All the shapes of spaces: a census of 9-small 3-manifolds *

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Abstract

In this work we present a complete (no misses, no duplicates) catalogue for closed, orientable and prime 3-manifolds induced by plane graphs with a bipartition of its edge set (blinks) up to 9 edges. Blinks form a universal encoding for such manifolds. We hope that this census becomes as useful for the study of concrete examples of 3-manifolds as the tables of knots are in the study of knots and links. Along the years we have made an issue in our computational work that it must be reproducible and independently checked by other researchers. Our software is available, but currently it lacks yet a good documentation and help is welcome to change this. An Wiki open source project is starting.

1 Introduction

John Hempel in his book *3-Manifolds* [2] said in opening Section 15 on Open Problems:
"The ultimate goal of the theory would be in providing solutions to: *The homeomorphism problem*: provide an *effective* procedure for determining whether two *given* 3-manifolds are homeomorphic, together with *The classification problem*: *effectively* generate a list containing exactly one 3-manifold from each homeomorphism class." We here provide a road for a segmented answer for these questions. The closed oriented 3-manifolds are partitioned by the number of edges in a minimum encoding of them by a certain class of plane graphs, shortly to be defined. We use lexicography to define a representative unique plane graph for each 3-manifold of that type. We explicitly solve the segmented problem up to 9 edges, see Theorem 1.1.

This work is in the confluency of two deep research passions of the authors, apparently very far apart: The study of closed orientable 3-manifolds and the study of plane graphs. By putting a bit of combinatorics on the graphs, namely, partitioning their edges into black and gray in an arbitrary way we get an object named a *blink*. We make here explicitly that each class of homeomorphic closed oriented 3-manifolds is a subtle class of blinks where two membres of each class is linked by means of a small sets of local simple moves. Thus, as simple as the blinks are, nevertheless they hold the mystery of 3-manifolds in their gist.

In references [3], [4] and [5] we have defined and show how a *blink*, that is, a plane graph with an arbitrary partition of its edges (here presented as colors black and gray) induces a well defined closed oriented 3-manifold. Moreover each such a manifold is induced by a blink (in fact, by infinite blinks). An *n-small* is a closed, orientable, and prime 3-manifold is a manifold induced by a blink with at most n edges. Relative to [4] the blinks of next theorem have receive two additions, the representative blinks $U[1563]$ and $U[2165]$. This is because the HGnQI-classes 9_{126} and 9_{199} of [4] split into two topological classes. An objective of the present work is to prove that the splitings indeed take place.

We observe that the blinks are enlarged in the appendix, showing them together with the corresponding blackboard framed links. The notation n_i attached to each blink below, is the name of its homeomorphism class, not merely its hgqi-class, as in [4].

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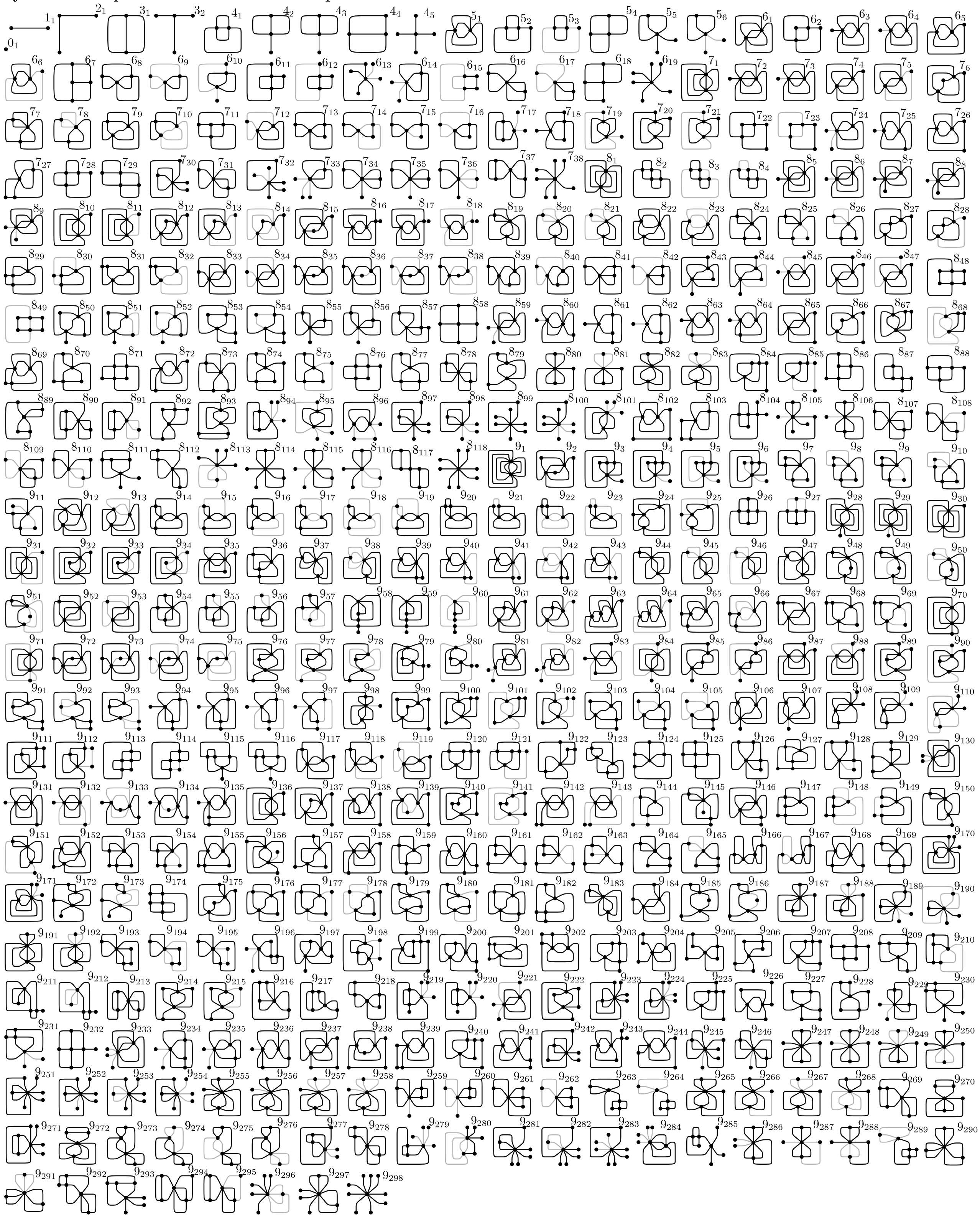
This paper concludes the proof of the following theorem:

(1.1) Theorem. *Let M^3 be a closed, oriented and prime 3-manifold induced by a blink with at most 9 edges. Then M^3 is homeomorphic to exactly one of the 3-manifolds induced by the 489 blinks below. Morever all of these are pairwise non-homeomorphic.*



Proof. The proof follows from L.Lins' thesis and from the discussion about the lengths of the smallest geodesics of the classes 9_{126} and 9_{199} $\square$

2 The resolution of the doubts left in L. Lins' thesis

The topological classification of the 9-small spaces was nearly completed in [4]. This work develops a theory for generating a distinguished set of blinks named U_n and indexed lexicographically, $U_n[i]$ is the i-th such blink. The relevance of U_n is that it misses no closed, orientable, prime and irreducible 3-manifold which is induced by a blink with n-edges.

The 3-manifolds of [4] are classified by homology and the quantum WRT_r-invariants $r = 3, \ldots, u$, up to d decimal digits forming $hgqi_u^d$-classes. Our algorithm for computing the WRT_r^d-invariants are based on the theory developed in [3].

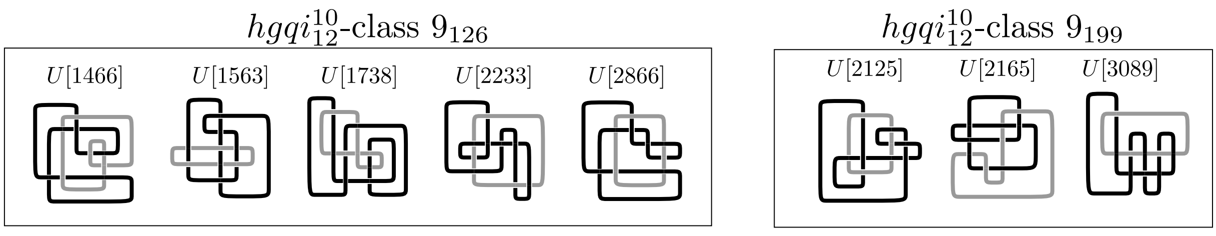


Figure 1: **Note's final challenge:** classify topologically 9_{126} and 9_{199}. Here, to classify has the following strict meaning: for each pair

After 6 years we have put our doubts as a Challenge to topologists and group algebraists, [6]. [7]

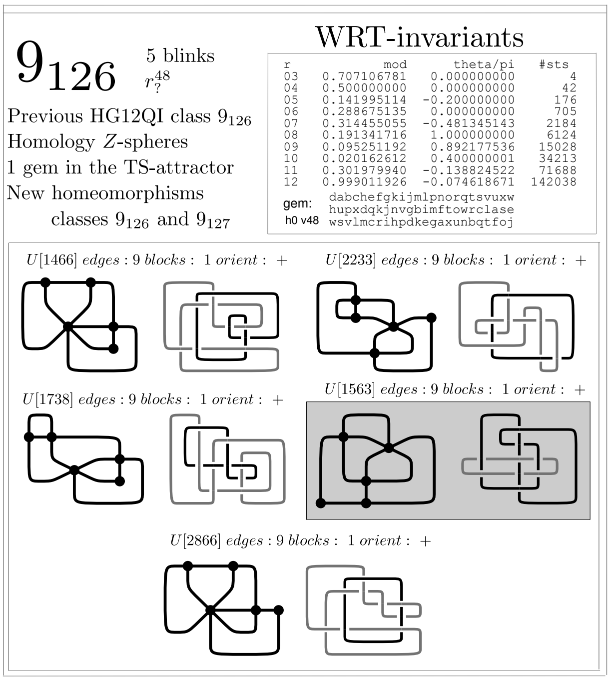


Figure 2: **Note's final challenge:** classify topologically 9_{126} and 9_{199}. Here, to classify has the following strict meaning: for each pair

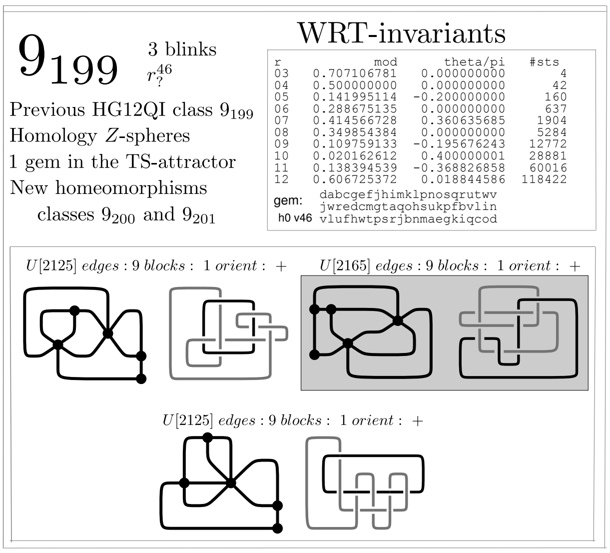
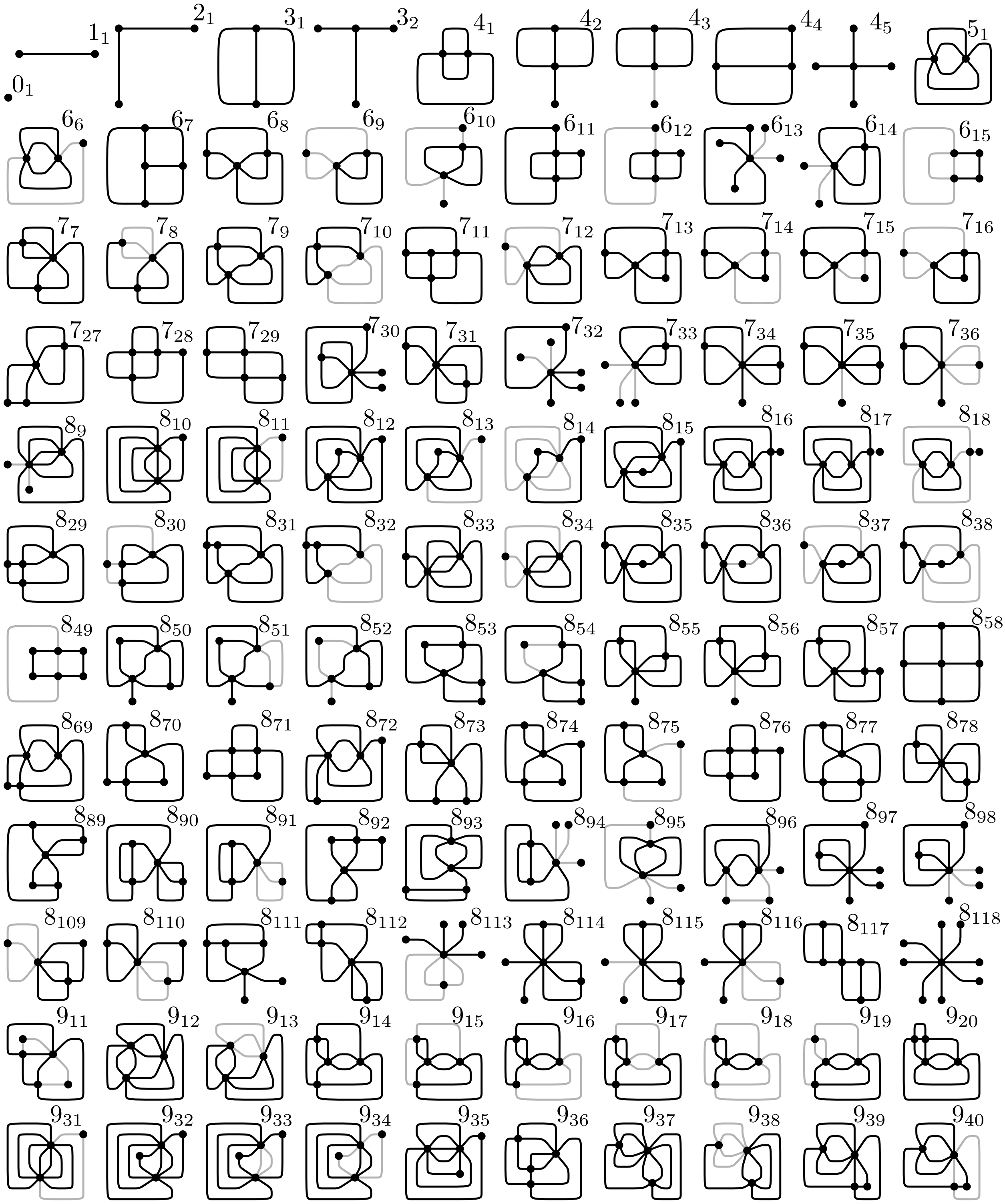


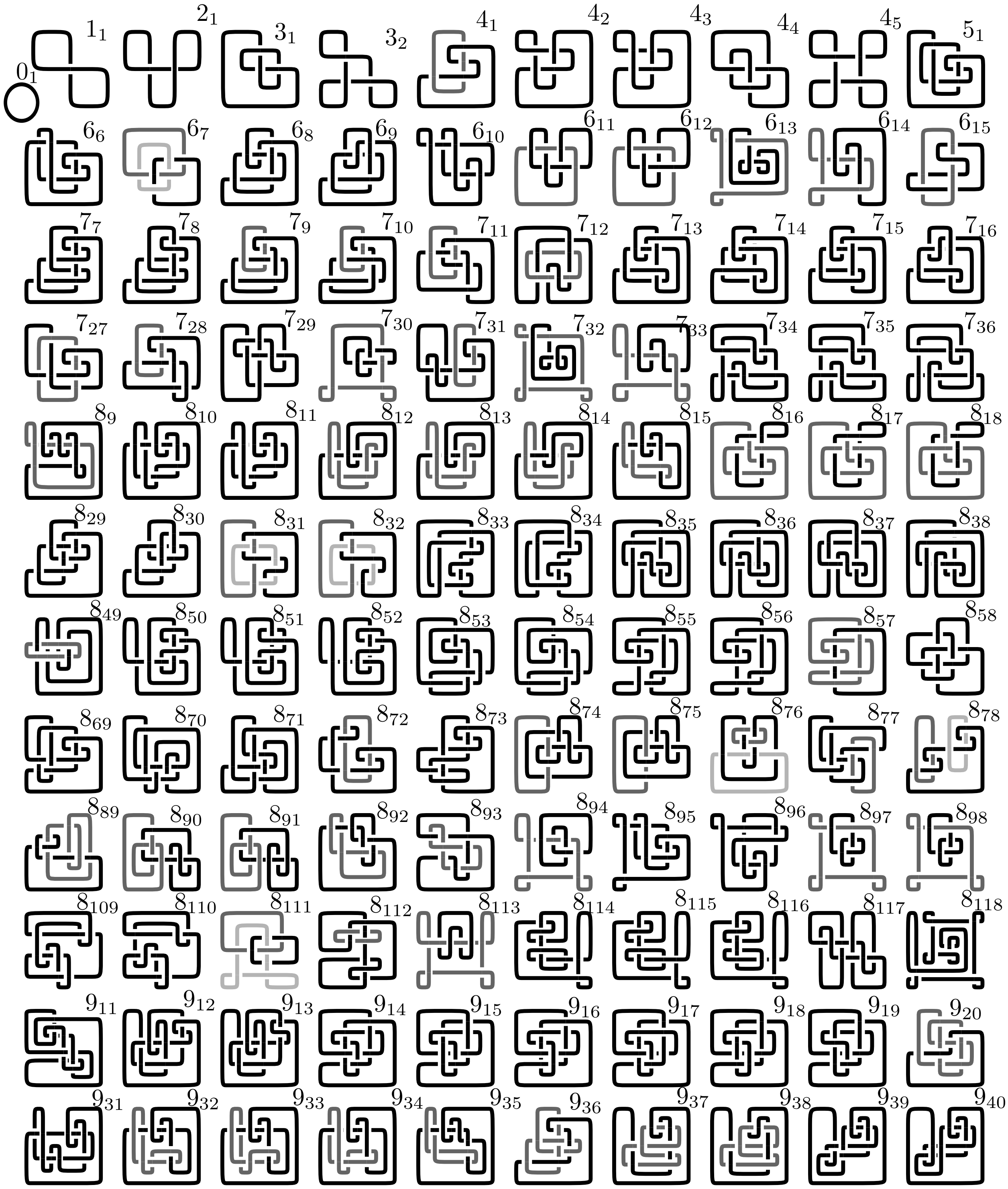
Figure 3: **Note's final challenge:** classify topologically 9_{126} and 9_{199}. Here, to classify has the following strict meaning: for each pair

3 Appendix: census (no misses, no duplicates) of 9-small 3-manifolds

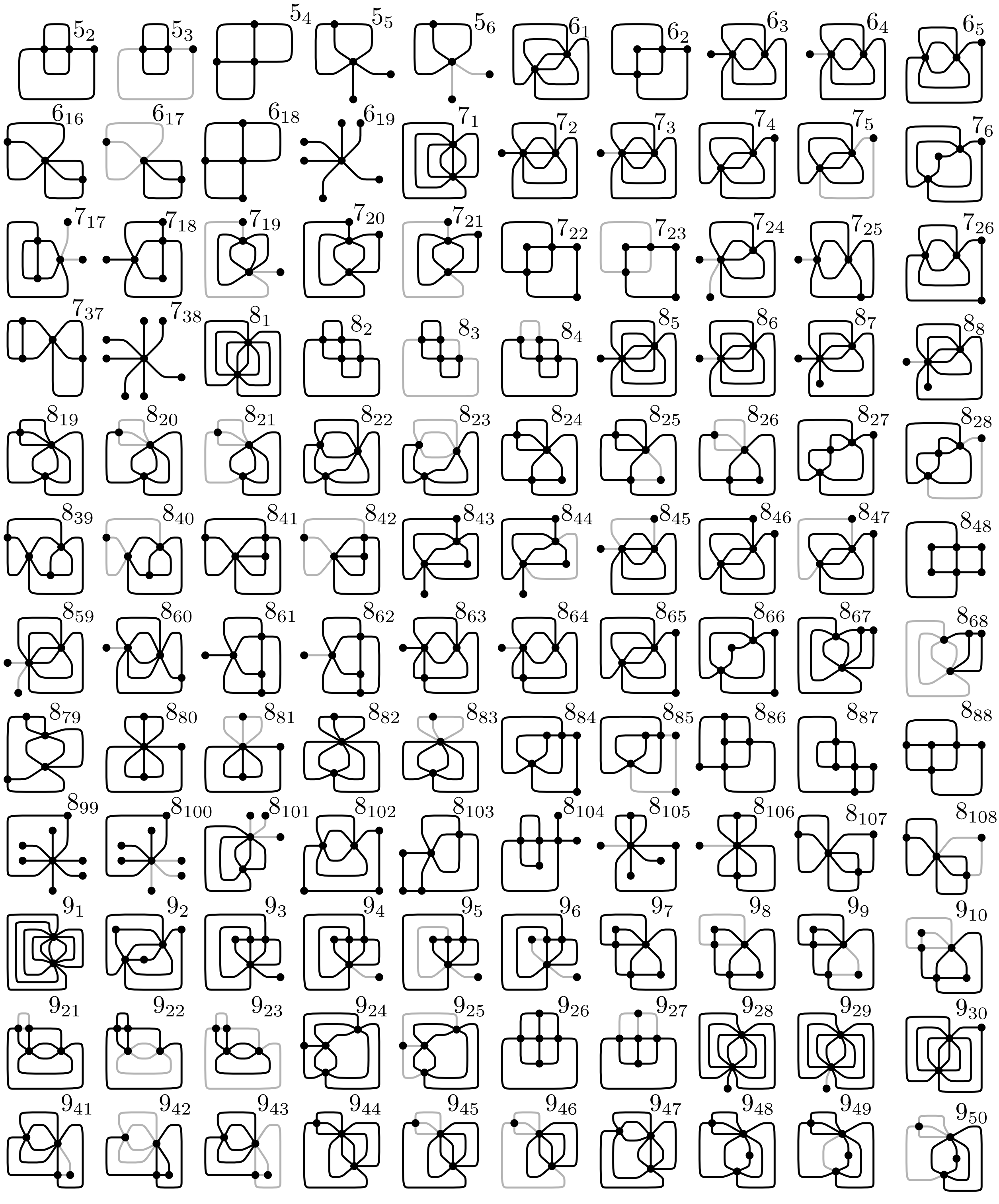
Part 1/4 in terms of blinks:



Part 1/4 in terms of blackboard framed links:



Part 2/4 in terms of blinks:



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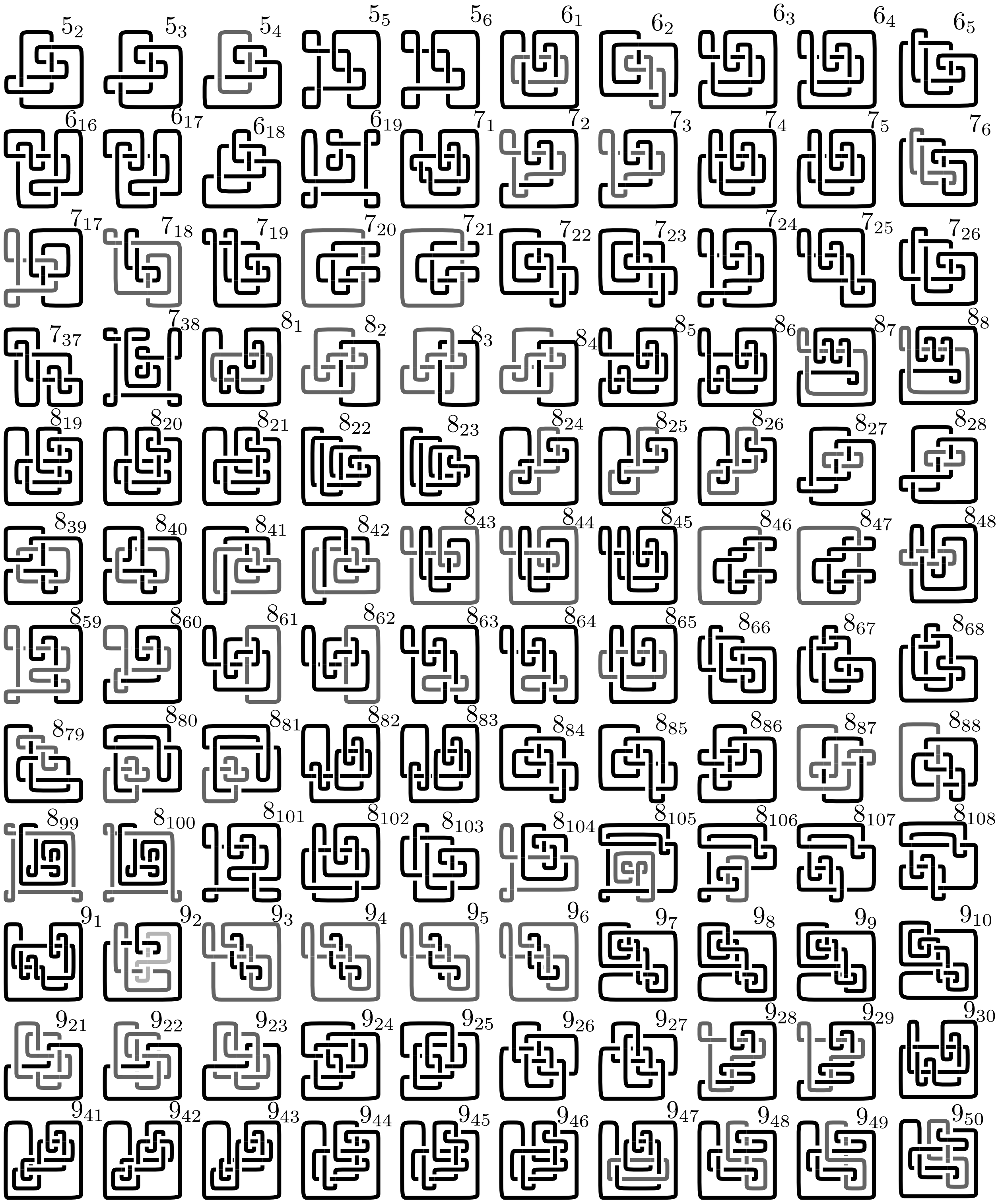
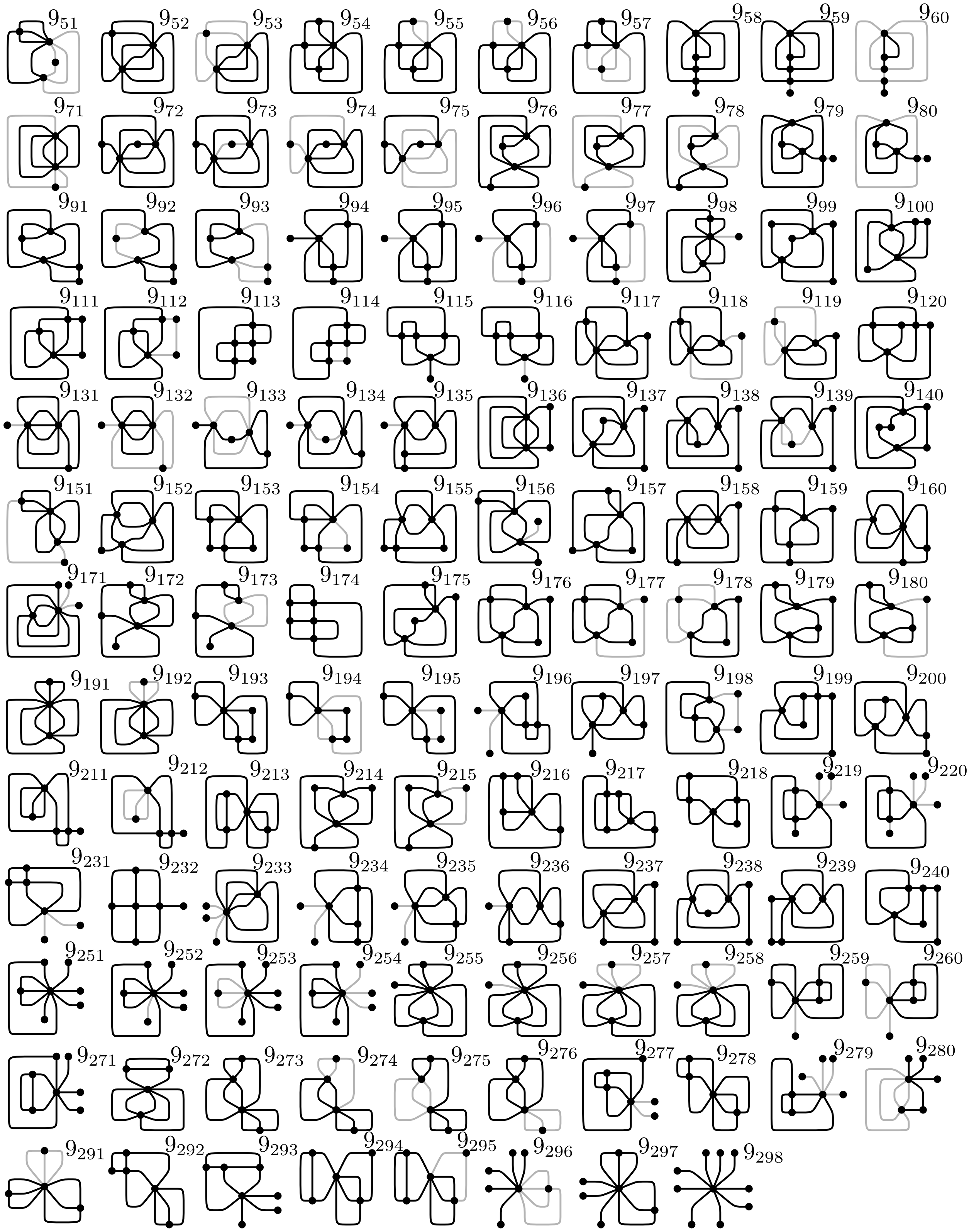


Part 3/4 in terms of blinks:



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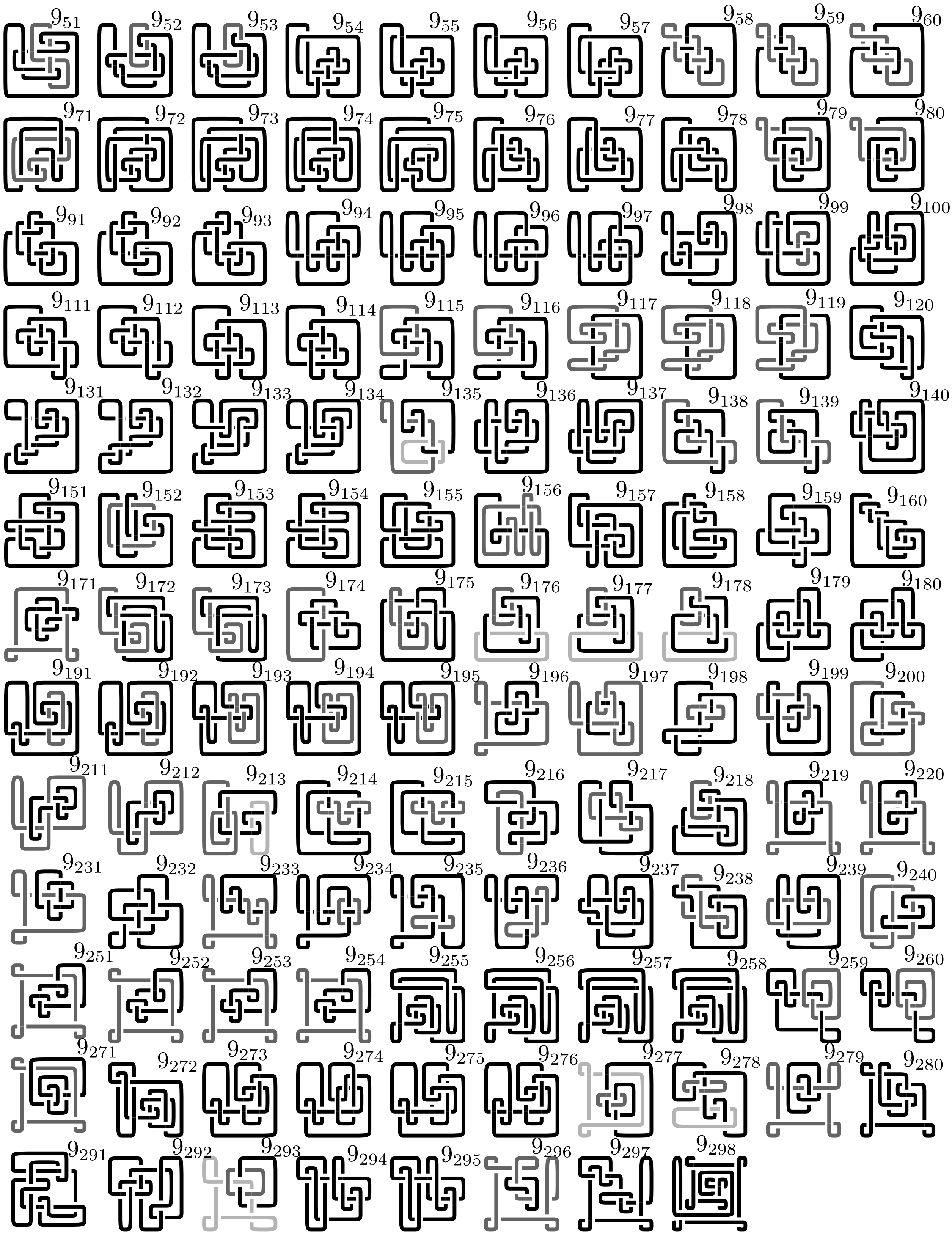
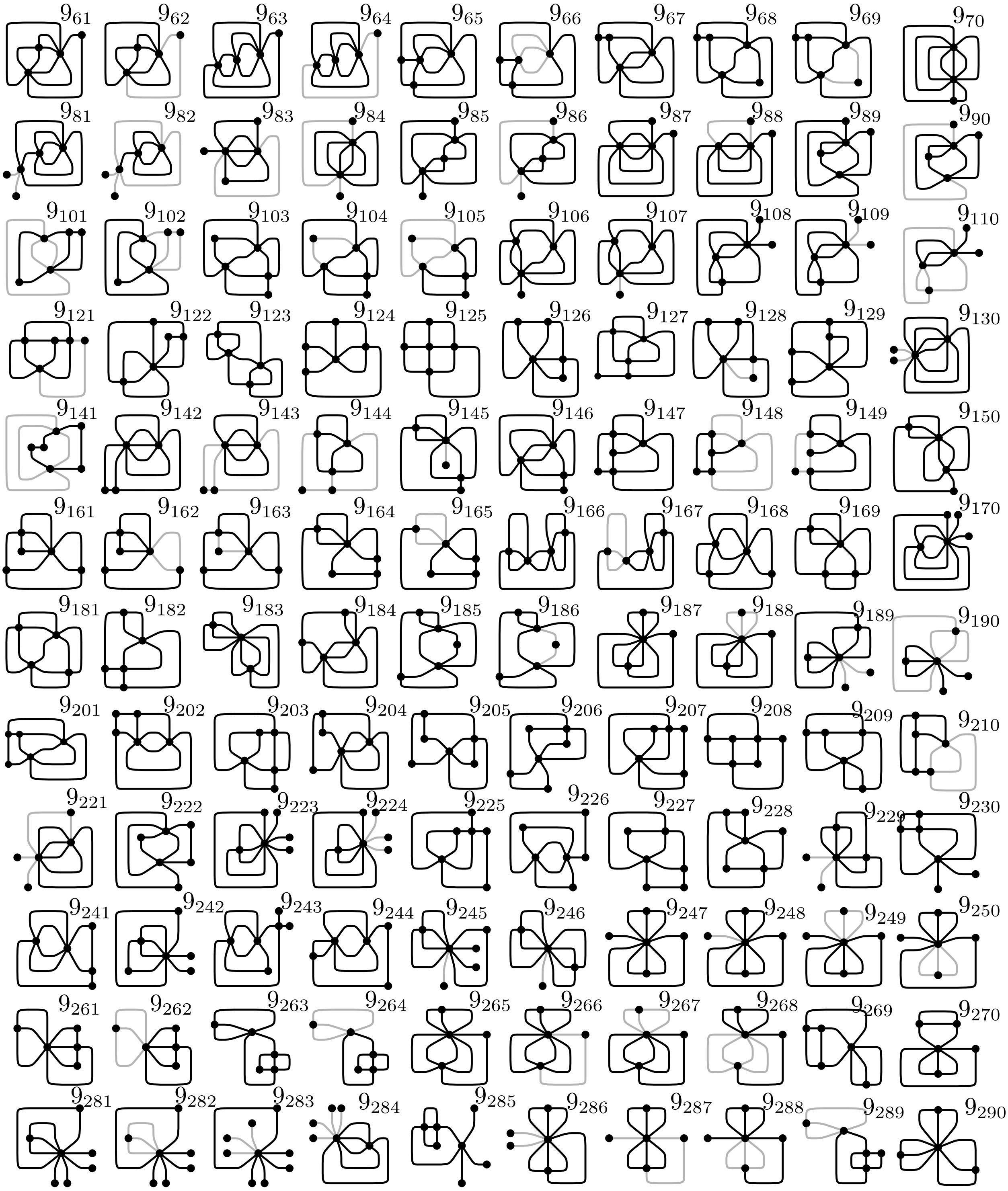
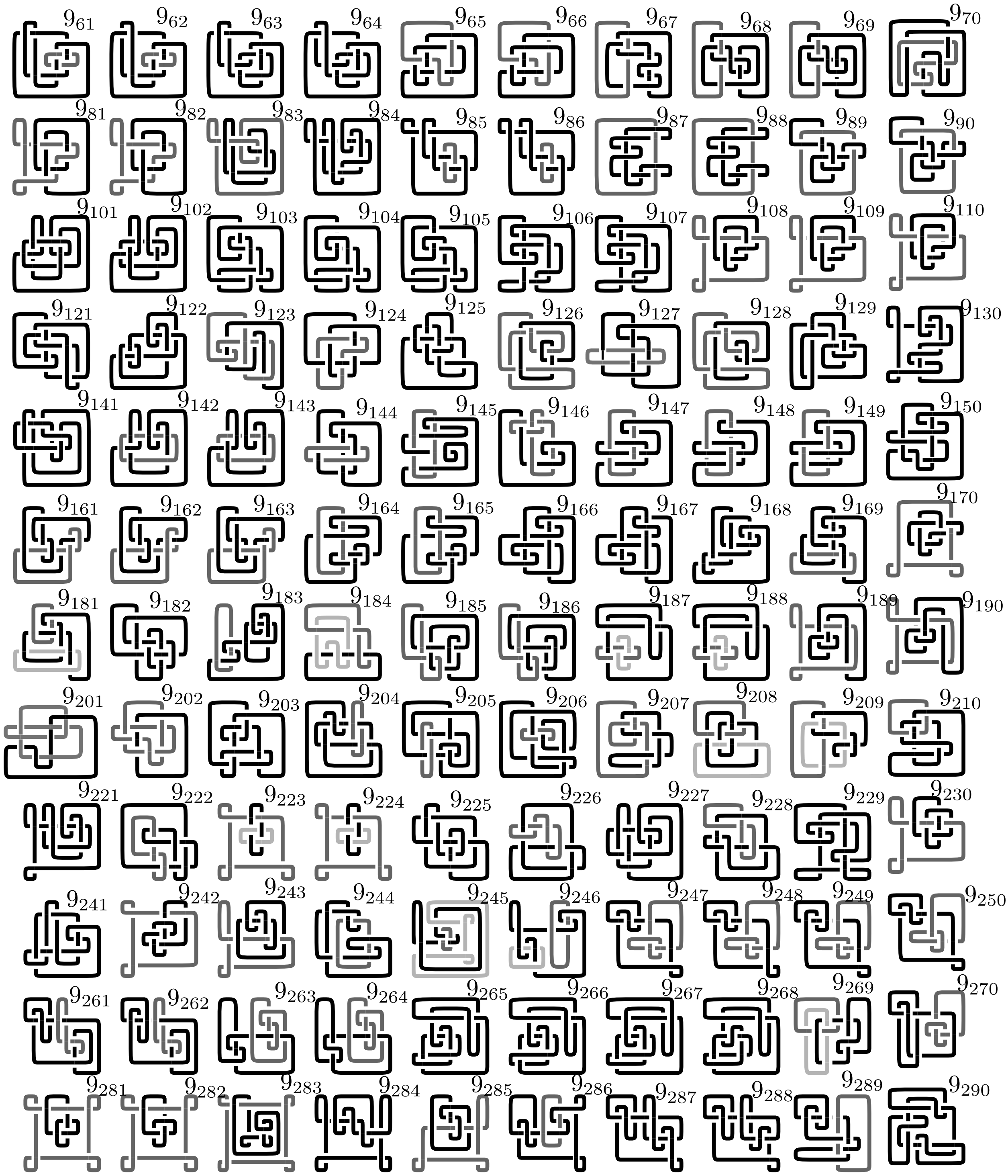


Part 4/4 in terms of blinks:



Part 4/4 in terms of blackboard framed links:



4 Definition of Gem

For completeness we briefly recall the basic definitions of gem theory, leading to its definition, [5]. A *4-graph* G is a finite bipartite 4-regular graph whose edges are partitioned into 4 colors, 0,1,2, and 3, so that at each vertex there is an edge of each color, a proper edge-coloration, [1]. For each $i \in \{0, 1, 2, 3\}$, let E_i denote the set of i-colored edges of G. A $\{j, k\}$-residue in a 4-graph G is a connected component of the subgraph induced by $E_j \cup E_k$. A 2-residue is a $\{j, k\}$-residue, for some distinct colors j and k. A *gem* is a 4-graph G such that for each color i, $G\backslash E_i$ can be embedded in the plane such that the boundary of each face is a 2-residue. From a gem there exists a straightforward algorithm to obtain a closed orientable 3-manifold, in two different, dual ways. Every such a manifold is obtainable in this way. An unecessary big gem is obtained from a triangulation T for a manifold by taking the dual of the barycentric subdivision of T. Here the colors corresponds to the dimensions. Doing simplifications in the gem completely destroys this correspondence.

5 Conclusion

A closed orientable 3-manifold is denoted *n-small* if it is induced by surgery on a blackboard framed link with at most n crossings. Our bet is that both pairs of 3-manifolds in the 2 first sections of this short note are not homeomorphic. This would mean that the 9-small manifolds are completely classified and that the combinatorial dynamics of Chapter 4 in [5] based on TS-moves which leads to a (small, in the case of hyperbolic 3-manifolds) number of minimal gems, named the *attractor of the 3-manifold* is successful. This induces an efficient algorithm which is capable of classifying topologically all the 3-manifolds given as a blackboard framed link with up to (so far) 9 crossings and maintains live the two Conjectures of page 15 of [5]: the TS- and u^n-moves yield an efficient algorithm to classify n-small 3-manifolds by explicitly displaying homeomorphisms, whenever they exist.

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