U.S. Treasury Zero-Coupon Curve (Feb 24, 2025)

Objective :
Using U.S. Treasury bond prices, our objective is to *develop a detailed automated algorithm to determine the continuously compounded discount rate for each bond payment date (coupons and principal).* We implement this both *in the form of a function* (which may require some manual experimentation) and *in the form of a graph.* We provide a *less-than-one-page* methodology write-up and show the results *in a table and a graph.*

Assumptions:
Our methodology relies on the following simplifying assumptions:

- **Bootstrapping:** Interpolation of log-discount factors across 797 coupon dates ensures internal consistency, but assumes that smooth discount factors exist between observed maturities.
- **Nelson–Siegel (NS):** Assumes the term structure can be captured by three components (level, slope, curvature), which may underfit complex shapes at longer maturities.
- **Nelson–Siegel–Svensson (NSS):** Extends NS by assuming a second curvature term captures any additional hump or twist in the yield curve, at the cost of higher risk of overfitting.
- **Chebyshev polynomial:** Assumes yields can be approximated by a degree-4 polynomial on the rescaled maturity domain. This ensures smoothness but has no financial interpretation and may extrapolate poorly outside the sample.

Methodology:
We began by bootstrapping the zero-coupon curve directly from U.S. Treasury bond prices. Using the Actual/Actual (ISDA) day-count convention, we expressed each cash flow in year fractions relative to the settlement date. Discount factors were solved sequentially by maturity, with intermediate coupon payments interpolated using log-discount factors. This interpolation was required for a total of 797 coupon dates, meaning that the estimation may not be exact at every point, but the methodology ensures global consistency. The advantage of bootstrapping is that it reflects market data exactly, while its drawback is sensitivity to noisy prices and occasional violations of monotonicity.

To obtain smoother, more interpretable curves, we also fitted three functional models:

- **Nelson–Siegel (NS):** A four-parameter model producing a smooth, parsimonious curve that captures level, slope, and curvature. The functional form is: $y(t) = \beta_0 + \beta_1 \frac{1-e^{-t/\tau_1}}{t/\tau_1} + \beta_2 \left(\frac{1-e^{-t/\tau_1}}{t/\tau_1} - e^{-t/\tau_1}\right)$.
  Its simplicity is an advantage, though flexibility at longer maturities is limited.
- **Nelson–Siegel–Svensson (NSS):** A six-parameter extension of NS that adds a second curvature term, improving the fit to humps or twists in the curve: $y(t) = \beta_0 + \beta_1 \frac{1-e^{-t/\tau_1}}{t/\tau_1} + \beta_2 \left(\frac{1-e^{-t/\tau_1}}{t/\tau_1} - e^{-t/\tau_1}\right) + \beta_3 \left(\frac{1-e^{-t/\tau_2}}{t/\tau_2} - e^{-t/\tau_2}\right)$.
  This flexibility can better match data but increases the risk of overfitting.
- **Chebyshev polynomial (degree 4):** A purely numerical approximation fitted on a rescaled maturity domain $x \in [-1, 1]$. The yield is expressed as:$y(t) = c_0T_0(x) + c_1T_1(x) + c_2T_2(x) + c_3T_3(x) + c_4T_4(x)$,
  where $T_k(x)$ are Chebyshev polynomials of the first kind and $x = 2\frac{t}{T_{\max}} - 1$. It provides a smooth and stable curve, but lacks financial interpretation and may behave poorly outside the sample range.

The final figure compares observed yields, the bootstrapped curve, and the three models. Bootstrapping provides the market-implied ground truth but is jagged, while the models deliver smooth approximations. Overall, NS and NSS strike a balance between interpretability and flexibility, whereas Chebyshev serves as a purely mathematical benchmark.

Interpretive Comment and Recommendation:
The fitted curves suggest a modest upward slope beyond short maturities, with yields stabilizing near 4.7–4.8% at the long end. A slight dip in the 2–5 year sector, visible in both the bootstrapped zeros and the NSS fit, likely reflects local demand or liquidity effects. The NSS model captures this feature well, thanks to its additional curvature term, while the simpler NS model smooths over it. The Chebyshev polynomial reproduces the shape numerically but lacks financial meaning.

For practical purposes, the NSS specification offers the best balance between flexibility and stability. NS remains useful as a parsimonious benchmark, and the polynomial is mainly a mathematical comparison. A pragmatic approach is to report both NSS and the bootstrap points: the first provides a smooth curve for valuation, the second serves as a direct market check.

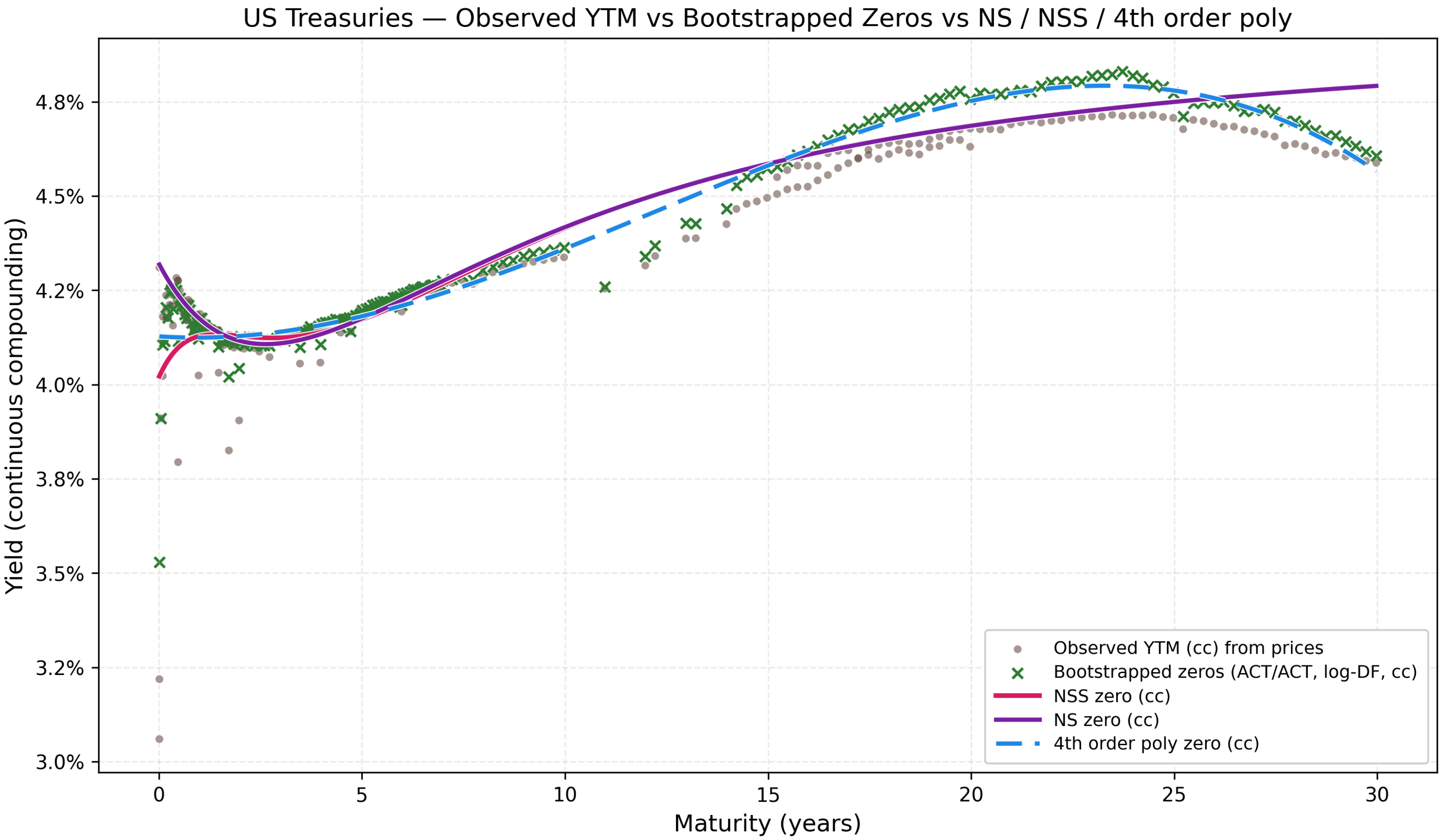

Table 1: Zero Rates by Maturity: Bootstrapped vs. Model Estimates

Maturity (yrs)	Bootstrapped Zero (cc)	NS Zero (cc)	NSS Zero (cc)	Poly-4 Zero (cc)
1	0.041593	0.041745	0.041286	0.041246
2	0.040901	0.041157	0.041287	0.041289
3	0.041232	0.041096	0.041242	0.041402
4	0.041443	0.041342	0.041392	0.041579
5	0.041944	0.041752	0.041726	0.041813
6	0.042316	0.042240	0.042176	0.042098
7	0.042753	0.042752	0.042680	0.042426
8	0.043051	0.043255	0.043193	0.042792
9	0.043416	0.043733	0.043687	0.043188
10	0.043604	0.044177	0.044148	0.043608
11	0.042609	0.044584	0.044569	0.044044
12	0.043431	0.044953	0.044951	0.044489
13	0.044285	0.045286	0.045293	0.044937
14	0.044734	0.045587	0.045599	0.045380
15	0.045735	0.045858	0.045874	0.045811
16	0.046212	0.046102	0.046119	0.046222
17	0.046766	0.046322	0.046339	0.046607
18	0.047234	0.046521	0.046538	0.046957
19	0.047557	0.046701	0.046716	0.047266
20	0.047589	0.046865	0.046878	0.047524
21	0.047750	0.047015	0.047026	0.047726
22	0.048025	0.047152	0.047160	0.047862
23	0.048180	0.047277	0.047283	0.047925
24	0.048184	0.047392	0.047395	0.047907
25	0.047676	0.047498	0.047499	0.047800
26	0.047477	0.047597	0.047595	0.047595
27	0.047284	0.047688	0.047684	0.047285
28	0.046983	0.047772	0.047767	0.046862
29	0.046591	0.047851	0.047844	0.046316

Table 2: Estimated Model Coefficients

Model	**Parameter**	**Value**
Nelson–Siegel	β_0	0.050063
	β_1	-0.006859
	β_2	-0.017615
	τ_1	2.621500
Nelson–Siegel–Svensson	β_0	0.049998
	β_1	-0.009801
	β_2	0.008867
	β_3	-0.027199
	τ_1	1.106362
	τ_2	2.259207
Chebyshev–4	c_0	0.044658
	c_1	0.003226
	c_2	-0.001169
	c_3	-0.001036
	c_4	-0.000020