Optical Fano Resonances

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Abstract:- This article will go through the optical Fano resonance first introduced in 1961 by U.Fano in the paper [1]. This explains the interference of discrete autoionized states with a continuum that gives rise to asymmetric peaks in excitation spectra. Here we also go through the the interaction of one discrete level with two or more continua and of a set of discrete levels with one continuum. We also looked into the classical analog of Fano resonance and some application and future prospects.

I. INTRODUCTION

Fano resonance is a phenomenon that occurs when two scattering processes interfere with each other, resulting in an asymmetric line shape. This interference arises due to the interaction between a broad, continuous background process and a narrow, resonant process. The resulting line shape is characterized by a sharp peak or dip, accompanied by an asymmetric tail.

This phenomenon was first observed by Ettore Majorana in 1935, but it was Ugo Fano who provided a theoretical explanation for it in 1961. Fano resonance is a weak coupling effect, meaning that the interaction between the two processes is not strong enough to create a new, hybridized state. Instead, the coupling modifies the properties of the resonant state, such as its energy and width, leading to the distinctive Fano line shape.

Fano resonance is a general wave phenomenon that can be observed in various fields of physics and engineering. It is particularly prominent in quantum mechanics, where it can occur in systems like atoms, molecules, and quantum dots. In optics, Fano resonance can be observed in the interaction of light with nanostructures, leading to sharp spectral features and enhanced light-matter interactions.

The original explanation of Fano resonance involved the inelastic scattering of electrons by helium atoms. In this process, an incident electron excites the helium atom to a higher energy state, known as an autoionization state. This state is unstable and quickly decays by emitting an electron. Fano showed that the interference between the direct scattering of the incident electron and the scattering through the autoionization state leads to the asymmetric Fano line shape.

In recent years, Fano resonance has gained significant attention due to its potential applications in various fields, including optical sensing, nonlinear optics, and quantum information processing. By carefully controlling the parameters that influence Fano resonance, researchers can manipulate the spectral properties of light and create novel devices with unique functionalities.

II. CLASSICAL ANALOG OF FANO RESONANCE

Here we will consider two coupled oscillators. Here we are driving one of the spring and the two springs are coupled to one another. We will study the behavior of the amplitudes after the transient motion decays. The equations of motion may be written as

$$\ddot{x_1} + \gamma_1 \dot{x_1} + \omega_1^2 x_1 + v_{12} x_2 = a_1 e^{i\omega t} \tag{1}$$

$$\ddot{x_2} + \gamma_2 \dot{x_2} + \omega_2^2 x_2 + v_{12} x_1 = 0 \tag{2}$$

where v_{12} describes the coupling of oscillators. the steady state solution is given as

$$x_1 = c_1 e^{i\omega t}, x_2 = c_2 e^{i\omega t} \tag{3}$$

Here, the amplitudes have the forms

$$c_1(\omega) = \frac{\omega_2^2 - \omega^2 + i\gamma_2\omega}{\left(\omega_1^2 - \omega^2 + i\gamma_1\omega\right)\left(\omega_2^2 - \omega^2 + i\gamma_2\omega\right) - v_{12}^2} a_1 \tag{4}$$

$$c_2(\omega) = -\frac{v_{12}}{\left(\omega_1^2 - \omega^2 + i\gamma_1\omega\right)\left(\omega_2^2 - \omega^2 + i\gamma_2\omega\right) - v_{12}^2} a_1 \tag{5}$$

the phases of the oscillators are defined through

$$c_1(\omega) = |c_1(\omega)|\, e^{-i\phi_1(\omega)}, \quad c_2(\omega) = |c_2(\omega)|\, e^{-i\phi_2(\omega)} \tag{6}$$

$$c_2(\omega) = -\frac{v_{12}}{(\omega_1^2 - \omega^2 + i\gamma\omega)(\omega_2^2 - \omega^2 + i\gamma_2\omega - v_{12}^2)} a_1 \tag{7}$$

The phases of the oscillator are defined through

$$c_1(\omega) = |c_1(\omega)|\, e^{-i\phi_1(\omega)}, \quad c_2(\omega) = |c_2(\omega)|\, e^{-i\phi_2(\omega)} \tag{8}$$

note that the phase difference between two oscillator is given by

$$\phi_2 - \phi_1 = \pi - \theta \tag{9}$$

where the extra phase shift θ is defined by the numerator of equation as

$$\theta = \arctan\left(\frac{\gamma_2\omega}{\omega_2^2 - \omega^2}\right). \tag{10}$$

When one of the oscillator has large damping then the we can find the following sets of equations:

$$|c_1(\omega)|^2 \approx |{a_1}^2| \frac{\gamma_1^2}{(\omega_1-\omega_2)^2+\gamma_1^2} \frac{(\Omega+q)^2}{\Omega^2+1} \tag{11}$$

where

$$\Omega = [\omega-\omega_2+(\frac{v_{12}^2}{\gamma_1})\frac{\omega_1-\omega_2}{1+q^2}]\frac{\gamma_1(1+q^2)}{v_{12}^2} \tag{12}$$

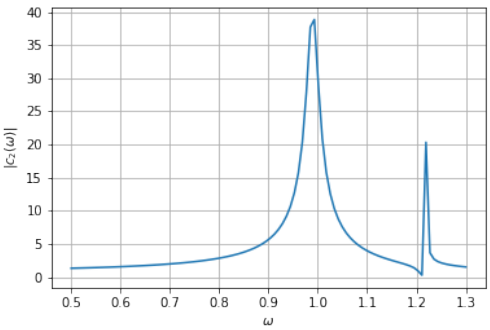

FIG. 1: Oscillation amplitude of the 1st particle

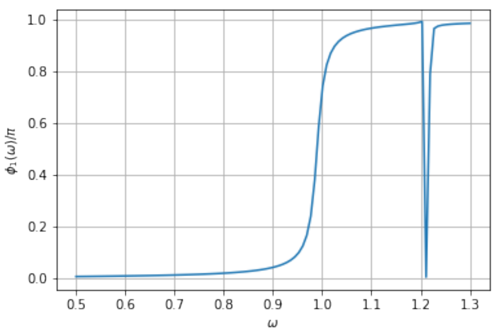

FIG. 2: Phase behaviour of the first particle

here $q = \cot\delta$ where δ is the phase of the response function $(\omega_2-\omega_1+i\gamma_1)^{-1}$. We will consider the case when $\gamma_2 = 0$(no friction for the 2nd oscillator).$\gamma_1 = 0.025$ and $v_{12} = 0.1$. we got two resonant peaks one symmetric at $\omega \approx 1$ and the other asymmetric at $\omega \approx 1.21$.the reason why the second resonant peak is asymmetric is due to the existence of the zero-frequency at $\omega_0 = \omega_2 = 1.2$ which is right near the peak position .it can be seen from equation that the amplitude of the first oscillator c_1 becomes zero at $\omega = \omega_2$ when $\gamma_2 = 0$.accordingly the line shape of the second resonance becomes distorted. note that the amplitude of the second oscillator c_2 tends to $\frac{a_1}{v_{12}}$ in equation at the zero-frequency.

Now we will analyze the behavior of the system.The dampled oscillation has a phase shift

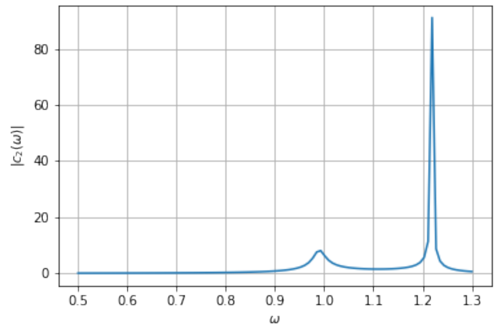

FIG. 3: oscillation amplitude of the 2nd particle

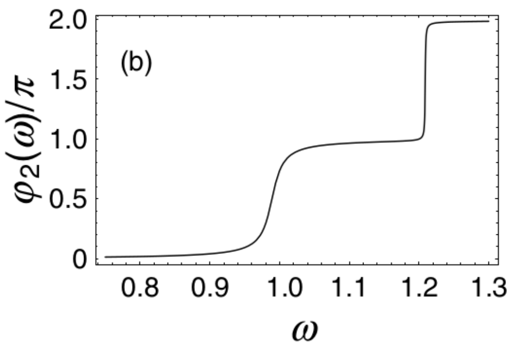

FIG. 4: Phase behaviour of the 2nd particle

$-\cot^{-1}((\omega^2-\omega_1^2)/\gamma_1\omega_2)$ with respect to the external oscillatory force. In the case of finite coupling v_{12} , the amplitude c_1 will give a characteristic resonance behaviour reflecting the presence of the second oscillator near the frequency ω_2.when the first oscillator is being driven near the resonance $\omega \leq Re[\omega_1]$, the amplitude quickly grows to a maximum and the displacement x_1 of the first oscillator gets the phase $\pi/2$ right across the resonance. After the frequency passes through the first resonance, but before it meets the zero-frequency ω_{zero} the first oscillator settles into steady state motion, and the displacement x_1 is eventually π out of phase wrt the external force. Next ,when the frequency sweeps through the second resonance, the oscillator gains the phase factor by π.The phase shift at $\omega = \omega_2$, $-\cot^{-1}((\omega_2^2-\omega_1^2)/\gamma_1\omega_2)$, of the damped oscillator will behave as the background phase shift for the resonance behaviour of $c_1(\omega)$ in the vicinity of $\omega = \omega_2$

for the second oscillator, two resonant peaks manifest the symmetric line shapes. The phase gain of the second oscillator by π is clearly seen in the graph at each time when the frequency passes through the resonances. The coupled amplitudes at the zero-frequency, we find that the first oscillator is out of phase with the second oscillator as ω goes through ω_0 and that, at this particular frequency, the motion of the first oscillator is quenched enforced effectively by the second oscillator. You can look in the paper [2] for more details.

III. FANO RESONANCE IN CASE OF ONE DISCRETE STATE AND ONE CONTINUUM

Consider an atomic system with a number of zero approximation states, and among these states one state called ϕ belongs to a discrete configuration and other called $\psi_{E'}$ belongs to a continuum of states. These states are non-degenerate i.e. we can specify each set with an adequate amount of quantum numbers[1].

Now our main goal is to diagonalize the portion of the energy matrix that belongs to the subset of states ϕ and $\psi_{E'}$. The elements of the portion of energy matrix corresponds to a square submatrix and is indicated by

$$\langle\phi| H |\phi\rangle = E_\phi \tag{13a}$$
$$\langle\psi_{E'}| H |\phi\rangle = V_{E'} \tag{13b}$$
$$\langle\psi_{E''}| H |\psi_{E'}\rangle = E'\delta(E''-E') \tag{13c}$$

Now, from Eq. 13c, we can imply that the submatrix belonging to the continuum state $\psi_{E'}$ has been previously diagonalized in the zero approximation. And it is also understood that the discrete energy level E_ϕ lies within the continuous range of values of E'.

Each energy value E within the range of E' is an eigenvalue of the matrix 13. Now from our basic understanding of quantum mechanics, say if ϕ_i is a set of discrete eigenstates with energy eigenvalue E, any state ψ_E with energy eigenvalue E can be written as,

$$\psi_E = \sum_i a_i\phi_i \tag{14}$$

Now, in our case, only $\phi_1 = \phi$ is a discrete state and others belong to continuum configuration, So our state Ψ_E can be written as,

$$\begin{aligned}\Psi_E &= a\phi + \sum_i b_i\psi_{E'_i}\\ &= a\phi + \int dE' \quad \underbrace{g(E')}_{\text{density of states}} \quad b_i\psi_{E'_i}\\ &= a\phi + \int dE'\; b_{E'}\psi_{E'}\end{aligned} \tag{15}$$

where both coefficient a and $b_{E'}$ are functions of E. Now these coefficient can easily be determined by the system of equations of the matrix 13. As Ψ_E is a eigenvector of energy matrix with eigenvector E, we can write,

$$H|\Psi_E\rangle = E|\Psi_E\rangle \tag{16}$$

By Eq. 13a,

$$\begin{aligned}&\langle\phi| H |\Psi_E\rangle = E\langle\phi|\Psi_E\rangle\\ \Rightarrow &a\langle\phi| H |\phi\rangle + \int dE'\; b_{E'}\langle\phi| H |\psi_{E'}\rangle\\ &= E\left(a\langle\phi|\phi\rangle + \int dE'\; b_{E'}\langle\phi|\psi_{E'}\rangle\right)\\ \Rightarrow &E_\phi a + \int dE'\; V^*_{E'}b_{E'} = Ea\end{aligned} \tag{17}$$

Similarly by Eq. 13b,

$$\begin{aligned}&\langle\psi_{E'}| H |\Psi_E\rangle = E\langle\psi_{E'}|\Psi_E\rangle\\ \Rightarrow &a\langle\psi_{E'}| H |\phi\rangle + \int dE''\; b_{E''}\langle\psi_{E'}| H |\psi_{E''}\rangle\\ &= E\left(a\langle\psi_{E'}|\phi\rangle + \int dE''\; b_{E''}\langle\psi_{E'}|\psi_{E''}\rangle\right)\\ \Rightarrow &V_{E'}a + \int dE''\; E''b_{E''}\delta(E'-E'')\\ &= E\int dE''\; b_{E''}\delta(E'-E'')\\ \Rightarrow &V_{E'}a + E'b_{E'} = Eb_{E'}\end{aligned} \tag{18}$$

So, finally we get the system of equations,

$$E_\phi a + \int dE'\; V^*_{E'}b_{E'} = Ea \tag{19a}$$
$$V_{E'}a + E'b_{E'} = Eb_{E'} \tag{19b}$$

The solution of this system of equations can be carried out exactly and thus the diagonalization of matrix 13 is achieved.

System 19 has some peculiarities due to the continuous spectrum. To solve it, we shall express $b_{E'}$ in terms of a with help of Eq. 57b. This procedure involves a division of Eq. 57b by $E-E'$ which may be zero. So, we will follow Dirac's procedure[3], and express the formal solution of Eq. 57b as ,

$$b_{E'} = \left[\frac{1}{E-E'} + z(E)\delta(E-E')\right]V_{E'}a \tag{20}$$

Now for the sake of simplicity, let's say $\psi_{E'}$ is represented by a wave function with asymptotic behaviour like $\sin(k(E')r+\delta)$. With the help of Eq 20, and using the fact that at large r, $\int dE'\; \frac{\sin[k(E')r+\delta]}{E-E'} \approx -\pi\cos[k(E)r+\delta]$, we can write

$$\int dE'\ b_{E'}\psi_{E'}$$
$$= \int dE' \Big(\frac{1}{E-E'} + z(E)\delta(E-E')\Big) V_{E'} a \psi_{E'}$$
$$\propto \int dE' \Big(\frac{\sin[k(E')r+\delta]}{E-E'} + z(E)\delta(E-E')\sin[k(E')r+\delta]\Big)$$
$$\propto -\pi\cos[k(E)r+\delta] + z(E)\sin[k(E)r+\delta]$$
$$\propto \sin[k(E)r+\delta+\Delta]$$

Where,

$$\Delta = -\arctan\left(\frac{\pi}{z(E)}\right) \tag{21}$$

This represents the phase shift due to configuration interaction between $\psi_{E'}$ and ϕ.

Now to determine the value of $z(E)$, we substitute Eq. 20 in Eq.57a,

$$E_\phi + \int dE'\ V^*_{E'}\Big[\frac{1}{E-E'} + z(E)\delta(E-E')\Big] V_{E'} a = Ea$$
$$\Rightarrow \int dE'\ |V_{E'}|^2 a\Big[\frac{1}{E-E'} + z(E)\delta(E-E')\Big] = (E-E_\phi)a$$
$$\Rightarrow F(E) + z(E)|V_E|^2 = E - E_\phi$$
$$\Rightarrow z(E) = \frac{E - E_\phi - F(E)}{|V_E|^2} \tag{22}$$

Where,

$$F(E) = P\int dE'\ \frac{|V_{E'}|^2}{E-E'} \tag{23}$$

P means " *principal part of* ". $|V_E|^2$ is the measure of strength of configuration interaction and has dimension of energy as $\psi_{E'}$ is normalized "per unit energy" owing to Eq. 13c. Notice that, as E transverses an interval $|V_E|^2$ about the "resonance" at $E = E_\phi + F$, a phase shift of π occurs. F corresponds to the shift of resonance position with respect to E_ϕ[1].

The coefficient a is determined by normalization. The ortho-normalization condition for the continuous spectrum must be expressed in terms of the coefficients a and $b_{E'}$ for a pair of values of E, indicated by E and $\bar{E}$, which may not coincide:

$$\langle\Psi_{\bar{E}}|\Psi_E\rangle = a(\bar{E})^* a(E) + \int dE'\ b^*_{E'}(\bar{E}) b_{E'}(E) = \delta(\bar{E}-E) \tag{24}$$

Substituting Eq. 20, we get,

$$a(\bar{E})^* a(E) + \int dE'\ |V_{E'}|^2 a^*(\bar{E})\Big[\frac{1}{\bar{E}-E'} + z(\bar{E})\delta(\bar{E}-E')\Big]\times \Big[\frac{1}{E-E'} + z(E)\delta(E-E')\Big]a(E) = \delta(\bar{E}-E) \tag{25}$$

To carry out the integral in Eq.25, we will use the following identity to deal with the point of double singularity $E = \bar{E}$.

$$\frac{1}{(\bar{E}-E')(E-E')} = \frac{1}{(\bar{E}-E)}\left(\frac{1}{E-E'} - \frac{1}{\bar{E}-E'}\right) + \pi^2\delta(\bar{E}-E)\delta[E' - \frac{1}{2}(\bar{E}+E)] \tag{26}$$

Now substituting Eq.26 in Eq. 25,

$$\int dE'\ |V_{E'}|^2 a^*(\bar{E})\Bigg[\frac{1}{(\bar{E}-E')(E-E')} + \frac{z(\bar{E})\delta(\bar{E}-E')}{E-E'} + \frac{z(E)\delta(E-E')}{\bar{E}-E'} + z(\bar{E})z(E)\underbrace{\delta(\bar{E}-E')\delta(E-E')}_{\delta(\bar{E}-E)\delta(E'-\frac{1}{2}(E+\bar{E}))}\Bigg]a(E) = \delta(\bar{E}-E)$$

$$\Rightarrow \int dE'\ |V_{E'}|^2 a^*(\bar{E})\Bigg[\underbrace{\frac{1}{\bar{E}-E}\left(\frac{1}{E-E'} - \frac{1}{\bar{E}-E'}\right)}_{I} + \underbrace{\pi^2\delta(\bar{E}-E)\delta[E'-\frac{1}{2}(\bar{E}+E)]}_{II} + \underbrace{z^2(E)\delta(\bar{E}-E)\delta[E'-\frac{1}{2}(\bar{E}+E)]}_{III}\Bigg]a(E)$$
$$+ a^*(\bar{E})\Bigg[\underbrace{\frac{|V_{\bar{E}}|^2 z(\bar{E})}{E-\bar{E}}}_{IV} + \underbrace{\frac{|V_E|^2 z(E)}{\bar{E}-E}}_{V}\Bigg]a(E) = \delta(\bar{E}-E)$$

$$\Rightarrow \underbrace{a^*(\bar{E})a(E)|V_E|^2[\pi^2 + z^2(E)]\delta(\bar{E}-E)}_{II+III} + \frac{1}{\bar{E}-E}a^*(\bar{E})\times \underbrace{[F(E) - F(\bar{E}) + z(E)|V_E|^2 + z(\bar{E})|V_{\bar{E}}|^2]}_{I+IV+V}a(E) = \delta(\bar{E}-E) \tag{27}$$

From Eq.24, the term $(I + IV + V)$ vanishes, So,

$$
\begin{aligned}
&a^*(\bar{E})a(E)|V_E|^2[\pi^2+z^2(E)]\delta(\bar{E}-E)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad=\delta(\bar{E}-E)\\
\Rightarrow&|a(E)|^2|V_E|^2[\pi^2+z^2(E)]\delta(\bar{E}-E)=\delta(\bar{E}-E)\\
\Rightarrow&|a(E)|^2=\frac{1}{|V_E|^2[\pi^2+z^2(E)]}\\
\Rightarrow&|a(E)|^2=\frac{|V_E|^2}{\pi^2|V_E|^4+(E-E_\phi-F(E))^2}
\end{aligned}
\tag{28}
$$

This result shows that due to configuration interaction, the probability of finding the discrete state ϕ is represented by a resonance curve with half width $\pi|V_E|^2$. In other words, if a state is prepared at any instant of time, then it will autoionize (become a continuum state) with a mean life $\frac{\hbar}{2\pi|V_E|^2}$.

From Eq.21,

$$
\begin{aligned}
\Delta&=-\arctan\left(\frac{\pi}{z(E)}\right)\\
\Rightarrow z(E)&=-\pi\cot(\Delta)
\end{aligned}
\tag{29}
$$

Using this in Eq. 28 , we get,

$$
\begin{aligned}
|a(E)|^2&=\frac{1}{|V_E|^2\pi^2[1+\cot^2\Delta]}\\
&=\frac{\sin^2\Delta}{|V_E|^2\pi^2}\\
\Rightarrow a(E)&=\frac{\sin\Delta}{|V_E|\pi}
\end{aligned}
\tag{30}
$$

From Eq. 20 and 30,

$$
\begin{aligned}
b_{E'}&=\left[\frac{1}{E-E'}+z(E)\delta(E-E')\right]V_{E'}a\\
&=\frac{V_{E'}}{\pi V_E}\left(\frac{\sin\Delta}{E-E'}-\pi\cos\Delta\delta(E-E')\right)\\
&=\frac{V_{E'}}{\pi V_E}\frac{\sin\Delta}{E-E'}-\cos\Delta\delta(E-E')
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
\Delta&=-\arctan\left(\frac{\pi}{z(E)}\right)\\
&=-\arctan\left(\frac{\pi|V_E|^2}{E-E_\phi-F(E)}\right)
\end{aligned}
\tag{32}
$$

Now to study about the variation of probability of variation of excitation of the stationary state Ψ_E, we will introduce a suitable transition operator T between state i and Ψ_E whose elements will correspond to the probabilities of the transitions.

With the help of Eq.15, Eq.30, Eq. 31 and Eq. 32, we can express,

$$
\begin{aligned}
&\langle\Psi_E|\,T\,|i\rangle\\
=&\frac{1}{\pi V_E^*}\langle\phi|\,T\,|i\rangle\sin\Delta\\
+&\frac{1}{\pi V_E^*}P\int dE'\,\frac{V_{E'}^*\langle\psi_{E'}|\,T\,|i\rangle}{E-E'}\sin\Delta\\
-&\langle\psi_E|\,T\,|i\rangle\cos\Delta\\
=&\frac{\sin\Delta}{\pi V_{E^*}}\langle\Phi|\,T\,|i\rangle-\langle\psi_E|\,T\,|i\rangle\cos\Delta
\end{aligned}
\tag{33}
$$

where

$$
|\Phi\rangle=|\phi\rangle+P\int dE'\,\frac{V_{E'}\,|\psi_{E'}\rangle}{E-E'}
\tag{34}
$$

which indicates the state ϕ modified by the interaction with the states of the continuum. As E passes through the resonance at $E=E_\phi+F$, a sharp variation of Δ causes a sharp variation of $\langle\psi_E|\,T\,|i\rangle$. As discussed earlier, when E passes $E=E_\phi+F$, Δ advances by a phase of π. Since, Δ is an odd function of variable $E-E_\phi-F(E)$, $\sin\Delta$ and $\cos\Delta$ are an odd and even function of variable $E-E_\phi-F(E)$ respectively. So, $\langle\Phi|\,T\,|i\rangle$ and $\langle\psi_E|\,T\,|i\rangle$ interfere with opposite phases on two sides of resonance[1].

In particular, the transition probability vanishes on one side of resonance, say, at $\Delta=\Delta_0$ i.e. $E=E_0$. In other words, $\tan\Delta_0=\frac{-\pi|V_{E_0}|^2}{E_0-E_\phi-F(E_0)}$.

Now,

$$
\begin{aligned}
&\langle\Psi_{E_0}|\,T\,|i\rangle=0\\
\Rightarrow&\frac{1}{\pi V_{E_0}^*}\langle\Phi|\,T\,|i\rangle\sin\Delta_0-\langle\psi_E|\,T\,|i\rangle\cos\Delta_0=0\\
\Rightarrow&\tan\Delta_0=\frac{\pi V_{E_0}^*\langle\psi_{E_0}|\,T\,|i\rangle}{\langle\Phi|\,T\,|i\rangle}=\frac{-\pi|V_{E_0}|^2}{E_0-E_\phi-F(E_0)}
\end{aligned}
\tag{35}
$$

Notice that from the experimental investigation of transition probability in the vicinity of autoionisation level will determine directly the resonance energy $E_\phi+F(E)$ rather than E_ϕ of the unperturbed discrete level, and similarly it will provide information on $\langle\Phi|\,T\,|i\rangle$(transition between modified state Φ and initial state i) rather than $\langle\phi|\,T\,|i\rangle$(transition between unperturbed state ϕ and initial state i).

Consider a new variable, reduced energy variable which is expressed as,

$$
\epsilon=-\cot\Delta=\frac{E-E_\phi-F(E)}{\pi|V_E|^2}=\frac{E-E_\phi-F(E)}{\frac{1}{2}\Gamma}
\tag{36}
$$

where $\Gamma=2\pi|V_E|^2$ is spectral width of autoionisation state ϕ and let's define a parameter q such that

$$
\begin{aligned}
q &= \frac{\langle\Phi|\,T\,|i\rangle}{\pi V_E^* \langle\psi_E|\,T\,|i\rangle} \\
&= \frac{\langle\phi|\,T\,|i\rangle + \int dE' \, \frac{V_{E'}^* \langle\psi_{E'}|\,T\,|i\rangle}{E-E'}}{\pi V_E^* \langle\psi_E|\,T\,|i\rangle} \\
&= \frac{\langle\phi|\,T\,|i\rangle + \int dE' \, \frac{\langle\phi|\,T\,|\psi_{E'}\rangle \langle\psi_{E'}|\,T\,|i\rangle}{E-E'}}{\pi \langle\phi|\,T\,|\psi_E\rangle \langle\psi_E|\,T\,|i\rangle}
\end{aligned} \tag{37}
$$

From Eq. 35, we can say that $q = \cot\Delta_0$ and is independent of any phase normalization at $E = E_0$.

From Eq.33,

$$
\begin{aligned}
&\langle\Psi_E|\,T\,|i\rangle = \frac{\sin\Delta}{\pi V_{E^*}} \langle\Phi|\,T\,|i\rangle - \langle\psi_E|\,T\,|i\rangle \cos\Delta \\
\Rightarrow &\frac{\langle\Psi_E|\,T\,|i\rangle}{\langle\psi_E|\,T\,|i\rangle} = \frac{\sin\Delta}{\pi V_{E^*}} \frac{\langle\Phi|\,T\,|i\rangle}{\langle\psi_E|\,T\,|i\rangle} - \cos\Delta \\
\Rightarrow &\frac{\langle\Psi_E|\,T\,|i\rangle}{\langle\psi_E|\,T\,|i\rangle} = (q - \cot\Delta)\sin\Delta = (q+\epsilon)\sin\Delta \\
\Rightarrow &\frac{|\,\langle\Psi_E|\,T\,|i\rangle\,|^2}{|\,\langle\psi_E|\,T\,|i\rangle\,|^2} = \frac{(q+\epsilon)^2}{cosec^2\Delta} = \frac{(q+\epsilon)^2}{1+\cot^2\Delta} \\
&= \frac{(q+\epsilon)^2}{1+\epsilon^2} = 1 + \frac{q^2 - 1 + 2q\epsilon}{1+\epsilon^2}
\end{aligned} \tag{38}
$$

So, the ratio of probability $|\,\langle\Psi_E|\,T\,|i\rangle\,|^2$ and $|\,\langle\psi_E|\,T\,|i\rangle\,|^2$ follows a family of curves which is a function of ϵ. Also, notice that

$$
\frac{1}{2}\pi q^2 = \frac{|\,\langle\Phi|\,T\,|i\rangle\,|^2}{2\pi |V_E|^2 |\,\langle\psi_E|\,T\,|i\rangle\,|^2} = \frac{|\,\langle\Phi|\,T\,|i\rangle\,|^2}{|\,\langle\psi_E|\,T\,|i\rangle\,|^2 \Gamma} \tag{39}
$$

which is the ratio of transition probabilities to the "modified" discrete state Φ and to a bandwidth Γ of unperturbed continuum state ψ_E. Now let's consider a case where ratio q , line

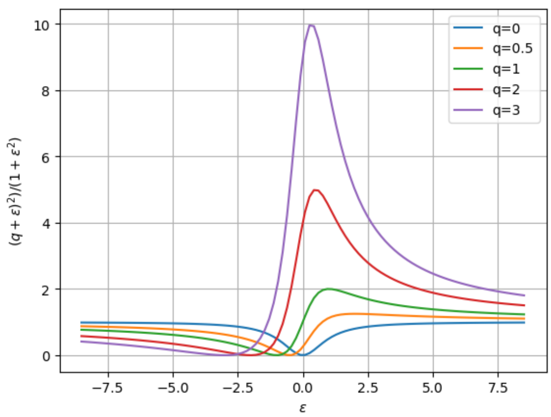

FIG. 5: Natural line shapes for different values of q

shift F and width $\Gamma = 2\pi|V_E|^2$ can be considered as independent of E for a sufficient range of energy, then by Eq. 38, we can express the excess transition probability due to discrete state ϕ as follows:

$$
\begin{aligned}
&\int dE \, [|\,\langle\Psi_E|\,T\,|i\rangle\,|^2 - |\,\langle\psi|\,T\,|i\rangle\,|^2] \\
= &\int dE \, |\,\langle\psi_E|\,T\,|i\rangle\,|^2 \left(\frac{|\,\langle\Psi_E|\,T\,|i\rangle\,|^2}{|\,\langle\psi_E|\,T\,|i\rangle\,|^2} - 1 \right) \\
= &|\,\langle\psi_E|\,T\,|i\rangle\,|^2 \int dE \, \frac{q^2 - 1 + 2q\epsilon}{1+\epsilon^2}
\end{aligned} \tag{40}
$$

As we assume that F and Γ is independent of E for a sufficient range, with the help of Eq.36, we can take $dE = \frac{1}{2}\Gamma d\epsilon$. If we continue from Eq. 40,

$$
\begin{aligned}
&\int dE \, [|\,\langle\Psi_E|\,T\,|i\rangle\,|^2 - |\,\langle\psi|\,T\,|i\rangle\,|^2] \\
= &|\,\langle\psi_E|\,T\,|i\rangle\,|^2 \int dE \, \frac{q^2 - 1 + 2q\epsilon}{1+\epsilon^2} \\
= &|\,\langle\psi_E|\,T\,|i\rangle\,|^2 \frac{1}{2}\Gamma \int d\epsilon \, \frac{q^2 - 1 + 2q\epsilon}{1+\epsilon^2} \\
= &|\,\langle\psi_E|\,T\,|i\rangle\,|^2 \frac{1}{2}\Gamma(q^2-1) \Bigg[\int_{-\infty}^{+\infty} d\epsilon \, \frac{1}{1+\epsilon^2} \\
&\qquad + 2q \underbrace{\int_{-\infty}^{+\infty} d\epsilon \, \frac{\epsilon}{1+\epsilon^2}}_{\text{vanishes as it is odd}} \Bigg] \\
= &|\,\langle\psi_E|\,T\,|i\rangle\,|^2 \frac{1}{2}\Gamma\pi(q^2-1) \\
= &|\,\langle\Phi|\,T\,|i\rangle\,|^2 - \frac{1}{2}\pi |\,\langle\psi_E|\,T\,|i\rangle\,|^2 \Gamma
\end{aligned} \tag{41}
$$

This above result should be equal to $|\,\langle\phi|\,T\,|i\rangle\,|^2$, because diagonalization of energy matrix carried out earlier is a unitary transformation so it cannot affect the total probability of transition. Any difference between LHS and RHS of the Eq.41 is caused by q,F and Γ having slight dependency on E[1].

We can connect this model with the classical model as we can give analogy between the discrete state and the oscillator that has zero damping coefficient and also between the continuum state and driven largely damped oscillator[4]. We can also say the reduced energy variable ϵ that we have used here is similar to the dimensionless frequency. The term q has similar significance in both cases by giving rise to asymmetric lineshape. This fact gives Fano resonance a universality as it can be explained by both classical and quantum intuition.

IV. FANO RESONANCE IN CASE OF ONE DISCRETE STATE AND TWO OR MORE CONTINUA

Consider a discrete state ϕ having interaction with states of different continua $\psi_{E'}$,$\chi_{E'}$,$\cdot$ distinguished by different quantum number. Here we will discuss the case of two continua states for simplicity.

The eigenvector ,in this case, will have the form:

$$\Psi_{hE} = a\phi + \int dE' \; [b_{E'}\psi_{E'} + c_{E'}\chi_{E'}] \qquad (42)$$

where h indicates a set of $(n-1)$ parameters to specify Ψ, since each value of E has a n-fold degenerate eigenvalue. (In case of $(n-1)$ continua states and one discrete state)[1]. In this case, there are one discrete and 2 continua states. So, we need 2 parameter for this problem. The coefficient a, $b_{E'}$, $c_{E'}$ are functions of E,h. Determination of the eigenvectors are trivial because $n-1$ continua are readily found, with $b_{E'}$ and $c_{E'}$, and such as to cancel the interaction with ϕ and, therefore, with $a = 0$ and another with $a \neq 0$ where we are considering the interaction of ϕ is not cancelled out.

The elements of the energy submatrix to be diagonalized is indicated by,

$$\langle\phi|\, H\, |\phi\rangle = E_\phi \qquad (43a)$$

$$\langle\psi_{E'}|\, H\, |\phi\rangle = V_{E'} \qquad \langle\chi_{E'}|\, H\, |\phi\rangle = W_{E'} \qquad (43b)$$

$$\langle\psi_{E''}|\, H\, |\psi_{E'}\rangle = \langle\chi_{E''}|\, H\, |\chi_{E'}\rangle = E'\delta(E'-E'') \qquad (43c)$$

$$\langle\chi_{E''}|\, H\, |\psi_{E'}\rangle = 0 \qquad (43d)$$

The last Eq.43d indicates that the matrix has been diagonalized with respect to the states $\psi_{E'}$ and $\chi_{E'}$.

$$\begin{aligned}
&\langle\phi|\, H\, |\Psi_{hE}\rangle = E\,\langle\phi|\Psi_{hE}\rangle \\
&\Rightarrow a\,\langle\phi|\, H\, |\phi\rangle + \int dE' \; (b_{E'}\,\langle\phi|\, H\, |\psi_{E'}\rangle \\
&\qquad + c_{E'}\,\langle\phi|\, H\, |\chi_{E'}\rangle) = E\Big(a\,\langle\phi|\phi\rangle \\
&+ \int dE' \; b_{E'}\,\langle\phi|\psi_{E'}\rangle + c_{E'}\,\langle\phi|\chi_{E'}\rangle\Big) \\
&\Rightarrow E_\phi a + \int dE' \; (V^*_{E'}b_{E'} + W^*_{E'}c_{E'}) = Ea
\end{aligned} \qquad (44)$$

Similarly by the technique we used in previous section, we get the two following equations,

$$\begin{aligned}
&\langle\psi_{E'}|\, H\, |\Psi_{hE}\rangle = E\,\langle\psi_{E'}|\Psi_{hE}\rangle \\
&\Rightarrow a\,\langle\psi_{E'}|\, H\, |\phi\rangle + \int dE'' \; (b_{E''}\,\langle\psi_{E'}|\, H\, |\psi_{E''}\rangle \\
&\qquad\qquad + c_{E''}\,\langle\psi_{E'}|\, H\, |\chi_{E''}\rangle) \\
&= E\Big(a\,\langle\psi_{E'}|\phi\rangle + \int dE'' \; b_{E''}\,\langle\psi_{E'}|\psi_{E''}\rangle \\
&\qquad\qquad + c_{E''}\,\langle\psi_{E'}|\chi_{E''}\rangle\Big) \\
&\Rightarrow V_{E'}a + \int dE'' \; E''b_{E''}\delta(E'-E'') \\
&\qquad\qquad = E\int dE'' \; b_{E''}\delta(E'-E'') \\
&\Rightarrow V_{E'}a + E'b_{E'} = Eb_{E'}
\end{aligned} \qquad (45)$$

$$\begin{aligned}
&\langle\chi_{E'}|\, H\, |\Psi_{hE}\rangle = E\,\langle\chi_{E'}|\Psi_{hE}\rangle \\
&\Rightarrow a\,\langle\chi_{E'}|\, H\, |\phi\rangle + \int dE'' \; (b_{E''}\,\langle\chi_{E'}|\, H\, |\psi_{E''}\rangle \\
&\qquad\qquad + c_{E''}\,\langle\chi_{E'}|\, H\, |\chi_{E''}\rangle) \\
&= E\Big(a\,\langle\chi_{E'}|\phi\rangle + \int dE'' \; b_{E''}\,\langle\chi_{E'}|\psi_{E''}\rangle \\
&\qquad\qquad + c_{E''}\,\langle\chi_{E'}|\chi_{E''}\rangle\Big) \\
&\Rightarrow W_{E'}a + \int dE'' \; E''c_{E''}\delta(E'-E'') \\
&\qquad\qquad = E\int dE'' \; c_{E''}\delta(E'-E'') \\
&\Rightarrow W_{E'}a + E'c_{E'} = Ec_{E'}
\end{aligned} \qquad (46)$$

So, for this case, the system of equations will be,

$$E_\phi a + \int dE' \; (V^*_{E'}b_{E'} + W^*_{E'}c_{E'}) = Ea \qquad (47a)$$

$$V_{E'}a + E'b_{E'} = Eb_{E'} \qquad (47b)$$

$$W_{E'}a + E'c_{E'} = Ec_{E'} \qquad (47c)$$

Now by $V^*_{E'} \times (47b) + W^*_{E'} \times (47c)$, we get,

$$(|V_{E'}|^2 + |W_{E'}|^2)a + E'(V^*_{E'}b_{E'} + W^*_{E'}c_{E'}) = E(V^*_{E'}b_{E'} + W^*_{E'}c_{E'}) \qquad (48)$$

Now $W_{E'} \times (47b) - V_{E'} \times (47c)$ yields,

$$E'(W_{E'}b_{E'} - V_{E'}c_{E'}) = E(W_{E'}b_{E'} - V_{E'}c_{E'}) \qquad (49)$$

Similar to what we have seen before for the system 19, Eq.48 and Eq.47a forms a pair of equation similar to system 19 and Eq.49 is decoupled from a. Now for solving the system of Eqs. 47a and 48, for the dependent variable a and $[V^*_{E'}b_{E'} + W^*_{E'}c_{E'}]$, we will use similar methods that we used in the previous section and replace the $|V_{E'}|^2$ term by $(|V_{E'}|^2 + |W_{E'}|^2)$.Also, Notice that $[W_{E'}b_{E'} - V_{E'}c_{E'}]$ behaves like $\delta(E'-E)$.

So, the solution of Eq.49 is: $[W_{E'}b_{E'} - V_{E'}c_{E'}] = N\delta(E' - E)$ where N is a suitable normalization factor[1].

From these solutions, first, we will write the solution of a, $b_{E'}$ and $c_{E'}$ considering $a \neq 0$. The solutions will be:

$$a_1 = \frac{\sin\bar{\Delta}}{\pi(|V_{E'}|^2 + |W_{E'}|^2)^{\frac{1}{2}}} \tag{50a}$$

$$b_{1E'} = \frac{V_{E'}}{(|V_{E'}|^2 + |W_{E'}|^2)^{\frac{1}{2}}}\Big[\frac{1}{\pi}\frac{\sin\bar{\Delta}}{E - E'} - \cos\bar{\Delta}\delta(E - E')\Big] \tag{50b}$$

$$c_{1E'} = \frac{W_{E'}}{(|V_{E'}|^2 + |W_{E'}|^2)^{\frac{1}{2}}}\Big[\frac{1}{\pi}\frac{\sin\bar{\Delta}}{E - E'} - \cos\bar{\Delta}\delta(E - E')\Big] \tag{50c}$$

Where

$$\bar{\Delta} = -\arctan\frac{\pi(|V_{E'}|^2 + |W_{E'}|^2)}{E - E_\phi - G(E)} \tag{51}$$

$$G(E) = P\int dE'\ \frac{|V_{E'}|^2 + |W_{E'}|^2}{E - E'} \tag{52}$$

Now for the case of $a = 0$, from system 47, we can say that both the solutions of $b_{E'}$ and $c_{E'}$ are $\delta(E' - E)$ with a appropriate normalization that follows $[V^*_{E'}b_{E'} + W^*_{E'}c_{E'}] = 0$(A condition which will be obtained by putting $a = 0$ in Eq.47a). So, the solutions will be:

$$a_2 = 0 \tag{53a}$$

$$b_{2E'} = \frac{W^*_E}{(|V_{E'}|^2 + |W_{E'}|^2)^{\frac{1}{2}}}\delta(E - E') \tag{53b}$$

$$c_{2E'} = -\frac{V^*_E}{(|V_{E'}|^2 + |W_{E'}|^2)^{\frac{1}{2}}}\delta(E - E') \tag{53c}$$

If there are $n > 2$ continua, the extension is very simple. In case of $a \neq 0$, the term $|V_E|^2 + |W_E|^2$, that appears in Eq.50, Eq.51 and Eq.52 will be replaced with some additional terms being added to it. Some additional coefficients $d_{1E'}, \cdots$, will be obtained which have similar expression like Eq.50b and Eq. 50c.

Similarly for case $a = 0$, the solution will be extended into a set of $n - 1$ orthogonal solutions of the n-variable homogenous equation $V^*_{E'}b_{E'} + W^*_{E'}c_{E'} + \cdots = 0$.

The transition probability of the initial state i to all stationary states of energy E is

$$|\langle\Psi_{1E}T|i\rangle|\,|^2 + |\,\langle\Psi_{2E}T|i\rangle|\,|^2 + \cdots \tag{54}$$

The first term of this sum varies as a function of energy in the vicinity of the resonance at $E_\phi + G$ according to the family of curves discussed earlier. The remaining terms are totally unaffected by the resonance as we have considered that they have no interaction with the discrete state. So, in presence of two or more continua, we are expected to observe an spectrum similar to the family of curves discussed in previous section with a superposition of smooth background and due to the presence of this background spectrum, spectral intensity would not drop to zero unlike what we have seen in the previously[1].

We can find various similarities between this model and the classical analogue that we have discussed earlier. In this case, we can say the damping coefficient of the second oscillator, which was zero for earlier case, has a finite value. This renders the system to couple with more continua so complete cancellation of oscillation is not possible[4].

V. FANO RESONANCE IN CASE OF NUMBER OF DISCRETE STATES AND ONE CONTINUUM

Now we will consider a situation when we have a set of discrete states $\phi_1, ..., \phi_n, ..$ experiences configuration interaction with a set of states $\Psi_{E'}$.so we can write

$$\langle\phi_m|\phi_n\rangle = E_n\delta_{mn}, \tag{55a}$$

$$\langle\Psi_{E'}|\,H\,|\phi_n\rangle = V_{E'n} \tag{55b}$$

$$\langle\Phi_{E''}|\,H\,|\Psi_{E'}\rangle = E'\delta(E'' - E') \tag{55c}$$

the eigenvector is in the form

$$|\Psi\rangle = \sum_n a_n\phi_n + \int dE' b_{E'}\phi(E'), \tag{56}$$

Let's say they have energy E. using equations 55 and 56 , we get the following sets of equations.

$$E_n a_n + \int dE' V_{nE'} bE' = Ea_n \tag{57a}$$

$$\sum_n V_{E'n}a_n + E'bE' = Eb_{E'} \tag{57b}$$

we can formally solve for b'_E the way its done in the first case .

$$b_{E'} = [\frac{1}{E - E'} + z(E)\delta(E - E')]\sum_n V_{E'n}a_n. \tag{58}$$

substituting eq 58 in we get .

$$E_n a_n + \sum_m F_{nm}a_m + z(E)V_{nE}\sum_m V_{Em}a_m = Ea_n. \tag{59}$$

Here we have ,

$$F_{nm} = P\int dE'\frac{V_{nE'}V_{E'm}}{E - E'} \tag{60}$$

. from this equation we see that interaction between the discrete states mediated by the interaction term F_{nm} due to coupling with intermediate continuum configuration. first we will diagonalize the matrix $E_n\delta_{mn} + F_{mn}$,ie we consider the effect of interaction on the discrete states.This effect perturbs the states ϕ_n and their energies E_n and replaces them with new states

$$\phi_\nu = \sum_n \phi_n A_{n\nu} \tag{61}$$

and with energies E_ν,which are obtained by solving the system of following equations,

$$E_n A_{n\nu} + \sum_m F_{nm} A_{m\nu} = A_{n\nu} E_\nu \tag{62}$$

For most of the case F_{nm} will be small , so we can solve the problem by perturbation theory.Asuuming that the coefficients $A_{n\nu}$ and energies E_ν have been obtained , we replace the coefficients a_n in by new coefficients $\bar{a}_\nu$, they are related by[1]

$$a_n = \sum_\nu A_{n\nu}\bar{a}_\nu \tag{63}$$

$$\begin{aligned}\sum_n a_n\phi_n &= \sum_n \left(\sum_\nu A_{n\nu}\bar{a}_\nu\right)\phi_n \\ &= \sum_\nu \bar{a}_\nu \left(\sum_n A_{n\nu}\phi_n\right) \\ &= \sum_\nu \bar{a}_\nu \bar{\phi}_\nu \end{aligned} \tag{64}$$

so Ψ_E can be written in the following way

$$\Phi_E = \sum_\nu \bar{a}_\nu\bar{\phi}_\nu + \int dE' b_{E'}\phi E' \tag{65}$$

now the matrix F_{nm} can now be eliminated from by means of and becomes

$$\begin{aligned}&\sum_\nu A_{n\nu}E_\nu\bar{a}_\nu + z(E)V_{nE}\sum_{m\nu} V_{Em}A_{m\nu}\bar{a}_\nu \\ &= E\sum_\nu A_{n\nu}\bar{a}_\nu\end{aligned} \tag{66}$$

multiplication by $(A^{-1})_{\mu n}$, summation over a and application of the orthonormality $\sum_n (A^{-1})_{\mu n}A_{n\nu} = \delta_{\mu\nu}$ yields finally

$$E_\nu\bar{a}_\nu + z(E)V_{\nu E}\sum_\mu V_{E\mu}\bar{a}_\mu = E\bar{a}_\nu \tag{67}$$

where

$$V_{E\mu} = \sum_n V_{En}A_{n\mu} \tag{68}$$

multiply both side of equation by $V_{E\nu}/(E-E_{nu})$ and summation over ν we get ,

$$z(E)\sum_\nu [|V_{E\nu}|^2/(E-E_\nu)] = 1 \tag{69}$$

then equation 67 can be written as

$$a_\nu = z(E)\frac{V_{\nu E}}{E-E_\nu}\sum_\mu V_{E\mu}\bar{a}_\mu \tag{70}$$

in terms of the expression

$$\sum_\mu = V_{E\mu}\bar{a}_\mu = \sum_n V_{En}a_n \tag{71}$$

this will play the role of normalization constant .we will represent z in terms of a phase shift Δ and rewrite equation 69 in the form

$$\frac{\pi}{z(E)} = -tan\Delta = \sum_\nu \frac{\pi|V_{E\nu}|^2}{E-E_\nu} = -\sum_\nu tan\Delta_\nu \tag{72}$$

Here Δ_ν represent the phase shift that would be contributed by the state ϕ_{nu}, if alone. For the determination of the normalization constant, we follow the same steps as previously done:

$$\begin{aligned}\langle\Psi_{\bar{E}}|\Psi_E\rangle &= \sum_n a_n^*(\bar{E})a_n(E) \\ &\quad + \int dE'\, b_{E'}^*(\bar{E})b_{E'}(E) \\ &= \delta(\bar{E}-E),\end{aligned} \tag{73}$$

substitution $b_{E'}$ from and using .we get

$$\begin{aligned}&\left|\sum_n V_{En}a_n(E)\right|^2 [\pi^2 + z(E)^2]\delta(\bar{E}-E) \\ &\quad + \sum_{nm} a_n^*(\bar{E})\left[\frac{1}{1+(\bar{E}-E)}\left(F_{nm}(E) - F_{nm}(\bar{E})\right)\right. \\ &\quad \left. + z(E)V_{nE}V_{Em} - z(\bar{E})V_{n\bar{E}}V_{\bar{E}m}\right]a_m(E) \\ &= \delta(\bar{E}-E).\end{aligned} \tag{74}$$

the $\sum_{mn}$ vanishes due to eq 59 ,so the normalization constant is given by

$$|\sum_n V_{En}a_n(E)|^2 = \frac{1}{\pi^2 + z(E)^2} = \frac{1}{\pi^2}sin^2\Delta \tag{75}$$

normalized solution of $\bar{a}_\nu$ and $b_{E'}$ is given as

$$\begin{aligned}\bar{a}_\nu &= \cos\Delta\tan\Delta_\nu/\pi V_{E\nu}, \\ b_{E'} &= cos\Delta[\sum_\nu \frac{V_{E'\nu}}{E-E'}\frac{\tan\Delta_\nu}{\pi V_{E\nu}} - \delta(E-E')]\end{aligned} \tag{76}$$

the transition matrix element for excitation of energy eigenstate Ψ_E is given by

$$\langle\Psi_E|T|i\rangle = \cos\Delta\Bigg[\sum_\nu \frac{\tan\Delta_\nu}{\pi V_{\nu E}}\langle\phi_\nu|i\rangle + \frac{\tan\Delta_\nu}{\pi V_{\nu E}} P\int dE' V_{\nu E'}\langle\Psi_{E'}|T|i\rangle\Bigg] \tag{77a}$$

$$= \cos\Delta\Bigg[\sum_\nu \frac{\tan\Delta_\nu}{\pi V_{\nu E}}\langle\bar{\Phi}_\nu|T|i\rangle - \langle\psi_E|T|i\rangle\Bigg] \tag{77b}$$

$$= \langle\psi_E|T|i\rangle\cos\Delta\left(\sum_\nu q_\nu\tan\Delta_\nu - 1\right). \tag{77c}$$

where

$$\begin{aligned}\bar{\Phi}_\nu &= \bar{\phi}_\nu + P\int dE' \frac{\psi_{E'}V_{E'\nu}}{E-E'},\\ q_\nu &= \frac{\langle\phi_\nu|T|i\rangle}{\pi V_{\nu E}\langle\psi_E|T|i\rangle}.\end{aligned} \tag{78}$$

here $\cos\Delta\tan\Delta_\nu$ has the same role as $sin\Delta$ in sec .Here $\cos\Delta\tan\Delta_\nu$ has a finite value at resonance points $\Delta_\nu = \pi/2$ as $cos\Delta$ vanishes there.also $\sum_\nu q_\nu tan\Delta_\nu = \sum_\nu q_\nu\pi|V_{E\nu}|^2/(E - E_\nu)$ varies from $-\infty$ to ∞ in each of the interval between two resonance which causes a rapid change in $\langle\Psi_E|\,T\,|i\rangle$.in fact in each of the interval $\langle\Psi_E|\,T\,|i\rangle$ vanishes[1].

VI. APPLICATION OF FANO RESONANCE

From the classical and quantum descriptions of Fano resonances, we found Fano resonances have many uses in various fields. These applications can be distinguished into two types of regimes[5], (a)Weak-coupling regime, (b)Strong-coupling regime. As we have discussed the basic classical and quantum mechanical picture of Fano resonances before, in this section, we will briefly discuss on how this understanding helps us to describe various physical phenomena.

Electromagnetically induced transparency or EIT is a coherent optical phenomenon that makes a medium transparent within a narrow spectral range around the absorption line. It can be described as the Fano resonance of a weak coupling regime. When the frequency of strongly damped oscillator ω_1 and weakly damped oscillator ω_2 (used in the classical description of Fano Resonance) match, the q parameter vanishes. In that scenario, the driven oscillator(largely damped) oscillates rapidly whereas the second oscillator does not oscillate at all. So, the driver sees as if the second oscillator does not exist i.e. second oscillator becomes transparent to the driver[5]. Similarly, EIT, in quantum description, describes a continuous state which transmits completely through the discrete state i.e. transition probability of initial state to modified discrete state($\langle\Phi|\,T\,|i\rangle$) is zero. In a much more detailed way, observation of EIT involves two optical fields which are tuned to interact with three quantum states of material. This is based on the destructive interference between atomic states. So, this phenomenon can be explained by Fano resonances by both pictures. Here the discrete and continuum state can be considered to be he weakly damped oscillator and strongly damped driven oscillator respectively for connecting both pictures together.

Borrmann effect is an anomalously strong transmission of waves through an absorbing crystal due to the Bragg diffraction. It is similar to EIT but here we use spatial domain rather than the time domain. Frequencies are replaced with wave numbers. x_1 and x_2 becomes amplitudes of left(incident) and right(diffracted) propagating plane wave, with $\omega_1 = -\omega_2$ and ω being their wave vectors and the Bloch wave vector calculated from the edge of Brillouin zone. As Bragg diffraction involves the interaction between waves and periodically spaced atoms in plane, the coupling coefficient g appears. The damping constants $\gamma_1 = -\gamma_2$ describe wave attenuation due to absorption and are of oppsite signs because left and right propagating wave decays in opposite direction. Now this left and right propagating plane wave i.e. incident and diffracted plane wave inferere with each other to give rise to standing waves. For a particular frequency, nodes of the standing wave coincide with the regions with lower electron density (lossy region) which suppresses the absorption. So, the transparency of X-ray is increased and it travels through further deeper into the lattice structure[5].

The Kerker effect describes a condition where the scattering amplitudes from spherical particles cancel in specific directions due to the differing spatial symmetries of electric and magnetic dipole scattering. It occurs even without direct coupling between oscillators and relies on interference of independent responses to external fields. Forward or backward scattering suppression is achieved when excitation amplitudes are equal. Unlike Fano or EIT effects, the oscillators can interact directly with the field, with the Kerker regime covering cases where damping rates dominate coupling[5].

The strong coupling regime emerges when the damping of both oscillators is weak, such that $\gamma_1, \gamma_2 \ll v_{12}$. In this regime, the real parts of the eigenfrequencies of the two coupled oscillators exhibit a splitting of $|\omega_+ - \omega_-| \approx 2v_{12}$ when the natural frequencies of the oscillators, ω_1 and ω_2, are tuned to match. This phenomenon, referred to as vacuum Rabi splitting or Autler–Townes splitting, draws an analogy with quantum optics.

Notably, while the real parts of ω_+ and ω_- begin to split at $v_{12} = |\gamma_1 - \gamma_2|/2$ this splitting can only be resolved spectrally when it surpasses the imaginary components of the eigenfrequencies, i.e., $v_{12} \gg \gamma_1, \gamma_2$, which defines the true strong coupling condition.

In a system of ring resonators coupled to a waveguide, transmission is suppressed at the eigenfrequencies $\omega_\pm$ and recovers in between. This can resemble the transmission spectrum of the EIT regime (panels j and d in Box 1). However, the central transmission maximum originates from different mechanisms: in the strong coupling regime, the resonators are not excited at the central frequency, whereas in the EIT regime, this frequency aligns with the resonant excitation of the weakly damped resonator. A key experimental signature of strong coupling is the avoided crossing of spectral resonances, observed as the oscillator frequencies are tuned relative to each other.

VII. FUTURE PROSPECT OF FANO RESONANCE

The concept of the Kerker effect, EIT, is important for designing robust optical devices with many applications in photonics and spectroscopy. Much research is done on Fano resonance in metasurfaces like lattices of nanospheres, hybrid graphene sheets, dielectric metasurfaces, and plasmonic metasurfaces. Fano profile has a dip into the continuum spectrum that goes to zero at one particular frequency, this opens up unexpected opportunities for applications, including cloaking. Fano resonance can be well engineered and the lineshape can be controlled, this provides a powerful tool for many applications like high-sensitivity sensing, all-optical switching,low-loss slow-light devices, invisibility cloaking, and many more.

Recent studies have uncovered a non-trivial manifestation of Fano resonance that is observable in both the time and frequency domains. This phenomenon has been theoretically analyzed and experimentally demonstrated in various photonic systems. A notable example is the trapping and confinement of electromagnetic energy through bound states in the continuum (BICs). To support BICs, a structure must extend infinitely in at least one direction. However, a trapped state can appear in the scattering spectrum in compact photonic structures as a sharp Fano resonance.

By adjusting the structure's parameters or the conditions of excitation, Fano resonances can become increasingly sharper and eventually vanish as their quality factor (Q) approaches infinity near the quasi-BIC point ($Q \approx 10^6$). This makes Fano resonances a precursor to BICs, exhibiting unique characteristics that could enable applications such as optical sensors, filters, waveguides, low-loss optical fibers, and large-area lasers[5].

Fano resonances have significant applications in advanced photonics, including ultrasmall lasers that leverage the interference between a continuum of waveguide modes and the discrete mode of a nanocavity. Other notable uses include molecular monolayer identification, tunable metamaterial absorbers for broadband mechanical resonance manipulation, optical spin angular momentum separation, nanofocusing beyond the diffraction limit, interferometric phase detectors, and plasmonic colorimetric sensors[5].

The development of extremely compact structures with simple geometry and symmetry that support Fano resonances holds promise for advancements in data storage, sensing technologies, and topological optics. Small plasmonic particles are particularly appealing due to the coexistence of Fano resonance and singular optics effects, enabling precise control of optical vortices at the nanoscale. This intricate interplay between far-field scattering and near-field Poynting vector flux underpins the recently introduced concept of nano-Fano resonances[5].

VIII. CONCLUSION

The article explores the phenomenon of Fano resonance, which arises from the interference between a discrete state and a continuum spectrum. This interaction produces a distinctive asymmetric resonance curve, a characteristic feature commonly observed in the excitation and absorption spectra of atoms. The study further delves into more complex scenarios, including the interaction between multiple discrete states and a single continuum, as well as a single discrete state coupled to multiple continua.

A classical analogy is also examined, highlighting similarities between quantum Fano resonance and classical systems, such as coupled oscillators. These analogies help bridge the gap between classical and quantum interpretations, offering deeper insight into the resonance behavior.

Finally, the article emphasizes the critical role of Fano resonance in advancing future technologies. In optics and photonics, it enables applications like plasmonic sensors, metamaterials, and ultra-fast optical switches, cloaking, solar cells. In spectroscopy, Fano resonance enhances precision in absorption and scattering measurements. This demonstrates the versatility and importance of fano resonance in modern science and engineering . The seminal work of Ugo Fano will remain one of the most cited papers in physics, as it is an influential source of exciting concepts for theoreticians, experimentalists and technologists with many possible applications proposed and yet to be implemented.

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