Closed, oriented, connected 3-manifolds are equivalence classes of plane graphs *

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Abstract

A *blink* is a plane graph with an arbitrary bipartition of its edges. As a consequence of a recent result of Martelli, I show that the homeomorphisms classes of closed oriented 3-manifolds are in 1-1 correspondence with specific classes of blinks. In these classes, two blinks are equivalent if they are linked by a finite sequence of local moves, where each one appears in a concrete list of 64 moves: they organize in 8 types, each beeing essentlially the same move on 8 simply related configurations. The size of the list can be substantially decreased at the cost of loosing symmetry, just by keeping a very simple move type, the *ribbon moves* denoted $\pm\mu_{11}^{\pm}$ (which are in principle redundant). The inclusion of $\pm\mu_{11}^{\pm}$ implies that all the moves corresponding to plane duality (the starred moves), except for $\mu_{20}^{\star}$ and $\mu_{02}^{\star}$, are redundant and the coin calculus is reduced to 36 moves on 36 coins. It is in the aegis of this work to find new important connections between 3-manifolds and plane graphs.

1 Statement of the Theorem

This paper proves the following theorem:

(1.1) Theorem. *The classes homeomorphisms of closed oriented connected 3-manifolds are in 1-1 correspondence with the equivalence classes of blinks where two blinks are equivalent if they are linked by a finite sequence of the local moves where each term is one of the 64 moves (not necessary distinct) below*

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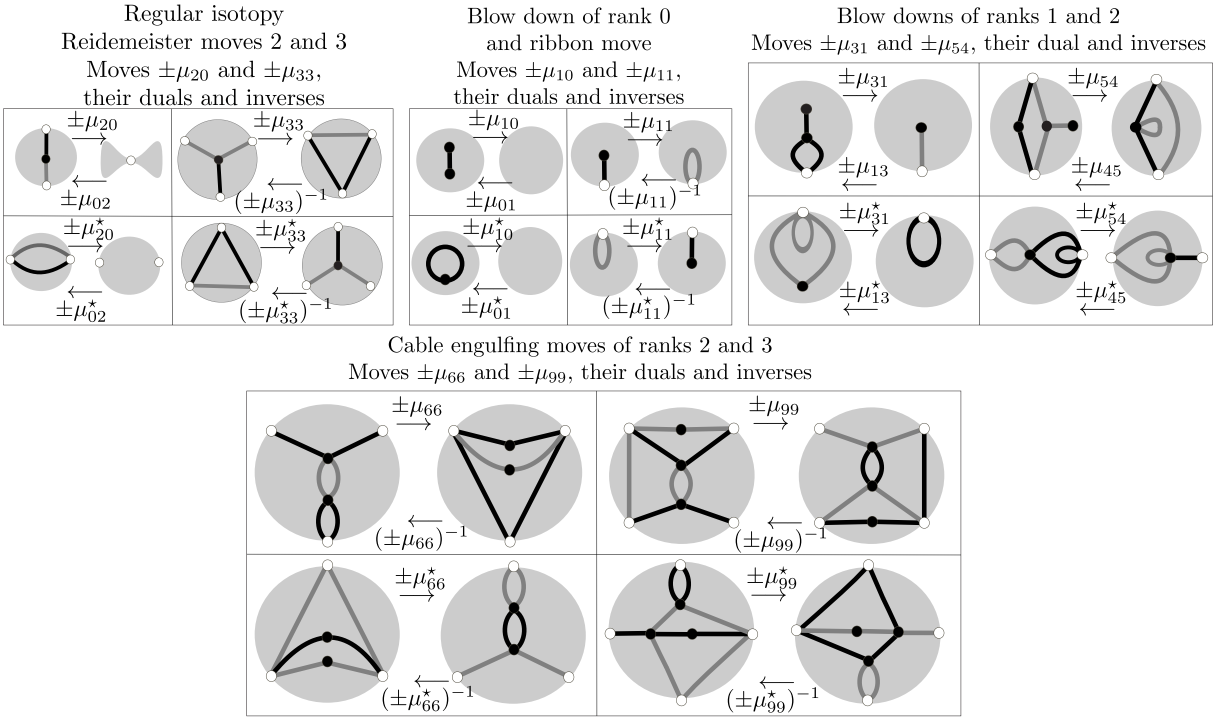

There are 64 local configurations of sub-blinks, each named a *coin*, divided into 8 families of 8 simply related coins and also divided into 32 pairs of left-right coins, A move replaces the left (right) coin of a pair by its right (left) coin. Thus, the number of moves is equal to the number of coins. Vertices internal to the coins are shown as small black disks. The complementary sub-blink in the exterior a coin is completely arbitrary; its intersection with the corresponding internal coin is a subset of the set of attachment vertices shown in the boundary of the coin as small white circles. The support of the coins are disks, except in the the right coin of μ_{20}, case in which is a pinched disk. The number k of the attachment vertices satisfies $k \in \{0, 1, 2, 3, 4\}$. A reduced but sufficient form of this coin calculus having 36 coins (and moves) is obtained in the final section.

2 Motivation

In his Appendix to part 0, J. H. Conway in his famous book *On Numbers and Games*, [2], says: *"This appendix is in fact a cry for a Mathematician Liberation Movement!*

Among the permissible kinds of constructions we should have:
(i) Objects may be created from earlier objects in any reasonable constructive fashion.
(ii) Equality among the created objects can be any desirable equivalence relation."

This paper is in the confluency of two deep research passions of the author, apparently very far apart: the topological study of closed orientable 3-manifolds and the combinatorial study of plane graphs. The result proved here provides a glimpse, in the spirit of Conway's quotation, to effectively enumerate once each closed orientable 3-manifolds. Blinks are easy to construct from simpler blinks, and their isomorphism problem can be solved by a polynomial algorithm which finds, via a few fixed conventions (lexicography), a numerical *code* for it. What can be more desirable than an equivalence relation on such simple mathematical objects that captures the subtle and difficult computational topological notion of factorizing any homeomorphism between two closed, oriented, connected 3-manifolds?

3 A brief historical review: a result by Lickorish and consequences by Kirby, Fenn-Rourke, Kauffman and Martelli

A *knot* is an embedding of a circle, an $\mathbb{S}^1$, into $\mathbb{R}^3$ or $\mathbb{S}^3$. The *unknot* is a knot which is the boundary of a disk. A *link with k components* is an embedding of a disjoint union of k copies of $\mathbb{S}^1$ into $\mathbb{R}^3$ or $\mathbb{S}^3$. In this way, a knot is a link with one component.

Knots and links can be presented by their *decorated general position projections* into a fixed plane $\mathbb{R}^2$. *General position* means that in the image of the link there is no triple points and that at each neighborhood of each double is the transversal crossing of two segments of the link, named *strands*. *Decorated* means that we keep the information of which strand is the upper one, usually by removing a piece of the lower strand. In this paper I use another way to decorate the link projections: the images of the link components are thich black curves and the upper strands are indicated by a thinner white segment inside the thich black curve at the crossing.

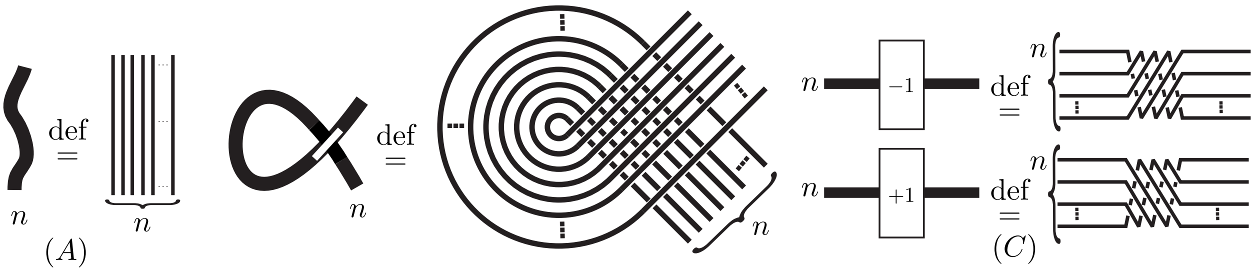

Figure 1: Notation for special disk neighborhoods of general position decorated link projections

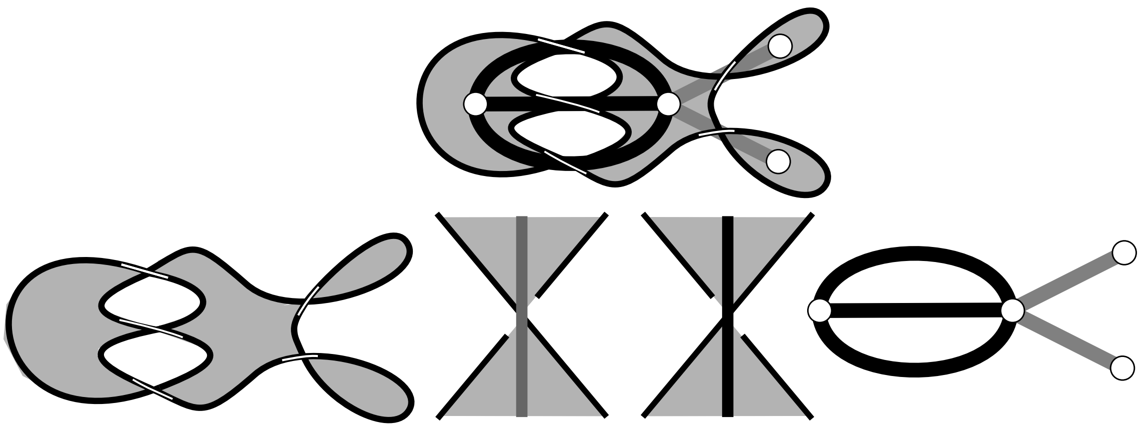

Figure 2: From BFL to blink and back: the projection of any link can be 2-face colorable into white and gray with the infinite face white so that each subcurve between two crossing have their incident faces receiving distinct colors. The above figure shows how to transform a link projection into a blink (with thicker edges than the curves representing the link projection). The vertices of the blink are distinguished fixed points represented by white disks in the interior of the gray faces. Each crossing of the link projection becomes an edge in the corresponding blink, An edge of the blink is gray if the upper strand that crosses it is from northwest to southeast, it is black if the the upper strand that crosses it is from northeast to southwest. The inverse procedure is clearly defined. In fact, the link is the so called *medial map* of the blink. Thus we have a 1-1 correspondence between general position decorated link projections and blinks. The (complete) blink at the right induces Poincaré's sphere, the spherical dodecahedron space. The expressibility of a blink is powerful: quite complicated 3-manifolds are induced by simple blinks.

In a grounding breaking work (1962), W.B.R. Lickorish, [13], proved that each closed orientable 3-manifold $\mathbb{M}^3$ can be encoded by a link in $\mathbb{S}^3$ where each one of its k components is endowed with an irreducible fraction (the framing) $\frac{\pm p}{q}$ where q could be 0, and in the case p must be 1 and the fraction becomes $\pm\infty$. To construct $\mathbb{M}^3$ from the framed link I act as follows: after removing from $\mathbb{M}^3$ an ϵ-neighborhood $(\mathbb{S}^1 \times \mathbb{D}^2)_i$ of the i-th link we are left with $\mathbb{M}^3 \backslash \bigcup_{i=1}^k (\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^k (\mathbb{S}^1 \times \mathbb{D}^2)_i$. The fraction specifies, in the toroidal boundary inside $\mathbb{S}^3$, the homology type $(\pm p, q)$ of the curve that is contractible in the solid torus inside $\mathbb{M}^3$. For each component, we then identify the simple curve given by the homological base pair with the meridian of a canonical copy of a solid torus in $\mathbb{R}^3$ so as to completely specify the pasting of the solid torus closing the toroidal hole. Lickorish's breakthrough was to prove that any $\mathbb{M}^3$ has inside it a finite number k of disjoint solid tori so that $\mathbb{M}^3 \backslash \bigcup_{i=1}^k (\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^k (\mathbb{S}^1 \times \mathbb{D}^2)_i$. Actually, this result had been proved 2 years before by A. H. Wallace [23] by using differential geometry. However it was the purely topological flavor of Lickorish's proof that spurs the subsequent developments.

In 1978 R. Kirby published his, to become famous, calculus of framed links, [11]. The gist of this paper is that two types of moves are enough to go from any framed link inducing a closed oriented 3-manifold to any other such link inducing the same manifold. One of the moves is absolutelly local: creating or cancellating an arbitrary new unknotted component with frame ± 1 separated from the rest of the link by an $\mathbb{S}^2$. The other type of move, *the band move*, [11, 9]) *(or handle sliding)* is non-local and infinite in number.

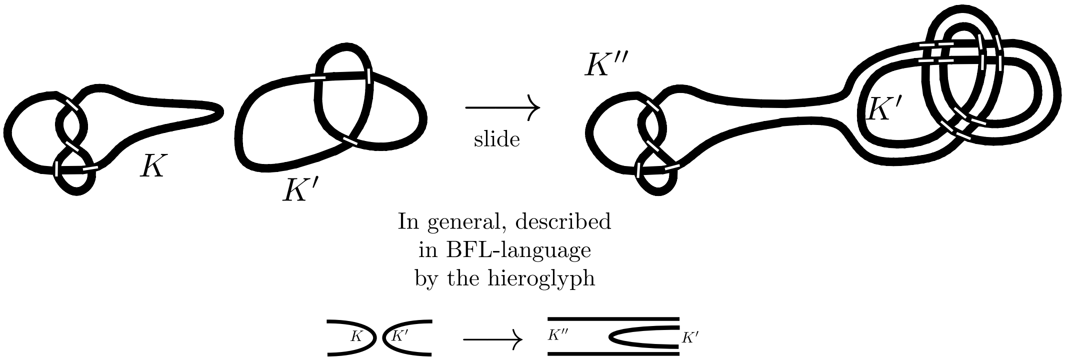

Figure 3: Kirby's handle slide move, also known as the band move. Given K and K' two distinct components of a link start by making a close parallel copy of K' in such a way that it has linking number 0 with its originator K'. Let K'' be the connected sum of K and the copy of K'. The connected sum is defined by a *band* which is a thin rectangle arbitrarily embedded into $\mathbb{S}^3$, so as to miss the link. The short sides of the band are attached to K and to the copy of K' and are recoupled in the other way. The band can be quite complicated because it may wander arbitrarily in $\mathbb{S}^3$ while connecting the two components. Framings are modified in a judicious way. The situation loses no generality and becomes particularly simple (particularly in respect to the new framings) if we have a blackboard framed link (shortly to be defined). More details in Kauffman's book, [9]. In blackboard framed language (BFL), this move is conveniently depicted via the hieroglyph shown above. See Section 12.3 of [10] .

In 1979 R. Fenn and C. Rourke ([7]) show that Kirby's moves could be replaced by an infinite sequence of a single type of move (a *blow down move*) indexed by n, which I depict at the left side of Fig. 4. In a blow down move the number of components decreases by 1. This has been a very useful reformulation with many applications, including Martelli's calculus (soon to be treated) which uses it instead of the direct moves of Kirby.

In the beginning of the 1990's L. Kauffman presented ([9]) a completely planar diagramatic way to deal with the calculus of Kirby and its reformulation by Fenn and Rourke. The basic idea comes from the fact that every 3-manifold is induced by surgery on a framed link which has *only finite integer framings*. This characterize the *handle surgeries*. According to Rolfsen Lickorish call each of these a *honest surgery*, page 262 of [18]. The proof that we can get any manifold by surgery on integer framed links uses, as a lemma, the fact that it is possible to modify the framed link maintaining the induced 3-manifold so that every component becomes unknotted. A proof of this lemma appears in Rolfsen's book. It also apears in page 137 of Kauffman-Lins monography, [10]. If a component is unknotted then it is simple to modify the link so that each component gets an integer framing, without distub the integrality of the framing of other components. So, without loss of generality we may suppose that all the components have finite integers as framings. Kauffman's proceeds by adjusting each component by attaching to it a judicious number of curls so that the required framing of a component coincides with the algebraic sum of its self-crossings. By specifying that the link is *blackboard framed*, we no longer need the integers to specifies the framing. They are a consequence. In this work I only use links given by blackboard framed projections: they are rather close of blinks, which are purely graph theoretical plane graphs with an edge bipartition.

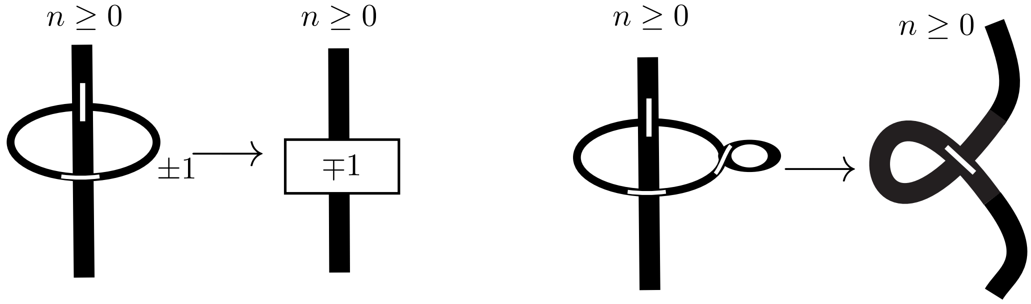

Figure 4: Fenn-Rourke infinite sequence of blown-down moves and their counterpart in Kauffman's blackboard framed links. These are infinite sequence of local moves. Cases $n = 0$ of these moves replace an isolated ± 1-framed component or the unknot with one crossing by nothing. Martelli replaced the infinite sequence by the first three and two new simple moves A_3 and A_4, described in Fig. 5.

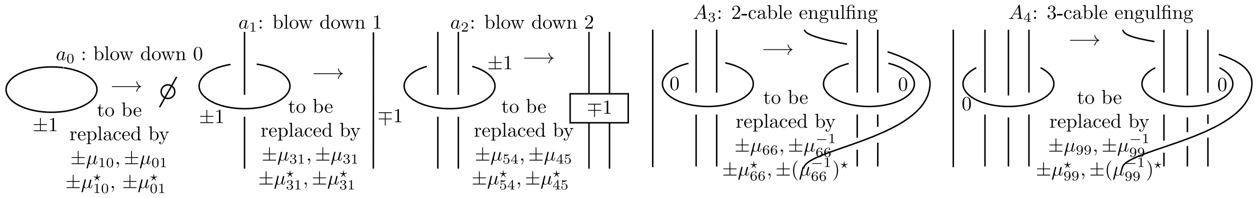

Figure 5: Martelli's calculus on fractional (or integer) framed links. He proves that by keeping only the blown down of ranks 0, 1 and 2 and replacing all the remaining infinite sequence by two new moves 2- and 3-cable engulfing (denoted by A_3 and A_4) a sufficient calculus for factorizing homeomorphisms between closed 3-dimensional 3-manifolds is achieved. Moves A_3 and A_4 do not translate into blink moves because their left sides are disconnected. What makes this work possible is the replacement of these non-connected configurations by equivalent moves a_3 and a_4 (see Fig. 6 where the diagrams for the moves appear -90^o-rotated) so that blink translations become available.

In an important recent paper B. Martelli [17] presented a local finite reformulation of the Fenn-Rourke version ([7]) of Kirby's calculus [11]. This calculus is presented in Fig. 5. It remains to be seen the consequences of Martelli' s result for obtaining new 3-manifold invariants.

4 Objective of the work

Our objective here is to further reformulate Martelli's moves so as to obtain a calculus of blinks, denominated *coin calculus*, which is an exact combinatorial counterpart for factorizing homeomorphisms of closed, orientable, connected 3-manifolds. It has the consequence that each 3-manifold becomes a subtle class of plane graphs. Our exposition is complete and elementary seeking to reach both audiences: topologists and combinatorialists. We feel that this result may be interesting for Combinatorics as well as to Topology and may enhance both areas: conceivably, some deep properties of plane graphs could be used to elucidate aspects of 3-manifolds and vice-versa.

Plane graphs are one of the most studied objects in Combinatorics. The role of planarity in finding polynomial efficient algorithm is well established. For instance, the *Max Cut Problem*,([8]), an NP-complete problem, becomes polynomial, if the graph is in the plane. This was a consequence of J. Edmonds's optimal maximum matching theory polyhedral theory: [4, 6]. Other NP-complete problems like the *Max Stable Vertex Set Problem*([8]), remain NP-complete when restricted to plane graphs. Also well established is the the role of plane graphs motivating and permitting useful generalizations in matroid theory, [22]. Matroids are a source of polynomial algorithms. A. Lehman used this theory to provide a solution for the Shannon Switching Game [12]. This solution was enhanced to a polynomial algorithm by J. Edmonds in [3]. Plane graphs and this paper were the motivation for his unexpected and amazing algorithm for polynomially finding a matroid partition into indepedent subsets, [5], with its various applications to scheduling problems. Yes, I do believe in *One Mathematics*, as advocated by L. Lovasz, in his famous essay, [16]. The area of combinatorics, particularly the area of efficient polynomial algorithms based on polyhedral methods had, ten years ago, its maturity declared by means of the publication of its *Magnum Opus*, in three volumes with more than 1800 mathematically dense pages, by A. Schrijver, [19, 20, 21]. It is my hope that some aspects of the polyhedral theory may have consequences on 3-manifolds algorithmic theory.

Two interesting open questions relating plane graphs and 3-manifolds are: (1) Which 3-manifolds correspond to the class of 3-connected monochromatic blinks? I have reasons to believe that the pair {blink, dual blink} (and the associated {graphic-cographic} matroids) is a complete invariant for these manifolds: a census of all the 242 blinks which are 3-connected, monochromatic and have up to 16 edges appear in [14]. In the domain of the census, the pair {graphical/cographical} matroids is a complete invariant. (2) The theorem established in this work brings closer of being true an old quest of mine: is there a way to associate a matroidal invariant to a general closed, orientable and connected 3-manifold?

Eleven years ago, I. Agol, J. Hass and W. Thurston proved that *3-manifold knot genus is NP-complete*, [1]. There seems to be relatively few results along this line. This one in particular suffer from the fact that it is very difficult to work combinatorially and to visualize a knot in an arbitrary 3-manifold. I think that discovering more NP-complete problems in 3-manifolds could arise from the result here presented. After all NP-complete problems abound in plane graphs. In my wildest dreams I see myself showing that reformulating one of these plane graph problem into the corresponding 3-manifold problem is polynomially solvable by topological means.

5 Proof of the Theorem

(5.1) Lemma. *In the presence of Reidemeister moves 2, generaly denoted by $\pm r_2^{\pm -1}$, moves $\pm a_3$ and $\pm A_3$ are equivalent and so are moves $\pm a_4$ and $\pm A_4$.*

Proof. We refer to Fig. 6. Its first line proves that $\pm a_3 \Rightarrow \pm A_3$. The second line proves that $\pm A_3 \Rightarrow \pm a_3$. The third line proves that $\pm a_4 \Rightarrow \pm A_a$. The last line proves that $\pm A_4 \Rightarrow \pm a_4$. □

Proof. **(of Theorem 1.1)**
In Fig. 7 we draw all the moves for the revised Martelli's moves in a blackboard framing language and the respective blinks superimposed. The result follows by removing the superimposed moves leaving only the blink moves which are shown again up to isotopy in the lower part of the figure. □

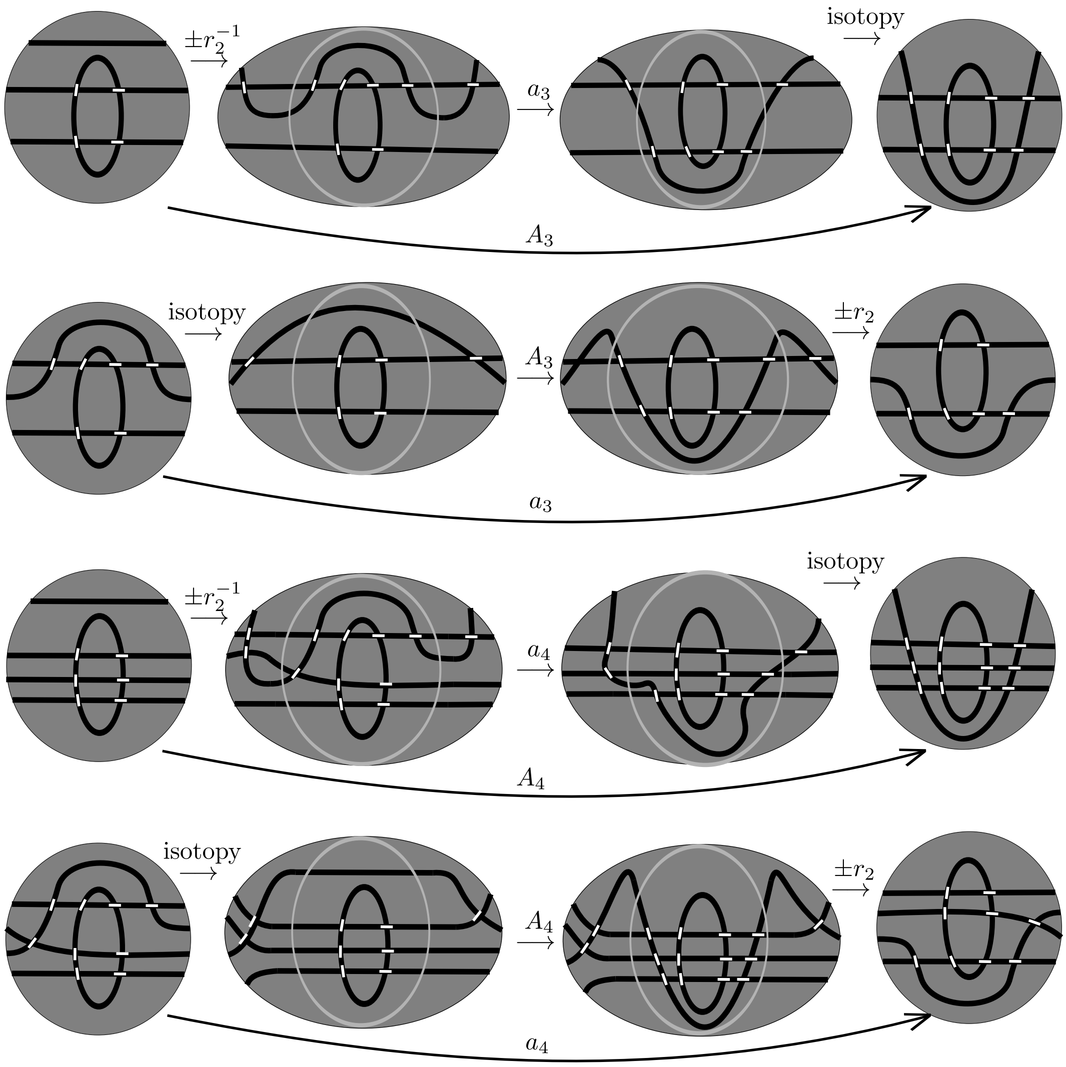

Figure 6: A proof that in the presence of $\pm r_2^{\pm 1}$, the equivalences $a_3 \equiv A_3$ and $a_4 \equiv A_4$ hold

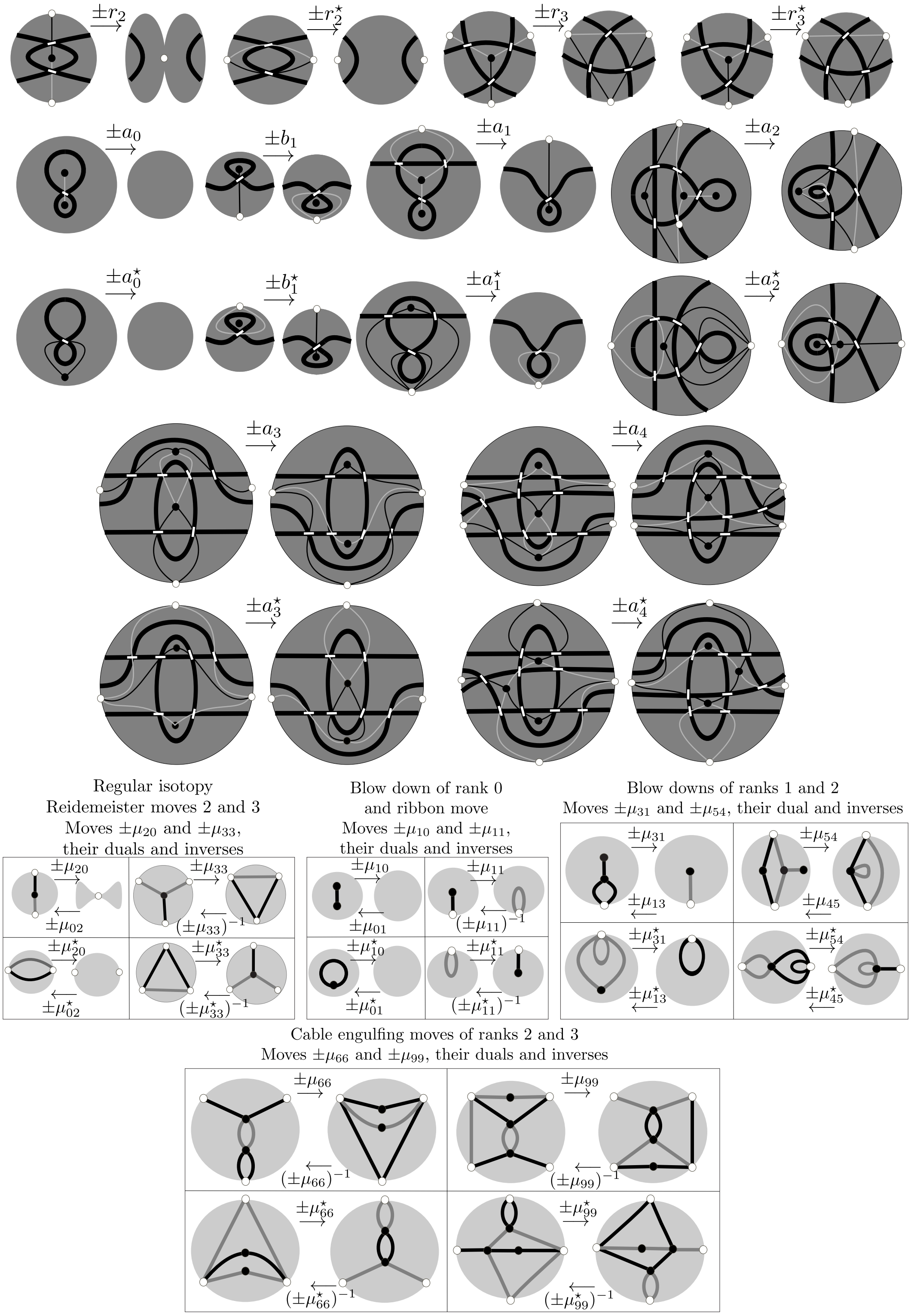

Figure 7: Blackboard framed version of Martelli's revised calculus with a_3, a_4 replacing A_3, A_4. In the upper part of the figure, links & blinks are superimposed implying the moves for the coin calculus in the lower part of the figure, concluding the proof of the Theorem 1.1.

6 The role of ribbon moves $\pm\mu_{11}^{\pm 1}$

The counterpart of the ribbon moves in the coin calculus, (also called ribbon moves) $\pm\mu_{11}^{\pm 1}$ are redundant because Martelli's calculus is in $\mathbb{R}^3$. We include them in our blackboard framed link calculus because with their inclusions all the dual moves, except $\mu_{20}^\star$ and $\mu_{02}^\star$, become redundant.

(6.1) Corollary. *The coin calculus can be simplified to include only the following set of 36 moves* $\{\pm\mu_{20}, \pm\mu_{02}, \pm\mu_{20}^\star, \pm\mu_{02}^\star, \pm\mu_{33}^{\pm 1}, \pm\mu_{01}, \pm\mu_{01}, \pm\mu_{11}^{\pm -1}, \pm\mu_{31}, \pm\mu_{13}, \pm\mu_{54}, \pm\mu_{45}, \pm\mu_{66}^{\pm}, \pm\mu_{99}^{\pm}\}$.

Proof. We work with the language of blackboard framed links which corresponds to the various coins. Moves corresponding to $(\pm\mu_{11}^{\pm 1})^\star$ and $(\pm\mu_{33}^{\pm 1})^\star$ are redundant because $\pm\mu_{11}^{\pm 1}$ $\pm\mu_{33}^{\pm 1}$ are self dual. Moves $\pm\mu_{10}^\star$ and $\pm\mu_{01}^\star$ are implied by a combination of moves $\pm\mu_{10}$, $\pm\mu_{01}$, $\pm\mu_{11}$ and $(\pm\mu_{11})^{-1}$. In particular, all the Reidemeister 2 and 3 moves (regular isotopy) are at our disposal. We can use this fact to change the external face of the link diagram to become any chosen adjacent face by using regular isotopy at the cost of creating two curls adjacent in the same component with distinct sign and the same rotation number. Use the the appropriate ribbon move to obtain two curls with the distinct signs and distinct rotation number. Now apply Whitney's trick (see page 30 of [14]) to cancel these curls. The net effect in the correponding final blink is that it is obtained from the initial blink by dualizing and interchanging black and gray edges. Having this double involutions at our disposal it is straighforward to obtain all the remaining dual moves. □

In Fig. 8 I present the 36 moves of the final reduced coin calculus.

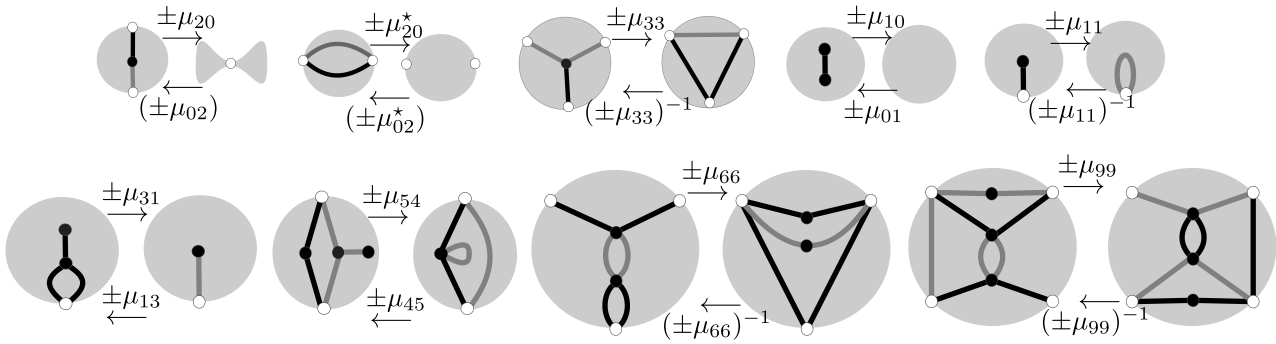

Figure 8: The 36 moves forming the reduced coin calculus

7 Conclusion

I finished this work by presenting below a complete census of the k-small 3-manifolds, for $k = 8$. These are the closed, oriented, connected and prime 3-manifolds induced by a blink with at most k edges.

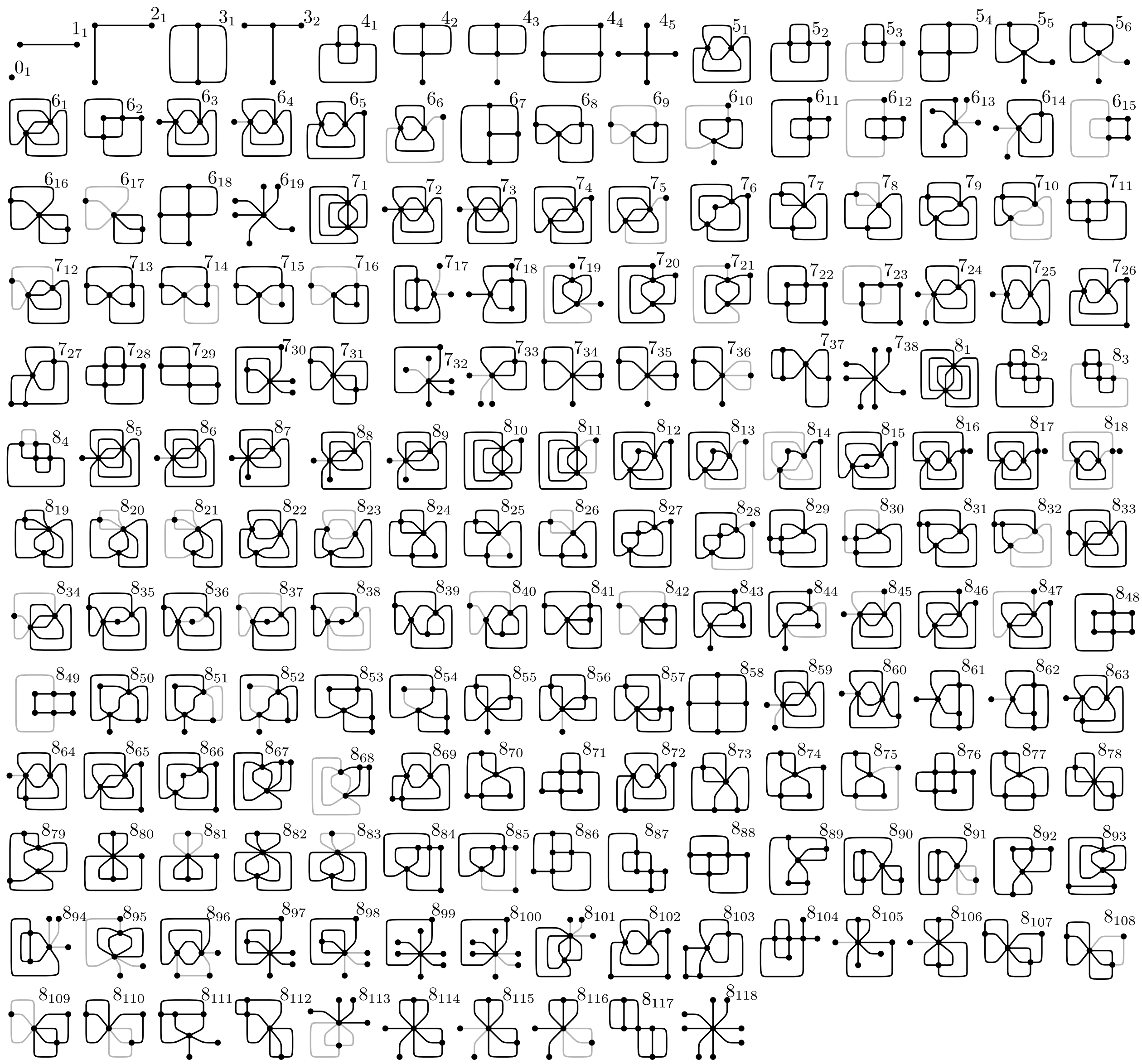

Figure 9: A complete census (no misses, no duplicates) of 8-small prime 3-manifolds. They correspond to the first 191 closed, oriented, connected and prime 3-manifolds. These are such 3-manifolds which are induced by blinks up to $k = 8$ edges. A blink is a finite plane graph with an (arbitrary) edge bipartition. Such census are possible by completely combinatorial methods: we generate a subset of blinks that misses no 3-manifold by lexicography and the theory in [14]; then we compute the homology and the WRT-invariants; at this level $k = 8$ these two invariants are seen to be complete. An entirely combinatorial recipe directly implementable to compute the WRT-invariants of a 3-manifold from a blink inducing it is given in Chapter 7 of [15]. This recipe, in its turn is justified by the (at the bottom, also combinatorial) theory developed in [10].

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