Closed oriented 3-manifolds are equivalence classes of plane graphs *

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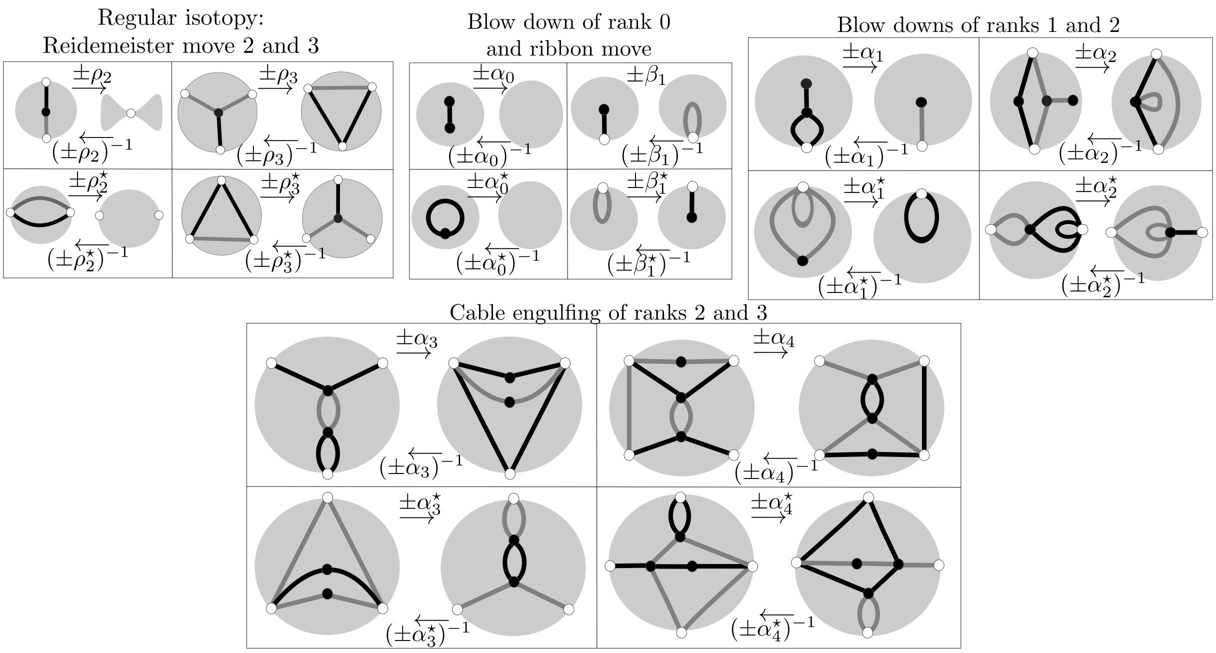
Abstract

A *blink* is a plane graph with an arbitrary bipartition of its edges. As a consequence of a recent result of Martelli, we show that the homeomorphisms classes of closed oriented 3-manifolds are in 1-1 correspondence with classes of blinks. Two blinks are equivalent if they are linked by a finite sequence of local moves, where each one appears in a concrete list of 64 moves: they organize in 8 types, each beeing essentlially the same move on 8 simply related configurations. The size of the list can be substantially decreased at the cost of loosing symmetry, just by introducing a very simple move, the ribbon move named β_1 (which is in principle redundant). Using β_1 makes all the moves coming from plane duality (the starred moves), except for $\rho_2^\star$, redundant.

1 Statement of the Theorem

This paper proves the following theorem:

(1.1) Theorem. *The classes homeomorphisms of closed oriented 3-maniflds are in 1-1 correspondence with the equivalence classes of blinks where two blinks are equivalent if they are linked by a finite sequence of the local moves where each term is one of the 64 moves below*



There are 64 local configurations divided into 32 pairs of left-right configurations. A move replaces the left (right) sub-blink of a pair by its right (left) counterpart. The exterior sub-blink to a fixed configuration is completely arbitrary, provided its intersection with the corresponding configurations are the attachment vertices shown in

*2010 Mathematics Subject Classification: 05C85 and 05C83 (primary), 57M27 and 57M15 (secondary) the boundary of the shaded disk. These vertices are shown as shown as small white circles in the boundary of the shaded disk (or pinched disk in the case of ρ_2) wherem the configuration lies. The number of such attachment vertices is the index of the label for the move.

2 Lickorish and moves by Kirby, Fenn-Rourke, Kauffman, Martelli

A *knot* is an embedding of a circle, an $\mathbb{S}^1$, into $\mathbb{R}^3$ or $\mathbb{S}^3$. The *unknot* is a knot which is the boundary of a disk. A *link with k components* is an embedding of a disjoint union of k copies of $\mathbb{S}^1$ into $\mathbb{R}^3$ or $\mathbb{S}^3$. In this way, a knot is a link with one component.

Knots and links can be presented by their *decorated general position projections* into a fixed plane $\mathbb{R}^2$. *General position* means that in the image of the link there is no triple points and that at each neighborhood of each double is the transversal crossing of two segments of the link, named *strands*. *Decorated* means that we keep the information of which strand is the upper one, usually by removing a piece of the lower strand. In this paper we use another way to decorate the link projections: the images of the link components are thich black curves and the upper strands are indicated by a thinner white segment inside the thich black curve at the crossing.

In a grounding breaking work, W.B.R. Lickorish in 1962, [3], proved that each closed orientable 3-manifold $\mathbb{M}^3$ can be encoded by a link in $\mathbb{S}^3$ where each one of its k components is endowed with an irreducible fraction (the framing) $\frac{\pm p}{q}$ where q could be 0, and in the case p must be 1 and the fraction becomes $\pm\infty$. To construct the 3-manifold $\mathbb{M}^3$ from the framed link we act as follows: after removing from $\mathbb{M}^3$ an ϵ-neighborhood $(\mathbb{S}^1 \times \mathbb{D}^2)_i$ of the i-th link we are left with $\mathbb{M}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i$. The fraction specifies, in the toroidal boundary inside $\mathbb{S}^3$, the homology type $(\pm p, q)$ of the curve that is contractible in the solid torus inside $\mathbb{M}^3$. For each component, we then identify the simple curve given by the homological base pair with the meridian of a canonical copy of a solid torus in $\mathbb{R}^3$ so as to completely specify the pasting of the solid torus closing the toroidal hole. Lickorish's breakthrough was to prove that any $\mathbb{M}^3$ has inside it a finite number k of disjoint solid tori so that $\mathbb{M}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times \mathbb{D}^2)_i$. Actually, this result had been proved 2 years before by A. H. Wallace [5] by using differential geometry. However it was the purely topological flavor of Lickorish's proof that spurs the subsequent developments.

In 1978 R. Kirby published his, to become famous, calculus of framed links, [2]. The gist of this paper is that a finite number of two types of moves are enough to go from any framed link inducing a closed oriented 3-manifold to any other such link inducing the same manifold. One of the moves is absolutelly local: creating or cancellating a an unknot with frame in $\{+1, -1, \infty, -\infty\}$. The other, *the band move* is non-local and infinite in number. Shortly after in 1979 R. Fenn and C. Rourke ([1]) show that Kirby's moves could be replaced by an infinite number a single type of moves parametrized by n. This has been a very useful reformulation with many applications, including Martelli's calculus (soon to be treated) which uses it instead of the direct moves of Kirby.

In a recent paper B. Martelli [4] presented a reformulation of the Fenn-Rourke version ([1]) of Kirby's calculus [2]. Our objective here is to further reformulate Martelli's moves so as to obtain a calculus of blinks, which is an exact combinatorial counterpart for factorizing homeomorphisms of closed orientable 3-manifolds. This goal is desirable because it has the consequence that each 3-manifold become a subtle class of plane graphs. Our exposition is complete and elementary seeking to reach both audiences: topologists and combinatorialists.

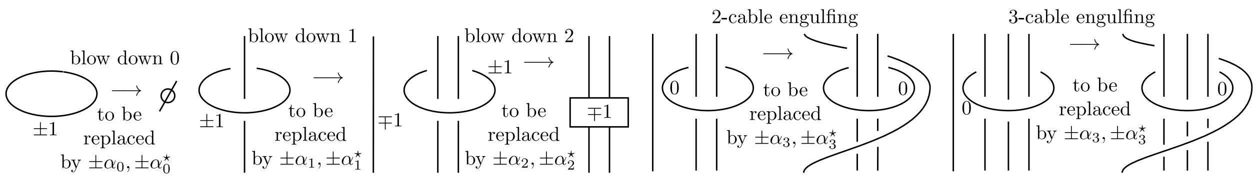


Figure 1: Martelli's calculus on fractional framed links

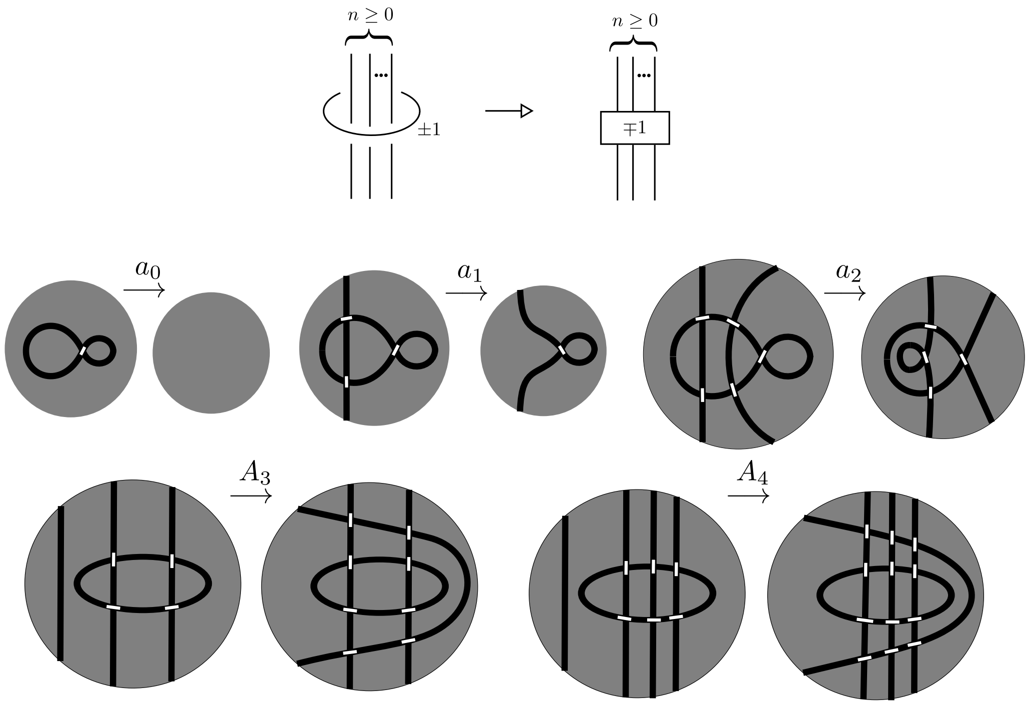


Figure 2: Fenn-Rourke infinite 1-parametrized blown-down moves and Martelli's moves in blackboard framed form

3 Proof of the Theorem

(3.1) Lemma. *In the presence of* $\rho_2^{\pm 1}$*, moves* $\pm a_3$ *and* $\pm A_3$ *are equivalent and so are* $\pm a_4$ *and* $\pm A_4$*.*

Proof. We refer to Fig. 3. Its first line proves that $\pm a_3 \Rightarrow \pm A_3$. The second line proves that $\pm A_3 \Rightarrow \pm a_3$. The third line proves that $\pm a_4 \Rightarrow \pm A_a$. The last line proves that $\pm A_4 \Rightarrow \pm a_4$. $\square$

References

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[5] A.H. Wallace. Modifications and cobounding manifolds. *Canad. J. Math*, 12:503–528, 1960.

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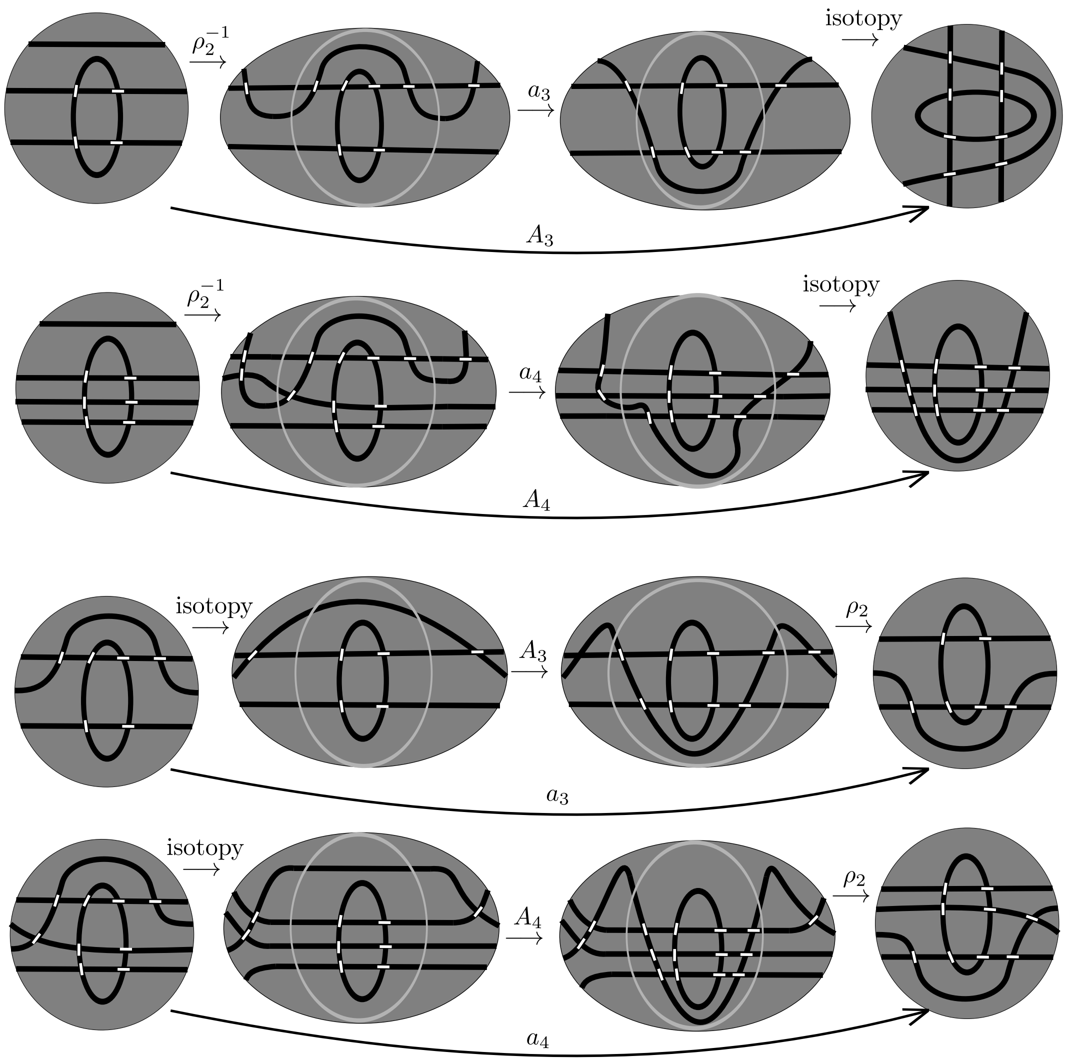

Figure 3: A proof that in the presence of $\pm\rho_2^{\pm 1}$, $a_3 \equiv A_3$ and $a_4 \equiv A_4$

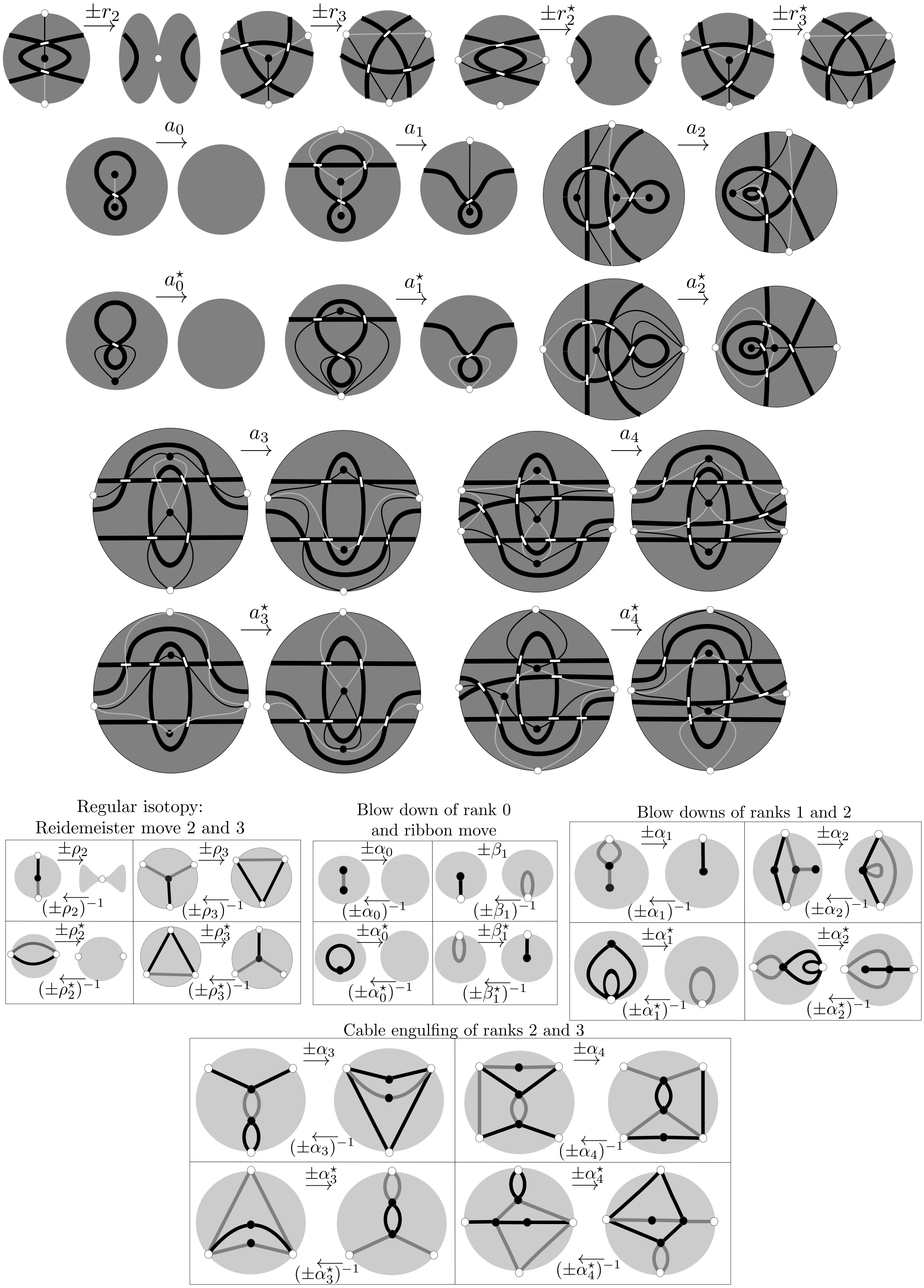

Figure 4: Links & blinks together implying moves for the blink calculus, concluding the proof of the Theorem 1.1