Fields, Meadows and Abstract Data Types

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To Boaz, until 120!

Abstract. Fields and division rings are not algebras in the sense of "Universal Algebra", as inverse is not a total function. Mending the inverse by any definition of 0^{-1} will not suffice to axiomatize the axiom of inverse $x^{-1} \cdot x = 1$, by an equation. In particular the theory of fields cannot be used for specifying the abstract data type of the rational numbers.

We define equational theories of *Meadows* and of *Skew Meadows*, and we prove that these theories axiomatize the equational properties of fields and of division rings, respectively, with $0^{-1} = 0$. Meadows are then used in the theory of Von Neumann regular ring rings to characterize strongly regular rings as those that support an inverse operation that turns it into a skew meadow. To conclude, we present in this framework the specification of the abstract type of the rational numbers, as developed by the first and third authors in [2]

1 Universal Algebra

Model theory is the study of general structures and their logical properties. The theory of Algebra studies the behavior of operations on a set. Abstract data types are syntactical objects that were created for representing mathematical objects to the computer in a language that it understands. The theory of Universal Algebras is the playground where these seemingly unrelated disciplines meet and interact.

Wikipedia defines Universal Algebra by: "Universal algebra studies common properties of all algebraic structures,... the axioms in universal algebra often take the form of equational laws" [15]. More traditional sources on the subjects are [1,3,4,7,10]. In Universal Algebra, as in general model theory, we specify a signature that declares the kind of constructs that are investigated. For universal algebras there are only constant names and function names with prescribed arity (and no relation names). We are interested in the class of all the structures where these names are interpreted as individuals and functions of the appropriate arity. Next we specify a collection T of (usually finitely many) equations among terms with variables. The class of algebras (models) for these equational theory, is the class $\mathcal{M}(T)$ of structures for which the equations hold universally.

There are two major features to an equational theory and its models:

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– The class $\mathcal{M}(T)$ is closed under substructures, under homomorphism, and under Cartesian products. In particular every algebra has a minimal subalgebra, in which every element interprets a fixed term (without variables). An algebra in which every element interprets a term is called *a prime algebra*.
– Among the prime algebras there is one that is called the *initial algebra*, and which is maximal among the prime algebras with respect to homomorphisms: every prime algebra is a homomorphic image of the initial algebra, via a unique homomorphism (in particular, the initial algebra is unique up to isomorphism). A canonical way to represent the initial algebra is as the term algebra modulo the congruence generated by the equations in the theory.

Here are some examples:

1. The signature has one constant "e" called "the unit element" one binary operation "$\cdot$" called "product", and one unary operation called "inverse". T_1 is the theory of groups, with the three equations: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $e \cdot x = x$ and $x^{-1} \cdot x = e$. The theory T_2 includes in addition the equation $x \cdot y = y \cdot x$. $\mathcal{M}(T_1)$ is the class of groups. $\mathcal{M}(T_2)$ is the class of commutative groups. The initial algebra in both cases is the trivial group, since the equations force all the terms to be equal to e.
2. Add to the signature n new constants, and don't change the axioms. The two new classes $\mathcal{M}(T_3)$ and $\mathcal{M}(T_4)$ are the same classes as $\mathcal{M}(T_1)$ and $\mathcal{M}(T_2)$, with arbitrary (not necessarily distinct) elements interpreting the new constants. The initial algebras are the free group with n generators, and the free commutative group with n generators, which are just the term algebras modulo the congruence generated by the group equations.
3. The signature has two constants 0 and 1, two binary operations, addition $+$ and multiplication $\cdot$, and one unary operation, "minus", $-$. T_5 is the set of ring equations, and T_6 is the set of equations for commutative rings. The classes of algebras are the class of rings and the class of commutative rings, respectively.

The initial algebra in both of the classes T_5 and T_6 is the ring Z of integers: It is easy to prove by structural induction on the constant terms (without variables) that the equational theory implies that each is equal to 0, some $\underline{n}$ or some $-\underline{n}$, where $\underline{n}$ is a sum of n copies of 1.

2 Algebraically Specified Abstract Data Types

The last example is suggestive: The initial structure for the ring equations is the structure Z of the integers. Every integer has a name in the language (actually many names), and the operations of addition, multiplication, and subtraction have names in the language. The syntactical rules for generating constant terms correspond to applying the operations on the integers. Thus we specified a notation system for the integers and their operations, in the setting of a universal algebra with its initial algebra. This is nice, as the equational theory on the one hand, and all the models of the theory, whether prime or general, can be used to study and better understand the structure that is behind the formal definition of the data type. Algebraic specification of abstract data types started with [5]; recent surveys are [8,9]; [16] is a comprehensive introduction to the subject.

Here are two more equational theories whose initial algebras are interesting data types:

Example 1 (Equational number theory). The signature has one constant "0", one unary function symbol, the successor symbol "S", and two binary function symbols for addition and multiplication. The equational theory says:

1. $x + 0 = x$
2. $x + S(y) = S(x + y)$
3. $x \cdot 0 = 0$
4. $x \cdot S(y) = x \cdot y + x$

(Following the formal rules of term construction would involve including more parentheses in each of the terms, but we do not bother).

For every n we denote by $\underline{n}$ the term

$$\underbrace{SS \cdots S}_{n \text{ times}}(0).$$

The proof that the initial algebra is the set of natural numbers, with the operations that we named, is done in two steps. First we show by induction on natural numbers that $\underline{n} + \underline{m} = \underline{n+m}$ and $\underline{n} \cdot \underline{m} = \underline{n \cdot m}$. Then we show by structural induction over closed terms that every closed term is equal to some $\underline{n}$ in every algebra in the class.

We can extend the signature by additional operations, and add equations that will ensure that the initial algebra is the set of natural numbers with the familiar operations. For example we can add predecessor P and the two equations:

$$P(0) = 0 \quad \text{and} \quad P(S(x)) = x$$

Or we can add the exponential function $E(x, y)$ and the equations:

$$E(x, 0) = S(0) \quad \text{and} \quad E(x, S(y)) = E(x, y) \cdot x$$

We note that this approach raises interesting questions about the types that are specified, and the class of models of the theory that specifies them. In particular, for the theory that specifies the integers (commutative ring theory) there are different prime algebras, the different rings Z_n of integers modulo n. The initial algebra is Z, because all the other prime algebras are homomorphic images of Z. In contrast, for the equational number theory, N is the unique prime structure. What is the significance of this fact? Are the other algebras in the class of models of equational number theory of any interest? Is the equational number theory complete, in that it entails every equation that holds in the natural numbers (or in all the first order models of Peano's axioms)? For the particular case of number theory the two last questions are items in an extensive body of research into the nature of the natural numbers and the limitations of formal logic as means to treat them.

Example 2 (String Equations (Lists)). Let $a_1, \cdots, a_n$ be n letters. The signature has one constant: Λ, and at least $n+1$ unary functions, Pop and a_i. Possibly also a binary function $Cat(,)$, and the equational theory says:

1. $Pop(\Lambda) = \Lambda$
2. $Pop(a_i(x)) = x$
3. $Cat(\Lambda, x) = x$
4. $Cat(a_i(x), y) = a_i(Cat(x, y))$

This is just a minor variation of the previous example, starting with Λ instead of 0, and with n different successor functions a_i instead of S, Pop instead of predecessor and Cat instead of $+$.

3 Fields, Meadows, and Skew Meadows

The most familiar abstract algebraic structure is a field. The most common abstract data type, alongside strings and the natural numbers, is the type of the rational numbers. Why didn't we say anything about them until now?

The sad truth is that the class of fields is not, and cannot be made into a class of models of an equational theory, simply because it is not closed under cartesian product, as the product of two fields has zero divisors.

We have no satisfactory account of who noticed this problem first and how it was dealt with in the past. Moore in 1920 [11] already discussed a weaker notion of inverse, and Penrose [12] made it into the "Moore Penrose Pseodoinverse". Independently Von Neuman rings were discussed [13] and a similar weak inverse was introduced in a different context. However the weak inverse was not introduced to address the problem that in a field 0 has no inverse, but the problem that in non fields elements that have no inverse may still poses something close to inverse. The fact that also 0 happens to have a pseodoinverse was not considered to be of interest. In the theory of algebraic type specifications the fact that a field was not a universal algebra was probably a major motive for the attempts to extend the notion of Universal Algebra to many sorted universal algebra, and try and replace equational theories by conditional equational theories, where simple implications are permitted. Both notions are very sensible, both as general concepts and with regards to fields: It is natural to treat 0 as a special kind, and on the other hand the axioms of fields involve only a very simple conditional statement. Unfortunately no modification of the notion of Universal Algebra can retain the main features of Universal Algebras stated in section 1. We prefer therefore to stay within the framework of universal algebra, weakening the field axioms to an equational theory. This leads to the following questions:

– What is the equational theory of fields, what is the class of its models, and what is the initial algebra in the class?
– Is this initial algebra the field of rational numbers? If not can we specify the rational numbers as the initial algebra for some equational theory?

Chronologically the second question was answered first, in [2], defining the notion of a meadow and using Lagrange four squares theorem. This is discussed in section 6. The first issue produced a new interesting algebraic theory, the theory of *meadows* (and of *skew meadows*), which fills in a gap between field (and skew field) theory, and advanced ring theory. Meadows (and skew meadows) are the main subject of the paper.

The commutative case of fields and meadows is simpler and it is the only part that is relevant to equational axioms for fields and to the specification of the type of rational numbers (or any other particular algebraic field). The non commutative case, that of skew meadows, is harder, interesting, relevant to division rings and to Von Neuman rings, and it includes the commutative case as a special case. We will investigate algebras that are not necessarily commutative.

3.1 Meadows and Skew Meadows

Strictly speaking a field is not an algebra at all, since the inverse function is not a total function, as 0 has no inverse. We intend to make the inverse function total by defining 0^{-1}. Since ring theory implies that $0 \cdot x = 0$ any definition of 0^{-1} will fails to satisfy the equation $x^{-1} \cdot x = 1$, and we must look for an alternative equational theory. The key observation is that we can weaken the inverse equation to one of "local inverse", which will hold for 0 because of the fact that $x \cdot 0 = 0$.

The signature for fields and of meadows has two constants 0 and 1, two unary functions , minus and inverse (written as x^{-1}), and two binary functions for addition and multiplication. We denote by $\mathcal{R}ing$ the ring equations. The addition of the axiom of commutativity $x \cdot y = y \cdot x$ makes it a commutative ring.

What can be said about the inverse function? The key observation is the fact that no matter how 0^{-1} is defined, the following equations will hold in every division ring:

$$(x^{-1} \cdot x) \cdot x = x \quad \text{and} \quad x \cdot (x^{-1} \cdot x) = x$$

Should we specify both of the equations, or does one of them imply the other? If so, does it make a difference which one of the two we choose? With hindsight we choose the first equation, and we will return to this question later.

From the rest of the properties of the inverse function in fields we choose to specify reflexivity: $(x^{-1})^{-1} = x$. We note that we are now forced to define $0^{-1} = 0$, because $0^{-1} = (0^{-1})^{-1} \cdot 0^{-1} \cdot 0^{-1} = 0 \cdot 0^{-1} \cdot 0^{-1}$ and $0 \cdot x = 0$ is implied by the ring equations.

This leads us to the following definition:

Definition 1. *The theory of* Skew meadows *has the following equations:*

1. *$\mathcal{R}ing$, the ring equations;*

2. *$\mathcal{R}ef$, reflexivity,* $(x^{-1})^{-1} = x$;
3. *$\mathcal{R}il$, a restricted inverse equation,* $(x^{-1} \cdot x) \cdot x = x$.

We denote by $\mathcal{R}il'$ the dual axiom $x \cdot (x^{-1} \cdot x) = x$.

The theory of meadows is the theory of skew meadows with the addition of the axiom of commutativity, $x \cdot y = y \cdot x$. A model of the axioms will be called *a skew meadow*, and a commutative skew meadow is called *a meadow*. Every division ring, and every product of division rings is a skew meadow. To make this statement precise, from now on, "a field" or " a division ring" means a *totalized algebra*, where the inverse function is is extended by the definition $0^{-1} = 0$. We will prove that an algebra in the signature of fields is a skew meadow if and only if it is isomorphic to a substructure of division rings. From this we will deduce that the equational theory of (skew) meadows equals to the equational theory of (skew) fields, and that finite skew fields are commutative (and in fact products of finite fields).

3.2 Some Properties of Skew Meadows

We list some properties that follow from the equations of skew meadows. Firstly, since $x^{-1} \cdot x \cdot x = x$, we know that if $x \cdot x = 0$ then $x = 0$, so that:

(1) There are no non trivial nilpotent elements in a skew meadow.

If e is idempotent ($e \cdot e = e$), and x is arbitrary then simple computation shows that $e \cdot x \cdot (1 - e)$ and $(1 - e) \cdot x \cdot e$ are idempotent. therefore $e \cdot x \cdot (1 - e) = 0$ and $(1 - e) \cdot x \cdot e = 0$, which shows that $e \cdot x = e \cdot x \cdot e$ and $x \cdot e = e \cdot x \cdot e$. We conclude that $e \cdot x = x \cdot e$, so that:

(2) Idempotent elements are central, they commute with every element.

In the following computation we underline the part that is modified according to the axiom $\mathcal{R}il$:

$$x \cdot x^{-1} \cdot \underline{x} \cdot x^{-1} = \underline{x \cdot x^{-1} \cdot x^{-1}} \cdot x \cdot x \cdot x^{-1} = \underline{x^{-1} \cdot x \cdot x} \cdot x^{-1} = x \cdot x^{-1}$$

and we conclude that

(3) $x \cdot x^{-1}$ (and substituting x^{-1} for x also $x^{-1} \cdot x$) is idempotent.

By (3) we have $x \cdot x^{-1} = x \cdot \underline{x^{-1} \cdot x} \cdot x^{-1}$, and also $x^{-1} \cdot x = x^{-1} \cdot \underline{x \cdot x^{-1}} \cdot x$. By (2) the first underlined expression commutes with its right neighbor and the second underlined expression commutes with its left neighbor, and in both cases we obtain the product $x \cdot x^{-1} \cdot x^{-1} \cdot x$. Therefore

(4) $x \cdot x^{-1} = x^{-1} \cdot x$

In particular $\underline{x \cdot x^{-1}} \cdot x = x^{-1} \cdot x \cdot x = x$, so that

(5) The equation $\mathcal{R}il'$, $x \cdot x^{-1} \cdot x = x$ holds in every skew meadow.

The following theorem will connect skew meadows (and in particular meadows) with products of division rings (respectively, with products of fields).

Theorem 1. *Let S be a skew meadow and $x \neq 0$ an element. There is a homomorphism from S onto a division ring that respects addition, multiplication* and inverse*, and that does not map x to* 0*.*

Proof (outline). The proof has three ingredients:

1. Write $e = x \cdot x^{-1}$. By the axioms and by what was already established we know that e is idempotent, $e \cdot x = x \cdot e = x$ and $e \cdot x^{-1} = x^{-1} \cdot e = x^{-1}$. It follows that:
   - $e \cdot R$ is a skew meadow in which x is invertible.
   - The map $H(z) = e \cdot z$ is a ring homomorphism from R onto $e \cdot R$, with $H(x) = x$.
2. If J is a maximal two sided ideal in a meadow then the quotient ring is a ring with division, and invertible elements are not mapped to 0. The significant fact here is that if a is not in J then there is some b such that $a \cdot b$ is equivalent to 1 modulo J, so that every element in the quotient is invertible. We denote by f the idempotent $a \cdot a^{-1}$. If $f \in J$ then also $a \notin J$. Hence $J + f \cdot R$ is a two sided ideal properly extending J, and therefore it is R. Therefore there are $i \in J$ and $r \in R$ such that $1 = i + f \cdot r = i + a \cdot a^{-1} \cdot r$, which shows that $a^{-1} \cdot r$ satisfies the requirement.
3. Composing H with the quotient mapping we obtain a ring homomorphism onto a division ring that does not map x to 0. We then note that by cancellation in division rings every ring homomorphism from a meadow into a division ring preserves also inverses. □

3.3 Skew Meadows and Division Rings

From the previous theorem we conclude:

Theorem 2

1. *A structure in the signature of meadows is a skew meadow if and only if it is isomorphic to a substructure of a product of division rings, and it is a meadow if and only if it is isomorphic to a substructure of a product of fields.*
2. *The equational theory of skew meadows entails all the equations that are true in division rings. The equational theory of meadows entails all the equations that are true in fields.*

Proof (outline). Every product of division rings, or substructure of one, is a skew meadow, since division rings are skew meadows and products and substructures preserve equations. On the other hand if R is a skew meadow, then for every $x \in R$ there is some division ring D_x and a homomorphism $h_x : R \longrightarrow D_x$, such that $h_x(x) \neq 0$. We combine these homomorphisms to a homomorphism into the Cartesian product:

$$H : R \longrightarrow \prod_{x \in R} D_x$$

Such that $H(z)$ has $h_x(z)$ at the x^{th} entry. Since $h_z(z) \neq 0$ the kernel of the homomorphism is trivial. This proves the first claim. Every equation that holds universally in division rings holds also in products of division rings, and in substructures of products of division rings. Therefore also in all the skew fields. □

We can do a little better:

Theorem 3. *Every finite skew meadow is commutative, and is in fact isomorphic to a finite product of finite fields.*

Proof (outline). Assume that the skew meadow $\mathcal{M}$ is finite, and check the division rings whose Cartesian product was used for the embedding of $\mathcal{M}$. Each was a homomorphic image of $\mathcal{M}$, and hence a finite division ring. By Wedderburn's theorem [**?**] it is necessarily commutative. There was one component corresponding to any element of $\mathcal{M}$, so that it is a finite product of fields. It probably can be shown that if the minimal number of components is taken then the isomorphism is onto the product. We took a different path, interesting for its own sake, showing that a finite meadow is the Cartesian product of its minimal ideals,and these ideals are fields (each with its private unit, which is an idempotent that generates the ideal in the meadow). □

4 Strongly Von Neumann Regular Rings

Long before Meadows were introduced, Von Neumann investigated function rings [13], and he found that some of them, in particular every ring of matrices over a field, satisfy the *(Von Neumann) Regularity property*:

$$\forall x \exists y (x \cdot y \cdot x = x)$$

Every skew meadow is necessarily a regular ring. The converse is not true: A ring of matrices over a finite field is a finite regular ring which is not commutative, and therefore it cannot support an inverse function that makes it a skew meadow.

Von Neumann regular rings were, and are, the subject of intensive investigation. Goodearl's book [6], is a good source on the subject. The property which is dual to regularity and which interests us is called *strong regularity* in [6]:

Definition 2. *A ring is* strongly regular *if it satisfies the axiom:*

$$\forall x \exists y (y \cdot x \cdot x = x)$$

Using ingredients from the proof of Theorem 3.5 of [6], we prove the analogue to Theorem 32

Theorem 4. *Let R be a strongly regular ring, and let $a \neq 0$ be an element. There is a ring homomorphism from R onto a division ring which does not map a to* 0.

Proof (outline). The proof is quite different from the proof of theorem 32. We note that in strongly regular rings there are no nilpotent elements; if $x^2 = 0$ then $x = 0$ by $\forall x \exists y (y \cdot x \cdot x = x)$. It follows that $a^n \neq 0$ for every n . Using Zorn's lemma we find a maximal two sided ideal J that does not contain any power of a, and we show that it is a *prime ideal*, i.e, if $x \cdot R \cdot y \subseteq J$ then $x \in J$ or $y \in J$. Indeed if $xRy \subseteq J$ and $x, y \notin J$ then a has a power in either of the two ideals generated by J and either one of x, y. Therefore there are j, j' in J, and $x_1, \cdots, x_n, x'_1, \cdots, x'_n$ and $y_1, \cdots, y_{n'}, y'_1, \cdots, y'_{n'}$ such that

$$a^m = j + x_1 \cdot x \cdot x'_1 + \cdots + x_n \cdot x \cdot x'_n$$

and

$$a^{m'} = j' + y_1 \cdot y \cdot y'_1 + \cdots + y_{n'} \cdot y \cdot y'_{n'}$$

Multiplying the two expressions on the right hand side we see that all the summands that do not include j or j' are of the form $(x_i \cdot x \cdot x'_i)(y_{i'} \cdot y \cdot y'_{i'})$. Since the inner subterm $x \cdot x'_i \cdot y_{i'} \cdot y$ is in J by $x \cdot R \cdot y \subseteq J$, these summands are also in J, contradicting the fact that $a^{m+m'}$ is not in J.

The image of a in R/J is not 0 and it remains to show that R/J is a division ring. We show first that it has no zero divisors. If $[x] \neq 0$ and $[y] \neq 0$ in R/J then $x \notin J$ and $y \notin J$. By primeness there is some z such that $x \cdot z \cdot y \notin J$, so that $[x \cdot z \cdot y] \neq 0$. Strong regularity is preserved under homomorphisms, so that there are no nilpotent elements, and the square is also not 0. I.e, $[x \cdot z \cdot y] \cdot [x \cdot z \cdot y] = [x] \cdot [z] \cdot [y] \cdot [x] \cdot [z] \cdot [y] \neq 0$. In particular $[y] \cdot [x] \neq 0$.

We conclude by observing that a strongly regular ring with no zero divisors is a division ring. For every $x \neq 0$ there is some y such that $y \cdot x \cdot x = x$ and therefore $(y \cdot x - 1) \cdot x = 0$. Since $x \neq 0$ we conclude that $y \cdot x - 1 = 0$. Thus every non zero element has an inverse. This concludes the proof. □

We note for further use that the last step in the proof shows that if in a ring R we have $y \cdot x \cdot x = x$ or $x \cdot y \cdot x = x$, and $h(x) \neq 0$ for a homomorphism onto a division ring, $h : R \to D$ then $h(y) = (h(x))^{-1}$.

We have now the following characterization for strongly regular rings:

Theorem 5

1. *A ring R is strongly regular if and only if it supports an inverse function that makes the ring into a skew meadow.*
2. *Such an inverse function is unique.*

Proof (outline). One direction is clear: A skew meadow is clearly a strongly regular ring.

As before We choose for every element x of the ring a division ring D_x, such that R is mapped onto D_x by a homomorphism h_x with $h_x(x) \neq 0$. We denote by H the monomorphism from R into the Cartesian product of the division rings, that maps every element z to the sequence that has $h(x)(z)$ as its x^{th} entry. Let R' be the image of R:

$$H : \ R \ \longrightarrow \ R' \ \subseteq \prod_{x \in R} D_x$$

R is isomorphic to a substructure of a product of division rings. But this does not conclude our quest! The isomorphism is not with respect to the inverse function, and it is not clear at all that R' is closed under the inverse function. However the particulars of the monomorphism are enough to prove the following claim:

Assume that $y \cdot x \cdot x = x$ in R. Then

- $x \cdot y \cdot x = x$.
- If we put $y' = y \cdot x \cdot y$ then $x \cdot y \cdot x = x$ and $y' \cdot x \cdot y' = y'$
- $H(y') = (H(x))^{-1}$

All these facts hold easily in R'. For every z, $h_z(y \cdot x)$ is 1 if $h_z(x) \neq 0$, and it is 0 if $h_z(x) = 0$. In either case $H(y \cdot x)$ is a sequence of zeros and ones that commutes with every element in the Cartesian product. The second item follows easily from the first one and we prove the third: For every z, if $h_z(x) \neq 0$ then $h_z(y') = (h_z(x))^{-1}$ by $x \cdot y \cdot x = x$. And if $h_z(x) = 0$ then $h_z(y') = 0$ by $y' = y' \cdot x \cdot y'$. Therefore in every entry $h_z(y') = (h_z(x))^{-1}$.

Thus R is ring isomorphic to a substructure of the product $\prod_{x \in R} D_x$ that is closed under the inverse function, and is therefore a skew meadow. The isomorphism induces an inverse operation on R and makes R into a skew meadow.

The uniqueness of the inverse function follows from the fact that a homomorphism of a skew meadow onto a division ring preserves also inverses. It follows that if there are two inverse function on R then the homomorphism on each D_x reduces both inverse functions to the inverse in D_x. Therefore under the ring monomorphism into $\prod_{x \in R} D_x$ both inverses are identified with the inverse function of $\prod_{x \in R} D_x$. □

5 What about Regular Rings and the Weaker Axiom, $\mathcal{R}il'$?

With hindsight, it would be natural to declare that $\mathcal{R}il$ is "the right axiom" and discard $\mathcal{R}il'$ as a corollary which curiously is quite weaker than $\mathcal{R}il$. Because of Von Neumann's theory emphasis on the condition $\forall x \exists y [x \cdot (x^{-1} \cdot x) = x\ ]$, and not on strong regularity, we cannot ignore the corresponding property $x \cdot (x^{-1} \cdot x) = x$. We know a little about it, and we mainly list open questions. We will call a ring with an inverse function a "*weak skew meadow*" if it satisfies the axioms of skew meadows with the axiom $\mathcal{R}il'$ replacing $\mathcal{R}il$.

1. Not every weak skew meadow is a skew meadow. In fact there are finite non commutative weak skew meadows: By a careful choice of inverses we can define a reflexive inverse function in the ring of two by two matrices, over a field of characteristics different from 2, which satisfies $x \cdot x^{-1} \cdot x = x$.
2. We do not know if a ring has at most one inverse function that makes it a weak skew meadow (we know that a skew meadow can not support a second unary operation that makes it into a weak skew meadow because the proof of Theorem 43 applies also in this case).

3. We do not know if every regular ring supports an inverse function that makes it into a weak skew meadow.

There are some natural properties that follow from the axioms of skew meadows, and their status with respect to weak skew meadow axioms is unclear. For each of these properties and for any combination of them, there are three possibilities: The property may be implied by the weak skew meadow axioms, or it may complete the weak axioms and imply the strong skew meadow axioms, or its addition may describe a new class that lies between the class of weak meadows and that of strong meadow. In the last case the new axiom may or may not imply the non existence of a finite non commutative algebra in the class. Here are some natural properties that are taken for granted in division rings and skew meadows and their status is unclear for weak skew meadows:

1. $x \cdot x^{-1} = x^{-1} \cdot x$.
2. $(x \cdot y)^{-1} = y^{-1}x^{-1}$.
3. $e^{-1} = e$ if e is idempotent (this property is not an equational property).
4. The inverse function is unique (not an equational property).

In weak skew meadows, item (1) implies, trivially, strong regularity. Items (2,3) together imply a little less trivially strong regularity. It is not clear if one of the items (2,3,4) by itself or if any combination other than (2,3) implies regularity, or at least prevents the theory from having finite non commutative models. We conclude that we are far from understanding the difference between the seemingly similar axioms $(x^{-1} \cdot x) \cdot x = x$ and $x \cdot x^{-1} \cdot x = x$.

6 Algebraic Specification of the Rational Field

We return to the question that started it all: Which equational theory has the rational field as its initial structure? The answer was given by the first and the third authors in [2]. We suggest to call it "*the theory of Lagrange meadows*". It is the equational theory of (commutative) skew meadows together with the following equation

$$(1 + x^2 + y^2 + z^2 + v^2) \cdot (1 + x^2 + y^2 + z^2 + v^2)^{-1} = 1 \ .$$

Theorem 6. *The field Q of rational numbers is the initial algebra in the class of Lagrange meadows, and moreover any prime algebra in the class is isomorphic to Q.*

Proof (outline). By Lagrange four squares theorem every natural number n can be written as the sum of four squares so that if $n \neq 0$ we can write $\underbrace{1 + \cdots + 1}_{n}$ in the form $(1 + x^2 + y^2 + z^2 + v^2)$. This assures that for every natural n the term $\underline{n} = \underbrace{1 + \cdots + 1}_{n}$ is not 0. It is not hard to prove by structural induction that in Lagrange meadows every fixed term equals to 0 or to a term of the form $\underline{n} \cdot (\underline{m})^{-1}$, which is different from 0. Therefore the prime algebra in every Lagrange meadow is the field of rational numbers, and we conclude that the field of rational numbers is the initial algebra for the equational theory of Lagrange meadows. □

Thus the abstract type of rational number is algebraically specified as an initial algebra.

7 Conclusion

The definition of meadows and skew meadow enabled us to contribute in three different areas:

- In the theory of *Universal Algebras* we identified the equational theory of fields and rings with division, and proved that it was just the equational theory for meadows and skew meadows.
- In *Ring Theory* we characterized the strongly regular Von Neuman rings as those that support a (necessarily unique) inverse function that makes it into a skew meadow. The inverse function is easily defined in term of any choice function that associates with every element x a partner y for which $y \cdot x \cdot x = x$.
- In the theory of *Abstract Types* we specified the type of the rational numbers as the algebra specified by the axioms of Lagrange meadows.

This seems like an invitation to an interesting domain of research that can shed light on the theories of universal algebras, of regular rings and of algebraic specification of data types. In particular, what is the significance of the fact that an algebraic theory has a single prime algebra, as in the case of Lagrange meadows that specify the rational field, but in contrast to the theory of meadows in general? Is there any additional importance to the equational theory of meadows, and of Lagrange meadows?

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