Closed oriented 3-manifolds are equivalence classes of plane graphs *

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Abstract

A *blink* is a plane graph with an arbitrary bipartition of its edges. As a consequence of a recent result of Martelli, we show that the homeomorphisms classes of closed oriented 3-manifolds are in 1-1 correspondence with classes of blinks. Two blinks are equivalent if they are linked by a finite sequence of local moves, where each one appears in a concrete list of 64 moves: they organize in 8 types, each beeing essentlially the same move on 8 simply related configurations. The size of the list can be substantially decreased at the cost of loosing symmetry, just by introducing a very simple move, the ribbon move named β_1 (which is in principle redundant). Using β_1 makes all the moves coming from plane duality (the starred moves), except for $\rho_2^\star$, redundant.

1 Statement of the Theorem

This paper proves the following theorem:

(1.1) Theorem. *The classes homeomorphisms of closed oriented 3-maniflds are in 1-1 correspondence with the equivalence classes of blinks where two blinks are equivalent if they are linked by a finite sequence of the local moves where each term is one of the 64 moves below*
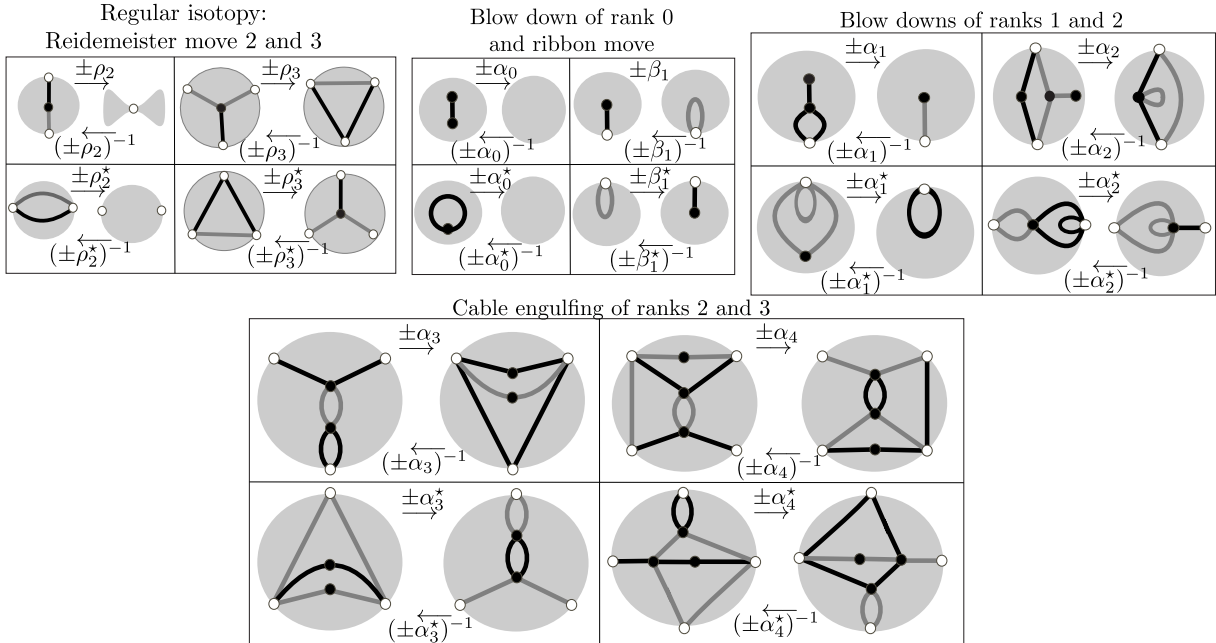
There are 64 local configurations divided into 32 pairs of left-right configurations, each one named a *coin*. A move replaces the left (right) coin of a pair by its right (left) coin. The complementary sub-blink in the exterior a coin is completely arbitrary; its intersection with the corresponding internal coin is the set of attachment vertices shown in the boundary of the disk (or pinched disk, in the case of ρ_2) where the coin lies. The number of such attachment vertices (one of 0,1,2,3,4) is the subscript for the label of the move.

2 Lickorish and moves by Kirby, Fenn-Rourke, Kauffman, Martelli

A *knot* is an embedding of a circle, an $\mathbb{S}^1$, into $\mathbb{R}^3$ or $\mathbb{S}^3$. The *unknot* is a knot which is the boundary of a disk. A *link with k components* is an embedding of a disjoint union of k copies of $\mathbb{S}^1$ into $\mathbb{R}^3$ or $\mathbb{S}^3$. In this way, a knot is a link with one component.

Knots and links can be presented by their *decorated general position projections* into a fixed plane $\mathbb{R}^2$. *General position* means that in the image of the link there is no triple points and that at each neighborhood of each double is the transversal crossing of two segments of the link, named *strands*. *Decorated* means that we keep the information of which strand is the upper one, usually by removing a piece of the lower strand. In this paper we use another way to decorate the link projections: the images of the link components are thich black curves and the upper strands are indicated by a thinner white segment inside the thich black curve at the crossing.

In a grounding breaking work, W.B.R. Lickorish in 1962, [5], proved that each closed orientable 3-manifold $\mathbb{M}^3$ can be encoded by a link in $\mathbb{S}^3$ where each one of its k components is endowed with an irreducible fraction (the framing) $\frac{\pm p}{q}$ where q could be 0, and in the case p must be 1 and the fraction becomes $\pm\infty$. To construct $\mathbb{M}^3$ from the framed link we act as follows: after removing from $\mathbb{M}^3$ an ϵ-neighborhood $(\mathbb{S}^1 \times \mathbb{D}^2)_i$ of the i-th link we are left with $\mathbb{M}^3 \setminus \bigcup_{i=1}^{k}(\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \setminus \bigcup_{i=1}^{k}(\mathbb{S}^1 \times \mathbb{D}^2)_i$. The fraction specifies, in the toroidal boundary inside $\mathbb{S}^3$, the homology type $(\pm p, q)$ of the curve that is contractible in the solid torus inside $\mathbb{M}^3$. For each component, we then identify the simple curve given by the homological base pair with the meridian of a canonical copy of a solid torus in $\mathbb{R}^3$ so as to completely specify the pasting of the solid torus closing the toroidal hole. Lickorish's breakthrough was to prove that any $\mathbb{M}^3$ has inside it a finite number k of disjoint solid tori so that $\mathbb{M}^3 \setminus \bigcup_{i=1}^{k}(\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \setminus \bigcup_{i=1}^{k}(\mathbb{S}^1 \times \mathbb{D}^2)_i$. Actually, this result had been proved 2 years before by A. H. Wallace [8] by using differential geometry. However it was the purely topological flavor of Lickorish's proof that spurs the subsequent developments.

In 1978 R. Kirby published his, to become famous, calculus of framed links, [4]. The gist of this paper is that two types of moves are enough to go from any framed link inducing a closed oriented 3-manifold to any other such link inducing the same manifold. One of the moves is absolutely local: creating or cancellating a an unknot with frame in $\{\infty, -\infty\}$. The other type of move, *the band move ([2]) (or handle sliding)* is non-local and infinite in number.

Shortly after in 1979 R. Fenn and C. Rourke ([1]) show that Kirby's moves could be replaced by an infinite number a single type of moves parametrized by n. This has been a very useful reformulation with many applications, including Martelli's calculus (soon to be treated) which uses it instead of the direct moves of Kirby.

In the beginning of the 1990's L. Kauffman presented ([2]) a completely plane diagramatic way to deal with the calculus of Kirby and of Fenn-Rourke. The basic idea comes from the fact that every 3-manifold is induced by surgery on a framed link which has *only finite integer framings*, which D. Rolfsen call *by honest surgery*, [7]. The proof of this result uses that it is possible to modify the framed link maintaining the induced 3-manifold so that every component becomes unknotted. A proof of this fact appears in Rolfsen's book. It also apears in page 137 of Kauffman-Lins monography, [3]. If a component is unknotted then it is simple to modify the link so that each component gets an integer framing, without distub the integrality of the framing of other components. So, without loss of generality we may supose that all the components have finite integers as framings. Kauffman's proceeds by adjusting each component by attaching to it a judicious number of curls so that the required framing of a component coincides with the algebraic sum of its self-crossings. By specifying that the link is *blackboard framed*, we no longer need the framings. In this work we only use blackboard framed links.

In an important recent paper B. Martelli [6] presented a local finite reformulation of the Fenn-Rourke version ([1]) of Kirby's calculus [4]. It remains to be seen the consequences of Martelli' s result for obtaining new 3-manifold invariants. Our objective here is to further reformulate Martelli's moves so as to obtain a calculus of blinks, which is an exact combinatorial counterpart for factorizing homeomorphisms of closed orientable 3-manifolds. This goal is desirable because it has the consequence that each 3-manifold become a subtle class of plane graphs. Our exposition is complete and elementary seeking to reach both audiences: topologists and combinatorialists. We feel that this result may be interesting for Combinatorics as well as to Topology and may enhance both areas. Plane graphs are one of the most studied objects in mathematics. Some of their properties could be used in 3-manifolds and vice-versa. At any rate I am using here a recipe of Conway, [?].
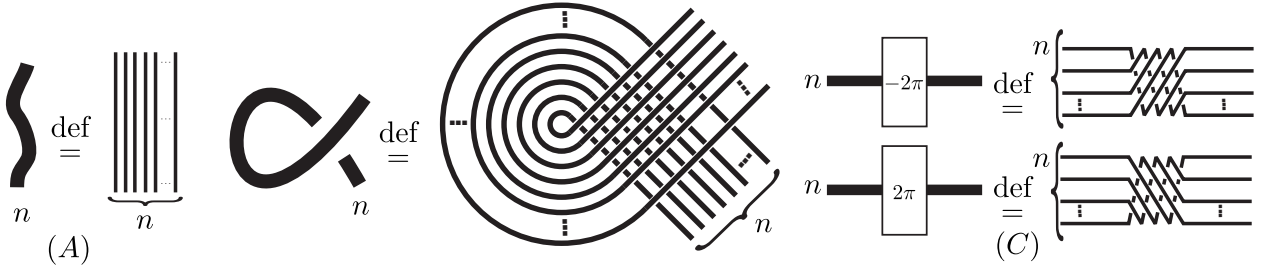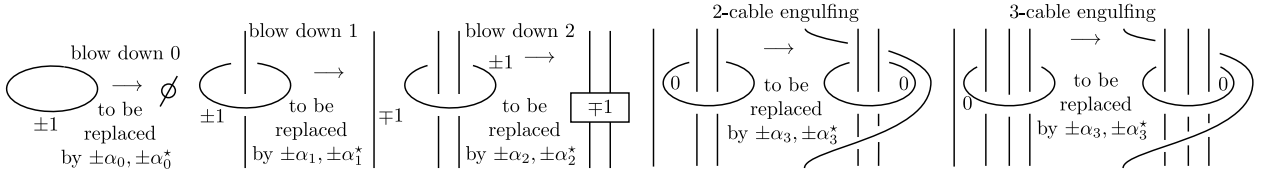
Figure 1: Some notation

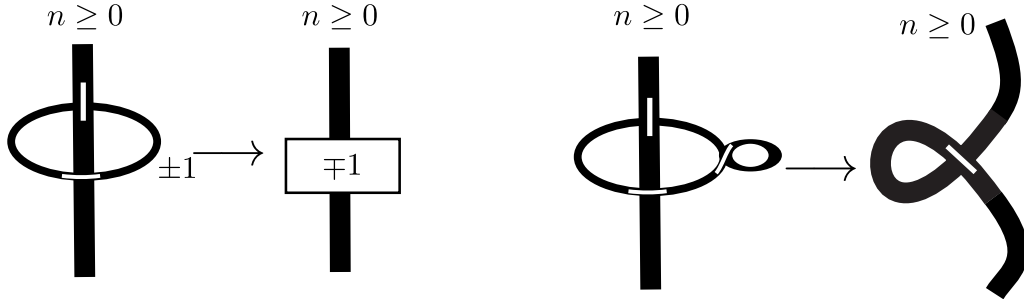Figure 2: Martelli's calculus on fractional framed links

Figure 3: Fenn-Rourke infinite sequence of blown-down moves and their counterpart in Kauffman's blackboard framed links. These are infinite sequence of local moves. Cases $n = 0$ of these moves replace an isolated ± 1-framed component or the unknot with one crossing by nothing. Martelli replaced the infinite sequence by the first three and two simple moves which we call *2- and 3- cable engulfing*, denoted by A_3 and A_4 shortly to be described. In our final form of the blink-coin calculus we use a revised versions of A_3 and A_4: the moves a_3 and a_4, which seem substantially simpler to put on the blink-coin calculus

3 Proof of the Theorem

(3.1) Lemma. *In the presence of $\rho_2^{\pm 1}$, moves $\pm a_3$ and $\pm A_3$ are equivalent and so are $\pm a_4$ and $\pm A_4$.*

Proof. We refer to Fig. 4. Its first line proves that $\pm a_3 \Rightarrow \pm A_3$. The second line proves that $\pm A_3 \Rightarrow \pm a_3$. The third line proves that $\pm a_4 \Rightarrow \pm A_a$. The last line proves that $\pm A_4 \Rightarrow \pm a_4$. □

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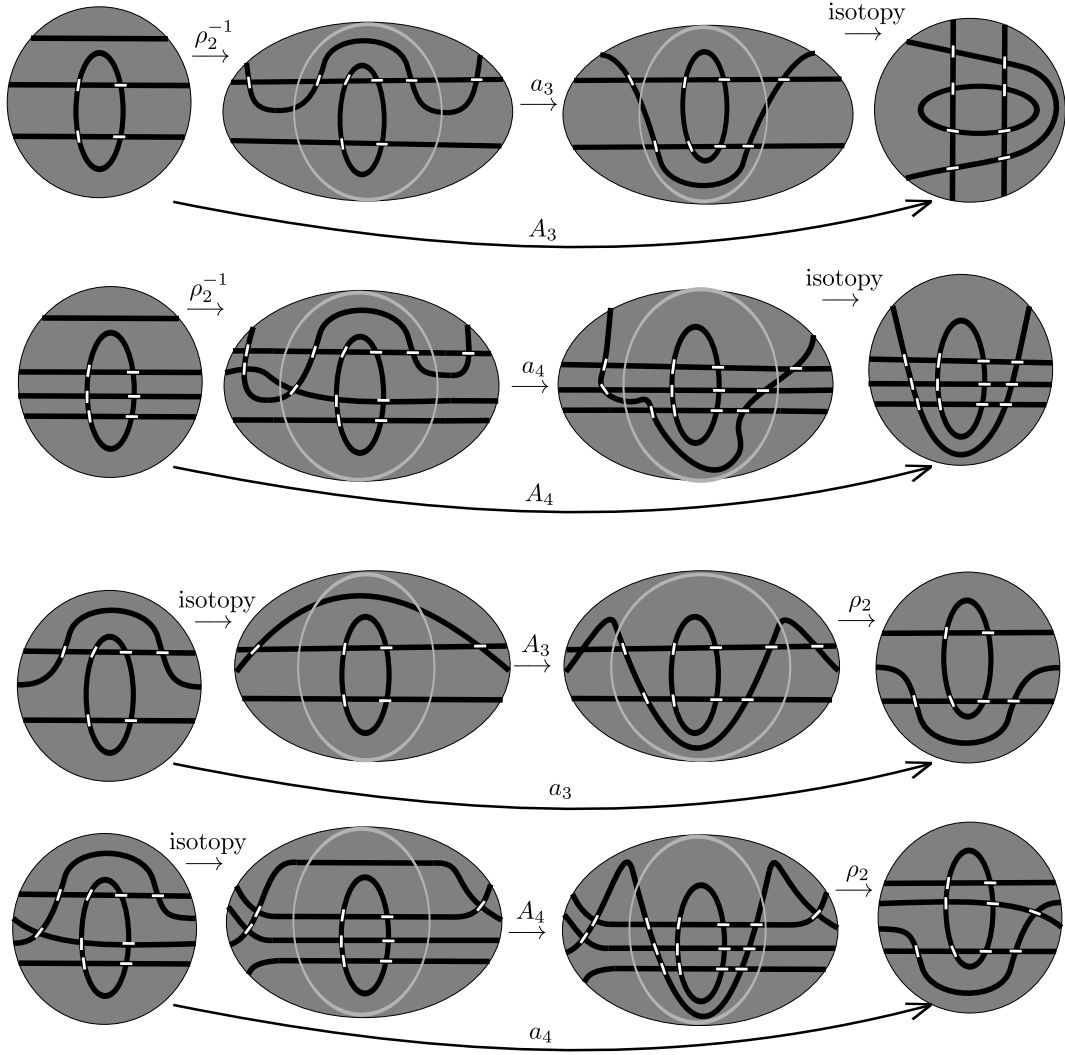
Figure 4: A proof that in the presence of $\pm\rho_2^{\pm 1}$, $a_3 \equiv A_3$ and $a_4 \equiv A_4$

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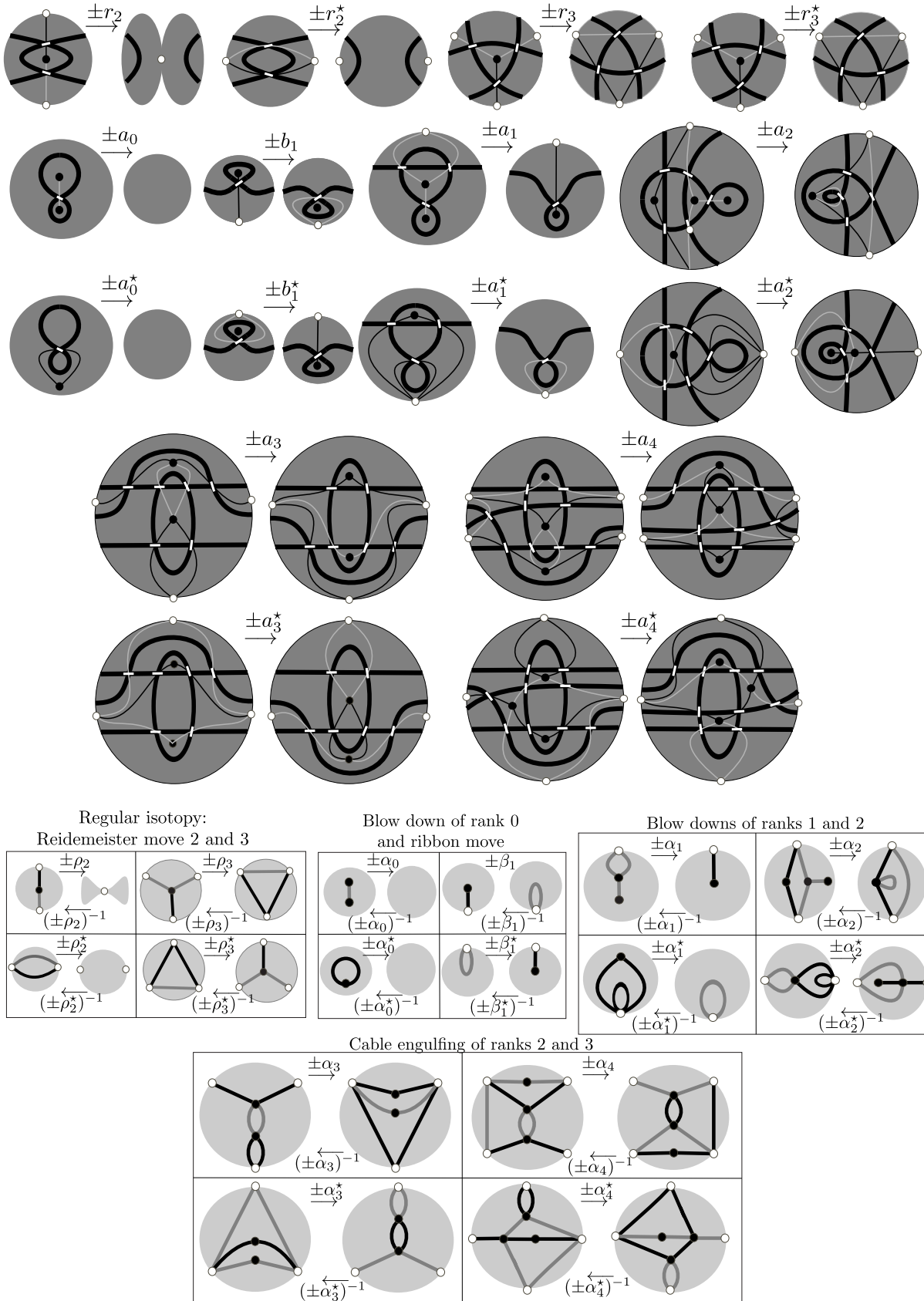Recife, PE 50740-560
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Figure 5: Links & blinks together implying moves for the blink calculus, concluding the proof of the Theorem 1.1