Applied Optimization

Lecture 01. Introduction

The Big Picture

Real-World Problem $\xrightarrow{\text{Modeling}}$ Mathematical Optimization Problem $\xrightarrow{\text{Algorithm}}$ Solution or Approximation

- Important questions during **modeling**
  - how accurately is the real-world problem represented?
  - which problems are easy/difficult to solve?
  - can we convert a difficult prob. into an equivalent easier one?
- Important questions when choosing **Algorithm**
  - which algorithm is " *best* " for specific problem?
  - strengths and weaknesses of available alternatives?
  - guarantees on "solution quality"?

"Mathematical" optimization problem

- Standard Form of continuous optimization problem

objective function

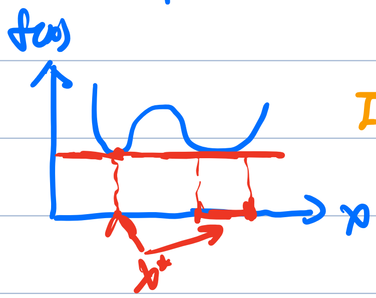
minimize $f_0(x): \mathbb{R}^n \to \mathbb{R}$ → rules in Bonnes's rules

subject to $f_i(x) \le 0, i=1,\dots,m$ → inequality constraint functions

$h_j(x) = 0, j=1,\dots,p$ → equality constraint functions

feasible set: $F = \{z \in \mathbb{R}^n \mid f_i(z) \le 0, i=1\dots m \wedge h_j(z)=0, j=1\dots p\}$

solution: a point in the feasible set: $x^* \in F$ and $f_0(x^*) \le f_0(z), \forall z \in F$



Important Notations of the class:

$\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}_{+} = \{x \in \mathbb{R} \mid x \ge 0\}$

Classes of Optimization Problems

linear ≠ constant

$f(x) = a^T x = \sum_{i=1}^n a_i x_i$

linear function satisfies: $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b), \forall \alpha, \beta \in \mathbb{R}$

- Linear Program (LP)

minimize $c^T x$
subject to $a_i^T x \le b_i$
$h_j^T x = d_j$

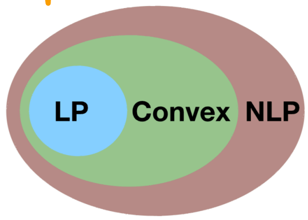
all f_0 and h_j are affine functions ⇒ $f(x) = a^T x + b = \sum_{i=1}^n a_i x_i + b$

- Nonlinear Program (NLP): f_0 and h_j can be arbitrarily nonlinear

• Convex Optimization Problem: f_0 are convex, h_j are affine

$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$, $\forall\, \alpha + \beta = 1$, $\alpha, \beta \ge 0$. ($\alpha, \beta \in R^+$)

→ Relationships: LP ⊂ Convex ⊂ NLP



Convex is GOOD because in convex functions, local minima is JUST global minima

Example: Linear Least Squares (LLS) → $x \in R^{n\times 1}$

minimize $\|Ax - b\|_2^2$, $A \in R^{k\times n}$, $b \in R^{k\times 1}$ (usually k is larger than n)

Recall: $\|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{x^T x} \Rightarrow \|x\|_2^2 = x^T x = \sum_i x_i^2$

Analytic Solution:

$$f_0(x) = \|Ax - b\|_2^2 = (Ax - b)^T \cdot (Ax - b) = (x^T A^T - b^T) \cdot (Ax - b)$$

$$= x^T A^T A x - \underbrace{x^T A^T b - b^T A x}_{\text{the same}} + b^T b = x^T A^T A x - 2 b^T A x + b^T b$$

Then, we are going to calculate the gradient $\frac{\partial f_0(x)}{\partial x} = 2A^T A x - 2A^T b = \vec{0} \in R^{n\times 1}$

n×1

Details: Matrix Calculus

If not sure, we can first do the derivations elements-wisely, and then check the dimension.

$\Leftrightarrow A^T A x = A^T b \Rightarrow \boxed{x^* = (A^T A)^{-1} A^T b}$ → analytic solution

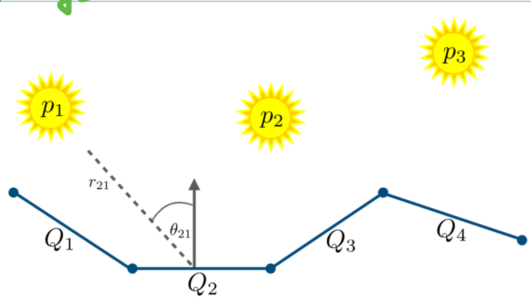
x^* is ALWAYS the minimum because: $\frac{\partial^2 f_0(x)}{\partial x^2} = 2A^T A \in R^{n\times n}$ is positive semi-definite (PSD)

always

Example - Illumination Problem (S. Boyd)

minimize $\max_{k=1\ldots n} |\log I_k - \log I_{des}|$

subject to $0 \le p_j \le P_{max}$, $j = 1 \ldots n$



How to solve?

① use uniform power: set $p_j = p$ and find the optimal p

pros: Easy calculation   cons: too stricted → often poor solution

② use least squares: minimize $\sum_{k=1}^{m} (I_k - I_{des})^2$, clamp p_j if $p_j < 0$ or $p_j > P_{max}$

↳ clamping might lead to large deviation

③ use weighted least squares: minimize $\sum_{k=1}^{m} (I_k - I_{des})^2 + \sum_{j=1}^{n} w_j (p_j - \frac{1}{2} P_{max})^2$

↳ heuristic

④ use linear programming: minimize: $\max_{k=1}^{m} |I_k - I_{des}|$ → change the objective function may lead to suboptimal solution

pros: Easy to solve

subject to: $0 \le p_j \le P_{max}, j=1,\dots,n$

⑤ (Best) use convex optimization: minimize $f_0(p) = \max_{k=1}^{m} h(I_k / I_{des})$ $h(u) = \max\{u, \frac{1}{u}\}$

which will be discussed in later sections ←   subject to: $0 \le p_j \le P_{max}, j=1,\dots,n$

Feasible set:

$\{x \in \mathbb{R}^n \mid f_i(x) \le 0, i=1,\dots,m \wedge h_i(x) = 0, i=1,\dots,p\}$

Visualizing Optimization Problems

↳ Geometrically, the intersection of m sublevel sets and p level sets

Consider the standard form of continuous optimization problem, the objective function can be shown as follows:

Notations:

level set: $L_c(f) = \{x \mid f(x) = c\}$

sublevel set: $L_c^-(f) = \{x \mid f(x) \le c\}$

superlevel set: $L_c^+(f) = \{x \mid f(x) \ge c\}$