# All the shapes of spaces: a census of 9-small 3-manifolds *

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**Abstract**

In this work we present a complete (no misses, no duplicates) catalogue for closed, orientable and prime 3-manifolds induced by plane graphs with a bipartition of its edge set (blinks) up to 9 edges. Blinks form a universal encoding for such manifolds. We hope that this census becomes as useful for the study of concrete examples of 3-manifolds as the tables of knots are in the study of knots and links. Along the years we have made an issue in our computational work that it must be reproducible and independently checked by other researchers. Our software BLINK is available, but currently it lacks yet a good documentation and help is welcome to change this. An Wiki open source project is starting.

## 1 Introduction

After presenting some instances of closed 3-manifolds, P. Alexandroff says in the English translation (1961) of his joint work with D. Hilbert [1], first published (1932) in German, [2]: "These few examples will suffice. Let it be remarked here that, at present, in contrast with the two-dimensional case, the problem of enumerating the topological types of manifolds of three and more dimensions is in an apparently hopeless state. We are not only far removed from the solution, but even from the first step toward a solution, a plausible conjecture".

John Hempel in his book (1976) *3-Manifolds* [5] writes at the opening of Section 15, titled *Open Problems:*
"The ultimate goal of the theory would be in providing solutions to: *The homeomorphism problem*: provide an *effective* procedure for determining whether two *given* 3-manifolds are homeomorphic, together with *The classification problem*: *effectively* generate a list containing exactly one 3-manifold from each homeomorphism class."

We mean to provide a road for a segmented answer for Hempel's questions. The closed oriented 3-manifolds are partitioned by the number of edges in a minimum encoding of them by a certain class of plane graphs, shortly to be defined. Lexicography is used to define a representative unique plane graph (a canonical form) for each closed oriented 3-manifold. We explicitly solve the segmented problem up to 9 edges, see Theorem 3.1. An important fact is that there is a polynomial algorithm to make available the canonical form of any manifold induced by plane graphs at the current level of the catalogue (currently 9 edges).

It is amazing how much the picture has changed in the 80 years since Alexandroff's-Hilbert book. The progress was due to the deep breakthroughs in the 1950's and 1960's, starting in the

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary) early 1980's of W. Thurston, in the early 2000's, G. Perelman, and more recently 2010's, I. Agol and many others. Currently there is a great amount of important reserch issues going on and these are exciting times for 3-manifold theory.

We present here our modest contribution to the topic. It is placed in the confluency of two deep research passions of the authors, apparently very far apart: The study of closed orientable 3-manifolds and the study of plane graphs. By putting a bit of combinatorics on the graphs, namely, partitioning their edges into black and gray in an arbitrary way we get an object named a *blink*. We make here explicit for the first time that each class of homeomorphic closed oriented 3-manifolds is a subtle class of blinks where two membres of each class is linked by means of a small number of local simple moves. These moves (see Fig. 1) are a slight reformulation for blinks of the one for framed links on Kirby's calculus ([8]) recently published by Martelli, [12]. Thus, as simple as the blinks are, nevertheless they hold, in their gist, the mystery of 3-manifolds.

# 2 A formal calculus on blinks

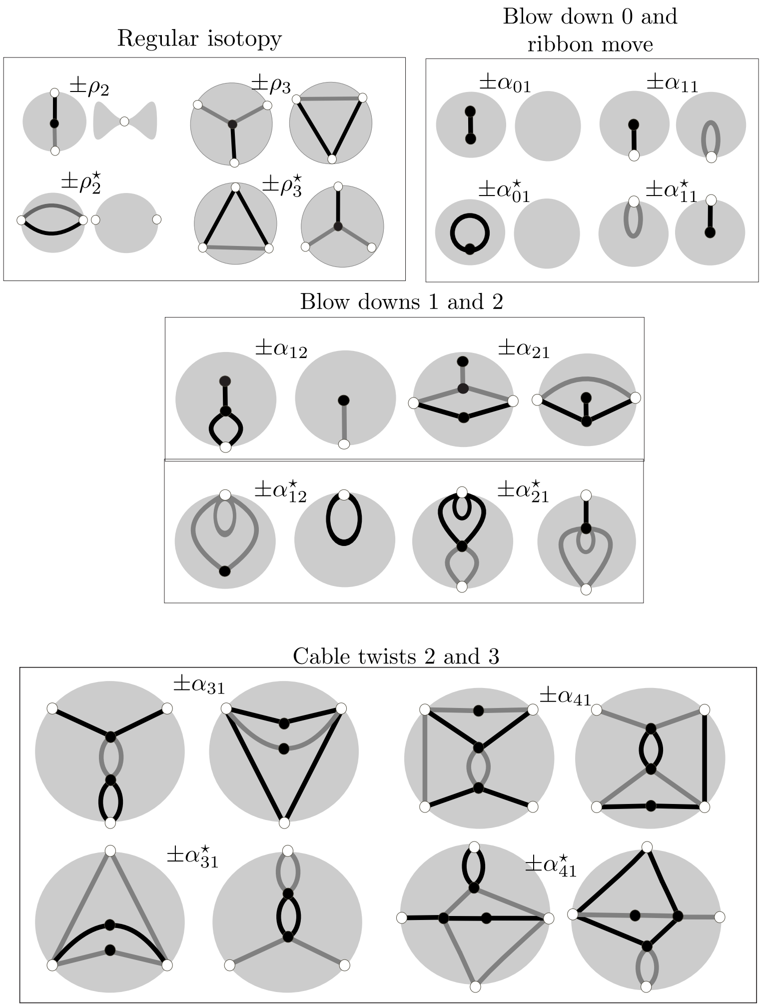



Figure 1: Blink reformulation of Martelli's calculus. Here the 20 moves are for *closed oriented* 3-manifolds. A few comments are in order. First, the white vertices on the boundary of the shaded disk are attachement vertices, the only connection with rest of the blink in the complement of the disk. The complementary configurations are entirely arbitrary. The internal configuration is exactly as shown, in particular the internal black vertices has their neighborhood as depicted. A *blink move* consists in replacing one of the 20 disks on the left by its counterpart on the right while leaving the exterior fixed, or vice versa. We made no effort in minimizing the number of moves. For instance moves $\pm\rho_{33}$ and $\pm\rho_{33}^\star$ are equivalent. The negative of a move is obtained by interchanging the colors gray and black. This corresponds to reversing the orientation. The starred moves correspond to planar duality and are nothing at the level of the associated bfl's. Strong regular isotopies are defined by strictly local disk replacements: some Reidemeister moves of type 2 causes difficulty by introducing a pinching in the disk. We avoid this at the cost of replacing the usual move 2 by six of the type disk replacements, in the spirit of paper [6] for finding invariants that depend on the faces of the bfl's.

**(2.1) Theorem.** *Two blinks represent the same oriented closed 3-manifold if and only if they are linked by a finite sequence where each term is one of the 20 moves of Fig. 1 or their inverses.*

***Proof.*** The moves are basically a reformulation of Martelli's calculus. The only difference is that we have made them entirely local as disks replacements and treat the oriented case at the expense of increasing the types of moves. □

Any plane drawing of a graph with an arbitrary bipartition of its edge set, a blink, corresponds to a unique closed oriented 3-manifold via the associated blackboard framed links. An important aspect about blinks is that each one possesses an easily obtainable *canonical form* inducing the same 3-manifold: it is named the *representative of the blink* and is obtained by lexicography form a small number of conventions, fixed in advance. This is explained in L. Lins' thesis, [9].

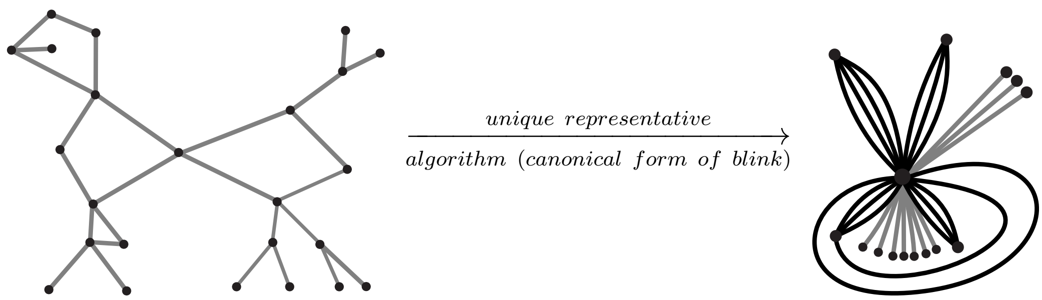



Figure 2: Blink representative algorithm: doglike blink and its representative

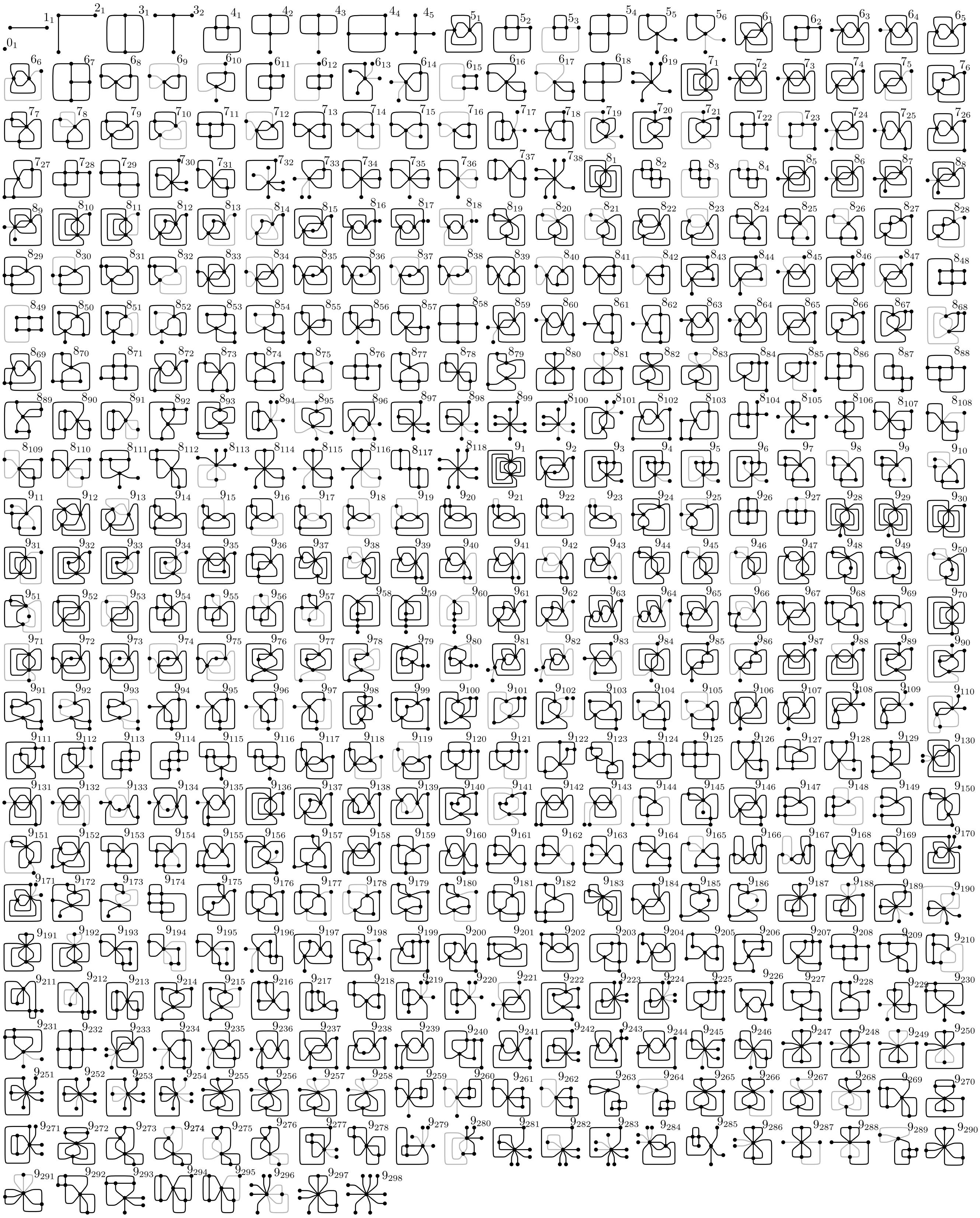
# 3 A complete duplicate free census of 9-small 3-manifolds

In references [7], [9] and [10] we have defined and show how a *blink*, that is, a plane graph with an arbitrary partition of its edges (here presented as colors black and gray) induces a well defined closed oriented 3-manifold. Moreover each such a manifold is induced by a blink (in fact, by infinite blinks). An *n-small* is a closed, orientable, and prime 3-manifold is a manifold induced by a blink with at most $n$ edges. Relative to [9] the blinks of next theorem have receive two additions, the representative blinks $U[1563]$ and $U[2165]$. This is because the HGnQI-classes $9_{126}$ and $9_{199}$ of [9] split into two topological classes. An objective of the present work is to prove that the splitings indeed take place.

We observe that the blinks are enlarged in the appendix, showing them together with the corresponding blackboard framed links. The notation $n_i$ attached to each blink below, is the name of its homeomorphism class, not merely its hgqi-class, as in [9].

This paper concludes the proof of the following theorem:

**(3.1) Theorem.** *Let $M^3$ be a closed, oriented and prime 3-manifold induced by a blink with at most 9 edges. Then $M^3$ is homeomorphic to exactly one of the 3-manifolds induced by the 489 blinks below. Morever all of these are pairwise non-homeomorphic.*



***Proof.*** The proof follows from L.Lins' thesis and from the discussion about the lengths of the smallest geodesics of the classes $9_{126}$ and $9_{199}$ $\square$

## 4 Definition of Gem

For completeness we briefly recall the basic definitions of gem theory, leading to its definition, [10]. A *4-graph* $G$ is a finite bipartite 4-regular graph whose edges are partitioned into 4 colors, 0,1,2, and 3, so that at each vertex there is an edge of each color, a proper edge-coloration, [3]. For each $i \in \{0, 1, 2, 3\}$, let $E_i$ denote the set of $i$-colored edges of $G$. A $\{j, k\}$-residue in a 4-graph $G$ is a connected component of the subgraph induced by $E_j \cup E_k$. A 2-residue is a $\{j, k\}$-residue, for some distinct colors $j$ and $k$. A *gem* is a 4-graph $G$ such that for each color $i$, $G\backslash E_i$ can be embedded in the plane such that the boundary of each face is a 2-residue. From a gem there exists a straightforward algorithm to obtain a closed orientable 3-manifold, in two different, dual ways. Every such a manifold is obtainable in this way. An unecessary big gem is obtained from a triangulation $T$ for a manifold by taking the dual of the barycentric subdivision of $T$. Here the colors corresponds to the dimensions. Doing simplifications in the gem completely destroys this correspondence.

## 5 The resolution of the doubts left in L. Lins' thesis

The topological classification of the 9-small spaces was nearly completed in [9]. This work develops a theory for generating a distinguished set of blinks named $U_n$ and indexed lexicographically, $U_n[i]$ is the $i$-th such blink. The relevance of $U_n$ is that it misses no closed, orientable, prime and irreducible 3-manifold which is induced by a blink with $n$-edges.

The 3-manifolds of [9] are classified by homology and the quantum $\mathrm{WRT}_r$-invariants $r = 3, \ldots, u$, up to $d$ decimal digits forming $hgqi^d_u$-classes. Our algorithm for computing the $WRT^d_r$-invariants are based on the theory developed in [7].

After 6 years we have put our doubts as a Challenge to topologists and group algebraists, [11]. [13]

The basic tool to differentiate the manifold is the software SnapPy, [4].

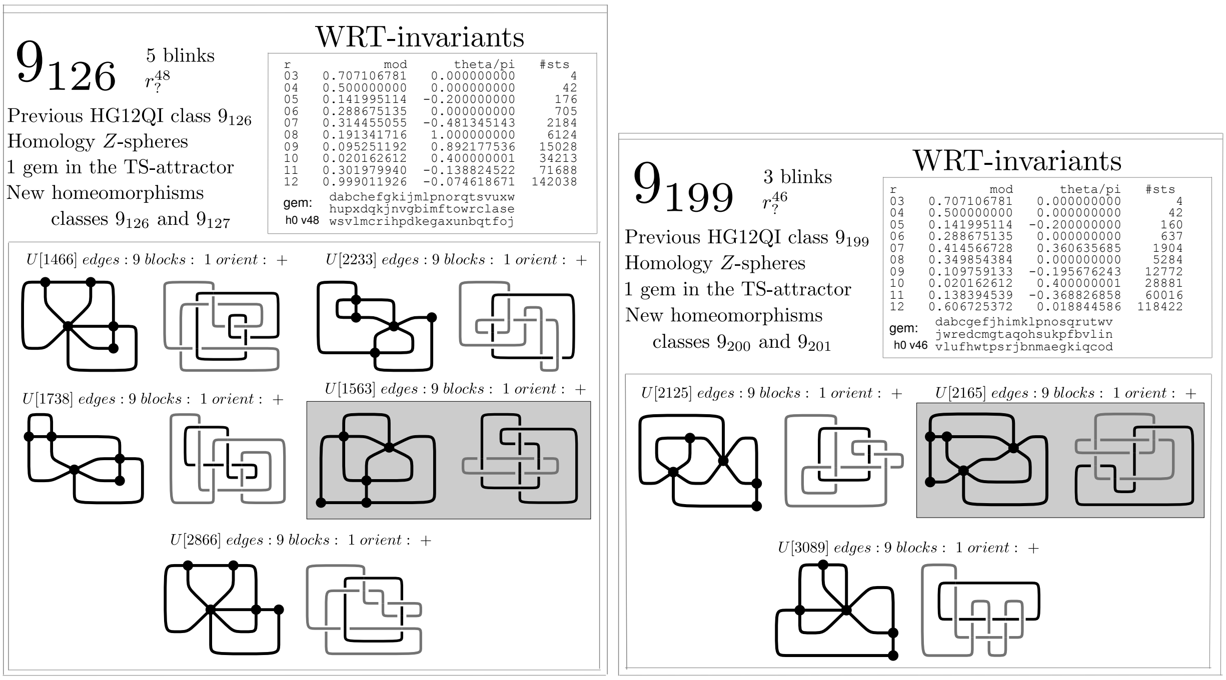



Figure 3: Solving the last two doubts in L. Lins's thesis. The latter are homeomorphic.

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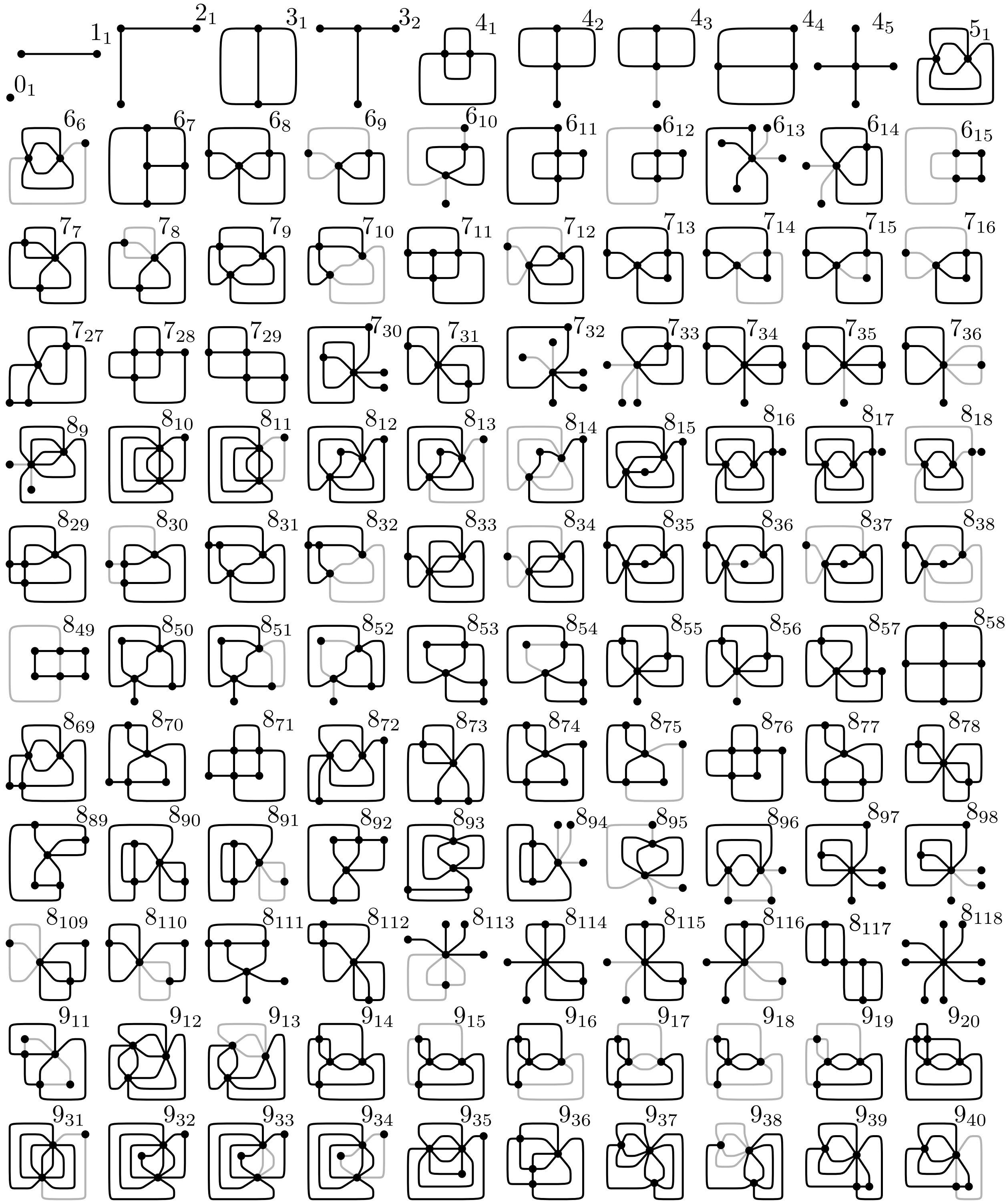
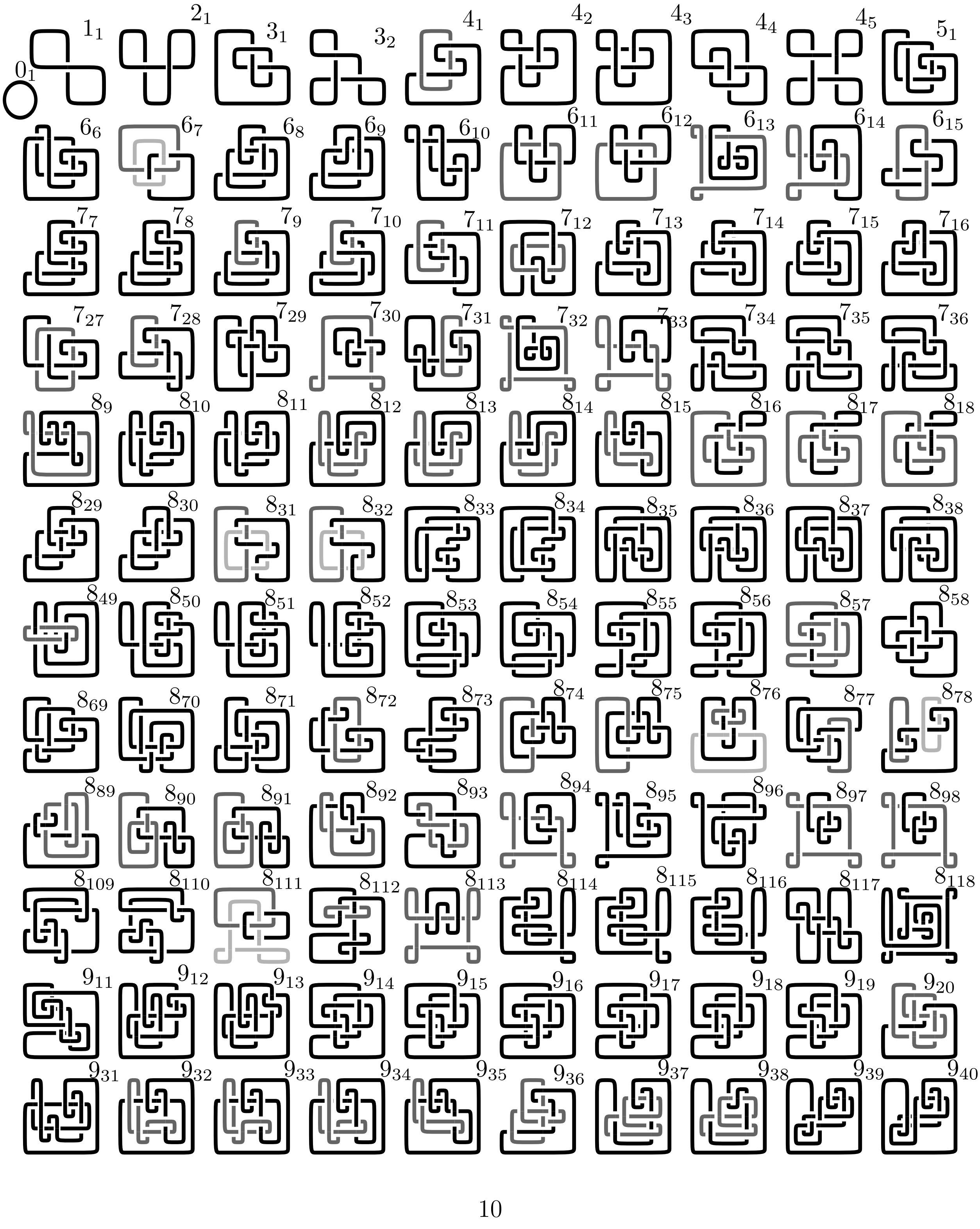
# 6 Appendix: census (no misses, no duplicates) of 9-small 3-manifolds

**Part 1/4 in terms of blinks:**



**Part 1/4 in terms of blackboard framed links:**



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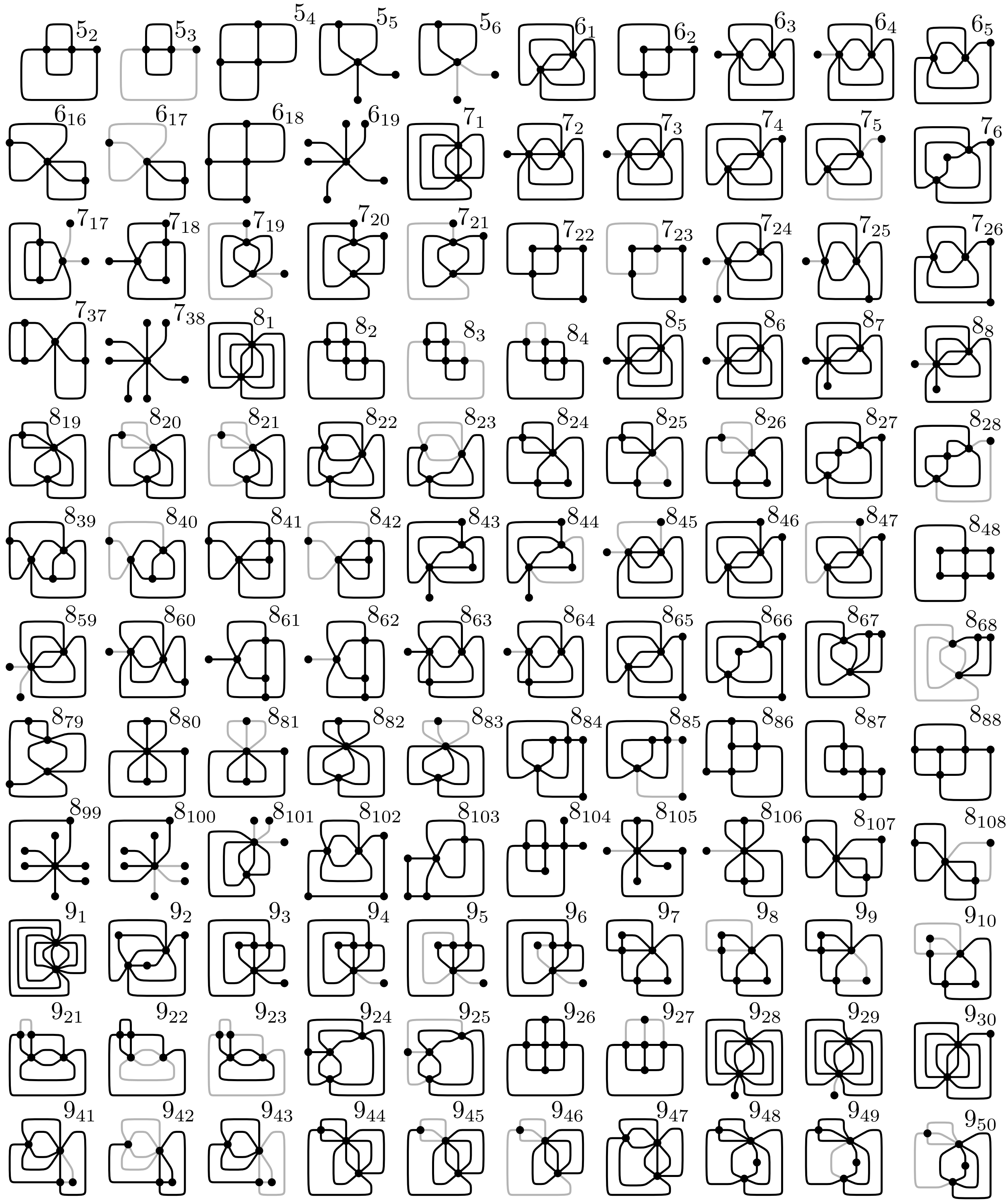

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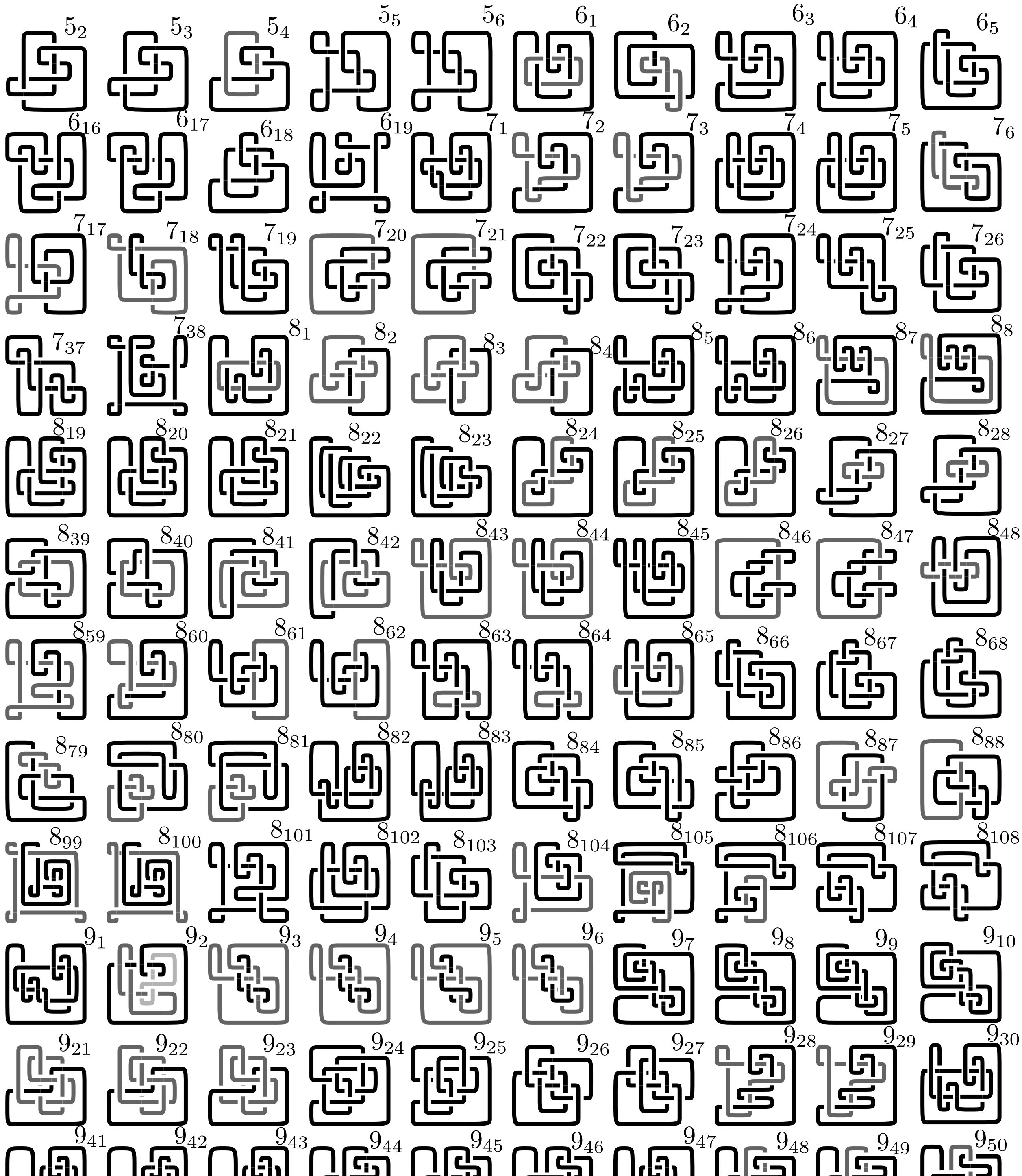

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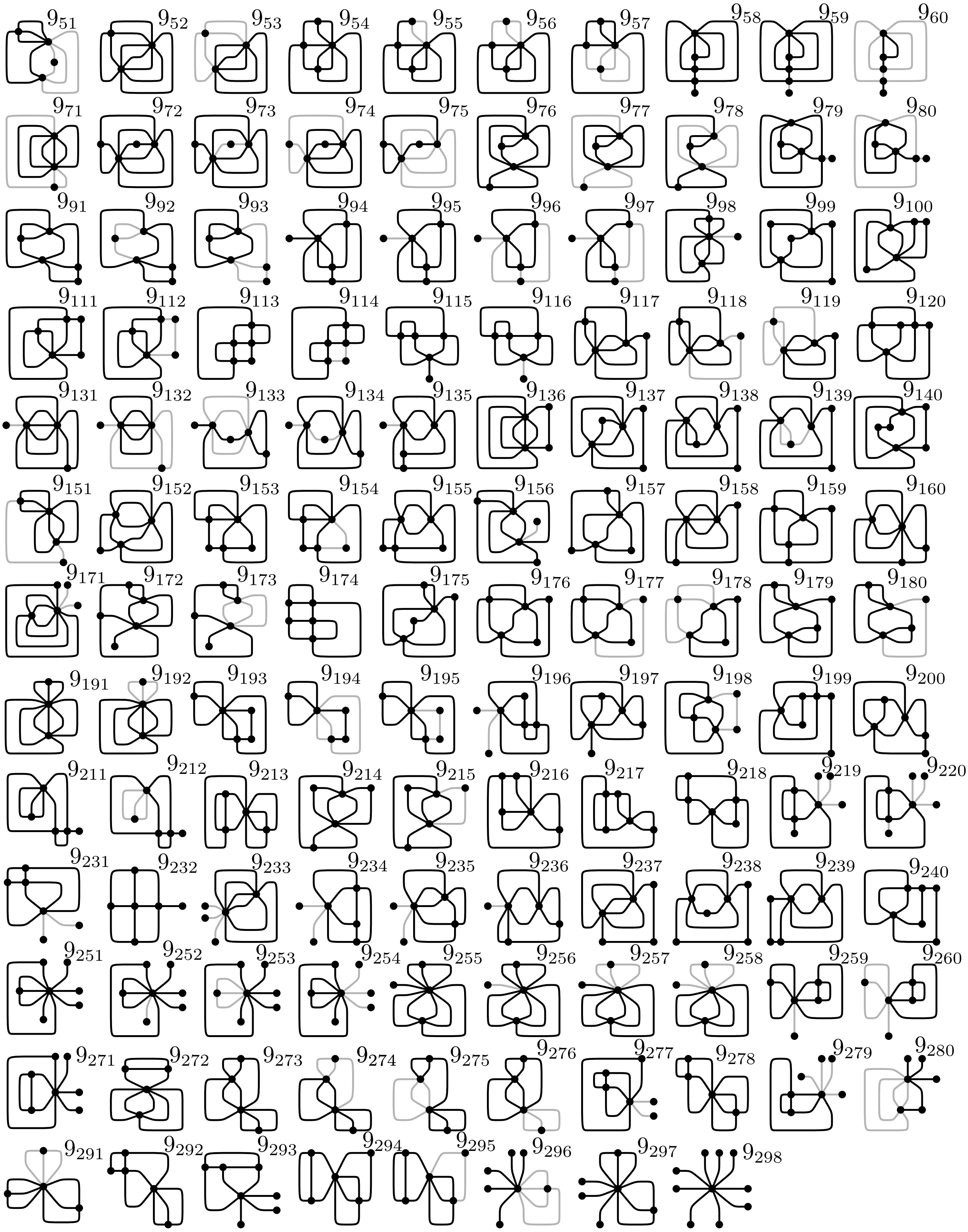

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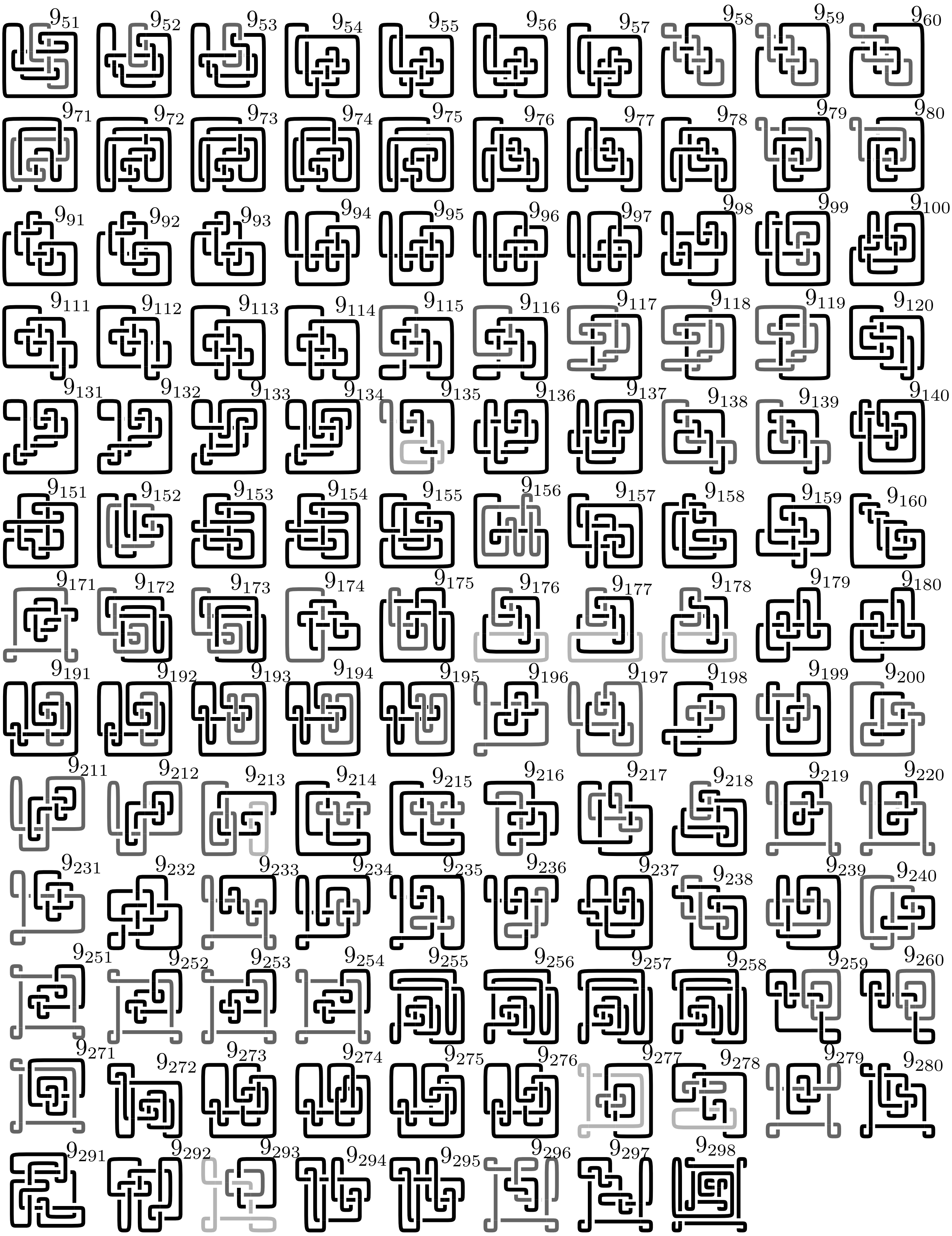

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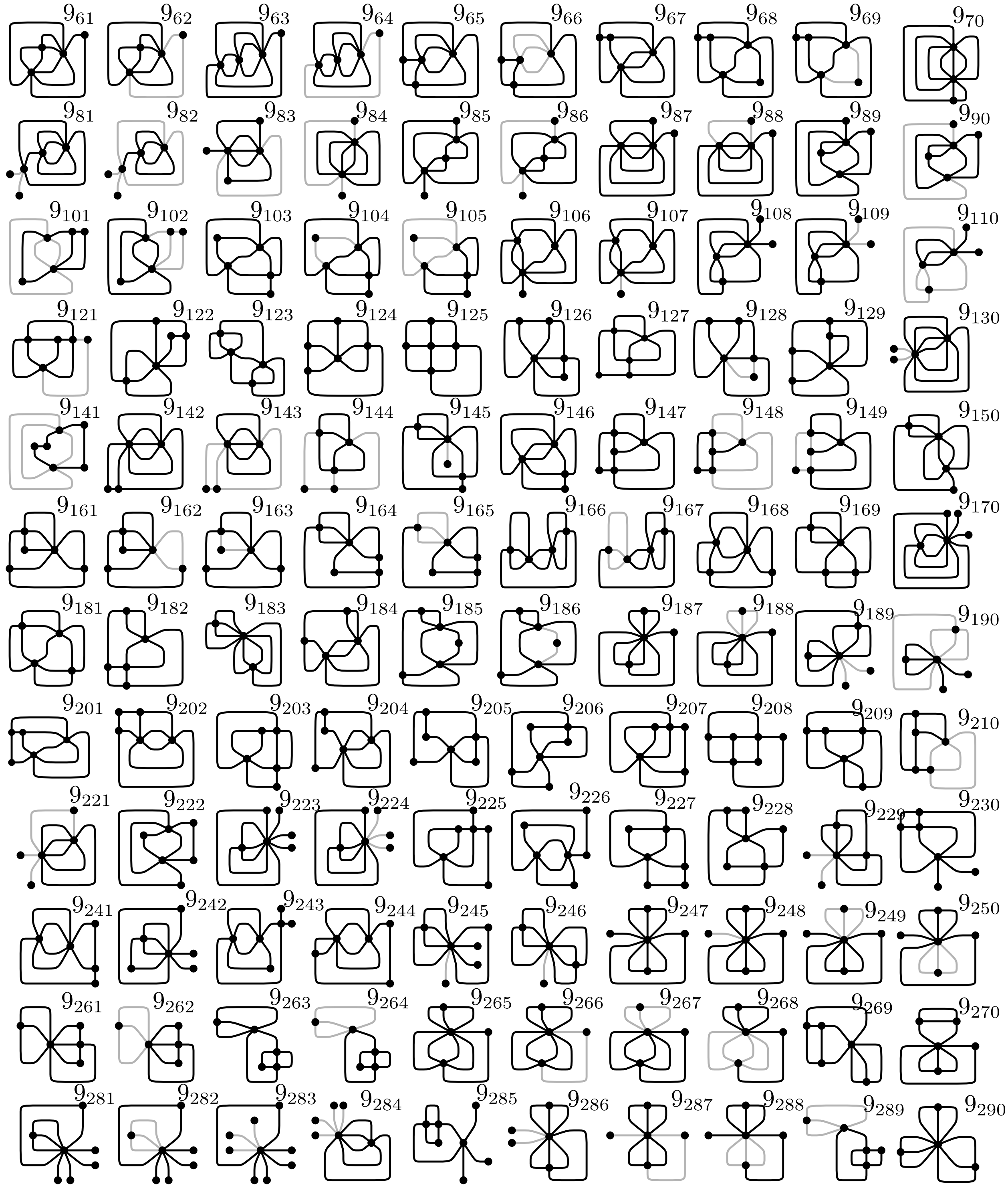
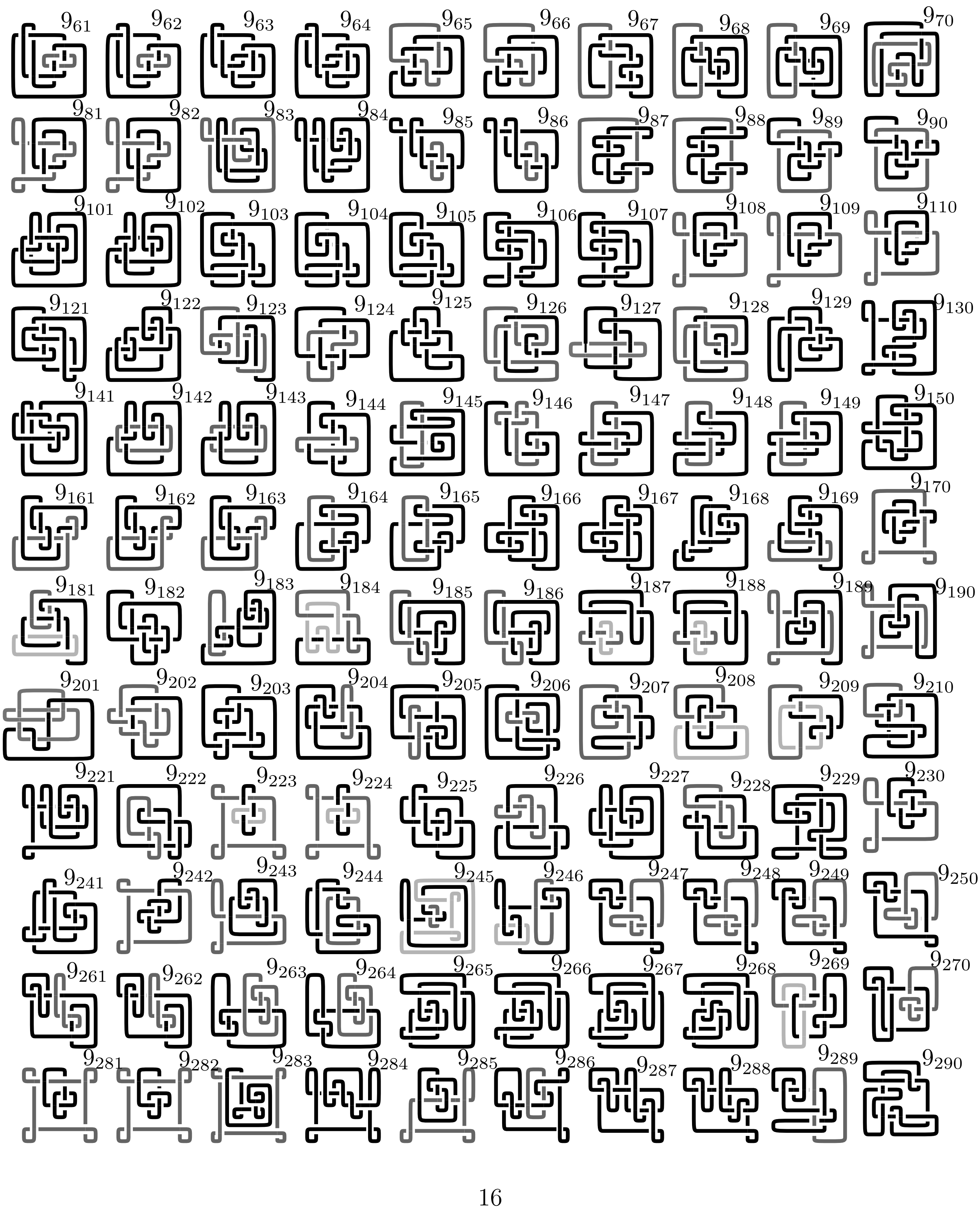


**Part 4/4 in terms of blackboard framed links:**



# 7 Conclusion

A closed orientable 3-manifold is denoted *n-small* if it is induced by surgery on a blackboard framed link with at most $n$ crossings. Our bet is that both pairs of 3-manifolds in the 2 first sections of this short note are not homeomorphic. This would mean that the 9-small manifolds are completely classified and that the combinatorial dynamics of Chapter 4 in [10] based on $TS$-moves which leads to a (small, in the case of hyperbolic 3-manifolds) number of minimal gems, named the *attractor of the 3-manifold* is successful. This induces an efficient algorithm which is capable of classifying topologically all the 3-manifolds given as a blackboard framed link with up to (so far) 9 crossings and maintains live the two Conjectures of page 15 of [10]: the $TS$- and $u^n$-moves yield an efficient algorithm to classify $n$-small 3-manifolds by explicitly displaying homeomorphisms, whenever they exist.

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