TP4 : Comp Stats

Exercice 3 : Bayesian analysis of a one-way random effects model.

$\boxed{1}$ We have the errors $\mathcal{E}$ are iid from the centered Gaussian $\mathcal{N}(0, \tau^2)$. ($\mathcal{E} = \{ \varepsilon_{ij},\ i \in \{1, \dots, N\},\ j \in \{1, \dots, k_i\} \}$)

The random effects $X = \{ X_i,\ i \in \{1, \dots, N\} \}$ are iid from a Gaussian $\mathcal{N}(\mu, \sigma^2)$ and independent of the errors $\mathcal{E}$.

* $\forall i \in \{1, \dots, N\},\ \forall j \in \{1, \dots, k_i\},\quad Y_{ij} = X_i + \varepsilon_{ij}$
and since $\varepsilon_{ij} \sim \mathcal{N}(0, \tau^2)$ we can deduce that
$\forall i \in \{1, \dots, N\},\ \forall j \in \{1, \dots, k_i\},$

$$Y_{ij} \sim \mathcal{N}(X_i, \tau^2)$$ (indeed, an affine transformation of a Gaussian variable is still a gaussian variable)

Moreover, the prior distribution is given by:

$$\pi_{prior}(\mu, \sigma^2, \tau^2) \propto \frac{1}{\sigma^{2(1+\alpha)}} \exp\left\{-\frac{\beta}{\sigma^2}\right\} \times \frac{1}{\tau^{2(1+\gamma)}} \exp\left\{-\frac{\beta}{\tau^2}\right\}$$

* The density of the a posteriori distribution $(X, \mu, \sigma^2, \tau^2)$ – up to a normalizing constant – is given by:

$$
\begin{aligned}
P(X, \mu, \sigma^2, \tau^2 \mid Y) &\propto P(Y, X, \mu, \sigma^2, \tau^2) \\
&\propto P(Y \mid X, \mu, \sigma^2, \tau^2) \cdot P(X \mid \mu, \sigma^2, \tau^2) \cdot P(\mu, \sigma^2, \tau^2) \\
&\propto \prod_{i=1}^{N} \prod_{j=1}^{k_i} P(Y_{ij} \mid X_i, \mu, \sigma^2, \tau^2) \times \prod_{i=1}^{N} P(X_i \mid \mu, \sigma^2, \tau^2) \\
&\quad \times \pi_{prior}(\mu, \sigma^2, \tau^2)
\end{aligned}
$$

$$\Rightarrow P(X, \mu, \sigma^2, \tau^2 \mid Y) \propto P(Y, X, \mu, \sigma^2, \tau^2)$$

$$P(X, \mu, \sigma^2, \tau^2 \mid Y) \propto \prod_{i=1}^{N} \prod_{j=1}^{k_i} \left[\frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{ -\frac{1}{2\tau^2} (X_i - Y_{ij})^2 \right\} \right]$$

$$\times \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} (X_i - \mu)^2 \right\}$$

$$\times \frac{1}{\sigma^{2(1+\alpha)}} \exp\left\{ -\frac{\beta}{\sigma^2} \right\} \times \frac{1}{\tau^{2(1+\gamma)}} \exp\left\{ -\frac{\beta}{\tau^2} \right\} .$$