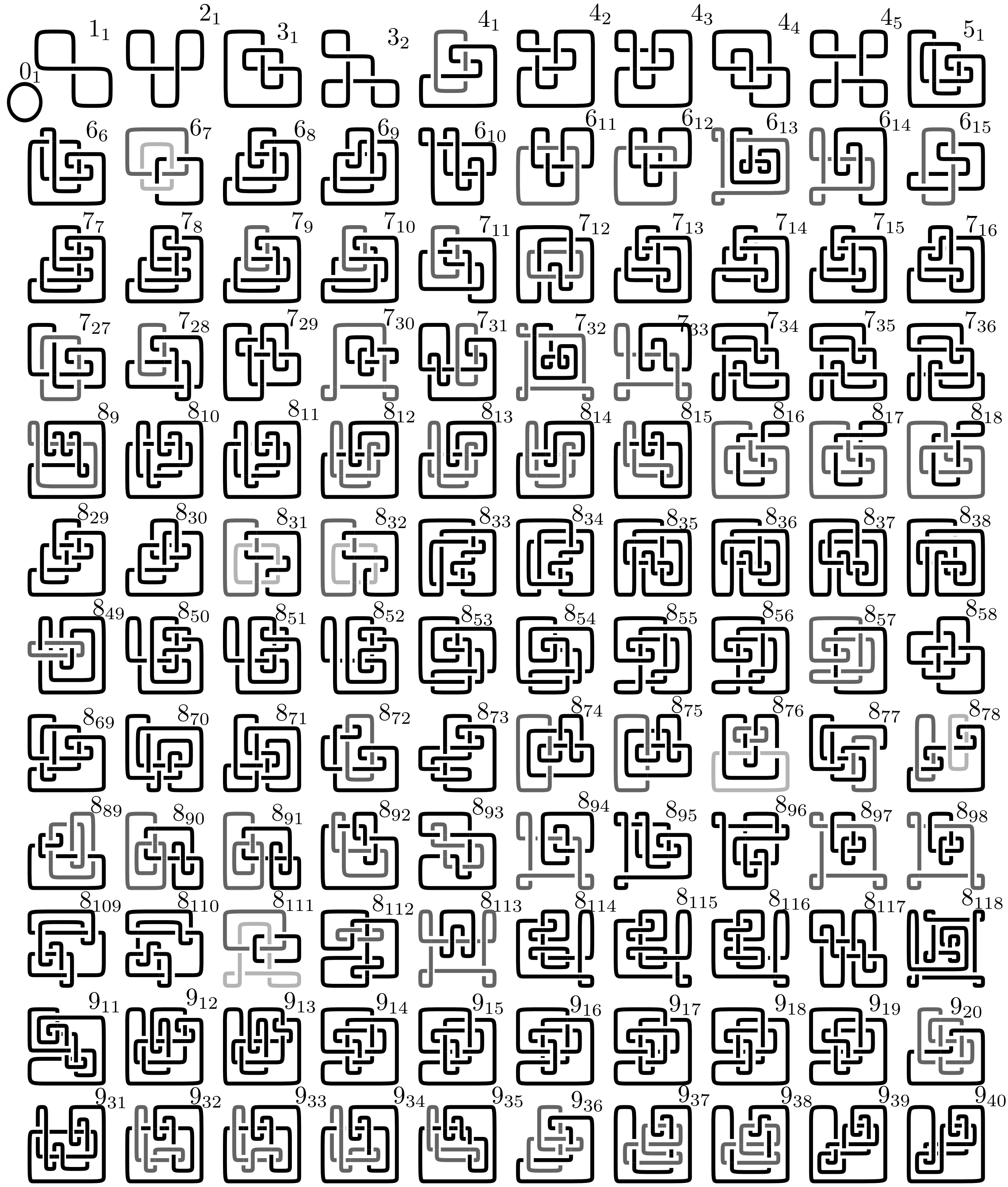
6 Appendix: census (no misses, no duplicates) of 9-small 3-manifolds

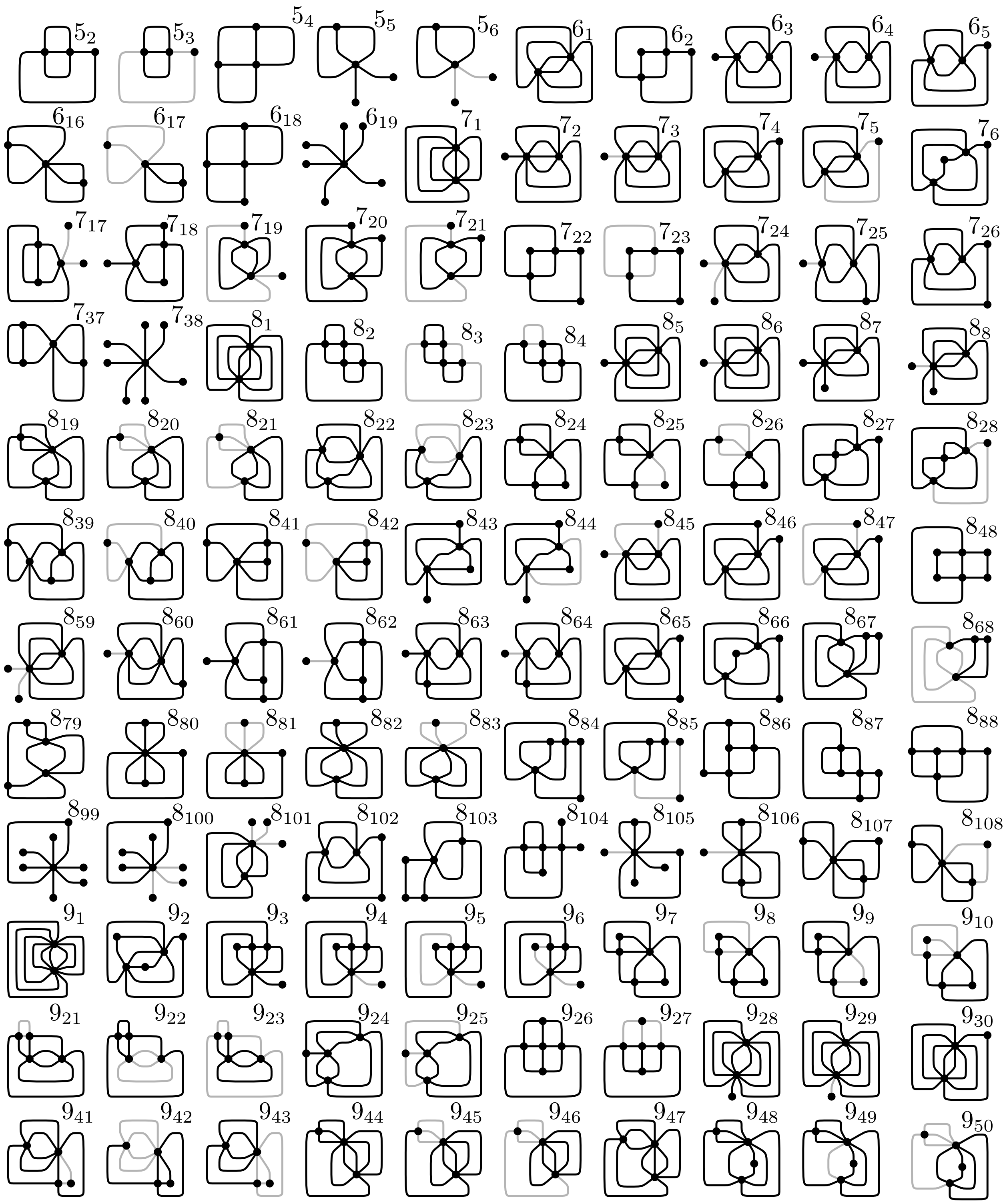
Part 1/4 in terms of blinks:



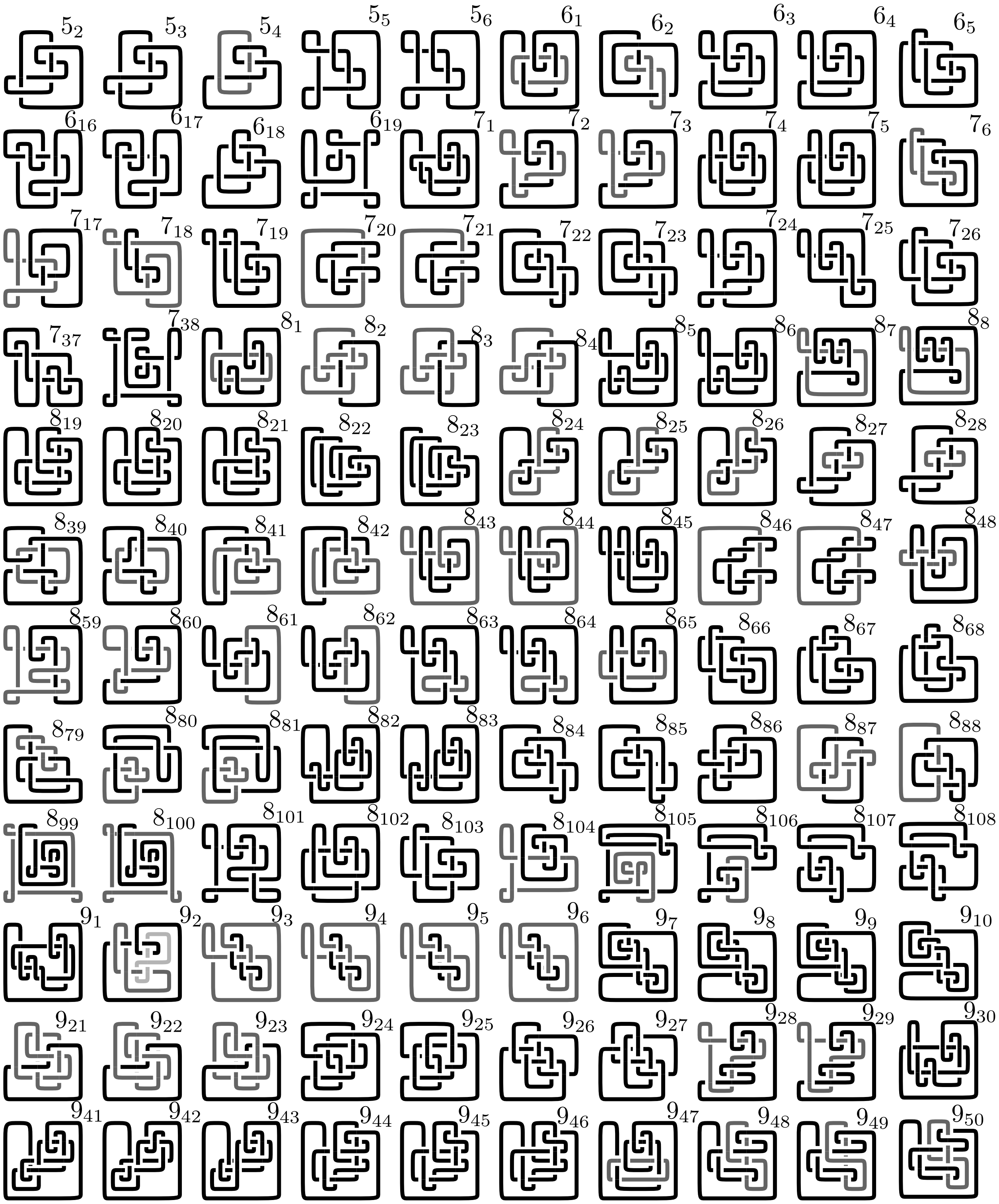
Part 1/4 in terms of blackboard framed links:



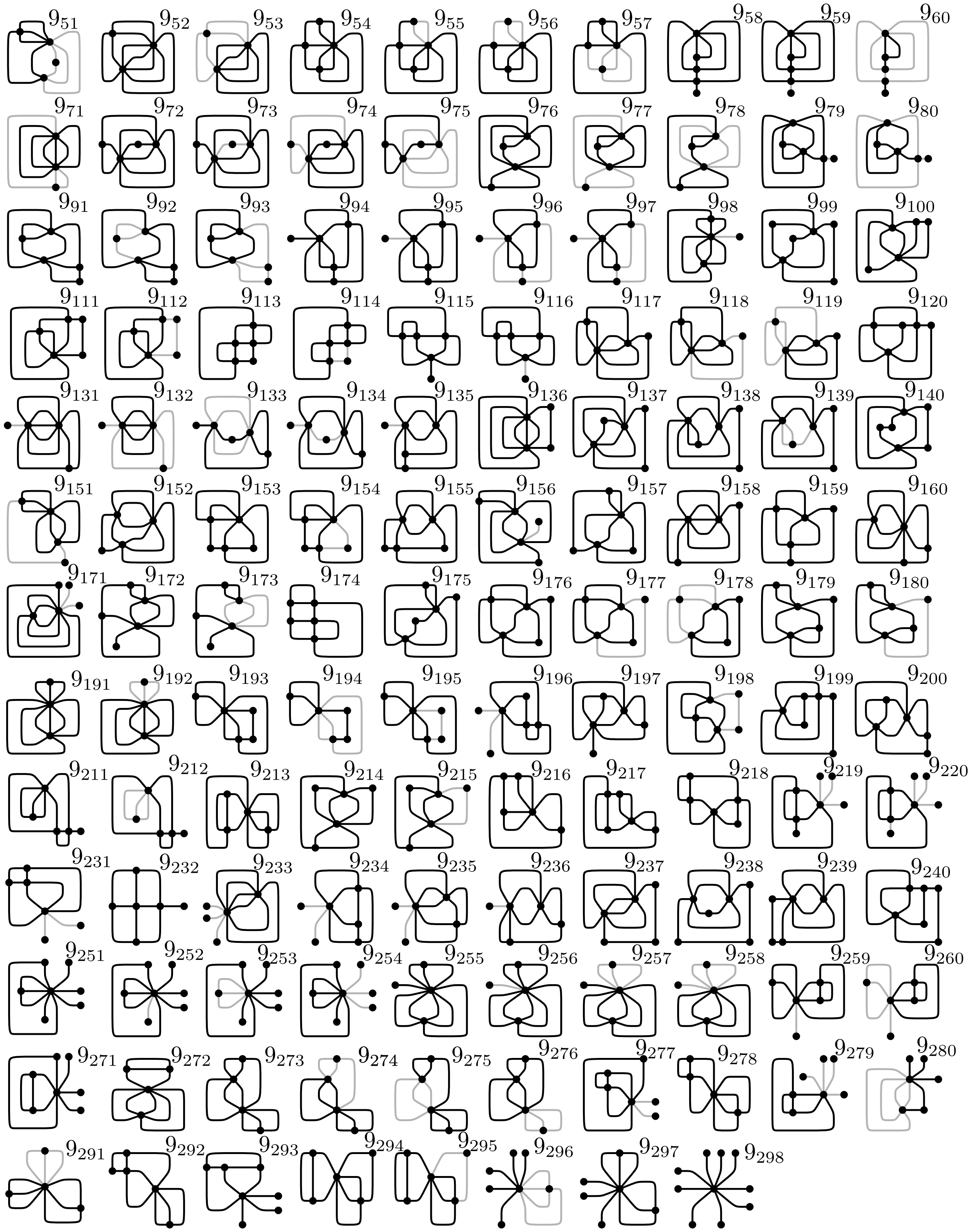
Part 2/4 in terms of blinks:



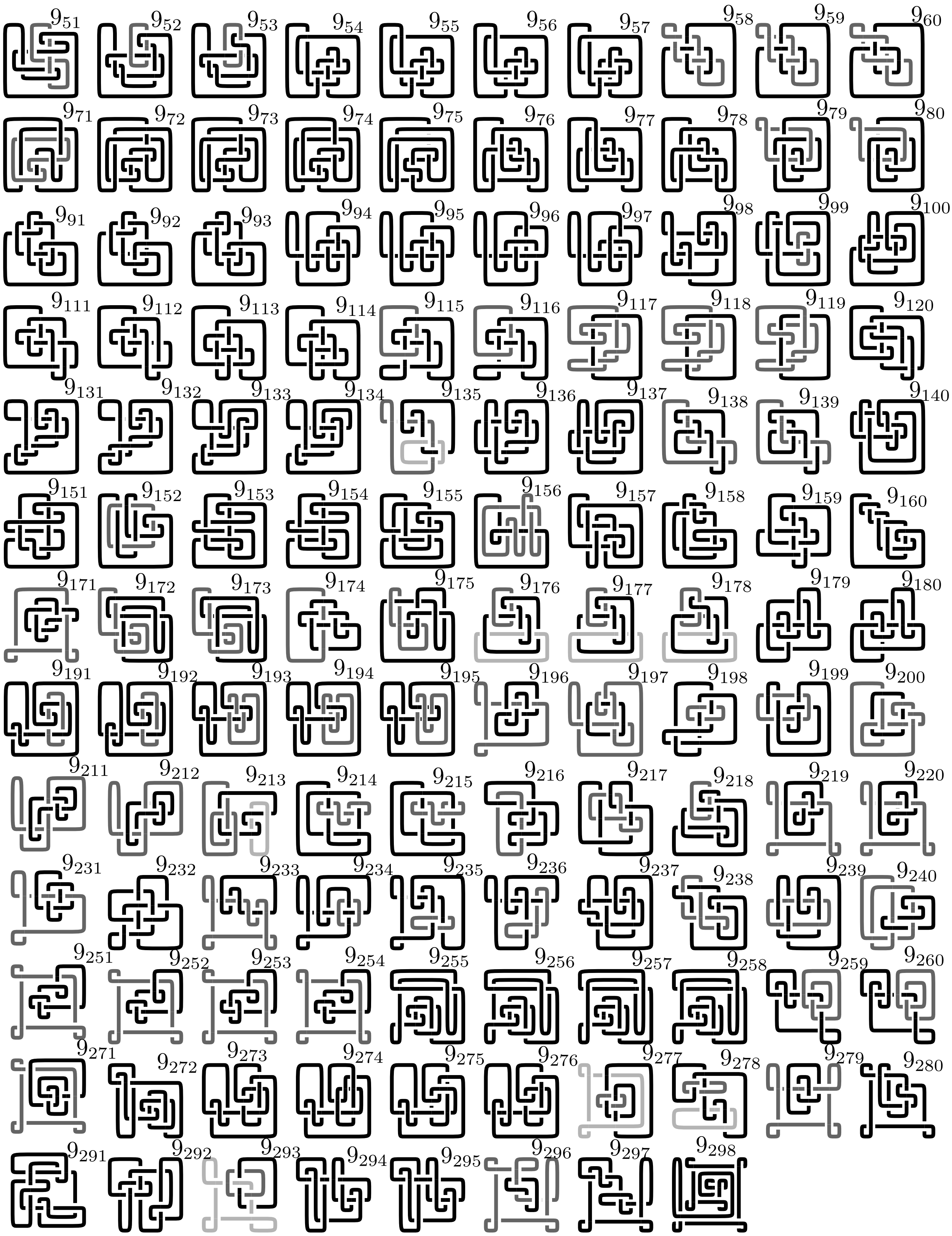
Part 2/4 in terms of blackboard framed links:



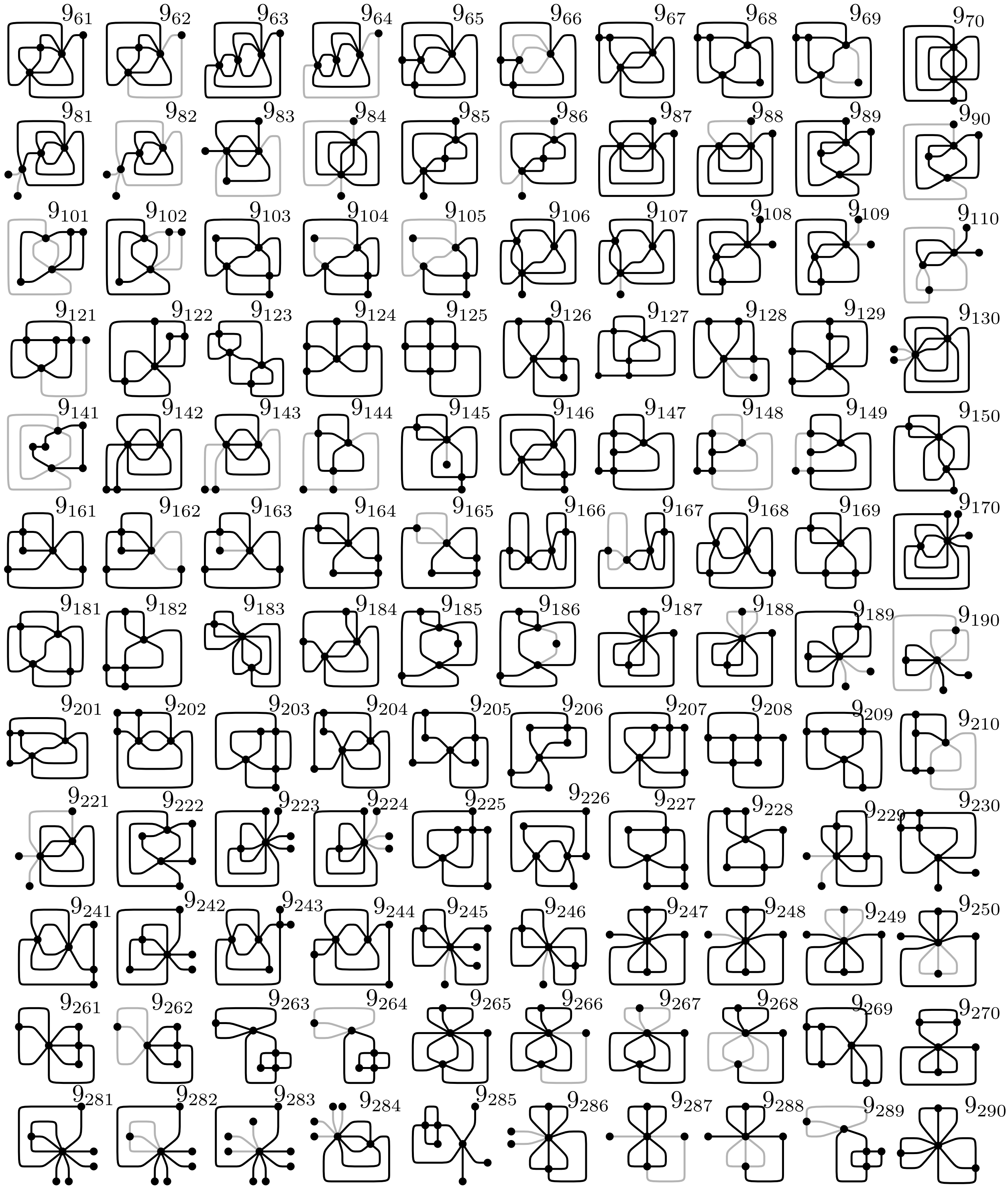
Part 3/4 in terms of blinks:



Part 3/4 in terms of blackboard framed links:



Part 4/4 in terms of blinks:



Part 4/4 in terms of blackboard framed links: