1 First Exercise

The first exercise given is the following:

Consider $X(k)$, given by

$$X(k) = W(k) - W(k-1), k = \pm 1, \pm 2, ...,$$

where $\{W(k)\}$ is a stationary stochastic process with independent, identically distributed (i.i.d) stochastic variables and $E\{W(k)\} = 0$, $E\{W^2(k)\} = 1$ and $E\{W^3(k)\} = 1$. The covariance sequence of $\{X(k)\}$ is given by:

$$\begin{aligned} c_2^x(\tau) = m_2^x(\tau) &= E\{X(k)X(k+\tau)\} \\ &= E\{(W(k) - W(k-1))(W(k+\tau) - W(k+\tau-1))\} \\ &= 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1) \end{aligned}$$

where $\delta(\tau)$ is the delta Kronecker function; hence,

$$c_2^x(\tau) = \begin{cases} 2, & \tau = 0 \\ -1, & \tau = 1,\ \tau = -1. \\ 0, & elsewhere \end{cases}$$

The corresponding Power Spectrum is given by

$$C_2^x(\omega) = \sum_{\tau=-1}^{1} c_2^x(\tau) e^{-j\omega\tau} = (2 - 2cos\omega).$$

1. Find the 3rd-order cumulants of $\{X(k)\}$, i.e., $c_3^x(\tau_1, \tau_2)$.
2. Find the skewness $\gamma_3^x = c_3^x(0,0)$. What do you observe?
3. Find the Bispectrum $C_3^x(\omega_1, \omega_2)$Is it complex, real or imaginary?
4. How the result of 2 affects the result of 3? Can you draw a general comment?

1.1 First Sub-task

At first, we compute the 3rd order cumulants of X(k).

$$\begin{aligned} c_3^x(\tau 1, \tau 2) &= m_3^x(\tau 1, \tau 2) - m_1^x[m_2^x(\tau 1) + m_2^x(\tau 2) + m_2^x(\tau 2 - \tau 1)] + 2(m_1^x)^3 \\ &= m_3^x(\tau 1, \tau 2) \text{ because } m_1^x = E[X(k)] = E[W(k)] - E[W(k-1)] = 0 \end{aligned}$$

Hence,

$$\begin{aligned}
c_3^x(\tau 1,\tau 2) &= m_3^x(\tau 1,\tau 2) = E(X(k)X(k+\tau 1)X(k+\tau 2) \\
&= E((W(k)-W(k-1))(W(k+\tau 1)-W(k+\tau 1-1)(W(k+\tau 2)-W(k+\tau 2-1)) \\
&= E(W(k)W(k+\tau 1)W(k+\tau 2) - W(k)W(k+\tau 1)W(k+\tau 2-1) - W(k)W(k+\tau 1-1)W(k+\tau 2) \\
&+ W(k)W(k+\tau 1-1)W(k+\tau 2-1) - W(k-1)W(k+\tau 1)W(k+\tau 2) + W(k-1)W(k+\tau 1-1)W(k+\tau 2) \\
&+ W(k-1)W(k+\tau 1)W(k+\tau 2-1) - W(k-1)W(k+\tau 1-1)W(k+\tau 2-1)) \\
&= E(W(k)W(k+\tau 1)W(k+\tau 2)) - E(W(k)W(k+\tau 1)W(k+\tau 2-1)) - E(W(k)W(k+\tau 1-1)W(k+\tau 2)) \\
&+ E(W(k)W(k+\tau 1-1)W(k+\tau 2-1)) - E(W(k-1)W(k+\tau 1)W(k+\tau 2)) \\
&+ E(W(k-1)W(k+\tau 1-1)W(k+\tau 2)) + E(W(k-1)W(k+\tau 1)W(k+\tau 2-1)) \\
&- E(W(k-1)W(k+\tau 1-1)W(k+\tau 2-1))
\end{aligned}$$

$W(k), W(k+\tau)$ are independent stochastic processes.Thus, $E(W(k)W(k+\tau) = E(W(k)E(W(k+\tau)$

Hence,

$$\begin{aligned}
c_3^x(\tau 1,\tau 2) &= -\delta(\tau 1,\tau 2-1) - \delta(\tau 1-1,\tau 2) \\
&+ \delta(\tau 1-1,\tau 2-1) - \delta(\tau 1+1,\tau 2+1) \\
&+ \delta(\tau 1+1,\tau 2-1) + \delta(\tau 1,\tau 2+1) \\
&= \begin{cases} -1, & \tau 1=0,\tau 2=1 \\ -1, & \tau 1=1,\tau 2=0 \\ +1, & \tau 1=1,\tau 2=1 \\ -1, & \tau 1=-1,\tau 2=-1 \\ +1, & \tau 1=-1,\tau 2=0 \\ +1 & \tau 1=0,\tau 2=-1 \end{cases}
\end{aligned}$$

1.2 Second Sub-task

To find the skewness we observe the 3rd order cumulants in $(\tau 1,\tau 2)=(0,0)$.
Hence,

$$\gamma_3^x = c_3^x(0,0) = 0$$

We can state that the stochastic process X(k) has a probability distribution where the observations are perfectly symmetrical around the mean.So, $(m_1^x = 0)$ and the observations are symmetrical around (0,0).

1.3 Third Sub-task

$$\begin{aligned}
C_3^x(\omega 1,\omega 2) &= \sum_{\tau 1=-1}^{\tau 1=+1}\sum_{\tau 2=-1}^{\tau 2=+1} c_3^x(\tau 1,\tau 2)e^{-j(\omega 1\tau 1+\omega 2\tau 2)} \text{ where, } |\omega 1|\le\pi, |\omega 1|\le\pi, |\omega 1+\omega 2|\le\pi \\
&= e^{-j(\omega 1+\omega 2)} - e^{-j(-\omega 1-\omega 2)} \\
&+ e^{j\omega 1} + e^{-j\omega 2} - e^{-j\omega 1} - e^{-j\omega 2} \\
&= -2j(-\sin\omega 1) - 2j(-\sin\omega 2) - 2j(\sin(\omega 1+\omega 2)) \\
&= 2j(\sin\omega 1 + \sin\omega 2 - \sin(\omega 1+\omega 2))
\end{aligned}$$

Hence, the bispectrum $C_3^x(\omega 1,\omega 2)$ is imaginary.

1.4 Fourth Sub-task

If $c_3^x(\tau 1,\tau 2)\neq 0$ there will be a real part in the bispectrum apart from the imaginary. So, we can assume that the skewness of X(k) determines the real part of the bispectrum.