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Homework Assignment 1
Advanced Signal Processing, 20/21

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1 First Exercise

The first exercise given is the following:

Consider $X(k)$, given by

$$X(k) = W(k) - W(k-1), k = \pm 1, \pm 2, ...,$$

where $\{W(k)\}$ is a stationary stochastic process with independent, identically distributed (i.i.d) stochastic variables and $E\{W(k)\} = 0$, $E\{W^2(k)\} = 1$ and $E\{W^3(k)\} = 1$. The covariance sequence of $\{X(k)\}$ is given by:

$$
\begin{aligned}
c_2^x(\tau) = m_2^x(\tau) &= E\{X(k)X(k+\tau)\} \\
&= E\{(W(k) - W(k-1))(W(k+\tau) - W(k+\tau-1))\} \\
&= 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1)
\end{aligned}
$$

where $\delta(\tau)$ is the delta Kronecker function; hence,

$$
c_2^x(\tau) = \begin{cases} 2, & \tau = 0 \\ -1, & \tau = 1, \ \tau = -1. \\ 0, & elsewhere \end{cases}
$$

The corresponding Power Spectrum is given by

$$
C_2^x(\omega) = \sum_{\tau=-1}^{1} c_2^x(\tau)e^{-j\omega\tau} = (2 - 2cos\omega).
$$

1. Find the 3rd-order cumulants of $\{X(k)\}$, i.e., $c_3^x(\tau_1, \tau_2)$.
2. Find the skewness $\gamma_3^x = c_3^x(0,0)$. What do you observe?
3. Find the Bispectrum $C_3^x(\omega_1, \omega_2)$ Is it complex, real or imaginary?
4. How the result of 2 affects the result of 3? Can you draw a general comment?

1.1 First Sub-task

At first, we compute the 3rd order cumulants of X(k).

$$
\begin{aligned}
c_3^x(\tau 1, \tau 2) &= m_3^x(\tau 1, \tau 2) - m_1^x[m_2^x(\tau 1) + m_2^x(\tau 2) + m_2^x(\tau 2 - \tau 1)] + 2(m_1^x)^3 \\
&= m_3^x(\tau 1, \tau 2) \text{ because } m_1^x = E[X(k)] = E[W(k)] - E[W(k-1)] = 0
\end{aligned}
$$

Hence,

$$
\begin{aligned}
c_3^x(\tau1, \tau2) = m_3^x(\tau1, \tau2) &= E(X(k)X(k+\tau1)X(k+\tau2)) \\
&= E((W(k) - W(k-1)(W(k+\tau1)) - W(k+\tau1-1))(W(k+\tau2) - W(k+\tau2-1))) \\
&= E(W(k)W(k+\tau1)W(k+\tau2) - W(k)W(k+\tau1)W(k+\tau2-1) - W(k)W(k+\tau1-1)W(k+\tau2) \\
&\quad + W(k)W(k+\tau1-1)W(k+\tau2-1) - W(k-1)W(k+\tau1)W(k+\tau2) + W(k-1)W(k+\tau1-1)W(k+\tau2) \\
&\quad + W(k-1)W(k+\tau1)W(k+\tau2-1) - W(k-1)W(k+\tau1-1)W(k+\tau2-1)) \\
&= E(W(k)W(k+\tau1)W(k+\tau2)) - E(W(k)W(k+\tau1)W(k+\tau2-1)) - E(W(k)W(k+\tau1-1)W(k+\tau2)) \\
&\quad + E(W(k)W(k+\tau1-1)W(k+\tau2-1)) - E(W(k-1)W(k+\tau1)W(k+\tau2)) \\
&\quad + E(W(k-1)W(k+\tau1-1)W(k+\tau2)) + E(W(k-1)W(k+\tau1)W(k+\tau2-1)) \\
&\quad - E(W(k-1)W(k+\tau1-1)W(k+\tau2-1))
\end{aligned}
$$

$W(k), W(k+\tau)$ are independent stochastic processes. Thus, $E(W(k)W(k+\tau)) = E(W(k))E(W(k+\tau))$

Hence,

$$
\begin{aligned}
c_3^x(\tau1, \tau2) &= -\delta(\tau1, \tau2-1) - \delta(\tau1-1, \tau2) \\
&\quad + \delta(\tau1-1, \tau2-1) - \delta(\tau1+1, \tau2+1) \\
&\quad + \delta(\tau1+1, \tau2-1) + \delta(\tau1, \tau2+1) \\
&= \begin{cases}
-1, & \tau1 = 0, \tau2 = 1 \\
-1, & \tau1 = 1, \tau2 = 0 \\
+1, & \tau1 = 1, \tau2 = 1 \\
-1, & \tau1 = -1, \tau2 = -1 \\
+1, & \tau1 = -1, \tau2 = 0 \\
+1 & \tau1 = 0, \tau2 = -1
\end{cases}
\end{aligned}
$$

1.2   Second Sub-task

To find the skewness we observe the 3rd order cumulants in $(\tau1, \tau2) = (0, 0)$.
Hence,

$$\gamma_3^x = c_3^x(0, 0) = 0$$

We can state that the stochastic process X(k) has a probability distribution where the observations are perfectly symmetrical around the mean. So, $(m_1^x = 0)$ and the observations are symmetrical around (0,0).

1.3   Third Sub-task

$$
\begin{aligned}
C_3^x(\omega1, \omega2) &= \sum_{\tau1=-1}^{\tau1=+1} \sum_{\tau2=-1}^{\tau2=+1} c_3^x(\tau1, \tau2) e^{-j(\omega1\tau1+\omega2\tau2)} \text{ where, } |\omega1| \leq \pi, |\omega2| \leq \pi, |\omega1 + \omega2| \leq \pi \\
&= e^{-j(\omega1+\omega2)} - e^{-j(-\omega1-\omega2)} \\
&\quad + e^{j\omega1} + e^{-j\omega2} - e^{-j\omega1} - e^{-j\omega2} \\
&= -2j(-\sin\omega1) - 2j(-\sin\omega2) - 2j(\sin(\omega1 + \omega2)) \\
&= 2j(\sin\omega1 + \sin\omega2 - \sin(\omega1 + \omega2))
\end{aligned}
$$

Hence, the bispectrum $C_3^x(\omega1, \omega2)$ is imaginary.

1.4   Fourth Sub-task

If $c_3^x(\tau1, \tau2) \neq 0$ there will be a real part in the bispectrum apart from the imaginary. So, we can assume that the skewness of X(k) determines the real part of the bispectrum.