Reverse-time and Kirchhoff migration

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M2 MVA

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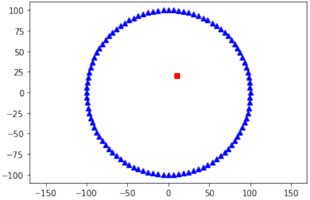
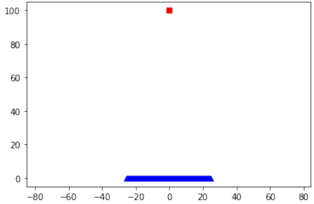
Introduction





- N transductors in $(\mathbf{x}_r)_{r=1}^N$
- One reflector in $\mathbf{x}_{\text{ref}}$
- <u>**Data set :**</u> For $\omega \in [\omega_0 - B, \omega_0 + B]$,

$$\widehat{U}(\omega) = \{\widehat{u}(\omega, \mathbf{x}_r; \mathbf{x}_s) = \widehat{u}_{rs}(\omega), r, s \in \{1, \dots, N\}\}$$

- By the Born approximation,

$$\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) = \widehat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \omega^2 \widehat{G}_0(\omega, \mathbf{x}, \mathbf{x}_{\text{ref}}) \rho(\mathbf{x}_{\text{ref}}) \widehat{G}_0(\omega, \mathbf{x}_{\text{ref}}, \mathbf{y})$$

- In 2-**D**, $\widehat{G}_0(\omega, \mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(\omega|\mathbf{x} - \mathbf{y}|)$

Imaging functions

With $\widehat{g}(\omega, \mathbf{z}) = \left(\widehat{G}_0(\omega, \mathbf{x}_r, \mathbf{z})\right)_{1 \leq r \leq N}$

- **Reverse-Time imaging function :**

$$\mathbf{I}_{\mathrm{RT}}(\mathbf{z}) = \overline{\widehat{g}(\omega, \mathbf{z})}^T \widehat{U} \overline{\widehat{g}(\omega, \mathbf{z})} = \left| \overline{\widehat{g}(\omega, \mathbf{z})}^T \overline{\widehat{g}(\omega, \mathbf{x}_{\mathrm{ref}})}^T \right|^2$$

- **Kirchoff-Migration imaging function :**

$$\mathbf{I}_{\mathrm{RT}}(\mathbf{z}) = \overline{k(\omega, \mathbf{z})}^T \widehat{U} \overline{k(\omega, \mathbf{z})} = \left| \overline{k(\omega, \mathbf{z})}^T \overline{\widehat{g}(\omega, \mathbf{x}_{\mathrm{ref}})}^T \right|^2$$

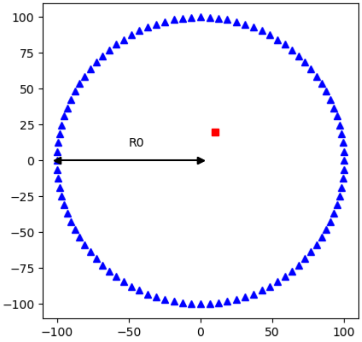
where $k(\omega, \mathbf{z}) = [\exp(i\omega \|\mathbf{x}_r - \mathbf{z}\|)]_{r=1}^N$

- **MUSIC imaging function :**

$$\mathbf{I}_{\mathrm{MUSIC}}(\mathbf{z}) = \langle \widehat{g}(\omega, \mathbf{z}), \mathbf{v}_1 \rangle|^2$$

where v_1 is the first singular vector of $\widehat{U}$

First experiment



Parameters :

- $N = 100$ transductors
- $\mathbf{x}_{\text{ref}}$
- $R_0 = 100$
- $\omega = 2\pi$
- (With noise) $\sigma = 0.03$

First experiment - $\mathbf{x}_{\text{ref}} = (10, 20)$

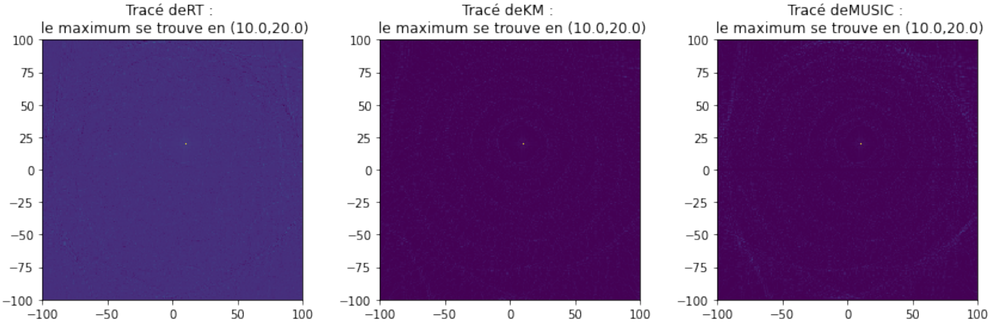


Figure: functions imaging, $N = 100, R_0 = 100, \mathbf{x}_{\text{ref}} = (10, 20), \omega = 2\pi$

First experiment - $\mathbf{x}_{\text{ref}} = (-50, -50)$

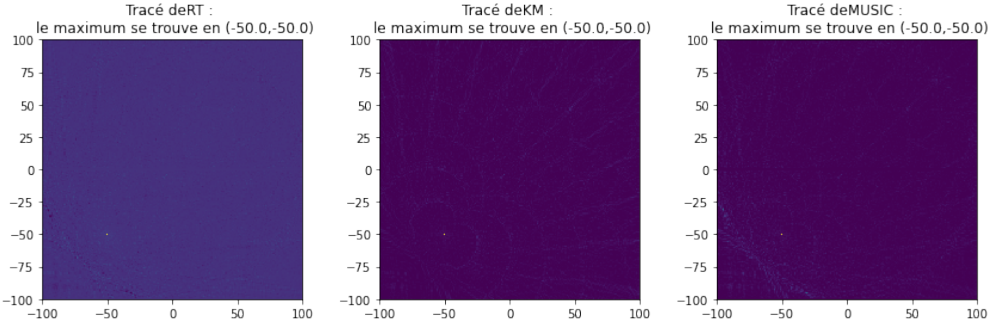


Figure: functions imaging, $N = 100, R_0 = 100, \mathbf{x}_{\text{ref}} = (-50, -50), \omega = 2\pi$

First experiment - $\mathbf{x}_{\mathbf{ref}} = (-100, -75)$

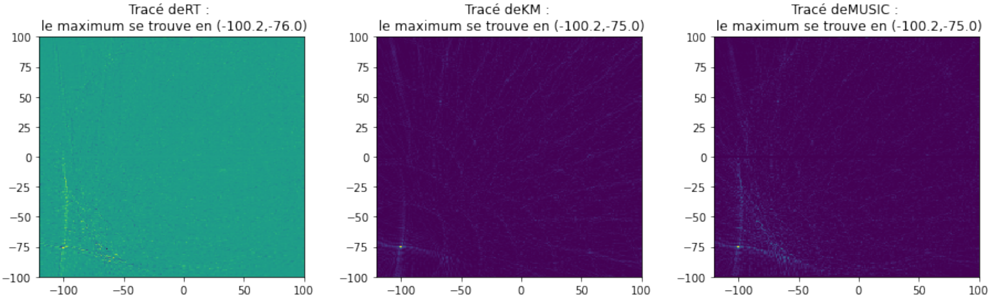


Figure: functions imaging, $N = 100, R_0 = 100, \mathbf{x}_{\mathbf{ref}} = (-100, -75), \omega = 2\pi$

First experiment - $\mathbf{x}_{\text{ref}} = (-100, 0)$

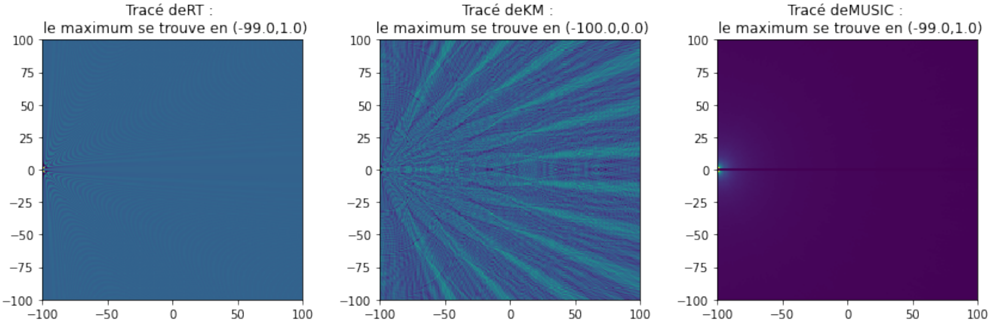


Figure: functions imaging, $N = 100, R_0 = 100, \mathbf{x}_{\text{ref}} = (-100, -75), \omega = 2\pi$

First experiment with noise - - $\mathbf{x}_{\mathbf{ref}} = (10, 20)$

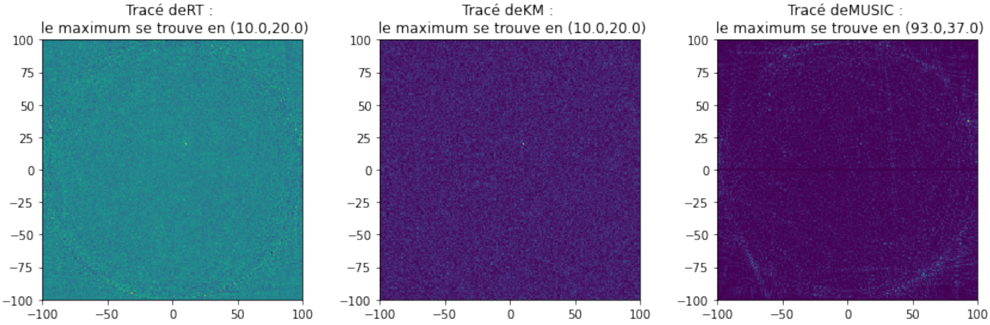


Figure: functions imaging, $N = 100, R_0 = 100, \mathbf{x}_{\text{ref}} = (10, 20), \omega = 2\pi$ with noise with $\sigma = 0.03$

First experiment with noise - $\mathbf{x}_{\text{ref}} = (100, 75)$

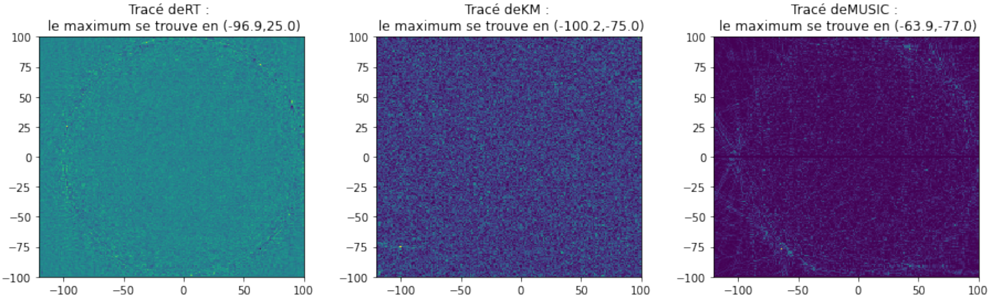


Figure: functions imaging, $N = 100, R_0 = 100, \mathbf{x}_{\text{ref}} = (-100, -75), \omega = 2\pi$ with noise with $\sigma = 0.03$

Conclusions on the first experiment

- Instability of **RT** and **MUSIC**.
- **MUSIC** is faster than the others.
- Moving the reflector don't really change the quality of the imaging functions.
- **MUSIC** does not work with noise

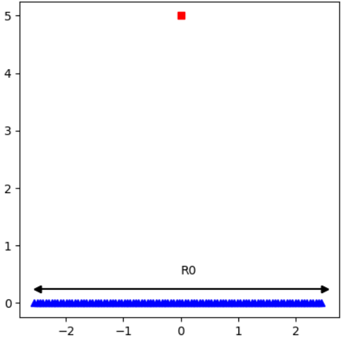
$$I_{\text{MUSIC}}(\mathbf{x}) = |\langle \widehat{g}(\omega, \mathbf{x}), \mathbf{v}_1 \rangle|^2$$

If $U = \widehat{g}(\omega, \mathbf{x}_{\text{ref}})^T \widehat{g}(\omega, \mathbf{x}_{\text{ref}})$,

$$v_1 = \frac{1}{\|\widehat{g}(\omega, \mathbf{x}_{\text{ref}})\|} \widehat{g}(\omega, \mathbf{x}_{\text{ref}})$$

But with noise, $U \neq \widehat{g}(\omega, \mathbf{x}_{\text{ref}})^T \widehat{g}(\omega, \mathbf{x}_{\text{ref}})$ and $\text{rank}(U) \neq 1$

Second experiment



Parameters :

- $N = 100$ transducors
- $\mathbf{x}_{\text{ref}}$
- R_0
- $\omega = 2\pi$

Second experiment - $R_0 = 50$

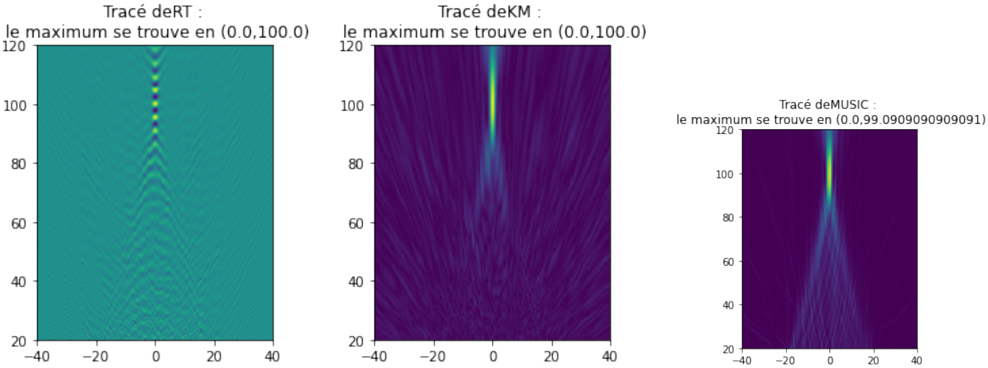


Figure: functions imaging, $N = 100, R_0 = 50, \mathbf{x}_{\text{ref}} = (0, 100), \omega = 2\pi$

Second experiment - $R_0 = 100$

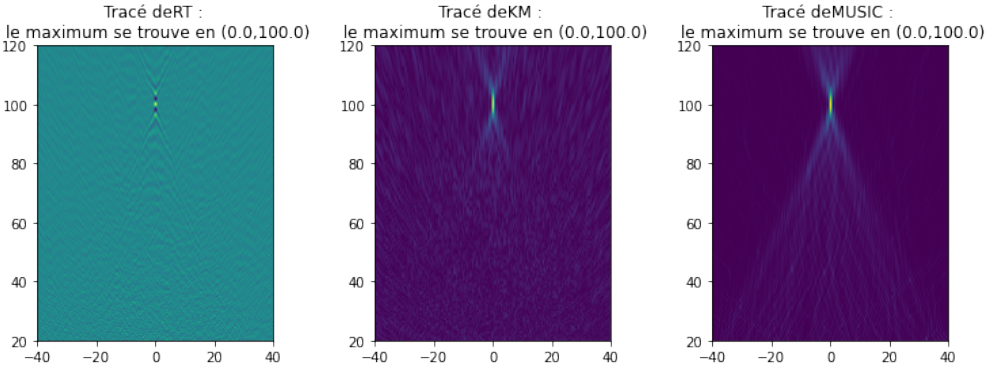


Figure: functions imaging, $N = 100, R_0 = 100, \mathbf{x}_{\text{ref}} = (0, 100), \omega = 2\pi$

Second experiment - $R_0 = 40$

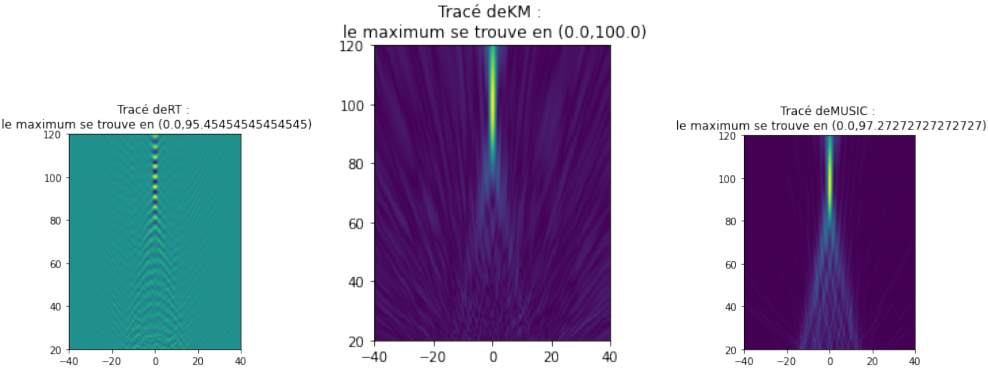


Figure: functions imaging, $N = 100, R_0 = 40, \mathbf{x}_{\text{ref}} = (0, 100), \omega = 2\pi$

Second experiment - $R_0 = 10$

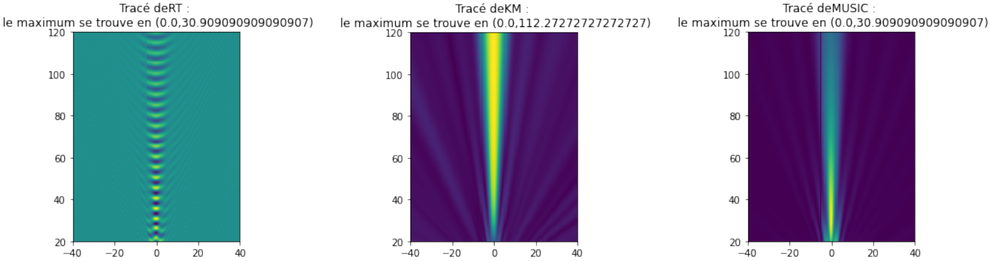
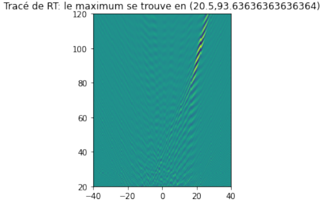
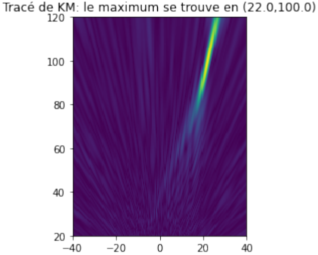


Figure: functions imaging, $N = 100, R_0 = 10, \mathbf{x}_{\mathbf{ref}} = (0, 100), \omega = 2\pi$

Second experiment - $\mathbf{x}_{\mathrm{ref}} = (22, 100)$





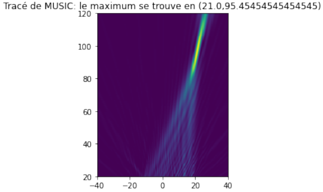
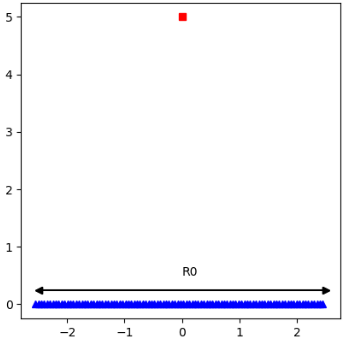


Figure: functions imaging, $N = 100, R_0 = 40, \mathbf{x}_{\mathrm{ref}} = (22, 100), \omega = 2\pi$

Conclusions on the second experiment

- Very sensitive to R_0
- **KM** more stable than the others

Third experiment



Parameters :

- $N = 40$ transductors
- $\mathbf{x}_{\text{ref}} = (0, 100)$
- $R_0 = 20$
- $\omega \in [\omega_0 - B, \omega_0 + B]$
- (With noise) $\sigma = 0.001$

Third experiment - $B = 0.05\omega_0$, 20 points

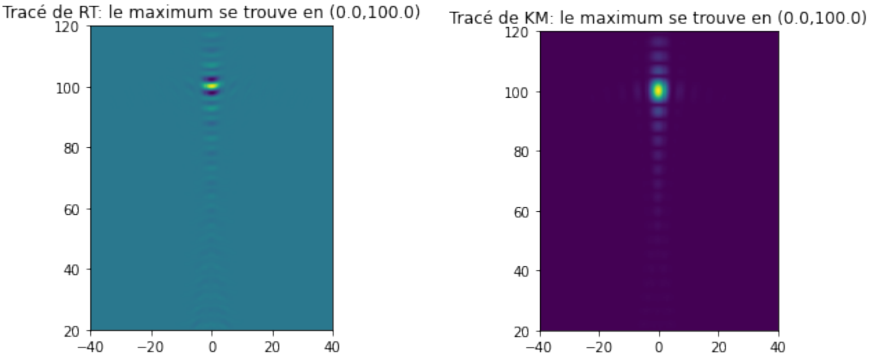


Figure: functions imaging, $N = 40, R_0 = 20, \mathbf{x}_{\text{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.05\omega_0$, 20 points in $[\omega_0 - B, \omega_0 + B]$

Third experiment- $B = 0.005\omega_0$, 20 points

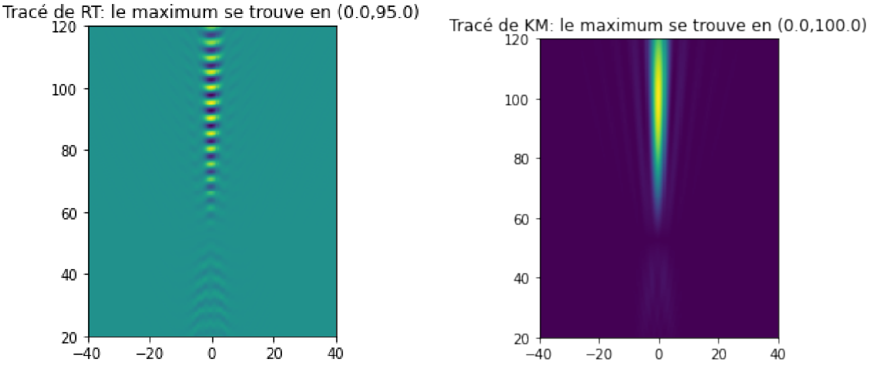


Figure: functions imaging, $N = 40, R_0 = 20, \mathrm{x}_{\mathrm{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.005\omega_0$, 20 points in $[\omega_0 - B, \omega_0 + B]$

Third experiment - $B = 0.1\omega_0$, 20 points

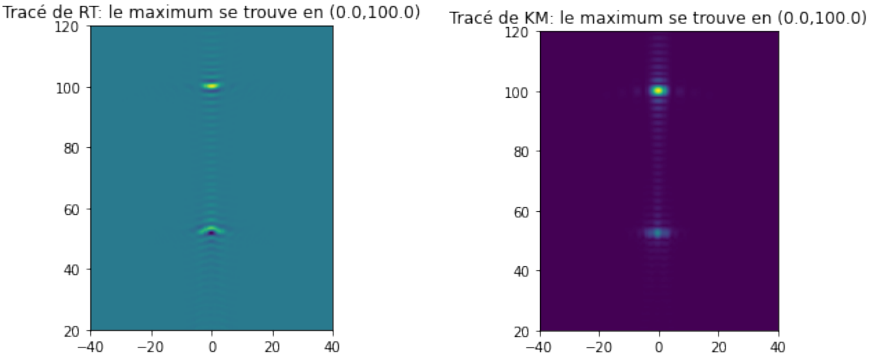


Figure: functions imaging, $N = 40, R_0 = 20, \mathbf{x}_{\text{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.1\omega_0$, 20 points in $[\omega_0 - B, \omega_0 + B]$

Third experiment - $B = 0.1\omega_0$, 20 points

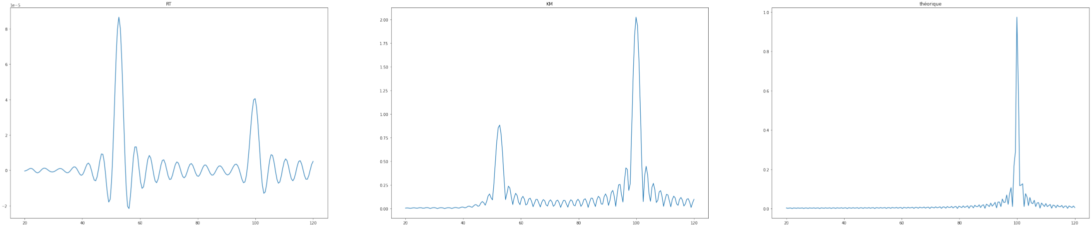


Figure: functions imaging, $N = 40, R_0 = 20, \mathbf{x}_{\text{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.1\omega_0$, 20 points in $[\omega_0 - B, \omega_0 + B]$

Third experiment- $B = 0.1\omega_0$, 40 points

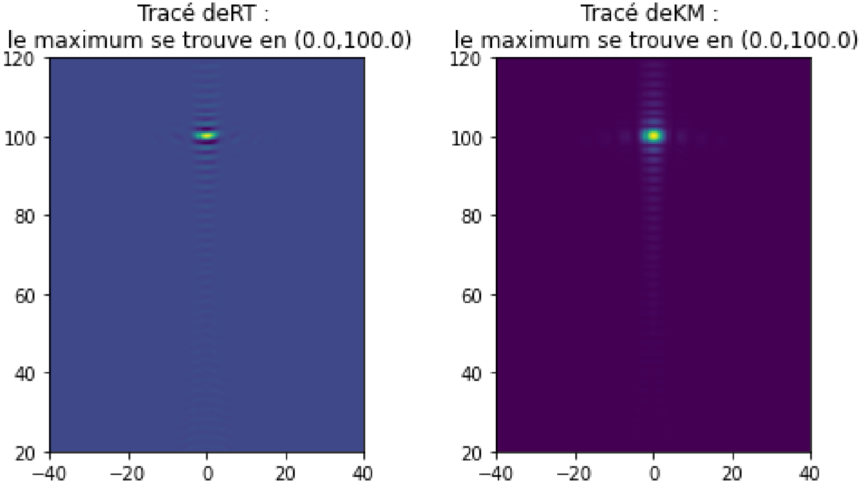


Figure: functions imaging, $N = 40, R_0 = 20, \mathbf{x}_{\text{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.1\omega_0$, 40 points in

Third experiment- $B = 0.1\omega_0$, 40 points

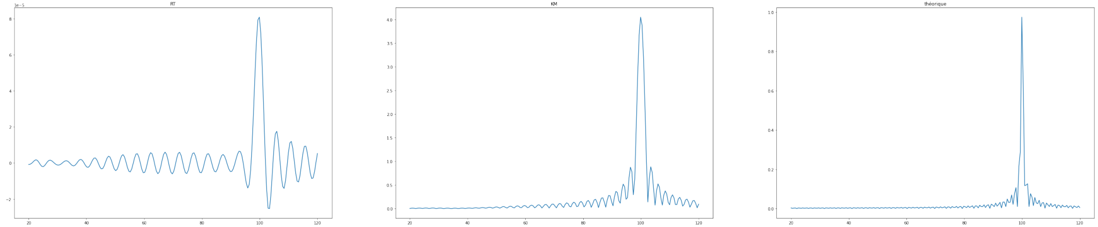


Figure: functions imaging, $N = 40, R_0 = 20, \mathbf{x}_{\text{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.1\omega_0$, 40 points in $[\omega_0 - B, \omega_0 + B]$

Third experiment with noise - $B = 0.1\omega_0$, 20 points

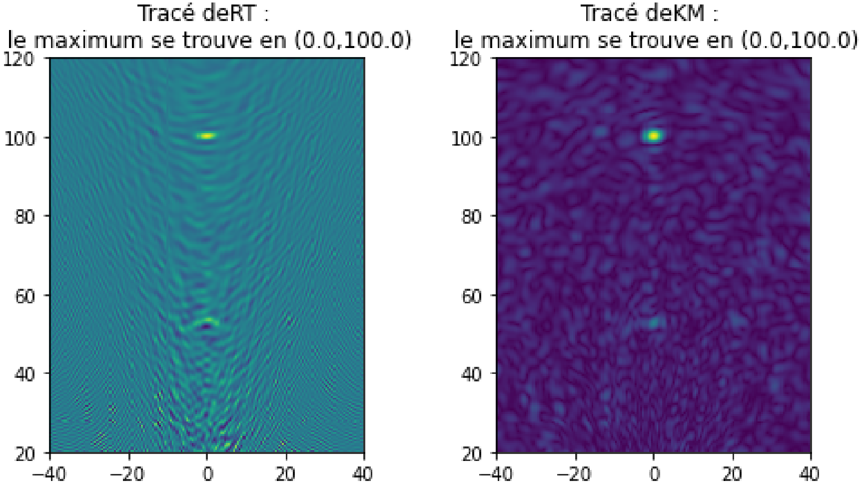


Figure: functions imaging, $N = 40, R_0 = 20, \mathbf{x}_{\text{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.1\omega_0$, 20 points in

Third experiment with noise- $B = 0.1\omega_0$, 40 points

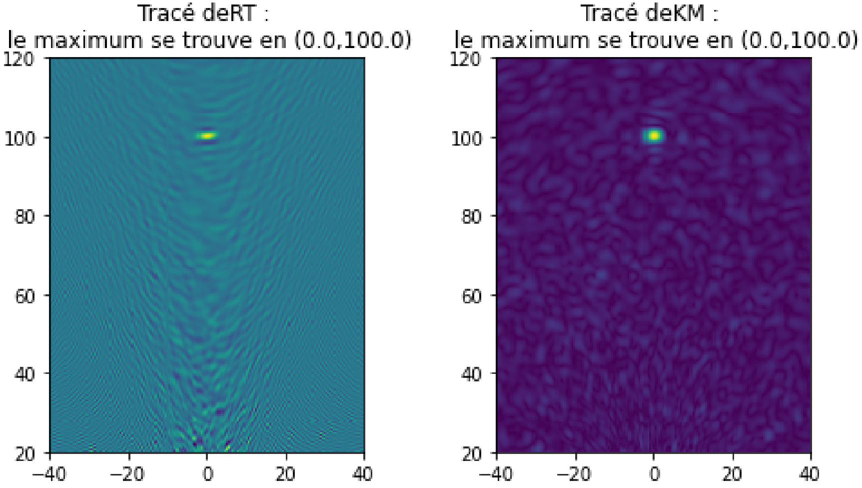


Figure: functions imaging, $N = 40, R_0 = 20, \mathbf{x}_{\text{ref}} = (0, 100), \omega_0 = 2\pi, B = 0.1\omega_0$, 40 points in

Conclusions on the third experiment

- If B increases, the number of points has to be increased

The end

Thank you for your attention !

Questions ?

Appendix

Hankel function

$$H_0^{(1)}(x) = J_0(x) + iY_0(x)$$

Born approximation

On fait l'hypothèse :

$$\frac{1}{c_{\mathrm{real}}^2(\mathbf{x})} = n_0^2(\mathbf{x}) + \rho_{\mathrm{real}}(\mathbf{x}), \text{ où } n_0 \text{ est connu}$$

On a :

$$\begin{aligned}
&\omega^2 n_0^2(\mathbf{z})\widehat{G}_0(\omega,\mathbf{z},\mathbf{x}) + \Delta_{\mathbf{z}}\widehat{G}_0(\omega,\mathbf{z},\mathbf{x}) = -\delta(\mathbf{z}-\mathbf{x}) \\
&\omega^2(n_0^2(\mathbf{z}) + \rho(\mathbf{z}))\widehat{G}(\omega,z,y) + \Delta_{\mathbf{z}}\widehat{G}(\omega,\mathbf{z},\mathbf{y}) = -\delta(\mathbf{z}-\mathbf{y})
\end{aligned}$$

Donc en multipliant la première ligne par $\widehat{G}(\omega,\mathbf{z},\mathbf{y})$ et la seconde par $\widehat{G}_0(\omega,\mathbf{z},\mathbf{x})$ et en faisant la différence :

$$\begin{aligned}
&\widehat{G}_0(\omega,\mathbf{z},\mathbf{x}) - \omega^2\widehat{G}_0(\omega,\mathbf{z},\mathbf{x})\rho(\mathbf{z})\widehat{G}(\omega,\mathbf{z},\mathbf{y}) - \widehat{G}_0(\omega,\mathbf{z},\mathbf{x})\Delta_{\mathbf{z}}\widehat{G}(\omega,\mathbf{z},\mathbf{y}) \\
&= -\delta(\mathbf{z}-\mathbf{x})\widehat{G}(\omega,\mathbf{z},\mathbf{y}) + \delta(\mathbf{z}-\mathbf{y})\widehat{G}_0(\omega,\mathbf{z},\mathbf{x})
\end{aligned}$$

Born approximation

Puis en intégrant selon $\mathbf{y}$ et utilisant la condition de Sommerfield,

$$\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) = \widehat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \omega^2 \int_\Omega \widehat{G}_0(\omega, \mathbf{x}, \mathbf{z}) \rho(\mathbf{z}) \widehat{G}(\omega, \mathbf{z}, \mathbf{y}) dz$$

$$\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) = \widehat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \omega^2 \int_\Omega \widehat{G}_0(\omega, \mathbf{x}, \mathbf{z}) \rho(z) \widehat{G}_0(\omega, \mathbf{z}, \mathbf{y}) d\mathbf{z}$$

Ici, $\Omega = \{\mathbf{x}_{\mathrm{ref}}\}$, donc l'approximation de Born donne :

$$\widehat{G}(\omega, \mathbf{x}, \mathbf{y}) = \widehat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \omega^2 \widehat{G}_0(\omega, \mathbf{x}, \mathbf{x}_{\mathrm{ref}}) \rho(\mathbf{x}_{\mathrm{ref}}) \widehat{G}_0(\omega, \mathbf{x}_{\mathrm{ref}}, \mathbf{y})$$

RT

Data set : $\{\widehat{u}_L(\omega, \mathsf{x}_r; \mathsf{x}_s) = \widehat{u}_{rs}(\omega) - \widehat{G}_0(\omega, \mathsf{x}_r, \mathsf{x}_s), r, s \in \{1, \dots, N\}, \omega \in B\}$

$$\operatorname{argmin}_c \sum_{r,s=1}^{N} \int_B d\omega \left| \widehat{u}_L(\omega, \mathsf{x}_r, \mathsf{x}_s) - \widehat{A_L}\rho(\omega, \mathsf{x}_r, \mathsf{x}_s) \right|^2$$

où

$$\widehat{A_L}\rho(\omega, \mathsf{x}_r, \mathsf{x}_s) = \int_\Omega \widehat{G}_0(\omega, \mathsf{x}_r, \mathsf{z})\rho(\mathsf{z})\widehat{G}_0(\omega, \mathsf{z}, \mathsf{x}_s) d\mathsf{z}$$

Ici, $\Omega = \{\mathsf{x}_{\mathrm{ref}}\}$, donc :

$$\widehat{A_L}\rho(\omega, \mathsf{x}_r, \mathsf{x}_s) = \widehat{G}_0(\omega, \mathsf{x}_r, \mathsf{x}_{\mathrm{ref}})\rho(\mathsf{x}_{\mathrm{ref}})\widehat{G}_0(\omega, \mathsf{x}_{\mathrm{ref}}, \mathsf{x}_s)$$

$$\rho_{\mathrm{RT}} = \widehat{A}_L^\star \widehat{u}_L$$

Focal spot

$$
\begin{aligned}
\rho_{\mathrm{RT}}(\mathbf{x}) &= \sum_{r,s=1}^{N} \overline{\widehat{G}_0(\omega, \mathbf{x}_s, \mathbf{x})\widehat{G}_0(\omega, \mathbf{x}_r, \mathbf{x})}\,\widehat{G}_0(\omega, \mathbf{x}_r, \mathbf{x}_{\mathrm{ref}})\widehat{G}_0(\omega, \mathbf{x}_s, \mathbf{x}_{\mathrm{ref}}) \\
&\approx \left(\int_{\partial B(0,R_0)} \overline{\widehat{G}_0(\omega, \mathbf{x}_s, \mathbf{x})}\,\widehat{G}_0(\omega, \mathbf{x}_r, \mathbf{x}_{\mathrm{ref}}) d\mathbf{z} \right)^2 \\
&\approx \left(\mathrm{Im}(\widehat{G}_0(\omega, \mathbf{x}_{\mathrm{ref}}, \mathbf{x})) \right)^2 \text{ par identité de Helmholtz-Kirchhoff} \\
&= \frac{1}{16} J_0(\omega|x - x_{\mathrm{ref}}|)^2
\end{aligned}
$$