Closed oriented 3-manifolds are equivalence classes of plane graphs *

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Abstract

A *blink* is a plane graph with an arbitrary bipartition of its edges. As a consequence of a recent result of Martelli, we show that the homeomorphisms classes of closed oriented 3-manifolds are in 1-1 correspondence with classes of blinks. Two blinks are equivalent if they are linked by a finite sequence of local moves, where each one appears in a concrete list of 64 moves: they organize in 8 types, each beeing essentlially the same move on 8 simply related configurations. The size of the list can be substantially decreased at the cost of loosing symmetry, just by keeping a very simple move, the *ribbon move* denoted μ_{11} (which is in principle redundant). Using μ_{11} makes all the moves coming from plane duality (the starred moves), except for $\mu_{20}^{\star}$, redundant.

1 Statement of the Theorem

This paper proves the following theorem:

(1.1) Theorem. *The classes homeomorphisms of closed oriented 3-maniflds are in 1-1 correspondence with the equivalence classes of blinks where two blinks are equivalent if they are linked by a finite sequence of the local moves where each term is one of the 64 moves (not necessary distinct) below*

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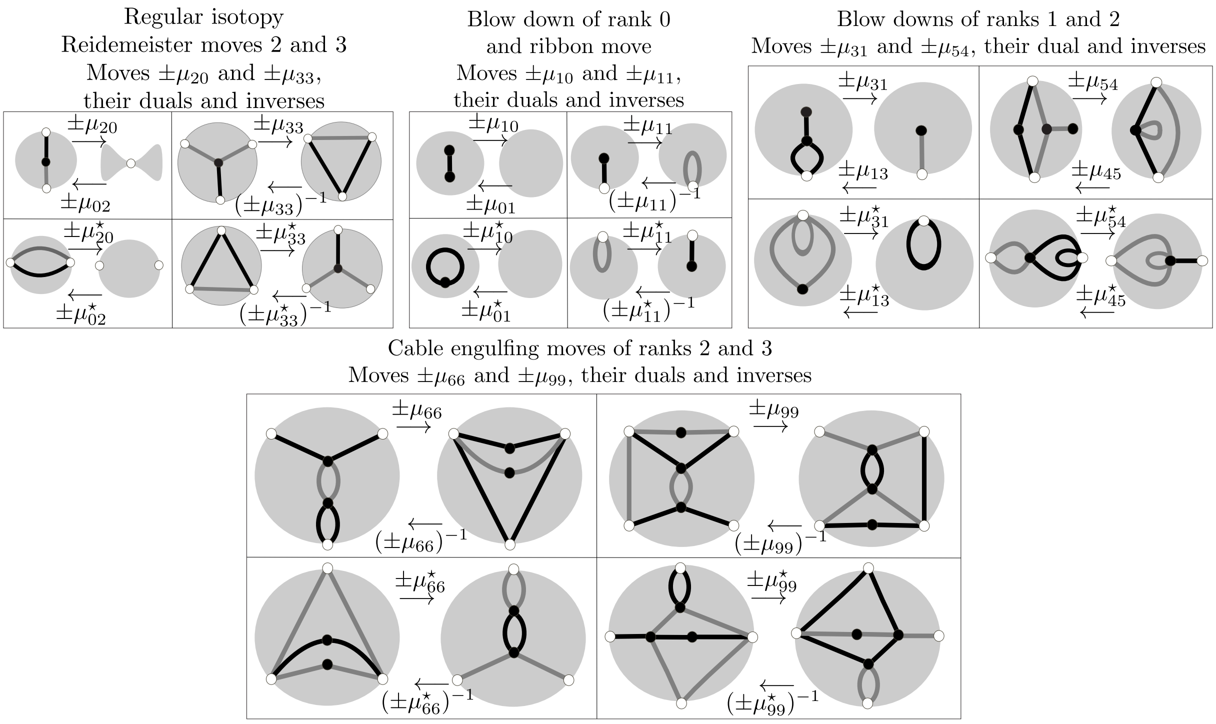

There are 64 local configurations divided into 32 pairs of left-right configurations, each one named a *coin*. A move replaces the left (right) coin of a pair by its right (left) coin. The complementary sub-blink in the exterior a coin is completely arbitrary; its intersection with the corresponding internal coin is the set of attachment vertices shown in the boundary of the disk (or pinched disk, in the case of ρ_2) where the coin lies as small white circles. The number k of such attachment vertices satisfies $k \in \{0, 1, 2, 3, 4\}$. Vertices internal to the coin are shown as small black disks.

In his Appendix to part 0, J.H.Conway in his famous book *On Numbers and Games*, [**?**], says: *"This appendix is in fact a cry for a Mathematician Liberation Movement!*

Among the permissible kinds of constructions we should have:
(i) Objects may be created from earlier objects in any reasonable constructive fashion.
(ii) Equality among the created objects can be any desirable equivalence relation."

This paper is in confluency of two deep research passions of the author, apparently very far apart: the topological study of closed orientable 3-manifolds and the combinatorial study of plane graphs. The result proved here provides a hint, in the spirit of Conway's quotation, to effectively enumerate once each closed orientable 3-manifolds. Blinks are easy to constuct, their isomorphism problem can be solved by a polynomial algorithm (by finding by means of a few fixed conventions (lexicography), a numerical *code* for it). Our *canonical form of a closed oriented 3-manifold* is defined to be the blink inducing it which has a minimum number of edges and minimum code.

2 A brief historical review: a result by Lickorish and consequences by Kirby, Fenn-Rourke, Kauffman and Martelli

A *knot* is an embedding of a circle, an $\mathbb{S}^1$, into $\mathbb{R}^3$ or $\mathbb{S}^3$. The *unknot* is a knot which is the boundary of a disk. A *link with k components* is an embedding of a disjoint union of k copies of $\mathbb{S}^1$ into $\mathbb{R}^3$ or $\mathbb{S}^3$. In this way, a knot is a link with one component.

Knots and links can be presented by their *decorated general position projections* into a fixed plane $\mathbb{R}^2$. *General position* means that in the image of the link there is no triple points and that at each neighborhood of each double is the transversal crossing of two segments of the link, named *strands*. *Decorated* means that we keep the information of which strand is the upper one, usually by removing a piece of the lower strand. In this paper we use another way to decorate the link projections: the images of the link components are thich black curves and the upper strands are indicated by a thinner white segment inside the thich black curve at the crossing.

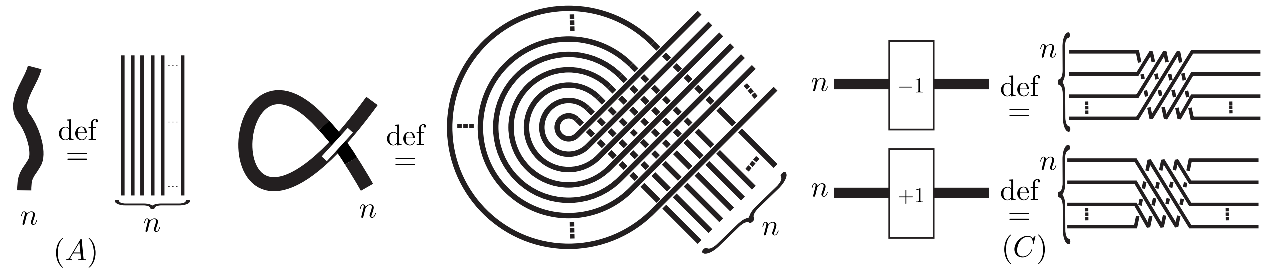

Figure 1: Notation for special disk neighborhoods of general position decorated link projections

In a grounding breaking work (1962), W.B.R. Lickorish, [?], proved that each closed orientable 3-manifold $\mathbb{M}^3$ can be encoded by a link in $\mathbb{S}^3$ where each one of its k components is endowed with an irreducible fraction (the framing) $\frac{\pm p}{q}$ where q could be 0, and in the case p must be 1 and the fraction becomes $\pm\infty$. To construct $\mathbb{M}^3$ from the framed link we act as follows: after removing from $\mathbb{M}^3$ an ϵ-neighborhood $(\mathbb{S}^1 \times \mathbb{D}^2)_i$ of the i-th link we are left with $\mathbb{M}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i$. The fraction specifies, in the toroidal boundary inside $\mathbb{S}^3$, the homology type $(\pm p, q)$ of the curve that is contractible in the solid torus inside $\mathbb{M}^3$. For each component, we then identify the simple curve given by the homological base pair with the meridian of a canonical copy of a solid torus in $\mathbb{R}^3$ so as to completely specify the pasting of the solid torus closing the toroidal hole. Lickorish's breakthrough was to prove that any $\mathbb{M}^3$ has inside it a finite number k of disjoint solid tori so that $\mathbb{M}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i$. Actually, this result had been proved 2 years before by A. H. Wallace [?] by using differential geometry. However it was the purely topological flavor of Lickorish's proof that spurs the subsequent developments.

In 1978 R. Kirby published his, to become famous, calculus of framed links, [?]. The gist of this paper is that two types of moves are enough to go from any framed link inducing a closed oriented 3-manifold to any other such link inducing the same manifold. One of the moves is absolutelly local: creating or cancellating an arbitrary new component with frame in

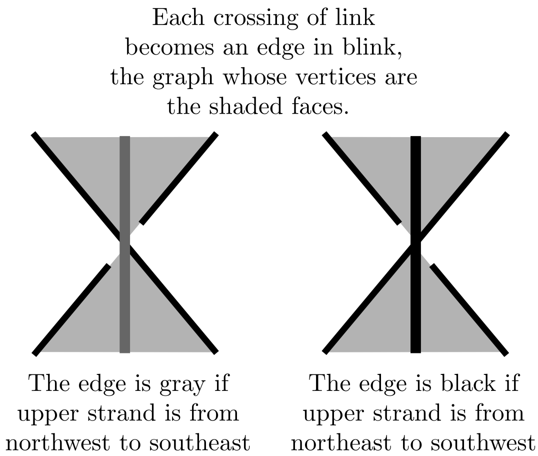

Figure 2: The projection of any link can be 2-face colorable into white and gray with the infinite face white so that each subcurve between two crossing have their incident faces receiving distinct colors. The above figure shows how to transform the projection into a blink (with thicker edges than the curves representing the link projection). The vertices of the blink are distinguished fixed points in the interior of the gray faces. The inverse procedure is clearly defined. Thus we have a 1-1 correspondence between general position decorated link projections and blinks.

$\{\infty, -\infty\}$. The other type of move, *the band move ([?]) (or handle sliding)* is non-local and infinite in number.

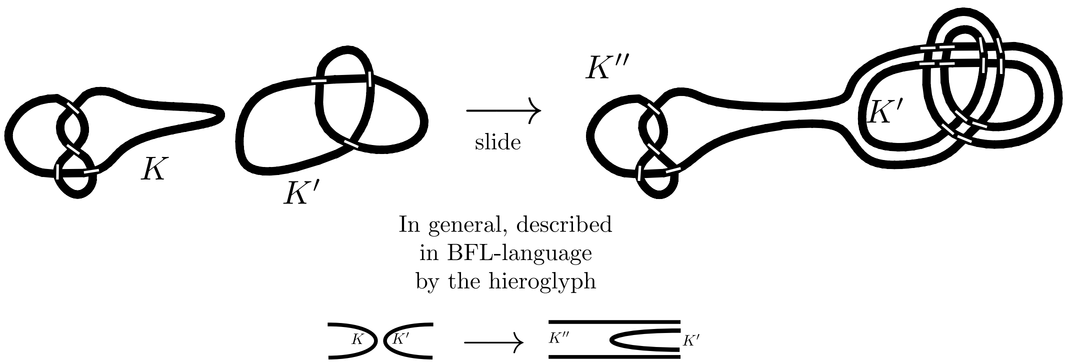

Figure 3: Kirby's handle slide move, also known as the band move. Given K and K' two distinct components of a link start by making a close parallel copy of K' in such a way that it has linking number 0 with its originator K'. Let K'' be the connected sum of K and the copy of K'. The connected sum is defined by a *band* which is a thin rectangle arbitrarily embedded into $\mathbb{S}^3$, so as to miss the link. The short sides of the band are attached to K and to the copy of K' and are recoupled in the other way. Framings are modified in a judicious way. The situation loses no generality and becomes particularly simple if we have a blackboard framed link (shortly to be defined). More details in Kauffman's book, [?]. In blackboard framed language (BFL), this move is conveniently depicted via the hieroglyph shown above. See Section 12.3 of [?] .

In 1979 R. Fenn and C. Rourke ([?]) show that Kirby's moves could be replaced by an infinite sequence of a single type of move (a *blow down move*) indexed by n, which we depict at the left side of Fig. 4. In a blow down move the number of components decreases by 1. This has been a very useful reformulation with many applications, including Martelli's calculus (soon to be treated) which uses it instead of the direct moves of Kirby.

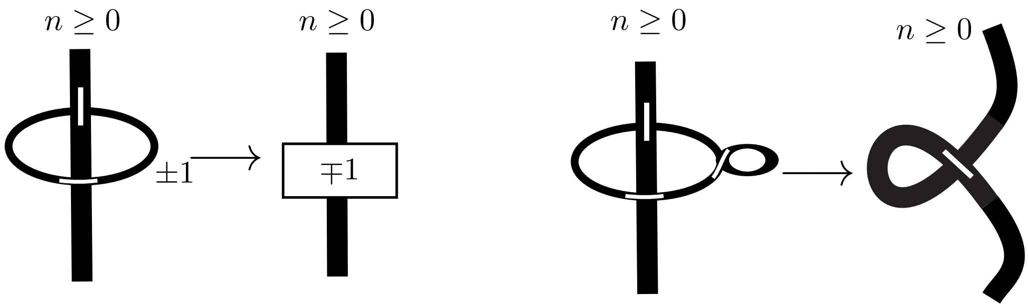

Figure 4: Fenn-Rourke infinite sequence of blown-down moves and their counterpart in Kauffman's blackboard framed links. These are infinite sequence of local moves. Cases $n = 0$ of these moves replace an isolated ± 1-framed component or the unknot with one crossing by nothing. Martelli replaced the infinite sequence by the first three and two new simple moves A_3 and A_4, described in Fig. 5.

In the beginning of the 1990's L. Kauffman presented ([?]) a completely planar diagramatic way to deal with the calculus of Kirby and its reformulation by Fenn and Rourke. The basic idea comes from the fact that every 3-manifold is induced by surgery on a framed link which has *only finite integer framings*. This characterize the *handle surgeries*. According to Rolfsen Lickorish call each of these a *honest surgery*, page 262 of [?]. The proof we can get any manifold by surgery on integer framed links uses, as a lemma, the fact that it is possible to modify the framed link maintaining the induced 3-manifold so that every component becomes unknotted. A proof of this lemma appears in Rolfsen's book. It also apears in page 137 of Kauffman-Lins monography, [?]. If a component is unknotted then it is simple to modify the link so that each component gets an integer framing, without distub the integrality of the framing of other components. So, without loss of generality we may supose that all the components have finite integers as framings. Kauffman's proceeds by adjusting each component by attaching to it a judicious number of curls so that the required framing of a component coincides with the algebraic sum of its self-crossings. By specifying that the link is *blackboard framed*, we no longer need the framings. In this work we only use links given by blackboard framed projections: they are rather close of blinks, which are purely graph theoretical plane graphs with an edge bipartition having no framing, see Fig. 2. The framings are internally reconstructible.

In an important recent paper B. Martelli [?] presented a local finite reformulation of the Fenn-Rourke version ([?]) of Kirby's calculus [?]. This calculus is presented in Fig. 5. It remains to be seen the consequences of Martelli' s result for obtaining new 3-manifold invariants.

3 Objective of the work

Our objective here is to further reformulate Martelli's moves so as to obtain a calculus of blinks, denominated *blink-coin calculus*, which is an exact combinatorial counterpart for factorizing homeomorphisms of closed orientable 3-manifolds. This goal is desirable because it has the

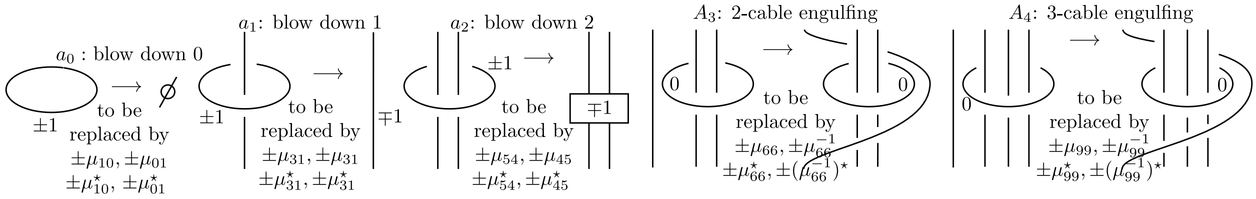

Figure 5: Martelli's calculus on fractional (or integer) framed links. He proves that by keeping only the blown down of ranks 0, 1 and 2 and replacing all the remaining infinite sequence by two new moves 2- and 3-cable engulfing (denoted by A_3 and A_4) a sufficient calculus for factorizing homeomorphisms between closed 3-dimensional 3-manifolds is achieved. Moves A_3 and A_4 do not translate into blink moves because their left sides are disconnected. What makes this work possible is the replacement of these non-connected configurations by equivalent moves a_3 and a_4 so that blink translations become available. See Fig. 6 where the diagrams for the moves appear -90^o-rotated.

consequence that each 3-manifold become a subtle class of plane graphs. Our exposition is complete and elementary seeking to reach both audiences: topologists and combinatorialists. We feel that this result may be interesting for Combinatorics as well as to Topology and may enhance both areas. Plane graphs are one of the most studied objects in Combinatorics. Some of their properties could be used in 3-manifolds and vice-versa. As an example of an interesting question relating the two areas is this: which 3-manifolds correspond to the class of 3-connected monochromatic blinks? We have reasons to believe that the blink itself is a complete invariant for these manifolds. A catalogue of all the 242 blinks which are 3-connected, monochromatic and have up to 16 edges appear in [**?**].

4 Proof of the Theorem

(4.1) Lemma. *In the presence of* $r_2^{\pm 1}$*, moves* $\pm a_3$ *and* $\pm A_3$ *are equivalent and so are* $\pm a_4$ *and* $\pm A_4$*.*

Proof. We refer to Fig. 6. Its first line proves that $\pm a_3 \Rightarrow \pm A_3$. The second line proves that $\pm A_3 \Rightarrow \pm a_3$. The third line proves that $\pm a_4 \Rightarrow \pm A_a$. The last line proves that $\pm A_4 \Rightarrow \pm a_4$. □

Proof. (**of Theorem 1.1**)
In Fig. 7 we draw all the moves for the revised Martelli's moves in a blackboard framing language and the respective blinks superimposed. The result follows by removing the superimposed moves leaving only the blink moves which are redraw up to isotopy in the lower part of the figure. □

5 The role of ribbon moves $\pm\mu_{11}^{\pm 1}$

Moves $\pm\mu_{11}^{\pm 1}$ are redundant because Martelli's calculus is in $\mathbb{R}^3$. We include them in our blackboard framed link calculus because with their inclusions all the dual moves, except $\mu_{20}^{\star}$ and $\mu_{02}^{\star}$, become redundant.

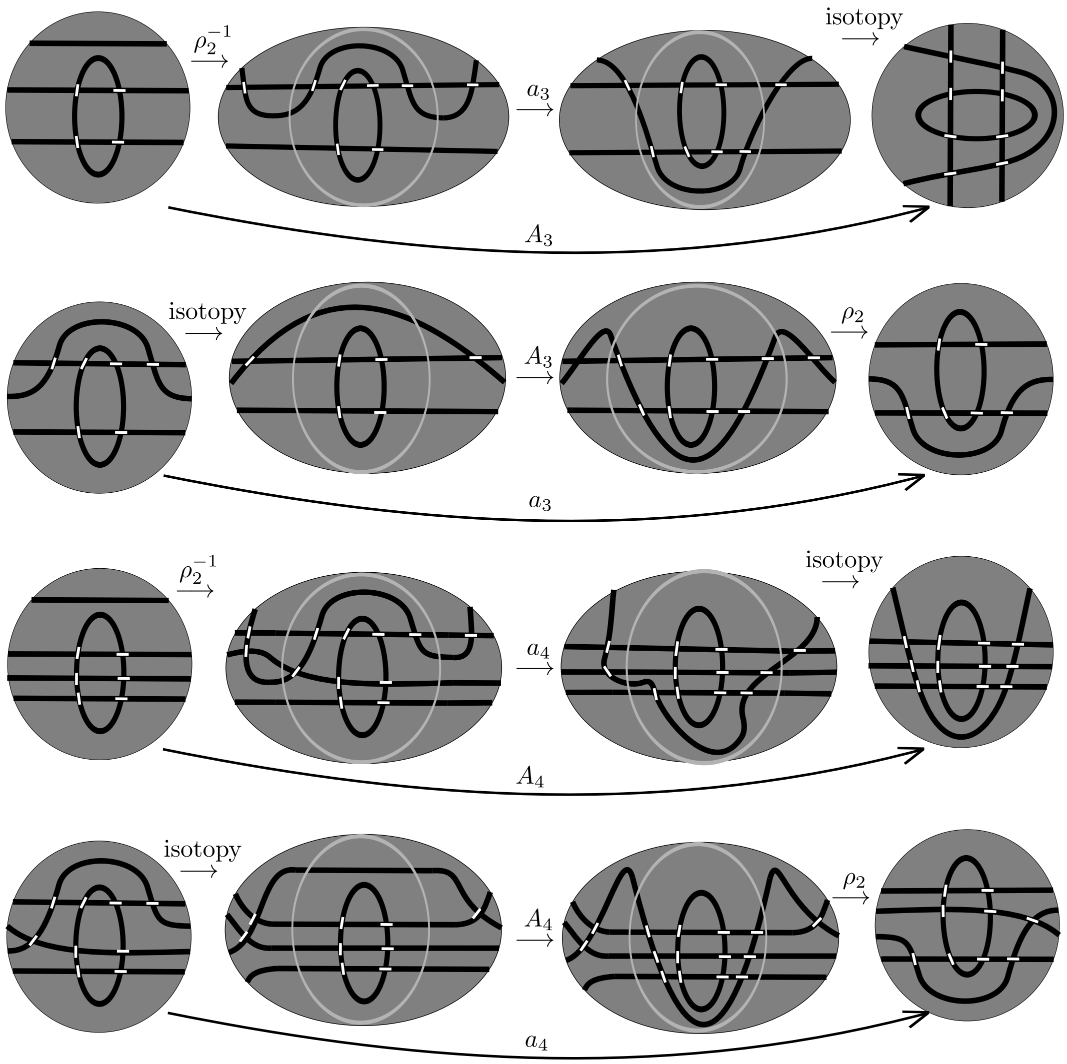

Figure 6: A proof that in the presence of $\pm r_2^{\pm 1}$, $a_3 \equiv A_3$ and $a_4 \equiv A_4$

(5.1) Theorem. *The blink-coin calculus can be simplified to include only the following set of 36 moves* $\{\pm\mu_{20}, \pm\mu_{02}, \pm\mu_{20}^{\star}, \pm\mu_{02}^{\star}, \pm\mu_{33}^{\pm 1}, \pm\mu_{01}, \pm\mu_{01}, \pm\mu_{11}^{\pm -1}, \pm\mu_{31}, \pm\mu_{13}, \pm\mu_{54}, \pm\mu_{45}, \pm\mu_{66}^{\pm}, \pm\mu_{99}^{\pm}\}$.

Proof. We work not a the blink level, but at the level of its associated blackboard framed link. Moves corresponding to $(\pm\mu_{11}^{\pm 1})^{\star}$ and $(\pm\mu_{33}^{\pm 1})^{\star}$ are redundant because $\pm\mu_{11}^{\pm 1}$ $\pm\mu_{33}^{\pm 1}$ are self dual. Moves $\pm\mu_{10}^{\star}$ and $\pm\mu_{01}^{\star}$ are implied by a combination of moves $\pm\mu_{10}$, $\pm\mu_{01}$, $\pm\mu_{11}$ and $(\pm\mu_{11})^{-1}$. In particular, all the Reidemeister 2 and 3 moves (regular isotopy) are at our disposal. We can use this fact to change the external face of the link diagram to become any chosen adjacent face by using regular isotopy at the cost of creating two curls with distinct sign and the same rotation number. Use the the appropriate ribbon move to obtain two curls with the distinct signs and distinct rotation number. Now apply Whitney's trick (see page 30 of [?]) to cancel these curls. The net effect in the correponding blink is that it is obtained from the initial blink by dualizing and interchanging black and gray edges. Having this double involutions at our disposal it is straighforward to obtain all the remaining dual moves. $\square$

In Fig. 8 we present the 36 moves of the final reduced coin-blink calculus.

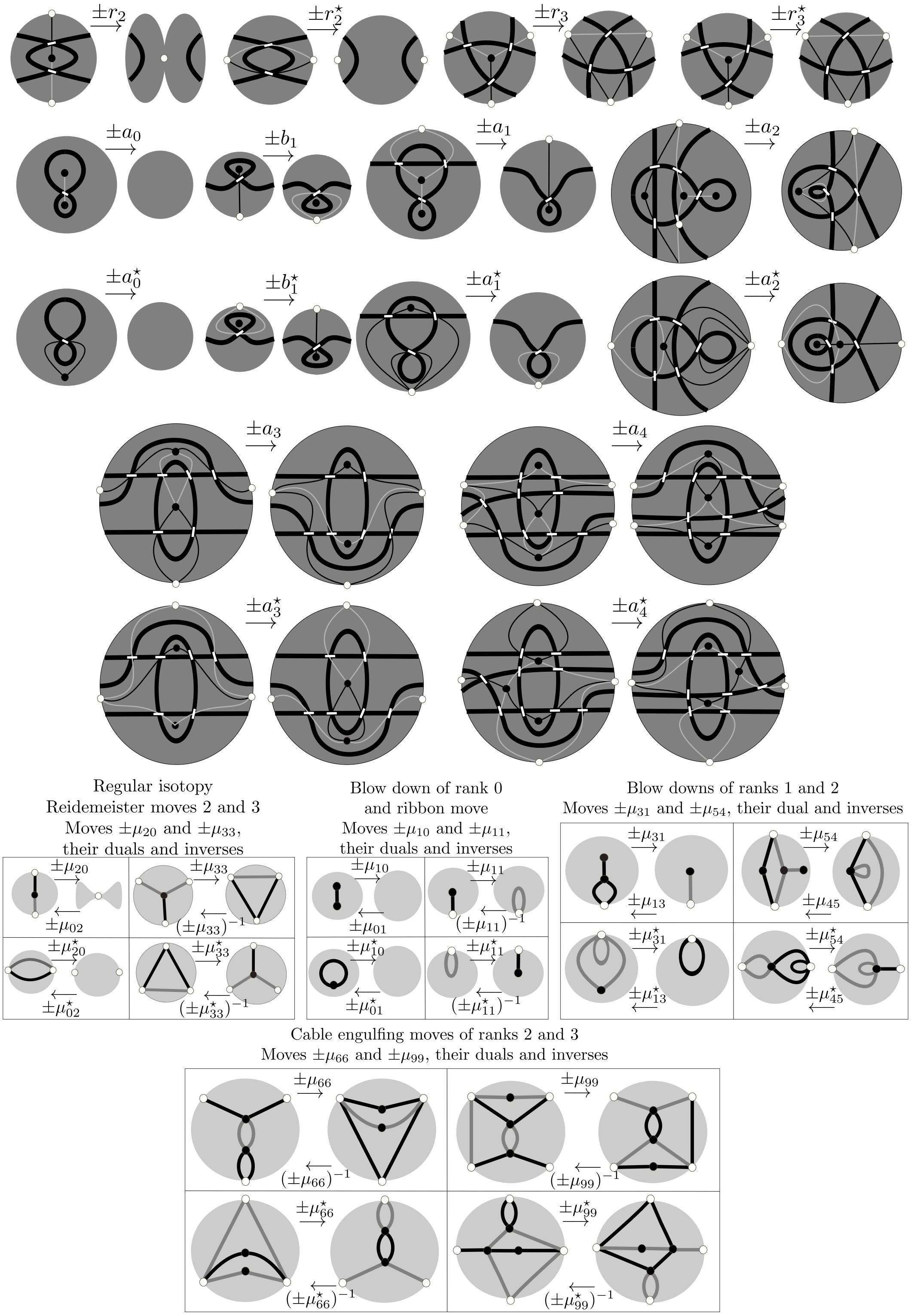

Figure 7: Blackboard framed version of Martelli's revised calculus with a_3, a_4 replacing A_3, A_4. In the upper part of the figure, links & blinks are superimposed implying moves for the blink-coin calculus in the lower part of the figure, concluding the proof of the Theorem 1.1

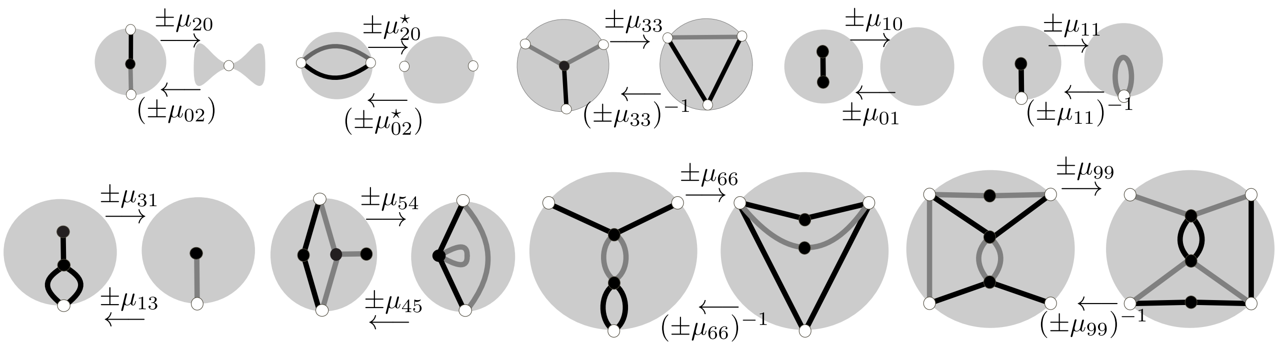

Figure 8: Reduced 36 move blink-coin calculus

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