TD - Queuing Theory
M/M/1 and M/M/c Queue Systems

Part I: M/M/1 Queue System

Exercise 1: Basic M/M/1 Analysis

A post office has a single service counter. Customers arrive at a rate of 20 per hour following a Poisson process. The service time is exponentially distributed with a mean of 2 minutes per customer.

  a) Calculate the server utilization ρ.

  b) Determine the average number of customers in the system (L).

  c) Find the average number of customers waiting in the queue (L_q).

  d) Calculate the average time a customer spends in the system (W).

  e) Calculate the average waiting time in the queue (W_q).

  f) Verify your results using Little's Law.

Exercise 2: Probability Calculations

For the post office in Exercise 1:

  a) What is the probability that the system is empty (P_0)?

  b) What is the probability that there are exactly 3 customers in the system?

  c) What is the probability that there are 5 or more customers in the system?

  d) What percentage of time is the server busy?

Exercise 3: ATM Machine

An ATM receives customers at a rate of 15 per hour. The average transaction time is 3 minutes.

  a) Is the system stable? Justify your answer.

  b) Calculate all performance measures (ρ, L, L_q, W, W_q).

  c) If the bank wants to reduce the average waiting time to less than 5 minutes, what should the service rate be?

Exercise 4: Sensitivity Analysis

A drive-through restaurant serves customers at a rate of $\mu = 20$ per hour. Currently, the arrival rate is $\lambda = 12$ per hour.

a) Calculate the current performance measures.

b) If the arrival rate increases to 16 per hour, recalculate all measures.

c) What is the percentage increase in average waiting time W_q?

d) At what arrival rate would the system become unstable?

Exercise 5: Variance and Standard Deviation

For an M/M/1 queue with $\lambda = 25$ and $\mu = 35$ customers per hour:

a) Calculate the variance of the number of customers in the system: $\text{Var}(N) = \frac{\rho}{(1-\rho)^2}$

b) Calculate the standard deviation σ.

c) Calculate the coefficient of variation $CV = \frac{\sigma}{L}$.

d) Interpret what this high variability means for system management.

Exercise 6: Service Time Distribution

A barber shop has exponentially distributed service times with mean 15 minutes.

a) What is the service rate μ in customers per hour?

b) What is the probability that a service takes less than 10 minutes?

c) What is the probability that a service takes more than 20 minutes?

d) If customers arrive at 3 per hour, calculate L and W.

Exercise 7: Utilization Impact

Complete the following table for an M/M/1 system with $\mu = 40$ per hour:

λ	ρ	L	W (minutes)	L_q
20	?	?	?	?
28	?	?	?	?
32	?	?	?	?
36	?	?	?	?
38	?	?	?	?

Comment on how performance deteriorates as ρ approaches 1.

Exercise 8: Optimal Service Rate

A small grocery store has customers arriving at 30 per hour. The current service rate is 35 per hour, but the owner can upgrade to a faster checkout system.

a) Calculate current average waiting time W_q.

b) If the owner wants $W_q < 2$ minutes, what minimum service rate is needed?

c) Calculate the percentage reduction in L if μ is increased to 50 per hour.

Part II: M/M/c Queue System

Exercise 9: Basic M/M/c Analysis

A bank has 4 tellers. Customers arrive at 48 per hour, and each teller can serve 15 customers per hour.

  a) Calculate the traffic intensity a.

  b) Calculate the utilization ρ.

  c) Is the system stable?

  d) Calculate the Erlang-C probability $C(4, 3.2)$.

  e) Calculate L_q, W_q, L, and W.

Exercise 10: Call Center Design

A call center receives 60 calls per hour. Each agent handles calls at a rate of 12 per hour. Management wants at most 20% of callers to wait.

  a) Calculate the minimum number of servers needed for stability.

  b) For $c = 6$, calculate $C(c, a)$ and check if it meets the 20% target.

  c) If not, try $c = 7$ and $c = 8$ until the target is met.

  d) For the optimal c, calculate all performance measures.

Exercise 11: Hospital Emergency Room

An ER has 3 doctors. Patients arrive at 24 per hour, and each doctor can treat 10 patients per hour on average.

  a) Calculate a and ρ.

  b) What percentage of patients must wait?

  c) What is the average number of patients waiting?

  d) What is the average total time in the ER?

  e) If a fourth doctor is added, recalculate $C(4, 2.4)$ and W_q.

Exercise 12: Server Comparison

Compare two configurations for serving 40 customers per hour:
Configuration A: 1 server at $\mu = 50$ per hour (M/M/1)
Configuration B: 5 servers at $\mu = 10$ per hour each (M/M/5)

  a) Calculate W and W_q for both configurations.

  b) Which configuration gives shorter total time?

  c) Which gives shorter waiting time?

  d) Discuss practical advantages of Configuration B despite longer total time.

Exercise 13: Optimal Staffing
A service center has the following costs:

- Cost per server: $50/hour

- Waiting cost: $20/hour per customer in system

With $\lambda = 45$ and $\mu = 15$ per hour per server:

a) Calculate total cost $TC(c)$ for $c = 4, 5, 6, 7$.

b) Which staffing level minimizes total cost?

c) Calculate the performance measures for the optimal c.

Exercise 14: Ticket Counter
An airport ticket counter has 3 agents. Passengers arrive at 36 per hour, and service time averages 4 minutes per passenger.

a) Set up the problem parameters (λ, μ, c, a, ρ).

b) Calculate $C(3, 2.4)$ and interpret.

c) Calculate queue length and waiting time measures.

d) If management wants average waiting time under 2 minutes, how many agents are needed?

Exercise 15: Server Pooling Effect
Compare these scenarios for $\lambda = 30$ per hour:
Scenario 1: Two separate M/M/1 queues, each with $\lambda = 15$ and $\mu = 20$
Scenario 2: One M/M/2 queue with $\lambda = 30$ and $\mu = 20$ per server

a) Calculate total L for Scenario 1 (sum of both queues).

b) Calculate L for Scenario 2.

c) Explain why pooling is beneficial (Scenario 2 should be better).

d) Calculate the percentage reduction in average system size.

Part III: Advanced Problems

Exercise 16: System Redesign
A customer service desk currently operates as M/M/1 with $\lambda = 18$ and $\mu = 24$ per hour. Management is considering two options:
Option A: Upgrade to faster service ($\mu = 30$), keeping M/M/1
Option B: Add a second server, each at $\mu = 24$ (M/M/2)

a) Calculate current performance measures.

b) Calculate performance for Option A.

c) Calculate performance for Option B.

d) Which option gives better W_q? Better L?

e) Recommend an option considering both performance and practical factors.

Exercise 17: Peak vs Off-Peak

A coffee shop experiences different arrival rates:

- Peak hours (8-10 AM): $\lambda = 40$ per hour

- Off-peak hours: $\lambda = 15$ per hour

- Service rate: $\mu = 25$ per hour per barista

a) For off-peak with 1 barista (M/M/1), calculate performance measures.

b) For peak with 1 barista, is the system stable? Why or why not?

c) How many baristas are needed during peak for stability?

d) For peak with 2 baristas (M/M/2), calculate $C(2, 1.6)$ and W_q.

e) Should a third barista be added during peak? Justify with calculations.

Exercise 18: Service Level Agreement

A help desk has a service level agreement (SLA) stating that 80% of customers should wait less than 5 minutes. Currently, $\lambda = 50$ per hour and $\mu = 15$ per hour per agent.

a) For $c = 4$, calculate average W_q. Does it meet SLA?

b) The probability of waiting less than t minutes in M/M/c is approximately:
$P(W_q < t) = 1 - C(c, a) \cdot e^{-c\mu(1-\rho)t/60}$
For $c = 4$, calculate $P(W_q < 5)$.

c) Try $c = 5$ and check if SLA is met.

d) Recommend the minimum staffing level to meet the SLA.

Exercise 19: Cost-Benefit Analysis

A supermarket is deciding between:
Plan A: 5 regular lanes at $\mu = 12$ per hour, cost $15/hour each
Plan B: 3 express lanes at $\mu = 20$ per hour, cost $25/hour each
Customer arrival rate is $\lambda = 48$ per hour. Waiting cost is $30/hour per customer.

a) For each plan, verify stability ($a < c$).

b) Calculate L for each plan.

c) Calculate total hourly cost (staff cost + waiting cost) for each plan.

d) Which plan is more economical?

e) Calculate the difference in average customer waiting time between plans.

Exercise 20: Comprehensive Analysis

A telecommunications company is designing a technical support center. They estimate:

- Call arrivals: 72 per hour (peak)

- Average handling time: 6 minutes

- Agent cost: $40/hour

- Waiting cost (customer dissatisfaction): $50/hour per customer

- Target: Less than 30% of callers should wait

a) Determine λ, μ, and a.

b) Find minimum c for stability, then test $c = 8, 9, 10$ for the 30% target.

c) For each feasible c, calculate total hourly cost.

d) Create a table showing c, $C(c, a)$, L, W (minutes), and Total Cost.

e) Recommend optimal staffing level, balancing cost and service quality.

f) Calculate the expected number of idle agents for your recommendation.