A family of accessibility measures derived from spatial interaction principles

Introduction

Historically, the focus of modern transportation planning has been to prioritize mobility while treating access to destinations as a by-product of movement. This has had problematic consequences: with the car seen as the ultimate mobility tool, this approach has led to the emergence and continued dominance of an automobility mono-culture (H. J. Miller 2011; Lavery, Páez, and Kanaroglou 2013). Decades of planning for this mono-culture (often characterized by road and highway expansion) have been marked by increased travel cost and environmental burdens, but often having only limited impact on the ease with which people can reach destinations (Steven Farber and Páez 2011; S. Handy 2002; Páez et al. 2010). In response to this situation, transportation researchers have increasingly advocated for the adoption of accessibility as a planning criterion (Silva et al. 2017; Paez et al. 2013; S. Handy 2020) using both positive and normative approaches (Paez, Scott, and Morency 2012; Levine 2020). Accessibility, a central concept in transport geography, planning, and engineering, is conceptually related to human mobility and the opportunity landscape. In simple terms, it is defined as the "potential of opportunities for interaction" (Hansen 1959). Compared to other measures of performance used in transportation that benchmark movement (e.g., VKT, PKT, etc.), accessibility brings a more holistic understanding of transportation and land use systems combined (S. L. Handy and Niemeier 1997).

An ascendant interest in accessibility has been accompanied by a remarkable boom on scholarly research, which has grown to include employment (e.g., Karst and Van Eck 2003; Grengs 2010; Páez et al. 2013; Merlin and Hu 2017; Tao et al. 2020), health care (e.g., Luo and Wang 2003; Páez et al. 2010; Wan, Zou, and Sternberg 2012; Delamater 2013; Boisjoly, Moreno-Monroy, and El-Geneidy 2017; Pereira et al. 2021), green spaces (Reyes, Paez, and Morency 2014; Rojas et al. 2016; Liang, Yan, and Yan 2024), schools (e.g., Williams and Wang 2014; Romanillos and Garcia-Palomares 2018; Marques, Wolf, and Feitosa 2021), social contacts (e.g., Neutens et al. 2007; S. Farber, Páez, and Morency 2012; S. Farber et al. 2013), and regional economic analysis (e.g., Vickerman, Spiekermann, and Wegener 1999; Lopez, Gutierrez, and Gomez 2008; Ribeiro, Antunes, and Páez 2010; Gutierrez et al. 2011) among many other domains of application. In other words, accessibility analysis is used today to broadly understand the potential to reach opportunities that are important to people (Ferreira and Papa 2020). However, despite its growth in popularity in scholarly works, challenges remain to more widespread adoption of accessibility in planning practice. Several barriers to bridging the scholarly-practice accessibility gap have been identified. For example, the diversity of accessibility definitions has been flagged by van Wee (2016), S. Handy (2020), and Kapatsila et al. (2023). Further, difficulties in the interpretability and communicability of outputs has also been noticed by Geurs and van Wee (2004), van Wee (2016), and Ferreira and Papa (2020).

Adoption of accessibility in planning practice is not necessarily made easier when potential adopters have to contend with a plethora of definitions, each seemingly more sophisticated but less intuitive than the last (Kapatsila et al. 2023). The menu of accessibility measures has grown to include gravity-based accessibility (e.g., Hansen 1959; Pirie 1979), cumulative opportunities (e.g., Wachs and Kumagai 1973; Pirie 1979; Ye et al. 2018), modified gravity (e.g., Schuurman, Berube, and Crooks 2010), 2-Step Floating Catchment Areas (e.g., Luo and Wang 2003), Enhanced 2-Step Floating Catchment Areas (e.g., Luo and Qi 2009), 3-Stage Floating Catchment Areas (e.g., Wan, Zou, and Sternberg 2012), Modified 2-Step Floating Catchment Areas (e.g., Delamater 2013), inverse 2-Step Floating Catchment Areas (e.g., F. Wang 2021), and n-steps Floating Catchment Areas (Liang, Yan, and Yan 2024). How is a practitioner to choose among this myriad options? What differences in accessibility scores should matter, and how should they be communicated? [see van Wee (2016); p. 14].

In this respect, Wu and Levinson (2020)'s contribution provides a unifying framework to think about accessibility measures, the most general of which they describe as S_i in Equation 1. By considering the key features of accessibility, namely travel-cost $f(cij)$ and the distribution of opportunities $g(O_i)$, Wu and Levinson (2020) demonstrate that a majority of the concepts found scattered throughout the accessibility literature can be seen as particular cases of a general accessibility formula (Equation 1). One needs only to judiciously change the way travel-cost and opportunities are formulated, to derive almost any known accessibility measure. In this way, Wu and Levinson (2020) have taken a considerable step towards clarifying the differences between various accessibility indicators.

$$S_i \propto \sum_j g(O_j) f(c_{ij}) \tag{1}$$

However, there is room to further clarify another relevant aspect: competition. Wu and Levinson (2020)'s unifying framework does not appear to accommodate many (or any) of the floating catchment area methods, a popular approach within the competitive accessibility literature. There have also been more recent developments in the literature: for instance, Soukhov et al. (2023) demonstrate that the introduction of a single constraint is sufficient to turn the general accessibility measure (i.e, Equation 1) into a competitive measure of accessibility in accordance with the 2-Step Floating Catchment Area approach. Moreover, the proportional allocation balancing factors in Soukhov et al. (2023) apply a singly-constraint to the accessibility measure, akin to the constraints introduced within the spatial interaction modelling framework outlined in Wilson (1971).

Motivated by the challenge of incorporating competition into a unified framework for accessibility, this paper contends that accessibility research must reconnect with its spatial interaction origins. Particularly, we argue that an important aspect of spatial interaction modelling–namely, constraining the results to match empirical observations–was never effectively reincorporated into accessibility analysis. Empirical constraints were embraced by early spatial interaction literature following the work of Wilson (1971), but this stream of literature tended to flow separately from research inspired by Hansen (1959)' accessibility. The application of Wilson (1971)'s empirical constraints supported the development of various spatial interaction models that remain relevant in research and practice today (Ortúzar and Willumsen 2011). However, the same cannot be said of the contemporary accessibility literature, where empirical constraints were not explicitly adopted. We argue that the absence of empirical constraints (and their attendant proportionality constants) has contributed to some of accessibility analysis' interpretability issues; for instance, the fuzziness of insights beyond simple proportional statements like 'higher-than' or 'lower-than' (E. J. Miller 2018).

These streams of literature share common headwaters. It is by looking to the past that we believe accessibility analysis can newly wade into the future. Hence, this work's primary focus is to show that the same empirical constraints used in Wilson (1971)'s family of spatial interaction models can be mapped onto accessibility. We begin by tracing the development of accessibility from its origins in spatial interaction, from the Newtonian gravitational expression in Ravenstein (1889) through to the seminal accessibility work of Hansen (1959). We then present evidence for a narrative highlighting the marked divergence between accessibility and spatial interaction modelling research after the work of Wilson (1971). Next, we hark back to Wilson (1971)'s spatial interaction models, and use it to derive a family of accessibility measures based on different types of constraints. We illustrate various members of this family with a simple numerical example and a real world data set. We then conclude by discussing the uses of these measure and their interpretation.

Newtonian's roots of human spatial interaction research

The patterns of people's movement in space have been a subject of scientific inquiry for at least a century and a half. From as far back as Henry C. Carey's *Principles of Social Science* (Carey 1858), a concern with the scientific study of human spatial interaction can be observed. It was in this work where Carey stated that "man [is] the molecule of society [and their interaction is subject to] the direct ratio of the mass and the inverse one of distance" (McKean 1883, 37–38). This statement shows how investigations into human spatial interaction have often been explicitly coloured by the features of Newton's Law of Universal Gravitation, first posited in 1687's *Principia Mathematica* and expressed as in Equation 2.

$$F_{ij} \propto \frac{M_i M_j}{D_{ij}^2} \tag{2}$$

To be certain, the expression above, a proportionality, is one of the most famous in all of science. In brief, it states that the force of attraction F between a pair of bodies i and j is directly *proportional* to the product of their masses M_i and M_j, and inversely *proportional* to the square of the distance between them D_{ij}. Direct proportionality means that as the product of the masses increases, so does the force. Likewise, inverse proportionality means that as the distance increases, the force decreases. Equation 2, however, does not quantify the magnitude of the force. To do so, an empirical constant is required to convert the proportionality into an equality, ensuring that values of the force F in Equation 2 match the observed force of attraction between masses. In other words, Equation 2 needs to be *constrained* using empirical data. Ultimately, the equation for the force is as seen in Equation 3, where G is an empirically calibrated proportionality constant:

$$F_{ij} = G\frac{M_i M_j}{D_{ij}^2} \tag{3}$$

Newton's initial estimate of G was based on a speculation that the mean density of earth was between five or six times that of water, an assumption that received support after Hutton's experiments of 1778 (Hutton 1778, 783). Still, it took over a century from the publication of *Principia* to refine the estimate of the proportionality constant to within 1% accuracy, with Cavendish's 1798 experiment (Cavendish 1798).

Early research on human spatial interaction: from Ravenstein (1889) to Stewart (1948)

Following Carey's *Principles* of 1858, research into human spatial interaction continued in different contexts. In the late 1880s, Ravenstein proposed some "Laws of Migration" based on his empirical analysis of migration flows in various countries (Ravenstein 1885, 1889). In these works, Ravenstein posited 1) a directly proportional relationship between migration flows and the size of destinations (i.e., centres of commerce and industry), and 2) an inversely proportional relationship between the size of flows and the separation between origins and destinations. As with Carey, these propositions echo Newton's gravitational laws. Over time, other researchers discovered similar relationships. For example, Reilly (1929) formulated a law of retail gravitation, expressed in terms of equal attraction to competing retail destinations. Later, Zipf proposed a $\frac{P_1 P_2}{D}$ hypothesis for the case of information (Zipf 1946a), intercity personal movement (Zipf 1946b), and goods movement by railways (Zipf 1946c). The $\frac{P_1 P_2}{D}$ hypothesis stated that the magnitude of flows was proportional to the product of the populations of settlements, and inversely proportional to the distance between them.

A common feature of these early investigations of human spatial interaction is that a proportionality constant similar to G in Equation 3 was never considered. Of the researchers cited above, only Reilly and Zipf expressed their hypotheses in mathematical terms. Reilly's hypothesis was presented in the following form:

$$B_a = \frac{(P_a \cdot P_T)^N}{D_{aT}^n} \tag{4}$$

Where B_a is the amount of business drawn to a from T, P_a and P_T are the populations of the two settlements, and D_{aT} is the distance between them. Quantity N was chosen by Reilly in a somewhat *ad hoc* fashion as 1, and he used empirical observations of shoppers to choose a value of $n = 2$.

Zipf, on the other hand, wrote his hypothesis in mathematical form as:

$$C^2 = \frac{P_1 \cdot P_2}{D_{12}} \tag{5}$$

where C is the volume of goods exchanged between 1 and 2, P_1 and P_2 are the populations of the two settlements, and D_{12} is the distance between them.

After Carey, it is in Stewart's work on the principles of demographic gravitation that we find the strongest connection yet to Newton's law (Stewart 1948). This may relate to academic backgrounds; where Ravenstein, Reilly, and Zipf were social scientists, Stewart was a physicist. Besides awareness of preceding research (he cites both Reilly and Zipf as predecessors in the analysis of human spatial interaction), Stewart appears to have been the first author to express his theorized relationships for human spatial interaction with a proportionality constant G, as follows:

$$F = G\frac{(\mu_1 N_1)(\mu_2 N_2)}{d_{12}^2} = G\frac{M_1 \cdot M_2}{d_{12}^2} \tag{6}$$

Where:

- F is the *demographic force*
- N_1 and N_2 are the numbers of people of in groups 1 and 2
- μ_1 and μ_2 are so-called *molecular weights*
- $M_1 = \mu_1 N_1$ and $M_2 = \mu_2 N_2$ are the demographic masses at 1 and 2
- d_{12}^2 is the distance between 1 and 2
- And finally G, a constant that Stewart "left for future determination" (1948, 34)

In addition to demographic force, Stewart defined a measure of the "population potential" of 2 with respect to 1 as follows:

$$V_1 = G\frac{M_2}{d_{12}} \tag{7}$$

For a system with more than two population bodies, Stewart formulated the population potential at i as follows (after arbitrarily assuming that $G = 1$):

$$V_i = \int \frac{D}{r} ds \tag{8}$$

where D is the population density over an infinitesimal area ds and r is the distance to i. In Equation 8, $D \cdot ds$ gives an infinitesimal count of the population, say dm, and so, after discretizing space, Equation 8 can be rewritten as:

$$V_i = \sum_j \frac{M_j}{d_{ij}} \tag{9}$$

Alerted readers will notice that Equation 9 is formally equivalent to our modern definition of accessibility.

Stewart's formulation of demographic force, developed in the context of what he called "social physics" (Stewart 1947), was problematic. It had issues with inconsistent mathematical notation. More seriously though, Stewart's work was permeated by a view of humans as particles following physical laws, but tinted by unscientific ideas that were unadultered racism. For instance, he assumed that the molecular weight μ of the average American was one, but "presumably…much less than one….for an Australian aborigine" [p. 35]. Stewart's ideas about "social physics" soon fell out of favour among social scientists, but not before influencing the nascent field of accessibility research, as detailed next.

Hansen's gravity-based accessibility to today

From Stewart (1948), we arrive to 1959 and Walter G. Hansen, whose work proved to be exceptionally influential in the accessibility literature (Hansen 1959). In his seminal paper, Hansen defined accessibility as "the potential of opportunities for interaction… a generalization of the population-over-distance relationship or *population potential* concept developed by Stewart (1948)" (p. 73). As well as being a student of city and regional planning at Massachusetts Institute of Technology, Hansen was also an engineer with the Bureau of Roads, and preoccupied with the power of transportation to shape land uses in a very practical sense. Hansen (1959) focused on Stewart (1948)'s *population potential* (expressed in Equation 9), leaving Stewart's other formulaic contributions and objectionable aspects of "social physics" behind. Hansen (1959) recast Stewart's population potential to reflect accessibility, a model of human behaviour useful to capture regularities in mobility patterns. Hansen (1959) replaced M_j in Equation 9 with *opportunities* to derive an *opportunity potential*, or more accurately, a *potential of opportunities for interaction* as follows:

$$S_i = \sum_j \frac{O_j}{d_{ij}^{\beta}} \tag{10}$$

A contemporary rewriting of Equation 10 accounts for a variety of impedance functions beyond the inverse power $d^{-\beta}$:

$$S_i = \sum_j O_j \cdot f(d_{ij}) \tag{11}$$

S_i in Equation 10 is a measure of the accessibility of site i. This is a function of O_j (the mass of opportunities at j), d_{ij} (the cost, e.g., distance or travel time, incurred to reach j from i), and β (a parameter that modulates the friction of cost). Today, Hansen is frequently cited as the father of modern accessibility analysis (e.g., Reggiani and Martín 2011), and Hansen-type accessibility is commonly referred to as the gravity-based accessibility measure.

Of note, however is that between Stewart (1948) and Hansen (1959) the proportionality constant G in Equation 7 vanished. There is some evidence that Hansen (1959) was aware of the importance of this constant as he wrote about *directly* and *inversely proportional* relationships when discussing population, opportunities, and their separation in space At any rate, those reading Hansen (1959) must recall that Stewart (1948) had set the proportionality constant G to 1, with a note that "G [was] left for future determination: a suitable choice of other units can reduce it to unity".

After Hansen (1959), accessibility analysis has been widely used in numerous disciplines but, to our knowledge, the proportionality constant has remained forgotten, with no notable developments to explicitly determine it. In this way, G continues to be implicitly set to 1, even when the fundamental relationship in accessibility is proportionality (e.g., $S_i \propto \sum_j g(O_j) f(d_{ij})$) and not equality (for instance, see the formula for accessibility at the top of Figure 1 in Wu and Levinson 2020). Alas, without a proportionality constant, the units of S_i remain unclear: the unit of "potential of opportunity for interaction" is left free to change as β is calibrated. For example, if c_{ij} is distance in meters, it will be number of opportunities per m^{β} when $f(c_{ij}) = d^{-\beta}$ but number of opportunities per $e^{-\beta \cdot m}$ when $f(c_{ij}) = e^{\beta \cdot m}$. Hansen-style accessibility, therefore, is better thought of as an ordinal measure of potential that can only be interpreted in terms of higher and lower accessibility (E. J. Miller 2018).

Wilson's family of spatial interaction models

In his groundbreaking work, Wilson (1971) defined a general spatial interaction model as follows:

$$T_{ij} = kW_i^{(1)} W_j^{(2)} f(c_{ij}) \tag{12}$$

The model in Equation 12 posits a quantity T_{ij} that represents a value in a matrix of flows of size $n \times m$, that is, between $i = 1, \cdots, n$ origins and $j = 1, \cdots, m$ destinations. The quantities $W_i^{(1)}$ and $W_j^{(2)}$ are proxies for the masses at $i = 1, \cdots, n$ origins and $j = 1, \cdots, m$ destinations. Finally, $f(c_{ij})$ is the $i - j$ element of an $n \times m$ matrix representing some function of travel cost c_{ij} which reflects travel impedance. In this way, T_{ij} explicitly measures *interaction* in the unit of trips, and the role of the single proportionality constant k is to ensure that the system-wide sum of T_{ij} represents the total flows in the data. In other words, k is a scale parameter that makes the overall amount of flows identical to the magnitude of the phenomenon being modeled.

Traditionally, development of the spatial interaction model put an emphasis on the interpretability of the results (Kirby 1970; Wilson 1967, 1971). But instead of relying on the heuristic of Newtonian gravity (e.g., some interaction between a mass at i and a mass at j separated by some distance), Wilson's approach was to maximise the entropy of the system. Entropy maximisation in this case achieves stable results as a statistical average that represents the population. The approach works by assuming undifferentiated individual interactions, and assessing their probabilities of making a particular journey. The result of Equation 12 then is a statistical average (Wilson 1971; Senior 1979).

To ensure that T_{ij} in Equation 12 is in the unit of trips, additional knowledge about the system is required. At the very least, this framework assumes that the total number of trips in the system T is known, and therefore:

$$\sum_i \sum_j T_{ij} = T \tag{13}$$

Additional information can be introduced. For example, when information is available about the total number of trips produced by each origin (O_i), the following constraint can be used:

$$\sum_j T_{ij} = O_i \tag{14}$$

Alternatively, if there is information available about the total number of trips attracted by each destination (D_j), the following constraint can be used:

$$\sum_i T_{ij} = D_j \tag{15}$$

It is also possible to have information about both O_i and D_j, in which case both constraints could be imposed on the model.

Parting from this, it is possible to derive a family of spatial interaction models. In Wilson's terms, there is the "unconstrained" (but actually total flow constrained) model in Equation 12. The proportionality constant k in this case is (see Cliff, Martin, and Ord 1974; Fotheringham 1984):

$$k = \frac{T}{\sum_i \sum_j T_{ij}} \tag{16}$$

When only one of Equation 14 or Equation 15 holds, the resulting models are, in Wilson's terms, singly-constrained. Entropy maximisation leads to the following production-constrained model:

$$T_{ij} = A_i O_i W_j^{(2)} f(c_{ij}) \tag{17}$$

Notice how, in this model, the proxy for the mass at the origin $W_i^{(1)}$ is replaced by the actual mass, as measured by the trips produced O_i. Also, there is no longer a single system-wide proportionality constant, but rather a set of proportionality constants (i.e., balancing factors) specific to origins. According to Wilson, these constants, namely A_i, are:

$$A_i = \frac{1}{\sum_j W_j^{(2)} f(c_{ij})} \tag{18}$$

The attraction-constrained model, in turn, takes the following form:

$$T_{ij} = B_j D_j W_i^{(1)} f(c_{ij}) \tag{19}$$

Notice how now the proxy for the mass at the destination $W_j^{(2)}$ is replaced by the actual mass, as measured by the trips attracted D_j. As before, destination-specific proportionality constants (i.e., balancing factors) B_j were derived by Wilson as:

$$B_j = \frac{1}{\sum_i W_i^{(1)} f(c_{ij})} \tag{20}$$

When both Equation 14 and Equation 15 hold, the resulting models is, in Wilson's terms, doubly-constrained, and takes the following form:

$$T_{ij} = A_i B_j O_i D_j f(c_{ij}) \tag{21}$$

In this model, both proxies for the masses are replaced with the known masses, that is, the trips produced by origin and the trips attracted by destination. And, now there are two sets of mutually dependent proportionality constants:

$$
\begin{aligned}
A_i &= \frac{1}{\sum_j B_j D_j f(c_{ij})} \\
B_j &= \frac{1}{\sum_i A_i O_i f(c_{ij})}
\end{aligned}
\tag{22}
$$

Derivation of these models is demonstrated in detail elsewhere (e.g., Ortúzar and Willumsen 2011; Wilson 1967). It is worth noting, however, that although Wilson's approach is built on a different conceptual foundation than the old reference to Newtonian gravity, the work succeeded at identifying the steps from proportionality to equality to yield variations of proportionality constants, including the one that eluded Stewart (1948) and that has been ignored in almost all subsequent accessibility research. Why was this key element of spatial interaction models ignored in accessibility research? In the next section we aim to address this question.

Accessibility and spatial interaction modelling: two divergent research streams

The work of Hansen (1959) and Wilson (1971) responded to important developments in the Anglo-American world at the time, in particular a need "to meet the dictates and needs of public policy for strategic land use and transportation planning" (Batty 1994). Said dictates and needs were far from trivial. In the US alone, the Federal-Aid Highway Act of 1956 authorized the creation of the US Interstate Highway System, with a budget of more than one hundred billion dollars (Weiner 2016; MDOT 2007). Spatial interaction modelling, with its ability to quantify trips, was incorporated into institutional modelling practices meant to "predict and provide", i.e., predict travel demand and supply transportation infrastructure (Kovatch, Zames, et al. 1971; Weiner 2016). Accessibility at the time did not quite have that power, as it did not quantify trips, but rather something somewhat more elusive: the less tangible "potential for interaction". In this way, where spatial interaction modelling became a key element of transportation planning practice, accessibility remained a somewhat more academic pursuit, and the two streams of literature only rarely connected.

To illustrate this point, we conducted a bibliographic analysis of the literature that cites Hansen (1959), Wilson (1971), or both. We retrieved all relevant documents using the Web of Science "Cited References" functionality, and the digital object identifiers of Hansen (1959) and Wilson (1971). As a result of this search we identified 1,788 documents that cite Hansen (1959), 258 documents that cite Wilson (1971), and 76 that cite both. The earliest document in this corpus dates to 1976 and the most recent is from 2025. The number of documents per year appears in Figure 1, where we see the frequency of documents over a span of almost fifty years. In particular, we notice the remarkable growth in the number of papers that cite Hansen (1959) compared to those that cite Wilson (1971) or both.

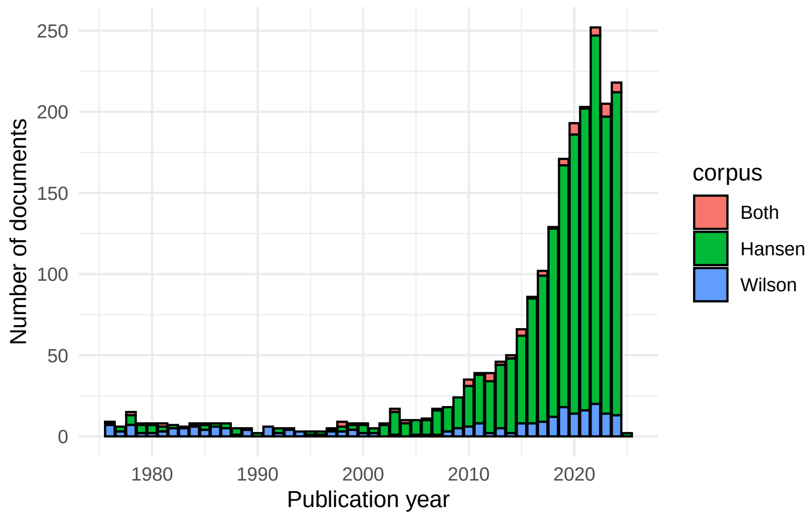

Figure 1: Historical pattern of publication: documents per year.

After compiling this corpus of documents, we used the {bibliometrix} package (Aria and Cuccurullo 2017) to create a bibliographical coupling matrix. In this matrix, two documents are coupled if they share at least one common reference, and the strength of the coupling increases with the number of references that the documents have in common. Figure 2 presents the results. In this figure, each symbol represents a document and their distance on the plot represents the strength of bibliographical coupling: more strongly coupled documents (i.e., those sharing more items in their lists of references) are plotted closer to each other.

Further examination of the bibliographical coupling matrix allows us to check the coupling strength within and between subgroups of documents (i.e., those that cite only Hansen, only Wilson, or both papers). First, we note that when we look at the three subgroups of documents pooled together, the average number of references shared by a pair of documents is 1.18. Table 1 presents the average number of items cited in common within each subgroup, as well as between subgroups. As seen in the table, the coupling within subgroup tends to be higher than when we consider the full corpus with all three groups together. The tightest coupling is seen for the group of documents that cite both Hansen and Wilson, with an average number of references in common of 3.36. The weakest coupling is between the Hansen and Wilson subgroups, with only 0.09 references in common on average, which indicates a large degree of decoupling (independence) between these bodies of work.

A family of accessibility measures

As we argued in the preceding section, the accessibility and spatial interaction modelling literature streams tended to evolve with relatively little contact after Hansen (1959) and Wilson

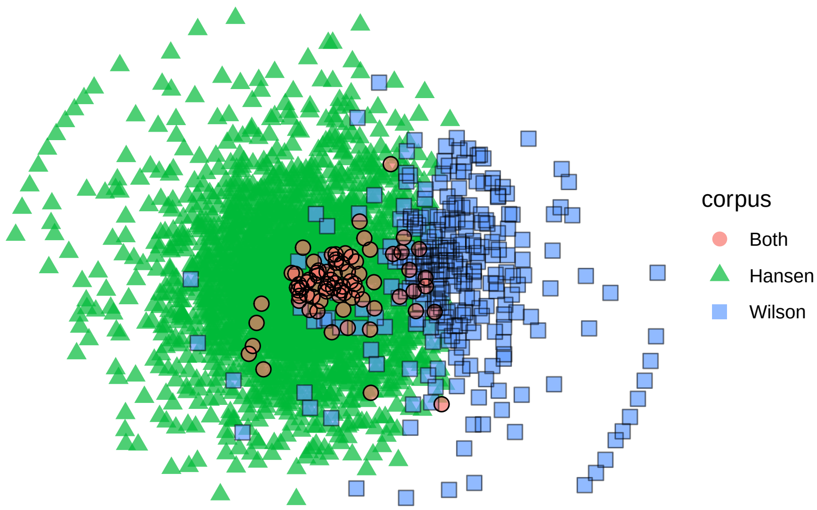

Figure 2: Bibliometric coupling of papers that cite Hansen (1959) and/or Wilson (1971).

Table 1: Average coupling strength within and between groups of documents. The average coupling strength is the number of references shared on average by a pair of documents.

corpus	Hansen	Wilson	Both
Hansen (n = 1788)	1.45	-	-
Wilson (n = 258)	0.09	1.31	-
Both (n = 76)	1.65	1.23	3.36

Average coupling for the pooled set of documents (n = 2122) is 1.18

(1971). This may explain the failure of the constraints/proportionality constant(s) of spatial interaction models to cross over to accessibility analysis. This is intriguing, considering that Wilson himself made an effort to connect his developments in spatial interaction modelling to accessibility, noting the similarity between the denominator of Equation 18 and Hansen's accessibility indicator. In fact, Wilson noted that Hansen's accessibility was the inverse of A_i, which offered a potential interpretation for the proportionality constants A_i (p. 10):

$$S_i = \frac{1}{A_i} = \sum_j W_j^{(2)} f(c_{ij}) \tag{23}$$

This, however, only helps to illustrate the ease with which confusion may arise in this context. The relationship in Equation 23 appears to be misguided when we note that Hansen, and Stewart before him, defined accessibility as a partial sum of the demographic force F, essentially going from the demographic force:

$$F = G\frac{M_1 \cdot M_2}{d_{12}^2}$$

to the pairwise population potential:

$$V_1 = G\frac{M_2}{d_{12}}$$

and from there to the system-wide population potential (after losing G):

$$V_i = \sum_j \frac{M_j}{d_{ij}}$$

It is useful to think of the accessibility as a partial sum of the spatial interaction, before we return to Wilson's general model:

$$T_{ij} = kW_i^{(1)} W_j^{(2)} f(c_{ij})$$

Then, a way to define the *potential for interaction* (i.e., accessibility) is as follows:

$$V_{ij} = kW_j^{(2)} f(c_{ij}) \tag{24}$$

where V_{ij} is the potential for interaction, and therefore:

$$V_i = k\sum_j W_j^{(2)} f(c_{ij}) \tag{25}$$

Similar to Equation 12, $W_j^{(2)}$ above is the mass at the destination, and the sub-indices are for $i = 1, \cdots, n$ origins and $j = 1, \cdots, m$ destinations.

Next, we show how various accessibility measures can be defined depending on how we choose to constrain them.

Unconstrained accessibility

Unconstrained accessibility results when we set the proportionality constant/balancing factor to one in Equation 24:

$$V_{ij}^0 = W_j^{(2)} f(c_{ij}) \tag{26}$$

Then, the partial sum becomes:

$$V_i^0 = \sum_j V_{ij}^0 = \sum_j W_j^{(2)} f(c_{ij}) = S_i \tag{27}$$

We can see that in this case we simply have Hansen's accessibility. In general, the following does not hold true:

$$\sum_i V_i^0 = O \tag{28}$$

where O is the total number of opportunities in the system. In other words, the sum of all accessibility scores will generally be different from the total of the opportunities available in the region. To complicate matters, the difference will depend on the choice of impedance function $f(c_{ij})$ and any associated parameters, as well as the number of locations n for which an accessibility score is calculated. For this reason, unconstrained accessibility can only be appropriately used as an ordinal variable to make comparisons of size (greater than, less than, equal to), but not to calculate intervals (the magnitude of differences) or ratios.

Total opportunity constrained accessibility

We can choose to retain the proportionality constant in Equation 24 to obtain the following:

$$V_{ij}^T = k^D W_j^{(2)} f(c_{ij}) \tag{29}$$

Our accessibility measure now becomes:

$$V_i^T = \sum_j V_{ij}^T = k^D \sum_j W_j^{(2)} f(c_{ij}) = k^D V_i^0 \tag{30}$$

The constraint that we impose in this case is the total number of opportunities D in the region (compare to Equation 13):

$$\sum_i V_i^T = \sum_i \sum_j V_{ij}^T = D \tag{31}$$

Substituting Equation 30 in Equation 31, and solving for k, we have that (compare to the constant in the total flow spatial interaction model, Equation 2.11 in Cliff, Martin, and Ord (1974)):

$$k^D = \frac{D}{\sum_i \sum_j V_{ij}^0} = \frac{D}{\sum_i \sum_j W_j^{(2)} f(c_{ij})} \tag{32}$$

Let us write our total opportunity constrained accessibility as follows:

$$V_i^T = k^D \sum_j W_j^{(2)} f(c_{ij}) = \frac{D}{\sum_i \sum_j V_{ij}^0} \sum_j W_j^{(2)} f(c_{ij}) = \sum_j W_j^{(2)} \frac{D \cdot f(c_{ij})}{\sum_i \sum_j W_j^{(2)} f(c_{ij})}$$

Further, we can see that, since O and $W_j^{(2)}$ are opportunities, the following term is dimensionless:

$$\kappa_{ij}^D = \frac{D \cdot f(c_{ij})}{\sum_i \sum_j W_j^{(2)} f(c_{ij})}$$

and therefore V_i^T is now in the units of $W_j^{(2)}$, that is, the mass at the destination:

$$V_i^T = \sum_j W_j^{(2)} \kappa_{ij}^D$$

Notice that the role of κ_{ij}^0 in this reformulation of accessibility is to adjust the number of opportunities accessible from i so that they represent a proportion of the total number of opportunities in the region. Constant κ_{ij}^0 then assigns opportunities in proportion to the impedance between i and j. For this reason we call it a proportional allocation factor.

Total population constrained accessibility: market potential

One might wish to measure accessibility to the mass of the *population* at i instead of the opportunities at j. This is equivalent to the concept of market potential:

$$M_{ij}^T = k^O W_i^{(1)} f(c_{ij}) = k^O M_{ij}^0 \tag{33}$$

with:

$$M_i^0 = W_i^{(1)} f(c_{ij})$$

The population-constrained market potential becomes:

$$M_j^T = \sum_i M_{ij}^T = k^O \sum_i W_i^{(1)} f(c_{ij}) \tag{34}$$

The constraint that we impose in this case is the total market potential equals the total population O in the region:

$$\sum_j M_j^T = \sum_i \sum_j M_{ij}^T = O \tag{35}$$

Substituting Equation 34 in Equation 35, and solving for k, we obtain:

$$k^O = \frac{O}{\sum_i \sum_i M_{ij}^T} = \frac{O}{\sum_i \sum_j W_i^{(1)} f(c_{ij})} \tag{36}$$

The constrained market potential then takes the following form:

$$M_i^T = k \sum_i W_i^{(1)} f(c_{ij}) = \frac{O}{\sum_i \sum_j M_{ij}^T} \sum_i W_i^{(i)} f(c_{ij})$$

Singly-constrained accessibility: opportunity-constrained

If we decide to introduce additional information into our analysis, say, the number of opportunities *by destination* (i.e., D_j), we can impose the following constraint (compare to Equation 15):

$$\sum_i V_{ij}^D = D_j \tag{37}$$

The underlying spatial interaction model is now the attraction-constrained model in Equation 19, and our accessibility measure becomes:

$$V_i^D = \sum_j B_j D_j W_i^{(1)} f(c_{ij}) \tag{38}$$

where $W_i^{(1)}$ is a measure of the mass at origin i (i.e., the opportunity-seeking population). The corresponding balancing factor, as per Wilson, is:

$$B_j = \frac{1}{\sum_i W_i^{(1)} f(c_{ij})} \tag{39}$$

If we introduce this balancing factor in Equation 38 we obtain:

$$V_i^D = \sum_j D_j \frac{W_i^{(1)} f(c_{ij})}{\sum_i W_i^{(1)} f(c_{ij})} \tag{40}$$

Further, we define the following proportional allocation factor:

$$\kappa_{ij}^D = \frac{W_i^{(1)} f(c_{ij})}{\sum_i W_i^{(1)} f(c_{ij})} \tag{41}$$

After this, it is possible to rewrite Equation 40 as:

$$V_i^D = \sum_j \kappa_{ij}^D D_j \tag{42}$$

Soukhov et al. (2023) have shown that the role of κ_{ij}^D is to allocate opportunities D_j proportionally to the mass at each origin i and the impedance between i and j. As before, κ_{ij}^D is dimensionless and V_i^D is in the units of opportunities D_j. The singly-constrained accessibility measure in Equation 42 is called spatial availability by Soukhov et al. (2023), because it represents the number of opportunities that can be reached *and* are available, in the sense that there is no competition for them. These authors also show that the following expression (accessibility per capita) is a constrained version of the popular two-stage floating catchment area measure of Shen (1998) and Luo and Wang (2003):

$$v_i^D = \frac{V_i^D}{W_i^{(1)}}$$

As an additional point, the constraint in Equation 37 ensures that the following also holds:

$$\sum_i V_i^D = D$$

For this reason, $\frac{V_i^D}{O}$ can be interpreted as the proportion of opportunities accessible *and* available to location i out of the total number of opportunities in the system

Singly-constrained accessibility: population-constrained

Not the most common application of accessibility analysis, but if the origins and destinations are transposed, we can define a measure of market potential that preserves the population, similar to Equation 36. The underlying spatial interaction model is now the production-constrained model in Equation 17, and our market potential measure V_i^P becomes:

$$M_j^P = \sum_i A_i O_i W_j^{(2)} f(c_{ij}) \tag{43}$$

In this case the measure is constrained by the population *by origin* (i.e., O_i). Compare to Equation 15:

$$\sum_j M_{ij}^P = O_i \tag{44}$$

The corresponding balancing factor, as per Wilson, is:

$$A_i = \frac{1}{\sum_j W_j^{(2)} f(c_{ij})} \tag{45}$$

Following the same logic as in the preceding section, one arrives at the following expression:

$$M_j^P = \sum_i \kappa_{ij}^O O_i \tag{46}$$

with:

$$
\kappa_{ij}^{O} = \frac{W_j^{(2)} f(c_{ij})}{\sum_i W_j^{(2)} f(c_{ij})} \tag{47}
$$

In addition to market potential, M_j^P can be interpreted as a constrained measure of "facility crowdedness" as in F. H. Wang (2018). This "crowdedness" now satisfies, in addition to Equation 37, the following:

$$
\sum_j M_j^P = O
$$

where O is the total population of the region. In this way, $\frac{M_j^O}{O}$ can be interpreted as the proportion of the total population serviced by location j.

Doubly-constrained

Readers will have noticed that the accessibility measures above have used either O_i or D_j but not both. When D_j, the opportunities, were used to constrain Equation 38, the mass of the population at the origin was given by $W_i^{(1)}$ (also in Equation 39). Contrariwise, When O_i, the population, was used as a constraint in Equation 43, the mass at the destination was given by $W_j^{(2)}$ (also in Equation 45). The reason is that the population and the opportunities are typically different things without a common metric. For example, on the side of the population we usually have people, but on the side of the opportunities we can have physicians, clinics, grocery stores, green space, schools, libraries, and so on. There are a few cases where there might be a one-to-one relationship between the population and the opportunities, with the most common one being employment, i.e., one person, one job.

A doubly-constrained approach can be implemented when such one-to-one relationship between population and opportunities. In this case, the two constraints in Equation 14 and Equation 15 are imposed. Furthermore, the two constraints must match:

$$
\sum_i O_i = \sum_j D_j
$$

This is because now the proportionality constants are mutually dependent and cannot be estimated if the constraints diverge. For this reason, the applicability seems to be relatively limited, because even in the case of jobs, there might be imbalances between O and D, say, where there are more jobs than people or vice-versa.

Table 2: Simple system with three zones. Population is in 10,000 persons and opportunities in number of physicians.

ID	Population[1]	Opportunities[2]
1	4	160
2	10	150
3	6	180

[1]Population is *Wi*[(1)] when used as a proxy for the mass at the origin, and *Oi* when used as a constraint.
[2]Opportunities is *Wj*[(2)] when used as a proxy for the mass at the destination, and *Dj* when used as a constraint.

Another important point to note about the doubly-constrained approach is as follows. Recall that accessibility is defined as "the potential for interaction". Additional constraints leave less room for that potential. In fact, a doubly constrained approach essentially converges on the spatial interaction model, where one person represents one trip to one opportunity, and it is probably better thought of as an estimate of "access" instead of "accessibility".

Calibration of the two sets of proportionality constants is accomplished by means of iterative proportional fitting, whereby the values of A_i are initialized as one for all i to obtain an initial estimate of B_j. The values of B_j are used to update the underlying trip matrix, before calibrating A_i. This process continues to update A_i and B_j until a convergence criterion is met (see Ortúzar and Willumsen 2011, 193–95).

Simple numerical example

In this section we use a simple numerical example to illustrate the various members of the family of accessibility measures introduced above. Consider a very simple system with three zones, as shown in Table 2. It can help to think of these numbers as population in 10,000s and number of physicians (for comparison, the number of physicians per 10,000 in Canada in 2022 was 24.97). If so, the Provider-to-Population-Ratio (PPR) in this system is 24.5. The population is $W_i^{(1)}$ when used as a proxy for the mass at the origin, and O_i when used as a constraint. Similarly, the opportunities are represented by $W_j^{(2)}$ when used as a proxy for the mass at the destination, and D_j when used as a constraint.

To complete the set-up of this example, the cost of movement between origins and destinations is as shown in Table 3.

Table 3: Cost matrix for system with three zones (travel time in minutes).

Origin ID	Destination ID 1	2	3
1	10	30	10
2	30	10	25
3	15	25	10

Table 4: Simple system: unconstrained accessibility.

Origin	V_i^0		
	$f_1\ (c_{ij}) = 1/c_{ij}^3$	$f_2\ (c_{ij}) = 1/c_{ij}^2$	$f_3\ (c_{ij}) = 1/c_{ij}^{0.1}$
	units: *physicians per min^3*	units: *physicians per min^2*	units: *physicians per min^0.1*
1	0.346	3.567	376.824
2	0.167	1.966	363.479
3	0.237	2.751	373.738

For comparison, we use three different impedance functions:

$$f_1(c_{ij}) = \frac{1}{c_{ij}^3}$$
$$f_2(c_{ij}) = \frac{1}{c_{ij}^2}$$
$$f_3(c_{ij}) = \frac{1}{c_{ij}^{0.1}}$$

Unconstrained accessibility

The following case is standard practice, the HAM. It is expressed as a value of the origin, a summary of the destination weight multiplied by the travel impedance. There is no explicit consideration of a proportionality constant (k), we can presume it is still set to 1 as in Stewart (1947). Neither are there origin weights $W_i^{(1)}$, we assume is also wrapped up in this proportionality and implicitly set to 1. Working backward from this expression and assumptions, S_{ij} can be expressed more generally as well.

S_i reflects the magnitude of opportunity accessibility but it is not scaled to equal the number opportunities O_j and/or population P_i (only in some case when k and $W_i^{(1)}$ equal 1).

In solving the example, each origin (row) all the opportunities at each destination are available. Likewise, each destination (column) offers all opportunities to each origin:

Compare the accessibility differences and between pairs of zone using the two different impedance functions:

```
[1] -0.1781096
```

```
[1] -1.600889
```

```
[1] -13.34517
```

```
[1] 0.4845702
```

```
[1] 0.5511526
```

```
[1] 0.9645852
```

This illustrates how the differences and ratios are not meaningful.

The total accessibility compared to the number of opportunities:

```
# A tibble: 1 x 3
  V_unc_1 V_unc_2 V_unc_3
    <dbl>   <dbl>   <dbl>
1   0.750    8.28   1114.
```

```
    O
1 490
```

And the ratio of accessibility to population compared to the ratio of opportunities to population:

```
# A tibble: 3 x 8
  oid   V_unc_i_1 V_unc_i_2 V_unc_i_3   O_i v_unc_1 v_unc_2 v_unc_3
  <chr>     <dbl>     <dbl>     <dbl> <dbl>   <dbl>   <dbl>   <dbl>
1 1         0.346      3.57      377.     4  0.0864   0.892    94.2
2 2         0.167      1.97      363.    10  0.0167   0.197    36.3
3 3         0.237      2.75      374.     6  0.0395   0.459    62.3
```

```
# A tibble: 1 x 3
  v_unc_1 v_unc_2 v_unc_3
    <dbl>   <dbl>   <dbl>
1  0.0375   0.414    55.7


   PPR
1 24.5
```

Notice the wild differences. Accessibility according to impedance function 3 basically says that everyone has access to all opportunities all at once. Accessibility according to function 1, on the other hand, says that each population has accessibility to around 3 opportunities, which is absurd: the number of opportunities available locally are in the order of 150-180. One should not have to adjust the impedance function to obtain "reasonable" results [EXAMPLE]; instead, the impedance function should reflect the travel behavior.

PREVIOUS TEXT FOLLOWS: REVISE DISCUSSION

We'd report origin 1 as having 148 accessibility (i.e., $S_{i=1} = 148$), origin 2 as 248 accessibility, and origin 3 as 172 accessibility. The total accessibility in the region is 568 ($\sum_i S_i = 568$), not equal to any meaningful value. This value is not often reported in the accessibility literature for this reason. One could also report the accessibility values that destinations offers (S_j), namely destination 1 offers 123, destination 2 offer 291, and destination 3 offers 154. However, these values are also not reported in the literature currently… as they aren't constrained to match up to anything known, including the number of opportunities at each j (i.e., 160, 437 and 193). Assuming $k = 1$, S_i and S_j represent magnitudes of accessibility concepts, but in an nonconstrained way. The population matrix is not incorporated in the S_{ij} calculation, so S_i cannot be directly compared to population. From a different perspective, S_j values also cannot be compared to known O_j values as the equation unconstrained.

OF NOTE: in the context of spatial interaction modelling, this nonconstrained version of model is seldom useful. While the resulting T_{ij} values are technically units of 'trips' - there are no constraints, so the assumption of k and $W_i^(1)$ equaling 1 is practically meaningless.

Accessibility, or the *potential* for interaction is different than interaction (V_{ij} vs. T_{ij}), namely since the number of opportunities that are accessible at an origin from a destination cannot be known in the same way as a trip from an origin to a destination. However, we would argue that getting closer to knowing this number and planning for it - is what accessibility research should strive to capture. An origin-destination based accessibility value V_{ij}. We demonstrate the following three cases sub-sequently.

END OF PREVIOUS TEXT

Table 5: Simple system: total opportunity constrained accessibility.

	V_i^T		
	$f_1\ (c_{ij}) = 1/{c_{ij}}^3$	$f_2\ (c_{ij}) = 1/{c_{ij}}^2$	$f_3\ (c_{ij}) = 1/{c_{ij}}^{0.1}$
Origin	units: *physicians*	units: *physicians*	units: *physicians*
1	225.760	210.980	165.742
2	109.397	116.282	159.873
3	154.843	162.737	164.385

Total opportunity-constrained accessibility

Proportionality constant for $f_1(c_{ij}) = 1/c_{ij}^3$:

$$
\begin{aligned}
k^D &= \frac{D}{\sum_i \sum_j O_j f(c_{ij})}\\
k^D &= \frac{D}{\frac{D_1}{c_{11}^3} + \frac{D_1}{c_{21}^3} + \frac{D_1}{c_{31}^3} + \cdots + \frac{D_3}{c_{31}^3} + \frac{D_3}{c_{32}^3} + \frac{D_3}{c_{33}^3}}\\
k^D &= \frac{490}{0.75}\\
k^D &= 653.3256
\end{aligned}
$$

Use these proportionality constants to calculate the total opportunity constrained accessibility:

Compare the accessibility differences and between pairs of zone using the two different impedance functions:

```
[1] -116.3636
```

```
[1] -94.69793
```

```
[1] -5.869738
```

```
[1] 0.4845702
```

```
[1] 0.5511526
```

```
[1] 0.9645852
```

This illustrates how the differences and ratios are now meaningful.

The total accessibility compared to the number of opportunities:

```
# A tibble: 1 x 3
  V_tot_1 V_tot_2 V_tot_3
    <dbl>   <dbl>   <dbl>
1     490     490     490
```

```
    O
1 490
```

This shows how the number of opportunities is preserved.

And the ratio of accessibility to population compared to the ratio of opportunities to population:

```
# A tibble: 3 x 8
  oid   V_tot_i_1 V_tot_i_2 V_tot_i_3   O_i v_tot_1 v_tot_2 v_tot_3
  <chr>     <dbl>     <dbl>     <dbl> <dbl>   <dbl>   <dbl>   <dbl>
1 1          226.      211.      166.     4    56.4    52.7    41.4
2 2          109.      116.      160.    10    10.9    11.6    16.0
3 3          155.      163.      164.     6    25.8    27.1    27.4
```

```
# A tibble: 1 x 3
  v_tot_1 v_tot_2 v_tot_3
    <dbl>   <dbl>   <dbl>
1    24.5    24.5    24.5
```

```
   PPR
1 24.5
```

Total population-constrained accessibility (market potential)

Proportionality constant for $f_1(c_{ij}) = 1/c_{ij}^3$:

$$
\begin{aligned}
k^O &= \frac{O}{\sum_i \sum_j O_i f(c_{ij})} \\
k^O &= \frac{O}{\frac{O_1}{c_{11}^3} + \frac{O_1}{c_{21}^3} + \frac{O_1}{c_{31}^3} + \cdots + \frac{O_3}{c_{31}^3} + \frac{O_3}{c_{32}^3} + \frac{O_3}{c_{33}^3}} \\
k^O &= \frac{490}{0.75} \\
k^O &= 653.3256
\end{aligned}
$$

Use these proportionality constants to calculate the total opportunity constrained accessibility:

Table 6: Simple system: total opportunity constrained accessibility.

Destination	M_i^D		
	$f_1\ (c_{ij}) = 1/c_{ij}^3$	$f_2\ (c_{ij}) = 1/c_{ij}^2$	$f_3\ (c_{ij}) = 1/c_{ij}^{0.1}$
	units: *population in 10,000s*	units: *population in 10,000s*	units: *population in 10,000s*
1	4.501	5.053	6.580
2	7.710	7.410	6.698
3	7.789	7.537	6.722

Singly-constrained accessibility: opportunity-constrained

Returning to our simple example, the opportunity-constrained case would yield the following B_j:

$$B_j = \frac{1}{\sum_i O_i f(c_{ij})}$$

$$B_1 = \frac{1}{\frac{4}{10^3} + \frac{10}{30^3} + \frac{6}{15^3}} = 162.6506$$
$$B_2 = \frac{1}{\frac{4}{30^3} + \frac{10}{10^3} + \frac{6}{25^3}} = 94.9474$$
$$B_3 = \frac{1}{\frac{4}{10^3} + \frac{10}{25^3} + \frac{6}{10^3}} = 93.9850$$

Balancing factors B_j:

```
# A tibble: 3 x 4
  did   B_j_1 B_j_2  B_j_3
  <chr> <dbl> <dbl>  <dbl>
1 1     163.  12.9  0.0672
2 2      94.9  8.77 0.0661
3 3      94.0  8.62 0.0658
```

Use these proportionality constants to calculate the singly-constrained opportunity-constrained accessibility:

Compare the accessibility differences and between pairs of zone using the two different impedance functions:

```
[1] -10.98879
```

Table 7: Simple system: singly-constrained opportunity constrained accessibility.

	V_i^D		
	$f_1\ (c_{ij}) = 1/{c_{ij}}^3$	$f_2\ (c_{ij}) = 1/{c_{ij}}^2$	$f_3\ (c_{ij}) = 1/{c_{ij}}^{0.1}$
Origin	units: *physicians*	units: *physicians*	units: *physicians*
1	173.876	150.200	100.041
2	162.887	179.212	241.157
3	153.238	160.587	148.802

```
[1] 29.01204
```

```
[1] 141.1168
```

```
[1] 0.9368008
```

```
[1] 1.193156
```

```
[1] 2.410595
```

This illustrates how the differences and ratios are now meaningful.

The total accessibility compared to the number of opportunities:

```
# A tibble: 1 x 3
  V_opp_1 V_opp_2 V_opp_3
    <dbl>   <dbl>   <dbl>
1     490     490     490
```

```
    O
1 490
```

This shows how the number of opportunities is preserved.

And the ratio of accessibility to population compared to the ratio of opportunities to population:

```
# A tibble: 3 x 8
  oid   V_opp_i_1 V_opp_i_2 V_opp_i_3   O_i v_opp_1 v_opp_2 v_opp_3
  <chr>     <dbl>     <dbl>     <dbl> <dbl>   <dbl>   <dbl>   <dbl>
1 1          174.      150.      100.     4    43.5    37.6    25.0
2 2          163.      179.      241.    10    16.3    17.9    24.1
3 3          153.      161.      149.     6    25.5    26.8    24.8
```

```
# A tibble: 1 x 3
  v_opp_1 v_opp_2 v_opp_3
    <dbl>   <dbl>   <dbl>
1    24.5    24.5    24.5
```

```
   PPR
1 24.5
```

Singly-constrained accessibility: population-constrained

Returning to our simple example, the production-constrained case would yield the following A_j when we use $f_1(c_{ij}) = 1/c_{ij}^3$:

$$A_i = \frac{1}{\sum_j O_j f(c_{ij})}$$

$$A_1 = \frac{1}{\frac{160}{10^3} + \frac{150}{30^3} + \frac{180}{15^3}} = 2.8939$$
$$A_2 = \frac{1}{\frac{160}{30^3} + \frac{150}{10^3} + \frac{180}{25^3}} = 5.9721$$
$$A_3 = \frac{1}{\frac{160}{15^3} + \frac{150}{25^3} + \frac{180}{10^3}} = 4.2193$$

```
# A tibble: 3 x 4
  oid   A_i_1 A_i_2   A_i_3
  <chr> <dbl> <dbl>   <dbl>
1 1      2.89 0.280 0.00265
2 2      5.97 0.509 0.00275
3 3      4.22 0.363 0.00268
```

Use these proportionality constants to calculate the singly-constrained population constrained accessibility:

Compare the accessibility differences and between pairs of zone using the two different impedance functions:

Table 8: Simple system: singly-constrained population-constrained accessibility (market potential).

Destination	M_j^P		
	$f_1\ (c_{ij}) = 1/{c_{ij}}^3$	$f_2\ (c_{ij}) = 1/{c_{ij}}^2$	$f_3\ (c_{ij}) = 1/{c_{ij}}^{0.1}$
	units: *population in 10,000s*	units: *population in 10,000s*	units: *population in 10,000s*
1	3.406	4.250	6.441
2	9.265	8.341	6.157
3	7.328	7.409	7.402

```
[1] 5.859314
```

```
[1] 4.091264
```

```
[1] -0.2845821
```

```
[1] 2.720221
```

```
[1] 1.962731
```

```
[1] 0.955818
```

This illustrates how the differences and ratios are now meaningful.

The total accessibility compared to the number of opportunities:

```
# A tibble: 1 x 3
  M_pop_1 M_pop_2 M_pop_3
    <dbl>   <dbl>   <dbl>
1      20      20      20
```

```
   O
1 20
```

This shows how the population is preserved.

And the ratio of accessibility to population compared to the ratio of opportunities to population:

Table 9: Simple system with three zones. Population is in 10,000 persons and opportunities in 10,000s physicians.

ID	Population[1]	Opportunities[2]
1	4	7
2	10	6
3	6	7

[1]Population is $Wi^{(1)}$ when used as a proxy for the mass at the origin, and Oi when used as a constraint.
[2]Opportunities is $Wj^{(2)}$ when used as a proxy for the mass at the destination, and Dj when used as a constraint.

```
# A tibble: 3 x 8
  did   M_pop_i_1 M_pop_i_2 M_pop_i_3   D_j m_pop_1 m_pop_2 m_pop_3
  <chr>     <dbl>     <dbl>     <dbl> <dbl>   <dbl>   <dbl>   <dbl>
1 1          3.41      4.25      6.44   160   0.852   1.06    1.61 
2 2          9.27      8.34      6.16   150   0.927   0.834   0.616
3 3          7.33      7.41      7.40   180   1.22    1.23    1.23 
```

```
# A tibble: 1 x 3
  m_pop_1 m_pop_2 m_pop_3
    <dbl>   <dbl>   <dbl>
1  0.0408  0.0408  0.0408
```

```
         PPR
1 0.04081633
```

Doubly-constrained case (population-opportunity constrained)

NOTE: The example needs to be revised in this case to have the population and opportunities match, as in, one person-one opportunity

In this case, V_{ij} would be:

```
Checking production, attraction balancing:
Production:  20
Attraction:  20
Production, attraction balancing OK.
```

```
Iteration: 1
Ai: 70.312 149.092 105.73
Bj: 1.907 0.649 0.989

Iteration: 2
Ai: 48.983 207.096 89.868
Bj: 2.313 0.473 1.152

Iteration: 3
Ai: 41.048 252.791 76.652
Bj: 2.537 0.39 1.272

Iteration: 4
Ai: 37.375 280.227 69.833
Bj: 2.649 0.353 1.337

Iteration: 5
Ai: 35.734 293.89 66.707
Bj: 2.7 0.337 1.367

Iteration: 6
Ai: 35.03 300.033 65.352
Bj: 2.722 0.33 1.381

Iteration: 7
Ai: 34.734 302.666 64.781
Bj: 2.731 0.327 1.386

Iteration: 8
Ai: 34.611 303.77 64.542
Bj: 2.735 0.326 1.389

Iteration: 9
Ai: 34.56 304.229 64.443
Bj: 2.736 0.326 1.39

Stopping Condition:  Error threshold met
Final Error: 0.063%

Checking production, attraction balancing:
Production:  20
Attraction:  20
```

```
Production, attraction balancing OK.

Iteration: 1
Ai: 6.818 12.662 9.033
Bj: 1.528 0.723 0.983

Iteration: 2
Ai: 5.537 15.089 8.11
Bj: 1.652 0.621 1.053

Iteration: 3
Ai: 5.168 16.16 7.629
Bj: 1.696 0.584 1.083

Iteration: 4
Ai: 5.04 16.566 7.452
Bj: 1.711 0.571 1.094

Iteration: 5
Ai: 4.995 16.711 7.389
Bj: 1.717 0.567 1.098

Iteration: 6
Ai: 4.979 16.761 7.368
Bj: 1.719 0.565 1.1

Iteration: 7
Ai: 4.974 16.778 7.36
Bj: 1.719 0.565 1.1

Stopping Condition:  Error threshold met
Final Error: 0.036%

Checking production, attraction balancing:
Production:  20
Attraction:  20
Production, attraction balancing OK.

Iteration: 1
Ai: 0.065 0.067 0.066
Bj: 1.013 0.994 0.992
```

```
Stopping Condition:  Error threshold met
Final Error: 0.016%
```

The A_i and B_j matrices:

```
          [,1]
[1,]  34.56044
[2,] 304.22911
[3,]  64.44338
```

```
         [,1]      [,2]     [,3]
[1,] 1.718552 0.5653009 1.099604
```

The trip matrix (notice, all summed up opportunities and populations are equivalent to our knowns)

Nice. But notice how the accessibility is simply the number of opportunities at the destination! In other words, it is simply the access.

Empirical examples

Use the TTS2016R package to illustrate every measure.

A note on interpretation and possible applications

When to use each of these measures?

Conclusions

Accessibility development has a rich history. There have been developments in the cases/breath of travel impedance functions, competition considerations, and in incorporating individual choice as part of activity-based accessibility. However, no work yet has examined accessibilities explicit connecting to SIM. We examine this connection, and provide a origin-destination (i to j) formulation of accessibility as part of the SIM framework. Novel stuff.

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