Reverse-time and Kirchhoff migration

The goal is to plot the images obtained by Reverse-Time migration in different (two-dimensional) configurations, and to study the resolution and stability properties of the method, and to compare with other methods. (Note: there are some elements of the resolution in the lecture notes).

1) Preliminaries.

We assume that the medium has speed of propagation $c_0 = 1$. The homogeneous two-dimensional Green's function $\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y})$ is solution of

$$\Delta_{\boldsymbol{x}} \hat{G}_0 + \omega^2 \hat{G}_0 = -\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^2$$

with the Sommerfeld radiation condition. It is given by

$$\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y}) = \frac{i}{4} H_0^{(1)}(\omega |\boldsymbol{x} - \boldsymbol{y}|)$$

where $H_0^{(1)}$ is the Hankel function

$$H_0^{(1)}(s) = J_0(s) + iY_0(s)$$

and J_0 is the Bessel function of the first kind of order zero and Y_0 is the Bessel function of the second kind of order zero (see `scipy.special`).

2) Time-harmonic localization - full aperture.

Consider N transducers on a circular array centered at $\mathbf{0}$ with radius R_0.

Consider a point-like reflector at $\boldsymbol{x}_{\rm ref}$.

Generate the data set, i.e. the matrix of the time-harmonic amplitudes $\hat{u}_{rs}(\omega)$ recorded by the rth receiver when the sth source emits a time-harmonic signal with unit amplitude and frequency ω. Use a Born approximation for the reflector to generate the data.

Plot the (two-dimensional) RT and KM imaging functional $\mathcal{I}_{\rm RT}(\boldsymbol{x})$ and $\mathcal{I}_{\rm KM}(\boldsymbol{x})$ using the data set.

Compare the focal spot with the theoretical function $J_0^2(\omega|\boldsymbol{x} - \boldsymbol{x}_{\rm ref}|)$ (give a proof of this formula).

Use $\omega = 2\pi$, $R_0 = 100$, $N = 100$, and $\boldsymbol{x}_{\rm ref} = (10, 20)$, and play with the numbers (in particular, move the reflector).

3) Time-harmonic localization - partial aperture.

We use the convention $\boldsymbol{x} = (x, z)$.

Consider N receivers $(\boldsymbol{x}_r)_{r=1,\dots,N}$ on a regular linear array (along the x-direction) centered at $\mathbf{0}$ with length R_0 (i.e. $\boldsymbol{x}_r = (x_r, 0)$, $x_r = -R_0/2 + R_0(r-1)/(N-1)$).

Generate the data set, i.e. the matrix of the time-harmonic amplitudes $\hat{u}_{rs}(\omega)$ recorded by the rth receiver when the sth source emits a time-harmonic signal with unit amplitude and frequency ω. Use a Born approximation for the point-like reflector.

Plot the (two-dimensional) RT and KM imaging functional $\mathcal{I}_{\rm RT}(\boldsymbol{x})$ and $\mathcal{I}_{\rm KM}(\boldsymbol{x})$ using the data set. Plot also the MUSIC-type imaging functional:

$$\mathcal{I}_{\rm MU}(\boldsymbol{x}) = \big|\langle \hat{\boldsymbol{g}}(\omega,\boldsymbol{x}), \boldsymbol{v}_1\rangle\big|^2$$

where $\hat{\boldsymbol{g}}(\omega,\boldsymbol{x})$ is the vector of Green's functions from the array to the search point $\boldsymbol{x}$:

$$\hat{\boldsymbol{g}}(\omega,\boldsymbol{x}) = \big(\hat{G}_0(\omega,\boldsymbol{x},\boldsymbol{x}_r)\big)_{r=1,\dots,N}$$

and $\boldsymbol{v}_1$ is the first singular vector of the response matrix $\hat{\mathbf{u}}$ (MUSIC means MUltiple SIgnal Classification).

Use $\omega = 2\pi$, $R_0 = 50$, $N = 100$, and $\boldsymbol{x}_{\rm ref} = (0, 100)$, and play with the numbers (reduce R_0, move the reflector).

Look at the focal spot in the (cross-range) x-direction and compare with the theoretical function ${\rm sinc}^2(\pi|x - x_{\rm ref}|/r_c)$, with $r_c = \lambda|\boldsymbol{x}_{\rm ref}|/R_0$ and $\lambda = 2\pi/\omega$.

Look at the focal spot in the (range) z-direction and compare with the theoretical function $\big|\int_0^1 \exp(-i\frac{\pi}{2}s^2\frac{|z-z_{\rm ref}|}{r_l})ds\big|^2$, with $r_l = 2\lambda|\boldsymbol{x}_{\rm ref}|^2/R_0^2$.

4) Time-dependent localization - partial aperture.

Consider N receivers on a linear array (along the x-direction) centered at $\mathbf{0}$ with length R_0.

We now assume that the sources emit a broadband signal with $\hat{f}(\omega) = \mathbf{1}_{[\omega_0-B,\omega_0+B]}(\omega)$.

Generate the data set, i.e. the matrices of the time-harmonic amplitudes $\hat{d}_{rs}(\omega)$ recorded by the rth receiver when the sth source emits a time-harmonic signal with unit amplitude and frequency ω, for ω sampled in $[\omega_0 - B, \omega_0 + B]$.

Plot the (two-dimensional) RT and KM imaging functional $\mathcal{I}_{\rm RT}(\boldsymbol{x})$ and $\mathcal{I}_{\rm KM}(\boldsymbol{x})$ using the data set.

Use $\omega_0 = 2\pi$, $B = 0.05\omega_0$, $R_0 = 20$, $N = 40$, and $\boldsymbol{x}_{\rm ref} = (0, 100)$, sample 20 frequencies over $[\omega_0 - B, \omega_0 + B]$, and play with the numbers (reduce the bandwidth B for instance).

Look at the focal spot in the (cross-range) x-direction and compare with the theoretical function ${\rm sinc}^2(\pi|x - x_{\rm ref}|/r_c)$, with $r_c = \lambda_0|\boldsymbol{x}_{\rm ref}|/R_0$ and $\lambda_0 = 2\pi/\omega_0$.

Look at the focal spot in the (range) z-direction and compare with the theoretical function $|{\rm sinc}(2B|z - z_{\rm ref}|)|$.

5) Stability with respect to measurement noise.

Consider measurement noise. The recorded signals are $\hat{u}_{rs}(\omega) + W_{rs}^{(1)}(\omega) + iW_{rs}^{(2)}(\omega)$, where the noise terms $W_{rs}^{(1)}(\omega)$, $W_{rs}^{(2)}(\omega)$ are independent and identically distributed Gaussian random variables with mean zero and variance $\sigma^2/2$.

Revisit the previous questions in the presence of measurement noise and study the stability issue (up to you: establish the statistics of the localization error, the dependence of the standard deviation in the localization error with respect to the number of sensors, with respect to the bandwidth, and so on, compare MUSIC and RT, ...).