Time crystal in periodically driven classical spin chain

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Abstract: In a periodically driven Floquet system, we have time-dependent Hamiltonian which has discrete time symmetry as the Hamiltonian is periodic in time. The breaking of this symmetry led to the novel state of matter called Discrete Time Crystal(DTC). In this article we recreate some of the results proposed in the paper [3] and [4]. we will also find a comparison with its quantum version as discussed in the paper [2] and look whether there is any periodic behavior of the appearance of DTC in the frequency domain in 1D,2D, and 3D lattice as observed in the quantum case[2].

Keywords: time crystal, Floquet systems, time-translation symmetry.

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1 Introduction

Phases of matter are generally described based on symmetries and symmetry-breaking [1]. One of the classic examples is crystal. the underlying Hamiltonian has continuous translation symmetry but in the ground state, the atom rearranges in a lattice structure that has discrete translation symmetry(i.e it has lower symmetry than its parent Hamiltonian). this breaking of spatial translation symmetry leds to the phase of matter that we are all familiar with, called Crystal. Frank Wilczek extended the concept of symmetry breaking to the time domain. No-go theorem rules out the possibility of quantum time crystals in equilibrium, but it leaves an open door full with potential in the non-equilibrium regime.

Recent discovery of discrete time crystal in periodically driven (Floquet)quantum many-body system has brought the field of research in forefront. Floquet systems are defined with a periodic Hamiltonian $H(t+T) = H(t)$. A discrete-time crystal is a non-equilibrium phase of matter that breaks this discrete-time symmetry. It also shows a stable subharmonic response of order parameters/observables. In floquet system, we look at the stroboscopic evolution of state or observable. let Φ be the time update rule and x be the initial condition. The state of the system after one time period is given as $x(T) = \Phi(x(0))$.

The dynamics exhibit m-fold time-translation symmetry breaking if there exists an observable O that exhibits periodic oscillation out to infinite times for a measurable volume of initial condition:

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{n=1}^{\tau}O(\Phi^{(mn)}(x)) \neq \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{n=1}^{\tau}O(\Phi^{(mn+p)}(x))$$

where $0 < p < m$, corresponds to the m phases of the orbit, with equality restored for $p = m$. A time crystal thus remembers which of the m initial conditions it is in. This is explained in detail in this paper [5].

2 Model

Consider a hypercubic lattice with linear size L and in dimension D=1,2 or 3. Each of the $N = L^D$ lattice sites and they are label as 1,2,...,N with classical spin $S_i = (S_i^x, S_i^y, S_i^z)$. The system is evolved with the following Hamiltonian

$$H(t) = \begin{cases} \sum_{j=1}^{N} JS_j^z S_{j+1}^z + hS_j^z & \text{for t}\in[0,\text{T}/2]\text{mod T} \\ \sum_{j=1}^{N} gS_j^x & \text{for t}\in[\text{T}/2,\text{T}]\text{mod T} \end{cases} \tag{1}$$

where J denotes the nearest-neighbor interaction strength while h and g are the magnetic field strengths along the z and x directions.The spin variable $\vec{S}_j, |\vec{S}_j| = 1$, on-site j satisfies the poisson bracket relation $\{S_i^\mu, S_j^\nu\} = \delta_{ij}\epsilon^{\mu\nu\rho}S_j^\rho$,where $\epsilon^{\mu\nu\rho}$ is the levi-civita symbol.

The time evolution of the system is governed by Hamilton's EOM $\dot{S}_j^\mu = \{S_j^\mu, H(t)\}$.We will be focusing on stroboscopic time evolution. Integrating the EOM over one period T, the evolved state is obtained from a successive application of a discrete map $\vec{S}_j(lT) = [\tau_2 \circ \tau_1]^l$, with l$\in \mathbb{N}$.During the first half-period, the time evolution follows the nonlinear rotation τ_1 about the z-axis:

$$\tau_1(\vec{S}_j) = \begin{pmatrix} S_j^x cos(k_jT/2) - S_j^y sin(k_jT/2) \\ S_j^x sin(k_jT/2) + S_j^y cos(k_jT/2) \\ S_j^z \end{pmatrix} \tag{2}$$

with spin-dependent natural frequency of rotation $K_j = J(S_{j-1}^z + S_{j+1}^z) + h$. In the second half cycle the dynamics follow the rotation

$$\tau_2(\vec{S}_j) = \begin{pmatrix} S_j^x \\ S_j^y cos(gT/2) - S_j^x sin(gT/2) \\ S_j^y sin(gT/2) + S_j^x cos(gT/2) \end{pmatrix} \tag{3}$$

This is the time update rule that we were talking about earlier.
The time-averaged Hamiltonian is given as

$$H_{avg} \equiv H_F^{(0)} = \frac{1}{2}\sum_{j=1}^{N} JS_j^z S_{j+1}^z + hS_j^z + gS_j^x \tag{4}$$

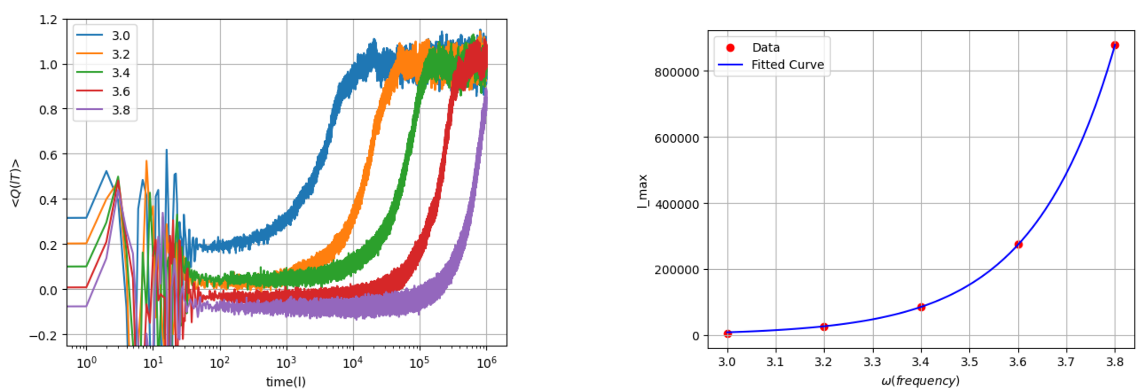
In my computation calculation, I have used J=1,g=0.9045,h=0.809. We set the system's initial state to be in antiferromagnetic ordering with respect to the z-direction. To bring out the many-body character of the model, we add noise to the azimuthal angle of each spin, independently drawn from a uniform distribution over $[-\pi/100, \pi/100]$. We then evolve the system for a long time and observe the energy the system absorbs. The quantities we measure are averaged over multiple noisy states.

H_{avg} constitutes a natural observable to measure the excess energy pumped into the system from the drive. let us define the expected energy over the initial ensemble of noisy AFM states:

$$< Q(lT) > \equiv \frac{< H_{avg}[\{\vec{S}_j(lT)\}] > -E_{initial}}{< H_{avg} >_{\beta=0} -E_{initial}} \tag{5}$$

the normalization is chosen with respect to an infinite temperature ensemble, where each spin points at a random direction, and hence $< H_{avg} >_{\beta=0} = 0$. Initializing the system in the ensemble of noisy AFM states, initially, we have $< Q(lT) > \approx 0$ and $< Q(lT) > = 1$ whenever the ensemble is heated to infinite temperature.

I am defining the $l_{max}(\Omega)$ to be the value of l when it crosses 0.8 mark of $< Q(lT) >$,(i.e when the system is about to thermalized to infinite temperature).here is the plot of l vs Ω.the data we got is fitted with the equation $d(g^{\alpha})e^{cx}$, where c, d, α are parameter. After optimizing, the value of the parameters are $\alpha \approx 29.18$,$c \approx 5.83$,$d \approx 0.0038$.



(a) (b)

Figure 1: a)Noise-average energy as a function of the number of driving cycles for different frequencies 3.0,3.2,3.4,3.6,3.8. It's averaged over 30 noisy samples.b)l_{max} vs frequency fitted with the equation $d(g^{\alpha})e^{cx}$

From the data, we can clearly see that the time between the onset of prethermal plateau to the infinite temperature state grows exponentially with frequency. Further ,we consider the magnetization of the system along the z direction,

$$m = \frac{1}{N}\sum_{i}^{N} S_i^z \tag{6}$$

To analyze the chaos in the dynamics, we introduce a measure of the distance between two initially very close copies of the system,$\{S_i\}$ and $\{S_i^{'}\}$ with $S_i(0) \approx S_i^{'}(0)$:

$$d(t) = \sqrt{\frac{1}{N}\sum_{i}^{N}[S_i(t) - S^{'}(t)]^2} \tag{7}$$

this measure is called "decorrelator" which directly probes the hallmark of chaos: sensitivity to initial condition.In the spherical coordinates, the copy of the system$\{S_i^{'}\}$ is initialized as

$$\begin{aligned} \theta_r^{'}(0) &= \theta_r(0) + 2\pi\Delta\delta_{\theta,r} \\ \phi_r^{'}(0) &= \phi_r(0) + 2\pi\Delta\delta_{\phi,r} \end{aligned} \tag{8}$$

with $\delta_{\theta,r}$ and $\delta_{\phi,r}$ are numbers drawn from Gaussian distribution with zero mean and standard deviation 1. The parameter Δ controls the initial distance between the two copies of the system $\{S_I\}$ and $\{S_i^{'}\}$.
the initial value of the decorrelator is small and is controlled by the perturbation strength $d(0) \sim \Delta$. In short times d is expected to grow exponentially, $d \approx d(0)e^{\lambda t}$ with the rate of growth controlled by λ and at long times when the system thermalizes it reaches its maximum value of $d_\infty = \sqrt{2}$, which corresponds to complete random spin orientation of both the copies and the system has lost all its initial memories .

Initial condition

We start with all spin aligned in the z-direction, corresponding to polar angles $\theta_i = 0$. To bring the many body characters into play, we perturb the polar angle with Gaussian noise with zero mean and standard deviation $2\pi W$, while the azimuthal angles ϕ_i are taken as uniformly distributed in the range from 0 to 2π.

$$\begin{aligned} &\theta_i(0) \sim Gauss(0, 2\pi W), && p(\theta) = \frac{e^{-\frac{1}{2}(\frac{\theta}{2\pi W})}}{2\pi W\sqrt{2\pi}} \\ &\phi(0) \sim Unif(0, 2\pi), && p(\phi) = \frac{1}{2\pi}\mathbf{1}_{[0\leq\phi<2\pi]} \end{aligned} \tag{9}$$

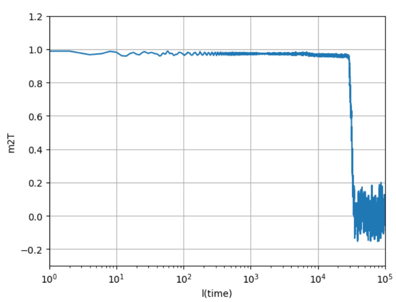
with p denoting the probability density function and $\mathbf{1}_{[0\leq\phi<2\pi]}$ is the indicator function equal to 1 when $0 < \phi \leq 2\pi$ and to zero otherwise. Since the perturbation has an axial symmetry with respect to the x-axis, for $N \to \infty$ we have$\frac{1}{N}\sum_{i}^{N} S_i^{x,y} = 0$. The magnetization $m = \frac{1}{N}\sum_{i}^{N} S_i^z$ along z instead goes from 1 for $W = 0$ to 0 for $W = \infty$. So physically we can think of W sort of temperature of the initial condition.

Results

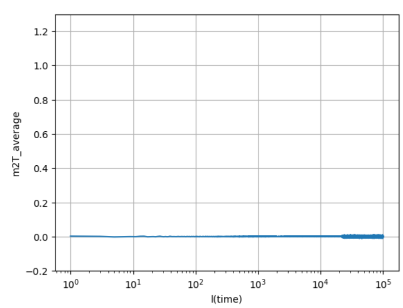
For $g = 0.515 \approx 1/2$ the spins are rotated at every period of an angle $\approx \pi$. At short time $t = 0, T, 2T, 3T, .. << \tau_{preth}$, intuitively we expect the spin to approximately follow this sequence $z, -z, z, -z$....but the remarkable thing is that we see this period double subharmonic response even in long time ie $\tau_{preth} < t < \tau_{ther}$. At very long time we don't see this sequence as the spin are aligned in a completely random direction .

More generally, a prethermal-n-DTC is characterized by two well-separated prethermalization and thermalization timescales,$\tau_{preth} \approx 1/\lambda$ and $\tau_{th} \approx e^{c\omega}$ For $t << \tau_{preth}$ the system exhibits a subharmonic response but the system is not stable in the transition region from t=0 to the beginning of the prethermal plateau The system has not yet equilibrated at stroboscopic time $t = nkT$ to the effective Hamiltonian H_{eff} that governs the dynamics at stroboscopic times t=T,nT,2nT,...This can be confirmed through d(decorrelator) as in this region $d \approx 0$, which means that the sensitivity to the initial condition is not yet expressed.For $\tau_{preth} < t < \tau_{th}$ the system at stroboscopic time $t = nkT$ has equilibrated to the effective Hamiltonian H_{eff} while the dynamics has remained subharmonic with periodicity nT, this is the distinctive feature of the prethermal-n-DTC. We can also see the sensitivity to the initial condition is prominent as $d/d\infty \sim 0.2$ but still remains correlated to the initial conditions as $d/d\infty < 1$. At later times $t >> \tau_{th}$, the system has reached the infinite temperature state, in which the spins have completely random orientation and memory of the initial condition has been completely lost($d/d_\infty = 1$).

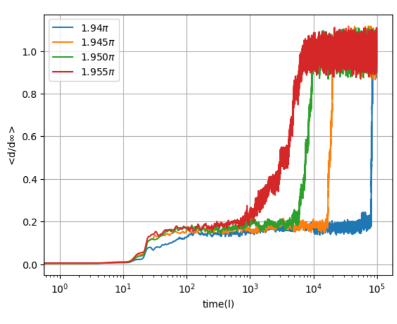
One key property for a prethermal DTC is that the prethermal plateau increases with ω which has been previously shown in 1D lattice, here we will show this in 2D lattice.
In our computer simulation, we have worked with g=0.515. So we simulated a 2-DTC. To ensure the existence of 2-DTC we used an order parameter which in our case is magnetization(m) along z direction. we measured magnetization after time t=0,2,4,..., which we call m2T. we also measured another quantity $m2T_{average}(2nT) = [m(2nT) + m((2n+1)T)]/2$. If there is a 2-DTC, we expect in the prethermal plateau the $m2T \approx 1$ and $m2T_{average} \approx 0$.



(a)



(b)



(c)

Figure 2: a)m2T vs time plot ,g=0.515 and $T = 1.945\pi$. b)$m2T_{average}$ vs time plot. c)decorrelator plot for different value of T

Clearly, our intuitive expectation is correct, as evident from the numerical results. You can also see that the prethermal plateau also increases with increase in the frequency(decrease in time period).

Motivated by the result in the paper [2] in which for the values of $JT \approx (2n+1)\pi$ we get a very stable time crystalline behavior. That naturally leds to the question, do we see the same periodic behavior in the classical case? To answer that question we have plotted the density plot of m2T(g vs T) Figure 3a and the density plot of decorrelator(g vs T) Figure 3b.

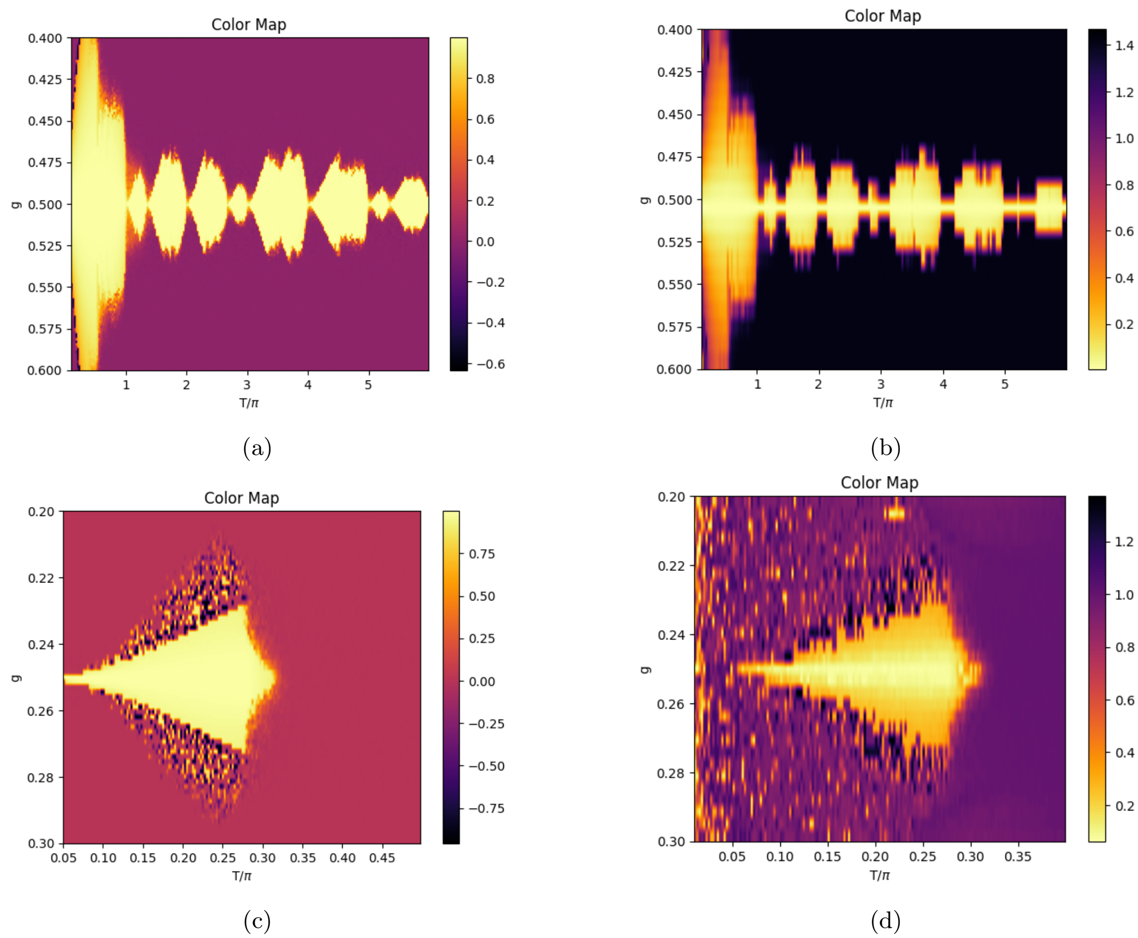
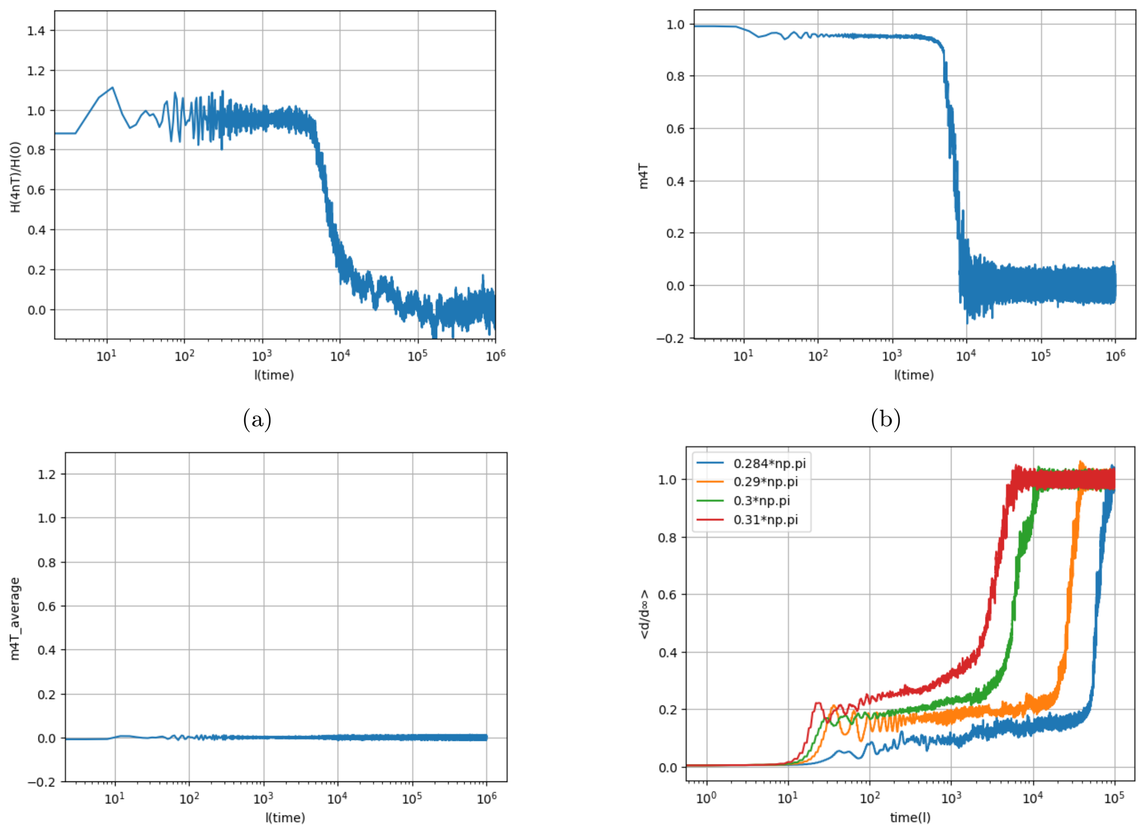


Figure 3: a)for 2-DTC , m2T(density plot)(g vs T/π),W=0.001.b)decorrelator density plot for 2-DTC (g vs T/π). c)for 4-DTC , m4T(density plot)(g vs T/π). d)decorrelator density plot for 4-DTC (g vs T/π) W=0.001 is constant for all.
All of these results are for 2D lattice

The results are very interesting. In Fig 3 a) we do see a periodic appearance of time crystal for 2-DTC with respect to T or frequency. Fig 3 b) gives the density plot for the decorrelator(g vs T), similar periodic behavior is seen here also. We also simulated a 4 DTC and analyzed the behavior through the plots in fig 3 c,d. Here m4T is defined in a similar way ie we measure the magnetization at 0,4T,8T,...and so on. We also simulated a 4-DTC in a 3D lattice structure . We set $g = 0.255 \approx \frac{1}{4}$, in this case the spin are rotated in each period by an angle $\pi/2$. Intuitively, the sequence of spin orientation is $\approx +z, -y, -z, +y, +z, \ldots$. similar to 2-DTC we plot the energy, magnetization and decorrelator after every 4T interval of time and define $m4T_{average}(4nT) = (m(4nT) + m((4n+1)T) + m((4n+2)T) + m((4n+3)T))/4$.if we have a stable 4-DTC then for $\tau_{preth} < t < \tau_{th}$ we expect $m4T \approx 1$ and $m4T_{average} \approx 0$ and we should get a stable $H(4nT)$ within this time period. The decorrelator should also increase and get stable at finite value $d < \sqrt{2}$ and attain the maximum value d_∞ when the system reaches its infinite temperature state, which depicts the sensitivity to its initial condition. Let us look at the plot and see what we have got



(a) (b) (c) (d)

Figure 4: a)for 4-DTC $T = 0.3\pi$ for fig 4 a,b,c, H(4nT)/H(0) vs Time .b)m4T vs Time .c) $m4T_{average}$ .d)decorrelator vs Time at different values of T averaged over 10 noisy sample All of these results are for 2D lattice

We see that our intuiative analysis matched the plots which ensures that we have a stable 4-DTC. In Fig 4.d we see that prethermal plateau(life time of 4-DTC) increases with increase in the frequency(decrease in time period).

Next, we looked at a 3D lattice structure and went through the same analysis. First, let's look at the plot of 2-DTC in 3D lattice

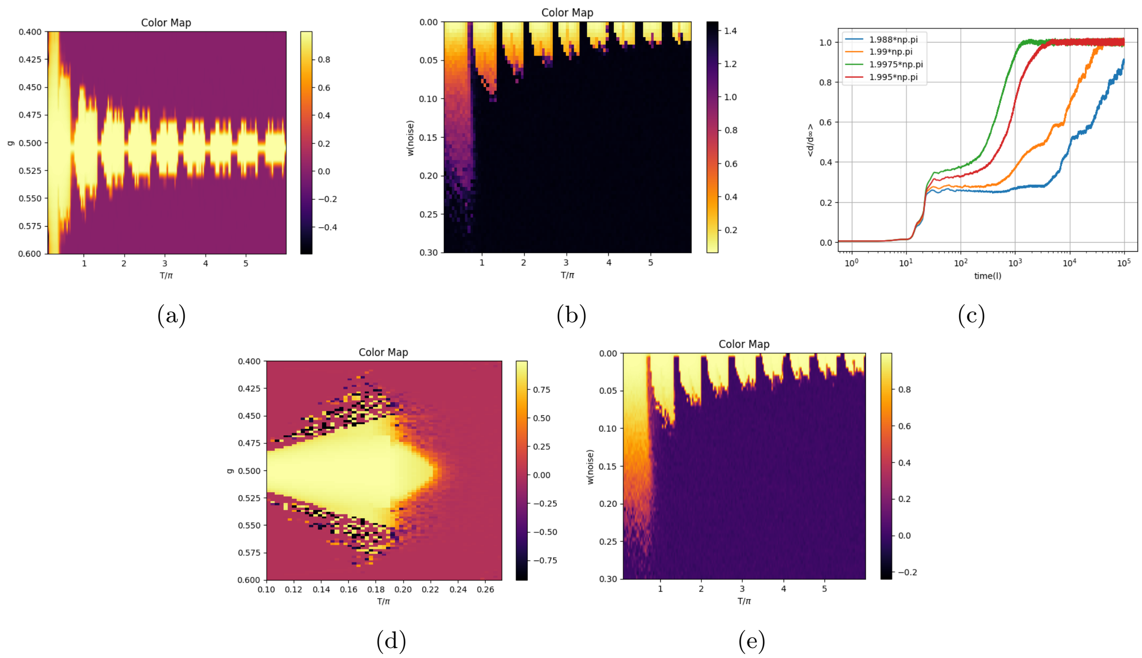


Figure 5: a)for 2-DTC , m2T(g vs T/π) density plot .b)decorrelator(w vs T/π) density plot . c) decorrelator plot for different value of T d)4-DTC in 3D, m4T density plot(g vs T/π).e)2-DTC m2t(w(noise) vs T/π) density plot
All of these results are for 3D lattice

Here we have some interesting results. Firstly in Fig 4 a) we see the periodic appearance of 2-DTC similar to what we observed in the case of 2-D lattice. but what is non-trivial is that there is a profile for finding the DTC which decays with increasing T. In Fig 4 b) we see how the noise affects the stability of DTC. With increased noise, DTC is destroyed, here also we see a decaying profile. finally, in Figure 3 c)we observe that the prethermal plateau increases with increased frequency.

There was also one thing that is different in the classical spin chain compared to the quantum case as discussed in [2]. In this paper, he has shown that the prethermal plateau increases with the system size but in the classical case we saw that the prethermal plateau remains unaffected by the system size. We also simulated a 4-DTC in a 3D lattice structure and the density plot of m4T (g vs T/π) and m4T(w(noise) vs T/π) is plotted in Fig 3 d,e respectively.

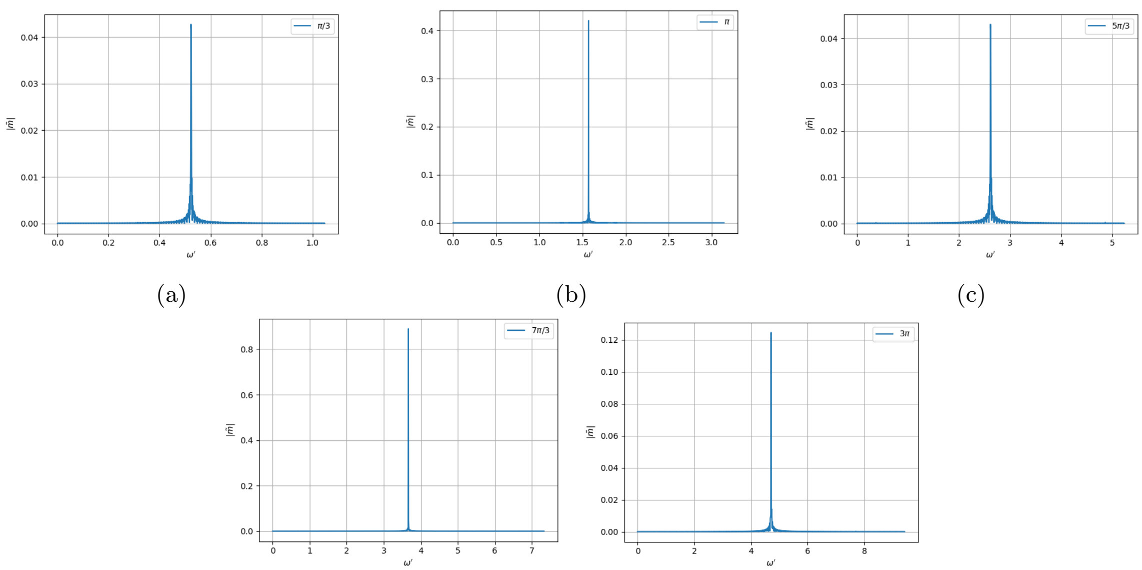
Now the next question is, can we distinguish between the different region in the density plot of m2T(g vs T) where we find DTC ? ie to say, can we distinguish between DTC that we get at $T = \pi/3, \pi, 5\pi/3$ and so on , from each other. To answer that question we look into the fourier spectrum around each of the T that is specified before.

The fourier spectrum of magnetization(m) is given as

$$\tilde{m}(\omega') = \frac{1}{M}\sum_{n=0}^{M-1} m(nT)e^{-i\omega' nT} \tag{10}$$

computed over a number of periods M.The choice of M is 10^4 in our case.

We find the fourier spectrum around $(0, \omega_0)$ for $\omega_0 \in \{\pi/3, \pi, 5\pi/3, 7\pi/3, 3\pi\}$

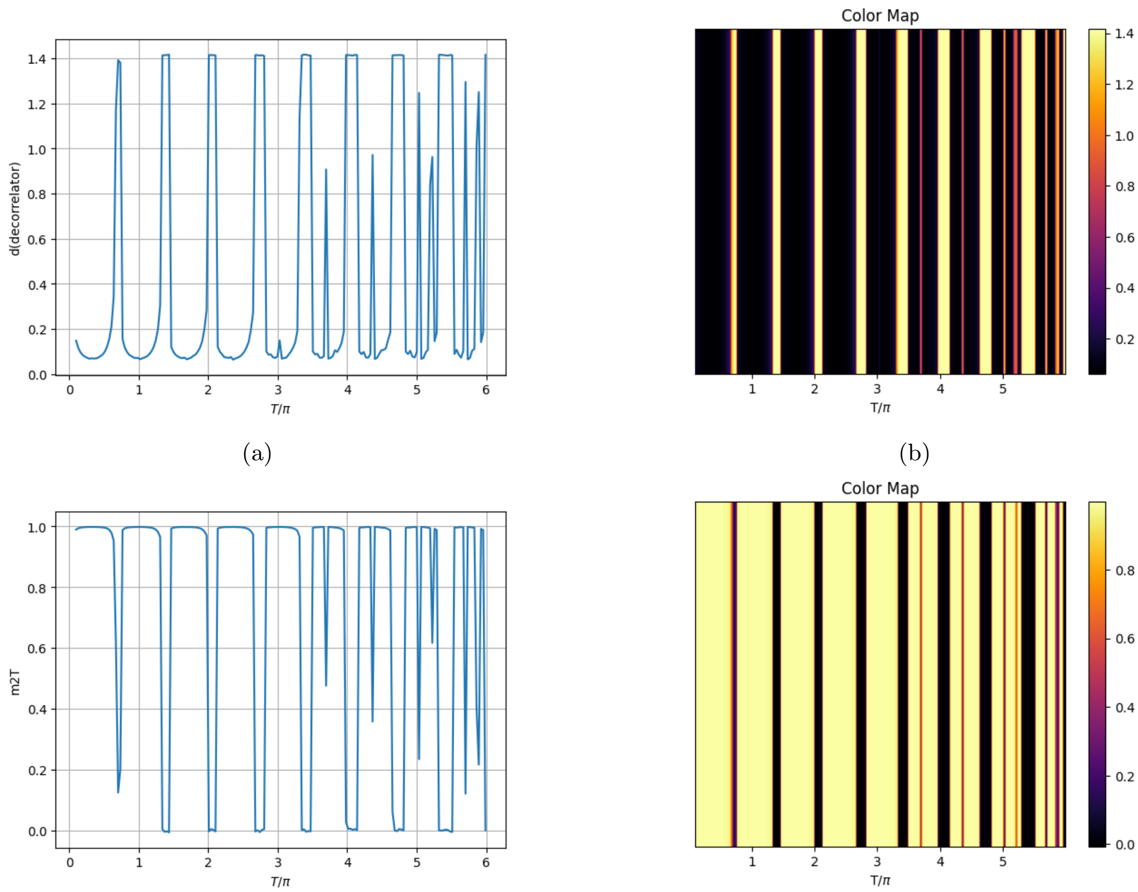


(a) (b) (c)

(d) (e)

Figure 6: Fourier spectrum about a)$T = \pi/3$,b)$T = \pi$,c)$T = 5\pi/3$,d)$T = 7\pi/3$,e)$T = 3\pi$, in the range (0,ω)

In each of the cases, we see that the maximum contribution comes from the frequency $\omega/2$.This is expected as the frequency of the order parameter(magnetization) in our case has a double period(half frequency).To explore more we will fix g and look at how magnetization and decorrelator varies with T. For 2-DTC(in 3D) these are the results . Here m2T is the average value(averaged over time) of magnetization at every 2T interval in the time range $10^3 - 10^5$.

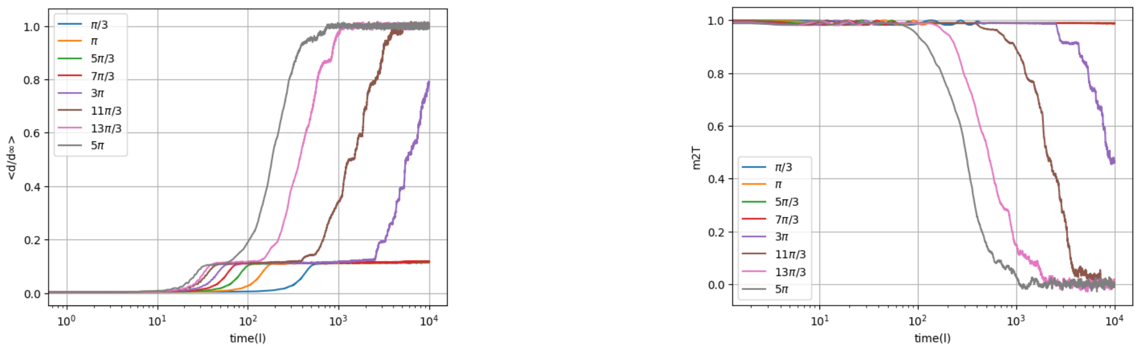


(a) (b) (c) (d)

Figure 7: For g=0.515 a)plot of decorrelator(d) vs T and b) is the density plot of a . c) plot of m2T vs T d)is the density plot of c
This results are for 2-DTC(in 3D-lattice)

We can see a periodic pattern here, also we see that for a higher value of T, in each blob(strip) there is some non-monotonous behavior. Here there is another interesting behavior. If you take say T_1 in the first strip and T_2 in the 2nd strip you will expect as $T_2 > T_1$, 2-DTC will be more stable(longer prethermal plateau) in the 1st strip than the 2nd strip. but that might not always be true. In fact we found that there is no correlation between T(or frequency) and length of prethermal plateau(stability of DTC) among different strips. But if you compare the value of T in the same strip then there is the trend that with increasing T(decreasing frequency) the prethermal plateau gets smaller i.e the DTC becomes less stable.

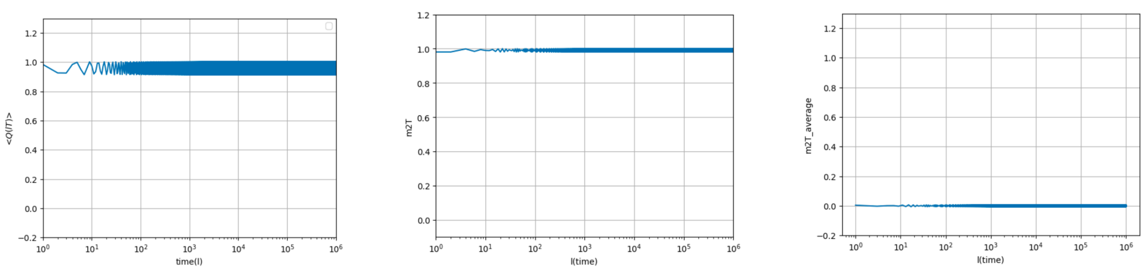
We can also study at the dynamics of the 2-DTC by plotting the decorrelator and m2T at the value of T$\in \{\pi/3, \pi, 5\pi/3, 7\pi/3, 3\pi, 11\pi/3, 13\pi/3, 5\pi\}$.



(a) (b)

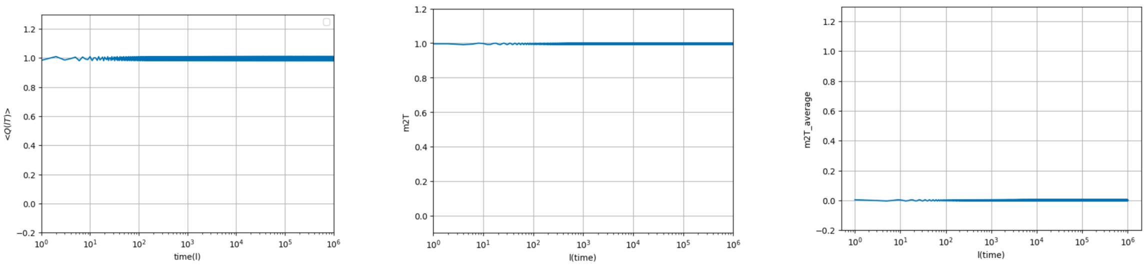
Figure 8: For g=0.53 a)plot of decorrelator(d) vs time for different values of T and b) is the plot of m2T vs time for different values fo T . These results are averaged over 40 noisy sample. This results are for 2-DTC(in 3D-lattice)

Next we tried to see weather we can distinguish between Time crystal that we get in different sector. To do that we took the mean field result by setting g=0.52,h=0(1 sample) we have all the spin pointing in the z direction with no error at time t=0 and have evolved the system from there.We will study the behaviour of heat absorbed over time and the magnetization .we looked at almost the center of each blob in the figure 5a in the time range $t = 0$ to $10^6 T$.All the result are for 2-DTC in 3D lattice



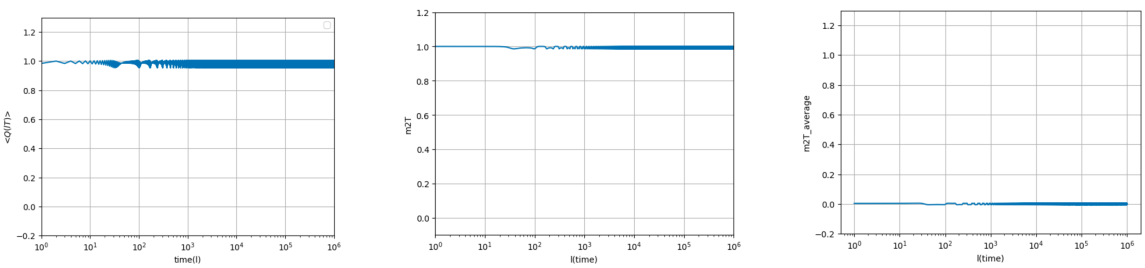
(a) (b) (c)

Figure 9: For 3D lattice $T = 1.2\pi$



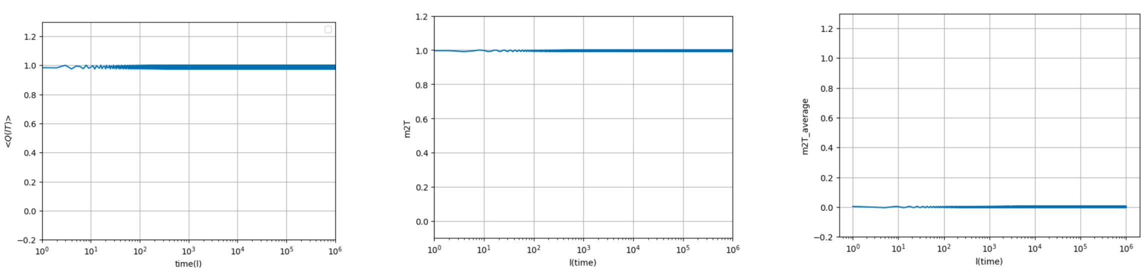
(a) (b) (c)

Figure 10: For 3D lattice $T = 1.6\pi$



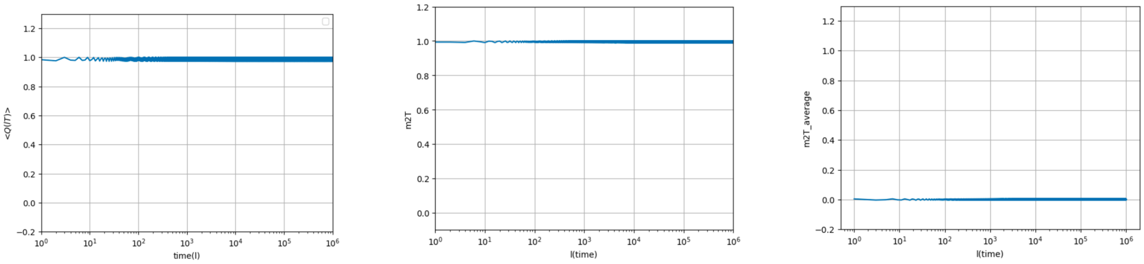
(a) (b) (c)

Figure 11: For 3D lattice $T = 2.35\pi$



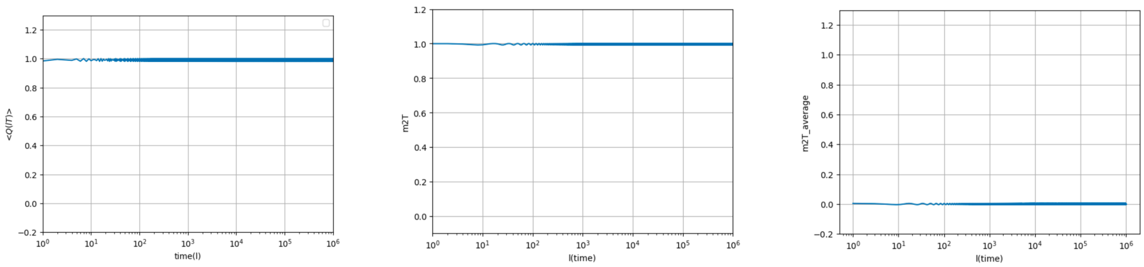
(a) (b) (c)

Figure 12: For 3D lattice $T = 3.1\pi$



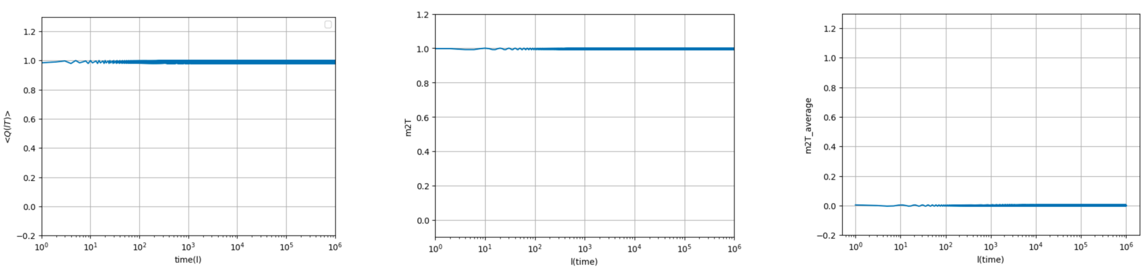
(a) (b) (c)

Figure 13: For 3D lattice $T = 3.8\pi$



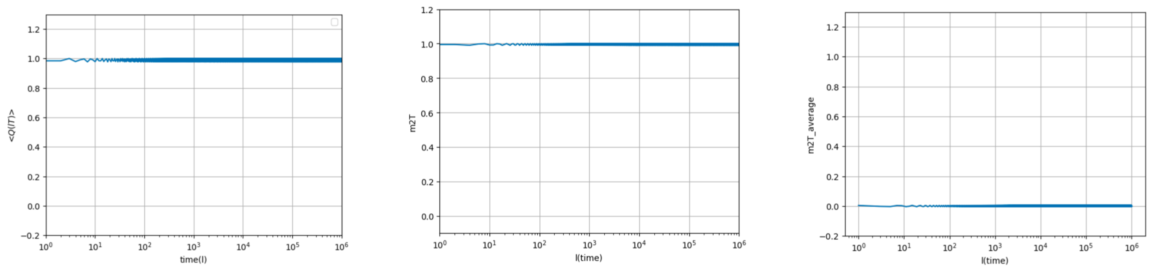
(a) (b) (c)

Figure 14: For 3D lattice $T = 4.4\pi$



(a) (b) (c)

Figure 15: For 3D lattice $T = 5.1\pi$



(a) (b) (c)

Figure 16: For 3D lattice $T = 5.8\pi$

As we can see all the graphs are very similar. We have very stable time-crystals which are hardly different from each. From this study we could not find any difference between the time crystal at different sectors in figure 5a.

Acknowledgement

I especially want to thank Prof Sayan Choudhury,who gave me an opportunity to work on this fascinating area of science - Time Crystal. His insights on this concept were pivotal for getting these results. He has been very helpful and open to discussion the entire time.

All my code are available in this link CODE

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