Closed, oriented, connected 3-manifolds are equivalence classes of plane graphs *

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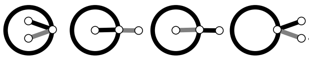
Abstract

A *blink* is a plane graph with an arbitrary bipartition of its edges. As a consequence of a recent result of Martelli, I show that the homeomorphisms classes of closed oriented 3-manifolds are in 1-1 correspondence with specific classes of blinks. In these classes, two blinks are equivalent if they are linked by a finite sequence of local moves, where each one appears in a concrete list of 64 moves: they organize in 8 types, each beeing essentlially the same move on 8 simply related configurations. The size of the list can be substantially decreased at the cost of loosing symmetry, just by keeping a very simple move type, the *ribbon moves* denoted $\pm\mu_{11}^{\pm}$ (which are in principle redundant). The inclusion of $\pm\mu_{11}^{\pm}$ implies that all the moves corresponding to plane duality (the starred moves), except for $\mu_{20}^{\star}$ and $\mu_{02}^{\star}$, are redundant and the coin calculus is reduced to 36 moves on 36 coins. It is in the aegis of this work to find new important connections between 3-manifolds and plane graphs.

1 Introduction

A *blink* is a plane graph with an arbitrary edge bipartition into two colors (black and gray). *Plane* means that it is given embedded in a plane. Two blinks B and B' are the same if there is an isotopy of the plane into itself so that the image of B is B'. The next four blinks are all distinct even tough they have the same subjacent graph: .

1.1 Statement of the Theorem

This paper proves the following theorem:

(1.1) Theorem. *The classes homeomorphisms of closed oriented connected 3-manifolds are in 1-1 correspondence with the equivalence classes of blinks where two blinks are equivalent if they are linked by a finite sequence of the local moves where each term is one of the 64 moves (not necessary distinct) below*

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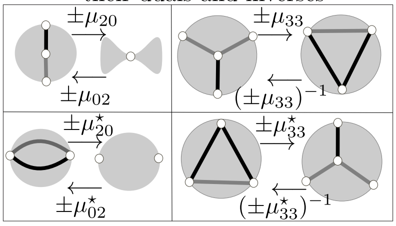
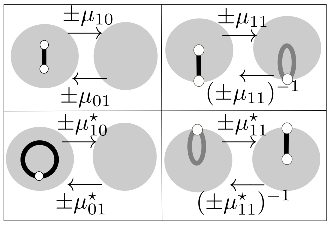
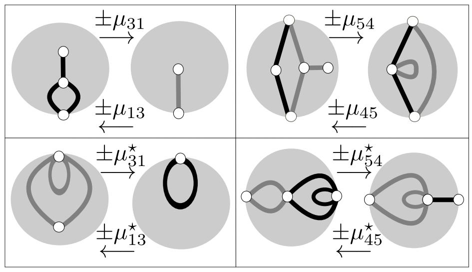
Regular isotopy
Reidemeister moves 2 and 3
Moves $\pm\mu_{20}$ and $\pm\mu_{33}$,
their duals and inverses



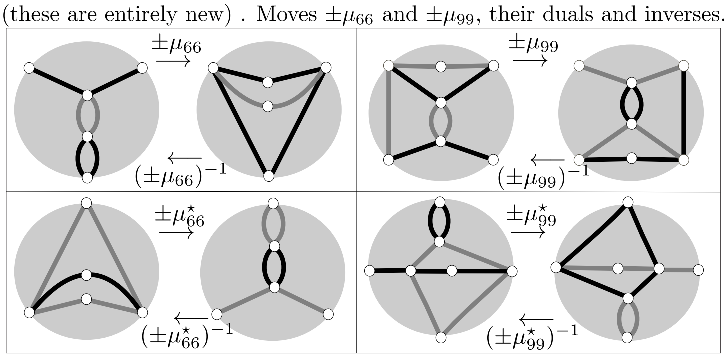
Blow down of rank 0
and ribbon move
Moves $\pm\mu_{10}$ and $\pm\mu_{11}$,
their duals and inverses



Blow downs of ranks 1 and 2
Moves $\pm\mu_{31}$ and $\pm\mu_{54}$, their dual and inverses



Blink substitutes for Martelli's cable engulfing moves of ranks 2 and 3
(these are entirely new) . Moves $\pm\mu_{66}$ and $\pm\mu_{99}$, their duals and inverses.



There are 64 local configurations of sub-blinks, each named a *coin*, divided into 8 families of 8 simply related coins and also divided into 32 pairs of left-right coins, A move replaces the left (right) coin of a pair by its right (left) coin. Thus, the number of moves is equal to the number of coins. Boundary and internal vertices of the coins are shown as small white disks. The complementary sub-blink in the exterior a coin is completely arbitrary; its intersection with the corresponding internal coin is a subset of the set of attachment vertices in the boundary of the coin. The support of the coins are disks, except in the the right coin of μ_{20}, case in which is a pinched disk. The number k of the attachment vertices satisfies $k \in \{0, 1, 2, 3, 4\}$.

1.2 Organization of the paper

In Section 2 the motivation, the topological preliminaries and an epistemological view of the work are discussed. The proof of the Theorem 1.1 is given in Section 3. A reduced but sufficient form of this coin calculus having 36 coins (and moves) is obtained in Section 4. In Section 5 I display a census (no misses, no duplicates) of the closed, oriented, connected, prime 3-manifold induced by blinks up to 8 edges.

2 Motivation and topological preliminaries

2.1 Motivation

In his Appendix to part 0, J. H. Conway in his famous book *On Numbers and Games*, [2], says: *"This appendix is in fact a cry for a Mathematician Liberation Movement!*

Among the permissible kinds of constructions we should have:

(i) Objects may be created from earlier objects in any reasonable constructive fashion.

(ii) Equality among the created objects can be any desirable equivalence relation."

This paper is in the confluency of two deep research passions of the author, apparently very far apart: the topological study of closed orientable 3-manifolds and the combinatorial study of plane graphs. The result proved here provides a glimpse, in the spirit of Conway's quotation, to effectively enumerate once each closed orientable 3-manifolds. Blinks are easy to construct from simpler blinks, and their isomorphism problem can be solved by a polynomial algorithm which finds, via a few fixed conventions (lexicography), a numerical *code* for it. What can be more desirable than an equivalence relation on such simple mathematical objects that captures the subtle and difficult computational topological notion of factorizing any homeomorphism between two closed, oriented, connected 3-manifolds?

2.2 Topological preliminaries

This subsection contains the basic topological material that I need. It is primarely intended to the combinatorially oriented readers, unfamiliar with the fundamental definitions of knots, links and framed lins. The topological oriented readers also should read it because I introduce some unfamiliar notation and definitions which are new and that will be used throughout the paper. Moreover, I present a short historic overview of the known results needed.

2.2.1 Knots and links into $\mathbb{R}^3$ and into $\mathbb{S}^3$

A *knot* K is an embedding of a circle, $\mathbb{S}^1$, into $\mathbb{R}^3$ (or $\mathbb{S}^3$, the boundary of a 4-dimensional ball). The *unknot* is a knot which is the boundary of a disk. A *link with k components* is an embedding of a disjoint union of k copies of $\mathbb{S}^1$, $\left(\cup_{i=1}^{k} \mathbb{S}^1_i\right)$, into $\mathbb{R}^3$ ((or $\mathbb{S}^3$) with disjoint images. In this way, a knot is a particular case of a link: one which has one component. Fig. 1 shows a 1-1 correspondence $K \leftrightarrow K'$ between knots into $\mathbb{R}^3$ and into $\mathbb{S}^3$. By abuse of language, I feel free to identify a knot with its image.

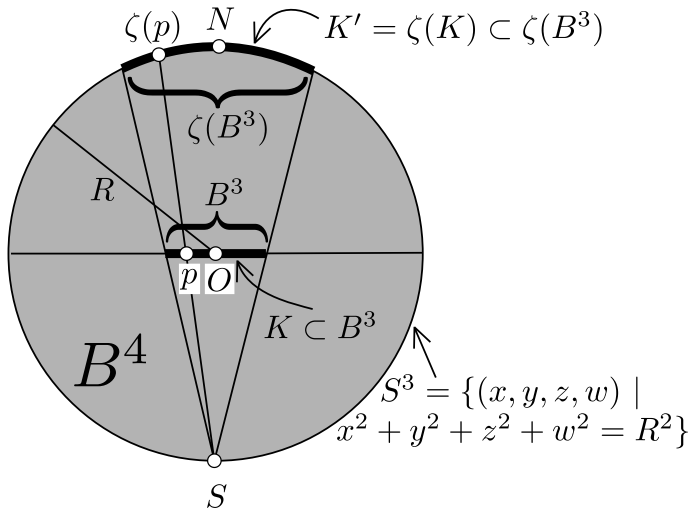


Figure 1: How a knot K into $\mathbb{R}^3$ is related to a knot K' into $\mathbb{S}^3$: the 3-sphere $\mathbb{S}^3$ is the boundary of a 4-ball $\mathbb{B}^4$; the knot K is contained in a ball $\mathbb{B}^3$ of radius r centered in the origin O and contained in the equator $\{(x, y, z, w) \in \mathbb{B}^4 \mid w = 0\}$; by making $\frac{R}{r}$ big knots K (which the south pole of the 4-ball) and K' are as close as being isometric as desired. K' is the image of K under the stereographic projection ζ centered at the south pole S. In this work I need to work with knots into $\mathbb{R}^3$ and into $\mathbb{S}^3$. Thus the easy correspondence between the two types is welcome.

Links can be presented with profit by their *decorated general position projections* into the xy-plane $\mathbb{R}^2$, by simply making 0 the z-coordinate. Recall that we have already make $w = 0$. Here *general position* means that in the image of the link there is no triple points and that at each neighborhood of a double point is the transversal crossing of two segments of the link, named *strands*. *Decorated* means that we keep the information of which strand is the upper one, usually by removing a piece of the lower strand. In this paper I use yet another way to decorate the link projections: the images of the link components are thich black curves and the upper strands are indicated by a thinner white segment inside the thich black curve at the crossing (see second definition in Fig. 6).

2.2.2 Framed knots, ribbons, framed links and blackboard framed links into $\mathbb{R}^3$

A *framed knot* is an embedding of $\mathbb{S}^1 \times [-\epsilon, +\epsilon]$ into $\mathbb{R}^3$ (or $\mathbb{S}^3$), for an arbitrarily fixed $\epsilon > 0$. A framed knot is also called *a ribbon.* The *base knot* of a ribbon is the ribbon restricted to $\mathbb{S}^1 \times \{0\}$. A *framed link* is a collection of ribbons with disjoint images. The ribbons that I use are prepared by isotopies so that their projections remain with constant width 2ϵ. Fig. 2 shows how to achieve this condition. A collection of these constant width ribbons is called a *blackboard framed link.*

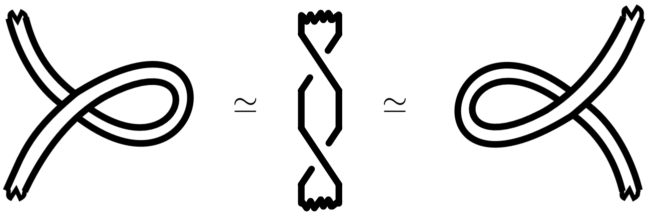
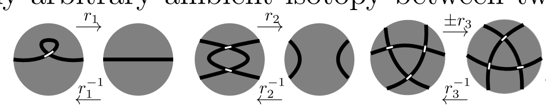


Figure 2: Getting a constant width immersion of a ribbon projection into $\mathbb{R}^2$. Each 360-degree rotation of the ribbon projection is ambient isotopic to a curl in the ribbon (in two different ways) so as to maintain constant the width immersion of the ribbon projection. In particular, after making these isotopies, the intersection of the images of $\mathbb{S}^1 \times \{-\epsilon\}$ and $\mathbb{S}^1 \times \{+\epsilon\}$ have 4 distinct points (it used to have 2 points) near each crossing of the base link and there are no other crossings in the immersed ribbons.

The isotopy class of a framed link is determined by assigning an integer to each component of the link which is equal to the *linking number* of the two components of the boundary of each of its bands oriented in the same way. The linking number can be obtained from an arbitrary decorated general position projection of the oriented band: it is half of the algebraic sum of the $(\pm)$-signs of the crossings of distinct boundary components. The convention is that  $\to +1$,  $\to -1$. The linking number is an invariant, that is, it does not depend on the particular projection used. This is proved by K. Reidemeister in its 1932 book on Knot Theory, [20]. He isolates three local moves r_1, r_2, r_3 that are enough to finitely factor any arbitrary ambient isotopy between two decorated general position projections of the same link: 

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As a matter of fact, Theorem 1.1 which I proved in this work is the counterpart for 3-manifolds of Reidemeister Theorem. Note that in the coin calculus depicted together with bflinks in Fig. 10 there are 8 versions of each of Reidemeister moves r_2 and r_3. Note also that Reidemeister move r_1 is not used at all. The equivalence class of decorated general position link projection generated by $r_2^{\pm 1}$ and $r_3^{\pm 1}$ is called *regular isotopy*. It plays an important role in the computation of the Jones invariants via Kauffman's bracket [9].

A *blackboard framed link* is a projection of a framed link prepared by isotopy so that the base link is projected as a decorated general position one and its bands are immersed with constant width 2ϵ, see Fig. 2. Moreover, we adjust the number of curls and their signs (in the base link) so that the framing of each component coincides with the linking number of the two boundary components of the band. The advantage of the blackboard framed link is that we no longer have to worry about assigning numbers to the components. These integer numbers are induced by the plane of projection. The importance of this concept was advocated by L. Kauffman in a number of works, including [10] and [11]. In this work, it is central: every framed link I use is blackboard framed.

A *bflink* is the projection of the base link of a blackboard framed link. Thus a bflink is a decorated general position link projection, with no numbers attached. Fig. 3 shows a 1-1 correspondence between bflinks and blinks.

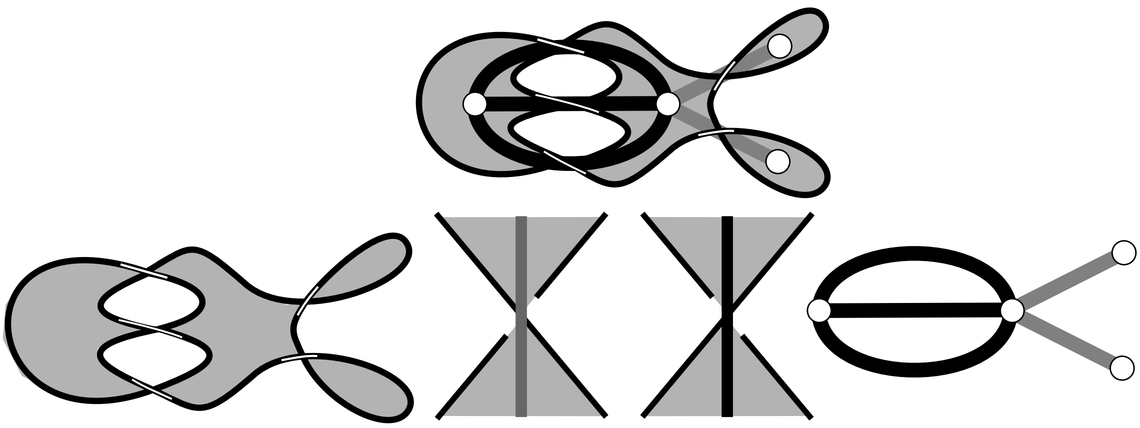


Figure 3: From bflink to blink and back: the projection of any link can be 2-face colorable into white and gray with the infinite face being white so that each subcurve between two crossing have their incident faces receiving distinct colors. The above figure shows how to transform a link projection into a blink (with thicker edges than the curves representing the link projection). The vertices of the blink are distinguished fixed points represented by white disks in the interior of the gray faces. Each crossing of the link projection becomes an edge in the corresponding blink, An edge of the blink is gray if the upper strand that crosses it is from northwest to southeast, it is black if the the upper strand that crosses it is from northeast to southwest. The inverse procedure is clearly defined. In fact, the link is the so called *medial map* of the blink. Thus we have a 1-1 correspondence between general position decorated link projections and blinks. The (complete) blink at the right induces Poincaré's sphere, the spherical dodecahedron space. The expressibility of a blink is powerful: quite complicated 3-manifolds are induced by simple blinks.

2.2.3 Lickorish's groundbreaking result and more link type objects

In a grounding breaking work (1962), W.B.R. Lickorish, [14], proved that any closed, oriented, connected $\mathbb{M}^3$ has inside it a finite number k of disjoint solid tori each in the form of a homeomorphic image of $\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta]$, denoted by $(\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta])_i$ so that, where $\mathbb{S}^3$ is the 3-dimensional sphere,

$$\mathbb{M}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta])_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta])_i.$$

As a consequence, each closed oriented 3-manifold can be obtained from $\mathbb{S}^3$ by removing a subset of disjoint solid tori and pasting them back in a different way. The most general parameters to identify the pasting is a pair of integers for each component of the link. A filling algorithm algorithm applies component by component, is named $(\pm p, q)$-*Dehn filling*, and is explained in Fig. 4. For proving the theorem, and this is important in the present work, the fillings used by Lickorish satisfy $q = 1$. He call these surgeries *honest*. Actually, Lickorish's result had been proved 2 years before by A. H. Wallace [27] by using differential geometry. However it was the purely topological flavor of Lickorish's proof that spurred the subsequent developments. Also, Lickorish does not state his theorem in this way. The form I use for the solid tori is convenient because of its simple relation with bflinks. It is inspired in the lucid account by J. Stillwell of the theorem given in [25].

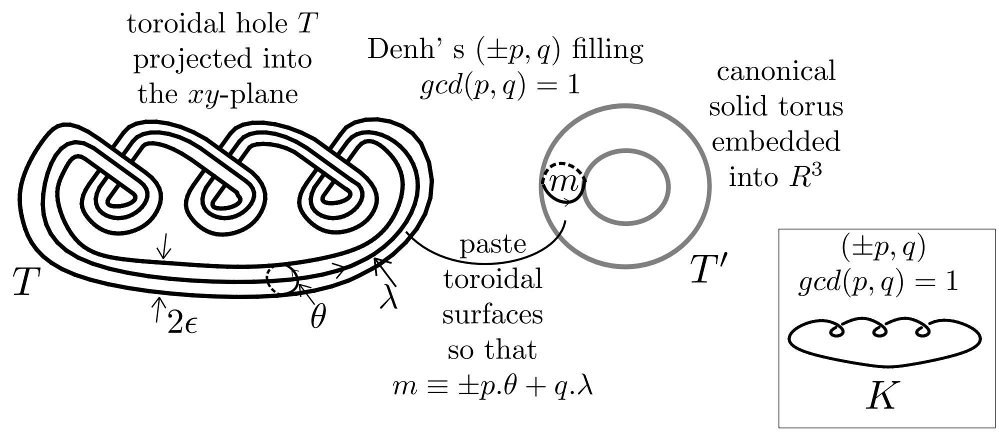


Figure 4: Dehn's $(\pm p, q)$-filling of a toroidal hole T and its particular case where $q = 1$, yielding a well defined bflink (or link) as input for an algorithm to produce a 3-manifold: suppose that $\frac{\pm p}{q}$ is an irreducible fraction, where the case q=0, $p = 1$ is permitted. Consider a solid torus in $\mathbb{S}^3$ as the homeomorphic image of $\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta]$, denoted by T, where ϵ and δ are arbitrarily small fixed positive constants. T can be changed by ambient isotopy so that for each $s \in \mathbb{S}^1$, T restricted to $\{s\} \times [-\epsilon, +\epsilon] \times [-\delta, +\delta]$, is a rectangle which is orthogonal to the projection xy-plane. Therefore, the projection of its restriction to $\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times \{\delta\}$ is a blackboard framed link whose base link λ is defined by the restriction of T to $\mathbb{S}^1 \times \{0\} \times \{\delta\}$. The important fact is that λ does not touch the boundary of its immersed band forming the projection of the blackboard framed link. Let θ be a curve that bounds a disk in T but not in its boundary, $\partial(T)$. The curve $\pm p.\theta + q.\lambda$ is isotopic to a simple curve in $\partial(T)$. Map this curve homeomorphically to m, a curve in a canonical solid torus T' in $\mathbb{R}^3$ so that it bounds a disk in T' but not in $\partial(T')$. This homeomorphism is univocally extensible to identify $\partial(T)$ and $\partial(T')$ thus closing the toroidal hole. Note that the whole situation is defined by the knot $K \equiv \mathbb{S}^1 \times \{0\} \times \{0\}$ and the surgery coeficients $(\pm p, q)$. In his original paper Lickorish only use Dehn fillings of type $(\pm p, 1)$, corresponding to what he calls *honesty surgeries*. For such surgeries each component K may be curl-adjusted so as the whole link becomes a bflink where each $\pm p$ is given by the linking number of the boundary components of the corresponding band. This establishes an algorithm to obtain a well defined closed, oriented, connected 3-manifold $\mathbb{M}^3$ from a bflink, or from a blink, as input. By Lickorish's theorem, every such manifold arises in this way. Even if the blink is not connected, $\mathbb{M}^3$ is connected: in general, it is the connected sum of the manifolds induced by the connected components of the blink which induces it.

An *hflink* is the base link of a framed link $L_1 \cup \ldots \cup L_k$ endowed with the surgery coefficients $(\pm p_1, q_1), \ldots (\pm p_k, q_k)$. A *residual bflink* of an hflink, or an *r-bflink*, is obtained by effecting an ambient isotopy in an hflink so that in the projected base link of the hflink all the surgery coefficients satisfy

$$\frac{\pm p_i}{q_i} = z_i + \frac{p'_i}{q_i},$$

where z_i is an integer equal to the linking number of boundary components of the i-th band of the hflink and $0 < p_i < q_i$. Of course this can be achieved only by inserting adequate curls.

2.2.4 Kirby's famous calculus of framed links

In 1978 R. Kirby published his, to become famous, calculus of framed links, [12]. The gist of this paper is that two types of moves are enough to go from any framed link inducing a closed oriented 3-manifold to any other such link inducing the same manifold. One of the moves is absolutelly local: creating or cancellating an arbitrary new unknotted component with frame ± 1 separated from the rest of the link by an $\mathbb{S}^2$. The other type of move, *the band move*, [12, 10]) *(or handle sliding)* is non-local and infinite in number.

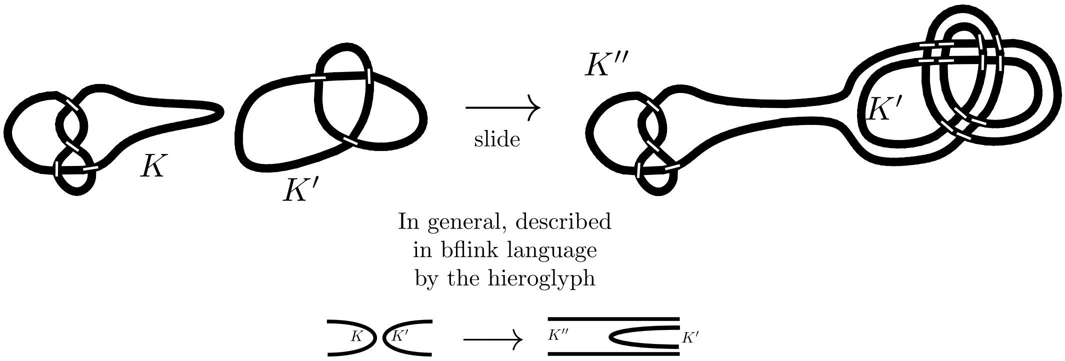


Figure 5: Kirby's *handle slide move*, also known as the *band move*. In this work the handle slide move is applied only to bflinks, so there is no issue about surgery coefficients. Given K and K' two distinct components of a bflink start by making a close parallel copy of K' in such a way that it forms an immersed band with its originator K. Let K'' be the connected sum of K and the copy of K'. The connected sum is defined by a em new band which is a thin rectangle arbitrarily embedded into $\mathbb{S}^3$, so as to miss the link. The short sides of this band are attached to K and to the copy of K' and then, recoupled in the other way. The new band can be quite complicated because it may wander arbitrarily (as long as it misses the link) in $\mathbb{R}^3$ in its way to connecting the two components. More details in Kauffman's book, [10]. In bflink language, this move is depicted in all its generality, via the hieroglyph shown in the bottom part of the Figure. See Section 12.3 of [11]. However, this hieroglyphic move is non-local because the outside of the hieroglyph changes, and since there are infinite exteriors, there are infinite Kirby's band moves.

2.2.5 Fenn-Rourk reformulation of Kirby's calculus

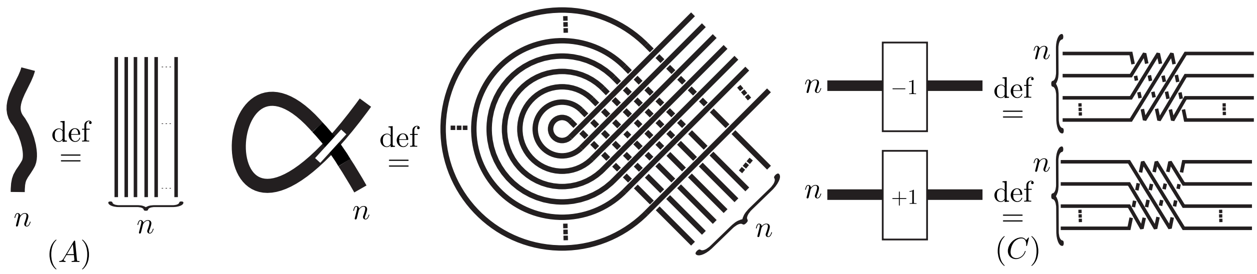


Figure 6: Notation for special disk neighborhoods of general position decorated link projections

In 1979 R. Fenn and C. Rourke ([7]) show that Kirby's moves could be replaced by an infinite sequence of a single type of move (a *blow down move*) indexed by n, which I depict at the left side of Fig. 7. In a blow down move the number of components decreases by 1. This has been a very useful reformulation with many applications, including Martelli's calculus (soon to be treated) which uses it instead of the direct moves of Kirby.

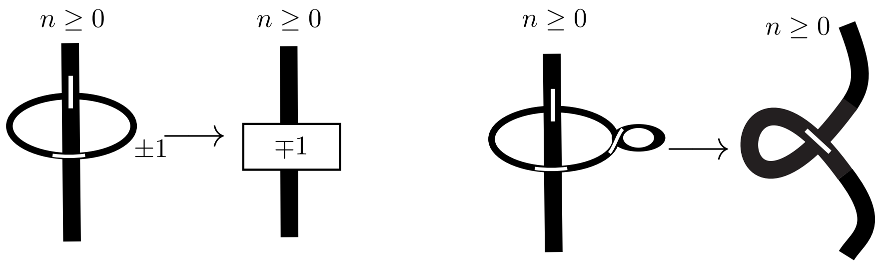


Figure 7: Fenn-Rourke infinite sequence of blown-down moves and their counterpart in Kauffman's blackboard framed links. These are infinite sequence of local moves. Cases $n = 0$ of these moves replace an isolated ± 1-framed component or the unknot with one crossing by nothing. Martelli replaced the infinite sequence by the first three and two new simple moves A_3 and A_4, described in Fig. 8.

2.2.6 Kauffman's idea to let the plane induce the integer framing

In the beginning of the 1990's L. Kauffman presented ([10]) a completely planar diagramatic way to deal with the calculus of Kirby and its reformulation by Fenn and Rourke. The basic idea comes from the fact that every 3-manifold is induced by surgery on a framed link which has *only finite integer framings*. This characterize the *handle surgeries*. According to Rolfsen Lickorish call each of these a *honest surgery*, page 262 of [21]. The proof that we can get any manifold by surgery on integer framed links uses, as a lemma, the fact that it is possible to modify the framed link maintaining the induced 3-manifold so that every component becomes unknotted. A proof of this lemma appears in Rolfsen's book. It also apears in page 137 of Kauffman-Lins monography, [11]. If a component is unknotted then it is simple to modify the link so that each component gets an integer framing, without distub the integrality of the framing of other components. So, without loss of generality we may suppose that all the components have finite integers as framings. Kauffman's proceeds by adjusting each component by attaching to it a judicious number of curls so that the required framing of a component coincides with the algebraic sum of its self-crossings. By specifying that the link is *blackboard framed*, we no longer need the integers to specifies the framing. They are a consequence. In this work I only use links given by blackboard framed projections: they are rather close of blinks, which are just purely graph theoretical plane graphs with an edge bipartition.

2.2.7 Martelli's finite calculus on fractionary framed link

In an important recent paper B. Martelli [19] presented a local and finite reformulation of the Fenn-Rourke version ([7]) of Kirby's calculus [12]. This calculus is presented in Fig. 8. It remains to be seen the consequences of Martelli' s result for obtaining new 3-manifold invariants. A possible door for obtaining such invariants are generalizations of the combinatorial approach to get WRT-invariants, justified in [11] and extensely used in [17] and in [16]. To find such a generalization one

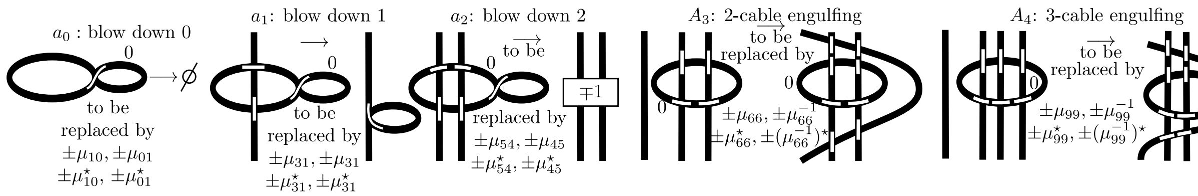


Figure 8: Martelli's calculus on r-bflinks. Each component of the r-bflink is endowed with an irreducible fraction $\frac{p}{q}$ assigned to it, $q > p \geq 0$. The surgery coeficients (p', q) of the solid torus obtained by δ-thickening the corresponding ribbon satisfies: $\frac{p'}{q} = z + \frac{p}{q}$, where z is the linking number of the boundary components of the ribbon. Note that for the internal components of all the above moves the residual surgery coefficient is 0. Martelli' proves that by keeping only the blown down of ranks 0, 1 and 2 and replacing all the remaining infinite sequence by two new moves, 2- and 3-cable engulfing (denoted by A_3 and A_4), a sufficient calculus for factorizing homeomorphisms between closed oriented and connected 3-manifolds is achieved solely in terms of the above 5 local moves plus the regular isotopies moves and the ribbon move. Moves A_3 and A_4 do not translate into blink moves because their left sides are disconnected. What makes this work possible is the replacement of these non-connected configurations by equivalent moves a_3 and a_4 so that blink translations become available. The equivalence is proved in Fig. 9 where the diagrams for the moves appear -90^o-rotated relative to this Figure.

has to take advantage of the specific sufficient local Martelli's moves now available (or the coin calculus on blinks). The WRT-invariants are obtained by hiding in the Temperley-Lieb algebra the infinite cases of Kirby's band move, as pioneered by Lickorish in [15]. See also page 144 of the join monography of L. Kauffman and myself, [11]. Finding such generalization of the WRT-invariants still seems to be a formidable task. However, Martelli's theorem and the coin calculus on blinks make it conceivable.

2.3 This work from an epistemological point of view

I proceed by further reformulating Martelli's moves so as to obtain a calculus of blinks, denominated *coin calculus*. It is is an exact combinatorial counterpart for factorizing homeomorphisms of closed, orientable, connected 3-manifolds. It has the consequence that each 3-manifold becomes a subtle class of plane graphs. This exposition has been and will remain complete and elementary. It seeks to reach both audiences: Topologists and Combinatorialists. We feel that this result may be interesting for Combinatorics as well as to Topology and may enhance both areas: conceivably, some deep properties of plane graphs could be used to elucidate aspects of 3-manifolds and vice-versa.

Plane graphs are one of the most studied objects in Combinatorics. The role of planarity in finding polynomial efficient algorithm is well established. For instance, the *Max Cut Problem*,([8]), an NP-complete problem, becomes polynomial, if the graph is in the plane. This was a consequence of J. Edmonds's optimal maximum matching theory polyhedral theory: [4, 6]. Other NP-complete problems like the *Max Stable Vertex Set Problem*([8]), remain NP-complete when restricted to plane graphs. Also well established is the the role of plane graphs motivating and permitting useful generalizations in matroid theory, [26]. Matroids are a source of polynomial algorithms. A. Lehman used this theory to provide a solution for the Shannon Switching Game [13]. This solution was enhanced to a polynomial algorithm by J. Edmonds in [3]. Plane graphs and this paper were the motivation for his unexpected and amazing algorithm for polynomially finding a matroid partition into indepedent subsets, [5], with its various applications to scheduling problems. Yes, I do believe in *One Mathematics*, as advocated by L. Lovasz, in his famous essay, [18]. The area of combinatorics, particularly the area of efficient polynomial algorithms based on polyhedral methods had, ten years ago, its maturity declared by means of the publication of its *Magnum Opus*, in three volumes with more than 1800 mathematically dense pages, by A. Schrijver, [22, 23, 24]. It is my hope that some aspects of the polyhedral theory may have consequences on 3-manifolds algorithmic theory.

Two interesting open questions relating plane graphs and 3-manifolds are: (1) Which 3-manifolds correspond to the class of 3-connected monochromatic blinks? I have reasons to believe that the pair {blink, dual blink} (and the associated {graphic-cographic} matroids) is a complete invariant for these manifolds: a census of all the 242 blinks which are 3-connected, monochromatic and have up to 16 edges appear in [16]. In the domain of the census, the pair {graphical/cographical} matroids is a complete invariant. (2) The theorem established in this work brings closer of being true an old quest of mine: is there a way to associate a matroidal invariant to a general closed, orientable and connected 3-manifold?

Eleven years ago, I. Agol, J. Hass and W. Thurston proved that *3-manifold knot genus is NP-complete*, [1]. There seems to be relatively few results along this line. This one in particular suffer from the fact that it is very difficult to work combinatorially and to visualize a knot in an arbitrary 3-manifold. I think that discovering more NP-complete problems in 3-manifolds could arise from the result here presented. After all NP-complete problems abound in plane graphs. In my wildest dreams I see myself showing that reformulating one of these plane graph problem into the corresponding 3-manifold problem is polynomially solvable by topological means.

3 Proof of the Theorem

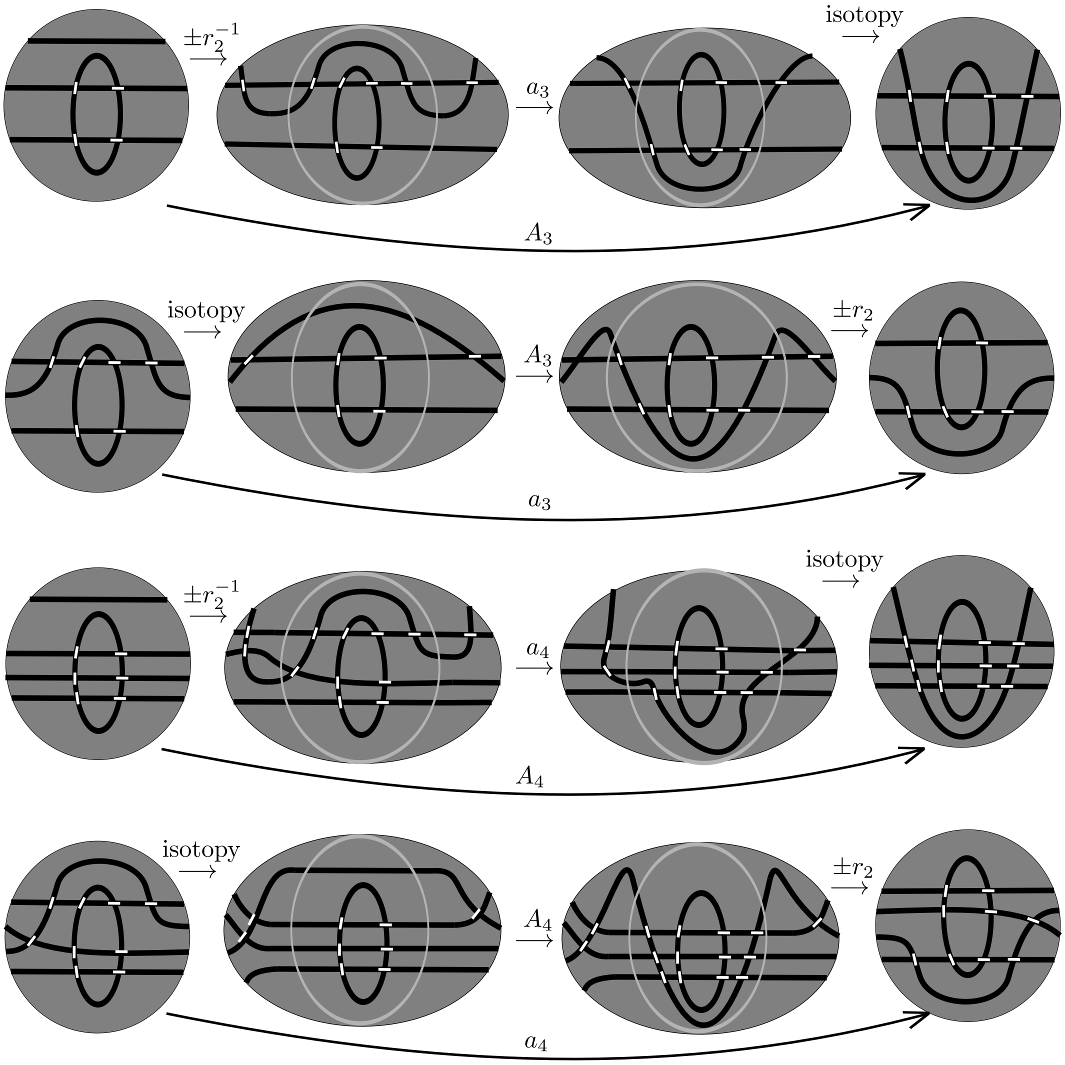


Figure 9: A proof that in the presence of $\pm r_2^{\pm 1}$, the equivalences $a_3 \equiv A_3$ and $a_4 \equiv A_4$ hold

(3.1) Lemma. *In the presence of Reidemeister moves 2, generaly denoted by $\pm r_2^{\pm -1}$, moves $\pm a_3$ and $\pm A_3$ are equivalent and so are moves $\pm a_4$ and $\pm A_4$.*

Proof. We refer to Fig. 9. Its first line proves that $\pm a_3 \Rightarrow \pm A_3$. The second line proves that $\pm A_3 \Rightarrow \pm a_3$. The third line proves that $\pm a_4 \Rightarrow \pm A_a$. The last line proves that $\pm A_4 \Rightarrow \pm a_4$. $\square$

Proof. **(of Theorem 1.1)**
In Fig. 10 we draw all the moves for the revised Martelli's moves on pairs of distinctly 2-colored bflinks and the respective blinks superimposed. The result follows by removing the bflink moves leaving only the blink moves which are redrawn up to isotopy, in the lower part of the figure. $\square$

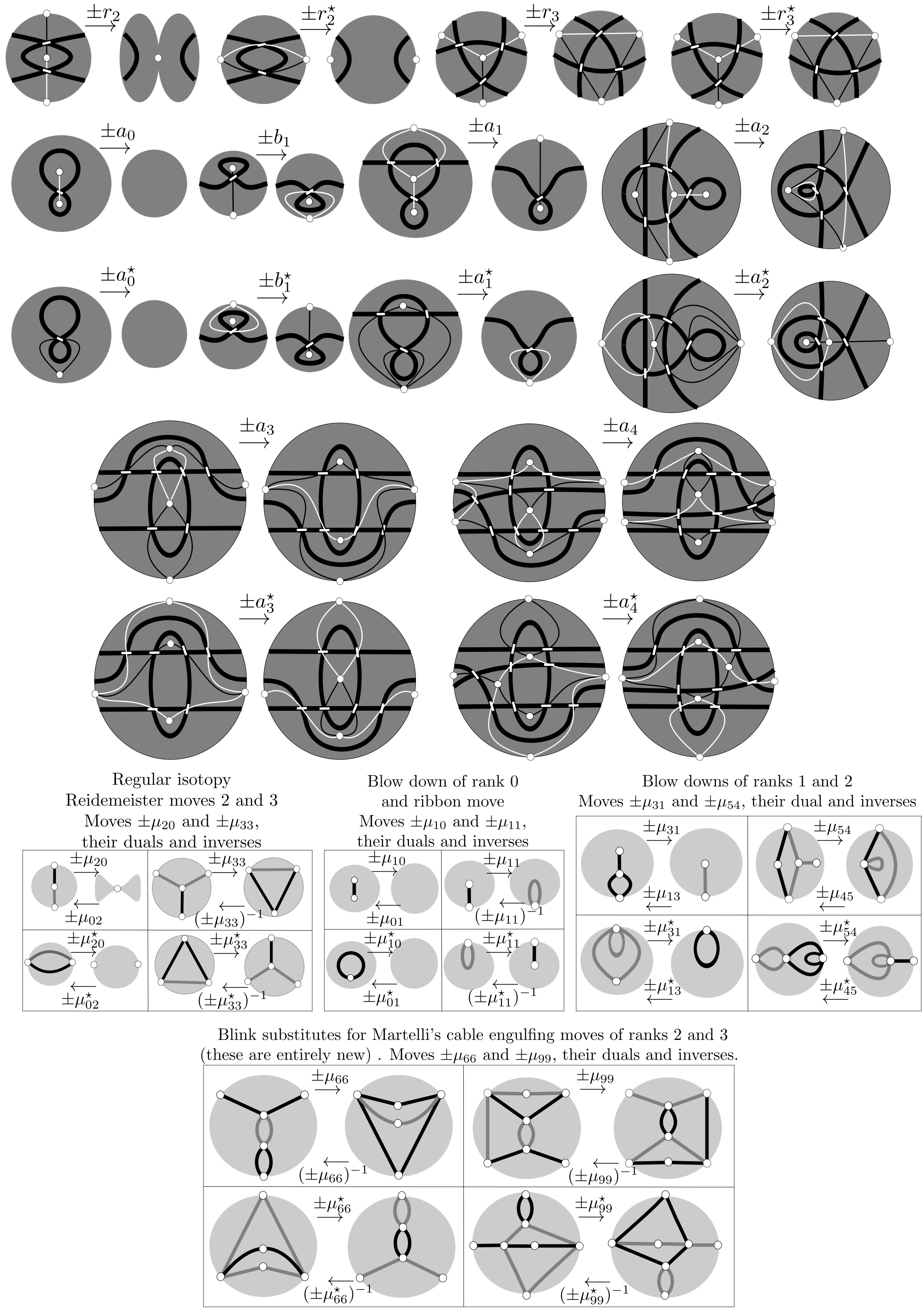


Figure 10: Bflink version of Martelli's revised calculus with a_3, a_4 replacing A_3, A_4. In the upper part of the figure, distinctly 2-face colored bflinks and respective blinks are superimposed implying the moves for the coin calculus in the lower part of the figure, concluding the proof of the Theorem 1.1.

4 The role of the ribbon moves $\pm\mu_{11}^{\pm 1}$

The counterpart of the ribbon moves in the coin calculus, (also called ribbon moves) $\pm\mu_{11}^{\pm 1}$ are redundant because Martelli's calculus is in $\mathbb{R}^3$. We include them in the coin calculus because with their inclusions all the dual moves, except $\mu_{20}^{\star}$ and $\mu_{02}^{\star}$, become redundant.

(4.1) Lemma. *Let f be the external infinite face of a decorated general position link projection and g be a face adjacent to f. Then it is possible to interchange f and g by means of regular isotopies and one finger move.*

Proof. The proof is given in Fig. 11. □

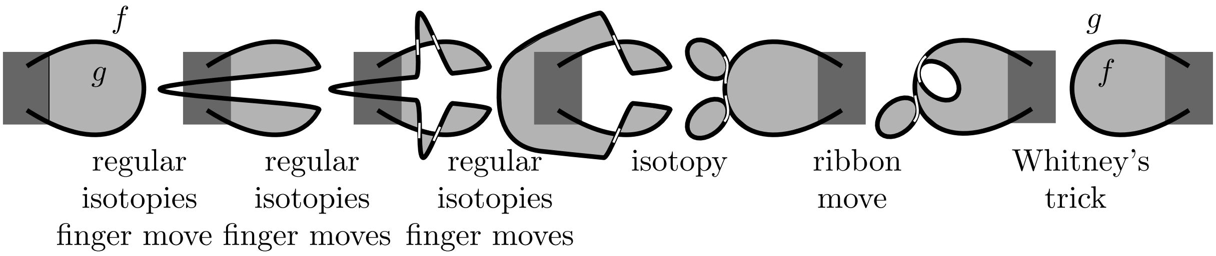
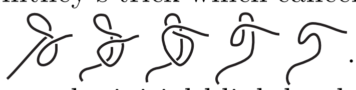


Figure 11: Changing the external face f to be any one of its adjacent faces g by means of r_2, r_3 (regular isotopies) and the ribbon move: the passage from the first to the second configurations is a *finger move* where a small segment of an edge is arbitrarily deformed as a *finger* passing over all the crossings of the dark gray rectangle; the finger move can be factored by regular isotopies; by other four finger moves one can go from the second to the third to the fourth configuration; the passage from the fourth to the fifth configuration is simply an isotopy; from the fifth to the sixth is a ribbon move; finally, the passage from the sixth to the seventh is acomplished by Whitney's trick, depicted in the proof below; Whitney's trick also factors as regular isotopies, as shown in the proof of Corollary 4.2. The net effect is to interchange the faces f and g: the initial infinite face is f and, at the end, the infinite face is g.

(4.2) Corollary. *The coin calculus can be simplified to include only the following set of 36 moves* $\{\pm\mu_{20}, \pm\mu_{02}, \pm\mu_{20}^{\star}, \pm\mu_{02}^{\star}, \pm\mu_{33}^{\pm 1}, \pm\mu_{01}, \pm\mu_{01}, \pm\mu_{11}^{\pm -1}, \pm\mu_{31}, \pm\mu_{13}, \pm\mu_{54}, \pm\mu_{45}, \pm\mu_{66}^{\pm}, \pm\mu_{99}^{\pm}\}$.

Proof. We work in the language of bflinks. Moves corresponding to $(\pm\mu_{11}^{\pm 1})^{\star}$ and $(\pm\mu_{33}^{\pm 1})^{\star}$ are redundant because $\pm\mu_{11}^{\pm 1}$ $\pm\mu_{33}^{\pm 1}$ are self dual. Moves $\pm\mu_{10}^{\star}$ and $\pm\mu_{01}^{\star}$ are implied by a combination of moves $\pm\mu_{10}$, $\pm\mu_{01}$, $\pm\mu_{11}$ and $(\pm\mu_{11})^{-1}$. In particular, all the Reidemeister 2 and 3 moves (regular isotopy) are at our disposal. We can use this fact to change the external face of the link diagram to become any chosen adjacent face by using regular isotopy at the cost of creating two curls adjacent in the same component with distinct sign and the same rotation number. Use the the appropriate ribbon move to obtain two curls with the distinct signs and distinct rotation number. Now apply Whitney's trick which cancel these opposite curls by using $r_2^{\pm 1}$ and $r_3^{\pm 1}$ moves (regular homotopies): . The net effect in the correponding final blink is that it is obtained from the initial blink by dualizing and interchanging black and gray edges. Having this double involutions at our disposal it is straighforward to obtain all the remaining dual moves. □

In Fig. 12 I present the 36 moves of the final reduced coin calculus.

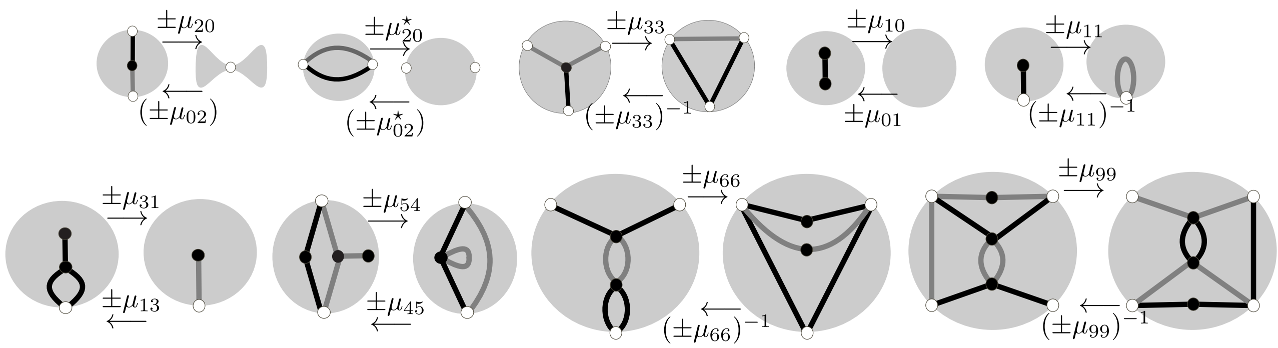


Figure 12: The 36 moves (and 36 coins not necessarily distinct) defining the reduced coin calculus.

5 Conclusion

I finish this work by presenting below a complete (no misses, no duplicates) census of the k-small 3-manifolds, for $k = 8$. These are the closed, oriented, connected and prime 3-manifolds induced by a blink with at most k edges.

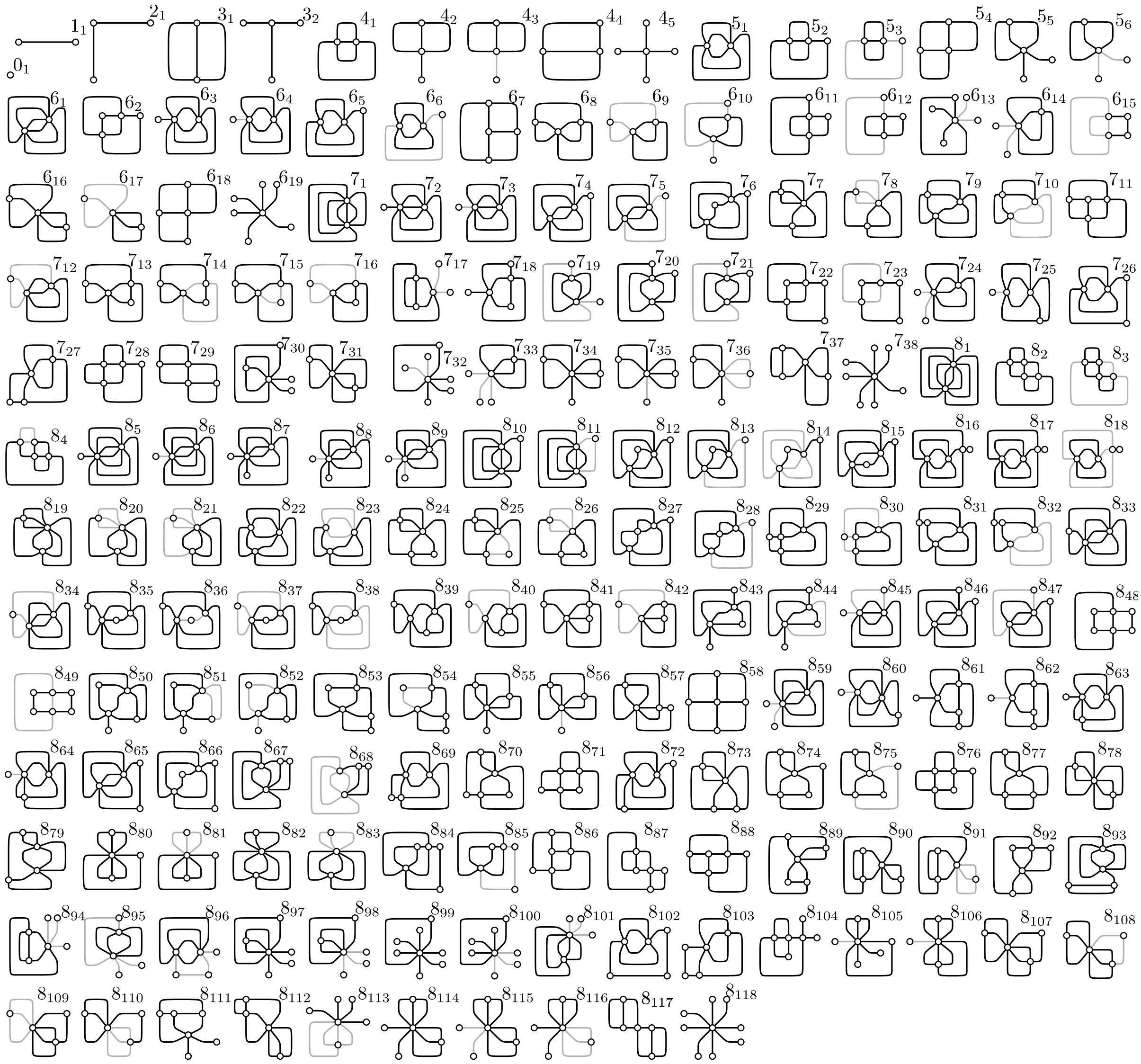


Figure 13: A complete census (no misses, no duplicates) of 8-small prime 3-manifolds. They correspond to the first (by lexicography) 191 closed, oriented, connected and prime 3-manifolds. These are such 3-manifolds which are induced by blinks up to $k = 8$ edges. A blink is a finite plane graph with an (arbitrary) edge bipartition. Such census are possible by completely combinatorial methods: we generate a subset of blinks that misses no 3-manifold by lexicography and the theory in [16]; then we compute the homology and the WRT-invariants; at this level $k = 8$ these two invariants are seen to be complete. An entirely combinatorial recipe directly implementable to compute the WRT-invariants of a 3-manifold from a blink inducing it is given in Chapter 7 of [17]. This recipe, in its turn is justified by at the very basic level, also by the combinatorial theory developed in [11]. There are 3 independent implementations of to obtain the Kauffman-Lins version (depending upon the Temperley-Lieb algebra) of the WRT-invariants. They were implemented by different people, in non-overlapping times: S. Lins (1990-1995), S. Melo (1999-2001) and L. Lins (2006-2007). The results agree. The software BLINK which compute these invariants and the above census by exhaustive generation of blinks is currently the object of an open source Github project, and is available upon request. The sequence of WRT-invariants of a closed oriented connected 3-manifold is an infinite sequence of complex numbers, indexed by $r \geq 3$. The r-th WRT-invariant of a manifold is directly obtained from a blink inducing it.

Figure 14: Bflink version of Martelli's revised calculus with a_3, a_4 replacing A_3, A_4. In the upper part of the figure, distinctly 2-face colored bflinks and respective blinks are superimposed implying the moves for the coin calculus in the lower part of the figure, concluding the proof of the Theorem 1.1.

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