Differential Form in Electromagnetism

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Abstract :- Electromagnetism is one of the well explored topic in physics. Over the years a lot of advance mathematical formalisms are developed to understand the topic in more depth and shine light into the non intuitive aspect of this subject. In this article we will explore the known result of electromagnetism through the lens of differential forms . We will see how this approach help us to simplify the math and give geometrical picture for better understanding.

Contents

1 Introduction

One of the most well established theory in physics is the theory of electromagnetism. James Clerk Maxwell discovered the full set of equations that govern the electromagnetic behavior. In his original there were 20 sets of coupled differential equations , but over the year with the advancement of mathematical tools ,elegant mathematical framework is developed which clubs all the 20 equations into 4 equations that we have all seen in any electromagnetic book in the market. But in almost all the book and lectures around the world vector calculus is the mathematical took which is used and taught to study electromagnetism. No doubt it is helpful in solving a lot of problem but there are other mathematical formalisms like dyadic, bivectors, tensors, quaternions, and Clifford algebras. Here we will be using differential form to study electromagnetism. What motivates us to do this? Differential forms notation simplifies some calculations and allows for elegant pictures that clarify some of the non-intuitive aspects of electromagnetic fields.

2 Differential Form

The use of differential form will not replace the vector notation but it will give more insight into the nontrivial concepts of electromagnetism.

The difference between vector fields and differential form is the different interpretations this two formalism gives . We can think of the electric field as a force on a test charge or the change in energy experienced by the test charge as it moves through the field . These two interpretation are not alien to each other but complements each other . In fact we can always translate back and forth between vectors and differential forms.

The energy W_{21} required to move a charged particle from $\vec{r_1}$ to $\vec{r_2}$ in the presence of electric field $\vec{E} = (E_x, E_y, E_z)$ is given by

$$W_{21} = -q \int_{\vec{r_1}}^{\vec{r_2}} \vec{E}.\vec{dl} \tag{1}$$

If we expand the integrand we get a linear combination of one form

$$\varepsilon = E_x dx + E_y dy + E_z dz \tag{2}$$

so eq1 can be written in the following way

$$W_{21}(t) = -q \int_{r_1}^{r_2} \varepsilon \tag{3}$$

In the differential forms notation the electric field is represented by a one-form.

One form can be graphically represented as surfaces in 3D and curves in 2D. In the above case each of surface represents equipotential surfaces . The integral value is the measure of how many surface the path cuts .The greater the electric field intensity ie the greater the component of the one form closer the surfaces are spaced.VIf the components are not constants then the surfaces are not flat they are curved.

Mathematically,vectors and differential forms are closely related in fact they are equivalent if they have the same components .

$$A_1 dx + A_2 dy + A_3 dz \leftrightarrow A_1 \hat{x} + A_2 \hat{y} + A_3 \hat{z} \tag{4}$$

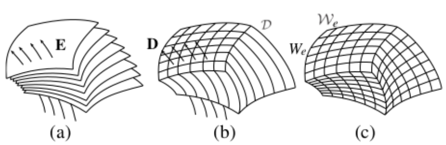
The one form A and the vector $\vec{A}$ are dual. In this new viewpoint we visualize the field via the change of the energy of a test charge as it moves through the field. The one form defined above ε is the surface of constant energy/electric potential. Now as the particle moves through space it hopes from one surface to another surface . This mental picture can be mathematically represented by a simple equation

$$\nu_{21} = -\int_1^2 \varepsilon \tag{5}$$

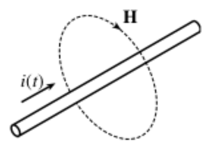
. In a similar way we can define the magnetic field intensity as one form

$$\mathcal{H} = H_x(x, y, z, t)\, dx + H_y(x, y, z, t)\, dy + H_z(x, y, z, t)\, dz \tag{6}$$

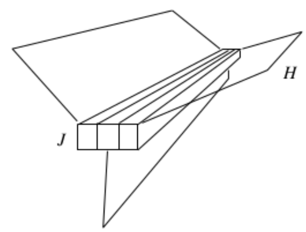
Mathematically , the differential form ε and H express the changes of the electric and magnetic potentials over an infinitesimal path element.



(a) Geometric representation of a a) one-form b) two form c) three form . the one form is represented by surfaces , the two form as tube s of lux or flow ,and the three form as cubes representing volumetric density



(b) Path of integration for the defination of current



(c) A graphical representation of the physical behavior of magnetic field and electric current:tubes of J producing surfaces of H

The magnetic field produced by a current is also an one form . The surfaces radiate outward from the line current figure 1(b) shows the path of integration for the defination of the current i. The integration path passes through a number of one form surfaces equal to the amount of current flowing through the path. In the figure you can see that the surfaces of H emanates form the tubes of current density J.

Next we will introduce the higher order differential forms. We will do so using examples . Let consider the current i flowing in x-direction through the surface A in the figure.To compute the current we have to integrate the x-component J_x of the current density over the surface A in the yz-plane.

$$i = \int_A J_x dy dz \tag{7}$$

Now while doing the integration we need to keep in mind the orientation of the area . Exterior differential form allow one do so in a rigorous way by the following wedge product property.

$$dy \wedge dz = -dz \wedge dy \tag{8}$$

in later we will using some the following notations interchangeable

$$\begin{aligned} dy \wedge dz &= -dz \wedge dy \\ dz \wedge dy &= -dy \wedge dz \end{aligned} \tag{9}$$

now equation 7 can be written in the following way

$$i = \int_A dy \wedge dz \tag{10}$$

A general two-form C is written as

$$C = C_1 dy \wedge dz + C_2 dz \wedge dx + C_3 dx \wedge dy \tag{11}$$

geometry we can visually as tube we represents a flow of some quantity in this case current .Similarly we can create forms of higher order using the wedge product.

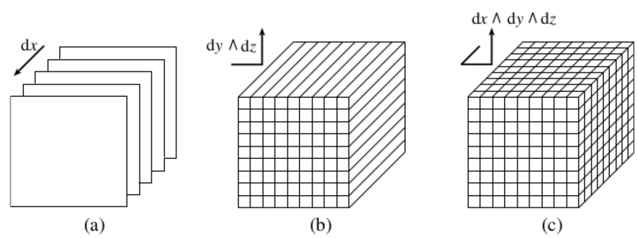


Figure 2: a)One-form b) two-form c) three form in Cartesian coordinates

3 Current density as two form

The flow of current in a conductor is defined via the current density vector field $J(\boldsymbol{x}) = [J_x(\boldsymbol{x}), J_y(\boldsymbol{x}), J_z(\boldsymbol{x})]^T$. The two-form is visualized as a bundle of tubes carrying the current . Each tube carries a unit amount current ,and the current density is inversely proportional to the cross-section area of the tubes .

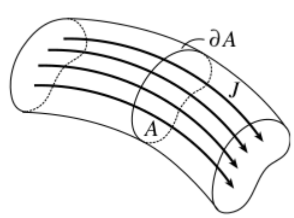


Figure 3: Current through the tube.Pictorial representation of two form J

The total current flowing through surface with area S is,

$$i = \int_S J_x dy \wedge + J_z dz \wedge dx + J_z dx \wedge dy \tag{12}$$

here we introduce the current density two-form $\mathcal{J}$

$$\mathcal{J} = J_x dy \wedge dz + J_z dz \wedge dx + J_z dx \wedge dy \tag{13}$$

the total current i flowing through S may be expressed in a compact notation as the integral of the differential form $\mathcal{J}$

$$i = \int_S \mathcal{J} \tag{14}$$

these differential forms represent the current or the flux through an infinitesimal area element.

4 Charge Density Three-Form

The electric charge q is given by the volume integral of the electric charge density ρ. For the electric charge density we introduce the electric charge density three form

$$\mathcal{Q} = \rho dx \wedge dy \wedge dz \tag{15}$$

graphing we can think of these differential as subdividing the volume into cells. The cell volume is inversely proportional to the charge density . each cell contain unit charge. We can obtain the total charge q by volume integrating the three form $\mathcal{Q}$

$$q = \int_V \mathcal{Q} \tag{16}$$

Quantity	Form	Type	Units	Vector/Scalar
Electric Field Intensity	$\mathcal{E}$	one-form	V	$\mathbf{E}$
Magnetic Field Intensity	$\mathcal{H}$	one-form	A	$\mathbf{H}$
Electric Flux Density	$\mathcal{D}$	two-form	C	$\mathbf{D}$
Magnetic Flux Density	$\mathcal{B}$	two-form	Wb	$\mathbf{B}$
Electric Current Density	$\mathcal{J}$	two-form	A	$\mathbf{J}$
Electric Charge Density	$\mathcal{Q}$	three-form	C	ρ

Table 1: Different Differential forms used in electromagnetism

$$\begin{aligned}
\mathcal{E} &= E_x\, dx + E_y\, dy + E_z\, dz \\
\mathcal{H} &= H_x\, dx + H_y\, dy + H_z\, dz \\
\mathcal{D} &= D_x\, dy \wedge dz + D_y\, dz \wedge dx + D_z\, dx \wedge dy \\
\mathcal{B} &= B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy \\
\mathcal{J} &= J_x\, dy \wedge dz + J_y\, dz \wedge dx + J_z\, dx \wedge dy \\
\mathcal{S} &= S_x\, dy \wedge dz + S_y\, dz \wedge dx + S_z\, dx \wedge dy \\
\mathcal{Q} &= \rho\, dx \wedge dy \wedge dz
\end{aligned} \tag{17}$$

5 Maxwell's Equations in Integral Form

The integral form of Maxwell's equation is given by

$$\oint_{\partial S} \mathcal{H} = \frac{d}{dt}\int_S \mathcal{D} + \int_S \mathcal{J}, \qquad \text{Ampere's Law} \tag{18}$$

$$\oint_{\partial S} \mathcal{E} = -\frac{d}{dt}\int_S \mathcal{B}, \qquad \text{Faraday's Law} \tag{19}$$

$$\oint_{\partial V} \mathcal{B} = 0, \qquad \text{No magnetic Dipole} \tag{20}$$

$$\oint_{\partial V} \mathcal{D} = \int_V Q. \qquad \text{Gauss's Law} \tag{21}$$

Here, ∂S is the boundary enclosing the surface S , and ∂V is the boundary enclosing the volume V.

Ampere's law relates the current(both conduction and displacement) to the magnetic field. Faraday's law tells a changing magnetic field produces electric field. Gauss law relates the electric flux through a closed volume to the charge enclosed by the volume .

6 Exterior derivative

The Exterior derivative operator (d) acts on a p-form and gives new (p+1) form.The operator can be written as

$$\mathbf{d} = \left(\frac{\partial}{\partial x}\,\mathrm{d}x + \frac{\partial}{\partial y}\,\mathrm{d}y + \frac{\partial}{\partial z}\,\mathrm{d}z\right)\wedge \tag{22}$$

when this operator act on a differential form the derivative act on the coefficients of the form and the differential combine with those of the form according to the properties of the exterior product. The sum and product rules for exterior differentiation are:

$$\mathrm{d}(\mathcal{U} + \mathcal{V}) = \mathrm{d}\mathcal{U} + \mathrm{d}\mathcal{V}, \tag{23}$$

$$\mathrm{d}(\mathcal{U} \wedge \mathcal{V}) = \mathrm{d}\mathcal{U} \wedge \mathcal{V} + (-1)^{\deg(\mathcal{U})}\mathcal{U} \wedge \mathrm{d}\mathcal{V}. \tag{24}$$

The exterior derivatives of p-form are

$$\begin{aligned}
\text{zero-form:}\quad & \mathrm{d}f(\mathbf{x}) = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z,\\
\text{one-form:}\quad & \mathrm{d}(\mathbf{u}(\mathbf{x})) = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathrm{d}z\wedge\mathrm{d}x + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y,\\
\text{two-form:}\quad & \mathrm{d}(\mathbf{v}(\mathbf{x})) = \left(\frac{\partial Q_z}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_x}{\partial z}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,\\
\text{three-form:}\quad & \mathrm{d}(\mathbf{Q}(\mathbf{x})) = 0.
\end{aligned} \tag{25}$$

7 Poincare's Lemma

A form $\mathcal{V}$ for which $d\mathcal{V} = 0$ is said to be closed , and a form $\mathcal{V}$ for which $\mathcal{V} = d\mathcal{U}$ is said to be exact . All exact forms are closed . For differential forms $\mathcal{V} = d\mathcal{U}$ implies $d\mathcal{V} = 0$ as $dd\mathcal{U} = 0$. It can be shown that all closed forms are exact . Poincare's lemma states that

$$d\mathcal{V} = 0 \leftrightarrow \mathcal{V} = d\mathcal{U}. \quad \text{for some } \mathcal{U} \tag{26}$$

If a differential form has zero exterior derivative , then it is itself the exterior derivative of another form

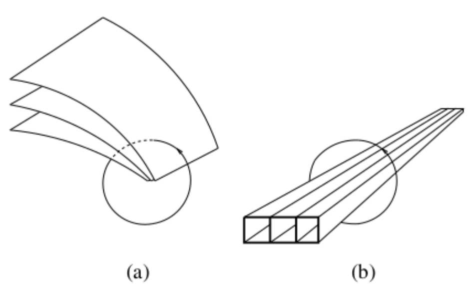
8 Stokes' Theorem

In equations 18 and 19 the integration on the boundary is related to the surface integral.The Strokes' theorem relates the integration of a p-form $\mathcal{U}$ over the closed p-dimension boundary $\partial\mathcal{V}$ of a (p+1)-dimensional volume V to the volume integral $d\mathcal{U}$ over V via

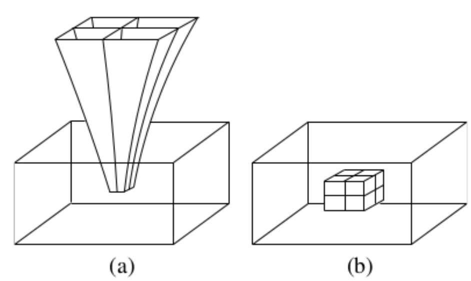
$$\oint_{\partial V} \mathcal{U} = \int_V d\mathcal{U} \tag{27}$$

If $\mathcal{U}$ is zero-form then Stokes' theorem states that $\int_a^b d\mathbf{U} = \mathbf{U}(b) - \mathbf{U}(a)$.This is the fundamental theorem in calculus. If $\mathbf{U}$ is a one form,then ∂V is a closed path and V is the surface enclosed by the closed path . Graphically , Stokes' theorem says that the number of surfaces of $\mathcal{U}$ pierced by the path is equal to he number of tubes of the two-form $d\mathbf{U}$ that passes through the surface.

If $\mathbf{U}$ is a two form ,then ∂V is closed surface and V is volume enclosed by the closed surface.Stokes' theorem requires that number of tubes of $\mathbf{U}$ that cross the surface is equal to the number of boxes of $d\mathbf{U}$ inside the volume.



(a) Stokes theorem for $\mathcal{U}$ a one form. a) the loop ∂V pierces three of the surfaces of $\mathcal{U}$ b) Three tubes of $d\mathcal{U}$ pass through a surfaces V bounded by the loop ∂V



(b) (i)The one-form $\mathcal{H}_1 - \mathcal{H}_2$.(ii) The one-form $\mathcal{J}_s$, represented by lines on the boundary.Current flowing along the lines

9 Maxwell's equations in local form

Applying Stokes's theorem to the integral form of Maxwell's equation 18 to 21 we obtain the differential representation of Maxwell's equations:

$$d\mathcal{H} = \frac{\partial}{\partial t}\mathcal{D} + \mathcal{J}, \quad \text{Ampere's Law} \tag{28}$$

$$d\mathcal{E} = \frac{\partial}{\partial t}\mathcal{B}, \quad \text{Faraday's Law} \tag{29}$$

$$d\mathcal{B} = 0, \quad \text{No magnetic monopoles} \tag{30}$$

$$d\mathcal{D} = \mathcal{Q}, \quad \text{Gauss's Law} \tag{31}$$

10 Boundary Conditions

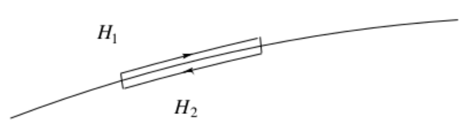
10.1 Boundary Condition on field intensity

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be magnetic field on two side of the surface as showing in the figure . If we choose an Amperian contour with one side just above and the other just below the boundary the left hand side of the Ampere's law becomes

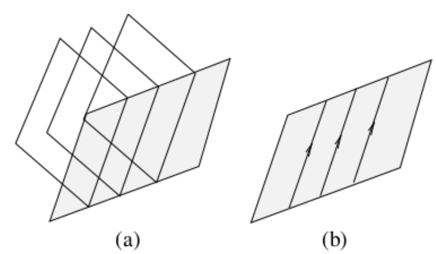
$$\oint_C \mathcal{H} = \int_P (\mathcal{H}_1 - \mathcal{H}_2) \tag{32}$$

In the limit when the width of the contour goes to zero and the sides of the contour meet each other along a common path P on the boundary.The right hand side is equal to the surface integral of current flowing across the path P. So the ampere equation becomes

$$\int_P (\mathcal{H}_1 - \mathcal{H}_2) = \int_P \mathcal{J}_s \tag{33}$$



(a) A discontinuity in the magnetic field above and below a boundary.The surface current flowing on the boundary can be found using an Amperian contour with infinitesimal width



(b) (i)The one-form $\mathcal{H}_1 - \mathcal{H}_2$.(ii) The one-form $\mathcal{J}_s$, represented by lines on the boundary.Current flowing along the lines

Now equation 33 holds for any arbitrary path P , so the integrand must be same .So we have ,

$$\mathcal{J}_s = (\mathcal{H}_1 - \mathcal{H}_2) \tag{34}$$

The one-form $\mathcal{J}_s$ is represented graphically by the lines along which the one-form $\mathcal{H}_1 - \mathcal{H}_2$ intersects the boundary .If the surfaces of $\mathcal{H}_1 - \mathcal{H}_2$ are parallel to the boundary, then the surfaces do not intersect, and the restriction is zero.Thus, 34 gives restriction on the tangential component of magnetic field intensity discontinuity.

In a similar manner, we can show that the electric field satisfies the boundary condition

$$\mathcal{E}_1 - \mathcal{E}_2 = 0 \tag{35}$$

This condition says that the tangential component of the electric field below and above the surface are the same ie their is a continuity across the boundary .

10.2 Boundary condition on Flux Density

Let the flux density above and below the boundary be $\mathcal{D}_1$ and $\mathcal{D}_2$. The left side of the Gauss's law becomes

$$\oint_{\partial V} D = \int_{\partial A} (\mathcal{D}_1 - \mathcal{D}_2) \tag{36}$$

in the limit when the height goes to 0 . A is the common area of the Gaussian pill box on the boundary when the height tends to 0.Gauss's law can now be written as

$$\int_{\partial A} (\mathcal{D}_1 - \mathcal{D}_2) = \int \mathcal{Q}_s \tag{37}$$

the above equation is true for any arbitrary surface so the integrand must me same.so the electric flux density satisfies the following boundary condition

$$(\mathcal{D}_1 - \mathcal{D}_2) = \mathcal{Q}_s \tag{38}$$

here $\mathcal{Q}_s$ is a two form representing the density of electric surface charge on the boundary.This two-form is represented graphically as boxes which are intersection of the tubes $\mathcal{D}_1 - \mathcal{D}_2$ with the boundary. If the tubes of the magnetic flux are parallel to the boundary, then the tubes do not intersect and the restriction is 0.The $\mathcal{D}_1 - \mathcal{D}_2$ two form tube in left side of the equation is normal component to the boundary as the parallel component does not have restriction.

The magnetic flux density satisfies the boundary condition

$$(\mathcal{B}_1 - \mathcal{B}_2) = 0 \tag{39}$$

this means the normal component of the magnetic flux above and below a boundary must be equal.

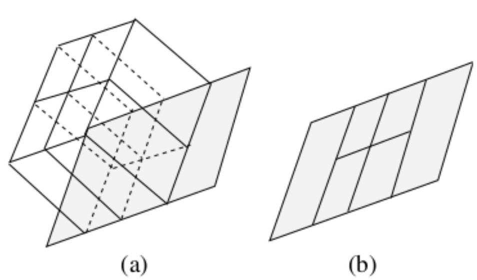


Figure 6: a) The two-form $\mathcal{D}_1 - \mathcal{D}_2$.b)The two-form $\mathcal{Q}_s$ represented by boxes on the boundary

Here we give you all the boundary condition that we have got so far

$$\begin{aligned}(\mathcal{E}_1 - \mathcal{E}_2)\big|_B &= 0 \\ (\mathcal{H}_1 - \mathcal{H}_2)\big|_B &= \mathcal{J}_s \\ (\mathcal{D}_1 - \mathcal{D}_2)\big|_B &= \mathcal{Q}_s \\ (\mathcal{B}_1 - \mathcal{B}_2)\big|_B &= 0\end{aligned} \tag{40}$$

11 Electromagnetic Potential

Introduction of scalar and vector potential in electromagnetism helps avoid working with 12 coupled partial differential maxwells equation given by equation 28 to 31.Once we solve the wave equation for potential we can derive all other quantities.

Due to equation 30 using Poincare's theorem we can write $\mathcal{B}$ as the external derivative of an one-form $\mathcal{A}$:

$$\mathcal{B} = d\mathcal{A} \tag{41}$$

$\mathcal{A}$ is called the magnetic vector potential. Inserting this equation in equation 29 we get

$$d(\mathcal{E} + \frac{\partial}{\partial t}\mathcal{A}) = 0 \tag{42}$$

Again using Poincare's lemma we write

$$\mathcal{E} = -d\Phi - \frac{\partial}{\partial t}\mathcal{A} \tag{43}$$

where Φ is a scalar potential.

Here $\mathcal{A}$ and Φ is not unique. We can change them in the following way and still it give us the same value of $\mathcal{E}, \mathcal{B}$.

$$\mathcal{A}_1 = \mathcal{A} + d\Psi, \tag{44}$$

$$\Phi_1 = \Phi + \frac{\partial}{\partial t}\Psi, \tag{45}$$

Both $(\mathcal{A}, \Phi)$ and $(\mathcal{A}_1, \Phi_1)$ represent the same physical state.This transformation is called a gauge transformation

12 Conclusion

After going through this article it is evident that the use of differential form in electromagnetism is advantageous in making the math relatively easy and shining light into the physics.Differential form provide a way to visualize fields and sources as potential surfaces and tubes of flux that is not available in vector notation.This makes the subject elegant and easy to understand.This also open a lot opportunities for the use of differential form in other area of physics like quantum mechanics , fluid dynamics and many more.

References

[1] Karl Warnick and Peter Russer. Differential forms and electromagnetic field theory. *Progress in Electromagnetics Research*, 148:83–112, 01 2014.