Bifurcation Analysis in Binary Neutron Star Inspirals: A Non-Linear Dynamics Approach

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Abstract

Binary neutron star systems are laboratories for studying gravitational fields. As these compact objects orbit and inspiral due to the emission of gravitational radiation, their evolution can exhibit sensitive dependence on initial conditions and system parameters. In this report, we try to investigate the dynamical behavior of a toy model Binary Neutron Star (BNS) system through the eyes of bifurcation theory, identifying critical symmetric mass ratio, and understand its orbital stability. Particular attention is given to establish a toy model to replicate some aspects of Post Newtonian approximations. Using numerical simulations, we try to observe how varying symmetric mass ratio (η) of the binary system can cause distinct dynamical changes, including stable orbits and chaotic transitions. Our findings offer deeper insights into the pre-merger inspiral phase of BNS system and contribute to improving waveform models used in GW detection and parameter estimation. Our bifurcation analysis also has applications for multi-messenger astronomy, it can help correlate gravitational and electromagnetic signals from events like GW170817.

I INTRODUCTION

When massive stars, several times the size of the Sun, exhaust their nuclear fuel, they collapse under their own gravity, triggering a supernova explosion that can be seen across the universe. Often, this explosion leaves behind a neutron star which is a dense stellar core with a mass greater than the Sun but compressed into a radius of less than 10 miles. They are the densest objects known apart from black holes. They can rotate at extraordinary speeds, completing a full spin in a fraction of a second. When they possess strong magnetic fields, they emit beams of radio waves, they appear as pulsars which are massive cosmic torches sweeping through the abyss of dark space.

In a binary star system, both stars can undergo supernovae which results in a pair of neutron stars orbiting each other. Many such binary systems have been discovered, beginning with the discovery of PSR 1913+16 in 1974 [**1**], a pulsar system whose study earned Hulse and Taylor the Nobel Prize. These binary neutron stars not only emit electromagnetic radiation but also gravitational waves which are detected by observatories like LIGO, VIRGO, etc. The first evidence of Binary Neutron Star merger was when Advanced LIGO and Advanced VIRGO made their first observation of a binary neutron star inspiral (GW170817)[**4**].

Bifurcation analysis is a mathematical framework used to study how small changes in system parameters can lead to qualitative shifts in behavior. Our work focuses on the inspiral phase of a binary neutron star system where we investigate the bifurcation points of the system using Post Newtonian equations and try to understand the orbital stability and see if small changes in symmetric mass ratio can lead to orbital instability or chaotic behavior of our binary system. Bifurcation theory will help us understand how the evolution of the binary system can diverge based on initial conditions, external influences, or nonlinear interactions. By incorporating bifurcation analysis into Binary Neutron Star(BNS) studies, we gain deeper insights into how small perturbations influence the final fate of these systems, which is essential for improving models of neutron star mergers and their astrophysical consequences.

II BASIC ASSUMPTIONS AND APPROXIMATIONS

Throughout this paper we will be assuming newtonian gravity as the base and adding 2.5PN toy model terms for relativistic corrections as it will simplify equations for us. We assume that the two neutron stars are co-rotating in a circular orbit and non-magnetized. From previously observed BNS systems like (GW170817) [**4**], the path was circularized due to GW emission long before merger. We treat orbital decay only driven by gravitational radiation, ignoring dynamical instabilities until the final merger phase. We focus on the inspiral dynamics up to the Innermost Stable Circular Orbit (ISCO), excluding merger or post-merger physics as merger requires full numerical relativity, which is beyond the scope of this study.

III Basic Equations and Methods

III.1 Post Newtonian Equations of motion

We will be taking 2.5PN corrections for our acceleration. Decomposing the acceleration of the system into $a_{conservative}$ (1PN+2PN) and $a_{dissipative}$ (2.5PN) parts. [**5**]

Conservative terms:

$$a_{conservative} = -\frac{GM}{r^2}[1 + \frac{A_1}{r} + \frac{A_2}{r^2}] \tag{1}$$

$\frac{A_1}{r}$ and $\frac{A_2}{r^2}$ are our 1PN and 2PN corrections respectively
Dissipative terms:

$$a_{dissipative} = \frac{B_0}{r^5}\dot{r} \tag{2}$$

Where,
$A_1 = \frac{3-\eta}{2}$ (1PN coefficient)
$A_2 = \frac{9-4\eta}{8}$ (2PN coefficient)
$B_0 = \frac{32}{5}\eta$
$\eta = \frac{q}{(1+q)^2}$ (Symmetric mass ratio), $q = \frac{M_1}{M_2}$ (Mass ratio of two neutron stars)

Therefore our system can be written as:

$$\frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ -\frac{1}{r^2}\left(1 + \frac{A_1}{r} + \frac{A_2}{r^2}\right) + \frac{B_0}{r^5}\dot{r} \end{bmatrix} \tag{3}$$

More accurate equations can be found in the section II of Lawrence E. Kidder [**6**]

III.2 Derivation of critical symmetric mass ratio

From Blanchet[5] we derive the critical value η (where $\eta = \frac{q}{(1+q)^2}$ is the symmetric mass ratio and $q = m_1/m_2$) using the effective potential for circular orbits in Post-Newtonian (PN) theory.

For a binary system Stability Condition of circular orbits is determined by the effective potential $V_{\text{eff}}(r)$:

$$\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_{\text{circ}}} \geq 0 \quad \text{(Stability condition).} \tag{4}$$

We get the first Post-Newtonian order (1PN Effective Potential) from Fock(1939)[**3**]:

$$V_{\text{eff}}(r) = -\frac{GM\mu}{r}\left[1 + \underbrace{\left(\frac{3-\eta}{2}\right)\frac{GM}{rc^2}}_{\text{1PN correction}} + \mathcal{O}(c^{-4})\right], \tag{5}$$

where $\mu = \frac{m_1 m_2}{M}$ is the reduced mass and $M = m_1 + m_2$.

The Innermost Stable Circular Orbit (ISCO) occurs when the potential's curvature vanishes:

$$\frac{d^2 V_{\text{eff}}}{dr^2} = 0. \tag{6}$$

Solving this yields the ISCO radius:

$$r_{\text{ISCO}} = \frac{6GM}{c^2}\left(1 + \sqrt{1-6\eta}\right)^{-1}. \tag{7}$$

Now, to find the critical symmetric mass ratio, we need to see when $\sqrt{1-6\eta}$ becomes zero:

$$\eta = \frac{1}{6}. \tag{8}$$

Thus we see that the circular orbit it stable for $\eta \leq 1/6$ and loses stability for $\eta \geq 1/6$ ($q \gtrsim 0.8$). Therefore we get:

- For $\eta < 1/6$: Stable circular orbits exist down to r_{ISCO}.
- For $\eta \geq 1/6$: No stable circular orbits; binaries plunge directly.

III.3 Numerical Analysis and Bifurcation Diagram

We use python modules such as numpy, scipy, odeint and matplotlib to solve the differential equations, draw the phase plot and bifurcation diagram.

IV Observations and Result

BNS Inspiral: Phase Plot (2.5PN Dynamics)

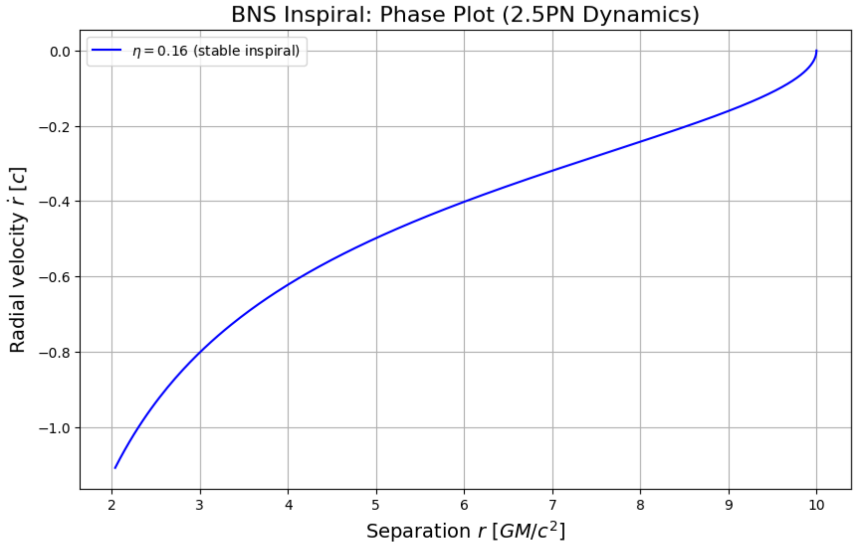

Figure 1: Phase plot of model BNS system dynamics for critial symmetric mass ratio $\eta = \dfrac{1}{6} \approx 0.16$

Bifurcation Diagram

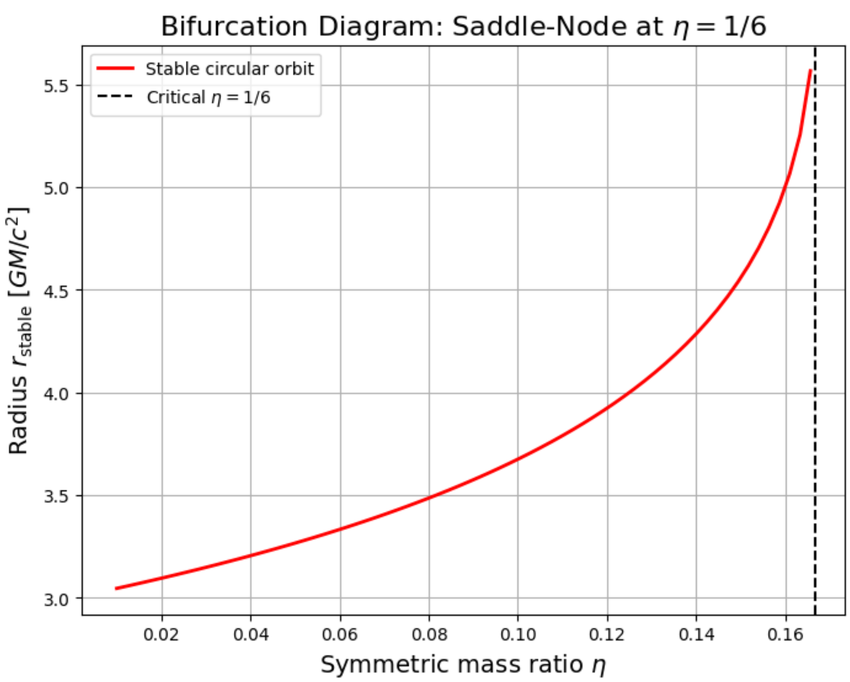

Figure 2: Bifurcation plot of model BNS system dynamics for $\eta = \dfrac{1}{6}$ showing saddle node bifurcation.

The model (PN) equations of motion are solved using the following initial conditions:

$$r_0 = 10\,,$$
$$\dot{r}_0 = 0\,.$$

These correspond to a quasicircular binary orbit at an initial separation of $r_0 = 10\,GM/c^2$, radial velocity $\dot{r}_0$ is initially zero. This models a binary neutron star system in a stable, circular orbit in early inspiral phase.

The Phase Plot curve in [Figure 1] shows how fast the neutron stars are approaching each other as the separation between them decreases. Our BNS system is inspiraling and the two neutron stars are slowly coalescing due to emission of gravitational radiation. The inspiral is very slow at large separations, as the gravitational radiation emission is weak. But as they get closer, the inspiral accelerates due to stronger emission of gravitational radiation which supports the theory and literature.

The vertical dotted line in [Figure 2] denotes a bifurcation point, a saddle node bifurcation. Our bifurcation diagram suggests that for small η(very unequal masses), our system supports more compact stable orbits. But as the mass ratio becomes more symmetric ($\eta \to \frac{1}{6}$) the stable orbit radius rapidly grows and the system becomes more sensitive to tidal deformability and relativistic corrections. At $\eta = \frac{1}{6}$ critical transition occurs, the circular orbit loses its stability. This might hints the inspiral instability or rapid merger of the system, which is the territory of Numerical Relativity.

V Applications of the study

Bifurcation analysis is an important aspect in the field of Gravitational Waves or Relativistic Astronomy. To study inspiralling binary neutron stars, binary black holes or neutron star - black hole merger, one has to appreciate the nonlinearity of the system. As it will help us study the effects of gravitation on these astrophysical bodies. Analyzing the bifurcation points of the system will help us model more accurate GW waveform templates and parameter estimation, it will help us study the effect of different initial conditions to better understand the system.

VI Conclusion

Throughout our study we explored the dynamical behavior of binary neutron star systems through the eyes of post newtonian dynamics and bifurcation theory by studying the stable circular orbits as a function of the symmetric mass ratio (η). After our numerical analysis we identified a saddle-node bifurcation occurring at the critical value $\eta = \frac{1}{6}$, beyond which the inspiral is no longer stable. This bifurcation point acts as a boundary between stable and unstable orbits, offering deeper understandings into the final stages of compact binary coalescence. The phase plot at $\eta = \frac{1}{6}$, confirms the presence of an inspiral trajectory, consistent with 2.5PN radiation reaction effects. Reflects the growing influence of GW emission as the system approaches merger. The initial conditions we choose are pretty good to study the transition from a stable orbit to instability, capturing the effects of both conservative and dissipative Post-Newtonian forces. At large separations, the GW emission (2.5PN term) is relatively weak, thus assuming circular orbit is a good approximation. Gradually the system starts to inspiral due to the GW emission, which leads to a decrease in orbital separation and a negative radial velocity over time. Overall, our analysis demonstrates how bifurcation analysis, when combined with relativistic post newtonian corrections, can provide valuable predictions about the orbital stability and merger thresholds of compact binary systems.

For future directions a more literature accurate BNS system can be modelled with PN correction terms and further simulate the system to understand the relationship between orbital stability & symmetric mass ratio and study the emission of Gravitational Radiation from it. We can also introduce the Tolman–Oppenheimer–Volkoff equation and vary parameters to study change in the tidal deformability of neutron stars.

Credits and References

[1] https://ui.adsabs.harvard.edu/abs/2015CQGra..32l4009D/abstract (Hulse & Taylor "PSR 1913+16")

[2] Higher order gravitational radiation losses in binary systems (Blanchet & Schäfer) : Mon. Not. R. astr. Soc. (1989) 239, 845-867

[3] Fock, V. A., "On motion of finite masses in general relativity", J. Phys. (Moscow), 1(2), 81–116 (1939)

[4] GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral (10.1103/PhysRevLett.119.161101)

[5] Blanchet, L. Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries. Living Rev. Relativ. 17, 2 (2014).

[6] Lawrence E. Kidder. Coalescing binary systems of compact objects to (post)5/2-Newtonian order. V. Spin effects Phys. Rev. D 52, 821

Appendix

The Python codes for the numerical analysis can be found here.