# Symmetry in physics

by Soudip Kundu

July 2024

---

## 1 Introduction

Here in this article i will try to summarize the concept of symmetry and Noether theorem in quantum mechanics.Everything i write over is inspire from [1] . for more detail read this lecture notes
**Defination 1.1** A state $|\psi\rangle = e^{i\phi}|\psi\rangle$ is said to be symmetric under a unitary transformation U if the transformed state is identical to the original state, up to a phase factor:

$$U|\psi\rangle = e^{i\phi}|\psi\rangle$$

## 2 Symmetries of hamiltonian

Like states,also the laws of nature , or equivalently the equations of motion,may be said to possesses symmetries . since the schrodinger's equation and the dynamics of motion follows directly from the hamiltonian , symmetries of the equation of motion correspond to symmetries of the Hamiltonian operator. In general ,any quantum operator A is called invariant under the unitary transformation U if [U,A]=0.If hamiltonian H is invariant under some unitary operator U, then U is called a symmetry of the Hamiltonian.the rationale behind this definition is that the expectation value of the Hamiltonian in any state is independent of transformation U :

$$(\langle\psi|U^\dagger)H(U|\psi\rangle) = \langle\psi|(U^\dagger H U)|\psi\rangle = \langle\psi|H|\psi\rangle$$

**Definition 1.2** A unitary transformation U is a symmetry of the Hamiltonian H if their commutator vanishes:[U,H]=0

## 3 Noether's theorem

The dynamics of many physical systems may be described almost entirely in terms of conservation laws, like the conservation of energy, momentum, angular momentum, and so on. One of the most profound insights in all of physics is the fact that the existence of such conserved quantities is always due to some underlying symmetry of the laws of nature. For instance translation invariance which means that the equation of motion remains invariant under translation in space, implies conservation of momemtum.

## 4 conserved quantities

The time evolution operator for time-independent Hamiltonian is given by $U(t) = e^{iHt}$. we saw in the previous section that U defines a symmetry of the Hamiltonian if it commutes with H. It follows that U also commutes with time evolution operator $U(t)$. As symmetry transformation is unitary,it can be written as the exponential $U = e^{iQ}$ of some Hermitian operator Q. The fact that U commutes with the time evolution operator then implies that $[Q, U(t)] = 0$. This in turn means that the expectation value of Q in any state $|\psi\rangle$ is conserved in time

$$\langle\psi(t)| Q |\psi(t)\rangle = (\langle\psi| U^{\dagger}(t))Q(U(t) |\psi\rangle) = \langle\psi| Q |\psi\rangle \tag{1}$$

so we found that the existence of symmetry operator U directly implies a conservation law for the observable Q, which is known as a conserved quantity or constant of motion.

## 5 continuity equations

the existence of any symmetry transformation implies a conservation law. for continuous symmetry transformation parametrised by a continuous variable,such as translation over a continuous distance or rotation over a continuous angle, there is an even stronger result known as Noether's theorem. this theorem states that each continuous symmetry is associated with a current $j^v(x,t)$ obeying the local conservation law or continuity equation $\partial_v j^v$.For discrete symmetries we have only a global constant of motion Q,but no local continuity equation.

later in the pdf you come across the continuity equation which is in the form

$$\partial_\nu j^\nu(x,t) = \partial j^t(x,t) + \nabla\vec{j}(x,t) = 0 \tag{2}$$

where $j^t$ is the charge density also written as $\rho$, written as the time component of the four-vector $j^\nu$.This leads to a quantity $Q(t) = \int \rho d\tau$(volume integration) which is conserved in time. we can argue in the following was

$$\frac{dQ(t)}{dt} = \int \frac{\partial\rho}{dt} d\tau = -\int \nabla.\vec{j}(x,t)d\tau$$

now by using the gauss theorem we can the above equation as

$$\frac{dQ(t)}{dt} = \int j(x,t)da \tag{3}$$

now this is a surface integral over the volume. now if we take the surface to infinity the current there goes to 0, so there is no contribution from the integration which leads us to $\frac{dQ(t)}{dt} = 0$

## 6 Proving Noether's theorem

Let $\phi_a(x) = \phi_a(\mathbf{x}, t)$. where the index a could also be for example label the two real components of a complex scalar field, or even different types of fields. The Lagrangian (density) is a function of both the fields and their derivatives $\mathcal{L} = \mathcal{L}[\phi_a(x), \partial_v \phi_a(x)]$, while the action,$S = \int dt d^D x \mathcal{L}$, is the integral of the lagrangian over space and time. A transformation of the fields can be written as

$$\phi_a(x) \to \phi_a^{'}(x) = \phi_a(x) + \delta_s \phi_a(x) \tag{4}$$

we will consider $\phi_a^{'}(x)$ to be the field resulting from the action of a symmetry transformation on $\phi_a(x)$, and the variation is assumed to be infinitesimally small.we can then use variation calculus to write the effect of the transformation on the lagrangian in terms of the transformations of the fields and their derivatives:

$$\mathcal{L} \to \mathcal{L}^{'} = \mathcal{L} + \delta_s \mathcal{L} \tag{5}$$

$$\delta_s \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} \delta_s \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_v \phi_a)} \delta_s(\partial_v \phi_a) \tag{6}$$

The expansion of $\delta_s \mathcal{L}$ in terms of infinitesimal variations of the fields is what makes Noether's theorem valid only for continuous and not discrete symmetries. if the variation $\delta_s \mathcal{L}$ happens to be zeros, the transformation that caused it can be called a symmetry of the action.in fact , even if $\delta_s \mathcal{L}$ is a total derivative $\partial_\nu K^\nu$ of some function $K^\nu$, we will call the transformation a symmetry. in that case, $\delta_s \mathcal{L}$ will only add a boundary term to the action, which does not affect the equations of motion. For most symmetry transformation the variation of the lagrangian will be zero.

notice that we have not specified whether or not the fields $\phi_a$ satisfy any equations of motion. the condition $\delta_s \mathcal{L} = \partial_\nu K^\nu$ defines what it means for a transformation to be a symmetry of action, regardless of the type of field it acts on.

Equation(6) can be written as

$$\delta \mathcal{L} = \partial_\nu (\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \delta_s \phi_a) + [\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\nu (\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)})] \delta_s \phi_a \tag{7}$$

The part in the square bracket is the euler-lagrange equations of motion prescribe to be zero. if we thus restrict attention to field configuration $\phi_a$ that satisfies these equations of motion.so we are left only the first term. notice that this condition of $\delta_s \mathcal{L}$ being a total derivative holds for specific field configuration $\phi_a$ , without putting any requirement on the transformation $\delta_s \phi_a$. conversely the previous result of $\delta_S \mathcal{L} = \partial_\nu K^\nu$ required the transformation to have the symmetry of the action.

Noether's theorem is now obtained by considering fields that obey the Euler-Lagrange equation of motion, and transformations that are symmetries of the action . We can then subtract the two condition on $\delta_S \mathcal{L}$, and obtain to the continuity equation

$$\partial_\nu (\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \delta_s \phi_a - K^\nu) \equiv \alpha \partial_\nu j^\nu = 0 \tag{8}$$

we introduce an infinitesimal parameter $\alpha$ and define the Noether current $j^\nu$ related to the symmetry transformation $\phi_a(x) \to \phi_a(x) + \delta_s \phi_a(x)$ as

$$\frac{1}{\alpha} (\partial_\nu (\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \delta_s \phi_a - K^\nu) \equiv \alpha \partial_\nu j_\nu) = \frac{\partial \mathcal{L}}{\partial(\partial_{nu} \phi_a} \Delta_s \phi_a - \frac{1}{\alpha} K^\nu \tag{9}$$

where $\Delta_s \phi = \delta_s \phi / \alpha$. the presence of a continuous symmetry implying the existence of a Noether current that is locally conserved, $\partial_\nu j^\nu = 0$, is the main result of Noether's theorem.

**Noether's Theorem:** To any continuous symmetry of an action corresponds a current $j^\nu(x,t)$ that is locally conserved. That is, it satisfies the continuity equation $\partial_\nu j^\nu(x,t) = 0$.

# 7 Noether charge

From the section continuity equation, we have seen that any local continuity equation implies the existence of a globally conserved quantity. In the context of Noether's theorem, this is called the Noether charge and is defined in a similar way as

$$Q(t) = \int d^D x j^t(x,t) = \int d^D x (\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)\Delta_s \phi} - \frac{1}{\alpha} K^t) \tag{10}$$

where $j^t = \rho$ is called the Noether charge density. Here Q(t) is independent of time/constant of motion.

## 7.1 Example of Noether currents and Noether charges

**Schrodinger field**
A complex scalar field $\psi(x,t)$ is called Schrodinger field when it has the action:

$$S[\psi,\psi^*] = \int dt d^D x (i\frac{\hbar}{2}(\phi^*(\partial_t \phi) - (\partial_t \phi^*)\phi) - \frac{\hbar^2}{2m}(\partial_n \phi^*)(\partial_n \phi) - V(x)\phi^* \phi) \tag{11}$$

The reason $\psi$ is called a Schrodinger field is that the Euler-Lagrange equation obtained by varying with respect to $\psi^*$ looks like the Schrodinger equation:

$$0 = \partial_t(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)}) + \partial_n(\frac{\partial \mathcal{L}}{\partial(\partial_n \phi^*))}) - \frac{\partial \mathcal{L}}{\partial \phi^*} - i\hbar \partial_t \psi - \frac{\hbar^2}{2m}\partial_n^2 \psi + V(x)\psi \tag{12}$$

The action has a continuous symmetry. It is invariant under phase rotations of the form

$$\phi(x) \rightarrow^{-i\alpha} \phi(x) \tag{13}$$

with $\alpha$ a real and continuous parameter. Notice that $\alpha$ does not depend on x,so that the phase rotation is a 'global' transformation, affecting all points in the system in the same way. Taking $\alpha$ to be infinitesimal, the exponent can be expanded and the variation of the field under phase rotation becomes $\Delta_s \psi(x) = -i\psi(x)$ and $\Delta_s \phi^*(x) = -i\phi^*(x)$.The Noether current and conserved Noether charge can now be identified:

$$j^t = \pi \Delta_s \psi + \pi^* \Delta_s \psi^* = \hbar \psi^* \psi, \qquad Q = \int d^D x \hbar \psi^* \psi, \tag{14}$$

$$j^n = i\frac{\hbar^2}{2m}((\partial_n \psi^*)\psi - \psi^*(\partial_n \psi))) \tag{15}$$

comparing with the schrodinger equation , the quantity $\int \psi^* \psi = \int |\psi|^2$ is a conserved quantity as it is the total probability ,when $\psi$ is normalized.In the lagrangian treatment , the conservation of the norm can be interpreted as a consequence of the invariance of the action under the global phase rotations. similarly, the local amplitude of the wave function,$|\psi(x)|^2$,can only change when it flows elsewhere in the form of a probability current $j^n$. Noether's theorem can then be interpreted as a continuity equation for the probability current.

# References


[1] Aron J. Beekman, Louk Rademaker, and Jasper van Wezel. An introduction to spontaneous symmetry breaking. *SciPost Phys. Lect. Notes*, page 11, 2019.