# A family of accessibility measures

## Introduction

ANAASTASIA: A brief intro describing the aims of the paper and anticipating the conclusions needs to go here.

This work aims to re-unify accessibility literature with its Newtonian roots by mapping it onto the family of gravity models. Not only to clarify accessibility's interpretation as a measure, but also to encourage future developments that draw from its physics roots, gravity modelling siblings, and spatial interaction history. We began by tracing the development of spatial interaction from Ravenstein (1889) to it's split into accessibility research from Hansen (1959) to today. In the following section, we will derive different types of accessibility measures, following the family of gravity models namely: non-constrained, constrained, unconstrained/Total constrained, singly constrained, composite singly constrained, doubly constrained. Then, we will provide empirical examples of the types of measures. Finally, we'll discuss the use of each measure and broadly the implications of ignoring the issue of proportionality in accessibility analysis.

## Background

The patterns of people's movement in space have been a subject of scientific inquiry for centuries. From as far back as 1858 with Carey's Principles of Social Science where he stated that "man [is] the molecule of society [and their interaction is subject to] the direct ratio of the mass and the inverse one of distance" (see McKean's 1883 Manual of Social Science, pp. 37-38), investigations of how people interact in space has often been explicitly coloured by the features of Newton's Law of Universal Gravitation, as posited in his *Principia Mathematica* of 1687 (see Equation 1):

$$F_{ij} \propto \frac{M_i M_j}{D_{ij}^2} \tag{1}$$

Equation 1 is one of the most famous equations in all of science, and it states that the force of attraction $F$ between a pair of bodies $i$ and $j$ is directly *proportional* to the product of their masses $M_i$ and $M_j$, and inversely *proportional* to the square of the distance between them $D_{ij}$. Direct proportionality only means that as the product of the masses increases, so does the force. Likewise, inversely proportional only means that as the distance increases, the force decreases. To quantify the magnitude of the force, an empirical factor is required to convert the proportionality into an equation, which ensures that the force $F$ matches empirical observations of the force of attraction between masses. Ultimately, the equation for the force is as seen in Equation 2:

$$F_{ij} = G\frac{M_i M_j}{D_{ij}^2} \tag{2}$$

where $G$ is a proportionality constant obtained empirically and called *Cavendish gravitational constant*, after Henry Cavendish's experimental results of 1798–more than a century after Newton's *Principia* was published–that brought it to within 1% accuracy level.

Following Carey, research into human spatial interaction continued apace in different contexts. In 1885 and 1889 Ravenstein (who appears to have been unaware of Carey's treatise) proposed his "Laws of Migration", based on his empirical analysis of migration flows in various countries. In these works, Ravenstein posited a directly proportional relationship with the size of the destination (i.e., centres of commerce and industry), and an inversely proportional relationship between the size of flows and separation between origin and destination, in clear similitude to Newton's gravitational laws. Over time other researchers kept discovering, seemingly independently, some principles of human spatial interaction that resembled Newton's universal law of gravitation. For example, Reilly et al. (1929) (who seems to have been unaware of Carey and Ravenstein) formulated a law of retail gravitation , expressed in terms of equal attraction to competing retail destinations. Later, Zipf (seemingly unaware of Carey, Ravenstein, and Rilley) proposed a $\frac{P_1 P_2}{D}$ hypothesis for the case of goods movement by railways Zipf (1946c), intercity personal movement Zipf (1946b),and information Zipf (1946a). This hypothesis stated that the magnitude of flows was proportional to the product of the populations of settlements, and inversely proportional to the distance between them.

A common theme among these early researchers of human spatial interaction is that a proportionality constant similar to $G$ in Equation 2 is missing. Of the researchers cited here, only Reilly and Zipf expressed their hypotheses in mathematical terms. Reilly's hypothesis was expressed as follows:

$$B_a = \frac{(P_a \cdot P_T)^N}{D_{aT}^n} \tag{3}$$

where $B_a$ is the amount of business drawn to $a$ from $T$, $P_a$ and $P_T$ are the populations of the two settlements, and $D_{aT}$ is the distance between them. Quantity $N$ was chosen by Reilly in a somewhat *ad hoc* fasion as 1, and he used empirical observations of shoppers to choose a value of $n = 2$.

Zipf, on the other hand, wrote his hypothesis in mathematical form as:

$$C^2 = \frac{P_1 \cdot P_2}{D} \tag{4}$$

where $C$ is the volume of goods exchanged between 1 and 2, $P_1$ and $P_2$ are the populations of the two settlements, and $D_{12}$ is the distance between them.

After Carey, it is in Stewart's (1948) work on the principles of demographic gravitation that we find the strongest connection yet to Newton's law (Stewart was a physicist, whereas Ravenstein, Reilly, and Zipf were social scientists), besides awareness of preceding research, as he cites Reilly and Zipf as predecessors in the analysis of human spatial interaction. Stewart appears to have been the first author to express his theorized relationships for human spatial interaction with a proportionality constant $G$, as follows:

$$F = G\frac{(N_1\mu_1)(N_2\mu_2)}{d_{12}^2} \tag{5}$$

where:

- $F$ is the *demographic force*
- $N_1$ and $N_2$ are the numbers of people of in groups 1 and 2
- $\mu_1$ and $\mu_2$ are so-called *molecular weights*
- $d_{12}^2$ is the distance between 1 and 2
- And finally $G$, a constant that Stewart "left for future determination" (1948, 34) but it is unclear whether he revisited

In addition to demographic force, Stewart defined a measure of "population potential" of 2 with respect to 1 as follows:

$$V_1 = G\frac{M_2}{d} \tag{6}$$

For a system with more than 2 population bodies, Stewart formulated the population potential at $i$ as:

$$V_i = \int \frac{D}{r} ds \tag{7}$$

where $D$ is population density over an infinitesimal area $ds$, and $r$ is the distance to $i$. In more contemporary terms, this can be rewritten as:

$$V_i = \sum_j \frac{M_j}{d_{ij}} \tag{8}$$

which astute readers will identify as our modern definition of accessibility.

Stewart's formulation of demographic force, developed in the context of what he called "social physics" (Stewart 1947), was problematic and had issues with inconsistent mathematical notation [e.g., in Stewart's 1948 paper, $G$ is used as a proportionality constant on p. 34 and again for the demographic energy on p. 54], but perhaps more seriously, it was permeated by a view of humans as particles and afflicted by unmitigated racism (Stewart assumed that the "molecular weight" $\mu$ was one for the average American, but "presumably…much less than one" for an Australian Aboriginal; p. 34). It did not take long for Stewart's ideas about "social physics" to fall out of favor among social scientists, but not before influencing Walter G. Hansen, then a student of city and regional planning at Massachusetts Institute of Technology.

Hansen's day job was as an engineer with the the Bureau of Roads in the United States, and preoccupied with transportation's power to shape land uses. As such, he was neither scientist nor social scientist, but rather an applied researcher. Hansen's (1959) paper in the Journal of the American Planning Association was instrumental in shedding some of the less convincing, or frankly objectionable, aspects of "social physics", to recast accessibility not as a model of physics, but as a model of human behavior useful to capture regularities in human mobility patterns. In his work, Hansen (1959) borrowed the idea of a "population potential" of Stewart (1948), which closely followed the Newtonian definition of energy potential, and applied it to the measurement of the potential of opportunities. The potential energy of one mass, say $M_i$, when applied to population instead of celestial bodies yields "potential population":

$$S_{ij} = \sum_j O_j \cdot d_{ij}^{\beta} \tag{9}$$

In this equation, Hansen (1959) replaced $M_j$ with *opportunities* to derive *opportunity potential*, or more accurately, *potential of opportunities for interaction*. A more general form of this expression is as follows, to account for different impedance functions to reflect variations in travel behavior:

$$S_{ij} = \sum_j O_j \cdot f(d_{ij}) \tag{10}$$

$S_{ij}$ in Equation 9 is is a relative measure of the accessibility at $i$ to an activity located at $j$, $O_j$ is the size of the activity at $j$ and $d_{ij}$ is the distance or travel time between $i$ and $j$ with $\beta$ describing the effect size. Today, Hansen (1959) is frequently cited as the father of modern accessibility analysis (e.g., Reggiani and Martín 2011) and Hansen-type accessibility is commonly referred to today as the gravity-based accessibility measure.

Of note, however is that between Stewart (1948) and Hansen (1959) the proportionality constant $G$ in Equation 2 vanished.

There is some evidence that Hansen (1959) was aware of the import of this constant, since in his description of the process of interest he wrote that "the formulation states that the accessibility at point 1 to a particular type of activity at area 2 (say employment) is *directly proportional* to the size of the activity at area 2 (number of jobs) and *inversely proportional* to some function of the distance separating point 1 from area 2" (p. 73; our emphasis). The reason for this treatment of the proportionallity constant could simply be the empirical and theoretical limitations of the state of knowledge at the time when Hansen (1959) drew from Stewart (1948). After all, in the latter the proportionality constant was set to 1 and that *"the constant G [was] left for future determination: a suitable choice of other units can reduce it to unity"*. In practice, this is still done today, and the proportionality constant continues to be implicitly set to 1 even when the relationship is clearly of proportionality and not equality (see for instance the formula for accessibility at the top of Figure 1 in Wu and Levinson 2020). Without a proportionality constant, however, the units of $S_ij$ are unclear: the unit of 'potential of opportunity for interaction' is left free to change as $\beta$ is calibrated for the population center with no discussion of a constant to balance the units.

Since Hansen (1959), accessibility analysis has been widely employed across planning disciplines, but to our knowledge, developments have not been concerned with explicitly determining a proportionality constant. The majority of studies develop on the breath of opportunity $O_j$ studied (i.e., work, shopping, schools, leisure () and their geographic case study selection, the accuracy of parameters that capture distance-decay $f(c_{ij})$, and additions to the opportunity $O_j$ attributes such as competitive demand considerations (Stouffer 1960; Weibull 1976; Shen 1998; Allen and Farber 2020; Soukhov et al. 2024). However, the difficulty in interpreting gravity-based accessibility's meaning as anything other than a relative index of potential interaction (higher accessibility/lower accessibility), plagues this type of analysis to the date, and contributes to resistance to its application in practice [REFERENCE MISSING??]. The quantity and variations on Equation 10 compounds the issue of interpretability across studies. A testament to this inconsistent interpretation are the unsettled classifications generated by influential review works of the accessibility literature; for instance, (**handyMeasuringAccessibilityExploration1997?**) classifies studies into gravity- and utility-based measures, Geurs and Wee (2004) introduces further classification by expanding categories into potential- (or gravity), utility-, location-, and person- based measures with different components. Wu and Levinson (2020) demonstrate that most or all of these concepts can be unified into its main features: the relationship between travel-cost and corresponding reachable opportunities. Indeed, these authors demonstrate how previous classifications are special cases of travel-cost or opportunities constraints. From the initial Newtonian analogy, we interpret Wu and Levinson (2020)'s work as offering a positive contribution by circling back to re-unify mass $M$ and separation distance $D$; however, an important component is still missing—the proportionality constant $G$.

The aim of this paper's focus is on the proportionality constant. We wish to demonstrate the importance of transiting from proportionality to equality in Equation 10, in order to balance the units of accessibility into more interpretable values with clear meanings. To address this matter we need not reinvent the wheel: instead we refer to the spatial interaction model as developed by Wilson (1971). In essence, in what follows, we reunite the accessibility indicators literature with the spatial interaction literature as developed by Wilson (1971) based on entropy maximization concepts. Although Wilson's approach is based on a different conceptual foundation than the old reference to Newtonian gravity, the work succeeded at identifying the steps from proportionality to equality to yield variations of proportionality constants: including the one that eluded Stewart (1948) and has been ignored in almost all subsequent accessibility research.

## Enter Wilson and entropy maximization

The gravity model traces back to these early 20th century empirical investigations (Carrothers 1956; Grigg 1977; Haynes and Fotheringham 1985; Oshan 2021); it also retains the proportionality constant and the resulting output is a tangible unit.

$$T_{ij} = kP_i^{\lambda}P_j^{\alpha}d_{ij}^{\beta} \tag{11}$$

Where $T_{ij}$ represents a $n$ x $m$ matrix of flows between $n$ origins $i$ and $m$ destinations $j$, $P$ are a vector of mass attributes at origin $i$ and $j$ and cost to overcome physical separation, and $d_{ij}$ is a $n$ x $m$ matrix. $d_{ij}$ is also frequently replaced by $f(c_{ij})$ in the literature, some function of $c_{ij}$ which decreases as $c_{ij}$ increases. Furthermore, the exponents are introduced to modulate the effect size of the variables, where $\lambda$ and $\alpha$ modulate $P$ and $\beta$ modulates $d$; when variables $T$, $P$ and $d$ are known, the exponents can be estimated to summarize the variable's effect size contribution to the known flows in the system. Finally and of most important note in this paper, $k$ is a scale parameter that makes the overall equation proportional to the rate characteristic of the modeled phenomenon. In other words, $T_{ij}$ explicitly measures *interaction* as a unit of trips and the proportionality constant $k$ ensures the total sum of $T_{ij}$ represents the total flows in the data.

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It's all about balancing factors! Lessons from the balancing factors of the gravity model (cool paper that explores this Kirby (1970))

Derive the various accessibility measures. In my mind:

- Non-constrained: it does not include a single proportionality constant.
- Total-constrained: uses a single proportionality constant
- Origin- or Destination-constrained: spatial availability
- Doubly constrained (only when the units of the marginals are the same, therefore of more limited application) ex. Horner (2004) and Allen and Farber (2020)

Origins in retail models, decision model for shops to optimally spatially locate to max. their potential consumer interaction. Starting with reilly's law of gravitational retail (1931) (can't find online), formalized later in Huff (1964) and Lakshmanan and Hansen (1965).

Independently formed in people spatial interaction: “intervening opportunities and competing migrants” Stouffer (1940) and Stouffer (1960), “axiomatically” Weibull (1976), “for multiple-modes, as a rate” Shen (1998);for multiple-modes, as the number of opportunities” Soukhov et al. (2024)

- Composite singly constrained: Balanced floating catchment areas (not truly doubly constrained)

## Examples

Use the TTS2016R package to illustrate every measure.

## Discussion

When to use each of these measures?

## Conclusions


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