Closed, oriented, connected 3-manifolds
are
subtle equivalence classes of plane graphs *
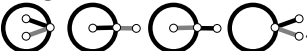
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Abstract

A *blink* is a plane graph with an arbitrary bipartition of its edges. As a consequence of a recent result of Martelli, I show that the homeomorphisms classes of closed oriented 3-manifolds are in 1-1 correspondence with specific classes of blinks. In these classes, two blinks are equivalent if they are linked by a finite sequence of local moves, where each one appears in a concrete list of 64 moves: they organize in 8 types, each being essentially the same move on 8 simply related configurations. The size of the list can be substantially decreased at the cost of loosing symmetry, just by keeping a very simple move type, the *ribbon moves* denoted $\pm\mu_{11}^{\pm}$ (which are in principle redundant). The inclusion of $\pm\mu_{11}^{\pm}$ implies that all the moves corresponding to plane duality (the starred moves), except for $\mu_{20}^{\star}$ and $\mu_{02}^{\star}$, are redundant and the coin calculus is reduced to 36 moves on 36 coins. A *residual fraction link* or a *flink*, is a new object which generalizes *blackboard-framed link*. It plays an important role in this work. I try to make the topological exposition as complete as possible: about half of the exposition deals with the topological preliminaries. The objective is to make it easier for the combinatorially oriented readers to understand the paper. It is in the aegis of this work to find new important connections between 3-manifolds and plane graphs.

1 Introduction

A *blink* is a plane graph with an arbitrary edge bipartition into two colors (black and gray). *Plane* means that it is given embedded in a plane. Two blinks B and B' are the same if there is an isotopy of the plane into itself so that the image of B is B'. The next four blinks are all distinct even tough they have the same subjacent graph: .
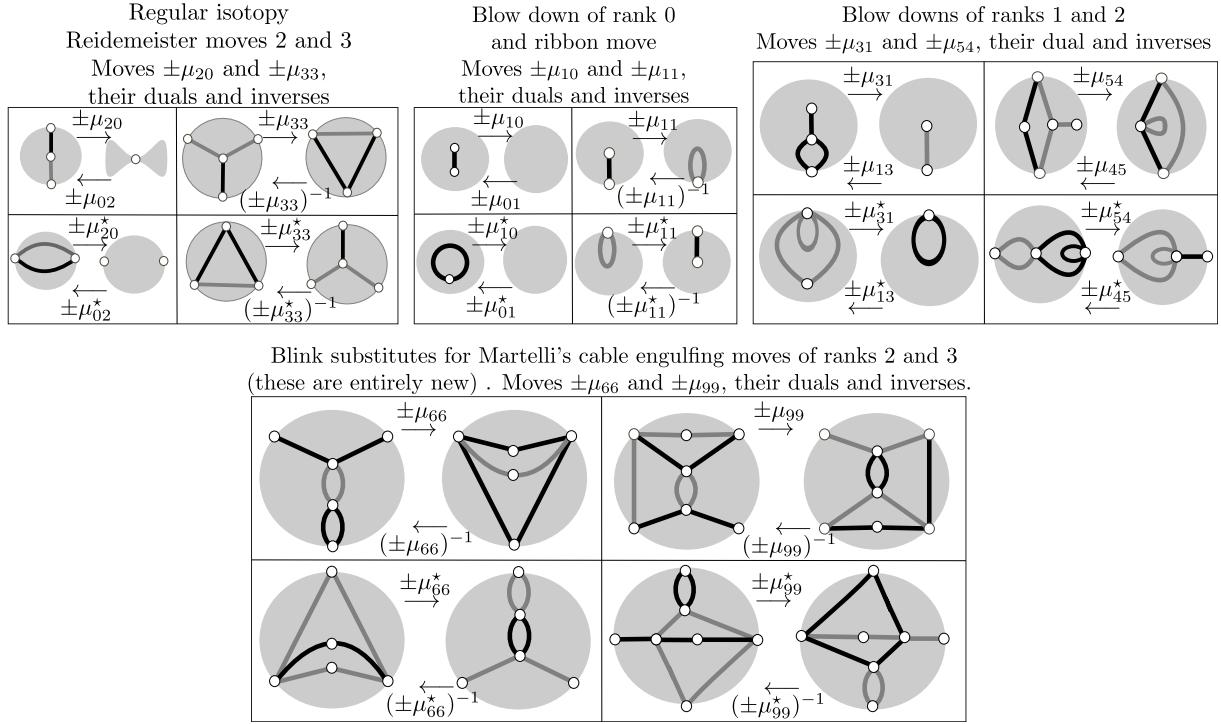
1.1 Statement of the Theorem

This paper proves the following theorem:

(1.1) Theorem. *The classes homeomorphisms of closed oriented connected 3-manifolds are in 1-1 correspondence with the equivalence classes of blinks where two blinks are equivalent if one is obtainable from the other by a finite sequence of the local moves where each term is one of the 64 moves (not*

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necessary distinct) below

Regular isotopy
Reidemeister moves 2 and 3
Moves $\pm\mu_{20}$ and $\pm\mu_{33}$,
their duals and inverses

Blow down of rank 0
and ribbon move
Moves $\pm\mu_{10}$ and $\pm\mu_{11}$,
their duals and inverses

Blow downs of ranks 1 and 2
Moves $\pm\mu_{31}$ and $\pm\mu_{54}$, their dual and inverses

Blink substitutes for Martelli's cable engulfing moves of ranks 2 and 3
(these are entirely new) . Moves $\pm\mu_{66}$ and $\pm\mu_{99}$, their duals and inverses.

There are 64 local configurations of sub-blinks, each named a *coin*, divided into 8 families of 8 simply related coins and also divided into 32 pairs of left-right coins, A move replaces the left (right) coin of a pair by its right (left) coin. Thus, the number of moves is equal to the number of coins. Boundary and internal vertices of the coins are shown as small white disks. The complementary sub-blink in the exterior a coin is completely arbitrary; its intersection with the corresponding internal coin is a subset of the set of attachment vertices in the boundary of the coin. The support of the coins are disks, except in the the right coin of μ_{20}, case in which is a pinched disk. The number k of the attachment vertices satisfies $k \in \{0, 1, 2, 3, 4\}$.

1.2   Organization of the paper

In Section 2 the motivation, the topological preliminaries and an epistemological view of the work are discussed. The proof of the Theorem 1.1 is given in Section 3. A reduced but sufficient form of this coin calculus having 36 coins (and moves) is obtained in Section 4. In Section 5 I display a census (no misses, no duplicates) of the closed, oriented, connected, prime 3-manifold induced by blinks up to 8 edges.

2   Motivation and topological preliminaries

2.1   Motivation

In his Appendix to part 0, J. H. Conway in his famous book *On Numbers and Games*, [2], says: *"This appendix is in fact a cry for a Mathematician Liberation Movement!*

Among the permissible kinds of constructions we should have:
(i) Objects may be created from earlier objects in any reasonable constructive fashion.
(ii) Equality among the created objects can be any desirable equivalence relation."

This paper is in the confluency of two deep research passions of the author, apparently very far apart: the topological study of closed orientable 3-manifolds and the combinatorial study of plane graphs. The result proved here provides a glimpse, in the spirit of Conway's quotation, to effectively enumerate once each closed orientable 3-manifolds. Blinks are easy to construct from simpler blinks, and their isomorphism problem can be solved by a polynomial algorithm which finds, via a few fixed conventions (lexicography), a numerical *code* for it. What can be more desirable than an equivalence relation on such simple mathematical objects that captures the subtle and difficult computational topological notion of factorizing any homeomorphism between two closed, oriented, connected 3-manifolds?

2.2   Topological preliminaries

This subsection contains the basic topological material that I need. It is primarely intended to the combinatorially oriented readers, unfamiliar with the fundamental definitions of knots, links and framed lins. The topological oriented readers also should read it because I introduce some unfamiliar notation and definitions which are new (like the flinks) and that will be used throughout the paper. Moreover, I present a short historic overview of the known results needed.

2.2.1   Knots and links into $\mathbb{R}^3$ and into $\mathbb{S}^3$

A *knot K* is an embedding of a circle, $\mathbb{S}^1$, into $\mathbb{R}^3$ (or into $\mathbb{S}^3$, the boundary of a 4-dimensional ball). The *unknot* is a knot which is the boundary of a disk. A *link with k components* is an embedding of a disjoint union of k copies of $\mathbb{S}^1$, $\left(\cup_{i=1}^{k}\mathbb{S}_i^1\right)$, into $\mathbb{R}^3$ ((or $\mathbb{S}^3$) with disjoint images. In this way, a knot is a particular case of a link: one which has one component. Fig. 1 shows a 1-1 correspondence $K \leftrightarrow K'$ between knots into $\mathbb{R}^3$ and into $\mathbb{S}^3$. By abuse of language, I feel free to identify a knot with its image.
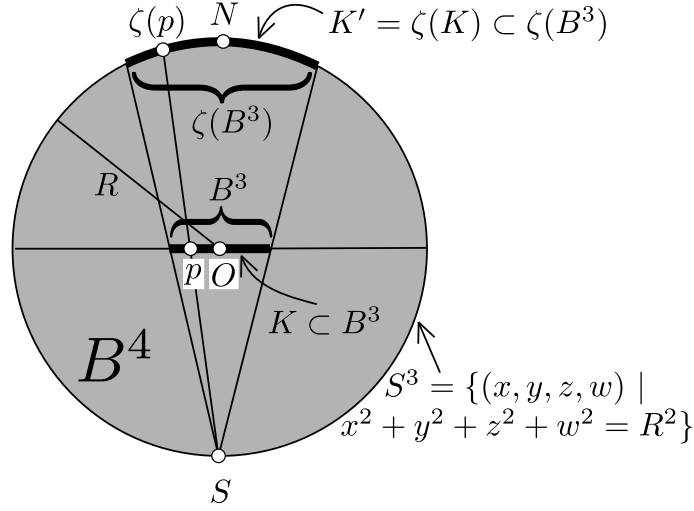
Figure 1: How a knot K into $\mathbb{R}^3$ is related to a knot K' into $\mathbb{S}^3$: the 3-sphere $\mathbb{S}^3$ is the boundary of a 4-ball $\mathbb{B}^4$; the knot K is contained in a ball $\mathbb{B}^3$ of radius r centered in the origin O and contained in the equator $\{(x, y, z, w) \in \mathbb{B}^4 \mid w = 0\}$; by making $\frac{R}{r}$ big knots K (which the south pole of the 4-ball) and K' are as close as being isometric as desired. K' is the image of K under the stereographic projection ζ centered at the south pole S. In this work I need to work with knots into $\mathbb{R}^3$ and into $\mathbb{S}^3$. Thus the easy correspondence between the two types is welcome.

Links can be presented with profit by their *decorated general position projections* into the xy-plane $\mathbb{R}^2$, by simply making 0 the z-coordinate. As in Fig. 1, the knot is inside $\mathbb{B}^3 \subset \mathbb{R}^3$, having made the 4th coordinate w equal to 0. Here *general position* means that in the image of the link there is no triple points and that at each neighborhood of a double point is the transversal crossing of two segments of the link, named *strands*. *Decorated* means that we keep the information of which strand is the upper one, usually by removing a piece of the lower strand. In this paper I use yet another way to decorate the link projections: the images of the link components are thich black curves and the upper strands are indicated by a thinner white segment inside the thich black curve at the crossing (see left side of Fig. 3).

2.2.2 Framed knots, ribbons, framed links and blackboard-framed links into $\mathbb{R}^3$

A *framed knot* is an embedding of $\mathbb{S}^1 \times [-\epsilon, +\epsilon]$ into $\mathbb{R}^3$ (or $\mathbb{S}^3$), for an arbitrarily fixed $\epsilon > 0$. A framed knot is also called *a ribbon*. The *base knot* of a ribbon is the ribbon restricted to $\mathbb{S}^1 \times \{0\}$. A *framed link* is a collection of ribbons with disjoint images. The ribbons that I use are prepared by isotopies so that their projections remain with constant width 2ϵ. Fig. 2 shows how to achieve this condition.
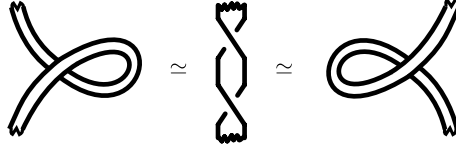
Figure 2: Getting a constant width immersion of a ribbon projection into $\mathbb{R}^2$. Each 360-degree rotation of the ribbon in the space is ambient isotopic to a curl in the ribbon (in two different ways) so as to maintain constant the width immersion of the ribbon projection. In particular, after making these isotopies, the intersection of the images of $\mathbb{S}^1 \times \{-\epsilon\}$ and $\mathbb{S}^1 \times \{+\epsilon\}$ have 4 distinct points (it used to have 2 points) near each crossing of the base link and there are no other crossings in the immersed ribbons.

Note that a blackboard-framed link is not a framed link. Indeed, the blackboard-framed is the base link of its framed link, see Fig. 3.
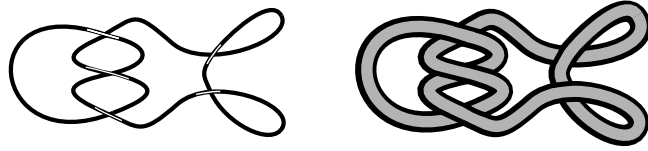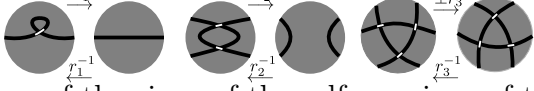

Figure 3: A knot and its corresponding framed knot (its ribbon) in an adequate projection, one which maintains the width of the ribbon. Taking the base knot of this ribbon produces back the blackboard-framed link, seeing on the left.

The isotopy class of a framed link is determined by assigning an integer to each component of the link which is equal to the *linking number* of the two components of the boundary of each of its ribbon oriented in the same way. The linking number can be obtained from an arbitrary decorated general position projection of the oriented band: it is equal to half of the algebraic sum of the ($\pm$)-signs of the crossings of distinct boundary components. The convention is that $\times_{\rightarrow +1,}$ $\times_{\rightarrow -1.}$ The linking number is an invariant, that is, it does not depend on the particular projection used. This is proved by K. Reidemeister in its 1932 book on Knot Theory, [20]. He isolates three local moves r_1, r_2, r_3 that are enough to finitely factor any arbitrary ambient isotopy between two decorated general position projections of the same link: . The *self-writhe of a projected link component* is the algebraic sum of the signs of the self-crossings of that component.
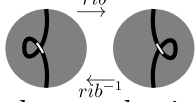
As a matter of fact, Theorem 1.1 which I proved in this work is the counterpart for 3-manifolds of Reidemeister Theorem. Note that in the coin calculus depicted together with flinks in Fig. 12 there are 8 versions of each of Reidemeister moves r_2 and r_3. Note also that Reidemeister move r_1 is not used at all. The equivalence class of decorated general position link projection generated by $r_2^{\pm 1}$ and $r_3^{\pm 1}$ is called *regular isotopy*. It plays an important role in the computation of the Jones invariants via Kauffman's bracket [9].

A *blackboard-framed link* is an *adequate projection* of a framed link prepared by isotopy so that the base link is projected as a decorated general position one and its ribbons are immersed with constant width 2ϵ, see Fig. 2. Moreover, we adjust the number of curls and their signs (in the base link) so that the self-writhe of each component coincides with the linking number of the two boundary components

of its ribbon. The advantage of the blackboard-framed link is that we no longer have to worry about assigning numbers to the components. These integer numbers are induced by the plane of projection. The importance of this concept was advocated by L. Kauffman in a number of works, including [10] and [11]. Next, I generalize it.

2.2.3   Flinks and its relation with blinks

A new object, defined in this work is a *flink*. It generalizes the notion of blackboard-framed link in an adequate way (an invariance under Kirby's handle sliding move) to be made clear, see Fig.6. Flink is a dutch word meaning *significantly*. It is also an acronym for residual **f**raction **link**. A *residual fraction* is either an irreducible fraction $\frac{p}{q}$ with $0 < p < q$, else $\pm 1/0 = \pm\infty$, else $0/1 = 0$. The *ribbon move* in a decorated general position link projection in $\mathbb{R}^2$ is the move defined by the following change in a pair of coins: $\overset{rib}{\longrightarrow}$ $\underset{rib^{-1}}{\longleftarrow}$. A *flink* is the equivalence class of fraction decorated general position link projections under regular isotopies and ribbon moves. *Fraction decorated* means decorated and having a residual fraction assigned to each component. A 0-*flink* is one so that each component has 0 as its residual fraction. Note that 0-flink and blackboard-framed link are the same concept. Fig. 4 shows that there is a 1-1 correspondence between 0-flinks and blinks.

The *surgery coefficients p' and q associated to a component of a flink* satisfy (by definition) $\frac{p'}{q} = w + \frac{p}{q}$ where w is the self-writhe of the component and $\frac{p}{q}$ is its residual fraction.
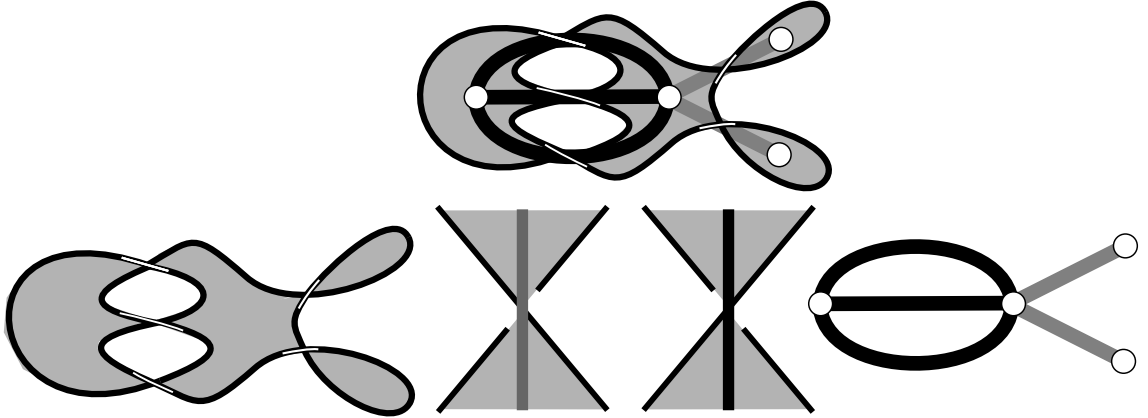

Figure 4: From 0-flink to blink and back: the projection of any link can be 2-face colorable into white and gray with the infinite face being white so that each subcurve between two crossing have their incident faces receiving distinct colors. The above figure shows how to transform a link projection into a blink (with thicker edges than the curves representing the link projection). The vertices of the blink are distinguished fixed points represented by white disks in the interior of the gray faces. Each crossing of the link projection becomes an edge in the corresponding blink, An edge of the blink is gray if the upper strand that crosses it is from northwest to southeast, it is black if the the upper strand that crosses it is from northeast to southwest. The inverse procedure is clearly defined. In fact, the link is the so called *medial map* of the blink. Thus we have a 1-1 correspondence between 0-flinks and blinks. The (complete) blink at the right induces Poincaré's sphere, the spherical dodecahedron space. The expressibility of a blink is powerful: quite complicated 3-manifolds are induced by simple blinks.

2.2.4 Lickorish's groundbreaking result

In a grounding breaking work (1962), W.B.R. Lickorish, [14], proved that any closed, oriented, connected $\mathbb{M}^3$ has inside it a finite number k of disjoint solid tori each in the form of a homeomorphic image of $\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta]$, denoted by $(\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta])_i$ so that, where $\mathbb{S}^3$ is the 3-dimensional sphere,

$$\mathbb{M}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta])_i = \mathbb{S}^3 \backslash \bigcup_{i=1}^{k} (\mathbb{S}^1 \times [-\epsilon, +\epsilon] \times [-\delta, +\delta])_i.$$

As a consequence, each closed oriented 3-manifold can be obtained from $\mathbb{S}^3$ by removing a subset of disjoint solid tori and pasting them back in a different way. The most general parameters to identify the pasting is a pair of integers for each component of the link, named *surgery coefficients*. A filling algorithm algorithm applies component by component, is named $(\pm p, q)$-*Dehn filling*, and is explained in Fig. 5. In this figure I explain how to get the surgery coefficients from the flink.
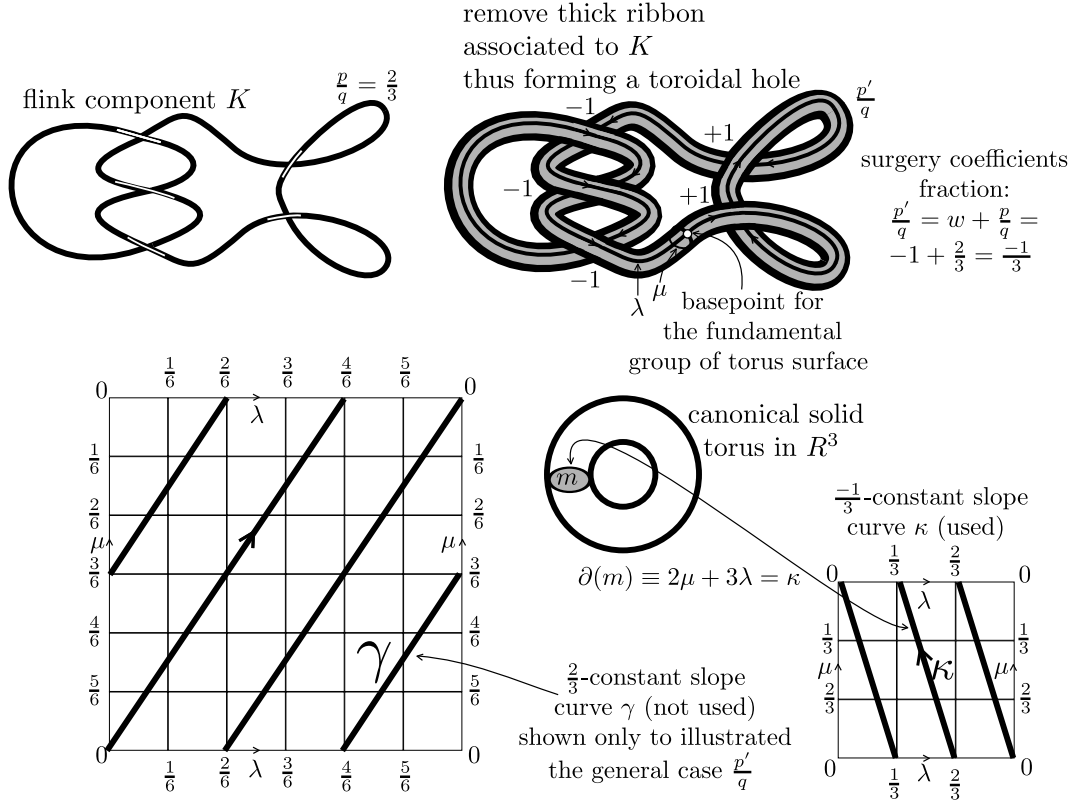
Figure 5: How to obtain a closed oriented connected 3-manifold from a flink in $\mathbb{S}^3$: remove the thick ribbons correponding to all components of the flink. Repeat the following (p', q)-Dehn filling for each component. Let $\lambda = K \times \{0\} \times \{+\delta\}$ and let μ be a closed curve that is contractible in the thick ribbon but not in its boundary. The pair of closed curves (μ, λ) form a basis for the fundamental group of the boundary of the thick ribbon. Orient μ arbitrarily and λ so that K equally oriented has linking number 1 with μ. Note that λ remains parallel to K and never touches the boundary of the ribbon. Let κ be the constant slope closed curve homotopic to $p\mu + q\lambda$. Consider a canonical solid torus embedded into $\mathbb{R}^3$ and let m be its meridian. Consider a homeomorphism that identify curves κ and the boundary of m, $\partial(m)$. This homeomorphism is univocally extensible to identify the boundaries of the thick ribbon and of the canonical solid torus, thus closing the toroidal hole: indeed, after identifying $m \times [-\zeta, +\zeta]$ (for a small $\zeta > 0$) in the solid torus and in the toroidal hole in the 3-manifold what remains to identify are two 3-balls. This has a unique solution up to isotopy.

For proving the theorem, and this is important in the present work, the fillings used by Lickorish satisfy $q = 1$. He call these surgeries *honest*. Actually, Lickorish's result had been proved 2 years before by A. H. Wallace [28] by using differential geometry. However it was the purely topological flavor of Lickorish's proof that spurred the subsequent developments. Also, Lickorish does not state his theorem in this way. The form I use for the solid tori is convenient because of its simple relation with flinks. It is inspired in the lucid account by J. Stillwell of the Lickorish's theorem given in [25]. Each each solid torus to be removed from $\mathbb{S}^3$ is a thich ribbon $K \times [-\epsilon, +\epsilon] \times [-\delta, +\delta]$, where K is (the image of) a component of the flink, with fixed small positive constants $\epsilon > \delta > 0$. Each section $k \in K$, $\{k\} \times [-\epsilon, +\epsilon] \times [-\delta, +\delta]$, is a rectangle whose δ-sides are parallel to the z-axis. Thus the projections of the thick ribbon and the of the ribbon coincide.

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2.2.5 Kirby's famous calculus of framed links

In 1978 R. Kirby published his, to become famous, calculus of framed links, [12]. The gist of this paper is that two types of moves are enough to go from any framed link inducing a closed oriented 3-manifold to any other such link inducing the same manifold. One of the moves is absolutelly local: creating or cancellating an arbitrary new unknotted component with frame ± 1 separated from the rest of the link by an $\mathbb{S}^2$. The other type of move, *the band move*, [12, 10]) *(or handle sliding)* is non-local and infinite in number.
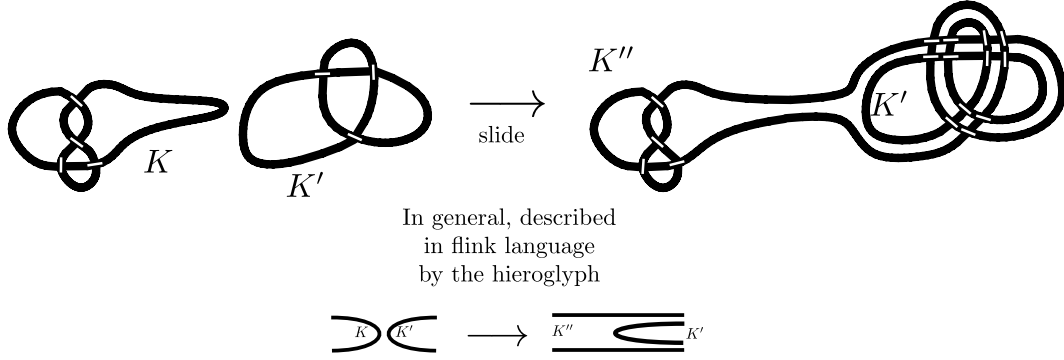

Figure 6: Kirby's *handle slide move*, also known as the *band move*. In this work the handle slide move is applied only to flinks, so there is no issue about surgery coefficients. Given K and K' two distinct components of a flink start by making a close parallel copy of K' in such a way that it forms an immersed band with its originator K. Let K'' be the connected sum of K and the copy of K'. The connected sum is defined by a em new band which is a thin rectangle arbitrarily embedded into $\mathbb{S}^3$, so as to miss the link. The short sides of this band are attached to K and to the copy of K' and then, recoupled in the other way. The new band can be quite complicated because it may wander arbitrarily (as long as it misses the link) in $\mathbb{R}^3$ in its way to connecting the two components. More details in Kauffman's book, [10]. The property of flinks that made me introduce the concept is that the **residual fractions of the flink are invariant under the Kirby's band move.** This move then can be depicted in all its generality, via the hieroglyph shown in the bottom part of the Figure. See Section 12.3 of [11]. However, this hieroglyphic move is non-local since the exterior of the hieroglyph changes, and beacause there are infinite exteriors, there are infinite Kirby's band moves. Fenn-Rourke reformulation, treated next, provide an infinite sequence of trully local moves.

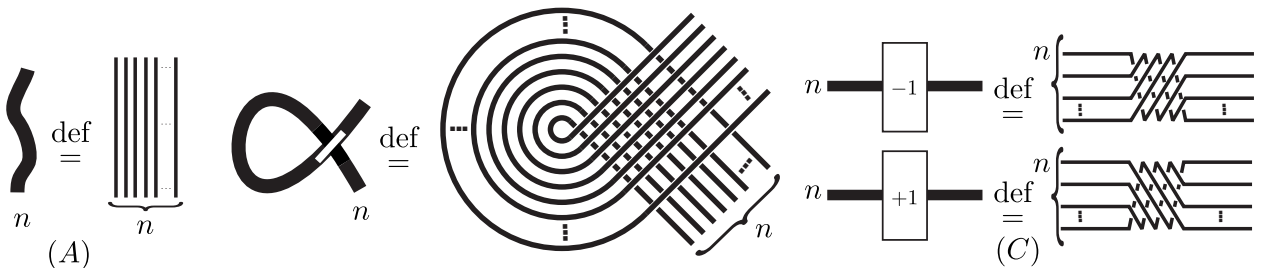### 2.2.6 Fenn-Rourke reformulation of Kirby's calculus

Figure 7: Notation for special disk neighborhoods of general position decorated link projections

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In 1979 R. Fenn and C. Rourke ([7]) show that Kirby's moves could be replaced by an infinite sequence of a single type of move (a *blow down move*) indexed by n, which I depict at the left side of Fig. 8. In a blow down move the number of components decreases by 1. This has been a very useful reformulation with many applications, including Martelli's calculus (soon to be treated) which uses it instead of the direct moves of Kirby.
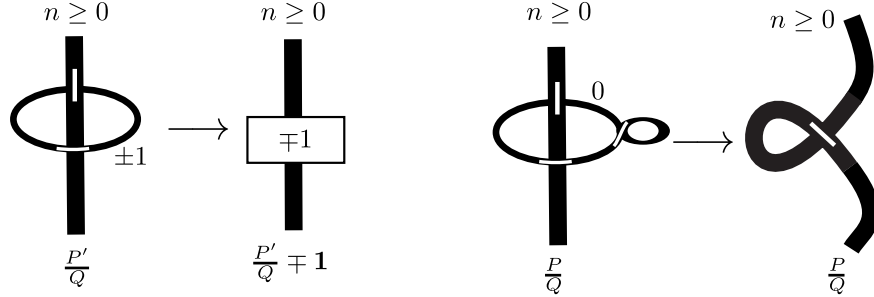

Figure 8: Fenn-Rourke infinite sequence of blown-down moves on fraction decorated links into $\mathbb{R}^3$ and their counterpart in terms of flinks, which generalizes Kauffman's blackboard-framed links. By definition, $\frac{P'}{Q} = \left(\frac{p_1'}{q_1}, \ldots \frac{p_n'}{q_n}\right)$ and $\frac{P}{Q} = \left(\frac{p_1}{q_1}, \ldots \frac{p_n}{q_n}\right)$, where $\frac{p_i'}{q_i}$ are generalized fractions associated with the i-th component of the link in $\mathbb{R}^3$ and $\frac{p_i}{q_i}$ is the associated residual fraction of the link projection into $\mathbb{R}^2$. The blown-downs constitute an infinite sequence of local moves. The special case $n = 0$ of these moves is also defined and replace an isolated ± 1-framed component of the unknot with one crossing by nothing. Martelli replaced the infinite sequence by the first three and two new simple moves A_3 and A_4, described in Fig. 9.

2.2.7 Kauffman's idea to let the plane induce the integer framing

In the beginning of the 1990's L. Kauffman presented ([10]) a completely planar diagramatic way to deal with the calculus of Kirby and its reformulation by Fenn and Rourke. The basic idea comes from the fact that every 3-manifold is induced by surgery on a framed link which has *only finite integer framings*. This characterize the *handle surgeries*. According to Rolfsen Lickorish call each of these a *honest surgery*, page 262 of [21]. The proof that we can get any manifold by surgery on integer framed links uses, as a lemma, the fact that it is possible to modify the framed link maintaining the induced 3-manifold so that every component becomes unknotted. A proof of this lemma appears 2.2.10. If a component is unknotted then it is simple to modify the link so that each component gets an integer framing, without distub the integrality of the framing of other components. See Theorem 2.2. So, without loss of generality we may suppose that all the components have finite integers as framings. Kauffman's proceeds by adjusting each component by attaching to it a judicious number of curls so that the required framing of a component coincides with the algebraic sum of its self-crossings. By specifying that the link is *blackboard-framed*, we no longer need the integers to specifies the framing. They are a consequence. In this work I only use only blackboard-framed projections or their generalization, flinks Flink is a convenient generalization of blackboard-framed link because their residual fractions remain constant in Kirby's calculus, in Fenn-Rourke reformulation, and in Martelli's calculus, treated next.

2.2.8 Martelli's finite calculus on fractionary framed link

In an important recent paper B. Martelli [19] presented a local and finite reformulation of the Fenn-Rourke version ([7]) of Kirby's calculus [12]. This calculus is presented in Fig. 9. It remains to be

seen the consequences of Martelli' s result for obtaining new 3-manifold invariants. A possible door for obtaining such invariants are generalizations of the combinatorial approach to get WRT-invariants, justified in [11] and extensely used in [17] and in [16]. To find such a generalization one has to take advantage of the specific sufficient local Martelli's calculus now available (or the coin calculus on blinks). The WRT-invariants are obtained by hiding in the Temperley-Lieb algebra the infinite cases of Kirby's band move, as pioneered by Lickorish in [15]. See also page 144 of the join monography of L. Kauffman and myself, [11]. Finding such generalization of the WRT-invariants still seems to be a formidable task. However, Martelli's theorem and the coin calculus on blinks makes it conceivable.
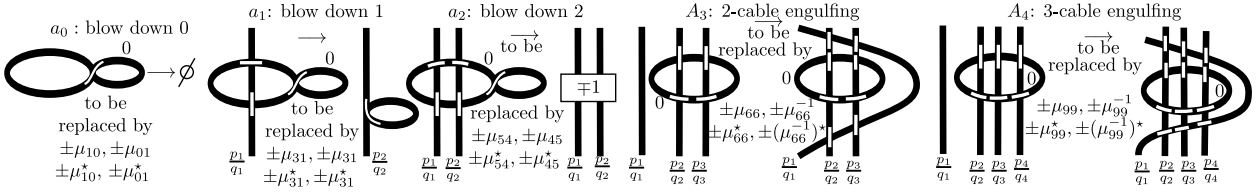

Figure 9: Martelli's calculus on flink language: Note that for the internal components of all the above moves the residual fraction are 0. Martelli' proves that by keeping only the blown-down of ranks 0, 1 and 2 and replacing all the remaining infinite sequence by two new moves, which I call 2- and 3-cable engulfing (denoted by A_3 and A_4), a sufficient calculus for factorizing homeomorphisms between closed oriented and connected 3-manifolds is achieved solely in terms of the above 5 local moves, their inverses, the regular isotopies moves and the ribbon moves. Moves A_3 and A_4 do not translate into blink moves because their left sides are disconnected. What makes this work possible is the replacement of these non-connected configurations by equivalent moves a_3 and a_4 so that blink translations become available. The equivalences $a_3 \equiv A_3$ and $a_4 \equiv A_4$ are proved in Fig. 11 where the diagrams for the moves appear right angle rotated relative to this figure. Observe that, in the flink language, all the residual fractions of Martelli's calculus remain invariant.

2.2.9   Unknotting a component

Any knot given by an decorated general position projection can be unknotted by a subset of crossing switches: starting in a non-crossing and going along the knot make sure that the first passage through a crossing becomes an upper strand (by switching the crossing if necessary). The result is clearly a projection of an unknot. Let F be a flink and F_i be one of its component which is knotted. It is possible to modify F at the cost of introducing new unknotted components so that F_i becomes unknotted and the induced 3-manifold does not change. This is a consequence of the above algorithm to unknot any knot and of Kirby's calculus on flinks, as shown in Fig. 2.2.10, which is adapted to fit the flink language, from [11].
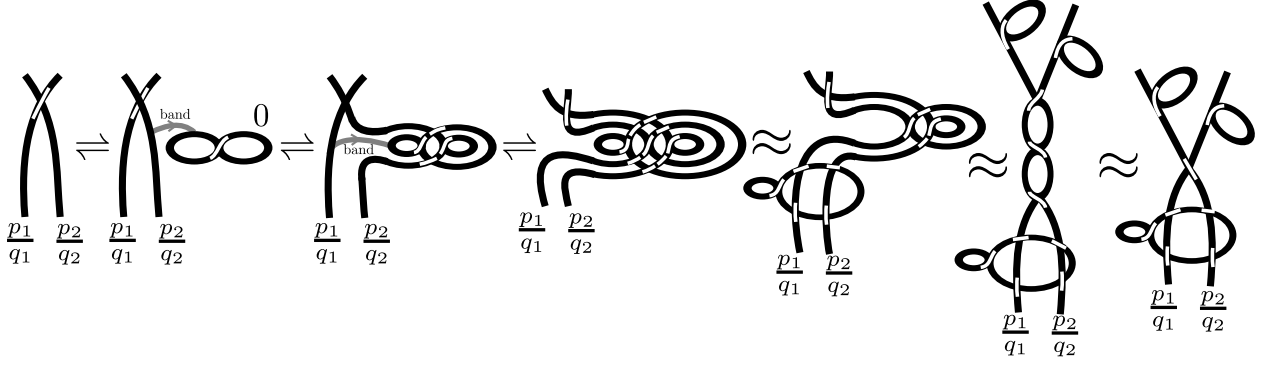
Figure 10: Switching a crossing at the cost of adding an unknoted component and some curls: this is achieved by applying Kirby's move of type 1 and twice Kirby's moves of type 2 as well as some isotopies. Note that the residual fractions stay invariant.

2.2.10 Obtaining a 0-flink inducing the same manifold as any input flink

Let $F = F_1 \cup F_2 \cup \ldots \cup F_k$ be a flink and let r'_j be the fraction of the surgery coefficients given, as already defined, by the sum of the residual fraction r_j and the self-writhe of the j-th component F_j. Let F_i be an unknotted component.

(2.1) Lemma. *The two following operations on a flink F maintain the induced 3-manifold:*

- *Create or cancell a component with residual fraction $\pm\infty$.*

- *Effect a t-full-twist about F_i changing judiciously the link and the surgery coefficient fractions r'_k's.*

    The surgery coefficient fractions change as follows:

    Component F_i of the twist: $r''_i = 1/(t + (1/r'_i))$

    Other components F_j: $r''_j = r'_j + t(\ell kn(F_i, F_j))^2$.

    The link changes as follows:

    Effect t positive or negative full twists, according to the $\mp$-sign of t, in the cable of parallel lines encircled by F_i.

Proof. The proof of this result is given in [21]. □

(2.2) Theorem. *Given any flink, there exsits a 0-flink inducing the same 3-manifold obtainable by a polynomial algorithm in the product of its residual fraction denominators.*

Proof. Let F_i be an knotted component of a flink F which have a non-null residual fraction $\frac{p}{q}$. Switch crossings as in so that F_i becomes unknotted, with the same residual fraction $\frac{p}{q}$ and respective surgery coefficient fraction $\frac{p'}{q}$. If $p' = \pm 1$ then use Lemma 2.2 with $t = \mp q$. The surgery coefficients fraction of F_i becomes $r''_i = 1/(t + (1/r'_i)) = 1/(\mp q + \pm q) = 1/0 = \infty$. And component F_i can be removed. Let $|p'| > 1$. We have $r''_i = 1/(t + (1/r'_i)) = 1/(t + (q/p'))$. Define t to be the integer so that $0 < t + q/p' = q'/p' < 1$. The new surgery coefficient fraction of F_i is $r''_i = p'/q'$, with $q' < q$. Repeating the procedure a number of times bounded by q we arrive at $q' = 1$ or $q' = 0$. Observe that the surgery coefficient fractions of the new components are integers and the one of the components F_j linked with F_i change by an integer. Thus we get to a flink whose all surgery coefficients fractions have $q = 1$. Therefore, by curl adjusting, a 0-flink inducing the same 3-manifold is obtained. □

2.3 This work from an epistemological point of view

I proceed by further reformulating Martelli's moves so as to obtain a calculus of blinks, denominated *coin calculus*. It is is an exact combinatorial counterpart for factorizing homeomorphisms of closed, orientable, connected 3-manifolds. It has the consequence that each 3-manifold becomes a subtle class of plane graphs. This exposition has been and will remain complete and elementary. It seeks to reach both audiences: Topologists and Combinatorialists. We feel that this result may be interesting for Combinatorics as well as to Topology and may enhance both areas: conceivably, some deep properties of plane graphs could be used to elucidate aspects of 3-manifolds and vice-versa.

Plane graphs are one of the most studied objects in Combinatorics. The role of planarity in finding polynomial efficient algorithm is well established. For instance, the *Max Cut Problem*,([8]), an NP-complete problem, becomes polynomial, if the graph is in the plane. This was a consequence of J. Edmonds's optimal maximum matching theory polyhedral theory: [4, 6]. Other NP-complete problems like the *Max Stable Vertex Set Problem*([8]), remain NP-complete when restricted to plane graphs. Also well established is the the role of plane graphs motivating and permitting useful generalizations in matroid theory, [27]. Matroids are a source of polynomial algorithms. A. Lehman used this theory to provide a solution for the Shannon Switching Game [13]. This solution was enhanced to a polynomial algorithm by J. Edmonds in [3]. Plane graphs and this paper were the motivation for his unexpected and amazing algorithm for polynomially finding a matroid partition into indepedent subsets, [5], with its various applications to scheduling problems. Yes, I do believe in *One Mathematics*, as advocated by L. Lovasz, in his famous essay, [18]. The area of combinatorics, particularly the area of efficient polynomial algorithms based on polyhedral methods had, ten years ago, its maturity declared by means of the publication of its *Magnum Opus*, in three volumes with more than 1800 mathematically dense pages, by A. Schrijver, [22, 23, 24]. It is my hope that some aspects of the polyhedral theory may have consequences on 3-manifolds algorithmic theory.

Two interesting open questions relating plane graphs and 3-manifolds are: (1) Which 3-manifolds correspond to the class of 3-connected monochromatic blinks? I have reasons to believe that the pair {blink, dual blink} (and the associated {graphic-cographic} matroids) is a complete invariant for these manifolds: a census of all the 242 blinks which are 3-connected, monochromatic and have up to 16 edges appear in [16]. In the domain of the census, the pair {graphical/cographical} matroids is a complete invariant. (2) The theorem established in this work brings closer of being true an old quest of mine: is there a way to associate a matroidal invariant to a general closed, orientable and connected 3-manifold?

Eleven years ago, I. Agol, J. Hass and W. Thurston proved that *3-manifold knot genus is NP-complete*, [1]. There seems to be relatively few results along this line. This one in particular suffer from the fact that it is very difficult to work combinatorially and to visualize a knot in an arbitrary 3-manifold. I think that discovering more NP-complete problems in 3-manifolds could arise from the result here presented. After all NP-complete problems abound in plane graphs. In my wildest dreams I see myself showing that reformulating one of these plane graph problem into the corresponding 3-manifold problem is polynomially solvable by topological means.
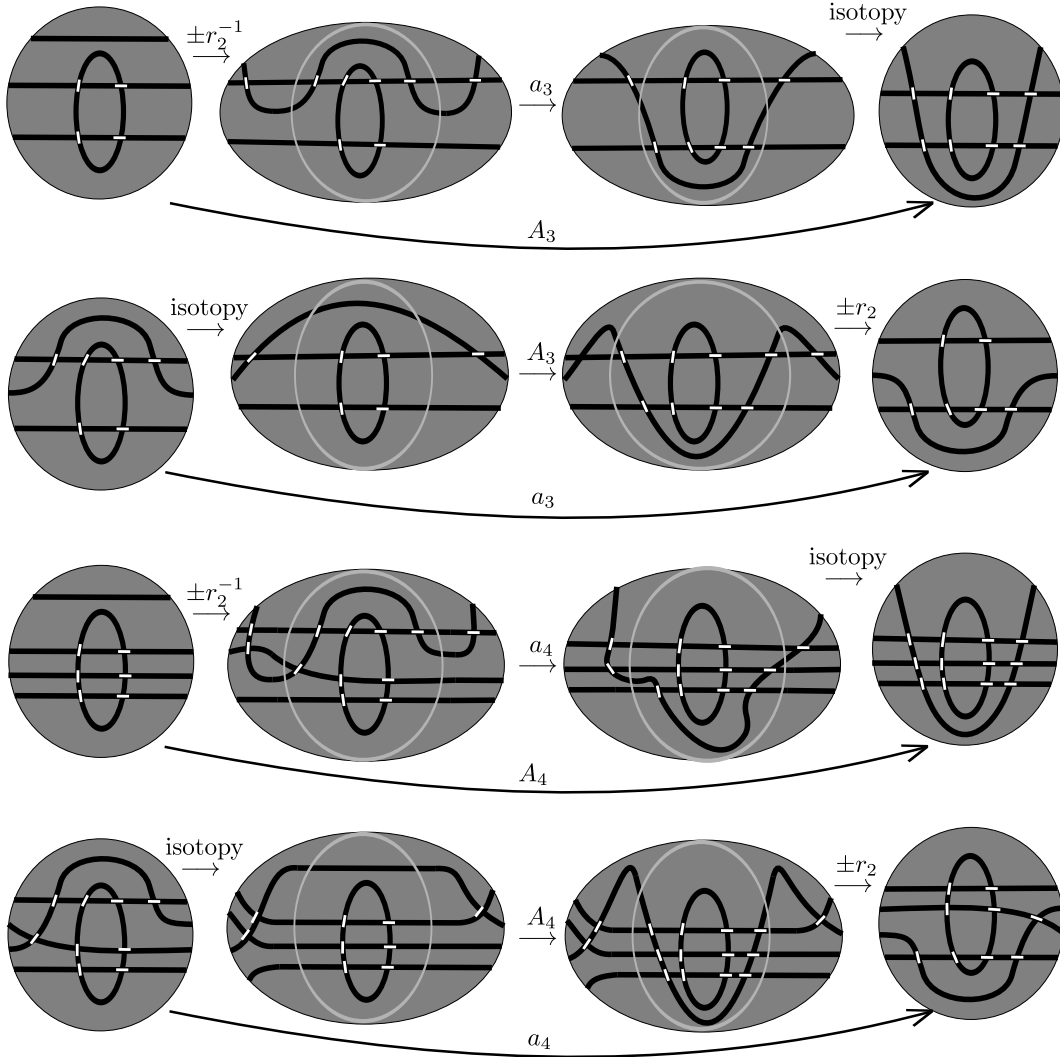
3 Proof of the Theorem

Figure 11: A proof that in the presence of $\pm r_2^{\pm 1}$, the equivalences $a_3 \equiv A_3$ and $a_4 \equiv A_4$ hold

(3.1) Lemma. *In the presence of Reidemeister moves 2, generaly denoted by $\pm r_2^{\pm 1}$, moves $\pm a_3$ and $\pm A_3$ are equivalent and so are moves $\pm a_4$ and $\pm A_4$.*

Proof. We refer to Fig. 11. Its first line proves that $\pm a_3 \Rightarrow \pm A_3$. The second line proves that $\pm A_3 \Rightarrow \pm a_3$. The third line proves that $\pm a_4 \Rightarrow \pm A_a$. The last line proves that $\pm A_4 \Rightarrow \pm a_4$.   □

Proof. (**of Theorem 1.1**)
In Fig. 12 we draw all the moves for the revised Martelli's moves on pairs of distinctly 2-colored 0-flinks and the respective blinks superimposed. The result follows by removing the 0-flink moves leaving only the blink moves which are redrawn up to isotopy, in the lower part of the figure.   □
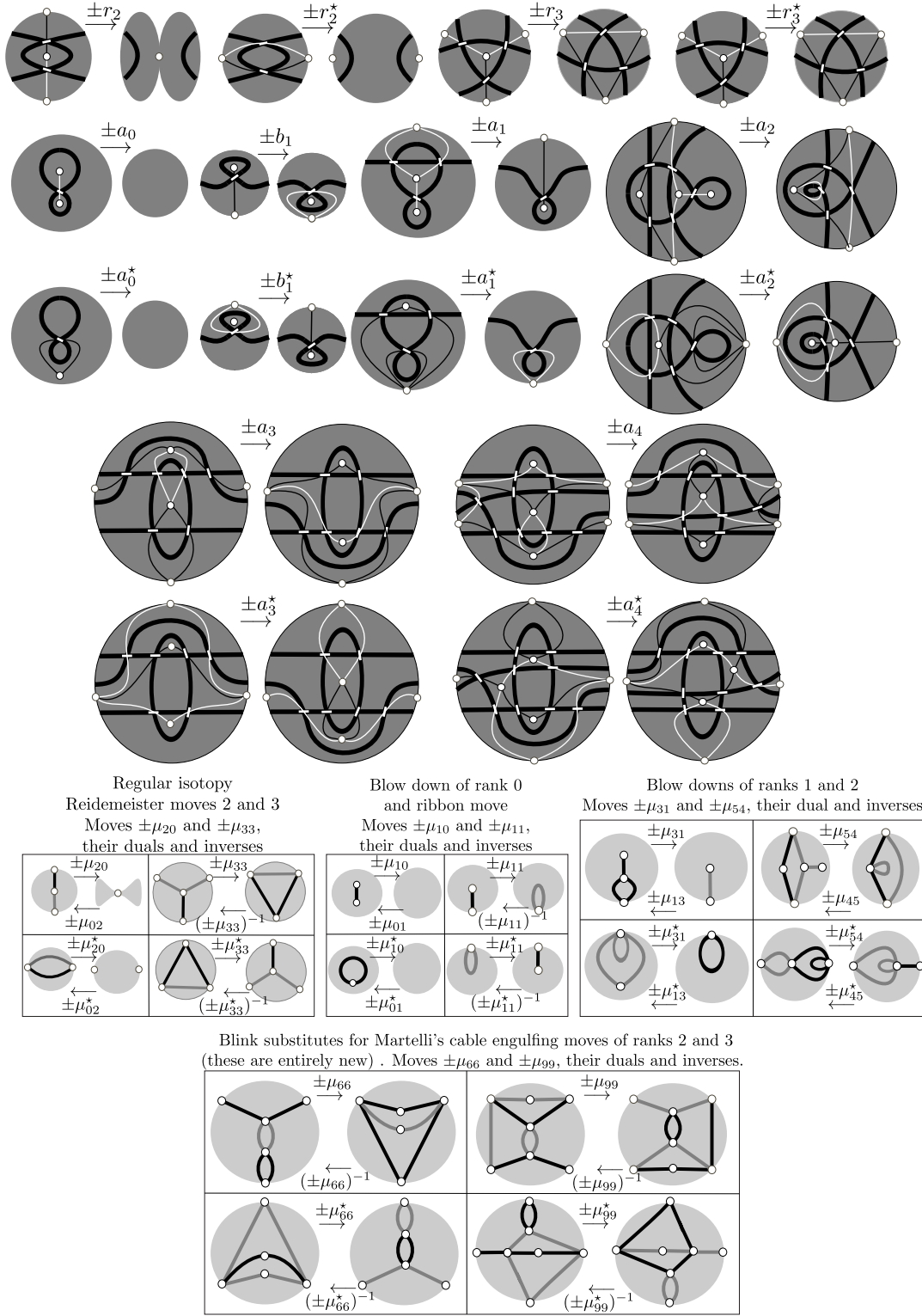
Figure 12: 0-Flink and blink versions of Martelli's revised calculus with a_3, a_4 replacing A_3, A_4: In the upper part of the figure, distinctly 2-face colored 0-flinks and respective blinks are superimposed. This implies the moves for the coin calculus on blinks in the lower part of the figure, concluding the proof of the Theorem 1.1.

4 The role of the ribbon moves $\pm\mu_{11}^{\pm 1}$

The counterpart of the ribbon moves in the coin calculus, (also called ribbon moves) $\pm\mu_{11}^{\pm 1}$ are redundant because Martelli's calculus is in $\mathbb{R}^3$. We include them in the coin calculus because with their inclusions all the dual moves, except $\mu_{20}^\star$ and $\mu_{02}^\star$, become redundant.

(4.1) Lemma. *Let f be the external infinite face of a decorated general position link projection and g be a face adjacent to f. Then it is possible to interchange f and g by means of regular isotopies and one ribbon move.*

Proof. The proof is given in Fig. 13. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$
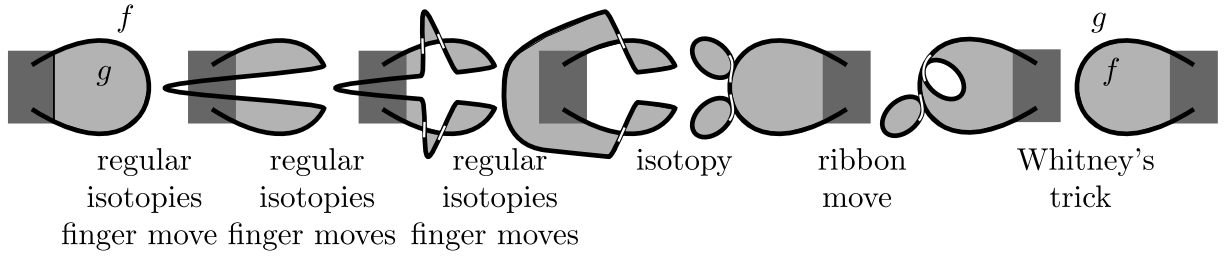

Figure 13: Changing the external face f to be any one of its adjacent faces g by means of r_2, r_3 (regular isotopies) and the ribbon move: the passage from the first to the second configurations is a *finger move* where a small segment of an edge is arbitrarily deformed as a *finger* passing over all the crossings of the dark gray rectangle; the finger move can be factored by regular isotopies; by other four finger moves one can go from the second to the third to the fourth configuration; the passage from the fourth to the fifth configuration is simply an isotopy; from the fifth to the sixth is a ribbon move; finally, the passage from the sixth to the seventh is acomplished by Whitney's trick, depicted in the proof below; Whitney's trick also factors as regular isotopies, as shown in the proof of Corollary 4.2. The net effect is to interchange the faces f and g: the initial infinite face is f and, at the end, the infinite face is g.

(4.2) Corollary. *The coin calculus can be simplified to include only the following set of 36 moves* $\{\pm\mu_{20}, \pm\mu_{02}, \pm\mu_{20}^\star, \pm\mu_{02}^\star, \pm\mu_{33}^{\pm 1}, \pm\mu_{01}, \pm\mu_{01}, \pm\mu_{11}^{\pm -1}, \pm\mu_{31}, \pm\mu_{13}, \pm\mu_{54}, \pm\mu_{45}, \pm\mu_{66}^\pm, \pm\mu_{99}^\pm\}.$

Proof. We work in the language of flinks. Moves corresponding to $(\pm\mu_{11}^{\pm 1})^\star$ and $(\pm\mu_{33}^{\pm 1})^\star$ are redundant because $\pm\mu_{11}^{\pm 1}$ $\pm\mu_{33}^{\pm 1}$ are self dual. Moves $\pm\mu_{10}^\star$ and $\pm\mu_{01}^\star$ are implied by a combination of moves $\pm\mu_{10}$, $\pm\mu_{01}$, $\pm\mu_{11}$ and $(\pm\mu_{11})^{-1}$. In particular, all the Reidemeister 2 and 3 moves (regular isotopy) are at our disposal. We can use this fact to change the external face of the link diagram to become any chosen adjacent face by using regular isotopy at the cost of creating two curls adjacent in the same component with distinct sign and the same rotation number. Use the the appropriate ribbon move to obtain two curls with the distinct signs and distinct rotation number. Now apply Whitney's trick which cancel these opposite curls by using $r_2^{\pm 1}$ and $r_3^{\pm 1}$ moves (regular homotopies): . The net effect in the correponding final blink is that it is obtained from the initial blink by dualizing and interchanging black and gray edges. Having this double involutions at our disposal it is straighforward to obtain all the remaining dual moves. $\qquad\qquad\square$

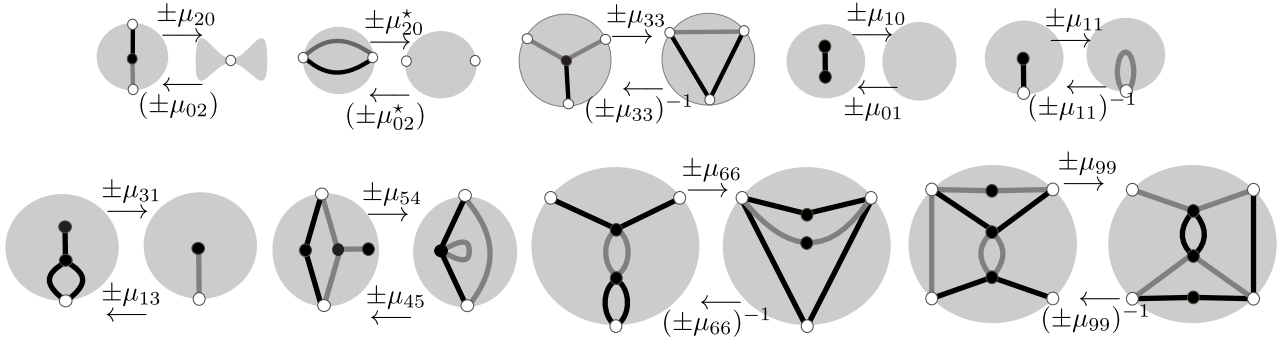In Fig. 14 I present the 36 moves of the final reduced coin calculus.

Figure 14: The 36 moves (and 36 coins not necessarily distinct) defining the reduced coin calculus.

5 Conclusion

I finish this work by presenting below a complete (no misses, no duplicates) census of the k-small 3-manifolds, for $k = 8$. These are the closed, oriented, connected and prime 3-manifolds induced by a blink with at most k edges.
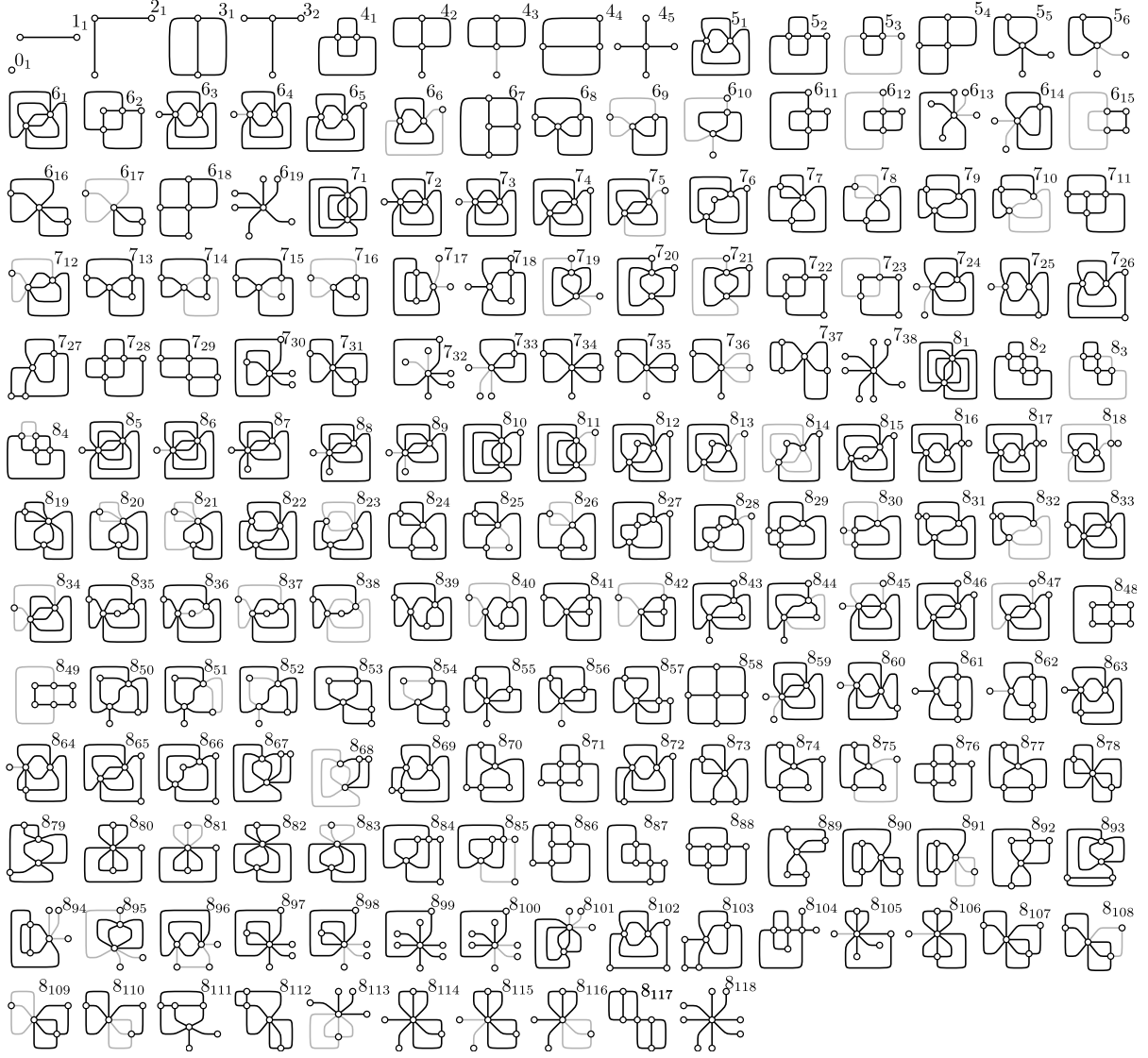
Figure 15: A complete census (no misses, no duplicates) of 8-small prime 3-manifolds. They correspond to the first (by lexicography) 191 closed, oriented, connected and prime 3-manifolds. These are such 3-manifolds which are induced by blinks up to $k = 8$ edges. The *sequence of WRT-invariants* of a closed oriented connected 3-manifold is an infinite sequence of complex numbers, indexed by $r \geq 3$. The r-th WRT-invariant of a manifold is directly obtained from a blink inducing it. An entirely combinatorial recipe directly implementable to compute the WRT-invariants of a 3-manifold from a blink inducing it is given in Chapter 7 of [17]. This recipe, in its turn is justified by at the very basic level, also by the combinatorial theory developed in [11]. Recall that a blink is a finite plane graph with an (arbitrary) edge bipartition. Such census are possible by completely combinatorial methods: we generate a subset of blinks that misses no 3-manifold by lexicography and the theory in [16]; then we compute the homology and the WRT-invariants; at this level $k = 8$ these two invariants are seen to be complete. There are 3 independent implementations of to obtain the Kauffman-Lins version (depending upon the Temperley-Lieb algebra) of the WRT-invariants. They were implemented by different people, in non-overlapping times: S. Lins (1990-1995), S. Melo (1999-2001) and L. Lins (2006-2007). The results agree. The software BLINK ([16]) computes the WRT-invariants and the above census by exhaustive generation of blinks. BLINK also draws links and 0-flinks in a grid, as a full strength application of network flow theory, using Tamassia's algorithm [26]. BLINK is currently the object of an open source Github project, and is available upon request.

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