Quantum Hall Effect

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Abstract :

We will learn about the reason behind the quantised value of conductivity that arises in quantum hall effect at high magnetic field .We will begin with the integer quantum hall effect. And explore how Laughlin wavefunction explain the fractional hall effect. We will study about the quasi holes and its properties like fractional charge , fractional statistics . We will see how a course graining description like Chern Simon can explain all of these features without microscopic detail.

Index Terms: Topology, Fractional charge, Anyons , Laughlin States, Chern Simon Theory

Preliminaries

We will go through preliminaries that we will be requiring throughout the text.First we will go through the Landau Levels

Landau Levels

In this section we will go through the phenomenon of free particles moving in a background magnetic field and the resulting phenomenon of Landau levels.Classically the lagrangian for a particle of charge -e and mass m moving in a background magnetic field $B = \nabla \times A$ is

$$L = \frac{1}{2} m\dot{x}^2 - e\dot{x}.A \tag{1}$$

Under a gauge transformation ,$A \longrightarrow A + \nabla\alpha$ the lagrangian changes by a total derivative $L \longrightarrow L - e\dot{\alpha}$. as a result the equation of motion remains unchanged under gauge transformation.

The canonical momentum is given as

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - eA \tag{2}$$

The hamiltonian is given by

$$H = \dot{x}.p - L = \frac{1}{2m}(p + eA)^2 \tag{3}$$

THe poison bracket relation between the x and p is given as

$$\{x_i, p_j\} = \delta_{ij} \text{ and } \{x_i, x_j\} = \{p_i, p_j\} = 0 \tag{4}$$

Quantisation

We have our Hamiltonian given by

$$H = \dot{x}.p - L = \frac{1}{2m}(p + eA)^2 \tag{5}$$

Here p and x are operator (we wont be putting hat).The magnetic field is perpendicular to the plane xy where the electrons are moving .$\nabla \times B = B\hat{z}$. The commutator relation is given by

$$[x_i, p_j] = i\hbar\delta_{ij} \text{ and } [x_i, x_j] = [p_i, p_j] = 0 \tag{6}$$

Mechanical momentum can be written as follows

$$\pi = p + eA = m\dot{x} \tag{7}$$

The commutator relation follows from the Poisson backet 4

$$[\pi_x, \pi_y] = -ie\hbar B \tag{8}$$

The raising and lowering operator is defined as

$$a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \text{ and } a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \tag{9}$$

The commutator bracket relation is given as

$$[a, a^\dagger] = 1 \tag{10}$$

The hamiltonian can now be written in terms of a and $a^\dagger$ as

$$H = \frac{1}{2m}\pi.\pi = \hbar\omega_B(a^\dagger a + \frac{1}{2}) \tag{11}$$

The rising and lowering operator act on the eigenstates of the hamiltonian as follows

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \text{ and } a|n\rangle = \sqrt{n}\,|n-1\rangle \tag{12}$$

The state $|n\rangle$ has energy

$$E_n = \hbar\omega_B(n + \frac{1}{2}) \tag{13}$$

These energy levels are called Landau levels.

First we will be working in Landau gauge and later in symmetric gauge. we work with the choice

$$A = xB\hat{y} \tag{14}$$

.Notice that the underlying magnetic is translational and rotational invariant but A is breaks translational(not in the y direction) and rotation symmetry.The Hamiltonian becomes

$$H = \frac{1}{2m}(p_x + (p_y + eBx)) \tag{15}$$

since we have translation invariance in the y direction because of our choice of A we can look for energy eigenstates which are eigenstates of py.By using the following ansatz we can find the eigenvalue and the eigenstates of this problem

$$\psi_k(x, y) = e^{iky} f_k(x) \tag{16}$$

After doing the calculation you will find the eigenfunction is given by

$$\psi_{n,k}(x, y) \sim e^{iky} H_n(\frac{x + kl_B^2}{l_B}) e^{-(x+kl_B^2)^2/2l_B^2} \tag{17}$$

where H_n is the Hermite polynomial and the eigenvalue is given as

$$E_n = \hbar\omega_B(n + \frac{1}{2}) \tag{18}$$

The wavefunction looks like strips , extended in teh y direction but exponentially localised around $x = -kl_B^2$ in the x direction.

Degeneracy

To find the Degeneracies to restrict ourselves to a finite region of the (x,y) plane . we pick a rectriangle of length L_x and L_y. Since the wavefunctions are localised around $x = -l_B^2$ and x is restricted $0 \leq x \leq L_x$ k must lie between $-L_x/l_B^2 \leq k \leq 0$ .The number of states is

$$\mathcal{N} = \frac{L_y}{2\pi} \int_{-L_x/l_B^2}^{0} dk = \frac{L_x L_y}{2\pi l_B^2} = \frac{eBA}{2\pi\hbar} = \frac{AB}{\phi_0} \tag{19}$$

where $\phi_0 = 2\pi\hbar/e$

we can do the similar calculation in the symmetric gauge which is given by

$$A = -\frac{1}{2} r \times B = -\frac{yB}{2}\hat{x} + x\frac{B}{2}\hat{y} \tag{20}$$

You will find that the lowest landau level wavefunction is given as

$$\psi_{LLL,m} \sim (\frac{z}{l_B}) e^{-|z|^2/4l_B^2} \tag{21}$$

, where m index the different degenerate eigenstates.Changing m does not changes the energy. These states are also eigenstates of the angular momentum operator

$$J = i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \tag{22}$$

Then,acting on these lowest Landau level states we have

$$J\psi_{LLL,m} = \hbar m \psi_{LLL,m} \tag{23}$$

We will be using these wavefunction later in our text so keep them in back of your mind.

Next we study about spectral flow

Spectral Flow

Suppose you have a solenoid carrying magnetic field B and flux $\phi = BA$. outside the solenoid the magnetic field is zero .by using strokes theorem you can write

$$\oint A.dr = \int B.ds = \phi \implies A_\phi = \frac{\Phi}{2\pi r} \tag{24}$$

Now consider a charged quantum particle restricted to lie in a ring of radius r outside the solenoid .The hamiltonian is

$$H = \frac{1}{2m}(p_\phi + eA_\phi) \tag{25}$$

The energy eigenstates are

$$\psi = \frac{1}{\sqrt{2\pi r}} e^{in\phi} \tag{26}$$

and the eigenvalues are

$$E = \frac{\hbar^2}{2mr^2}(n + \frac{\phi}{\phi_0})^2,\ n \in Z \tag{27}$$

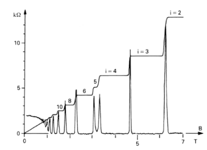
Now suppose we turn off the solenoid and place the particle in the n=0 ground state. if we increase the flux then , by the time we have reached $\phi = \phi_0$ the n=0 state has transformed into the state that we previously labeled as n=1. similarly state n is shifted to n+1. This is example of the phenomenon of spectral flow. We will using this idea later in the text to explain the quantised conductivity and fractional charge of quasi-holes.

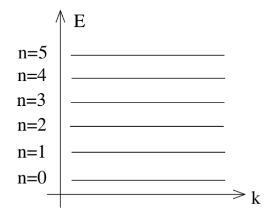
Integer Quantum Hall Effect

Introduction

To understand this we can assume no interaction between atoms. The only way electrons "know" about each other is through the Pauli exclusion principle. Experimental data for the Hall resistivity shows a number of plateaux labeled by an integer ν. The energy spectrum also forms Landau levels, each labeled by an integer. Each level can accommodate a large but finite number of electrons.

Visual Illustration: Below, we show two illustrations—one of the Hall resistivity plateaux and another of the energy spectrum showing Landau levels.



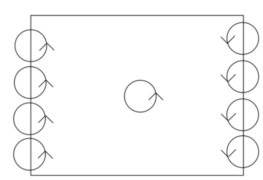


Here in the right picture you see the quantised value of the resistivity as we increase the magnetic field. and to the right you see the landau levels.

Edge Modes

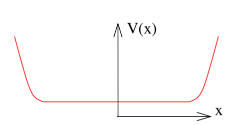
Classically, the particles move in circles in a magnetic field. At the edges, however, the particles cannot move in perfect circles. In the presence of the magnetic field, they move along one direction (anti-clockwise). At the edge, the particles collide with the boundary, and the result is what we call **skipping modes**. The particles move in one direction on one edge and in the opposite direction on the other edge, resulting in different chirality on the two edges.

The illustration below shows this edge behavior.



Quantum Picture

The edge modes are modeled by a potential, as shown in the following illustration.



The Hamiltonian governing the system is given by the expression:

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2 + V(x)\right)$$

The potential $V(x)$ can be Taylor expanded to include only the linear term as follows:

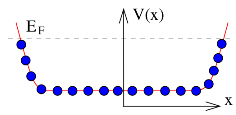
$$V(x) \approx V(X) + \frac{\partial V}{\partial x}(x - X)$$

The drift velocity of the particles is given by:

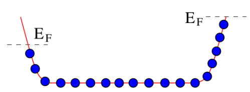
$$v_y = -\frac{1}{eB}\frac{\partial V}{\partial x}$$

Clearly, $v_y > 0$ on the left edge and $v_y < 0$ on the right edge.

Next, we fill the available states and observe the behavior. The states are labeled by the momentum $\hbar k$, but we can think of this in terms of the position of the state in the x direction. This allows us to represent the filled states as shown below, where we have introduced a chemical potential E_F.



Next, we introduce a potential difference $\Delta\mu$. This means that we fill up more states on the right-hand edge than on the left-hand edge, as shown below:



The current is given by the equation:

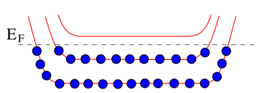
$$I_y = -e\int \frac{dk}{2\pi} v_y(k) = \frac{e}{2\pi l_B^2}\int dx \frac{1}{eB}\frac{\partial V}{\partial x} = \frac{e}{2\pi\hbar}\Delta\mu$$

where the Hall voltage is $eV_H = \Delta\mu$. The Hall conductivity is:

$$\sigma_{xy} = \frac{I_y}{V_H} = \frac{e^2}{2\pi\hbar}$$

This is the conductivity for a single Landau level.

The conductivity derived here does not depend on the shape of the potential as long as it does not cross the chemical potential E_F. Generalizing this to multiple Landau levels, as long as the chemical potential E_F lies between Landau levels, we have n filled Landau levels, as shown in the figure below.

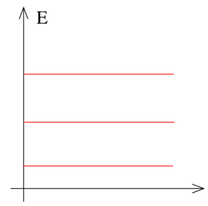


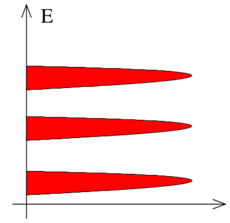
Correspondingly, there are n types of chiral modes on each edge, causing the conductivity to be n times that of the previous value. Up to this point, we have seen how the conductivity takes discrete values at the plateaus. Next, we will look into the very existence of these plateaus and explore why we observe jumps from one plateau to another.

The Role of Disorder

Every sample has impurities that can be modeled with some random potential $V(x)$. We consider $V \ll \hbar\omega_B$ to ensure there is no crossover of energy levels.

Quantum perturbation theory will split the degeneracy of the system and broaden the discrete energy spectrum.





In the semiclassical picture, electrons move in circular orbits. The quantum operators for the center of the orbit are

$$X = x - \frac{\pi_y}{m\omega_B} \quad \text{and} \quad Y = y + \frac{\pi_x}{m\omega_B}$$

where $\pi = p + eA = m\dot{x}$.

The time evolution of these operators is given by

$$i\hbar\dot{X} = [X, H+V] = il_B^2\frac{\partial V}{\partial Y},$$
$$i\hbar\dot{Y} = [Y, H+V] = -il_B^2\frac{\partial V}{\partial X}.$$

Define the velocity of the center of the circle and gradient of potential as

$$v = (\dot{X}, \dot{Y}) \quad \text{and} \quad \nabla V = \left(\frac{\partial V}{\partial X}, \frac{\partial V}{\partial Y}\right).$$

You can show that $v \cdot \nabla V = 0$. This implies that the center of mass drifts in a direction $(\dot{X}, \dot{Y})$ which is perpendicular to ∇V, i.e., the motion is along equipotentials.

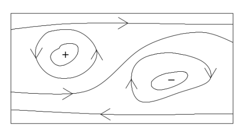


Figure 18: The localisation of states due to disorder.

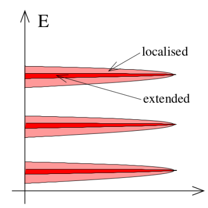


Figure 19: The resulting density of states.

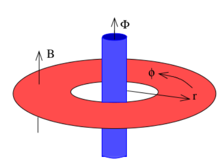
We consider a random potential with various peaks and troughs. "+" and "−" mark the maxima and minima. The particles are trapped at the extrema.

The far edges of a band—either at high or low energy—are localized. Only the states close to the center of the band are extended.

Suppose we have filled all the extended states in a given Landau level and consider what happens as we decrease B with fixed n. Each Landau level can accommodate fewer electrons. Rather than jumping up to the next Landau level, we now begin to populate the localized states. Since these states can't contribute to the current, the conductivity doesn't change.

Electron on an Annulus

We consider electrons moving in an annulus.



Now we pass an additional flux Φ through the center of the ring. We slowly increase Φ from 0 to $\Phi_0 = 2\pi\hbar/e$. This induces an electric field around the ring $\mathcal{E} = -\partial\phi/\partial t = \Phi_0/T$.

For the ring geometry, it is best to use the symmetric gauge and radial coordinates, $z = x + iy = re^{i\theta}$. If we evolve the system slowly, then the adiabatic theorem ensures that the final energy eigenstates must lie in the same Landau level as the initial state.

The wavefunctions in the lowest Landau level are

$$\psi_m \sim z^m e^{-|z|^2/4l_B^2} = e^{im\phi} r^m e^{-r^2/4l_B^2}.$$

The mth wavefunction is strongly peaked at a radius $r \approx \sqrt{2ml_B^2}$, where m is an integer. If we increase the flux from $\Phi = 0 \longrightarrow \Phi_0$, then from spectral flow we know the wavefunction shifts. Mathematically,

$$\psi_m(\Phi = 0) \longrightarrow \psi_m(\Phi = \Phi_0) = \psi_{m+1}(\Phi = 0).$$

This means each state moves outward, from radius $r = \sqrt{2ml_B^2}$ to $r = \sqrt{2(m+1)l_B^2}$. If n Landau levels are filled, then n electrons are transferred from the inner ring to the outer ring in time T. So the current is $I = -ne/T$, and the resistivity is given by

$$\rho_{xy} = \frac{\mathcal{E}}{I_r} = \frac{2\pi\hbar}{e^2} \cdot \frac{1}{n}.$$

Role of Topology

We begin with the Kubo formula for the conductivity:

$$\sigma_{xy} = i\hbar \sum_{n\neq 0} \frac{\langle 0| J_y |n\rangle \langle n| J_x |0\rangle - \langle 0| J_x |n\rangle \langle n| J_y |0\rangle}{(E_n - E_0)^2}.$$

We thread a uniform magnetic field B through the torus. The magnetic field must satisfy the Dirac quantization condition:

$$BL_xL_y = \frac{2\pi\hbar}{e} n, \quad n \in \mathbb{Z}.$$

Let us see how we get this from a topological argument.

We consider the wavefunction on the torus and introduce magnetic translation operators:

$$T(\mathbf{d}) = e^{-i\mathbf{d}\cdot\mathbf{p}/\hbar} = e^{i\mathbf{d}\cdot(i\nabla + e\mathbf{A}/\hbar)}.$$

This operator translates a state $\psi(x, y)$ by a position vector on the torus $\mathbf{d}$.

We choose the gauge $\mathbf{A}$ with $A_x = 0$ and $A_y = Bx$. If we take a state $\psi(x, y)$ and translate it around a cycle of the torus, we find:

$$T_x\psi(x, y) = \psi(x + L_x, y) = \psi(x, y),$$
$$T_y\psi(x, y) = e^{-ieBL_y x/\hbar}\psi(x, y + L_y) = \psi(x, y).$$

If we translate around the x-cycle followed by the y-cycle, or in the opposite order, the result should be the same:

$$T_yT_x = e^{-ieBL_xL_y/\hbar} T_xT_y.$$

This leads to the condition:

$$\frac{eBL_xL_y}{\hbar} \in 2\pi\mathbb{Z},$$

which is the Dirac quantization condition.

Adding Flux

We now add two fluxes Φ_x and Φ_y through the x and y-cycles of the torus, respectively. The gauge potential becomes:

$$A_x = \frac{\Phi_x}{L_x}, \quad A_y = \frac{\Phi_y}{L_y} + Bx.$$

This gives an additional term in the Hamiltonian:

$$\Delta H = -\sum_{i=x,y} \frac{J_i\Phi_i}{L_i}.$$

Using perturbation theory, the new ground state is given by:

$$|\psi_0\rangle' = |\psi_0\rangle + \sum_{n\neq 0} \frac{\langle n| \Delta H |\psi_0\rangle}{E_n - E_0} |n\rangle.$$

Differentiating this with respect to Φ_i, we obtain:

$$|\frac{\partial\psi_0}{\partial\Phi_i}\rangle = -\frac{1}{L_i} \sum_{n\neq 0} \frac{\langle n| J_i |\psi_0\rangle}{E_n - E_0} |n\rangle.$$

These are exactly the terms appearing in the Kubo formula. Plugging this into the Kubo formula, we get:

$$\sigma_{xy} = i\hbar \left[\frac{\partial}{\partial\Phi_y} \langle\psi_0| |\frac{\partial\psi_0}{\partial\Phi_x}\rangle - \frac{\partial}{\partial\Phi_x} \langle\psi_0| |\frac{\partial\psi_0}{\partial\Phi_y}\rangle \right].$$

In the perturbed Hamiltonian, Φ_x and Φ_y are the two parameters, and the spectrum depends only on $\Phi_i \mod \Phi_0$ due to the spectral theorem. So we can think of these parameters as periodic. The space of the parameters is itself a torus T_Φ^2.

The Berry connection and Berry curvature on the torus T_Φ^2 are given by:

$$\mathcal{A}_i(\Phi) = -i \langle\psi_0| \frac{\partial}{\partial\theta_i} |\psi_0\rangle,$$

$$\mathcal{F}_{xy} = \frac{\partial\mathcal{A}_x}{\partial\theta_y} - \frac{\partial\mathcal{A}_y}{\partial\theta_x} = -i \left[\frac{\partial}{\partial\theta_y} \langle\psi_0| \frac{\partial\psi_0}{\partial\theta_x}\rangle - \frac{\partial}{\partial\theta_x} \langle\psi_0| \frac{\partial\psi_0}{\partial\theta_y}\rangle \right].$$

Then the conductivity is given by:

$$\sigma_{xy} = -\frac{e^2}{\hbar}\mathcal{F}_{xy}.$$

If we average over all fluxes(basically we are using Gauss Bonnet theorem), we get:

$$\sigma_{xy} = -\frac{e^2}{\hbar} \int_{T_\Phi^2} \frac{d^2\theta}{(2\pi)^2} \mathcal{F}_{xy}.$$

The Chern number is defined as:

$$C = \frac{1}{2\pi} \int_{T_\Phi^2} d^2\theta \mathcal{F}_{xy},$$

and this is always an integer. Thus, the Hall conductivity is:

$$\sigma_{xy} = -\frac{e^2}{2\pi\hbar} C.$$

This is the Integer Quantum Hall Effect.

Fractional Hall Effect

Laughlin States

To address the fractional quantum Hall states, Laughlin proposed a wavefunction that successfully describes filling fractions of the form $\nu = \frac{1}{m}$. To gain insight into this, we consider a toy model consisting of two particles interacting via a central potential, i.e., $V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$. Since the Hamiltonian for a central potential commutes with the angular momentum operator, it is natural to work in the basis of angular momentum eigenstates, which also diagonalize the Hamiltonian. Additionally, we employ the symmetric gauge, as the lowest Landau level wavefunctions in this gauge are themselves eigenstates of angular momentum.

The single-particle eigenstates in the lowest Landau level take the form

$$\psi_m \sim z^m e^{-|z|^2/4l_B^2}, \tag{28}$$

where $z = x - iy$, and l_B is the magnetic length. These states are localized on rings of radius $r = \sqrt{2m}l_B$. The angular momentum operator J acts on these states as

$$J = i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \implies J\psi_m = \hbar m\psi_m, \tag{29}$$

with the integer m labeling angular momentum of the lowest Landau wavefunction.

If we neglect mixing between Landau levels, the eigenstate for two particles in a central potential takes the form

$$\psi \sim (z_1 + z_2)^M (z_1 - z_2)^m e^{-(|z_1|^2+|z_2|^2)/4l_B^2}, \tag{30}$$

where M is the angular momentum of the center of mass and m is the relative angular momentum between the particles.

Generalizing this to a many-body system, Laughlin proposed that the ground state wavefunction at filling fraction $\nu = \frac{1}{m}$ is given by

$$\psi(z_i) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^{N}|z_i|^2/4l_B^2}. \tag{31}$$

In this expression, m must be odd for the wavefunction to be antisymmetric (for fermions), and even for symmetric wavefunctions (bosons).

To understand the implications of this form, consider the dependence of the wavefunction on a single particle coordinate, say z_1. The prefactor involving z_1 is

$$\prod_{i=2}^{N}(z_1 - z_i)^m, \tag{32}$$

which implies that the power of z_1 is $m(N-1)$. This indicates that the maximum angular momentum of the first particle is $m(N-1)$.

The corresponding spatial extent of the wavefunction leads to a maximum radius approximately given by $R \approx \sqrt{2mN}\, l_B$. Hence, the area occupied by the system is $A \approx 2\pi mNl_B^2$. The degeneracy of each Landau level is given by $AB/\Phi_0 \approx mN$, which leads to the filling fraction

$$\nu = \frac{1}{m}. \tag{33}$$

The Fully Filled Landau Level

The many-particle wavefunction for non-interacting electrons must be antisymmetric under the exchange of any two particles, since electrons are fermions.

In general the many body wavefunction can be written as

$$\psi(z_1, ..., z_N) = f(z_1, ..., z_N)e^{-\sum_{i=1}^{N}|z_i|^2 4l_B^2} \tag{34}$$

where $f(z)$ is an analytic function. This requirement is satisfied by expressing the wavefunction as a Slater determinant:

$$\psi(x_i) = \begin{vmatrix} \psi_1(x_1) & \psi_1(x_2) & \cdots & \psi_1(x_N) \\ \psi_2(x_1) & \psi_2(x_2) & \cdots & \psi_2(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(x_1) & \psi_N(x_2) & \cdots & \psi_N(x_N) \end{vmatrix}. \tag{35}$$

The single-particle states in the lowest Landau level are given by

$$\psi_m(z) \sim z^{m-1}e^{-|z|^2/4l_B^2}, \quad m = 1, 2, \ldots, N. \tag{36}$$

If you put 34 and 36 The prefactor of the Slater determinant in this case becomes the Vandermonde determinant:

$$f(z_i) = \begin{vmatrix} z_1^0 & z_2^0 & \cdots & z_N^0 \\ z_1^1 & z_2^1 & \cdots & z_N^1 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} = \prod_{i<j}(z_i - z_j). \tag{37}$$

This is precisely the Laughlin wavefunction for $m = 1$, showing that the Laughlin state at $m = 1$ corresponds exactly to a completely filled lowest Landau level.

Plasma Analogy

Calculating observables from the Laughlin wavefunction directly is extremely difficult due to the presence of a macroscopic number of integrals over particle coordinates $\int d^2z_i$. To gain insight, Laughlin introduced an analogy with a classical system—a two-dimensional plasma.

Suppose we wish to compute the expectation value of the density:

$$n(z) = \sum_{i=1}^{N}\delta(z - z_i). \tag{38}$$

Its expectation value in the Laughlin state is given by:

$$\langle n(z)\rangle = \frac{\int \prod_{i=1}^{N} d^2z_i\, n(z)P[z_i]}{\int \prod_{i=1}^{N} d^2z_i\, P[z_i]}, \tag{39}$$

where the probability distribution is

$$P[z_i] = \prod_{i<j}\frac{|z_i - z_j|^{2m}}{l_B^{2m}}e^{-\sum_i |z_i|^2/2l_B^2}. \tag{40}$$

This probability distribution can be interpreted as a Boltzmann factor:

$$P[z_i] = e^{-\beta U(z_i)}, \tag{41}$$

with an effective potential energy

$$\beta U(z_i) = -2m\sum_{i<j}\log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{1}{2l_B^2}\sum_{i=1}^{N}|z_i|^2. \tag{42}$$

Here, $\beta = \frac{2}{m}$, playing the role of an inverse temperature, though in this context it is not associated with any real thermal environment—it is just a mathematical analogy.

Rewriting the effective potential $U(z_i)$, we obtain:

$$U(z_i) = -m^2 \sum_{i<j} \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{m}{4l_B^2} \sum_{i=1}^{N} |z_i|^2. \tag{43}$$

This corresponds to the potential energy of a classical two-dimensional plasma, where each particle carries a charge $q = -m$.

From Poisson's equation in two dimensions:

$$-\nabla^2 \Phi = 2\pi q \delta^2(r) \quad \Rightarrow \quad \Phi = -q \log\left(\frac{r}{l_B}\right), \tag{44}$$

we see that the first term in U corresponds to the interaction energy between two particles of charge q. The second term arises from a neutralizing background with constant charge density. If this background has charge density ρ_0, the Poisson equation becomes:

$$-\nabla^2 \phi = 2\pi \rho_0, \tag{45}$$

and the potential $\phi = \frac{|z|^2}{4l_B^2}$ obeys

$$-\nabla^2 \left(\frac{|z|^2}{4l_B^2}\right) = -\frac{1}{l_B^2}. \tag{46}$$

Thus, the background charge density must be

$$\rho_0 = -\frac{1}{2\pi l_B^2}. \tag{47}$$

The benefit of analysing this analogous model is that we can use our intuition for the plasma model. To minimize the energy, the system tends to neutralize the background charge.So the compensating density of charge will be $mn = \rho_0$

$$n = \frac{1}{2\pi l_B^2 m} \tag{48}$$

This is expected density of state at filling fraction $\nu = \frac{1}{m}$.This also tells us that the average density of particle is constant At low density (large m) we have solid and high density(low m) we have liquid.(keeping constant temperature). Numerically it is seen for m > 70 we have solid and for low m like m=3 or 5 laughlin wavefunction describes a liquid. Intuitively this also makes sense as the electrons are trying to move away from each other, if they have more space to move they can configure in a lattice structure so as to minimise the repulsion between the particles and minimise the energy. on the other hand if they are close to each other the energy cost is very high so they move around trying to find the position to reduce the interaction leading to liquid behaviour.

Quasi-Holes and Quasi-Particles

So far we have discussed the ground state of the $\nu = \frac{1}{m}$ quantum Hall systems. There are two types of charged excitations known as quasi-holes and quasi-particles.

Quasi-Holes The wavefunction describing a quasi-hole at position $\eta \in \mathbb{C}$ is

$$\Psi_{\text{hole}}(z, \eta) = \prod_{i=1}^{N} (z_i - \eta) \prod_{k<l} (z_k - z_l)^m e^{-\sum_{i=1}^{N} |z_i|^2/4\ell_B^2} \tag{49}$$

We see that the electron density vanishes at the point η. We can introduce M quasi-holes in the quantum Hall fluid at positions η_j and the wavefunction is given as

$$\Psi_{M\text{-holes}}(z, \eta) = \prod_{j=1}^{M} \prod_{i=1}^{N} (z_i - \eta_j) \prod_{k<l} (z_k - z_l)^m e^{-\sum_{i=1}^{N} |z_i|^2/4\ell_B^2} \tag{50}$$

The quasi-holes carries a fractional charge. We will now see this using the plasma analogy.

The resulting plasma potential energy has a new term:

$$U(z_i) = -m^2 \sum_{i<j} \log\left(\frac{|z_i - z_j|}{\ell_B}\right) - m \sum_{i} \log\left(\frac{|z_i - \eta|}{\ell_B}\right) + \frac{m}{4\ell_B^2} \sum_{i=1}^{N} |z_i|^2$$

This extra term looks like an impurity in the plasma with charge 1.The impurity carries a charge $-1/m$ the charge of the electron Each particles in the plasma correspond to electrons but have effective charge $-m$. The particles in the plasma will swarm around the impurity to screen it. so the net missing charge is $+\frac{1}{m}$, indicating the quasi-hole has charge $+e/m$ if we write the charge of the electron as $-e$.

Quasi-Particles These are excitations which carry charge $e^* = -\frac{e}{m}$, i.e., the same sign as the electron charge. Unlike quasi-holes (which decrease charge density), quasi-particles increases charge density in the hall fluid and decreases the maximum angular momentum. we can simply divide the wavefunction by z to reduce the highest power and also reduce the maximum angular momentum. reducing the maximum angular momentum will also reduce the maximum radius which in turn will reduce the area and increase the charge density. but dividing by z makes the wavefunction singular. Instead, we apply differential operators to reduce the highest power of z.

The wavefunction for a quasi-particle at η is then given by:

$$\Psi_{\text{particle}}(z, \eta) = \left[\prod_{i=1}^{N} \left(\ell_B^2 \frac{\partial}{\partial z_i} - \frac{1}{2}\bar{\eta}\right) \prod_{k<l} (z_k - z_l)^m\right] e^{-\sum_{i=1}^{N} |z_i|^2/4\ell_B^2} \tag{51}$$

Hall Conductivity from Laughlin States

Consider the annulus geometry and increase the flux through the hole from 0 to ϕ_0.

By spectral flow, the angular momentum of each electron increases by 1. This is achieved by multiplying the wavefunction by $\prod_i z_i$, which corresponds to adding a quasi-hole at $\eta = 0$.

As ϕ increases from 0 to ϕ_0, a particle of charge $-e/m$ is transferred from the inner circle to the outer circle. A full electron is transferred only when the flux increases by $m\phi_0$. This gives the Hall conductivity:

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \cdot \frac{1}{m} \tag{52}$$

Anyons

In 3D, particles are either bosons or fermions. But in 2D, a loophole allows for other statistics.

Let $\psi(r_1, r_2)$ describe two identical particles. Upon exchange: $|\psi(r_1,r_2)|^2 = |\psi(r_2,r_1)|^2$, so

$$\psi(r_1, r_2) = e^{i\pi\alpha}\psi(r_2, r_1) \tag{53}$$

Exchanging again gives:

$$\psi(r_1, r_2) = e^{2\pi i\alpha}\psi(r_1, r_2) \Rightarrow e^{2\pi i\alpha} = 1$$

If we want the wavefunction to be exactly same after exchanging two particles then this restricts the value of alpha. In 3D this restricts α to 0 (bosons) or 1 (fermions), but in 2D this condition that the wavefunction should look exactly the same does not apply—allowing alpha to take other values α. This are anyons.

Fractional Statistics of Quasi-Holes

Now we will compute the quantum statistics of quasi holes in the $\nu = 1/m$ Wavefunction for M quasi-holes is:

$$\langle z, \bar{z}|\eta_1, \ldots, \eta_M\rangle = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m e^{-\sum_{i=1}^{N}|z_i|^2/4\ell_B^2} \tag{54}$$

Normalize the wavefunction:

$$|\psi\rangle = \frac{1}{\sqrt{Z}}|\eta_1, \ldots, \eta_M\rangle \tag{55}$$

where the normalization factor is defined as $Z = \langle\eta_1, ..., \eta_M|\eta_1, ..., \eta_M\rangle$,which is given as

$$Z = \int\prod d^2z_i\ \exp\left(\sum_{i,j}\log|z_i - \eta_j|^2 + m\sum_{k<l}\log|z_k - z_l|^2 - \frac{1}{2\ell_B^2}\sum_i|z_i|^2\right)$$

Holonomic and Non holonomic Berry connections is given as :

$$\mathcal{A}_\eta = -i\langle\psi|\frac{\partial}{\partial\eta}|\psi\rangle = -\frac{i}{2}\frac{\partial\log Z}{\partial\eta}, \tag{56}$$

similarly

$$\mathcal{A}_{\bar{\eta}} = \frac{i}{2}\frac{\partial\log Z}{\partial\bar{\eta}} \tag{57}$$

This is hard to compute directly, so we use plasma analogy.

In this analogy, the quasi-hole acts like a charged impurity. Screening causes charges to rearrange around the impurity. Due to this the potential due to the impurity falls off exponentially as $e^{-r/\lambda}$, where λ is the Debye length.

Total plasma energy:

$$U(z_k; \eta_i) = -m^2\sum_{k<l}\log\left(\frac{|z_k - z_l|}{\ell_B}\right) - m\sum_{k,i}\log\left(\frac{|z_k - \eta_i|}{\ell_B}\right) - \sum_{i<j}\log\left(\frac{|\eta_i - \eta_j|}{\ell_B}\right) + \frac{m}{4\ell_B^2}\sum_k|z_k|^2 + \frac{1}{4\ell_B^2}\sum_i|\eta_i|^2$$

The corrected plasma partition function is then given as

$$\int\prod d^2z_i e^{-\beta(z_i,\eta)U} = \exp(\frac{-1}{m}\sum_{i<j}log|\eta_i - \eta_j|^2 + \frac{1}{2ml_B^2}\sum|\eta_i|^2)Z \tag{58}$$

if the distance between the holes $|\eta_i - \eta_j|$ are greater than the Debye length(λ) then the integral is independent of the position η_i for high enough temperature.so we can write

$$Z = C\exp\left(\frac{1}{m}\sum_{i<j}\log|\eta_i - \eta_j|^2 - \frac{1}{2m\ell_B^2}\sum_i|\eta_i|^2\right)$$

Then,the berry connection for M quasi holes is given as ,

$$\mathcal{A}_{\eta_i} = -\frac{i}{2m}\sum_{j\neq i}\frac{1}{\eta_i - \eta_j} + i\frac{\bar{\eta}_i}{4m\ell_B^2}$$

$$\mathcal{A}_{\bar{\eta}_i} = \frac{i}{2m}\sum_{j\neq i}\frac{1}{\bar{\eta}_i - \bar{\eta}_j} - i\frac{\eta_i}{4m\ell_B^2}$$

Fractional Charge

Take a quasi-hole and move it in a closed path so that it encloses no other quasi-holes. only the 2nd term in the berry connection contributes to the berry phase .

$$\mathcal{A}_\eta = i\frac{\bar{\eta}}{4m\ell_B^2}, \quad \mathcal{A}_{\bar{\eta}} = -i\frac{\eta}{4m\ell_B^2}$$

After traversing a closed path it gain a phase γ which is given by the berry phase

$$e^{i\gamma} = \exp\left(-i\oint\mathcal{A}_\eta d\eta + \mathcal{A}_{\bar{\eta}}d\bar{\eta}\right)$$

This gives the berry phase to be

$$\gamma = \frac{e\phi}{m\hbar} \tag{59}$$

where ϕ is the total magnetic flux enclosed by the path C.This the usual Aharonov Bohm phase picked up by the particle ie if you have a charge e* , it will pick a phase $\gamma = e^*\phi/\hbar$, where ϕ is the flux the closed path encloses. comparing this with equation 59. we get the charge of the quasi holes to be

$$e^* = \frac{e}{m} \tag{60}$$

Fractional Statistics

Now we take two quasi holes and move one quasi-hole around another in a closed loop. The second term will once again give the Aharonov phase . The first term in $\mathcal{A}_\eta$ will give us the statistics

$$e^{i\gamma} = exp(-\frac{1}{2m}\oint\frac{d\eta_1}{\eta_1 - \eta_2} + \text{h.c.}) = e^{\frac{2\pi i}{m}} \tag{61}$$

Now when we exchange two object we rotate one object by 180 and not 360. that if we exchange the particle once again we can write as follows

$$e^{2\pi i\alpha} = e^{2\pi i/m} \Rightarrow \alpha = \frac{1}{m} \tag{62}$$

For fully filled landau level $m = 1$, quasi-holes are fermions. Bur for a fractional quantum Hall state, the quasi holes are anyons.

Chern-Simons Theory and the IQHE

So far, we have approached the quantum Hall states from a microscopic perspective. Here, we describe the quantum Hall effect on a more coarse-grained level. Our goal is to construct effective field theories that capture the response of the quantum Hall ground state to low-energy perturbations. We treat the gauge potential A_μ of electromagnetism as a background gauge field. In field theory, A_μ typically couples to the current J_μ, so the action includes the term

$$S_A = \int d^3x\, J^\mu A_\mu. \tag{63}$$

We are working in $d = 2+1$ dimensions. This action is invariant under gauge transformations due to the conservation of current, $\partial_\mu J^\mu = 0$. This will be our starting point to find the effective field theory that tells us how the system responds when we perturb the background electric or magnetic field.

Integer Quantum Hall Effect

We assume that at low energies, there are no degrees of freedom that can affect the physics when the system is perturbed. This means there is a gap to the first excited state, so our system behaves as an insulator rather than a conductor. The partition function is schematically given as

$$Z[A_\mu] = \int D(\text{fields})\, e^{iS[\text{fields};A]/\hbar},$$

where the "fields" refer to all dynamical degrees of freedom. If we integrate out all these dynamical degrees of freedom, the partition function becomes

$$Z[A_\mu] = e^{iS_{\text{eff}}[A_\mu]/\hbar}.$$

From 63, we can derive

$$\frac{\delta S_{\text{eff}}[A_\mu]}{\delta A_\mu(x)} = \langle J^\mu(x)\rangle. \tag{64}$$

This equation tells us that the effective action encodes the response of the current to the electric and magnetic fields.

Chern-Simons Effective Action

If we are only interested in large-distance behavior, the effective action will be a local function. Hence, we can write $S_{\text{eff}}[A] = \int d^dx \ldots$. This locality is justified because our theory has a gap ΔE, and any non-locality arises only at distances comparable to or smaller than $v\hbar/\Delta E$. In $d = 2+1$ dimensions, the effective action is given by the Chern-Simons term, $S_{\text{eff}}[A] = S_{\text{CS}}[A]$, where

$$S_{\text{CS}}[A] = \frac{k}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{65}$$

Here, k is called the level of the Chern-Simons term. We get the current from the Chern-Simons and 64 term and it is given by

$$J_i = \frac{\delta S_{\text{CS}}[A]}{\delta A_i} = -\frac{k}{2\pi}\epsilon_{ij}E_j.$$

From this, we get the Hall conductivity as

$$\sigma_{xy} = \frac{J_x}{E_y} = \frac{k}{2\pi},$$

which matches with the Hall conductivity of ν filled Landau levels if we identify the Chern-Simons level with $k = \frac{e^2\nu}{\hbar}$.

Now the question is why k should be quantised?

Calculating the partition function

Before answering the above question we need to learning this paragraph.

Consider a quantum particle of mass m moving in one direction with coordinate q . Suppose it moves in a potential $V(q)$. In statistical mechanics the partition function is given as

$$Z[\beta] = Tre^{-\beta H} \tag{66}$$

where H is the Hamiltonian operator,$\beta = 1/T$(take $k_B = 1$)

Suppose the particle is at q_i at t=0 . then the amplitude of finding the particle at $q = q_f$ at time t is given as

$$\langle q_f|\, e^{-iHt}\, |q_i\rangle = \int_{q(0)=q_i}^{q(t)=q_f} Dqe^{iS} \tag{67}$$

where S is the classical action

$$S = \int_0^t dt'[\frac{m}{2}(\frac{dq}{dt})^2 - V(q)] \tag{68}$$

Our goal is the use 67 to find the partition function.we take Euclidean $\tau = it$ after substitution the action becomes

$$iS = \int_0^{-i\tau}[\frac{-m}{2}(\frac{dq}{d\tau})^2 - V(q)] \equiv -S_E \tag{69}$$

where S_E is Euclidean action.Next we take $\tau = \beta$. then the quantum amplitude becomes

$$\langle q_f|\, e^{-H\beta}\, |q_i\rangle \tag{70}$$

We are almost there . the partition function can now be written

$$\begin{aligned} Tre^{-\beta H} &= \int dq_i \,\langle q_i|\, e^{-H\beta}\, |q_i\rangle \\ &= \int dq_i \int_{q(0)=q_i}^{q(\beta)=q_i} Dqe^{-S_E} \\ &= \int_{q(0)=q(\beta)} Dqe^{-S_E} \end{aligned}$$

. here we basically integrate over all the trajectories that starts and end at the same point after Euclidean time $\tau = \beta$. This is summarised by saying that Euclidean time τ is parameterising a circle it is periodic with periodcity β

$$\tau \equiv \tau + \beta \tag{71}$$

The take away is that if we want to want to take a quantum field that lives on $R^{d-1,1}$ and want to compute the partition function then you can consider the Euclidean path integral but the theory now formulated on Euclidena space $R^{d-1}\times S^1$, where the circle is parametrised by $\tau \in [0,\beta)$

Quantisation of the Chern-Simons Level

To find the partition function at finite temperature, we take time to be Euclidean S^1, parameterised by τ with periodicity. When we perform the gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu\omega$, the wavefunction and the field transform as $e^{i\omega/\hbar}$, and we want this to be single-valued. If we set

$$\omega = \frac{2\pi\hbar\tau}{e\beta},$$

this leaves the exponential $e^{i\omega/\hbar}$ single-valued. We work on a sphere rather than $\mathbb{R}^2$. If we put a Dirac magnetic monopole inside the S^2, then Dirac quantisation tells us the minimum flux that we get is given by

$$\frac{1}{2\pi}\int_{S^2} F_{12} = \frac{\hbar}{e},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We choose $A_0 = a$, some constant and we consider that there is no time dependence . Expanding the Chern-Simons term we get

$$S_{CS} = \frac{k}{4\pi}\int d^3x A_0F_{12} + A_1F_{20} + A_2F_{01}$$

Focusing on the last two term if we through away ∂_0 term we are left with $A_1\partial_2A_0 - A_2\partial_1A_0$ after integration by parts this will become $A_0(\partial_1A_2 - \partial_2A_1) = A_0F_{12}$. So a factor 2 will come out of the integral.Now we have

$$S_{\text{CS}} = \frac{k}{2\pi}\int d^3x\, A_0F_{12} \implies S_{\text{CS}} = \beta\alpha\frac{\hbar k}{e}.$$

Under a large gauge transformation, i.e., we shift only the temporal component of the gauge field,

$$A_0 \longrightarrow A_0 + \frac{2\pi\hbar}{e\beta},$$

the Chern-Simons term is not gauge invariant:

$$S_{\text{CS}} \longrightarrow S_{\text{CS}} + \frac{2\pi\hbar^2 k}{e^2}.$$

This is not a problem because what we really want is for the partition function to remain invariant:

$$Z[A_\mu] = e^{iS_{\text{eff}}[A_\mu]/\hbar}.$$

We can ensure this if

$$\frac{\hbar k}{e^2} \in \mathbb{Z}.$$

From here we get the qunatised value of k that we have been seeking for . we have $k = \frac{e^2\nu}{\hbar}$ with $\nu \in \mathbb{Z}$. Finally the Hall conductivity is given by

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu.$$

Fractional Quantum Hall Effect

It turns out that there are degrees of freedom, also known as topological degrees of freedom, that can alter the low-energy physics of a gapped system. Similar to our previous case, the partition function is given as

$$Z[A_\mu] = \int D(\text{fields})\, e^{iS[\text{fields;A}]/\hbar}$$

but this time we will not integrate, to retain the topological degree of freedom. The topological degree of freedom in our system is an emergent gauge field a_μ which arises from the collective behaviour of many underlying electrons. For massless degrees of freedom, the dynamics is given by the Maxwell action

$$S_{\text{Maxwell}}[a] = -\frac{1}{4g}\int d^3x\, f_{\mu\nu}f^{\mu\nu}$$

The equation of motion is $\partial_\mu f^{\mu\nu} = 0$. They admit wave solutions. In 2 + 1 dimensions, there is only one polarization, so we have a single massless degree of freedom.

Now we also have the Chern-Simons term

$$S_{\text{CS}}[a] = \frac{k}{2\pi}\int d^3x\, \epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho$$

If we add both these terms in the action, we get something interesting:

$$S = S_{\text{Maxwell}} + S_{\text{CS}}.$$

The equation of motion for a_μ becomes

$$\partial_\mu f^{\mu\nu} + \frac{kg^2}{4\pi}\epsilon^{\nu\rho\sigma}f_{\rho\sigma} = 0.$$

This describes a photon with mass, and the spectrum has an energy gap $E_{\text{gap}} = kg^2/2\pi$. As $g^2 \to \infty$, the photon becomes infinitely massive and we are left with no physical excitations.

1 Effective Theory for the Laughlin States

The partition function is given as

$$Z[A_\mu] = \int Da_\mu\, e^{iS_{\text{eff}}[a,A]/\hbar}.$$

Our goal is to write $S_{\text{eff}}[a, A]$. We know from our previous discussion that the background field needs to couple to the dynamic field, so there should be a term coupling A_μ and a_μ. We also know there are terms that couple A_μ and J_μ. So there must be a relation between a_μ and J_μ. The relation between J_μ and a_μ is given by

$$J^\mu = \frac{e^2}{2\pi\hbar}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho,$$

which satisfies the conservation of current, $\partial_\mu J^\mu = 0$. The effective action is given as

$$S_{\text{eff}}[a; A] = \frac{e^2}{\hbar}\int d^3x\, \frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{m}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \dots$$

The first term is the Chern-Simons term that comes from the coupling $A_\mu J^\mu$. The second term is the new term that we add. The equation of motion is given as

$$f_{\mu\nu} = \frac{1}{m}F_{\mu\nu}.$$

The solution of this is $a_\mu = A_\mu/m$. Putting this back in the expression for S_{eff}, we get

$$S_{\text{eff}}[A] = \frac{e^2}{\hbar}\int d^3x\, \frac{1}{4\pi m}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.$$

This is almost the same as 65 except we have a factor of m in the denominator. This gives us the Hall conductivity as

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\frac{1}{m}.$$

Quasiholes and Quasi-Particles

Now we add another term to the action that couples the effective gauge field a_μ to its own current j^μ. To ensure gauge invariance we must have $\partial_\mu j^\mu = 0$. This current will describe quasi-holes and quasi-particles in the system:

$$\Delta S = \int d^3x \, a_\mu j^\mu.$$

We set the background gauge field to be zero. The equation of motion is then given by

$$\frac{e^2}{2\pi\hbar} f_{\mu\nu} = \frac{1}{m}\epsilon_{\mu\nu\rho} j^\rho.$$

We take a static charge and put it at the origin, so $j^1 = j^2 = 0$ and $j^0 = e\delta^2(x)$. Then the equation of motion becomes

$$\frac{f_{12}}{2\pi} = \frac{\hbar}{em}\delta^2(x).$$

We see the effect of the Chern-Simons term is to attach magnetic flux $\hbar/em$ to the charge e.

On calculating the current density in the background, we get

$$J^0 = \frac{e^2}{2\pi\hbar} f_{12} = \frac{e}{m}\delta^2(x).$$

We see that the charge of this stationary particle is e/m.

Fractional statistics also emerges from here. From Aharonov-Bohm physics, we know that if a charge q particle moves around a flux ϕ, it picks up a phase $e^{iq\phi/\hbar}$. Here our quasi-particles carry both charge $q = e$ and flux $\phi = 2\pi\hbar/em$. If we move one particle around the other, it will pick up a phase $e^{iq\phi/\hbar}$. When we exchange two particles, we are rotating each particle not by 360° but by 180°, so we get $e^{i\pi\alpha} = e^{iq\phi/2\hbar}$. By comparing, we get

$$\alpha = \frac{1}{m}.$$

This is the expected statistics of quasi-holes in the Laughlin states.

2 Conclusion

So in this report, we took a long tour from the Landau theory and implemented it in understanding the integer quantum Hall effect. We also saw how disorder is important to have these phenomena and how the robustness of the Hall effect is closely linked with topology. Later, we invoked the electron interaction in the system and saw how Laughlin's wavefunction explains the fractional quantum hall effect. We also learned about quasi-holes and particles, and their properties like fractional charge and fractional statistics. Next we moved onto a coarse-grained description of Hall effect, where we used Chern-Simon theory. we saw how powerful this method is . Just by finding a clever way to find the partition function we were able to describe all the features that we saw before without the knowledge of the microscopic details.

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