A family of accessibility measures derived from spatial interaction principles

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Abstract

Transportation planning has long prioritized the efficiency of movement, or mobility. However, the concept of accessibility represents a more comprehensive evolution, shifting focus from mere movement to the potential to reach (i.e., spatially interact) with desired destinations. Despite growing recognition of accessibility-based planning approaches, the concept remains fragmented, with inconsistent definitions and unclear interpretations. This work's aim is to clarify and unify the concept of accessibility by connecting it into spatial interaction modeling. We demonstrate that widely used mobility and accessibility models, such as gravity-based accessibility and spatial interaction models, share common theoretical roots. From this foundation, this paper offers three contributions: (A) we introduce a family of accessibility measures within the principles of spatial interaction, and (B) formally define four members of the family, namely the 'unconstrained' measure (i.e., Hansen-type accessibility), the 'total constrained' measure (i.e., a constrained version of the Hansen-type accessibility), the 'singly constrained' measure (i.e., related to the popular two step floating catchment approach - 2SFCA), and the 'doubly constrained' measure representing realized interactions or 'access', effectively equal to the doubly constrained spatial interaction model; and (C) we demonstrate the interpretability advantages of the family, as these constrained accessibility measures yield values in units of the number of potential "opportunities for spatial interaction" or "population for spatial interaction" for each zone and zonal flow. The family of accessibility measures proposed here clarifies the concept of 'potential' in accessibility, demonstrates theoretical and formulaic linkages across popular accessibility and spatial interaction models, and reintroduces measurement units into accessibility measures. By doing so, we believe this family of measures can unlock a clearer, more interpretable, and cohesive foundation for accessibility analysis.

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Introduction

In the early nineteenth century, industrializing cities grew rapidly: so did traffic congestion and the development of transportation planning practice focused primarily on mobility. In this newly founded practice, access to destinations was treated as a by-product of movement. Following the rise of the automobile and significant investments in transportation infrastructure after World War II, this mobility-oriented approach has led to some problematic outcomes. Specifically, the car became seen as the ultimate mobility tool, helping to foster the development of low-density, single-use residential neighborhoods and entrenching an automobility mono-culture within transportation systems (H. J. Miller 2011; Lavery, Páez, and Kanaroglou 2013). Decades of planning for this automobility mono-culture, still often characterized by road and highway expansion, have been marked by increased travel costs and environmental burdens, with limited impact on enhancing the ease with which people can reach destinations (Steven Farber and Páez 2011; S. Handy 2002; Páez et al. 2010). In response, transportation researchers have increasingly advocated for the adoption of accessibility as a planning criterion, in contrast to traditional mobility-oriented transportation planning approaches which translate into indicators that benchmark movement (e.g., vehicle kilometres traveled, intersection through traffic, etc.) which are not necessarily linked to improved accessibility (Silva et al. 2017; Paez et al. 2013; S. Handy 2020; El-Geneidy and Levinson 2022).

Accessibility may be defined as the "potential of opportunities for [spatial] interaction" (Hansen 1959), in contrast to mobility which is simply, movement in itself or 'spatial interaction'. Mobility has been the basis of traditional transportation planning approaches (Ortúzar and Willumsen 2011). While accessibility brings a more holistic understanding of combined transportation and land use systems, incorporating both the explicit consideration of destination as opportunity and the concept of *potential* (S. L. Handy and Niemeier 1997).

The growing interest in accessibility has been accompanied by a boom in scholarly research using different methods and focusing on different research contexts; it has grown to include studies of access to employment (e.g., Karst and Van Eck 2003; Grengs 2010; Páez et al. 2013; Merlin and Hu 2017; Tao et al. 2020), health care (e.g., Luo and Wang 2003; Páez et al. 2010; Wan, Zou, and Sternberg 2012; Delamater 2013; Boisjoly, Moreno-Monroy, and El-Geneidy 2017; Pereira et al. 2021), green spaces (Reyes, Paez, and Morency 2014; Rojas et al. 2016; Liang, Yan, and Yan 2024), schools (e.g., Williams and Wang 2014; Romanillos and Garcia-Palomares 2018; Marques, Wolf, and Feitosa 2021), social contacts (e.g., Neutens et al. 2007; S. Farber, Páez, and Morency 2012; S. Farber et al. 2013), and regional economic analysis (e.g., R. Vickerman, Spiekermann, and Wegener 1999; Lopez, Gutierrez, and Gomez 2008; Ribeiro, Antunes, and Páez 2010; Gutierrez et al. 2011) among many other domains of application. In other words, accessibility analysis is used today to broadly understand the potential to reach (or spatially interact with) opportunities that are important to people (Ferreira and Papa 2020). However, despite its growth in popularity in scholarly works, challenges remain with respect to the more widespread adoption of accessibility in planning practice. For instance, the diversity of accessibility definitions has been flagged by van Wee (2016), S. Handy (2020), and Kapatsila et al. (2023). Further, difficulties in the interpretability and communicability of outputs has also been noticed by many authores, including Karst T. Geurs and van Wee (2004), van Wee (2016), and Ferreira and Papa (2020).

The adoption of accessibility in planning practice is not necessarily made easier when potential adopters have to contend with a plethora of definitions, each seemingly more sophisticated but less intuitive than the last (Kapatsila et al. 2023). At a high level, Karst T. Geurs and van Wee (2004) identify four families of accessibility measures: infrastructure-, place-, person-, and utility-based. Of the place-based family, which is the focus of this paper, the menu has grown to include gravity-based accessibility (e.g., Hansen 1959; Pirie 1979), cumulative opportunities (e.g., Wachs and Kumagai 1973; Pirie 1979; Ye et al. 2018), modified gravity (e.g., Schuurman, Berube, and Crooks 2010), 2-Step Floating Catchment Areas (e.g., Luo and Wang 2003), Enhanced 2-Step Floating Catchment Areas (e.g., Luo and Qi 2009), 3-Stage Floating Catchment Areas (e.g., Wan, Zou, and Sternberg 2012), Modified 2-Step Floating Catchment Areas (e.g., Delamater 2013), inverse 2-Step Floating Catchment Areas (e.g., F. Wang 2021), and n-steps Floating Catchment Areas (Liang, Yan, and Yan 2024). How is a practitioner to choose among this myriad options? What differences in accessibility scores should matter, and how should they be communicated? [see van Wee (2016); p. 14].

Here, we seek to address the breath of place-based accessibility's definitions and lack of interpretability by demonstrating that mobility-oriented models–such as the commonly used gravity-based accessibility (e.g., Hansen 1959) and the spatial interaction model (e.g., Wilson 1971)– are rooted in the same spatial interaction modeling foundation. In fact, we propose that accessibility can be specified using spatial interaction principles as a *family of measures*, akin to the family of spatial interaction models introduced in Wilson (1971) that can be defined using balancing factors that constrains values based on known information about the land-use.

This work aims to offer three contributions: (1) it introduces a family of accessibility measures within the principles of spatial interaction; (2) it formally defines three *constrained* accessibility measures by reintroducing Wilson-analogous balancing factors. These measures are the total constrained accessibility measure, the singly constrained accessibility measure, and the doubly constrained measure, which are explicitly connected to popular measures such as the Hansen-type accessibility (Hansen 1959), the popular competition approach of the 2-Step Floating Catchment Area (2SFCA) (Q. Shen 1998a; Luo and Wang 2003), and the concept of market potential (C. D. Harris 1954; R. W. Vickerman 1974). These Wilson-analogous balancing factors introduced are also defined in order of increasing restrictiveness to shed light on the role of *potential* in accessibility and access. (3) This work also demonstrates that the introduction of balancing factors makes accessibility measures easier to interpret and to communicate by restoring the measurement units to the resulting raw accessibility values. Each zonal and zonal flow value from a constrained accessibility measure is always in interpretable units, namely the number of 'opportunities for spatial interaction' or 'population for spatial interaction'. This is in contrast to conventional accessibility measures, particularly gravity based measures, that yield values in units of 'opportunities weighted by some representation of travel friction'.

To achieve these stated objectives, this paper contends that accessibility research must reconnect with its spatial interaction origins. Particularly, we argue that an important aspect of spatial interaction modelling–namely, constraining the results to match empirical observations–was never effectively reincorporated into accessibility analysis. Empirical constraints were embraced by early spatial interaction literature following the work of Wilson (1971), but this stream of literature tended to flow separately from research inspired by Hansen (1959)' accessibility. The application of Wilson (1971)'s empirical constraints supported the development of various spatial interaction models that remain relevant in research and practice today (Ortúzar and Willumsen 2011). However, the same cannot be said of the contemporary accessibility literature, where empirical constraints were not explicitly adopted. We argue that the absence of empirical constraints (and their attendant proportionality constants) has contributed to some of accessibility analysis' interpretability issues; for instance, the fuzziness of insights beyond simple proportional statements like 'higher-than' or 'lower-than' (E. J. Miller 2018). Moreover, without constraints we lose track of what accessibility measures, that is, the number of opportunities that can be spatially interacted with. Without a clear sense of measurement units, comparability between accessibility measures, across cities, and transport modes, may also become compromised.

Fortunately, accessibility and spatial interaction modelling literature share common headwaters, and the latter has given careful attention to measurement units and their interpretability. It is by looking to the past that we believe accessibility analysis can newly wade into the future. We continue this work in the following section by tracing the development of accessibility from its origins in spatial interaction: from the Newtonian gravitational expression in Ravenstein (1889) through to the seminal accessibility work of Hansen (1959). We then present evidence for a narrative highlighting the marked divergence between accessibility and spatial interaction modelling research after the work of Wilson (1971). Next, we hark back to Wilson (1971)'s spatial interaction models, and use them to derive a family of accessibility measures based on analogous constraints. We illustrate members of this family of constrained accessibility measures with a simple numerical example. We then conclude by discussing the uses of these measure and their interpretation.

Newtonian's roots of human spatial interaction research

The patterns of people's movement in space have been a subject of scientific inquiry for at least a century and a half. From as far back as Henry C. Carey's *Principles of Social Science* (Carey 1858), a concern with the scientific study of human spatial interaction can be observed. It was in this work where Carey stated that "man [is] the molecule of society [and their interaction is subject to] the direct ratio of the mass and the inverse one of distance" (McKean 1883, 37–38). This statement shows how investigations into human spatial interaction have often been explicitly coloured by the features of Newton's Law of Universal Gravitation, first posited in 1687's *Principia Mathematica* and expressed as in Equation 1.

$$F_{ij} \propto \frac{M_i M_j}{D_{ij}^2} \tag{1}$$

To be certain, the expression above is one of proportionality, and is also the most famous in all of science. It states that the force of attraction F between a pair of bodies i and j is directly *proportional* to the product of their masses M_i and M_j, and inversely *proportional* to the square of the distance between them D_{ij}. Direct proportionality means that as the product of the masses increases, so does the force. Likewise, inverse proportionality means that as the distance increases, the force decreases. Equation 1, however, does not quantify the magnitude of the force. To do so, an empirical constant, a.k.a. the gravitational constant, is required to convert the proportionality into an equality, ensuring that values of the force F in Equation 1 match the observed force of attraction between masses. In other words, Equation 1 needs to be *constrained* using empirical data. Ultimately, the equation for the force is as seen in Equation 2, where G is an empirically calibrated proportionality constant:

$$F_{ij} = G\frac{M_i M_j}{D_{ij}^2} \tag{2}$$

Newton's initial estimate of G was based on a speculation that the mean density of earth was between five or six times that of water, an assumption that received support after Hutton's experiments of 1778 (Hutton 1778, 783). Still, it took over a century from the publication of *Principia* to refine the estimate of the proportionality constant to within 1% accuracy, with Cavendish's 1798 experiment (Cavendish 1798).

Early research on human spatial interaction: from Ravenstein (1889) to Stewart (1948)

Since the 1880s to the 1940s, a number of researchers theoretically and empirically attempted to characterise human spatial interaction as some force of attraction F that is directly proportion to the masses M_i and M_j and inversely proportional by their separation distance. This concept was captured with different expressions, but all tie back to the same Newtonian gravity analogy, although not all of them included a proportionality constant in their formulation.

Following Carey's *Principles* of 1858, research into human spatial interaction continued in different contexts. In the late 1880s, Ravenstein proposed some "Laws of Migration" based on his empirical analysis of migration flows in various countries (Ravenstein 1885, 1889). In these works, Ravenstein posited 1) a directly proportional relationship between migration flows and the size of destinations (i.e., centres of commerce and industry), and 2) an inversely proportional relationship between the size of flows and the separation between origins and destinations. As with Carey, these propositions echo Newton's gravitational laws. Over time, other researchers discovered similar relationships. For example, Reilly (1929) formulated a law of retail gravitation, expressed in terms of equal attraction to competing retail destinations that could be understood as 'potential trade territories'. Later, Zipf proposed a $\frac{P_1 P_2}{D}$ hypothesis for the case of information (Zipf 1946a), intercity personal movement (Zipf 1946b), and goods movement by railways (Zipf 1946c). The $\frac{P_1 P_2}{D}$ hypothesis stated that the magnitude of flows was proportional to the product of the populations of settlements, and inversely proportional to the distance between them.

A common feature of these early investigations of human spatial interaction is that a proportionality constant similar to G in Equation 2 was never considered. Of the researchers cited above, only Reilly and Zipf expressed their hypotheses in mathematical terms. Reilly's hypothesis was presented in the following form:

$$B_a = \frac{(P_a \cdot P_T)^N}{D_{aT}^n} \tag{3}$$

where B_a is the amount of business drawn to a from T, P_a and P_T are the populations of the two settlements, and D_{aT} is the distance between them. Quantity N was chosen by Reilly in a somewhat *ad hoc* fashion as 1, and he used empirical observations of shoppers to choose a value of $n = 2$.

Zipf, on the other hand, wrote his hypothesis in mathematical form as:

$$C^2 = \frac{P_1 \cdot P_2}{D_{12}} \tag{4}$$

where C is the volume of goods exchanged between 1 and 2, P_1 and P_2 are the populations of the two settlements, and D_{12} is the distance between them.

After Carey, it is in Stewart's work on the principles of demographic gravitation that we find the strongest connection yet to Newton's law (Stewart 1948). This may relate to academic backgrounds; where Stewart was a physicist while Ravenstein, Reilly, and Zipf were social scientists. Besides awareness of preceding research (he cites both Reilly and Zipf as predecessors in the analysis of human spatial interaction), Stewart appears to have been the first author to express his theorized relationships for human spatial interaction with a proportionality constant G, as follows:

$$F = G\frac{(\mu_1 N_1)(\mu_2 N_2)}{d_{12}^2} = G\frac{M_1 \cdot M_2}{d_{12}^2} \tag{5}$$

Where:

- F is the *demographic force*
- N_1 and N_2 are the numbers of people of in groups 1 and 2
- μ_1 and μ_2 are so-called *molecular weights*, the attractive weight of groups 1 and 2
- $M_1 = \mu_1 N_1$ and $M_2 = \mu_2 N_2$ are the demographic masses at 1 and 2
- d_{12}^2 is the distance between 1 and 2
- And finally G, a constant that Stewart "left for future determination" (1948, 34)

In addition to demographic force, Stewart defined a measure of the "population potential" of group 2 with respect to group 1. In other words, the potential number of people from location 2 that could visit location 1, as follows:

$$V_1 = G\frac{M_2}{d_{12}} \tag{6}$$

For a system with more than two population bodies, Stewart formulated the population potential at i as follows (after arbitrarily assuming that $G = 1$):

$$V_i = \int \frac{D}{r} ds \tag{7}$$

where D is the population density over an infinitesimal area ds and r is the distance to i. In Equation 7, $D \cdot ds$ gives an infinitesimal count of the population, say dm, and so, after discretizing space, Equation 7 can be rewritten as:

$$V_i = \sum_j M_j d_{ij}^{-1} \tag{8}$$

Alerted readers will notice that Equation 8, with some re-organization of terms, is formally equivalent to our modern definition of accessibility popularized by Hansen (1959) in the late 1950s.

Stewart's formulation of demographic force, developed in the context of what he called "social physics" (Stewart 1947), was problematic. It had issues with inconsistent mathematical notation. More seriously though, Stewart's work was permeated by a view of humans as particles following physical laws, but tinted by unscientific and racist ideas. For instance, he assumed that the molecular weight μ of the average American was one, but "presumably…much less than one….for an Australian aborigine" [p. 35]. Stewart's ideas about "social physics" soon fell out of favour among social scientists, but not before influencing the nascent field of accessibility research, as detailed next.

Hansen's gravity-based accessibility to today

From Stewart (1948), we arrive to 1959 and Walter G. Hansen, whose work proved to be exceptionally influential in the accessibility literature (Hansen 1959). In this seminal paper, Hansen defined accessibility as "the potential of opportunities for interaction… a generalization of the population-over-distance relationship or *population potential* concept developed by Stewart (1948)" (p. 73). As well as being a student of city and regional planning at the Massachusetts Institute of Technology, Hansen was also an engineer with the Bureau of Roads, and preoccupied with the power of transportation to shape land uses in a very practical sense. Hansen (1959) focused on Stewart (1948)'s *population potential* (expressed in Equation 8), and not on the other formulaic contributions and objectionable aspects of "social physics". Hansen (1959) recast Stewart's population potential to reflect accessibility, a model of human behaviour useful to capture regularities in mobility patterns. Hansen (1959) replaced M_j in Equation 8 with *opportunities* to derive an *opportunity potential*, or more accurately, a *potential of opportunities for interaction* as follows:

$$S_i = \sum_j \frac{O_j}{d_{ij}^{\beta}} \tag{9}$$

A contemporary rewriting of Equation 9 accounts for a variety of impedance functions beyond the inverse power $d^{-\beta}$:

$$S_i = \sum_j O_j \cdot f(d_{ij}) \tag{10}$$

S_i in Equation 9 is a measure of the accessibility from site i. This is a function of O_j (the mass of opportunities at j), d_{ij} (the cost, e.g., distance or travel time, incurred to reach j from i), and β (a parameter that modulates the friction of cost). Today, Hansen is frequently cited as the father of modern accessibility analysis (e.g., Reggiani and Martín 2011), and Hansen-type accessibility is commonly referred to as the gravity-based accessibility measure.

However, Hansen's use of Stewart's *population potential* measure included one crucial omission that afflicts the literature to this day. The omission is that between Stewart (1948) and Hansen (1959), the proportionality constant G in Equation 6 vanished. This constant was not explicitly addressed in Hansen (1959) and accessibility research continues to evolve without it. Since Hansen (1959), accessibility analysis has been widely used in numerous disciplines but, to our knowledge, the proportionality constant has remained forgotten, with no notable developments to explicitly acknowledge or determine it.

The omission of this constant generates a fundamental problem for the measurement unit of accessibility estimates, which undermines the interpretation, communication and comparability of accessibility analysis. Those reading Hansen (1959) must recall that Stewart (1948) had set the proportionality constant G to 1, with a note that "G [was] left for future determination: a suitable choice of other units can reduce it to unity" [p. 34]. In practice, the persistent omission of the constant in accessibility analyses means that G continues to be implicitly set to 1, even when the fundamental relationship in accessibility is proportionality (e.g., $S_i \propto \sum_j g(O_j)f(d_{ij})$) and not equality (for instance, see the formula for accessibility at the top of Figure 1 in Wu and Levinson 2020). The direct consequence is that without a proportionality constant, the units of S_i remain unclear: the unit of "potential of opportunity for interaction" is left free to change as β is calibrated. For example, if c_{ij} is distance in meters, it will be number of opportunities per m^{β} when $f(c_{ij}) = d^{-\beta}$ but number of opportunities per $e^{-\beta \cdot m}$ when $f(c_{ij}) = e^{-\beta \cdot m}$. This undermines the comparability of accessibility metrics with different decay functions, and renders their results difficult to understand and communicate. The Hansen-style accessibility estimates found in the literature, therefore, are better thought of as ordinal measures of potential that can only be interpreted in terms of higher and lower accessibility, but which has not palpable meaning (E. J. Miller 2018).

Wilson's family of spatial interaction models

On the other side of the Atlantic, Alan G. Wilson was developing related, yet parallel work. In his groundbreaking study (Wilson 1971), Wilson defined a general spatial interaction model as follows. While accessibility was characterised as an associated concept of 'potential', the primary focus was on modelling observed spatial interaction:

$$T_{ij} = kW_i^{(1)}W_j^{(2)}f(c_{ij}) \tag{11}$$

The model in Equation 11 posits a quantity T_{ij} that represents a value in a matrix of flows of size $n \times m$, that is, between $i = 1, \cdots, n$ origins and $j =$

$1, \cdots, m$ destinations. The quantities $W_i^{(1)}$ and $W_j^{(2)}$ are proxies for the masses at $i = 1, \cdots, n$ origins and $j = 1, \cdots, m$ destinations. The super-indices (1) and (2) are meant to indicate that these masses could be different things, i.e., $W_i^{(1)}$ could be populations, and $W_j^{(2)}$ hectares of park space. Finally, $f(c_{ij})$ is some function of travel cost c_{ij} which reflects travel impedance. In this way, T_{ij} explicitly measures *interaction* in the unit of trips, and the role of k is to ensure that the system-wide sum of T_{ij} represents the total flows in the data. In other words, k is a scale parameter that makes the overall amount of flows identical to the magnitude of the phenomenon being modeled. In other words, it balances the units, in a conceptually similar sense as the gravitational constant in Newton's Law of Universal Gravitation.

Traditionally, the development of the spatial interaction model put an emphasis on the interpretability of the results (Kirby 1970; Wilson 1967, 1971). But instead of relying on the heuristic of Newtonian gravity (e.g., some interaction between a mass at i and a mass at j separated by some distance), Wilson's approach was to maximise the entropy of the system. Entropy maximisation in this case achieves stable results as a statistical average that represents the population. The approach works by assuming undifferentiated individual interactions, and assessing their probabilities of making a particular journey. The result of Equation 11 then is a statistical average (Wilson 1971; Senior 1979).

To ensure that T_{ij} in Equation 11 is in the unit of trips (our unit of origin-destination spatial interaction), additional knowledge about the system is required. At the very least, this framework assumes that the total number of trips in the system T is known, and therefore:

$$\sum_i \sum_j T_{ij} = T \tag{12}$$

Additional information can be introduced. For example, when information is available about the total number of trips produced by each origin, $W_i^{(1)}$ is represented as O_i and the following constraint can be used:

$$\sum_j T_{ij} = O_i \tag{13}$$

Alternatively, if there is information available about the total number of trips attracted by each destination, $W_j^{(2)}$ is represented as D_j and the following constraint can be used:

$$\sum_i T_{ij} = D_j \tag{14}$$

It is also possible to have information about both O_i and D_j, in which case both constraints could be imposed on the model at once.

Using information about the system that satisfy these constraints fully, partially or not at all, a family of spatial interaction models can be derived based on Equation 11. K is specified depending on the applied constraint(s). In the framework introduced in Wilson (1971), three constrained versions are specified: the first being a case where to results only match the total volume of interaction, the second being a singly constrained case, and the third a doubly constrained case.

In the first, Equation 13 and Equation 14 do not hold. In practical terms, this means that the total number of trips predicted by the model must be equal to sum of all flows from origins i to destinations j. The balancing constant K in this case is (see Cliff, Martin, and Ord 1974; A. S. Fotheringham 1984):

$$K = \frac{T}{\sum_i \sum_j T_{ij}} \tag{15}$$

In the second case, only one of Equation 13 or Equation 14 hold. The resulting models are, in Wilson's terms, singly constrained. When only Equation 13 holds, entropy maximisation leads to the following production-constrained model:

$$T_{ij} = A_i O_i W_j^{(2)} f(c_{ij}) \tag{16}$$

Notice how, in this model, the proxy for the mass at the origin $W_i^{(1)}$ is replaced with O_i, representing what we know about the system, the spatial interaction outbound flow, i.e., outbound trips produced at i. Also, there is no longer a single system-wide proportionality constant, but rather a set of proportionality constants (i.e., balancing factors) specific to origins. For this model, the balancing factors ensure that Equation 13 is satisfied, meaning that the sum of predicted flows from one origin going to all destinations must equal the known mass at that origin O_i i.e., the total number of outbound trips. Satisfying this constraint also implicitly fulfills the total constraint (Equation 12), since the sum of O_i values across all origins equals the total number of trips. This model is useful when trips ends are unknown but the number of trips originating from each location is known and the total of these trips represents all trips in the system. The balancing factors for the production-constrained model are solved for each origin A_i, and according to Wilson are:

$$A_i = \frac{1}{\sum_j W_j^{(2)} f(c_{ij})} \tag{17}$$

The attraction-constrained model is similar to the production-constrained model as it is also singly constrained but from the perspective of the mass at the destination. From the attraction-constrained model, the proxy for the mass at the destination $W_j^{(2)}$ is now replaced with D_j, representing the spatial interaction inbound flow, i.e., trips attracted at the destination and takes the following form:

$$T_{ij} = B_j D_j W_i^{(1)} f(c_{ij}) \tag{18}$$

In this model (Equation 18), the balancing factors ensure that Equation 14 is satisfied (hence the total constraint Equation 12 is as well), meaning that the sum of predicted flows going to one destination from all origins must equal the known mass of that destination D_j i.e., the total number of inbound trips to j. This should hold for all destinations. As before, destination-specific proportionality constants (i.e., balancing factors) B_j were derived by Wilson as:

$$B_j = \frac{1}{\sum_i W_i^{(1)} f(c_{ij})} \tag{19}$$

The third case in the family of spatial interaction models is the production-attraction constrained model. In this case, both Equation 13 and Equation 14 hold simultaneously. These constraints ensure that the sum of predicted flows from one origin to all destination, and the predicted flows going to one destination from all origins must equal the known mass of the origin O_i and of the destination D_j. This should hold for all origins and destinations. The resulting model is, in Wilson's terms, doubly constrained, and takes the following form:

$$T_{ij} = A_i B_j O_i D_j f(c_{ij}) \tag{20}$$

In this model, both proxies for the masses are replaced with the known masses, that is, the trips produced by the origin and the trips attracted by the destination. There are now two sets of mutually dependent proportionality constants:

$$\begin{aligned} A_i &= \tfrac{1}{\sum_j B_j D_j f(c_{ij})} \\ B_j &= \tfrac{1}{\sum_i A_i O_i f(c_{ij})} \end{aligned} \tag{21}$$

Derivation of these models is demonstrated in detail elsewhere (e.g., Ortúzar and Willumsen 2011; Wilson 1967). It is worth noting, however, that although Wilson's approach is built on a different conceptual foundation than the old reference to Newtonian gravity, the work succeeded at identifying the steps from proportionality to equality to yield variations of proportionality constants, including the one that eluded Stewart (1948) and that has been overlooked in almost all subsequent accessibility research. Why was this key element of spatial interaction models potentially ignored in accessibility research? In the next section we aim to address this question.

Accessibility and spatial interaction modelling: two divergent research streams

The work of Hansen (1959) and Wilson (1971) responded to important developments, in particular a need "to meet the dictates and needs of public policy for strategic land use and transportation planning" (Michael Batty 1994). These dictates and needs were far from trivial. In the United States alone, the Federal-Aid Highway Act of 1956 authorized the creation of the U.S. Interstate Highway System, with a budget that ultimately exceeded one hundred billion dollars (equivalent to over $600 billion in 2023) (Weiner 2016; MDOT 2007). Spatial interaction modelling was incorporated into institutional modelling practices meant to "predict and provide", i.e., predict travel demand and supply transportation infrastructure (Kovatch, Zames, et al. 1971; Weiner 2016). Accessibility, at the time, did not quite have that power, as it did not quantify trips, but rather something somewhat more elusive: it predicts the less tangible "potential" for spatial interaction with opportunities. In this way, where spatial interaction modelling became a key element of transportation planning practice, accessibility remained a somewhat more academic pursuit, and the two streams of literature only rarely connected.

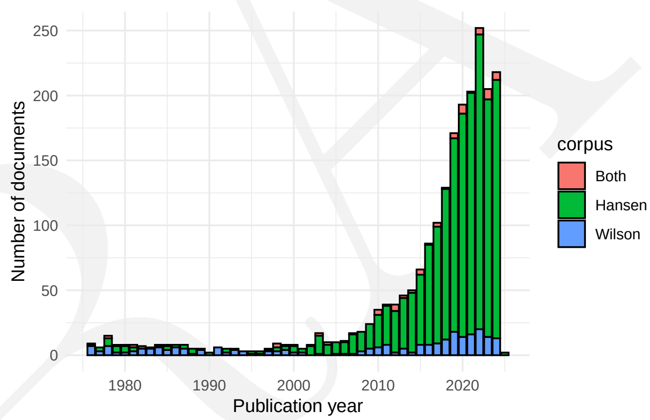

Figure 1: Historical pattern of publication: documents per year.

To illustrate this point, we conducted a bibliographic analysis of the literature that cites Hansen (1959), Wilson (1971), or both. We retrieved all relevant documents using the Web of Science "Cited References" functionality, and the digital object identifiers of Hansen (1959) and Wilson (1971). As a result of this search, we identified 1,788 documents that cite Hansen (1959), only 258 documents that cite Wilson (1971), and 76 that cite both. The earliest document in this corpus dates to 1976 and the most recent is from 2025. The number of documents per year appears in Figure 1, where we see the frequency of documents over a span of almost fifty years. In particular, we notice the remarkable growth in the number of papers that cite Hansen (1959) compared to those that cite Wilson (1971) since the year 2000.

As noted, literature that cites both Wilson (1971) and Hansen (1959) are sparse (only 3.6% of the corpus, visualised in pink in Figure 1). After reading the works, we can also discern that they too are divergent, with one stream focused on developing accessibility (i.e., *potential for* spatial interaction) and another on spatial interaction. The focuses of these divergent streams contribute this paper's broader hypothesis that the concepts of accessibility and spatial interaction have remained largely disconnected, and at times, improperly conflated.

On one hand, the stream of literature focused on spatial interaction models inspired by Wilson (1971) and which cite both Hansen (1959) and Wilson (1971), tends to contribute to understanding how accessibility is interpreted and incorporated in spatial interaction models. These works treat separate spatial interaction and accessibility as a seperate but related phenomenon. Specifically, some early works interpret the spatial interaction model's balancing factors (Equation 17 or Equation 21) as the inverse of Hansen (1959)'s model (B. Harris and Wilson 1978; G. Leonardi 1978; A. Stewart Fotheringham 1981; A. S. Fotheringham 1985), recognizing it as a "common sense" approach (Morris, Dumble, and Wigan 1979, 99) to including accessibility in the spatial interaction model, though further exploration of its relationship is warranted (M. Batty and March 1976). Some authors have explored this relationship, for instance as in A. S. Fotheringham (1985) who demonstrates how the spatial interaction model may insufficiently explain spatial patterns, and suggests that explicitly defining destinations' accessibility as a variable within the model may remedy the issue (e.g., the *competition destination* model). Other works used both Hansen (1959) and Wilson (1971)'s framework in conjunction, such as in defining location-allocation problems in operations research (G. Leonardi 1978; Beaumont 1981), estimating trip flows (or some other spatial interaction flows) alongside accessibility (e.g., Clarke, Eyre, and Guy 2002; Grengs 2004; Türk 2019), or considering accessibility within spatial interaction models, in line with A. S. Fotheringham (1985)'s demonstration (e.g., Beckers et al. 2022). Other works departed from Hansen (1959)'s definition and aligned with spatial interaction in different ways, such as using micro-economic consumer behaviour concepts to express potential for spatial interaction (Morris, Dumble, and Wigan 1979; Giorgio Leonardi and Tadei 1984).

On the other hand, there is another subset of literature that cite both Hansen (1959) and Wilson (1971) that is accessibility-focused. We categorise their citation of Wilson (1971) for three general reasons. Firstly, a group of these works cite Wilson (1971) as attribution for using context-dependent travel cost functions (e.g., Weibull 1980; S. L. Handy and Niemeier 1997; Kwan 1998; Q. Shen 1998a; Ashiru, Polak, and Noland 2003; Rau and Vega 2012; Pan 2013; Margarida Condeço Melhorado et al. 2016; Caschili, De Montis, and Trogu 2015; Grengs 2015; Pan, Jin, and Liu 2020; Chia and Lee 2020; Roblot et al. 2021; Sharifiasl, Kharel, and Pan 2023; Kharel, Sharifiasl, and Pan 2024). These works do not necessarily comment on spatial interaction explicitly. Secondly, another group of this accessibility-focused "both" citing literature *does* associate spatial interaction as defined in Wilson (1971) with accessibility's potential for spatial interaction more explicitly (e.g., H. J. Miller 1999; Giuliano et al. 2010; Grengs et al. 2010; Grengs 2010, 2012; Levine et al. 2012; Levinson and Huang 2012; Tong, Zhou, and Miller 2015; X. Liu and Zhou 2015; He et al. 2017; Wu and Levinson 2020; Ng et al. 2022; Naqavi et al. 2023; Suel et al. 2024). We agree accessibility and spatial interaction are related topics: accessibility is an expression of its *potential* and the Wilson (1971) paper briefly touches on the concept. However, in some of this literature, Hansen (1959) and Wilson (1971) are co-cited as both being 'gravity models' (e.g., S. Liu and Zhu 2004; Dai, Wan, and Gai 2017; Y. Shen 2019; Chia and Lee 2020), perhaps revealing the murkiness of the distinction between spatial interaction and the *potential for* spatial interaction in the literature. Thirdly, there is a group of accessibility-focused works that interpret Hansen (1959)'s model as the singly- or doubly- constrained spatial interaction model's inverse balancing factor (e.g., R. W. Vickerman 1974). This group cities the earlier spatial interaction works that make this connection and is especially prominent in the investigation of competitive accessibility topics (e.g., Karst and Van Eck 2003; Karst T. Geurs, van Wee, and Rietveld 2006; Willigers, Floor, and van Wee 2007; El-Geneidy and Levinson 2011; Curtis and Scheurer 2010; Manaugh and El-Geneidy 2012; Chen and Silva 2013; Alonso et al. 2014; Albacete et al. 2017; Sahebgharani, Mohammadi, and Haghshenas 2019; Mayaud et al. 2019; Allen and Farber 2020; Levinson and Wu 2020; Marwal and Silva 2022; Su and Goulias 2023). As outlined in preceding sections, we argue interpreting the singly- or doubly-constrained spatial interaction model's balancing factor as accessibility yields output values that are similarly plagued by interpretability issues.

Lastly, as an extension of the third reason within this group, only the works of Soukhov et al. (2023) and Soukhov et al. (2024) use Wilson (1971)'s balancing factors as a method for maintaining constraints on opportunities within the context of competitive accessibility. These works introduce the balancing factors as a mechanisms to ensure that opportunities at each destination are proportionally allocated to each zone (based on the proportion of population seeking opportunities and the relative travel impedance). This is to ensure that each zonal accessibility value is the sum of this proportional allocation from each destination, and that all zonal values ultimately sum to the total number of opportunities in the region. However, these balancing factors were deduced intuitively. These works do not explicitly state that the mathematical formulation of the equations are effectively equivalent to Wilson's singly constrained model (derived from entropy maximization). This equivalence is only discovered in hindsight, as will be demonstrated in this study. These two works also do not discuss other constrained cases that will be addressed in the next section.

A family of accessibility measures

In this section, we introduce a family of accessibility measures inspired by Wilson's spatial interaction modelling framework. At the end of this section, Table 1 presents a summary, where each member of the family of accessibility measure is explained in plain language, alongside their balancing factor, proportional allocation factor and mathematical equation and value interpretations.

As argued in the preceding section, the streams of research on accessibility and spatial interaction modelling have evolved as largely separate streams with little contact since Hansen (1959) and Wilson (1971). This may explain why the constraints and associated balancing factors of spatial interaction models did not cross over to accessibility analysis. This is intriguing since Wilson made an effort to connect developments in spatial interaction modelling to accessibility, noting for instance, the denominator of the proportionality constants specific to origins (Equation 17) is the inverse of balancing factor A_i (Wilson 1971, 10):

$$S_i = \frac{1}{A_i} = \sum_j W_j^{(2)} f(c_{ij}) \tag{22}$$

Understanding S_i as the inverse of Wilson's A_i does not uncover any new meaning for S_i itself. Indeed, mathematically it is true, A_i's role in Wilson's general model $T_{ij} = kW_i^{(1)} W_j^{(2)} f(c_{ij})$ is that of a balancing factor k i.e., keeping units balanced and proportionality based on constraints. Understanding S_i itself as a balancing factor is not wholly helpful as Hansen and (Stewart before him) defined accessibility as a partial sum of the demographic force F or the system-wide population potential of opportunities for spatial interaction (i.e., (**q-stewart-population-potential-sum?**) representing $V_i = \sum_j \frac{M_j}{d_{ij}}$ after losing G).

Hence, we propose stepping back to introduce a revised definition of accessibility: the *constrained* potential for spatial interaction. Once we bring back Wilson's proportionality constant k into the picture, we can define the *potential for spatial interaction* between two locations i and j is as follows:

$$V_{ij} = kW_j^{(2)} f(c_{ij}) \tag{23}$$

where V_{ij} is the potential for interaction from i to j. The accessibility from origin i can then be summarised as as a partial sum of the potential at i:

$$V_i = k \sum_j W_j^{(2)} f(c_{ij}) \tag{24}$$

Similar to Equation 11, $W_j^{(2)}$ above is the mass at the destination and the sub-indices are for $i = 1, \cdots, n$ origins and $j = 1, \cdots, m$ destinations.

The market potential variant can also be generally defined, which is effectively the transpose of i and j in Equation 23 and Equation 24 as follows:

$$M_{ji} = kW_i^{(1)} f(c_{ij}) \tag{25}$$

$$M_j = k \sum_j W_i^{(1)} f(c_{ji}) \tag{26}$$

Similar to Equation 11, $W_j^{(2)}$ above is the mass at the destination and the sub-indices are for $i = 1, \cdots, n$ origins and $j = 1, \cdots, m$ destinations.

To detail the anatomy of V_{ij} and M_{ji} along with the partial sums of V_i and M_j, Figure Figure 2 illustrates the accessibility analytical framework we propose using a simple 3 zone system. Each measure's most disaggregate value is X_{ij}, potential for spatial interaction from i to j. The X is a stand in for the ij values of all the cases and their variants that will be described (i.e., $V_{ij}^0, M_{ji}^0, V_{ij}^T, M_{ji}^T, V_{ij}^S, M_{ji}^S$ and V_{ij}^D, M_{ji}^D). The single marginals represent the origin-side and destination-side weights of the zones. The total marginal represent the sum of a single marginal.

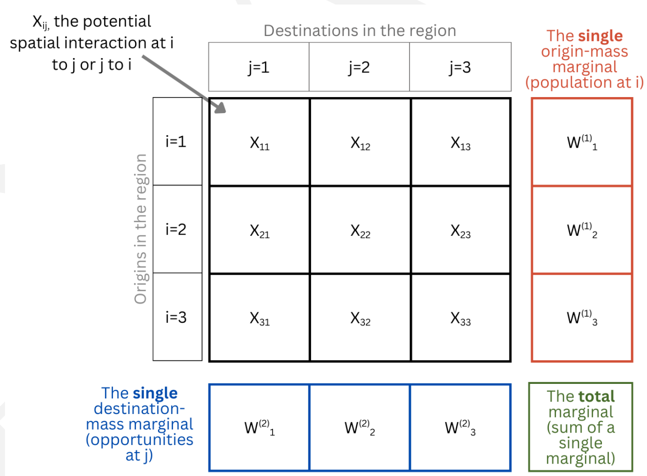

Figure 2: The family of accessibility measures analytical framework: labelling and associating ij flows, zonal weights, the single marginals, and the total marginal.

As an additional overview, we define four distinct members of the constrained accessibility measure family, all delineated based on their constant k, which takes the form of either balancing factors K^T, B_j, A_i depending on the indicator:

1. **The unconstrained case** with variants V_i^0 and M_i^0. This case is equivalent to Hansen (1959)'s formulation (first variant) and Reilly (1929)'s

market potential formulation (second variant), whereby a balancing factor is neglected, so units of zonal values are in units of 'opportunities-by-travel-impedance-value' and 'population-by-travel-impedance-value'. In this case, no constraints are introduced to ensure that values of marginals are preserved.

2. **The total constrained case** with variant V_i^T and variant M_j^T. The first variant V_i^T–the *total constrained accessible opportunities measure*–resembles Hansen (1959)'s formulation but with an additional regional balancing factor K^T term, defined as the ratio of the total number of opportunities in the region to the total sum of unconstrained accessibility values in the region. In this way, K^T ensures that each zonal accessibility value is a proportion of the total opportunities in the region (i.e., the total marginal in Figure 2), requires no information about the population seeking opportunities, and is in units of 'opportunities' accessible. The second variant M_j^T is the transpose of i to j of the first variant, effectively a constrained version of market potential and is referred to as *total constrained accessible population measure*. In this variant, each zonal accessibility value is a proportion of the total population in the region (maintained by $\hat{K}^T$), requires no information about the opportunities that are sought, and each zonal value is in units of 'population' accessible.

- When conceptualising *potential* for spatial interaction–whether with opportunities or population–the total constrained case reflects the lowest level of restriction and hence maximum potential. The balancing factors K^T and $\hat{K}^T$ only ensure that V_{ij}^T and M_{ji}^T values end up matching the *total sum* of one of the single marginals, not the individual marginals themselves. For example, K^T does not guarantee that all the opportunities accessibility at $V_{i=1}^T$ reflect a proportional allocation of the destination mass (i.e., *number* of opportunities) at $j = 1, 2, 3$. In some cases, an allocation of opportunities from a destination j to $i = 1$ could *exceed* the number of opportunities at that j (i.e., meaning that destination j is very attractive and reachable for an i, relative other flows in the region). However, what K^T *does* ensure is that each V_{ij}^T does not exceed the overall number of opportunities in the region–the total marginal. In other words, $V_{i=1}^T$ expresses the number of opportunities that origin $i = 1$ could potentially interact with, as drawn from the entire regional opportunity mass, rather than constrained by individual destination totals. In this way, the *total constrained* case reflects the maximum amount of *potential* while still maintaining interpretable units.

3. **The singly constrained case**, with two variants V_i^S (opportunity-constrained) and M_j^S (population-constrained). The first variant is mathematically equivalent to the spatial availability measure (Soukhov et al. 2023), and this variant's per capita form is equivalent to the popular 2SFCA measure (Luo and Wang 2003; Q. Shen 1998b). Both variants can also be defined using either Hansen (1959)'s or the market

potential formulation but with balancing factor B_j for V_i^S or A_i for M_j^S. These balancing factors ensure that the total marginal (green box in Figure 2) is maintained *as well as* the values at one of the single marginals (destination-mass marginal for opportunity-constrained and origin-mass marginal for population constrained in Figure 2). In this way, the singly constrained case reflects a medium amount of potential, restricted by either values fitting the single destination-mass marginal or the single origin-mass marginal.

- The first variant V_i^S includes a set of destination-side balancing factors B_j which ensure opportunities (or destination masses) are allocated to each origin i based on the population (i.e, origin mass) of the origin i and travel impedance from that j to all is. Hence, B_j ensures that each V_i^S is the product sum of a *proportional share* of opportunities allocated from each destination j implicitly also being a share of the total opportunities in the region. Similar to the total constrained case, each V_i^S and $V_i j^S$ accessibility value is expressed in units of 'opportunities' accessible. But unlike the total constrained case, the singly constrained case explicitly incorporates population by allocating a balanced share of a destination's total opportunities to populations at reachable origins.
- The second variant M_j^S includes a set of opportunity-side balancing factors A_i which ensure population (or origin masses) are allocated to each destination j based on the destination mass of the destination j and all possible travel impedance from that i to all js. A_i ensures similar, but transposed, so constraints are satisfied. Specifically, each M_j^S represents both a share of the total regional population and the sum of balanced proportions of population from all origins, allocated to each destination based on the opportunities sought and travel impedance. Each zonal value is expressed in units of 'population' accessible.

4. **The doubly constrained case**. It is constrained simultaneously by population (origin masses) and opportunities (destination masses), so each V_{ij}^D (equivalent to M_{ji}^D) is expressed in units of 'population-opportunity capacity' that is accessible between i to j. This case requires the number of opportunities and population to match, e.g., the analyst must know the matching spatial interaction capacity of the population (demand) and opportunities (supply). By using this one-to-one matching data and the double constraints (i.e., A_i and B_j at once, ensuring both the double constraint maintains both marginals in Figure 2), this case restricts the *potential* to spatially interact completely. In other words, the doubly constrained accessibility values reflect the number of predicted interactions between the opportunities and population, effectively, this case is equivalent formulaically as Wilson's doubly constrained spatial interaction model (i.e., *attraction-production constrained*). It can be understood as predicting a value of 'access', and not accessibility.

And as a summary: each member of the family of accessibility measure is named, explained in plain language, alongside their balancing factor(s), proportional allocation factor(s), and mathematical equation and value interpretations in Table 1.

Name of Case and Variant	**Constraint Explanation and Balancing Factor**	**Proportional Allocation Factor**	**Measure Equation**	**Interpretation**
Unconstrained Accessible Opportunities (V_i^0) and Unconstrained Accessibile Population (M_j^0)	No constraints; marginals not equal to any regional or zonal knowns.	None	$V_i^0 = \sum_j O_j f(c_{ij})$; $M_j^0 = \sum_i P_i f(c_{ij})$	Values in various units depending on the impedance and destination-mass (e.g., "opportunities x decay") for V_i^0 andimpedance and origin-mass (e.g., "population x decay"); no total or marginal constraint
Total Constrained Accessible Opportunities (V_i^T) and Total Constrained Accessibile Population (M_j^T)	Balancing factors K^T and $\hat{K}^T$ ensures the sum of ij values equals the total marginal, where: $K^T = \frac{D}{\sum_i V_i^0}$; $\hat{K}^T = \frac{O}{\sum_j M_j^0}$	Allocates the total marginal as opportunities based on $\kappa_i^T = \sum_j K^T f(c_{ij})$ and as population based on $\hat{\kappa}_j^T = \sum_i \hat{K}^T f(c_{ij})$	$V_i^T = \kappa_i^T \sum_j W_j^{(2)}$; $M_j^T = \hat{\kappa}_j^T \sum_i W_i^{(1)}$	Values reflect a share of total regional opportunities (V_i^T) or population (M_j^T).
Singly Constrained Accessible Opportunities (V_i^S) and Singly Constrained Accessible Population (M_j^S)	Single balancing factor B_j (for V_i^S) that ensures the destination-mass marginal is constrained, and A_i (for M_j^S) ensures the origin-mass marginal is constrained: $B_j = \frac{1}{\sum_i W_i^{(1)} f(c_{ij})}$; $A_i = \frac{1}{\sum_j W_j^{(2)} f(c_{ij})}$	Allocates the single opportunities marginal proportionally based on $\kappa_i^S = W_i^{(1)} B_j$ in the case of V_i^S and the single population marginal proroportionally based on $\hat{\kappa}_j^S = W_j^{(2)} A_i$ in the case of M_j^S	$V_i^S = \kappa_i^S \sum_j D_j$; $M_j^S = \hat{\kappa}_j^S \sum_i P_i$	V_i^S values reflect a share of the opportunities at each destination based on origin population 'demand' and impedance; M_j^S values reflect a share of the population at each origin based on destination opportunities 'supply' and impedance.
Doubly Constrained Access (V_{ij}^D or M_{ij}^D)	Values reflect both single marginals simutaneously, maintained via A_i and B_j.	—	$V_{ij}^D = A_i B_j P_i O_j f(c_{ij})$	The spatial interactions between population and opportunities (i.e., access).

Table 1: Summary of constrained accessibility measure types and interpretations

Setup of the simple numeric example

Consider a simple region as shown in Figure 2, with zone IDs 1, 2 and 3. Each zone is both an origin i and a destination j. The following three pieces of information are defined: zonal population and opportunities, zonal cost matrix, and travel impedance functions for three types of travel behaviour. In this example, the population are people, the opportunities are physicians, and the region can be described by three possible travel behaviours.

Firstly, Table 2 summarises the population (in units of 10,000s of people) and the opportunities (the number of physicians) per zone. [^1]. Considering Table 2's values, the Provider-to-Population-Ratio (PPR) in this system is 24.5. For reference, the number of physicians per 10,000 in Canada in 2022 was 24.97 (WHO 2025). Secondly, to pair the zonal population and opportunity information, the assumed cost of movement (in minutes of travel time) between origins and destinations is as shown in Table 3.

Table 2: Simple system with three zones (ID 1, 2 and 3). Population is in 10,000 persons and opportunities in number of physicians.

ID (i or j)	Population[1]	Opportunities[2]
1	4	160
2	10	150
3	6	180

[1]Population is $Wi^{(1)}$ when used as a proxy for the mass at the origin, and *Oi* when used as a constraint.
[2]Opportunities is $Wj^{(2)}$ when used as a proxy for the mass at the destination, and *Dj* when used as a constraint.

Table 3: Cost matrix for system with three zones (travel time in minutes).

	Destination ID		
Origin ID	1	2	3
1	10	30	15
2	30	10	25
3	15	25	10

From both Table 2 and Table 3, Zone 3 and 1 can be interpreted as of an urban core with major healthcare institutions and a healthcare cluster on the edge of the city respectively, each with a moderate residing population (with zone 3 having both a higher population and physician count). Zone 2, can be interpreted as a more distant bedroom community, with a relatively high population and fewer physicians. In sum, Zones 1 and 3 are more proximate to each other than to Zone 2, and together match Zone 2's population while offering more than twice the physician availability.

And lastly, in Equation 27 we distinguish accessibility measure values for the following three impedance functions that represent the potential for spatial interaction travel behaviour of the population to opportunities. Accessibility will be calculated three times for each case, one assuming the most decay ($f_1(c_{ij})$), another assuming medium decay ($f_2(c_{ij})$), and a third assuming the least decaying travel behaviour ($f_3(c_{ij})$) for the entire region. A helpful analogy may be tying travel behaviour to the used mode's mobility potential, i.e., the most decaying travel behaviour ($f_1(c_{ij})$) would assume all travel in the region being done by foot, while calculating accessibility assuming the least decay ($f_3(c_{ij})$) would assume unfettered automobility. Or alternatively, these functions could represent travel behaviour on snowstorm-affected day ($f_1(c_{ij})$) for the entire region versus a clear, ideal travel day ($f_3(c_{ij})$). As an example of a discussion on how travel behaviour has been considered in accessibility measures cost of travel see Paez, Scott, and Morency (2012).

$$
\begin{aligned}
f_1(c_{ij}) &= \tfrac{1}{c_{ij}^3} \\
f_2(c_{ij}) &= \tfrac{1}{c_{ij}^2} \\
f_3(c_{ij}) &= \tfrac{1}{c_{ij}^{0.1}}
\end{aligned}
\tag{27}
$$

Any set of concepts representing population, opportunities, and their associated travel behaviour, whether representing the entire region uniformly (as we will demonstrate) or representing specific subgroups, can be substituted into our simple example, depending on the research question. The purpose of the following simple example is to demonstrate the calculation of each member of the accessibility measure family, interpret the values, and compare them both within and across travel behaviour groups and members of the family of accessibility measures.

Unconstrained accessibility

Setting the balancing factor k to 1 or omitting it completely in Equation 23 results in the unconstrained accessibility case:

$$V_{ij}^0 = 1 * W_j^{(2)} f(c_{ij}) \tag{28}$$

In this case, the partial sum of spatial interaction is simply identical to Hansen's accessibility S_i (Hansen 1959), the current standard practice in accessibility measurement:

$$V_i^0 = \sum_j V_{ij}^0 = \sum_j W_j^{(2)} f(c_{ij}) = S_i \tag{29}$$

The sum of the unconstrained accessibility values for each origin V_i^0 generally does not equal the total number of opportunities O (e.g., $\sum_i V_i^0 \neq O$), since arbitrarily setting k to 1 (or neglecting k all together) strips the values of any meaningful unit-based interpretation, the units are in 'summed opportunities by some travel impedance value'. Moreover, comparisons of these V_i^0 values across different contexts such as different impedance functions $f(c_{ij})$ or varying number of zones exacerbates this issue as the units between V_i^0 values change i.e., comparisons between a value of 'summed opportunities by a travel impedance value' to a value of 'summed opportunities by another travel impedance value' are not directly interpretable. From this perspective, the raw unconstrained accessibility scores are not intuitively comparable across different contexts and decay functions. They should more appropriately be used as an ordinal variable to make comparisons of size (i.e., greater than, less than, equal to), not to calculate ratios or intervals (i.e., the magnitude of differences).

Returning to our numeric example, the calculated unconstrained accessibility V_i^0 for each origin, a sum of all the travel impedance weighted opportunities at each destination ($\sum_i V_i^0$), in displayed in Table 4.

Table 4: Simple system: unconstrained accessibility.

	V_i^0		
	$f_1\ (c_{ij}) = 1/c_{ij}^3$	$f_2\ (c_{ij}) = 1/c_{ij}^2$	$f_3\ (c_{ij}) = 1/c_{ij}^{0.1}$
Origin	units: *physicians-minute^-3*	units: *physicians-minute^-2*	units: *physicians-minute^-0.1*
1	0.219	2.567	371.143
2	0.167	1.966	363.479
3	0.237	2.751	373.738
Sum	0.6233422	7.283556	1108.361

As the different impedance functions represent different travel behaviours, comparing the raw unconstrained accessibility values across groups is meaningless beyond notions of higher or lower. For instance, at zone 1 the difference between the least decay ($f_3(c_{ij})$) and most decay ($f_1(c_{ij})$) groups is 370.92, but in what units? These two values are a product of different impedance functions, making the comparison uninterpretable in absolute terms. Likewise, we could compare values within the same travel behaviour scenario across different zones, as they are in the same units, however the issue of unit interpretability will also be apparent. Considering the most decaying scenario $f_1(c_{ij})$ and zone 1 (the zone with a healthcare cluster at the edge of the city): zone 1 captures 0.0181185 fewer physicians-minute^{-1} than zone 3 (urban core). Again, the fundamental uninterpretability of what is a *physicians-minute*$^{-1}$ or *opportunity-weighted-travel-impedance* unit remains.

While one could attempt to adjust the units post-calculation (e.g., scaling, population normalization) or select impedance functions to facilitate comparison across scenarios (potentially at the expense of accurately reflecting travel behavior), such adjustments may introduce bias. Hence, the raw unconstrained accessibility values themselves are challenging to compare due to their units. To enable more meaningful comparison, the following sections will detail the introduction of constraining constants to ensure consistent units across scenarios and demonstrate results on the same numeric example.

Total constrained accessibility

In the total constrained accessibility case, a total balancing factor proportionally adjusts unconstrained zonal accessibility values V_i^0 based on the regional sum of V_i^0 and the total population or opportunities in the region. Alternatively, we reformulate this case using a proportional allocation constant, which allocates opportunities (or population) proportionally based on the the travel impedance and the total population or opportunities in the region. In both formulations, all zonal values become a proportion of a known system total, be it the regional opportunities or regional population depending on the variant.

We define two variants for this case: (a) V_i^T where accessibility is constrained by the total number of opportunities (total constrained accessible opportunity) and which is interpreted as Hansen's accessibility with a constraining constant, and (b) M_j^T, where i and j of the first variant is transposed, yielding a measure constrained by the total number of population and to be interpreted as constrained 'market potential'.

Total constrained accessible opportunities: Hansen's accessibility with a total constraint

Unlike in Equation 23, the proportionality constant k is retained. For the total constrained case, it is represented as K^T:

$$V_{ij}^T = K^T \cdot W_j^{(2)} \cdot f(c_{ij}) \tag{30}$$

In this way, the total constrained accessibility measure now becomes Hansen's accessibility with a balancing factor K^T:

$$V_i^T = \sum_j V_{ij}^T = K^T \sum_j W_j^{(2)} f(c_{ij}) = K^T \cdot V_i^0 \tag{31}$$

Imagine that the only system known is the total number of opportunities D in the region. Accordingly, the constant we impose in this case ensures the regional sum of total constrained accessibility is equal to the total number of opportunities as follows:

$$\sum_i V_i^T = \sum_i \sum_j V_{ij}^T = D \tag{32}$$

This constraint is analogous to the total constraint of Equation 12, congruent with Wilson's framework. Given the total number of opportunities in the region, we can then substitute Equation 31 in Equation 32 to solve for K^T:

$$K^T = \frac{D}{\sum_i \sum_j V_{ij}^0} = \frac{D}{\sum_i \sum_j W_j^{(2)} f(c_{ij})} \tag{33}$$

Which is also congruent with Wilson's framework as it comparable to the total flow spatial interaction model (e.g., Equation 2.11 in Cliff, Martin, and Ord (1974)). Hence, rearranging the equation to have opportunities and the proportional constant distinctly represented, our total constrained accessibility model is:

$$V_i^T = K^T \sum_j W_j^{(2)} f(c_{ij}) = \sum_j W_j^{(2)} \frac{D \cdot f(c_{ij})}{\sum_i \sum_j W_j^{(2)} f(c_{ij})}$$

Further, we can see that, since D and $W_j^{(2)}$ are both in units of opportunities, the proportional allocation factor for the total constrained opportunity case κ_i^T is dimensionless:

$$\kappa_i^T = \frac{\sum_j D \cdot f(c_{ij})}{\sum_i \sum_j W_j^{(2)} f(c_{ij})}$$

and therefore V_i^T is now in the units of $W_j^{(2)}$, that is, the mass at the destination ($V_i^T = \kappa_i^T \sum_j W_j^{(2)}$). The role of κ_i^T in this reformulation of accessibility is to transform between units and to adjust the number of opportunities accessible from i so they represent a proportion of the total number of opportunities in the region. κ_i^T then assigns opportunities in proportion to the impedance between i and j. This is why we refer to κ_i^T as a proportional allocation factor. On the other hand, the proportionality constant K^T balances the units of V_i^0, the Hansen-type accessibility values, and is an alternative expression of the total constrained accessibility measure.

Referring back to our simple numeric example, balancing factor K^T for the most decay travel behaviour scenario $f_1(c_{ij}) = 1/c_{ij}^3$ would then be:

$$\begin{aligned}
K^T &= \frac{D}{\sum_i \sum_j W_j^{(2)} f(c_{ij})} \\
K^T &= \frac{D}{\frac{W_1^{(2)}}{c_{11}^3} + \frac{W_1^{(2)}}{c_{21}^3} + \frac{W_1^{(2)}}{c_{31}^3} + \cdots + \frac{W_3^{(2)}}{c_{31}^3} + \frac{W_3^{(2)}}{c_{32}^3} + \frac{W_3^{(2)}}{c_{33}^3}} \\
K^T &= \frac{490}{0.6233422} \\
K^T &= 786.085
\end{aligned}$$

Using the calculated proportionality constants for all zones and multiplying them by the unconstrained accessibility value V_i^0, the total opportunity constrained accessibility values for all zones and different travel behaviour scenarios is presented in Table 5. κ_i^T for each zone is not shown, but can be understood to be the unitless proportion of opportunities (of the total opportunities) allocated to each zone based on travel impedance.

Table 5: Simple system: total opportunity constrained accessibility.

	V_i^T		
	$f_1\ (c_{ij}) = 1/c_{ij}{}^3$	$f_2\ (c_{ij}) = 1/c_{ij}{}^2$	$f_3\ (c_{ij}) = 1/c_{ij}{}^{0.1}$
Origin	units: *physicians*	units: *physicians*	units: *physicians*
1	172.065	172.672	164.080
2	131.627	132.247	160.692
3	186.308	185.081	165.228
Sum	490	490	490

In contrast to unconstrained accessibility V_i^0, imposing a constraint -in this case a total opportunity constraint for this variant- allows for the comparison of differences and ratios between regions and across different travel behaviour scenarios as well. Each value is effectively in units of physicians, with the impedance units already accounted for by κ_i^T.

Considering the highest decay scenario ($f_1(c_{ij})$), zone 1 (a healthcare cluster at the edge of the city) captures an intermediate amount of physicians (172.0652825) like in the unconstrained accessibility case. However, unlike in the unconstrained case, we can say that this value is out of the 490 physicians in the region, which allows us also to deduce that zone 1 captures 1.3072213 and 0.9235529 times more than zone 2 and 3. Values for the lesser decay ($f_2(c_{ij})$) and lowest decay ($f_3(c_{ij})$) scenarios are calculated separately, with decay scenario values also summing to equal 490 physicians accessible in the region.

One can also directly compare values at a specific zone due to the consistent units. For instance, zone 1 remains intermediate in capturing accessible physicians relative to zones 2 and 3 across scenarios, similar to the unconstrained case. However, the difference between travel behaviour scenarios differ in direction. Specifically, Zone 1 captures 0.6067946 more and 7.9850478 fewer opportunities than the lesser decay scenarios $f_2(c_{ij})$ and $f_3(c_{ij})$ respectively. Why? κ_i^T ensures proportional allocation for each travel behaviour scenario. Meaning, while the unconstrained accessibility increases, κ_i^T adjusts the values to remain proportional to the total number of opportunities (490 physicians accessible in the region). As the decay behaviour decreases, more opportunities are accessible for all zones. In the medium decay scenario $f_2(c_{ij})$, zone 1 sees a slight increase in values (relative to the highest decay scenario) as the zone can accessible more opportunities relative to increases seen in other zones. However, in the lowest decay scenario, zone 1 sees a decrease, as it is outpaced by increases in other zones - namely zone 2 (recall: zone 2 has the lowest number of opportunities, hence the increases in opportunity gains is much higher in a low decay scenario).

Using the total opportunity constrained formulation of accessibility offers a solution to the unit interpretability issue of Hansen (1959)'s accessibility measure. Intuitively, the use of the constraint illustrates how the differences and ratios of values between zones and decay groups can be compared. This is true for other constrained cases of the family of accessibility measures.

Total constrained accessible population: Reilly's potential trade territories with a total constraint

Another variant of the total constrained accessibility measure is the *total constrained accessible population measure*, which represents the transpose of i to j of the total constrained accessible opportunities measure. This variant, expressed in Equation 34, represents an expression of the concept of market potential (i.e., potential users) as proposed in C. D. Harris (1954) and R. W. Vickerman (1974), and which Reilly earlier referred to as 'potential trade territories' (Reilly 1929). This unconstrained form of market potential M_j^0 (Equation 35), effectively the i to j transpose of V_{ij}^0, has been used in recent research to express the potentially accessible population (i.e., users) as a result of regional transportation infrastructure investment projects (e.g., Gutiérrez 2001; Holl 2007; Condeço-Melhorado and Christidis 2018). Put another way, market potential can also be thought of as a form of *passive accessibility*, indicating the number of people that can reach each destination. However, like V_{ij}^0, issues of unit interpretability arise in M_j^0's unconstrained form. To address this, the constrained variant, the total constrained accessible population measure M_j^T, is introduced. To formulate this variant, the total balancing factor K^T is applied to the mass of the *population* at i ($W_i^{(1)}$) instead of the opportunities at j.

$$M_{ji}^T = \hat{K}^T \cdot W_i^{(1)} f(c_{ij}) = \hat{K}^T \cdot M_{ji}^0 \tag{34}$$

with M_j^0 being the i j transpose of V_{ij}^0:

$$M_j^0 = W_i^{(1)} f(c_{ij}) \tag{35}$$

The total constrained accessible population measure (market potential) then becomes:

$$M_j^T = \sum_i M_{ji}^T = \hat{K}^T \sum_i W_i^{(1)} f(c_{ij})$$

Where we impose the total system known as a constraint, i.e., that the total market potential equals the total population O in the region:

$$\sum_j M_j^T = \sum_i \sum_j M_{ji}^T = O \tag{36}$$

Substituting Equation 34 in Equation 36, and solving for $\hat{K}^T$, we obtain:

$$\hat{K}^T = \frac{O}{\sum_i \sum_i M_{ji}^T} = \frac{O}{\sum_i \sum_j W_i^{(1)} f(c_{ij})} \tag{37}$$

The constrained market potential then takes the following form:

$$M_j^T = \hat{K}^T \sum_i W_i^{(1)} f(c_{ij}) = \sum_i W_i^{(2)} \frac{O \cdot f(c_{ij})}{\sum_i \sum_j W_i^{(2)} f(c_{ij})}$$

Where the following $\hat{\kappa}_i^T$ proportional allocation factor is dimensionless:

$$\hat{\kappa}_i^T = \frac{\sum_i O \cdot f(c_{ij})}{\sum_i \sum_j W_i^{(2)} f(c_{ij})}$$

Returning back to the numerical example, the proportionality constant $\hat{K}^T$ is solved for each travel behaviour scenario, and the market potential of each zone M_j^T is expressed as units of population (e.g., the number of people accessible from each origin at that destination) in Table 6.

Table 6: Simple system: total opportunity constrained accessibility.

	$\mathrm{M_i}^{\mathrm{S}}$		
	$\mathrm{f_1\ (c_{ij}) = 1/c_{ij}}^3$	$\mathrm{f_2\ (c_{ij}) = 1/c_{ij}}^2$	$\mathrm{f_3\ (c_{ij}) = 1/c_{ij}}^{0.1}$
Destination	units: *population in 10,000s*	units: *population in 10,000s*	units: *population in 10,000s*
1	5.018	5.447	6.598
2	8.596	7.986	6.717
3	6.386	6.567	6.684
Sum	20	20	20

Readers may note the difference in trends in accessible population (Table 6) and accessible physicians (i.e., the preceding subsection, Table 5). In Table 5, zone 1, 2, 3 represent destinations and the accessibility values reflect the number of accessible people from the vantage of physicians. Zone 1, in its role as a destination, is no longer intermediately-ranked relative to other zones; it now attracts the fewest number of people across all three travel behaviour scenarios. However, similar to the total constrained opportunity case, as travel decay reduces, the availability of population begins to converge (though Zone 1 continues as the lowest-ranked) for similar reasons. As decay reduces, the population's travel impedance to all zones become more similar, making the relative location of the zones less important and all people in the region more equally accessible. Overall: like the total constrained accessible opportunities case, this variant allows for the interpretation of both ordinal and interval comparisons of the raw values themselves.

Singly constrained accessibility

Similar to the total constrained accessibility measure, the singly constrained case includes a balancing factor that adjusts the unconstrained zonal accessibility values V_i^0 such that a *single* constraint is satisfied. Two variants are defined: (a) the *singly constrained accessible opportunities* case (an alternative formula to spatial availability in Soukhov et al. (2023)), and (b) the *singly constrained accessible population* case, its transpose, or singly constrained market potential.

Unlike the total constraint (i.e., Equation 32), the single constraint–as will be defined in Equation 38 and Equation 46 for the first and second variants– incorporates additional information. In the opportunities-accessible variant, the associated balancing factor constrains the 'potential' for spatial interaction by ensuring that only a proportional amount of opportunities at each destination are allocated to 'demanding' origins. This allocation is informed by demand, or the relative amount of population, and associated travel impedance connecting the zones. In the population-accessible variant (i.e., the i -> j transpose of the first variant), population at each origin is proportionally allocated to destinations based on the share of opportunities and travel impedance.

In both variants, the singly constrained accessibility measure introduces population-based (or opportunity-based) competition at the zonal level, unlike the total constraint, which more simply allocates a fixed regional total of opportunities (or population depending on the variant). In sum, all singly constrained zonal accessibility values are both a proportion of the known regional opportunity (or population) total *and* a sum of a *balanced* proportion of opportunities allocated from each destinations (or population allocated from each origin). Each zonal value remains in units of opportunities accessible (or population accessible).

Singly constrained accessible opportunities: spatial availability

To demonstrate this variant formulaically, we begin with the opportunity constraint (Equation 38) as our known piece of information. Namely, the sum of accessible opportunities from a destination should equal the number of opportunities D_j at that destination. As the number of opportunities at each j are known, it is represented as D_j instead of $W_j^(2)$ as in the total constrained case. This constraint should hold for all destinations in the region. This is comparable to the single attraction-constraint (Equation 14) from Wilson's framework.

$$\sum_i V_{ij}^S = D_j \tag{38}$$

The underlying spatial interaction model is now the attraction-constrained model in Equation 18, and our accessibility measure becomes:

$$V_i^S = \sum_j B_j D_j W_i^{(1)} f(c_{ij}) \tag{39}$$

where $W_i^{(1)}$ is a measure of the mass at origin i (i.e., the opportunity-seeking population). The corresponding balancing factor, as per Wilson, is:

$$B_j = \frac{1}{\sum_i W_i^{(1)} f(c_{ij})} \tag{40}$$

Introducing the balancing factor in Equation 39, we obtain:

$$V_i^S = \sum_j D_j \frac{W_i^{(1)} f(c_{ij})}{\sum_i W_i^{(1)} f(c_{ij})} \tag{41}$$

Further, we define the following proportional allocation factor:

$$\kappa_i^S = \frac{W_i^{(1)} f(c_{ij})}{\sum_i W_i^{(1)} f(c_{ij})} \tag{42}$$

After this, it is possible to rewrite Equation 41 as an origin summary expression of proportionally allocated known opportunities (i.e., D_j).

$$V_i^S = \kappa_i^S \sum_j D_j \tag{43}$$

Soukhov et al. (2023) have shown that the role of κ_i^S is to allocate opportunities D_j proportionally to the mass at each origin i and the impedance between i and j. As in the total constrained opportunity case, κ_i^S is dimensionless and V_i^S is in the units of opportunities D_j. The singly constrained accessibility measure in Equation 43 is called spatial availability by Soukhov et al. (2023), because it represents the number of opportunities that can be reached *and* are available, in the sense that accessible opportunities have been proportionally allocated based on relative demand, travel impedance and the regional total number of opportunities, i.e., spatial competition for them has been considered. These authors also show that the following expression (accessibility per capita) is a constrained version of the popular two-stage floating catchment area measure of Q. Shen (1998b) and Luo and Wang (2003):

$$v_i^S = \frac{V_i^S}{W_i^{(1)}} \tag{44}$$

Returning to the simple numeric example, the opportunity-constrained case would yield the following B_j for $f_1(c_{ij})$:

$$
\begin{aligned}
B_j &= \frac{1}{\sum_i W_i^{(1)} f(c_{ij})} \\
B_1 &= \frac{1}{\frac{4}{10^3}+\frac{10}{30^3}+\frac{6}{15^3}} = 162.6506 \\
B_2 &= \frac{1}{\frac{4}{30^3}+\frac{10}{10^3}+\frac{6}{25^3}} = 94.9474 \\
B_3 &= \frac{1}{\frac{4}{10^3}+\frac{10}{25^3}+\frac{6}{10^3}} = 93.9850
\end{aligned}
$$

The balancing factors B_j for the $f_2(c_{ij})$ decay group for zones 1, 2 and 3 is 12.8571429, 8.7685113 and 10.6635071, and for $f_3(c_{ij})$ decay group is 0.0672461, 0.0660559 and 0.0663798. Using these these balancing constants, we can calculate the singly constrained opportunity accessibility:

Table 7: Simple system: singly constrained opportunity accessibility.

		V_i^S		
		$f_1\ (c_{ij}) = 1/c_{ij}^3$	$f_2\ (c_{ij}) = 1/c_{ij}^2$	$f_3\ (c_{ij}) = 1/c_{ij}^{0.1}$
Origin	Population (units: *people in 10,000s*)	units: *physicians*	units: *physicians*	units: *physicians*
1	4	133.469	122.255	98.848
2	10	166.781	185.096	241.877
3	6	189.750	182.650	149.275
Sum	—	490	490	490

Imposing the single proportional allocation factor κ_i^S allows for the comparison of differences and ratios of the accessibility values, like previously discussed in the total constrained accessible opportunities case. The proportional allocation factor ensures that resulting values are in units of *physicians*, with the impedance units already accounted for in the allocation process.

However, unlike the total constrained opportunity case, κ_i^S reflects zonal competition based on the mass of the origin, i.e., population. Again, consider the highest decay scenario $f_1(c_{ij})$. Under this scenario, zone 1 no longer captures a medium amount of physicians as in the total constrained opportunity case: it now captures the fewest in the region i.e., 133.4687282 at zone 1, 166.7813387 at zone 2, and 189.7499331 at zone 3. Why may zone 1 (healthcare cluster at the edge of the city) capture 27% of the physicians regionally while this same zone captures 35% in the total opportunity constrained case?

This difference is due to the single-opportunity factor κ_i^S's role in V_i^S. The only inputs required in the total-opportunity factor κ_i^T is the total number of opportunities in the region D as well as the associated opportunities at each j and travel impedance. Opportunities are allocated to is, regardless of the mass weights of origin. Whereas the single opportunity constraint κ_i^S requires the population at i as an input. In fact, κ_i^S is calculated as the proportion of impedance-weighted population at an i to the sum of impedance-weighted population for the entire region. Hence, since zone 1 has the lowest population in the region, is in close proximity to a more populated zone (zone 3, the 'urban core'), and is not well connected (in terms of travel impedance) to other zones with opportunities, V_1^S values, or the number of physicians accessible, is lowest.

Readers may also notice the change in the proportion of opportunities drawn from different zones depending on the travel behaviour scenarios considered. For instance, consider Zone 2 which has the highest population. It is more evident in the $f_3(c_{ij})$ scenario than in higher decay scenarios that this zone does not have an exceptionally large population for the region - Zone 2 only represents 50% of the population in the three zone region. In this sense, other zones are not *that* disadvantaged, and in this scenario with unfettered travel cost, Zones 1 and 3 also take opportunities from Zone 2 (i.e., Zone 1 and Zone 3 takes 6% and 8% more from Zone 2 between $f_3(c_{ij})$ and $f_1(c_{ij})$ scenarios hence $\kappa_{2,2}^S$ decreases by 14%). Zones 1 and 3 are allocated opportunities at relates similar to their relative population size.

Readers may also notice the change in the proportion of opportunities drawn from different zones depending on the travel behaviour scenarios considered. For instance, Zone 2 has the highest zonal population, representing 50% of the 200,000 regional population. However, due to its high relative travel distance from the other zones, its population is less competitive in capturing opportunities in the high-decay travel scenario ($f_1(c_{ij})$): Under this scenario, zone 2 captures almost exclusively opportunities from its own zone. However, in $f_3(c_{ij})$, the scenario with unfettered travel cost, Zone 2 captures by far the most number of physicians. But in this scenario, Zones 1 and 3 also take opportunities from Zone 2 (i.e., Zone 1 and Zone 3 takes 6% and 8% more from Zone 2 between $f_3(c_{ij})$ and $f_1(c_{ij})$ scenarios hence $\kappa_{2,2}^S$ decreases by 14%). Zones 1 and 3 are allocated opportunities at rates similar to their relative population size.

In this way, the consideration of constrained accessibility *per capita* may be clarifying. Often, accessibility values are reported as raw scores without the consideration for population. But, as we introduced constraints, these constrained accessibility values can be normalized using anything that is relevant to the zone. In Table 8, we present per capita accessibility for the numeric example, simply in units of number of physicians accessible per population at each zone. Notably, these per capita rates are equivalent to the 2SFCA values.

Table 8: Simple system: singly constrained opportunity accessibility per capita.

		v_i^S		
		$f_1(c_{ij}) = 1/{c_{ij}}^3$	$f_2(c_{ij}) = 1/{c_{ij}}^2$	$f_3(c_{ij}) = 1/{c_{ij}}^{0.1}$
Origin	Population (units: *people in 10,000s*)	units: *physicians per capita*	units: *physicians per capita*	units: *physicians per capita*
1	4	33.367	30.564	24.712
2	10	16.678	18.510	24.188
3	6	31.625	30.442	24.879

This simple example was constructed so that the regional average equals 24.5 physicians per 10,000 people. As distance decay decreases and becomes *relatively* uniform (all zones can reach all zones), the effect of population drives the proportional allocation of opportunities. Consequently, per capita accessibility values begin to stabilise to the regional per capita average (e.g., in the lowest distance decay $f_3(c_{ij})$, per capita values are all around 24 physicians accessible per capita).

This trend mirrors how the accessibility values in the total constrained opportunity case stabilises to the V_i^T regional average (e.g., the accessible opportunities allocated to each of the three zones approaches a third of 490 physicians, or 163.33, under the unfettered mobility scenario $f_3(c_{ij})$). These patterns make intuitive sense: the balancing factors act as regional and/or zonal averaging mechanisms. In distance decay travel behaviour scenarios that are more relatively uniform (i.e., low for all zones like in $f_3(c_{ij})$), what remains is the relative effect of the other variables in the balancing factor. In the total constrained case, this is the proportion of opportunities relative to the regional opportunities, and in the case of the single opportunity constrained case, this is the population at a zone relative to the regional population.

Singly constrained accessible population: market availability

Similar to Equation 37 in transposing the origins and destinations, we can define a *singly constrained* measure of market potential that preserves the known population (i.e., the mass weight at the origin $W_i^{(1)}$ is now represented by O_i). In it's per-capita expression, i.e., equivalent to 2SFCA, this constrained concept of market potential been used to express “facility crowdedness” as in F. H. Wang (2018).

The underlying spatial interaction model is now the production-constrained model in Equation 16, and our market potential measure M_j^S becomes:

$$M_j^S = \sum_i A_i O_i W_j^{(2)} f(c_{ij}) \tag{45}$$

In this case, the measure is singly constrained by the population *by origin* (i.e.,

O_i), like Equation 14 from Wilson's framework:

$$\sum_j M_{ji}^S = O_i \tag{46}$$

And the corresponding balancing factor, as per Wilson, is:

$$A_i = \frac{1}{\sum_j W_j^{(2)} f(c_{ij})} \tag{47}$$

Following the same logic as in the preceding section on total constrained market potential, one arrives at the following expression:

$$M_j^S = \hat{\kappa}_j^S \sum_i O_i \tag{48}$$

with:

$$\kappa_j^S = \frac{W_j^{(2)} f(c_{ij})}{\sum_i W_j^{(2)} f(c_{ij})} \tag{49}$$

As well, the single (population) constraint in Equation 46 ensures that the the total constraint (e.g., $\sum_j M_j^S = \sum_i \sum_j M_{ji}^S = O$) is maintained.

With these constraints, $\frac{M_j^S}{O}$ can be interpreted as the proportion of the total population serviced by location j.

For the sake of brevity, we'll move forward onto the doubly constrained case.

Doubly constrained accessibility

This accessibility case requires zonal populations and opportunities to match one-to-one, like the doubly constrained spatial interaction model. In this way, doubly constrained accessibility can be thought as "access" or simply spatial interaction (no potential).

To contextualize this point, the total and singly constrained accessibility measures discussed thus far have used either O_i, D_j, or the regional sums of either, but never both simultaneously. For example, when opportunities D_j are used to constrain Equation 39 in the *singly constrained accessible opportunities measure*, the specific mass of the population at origin i demanding only those opportunities at that j is unknown. Instead, only the population at i demanding opportunities in the region is known, and this is more generally represented as $W_i^{(1)}$ (as in Equation 40). Similarly, when the population O_i is used as a constraint in Equation 45 in the *singly constrained accessible population measure*, the mass at the destination is given by $W_j^{(2)}$ (also in Equation 47) since only the mass of opportunities at each j is known and information on what opportunities are allocated to what zone is unknown.

By contrast, the double-constrained accessibility case requires that both populations and opportunities match. Meaning, opportunities at each destination can be accessed by all populations, while at the same time, the population at each origin can be accessed by all opportunities. However, this requirement is often unintuitive in traditional accessibility analysis. Namely, the distinction between population (origin masses) and opportunities (destination masses) typically represent different entities without a shared unit of measurement. On the population side, we usually count people; on the opportunity side, we may be referring to physicians, clinics, grocery stores, schools, parks, or libraries. In a few cases, a one-to-one correspondence or their own capacities at which they interact and are interacted with may exist e.g., one person with one job. Another similar opportunity type example is healthcare, e.g., one person and one unit of capacity, e.g. a vaccine shot.

A doubly constrained approach to accessibility calculation needs a one-to-one relationships between population and opportunities to be present. Mathematically, this model requires the simultaneous imposition of both the population- and opportunity- constraints in the preceding singly constrained variants (Equation 38 and Equation 46), namely the sum of population in all origins should match the sum of opportunities in all destinations (Equation 50):

$$\sum_i O_i = \sum_j D_j \tag{50}$$

As before, the simultaneous imposition of both constraints ensures the total system constraint is maintained i.e., the sum of all doubly constrained accessibility values $\sum_i V_i^D = \sum_i \sum_j V_{ij}^D = D$ remains equal to the total number of opportunities in the region O as shown in Equation 32.

As the doubly constrained accessibility measure V_{ij}^D takes the form of the production-attraction (doubly constrained) spatial interaction model, as shown in Equation 20, V_i^D is as follows:

$$V_{ij}^D = A_i B_j O_i D_j f(c_{ij}) \tag{51}$$

where the corresponding balancing factors A_i and B_j, as per Wilson, are:

$$A_i = \frac{1}{\sum_j B_j D_j f(c_{ij})}$$
$$B_j = \frac{1}{\sum_i A_i O_i f(c_{ij})}$$

Calibration of the two sets of proportionality constants is accomplished by means of iterative proportional fitting, whereby the values of A_i are initialized as one for all i to obtain an initial estimate of B_j. The values of B_j are used to update the underlying V_{ij}^D matrix, before calibrating A_i. This process continues to update A_i and B_j until a convergence criterion is met (see Ortúzar and Willumsen 2011, 193–95). The proportional allocation factor κ_{ij}^D would then be:

$$\kappa_{ij}^D = \sum_j \frac{1}{\sum_j B_j D_j f(c_{ij})} \frac{1}{\sum_i A_i O_i f(c_{ij})} O_i f(c_{ij})$$

One could rewrite Equation 51 as an origin summary expression of proportionally allocated opportunities:

$$V_i^D = \kappa_{ij}^D \sum_j D_j \tag{52}$$

However, V_i^D is not wholly helpful, as it will simply equal the origin-mass marginal. In this way, only $V^D ij$ values are interpretable. Furthermore, unlike in the total and singly constrained cases, the doubly-constrained case does not have an interpretable per capita version. For instance, representing V_{ij}^D per capita is not meaningful, as the value *already* matches population to opportunities. Following this logic, the market potential form M_{ij}^D is effectively equivalent to V_{ij}^D, but can be read with a different interpretation: i.e., the opportunities accessed from j at an i vs. the population accessed from i at a j. The inputs of 'opportunities accessed' and 'accessed population' can already be interpreted as inherently being sensitive to both opportunities and population.

To calculate doubly constrained accessibility, the interpretation of the population data and the counts of the opportunity data in the numeric example must be modified. Namely, a count of physician *capacity* per destination is needed instead of just the number of physicians, as used to calculated total and singly constrained cases. We also must be able to clearly state that the population is a count of people seeking opportunities at the new capacities, i.e., the population must reflect the *capacity* of the population to interact with opportunities.

So, this adjusted simple example is summarised in Table 9: with the population (in units of 10,000s of people seeking physicians) and the opportunities (in units of 10,00s of physician-capacity) per zone. For the population, we leave this unchanged numerically but theoretically know that each person interacts with one physician capacity (i.e., our opportunities with a *capacity*). Hence, the number of opportunities per destination is new: the example is modified such that the physician-capacity at each zone is an approximately scaled version of the number of destination-side physicians at each zone from the unmodified example (Table 2). To emphasis the new definition of 'provider' as physician capacity, the new system PPR is simply 1, this is compared to the unmodified example which yields system PPR of24.5. We keep the same zonal cost matrix, and travel impedance functions for three types of travel behaviour as before (Table 3 and Equation 27).

Table 9: Modified simple system with three zones reflecting matched population and opportunities. Population is in 10,000 persons and opportunities in 10,000 of physician-capacity.

ID (i or j)	Population	Opportunities
1	4	7
2	10	5
3	6	8

However, despite the modifications to the example, our objective remains the same as in previous cases: to measure accessibility under different travel behavior scenarios. Specifically, we aim to quantify the number of *potential* spatial interactions between the physician-seeking population in each zone i to the physician capacity in a zone j i.e., V_{ij}^D. The highest decay travel behaviour scenario ($f_1(c_i j)$) is presented in Table 10.

Table 10: Doubly constrained opportunity accessibility assuming the most decay travel behaviour scenario in the modified simple system.

		Destination ID			
	Origin ID	1	2	3	sum
	1	3.235859	0.01032226	0.7556568	4
	2	2.132602	4.95932483	2.9044391	10
	3	1.631539	0.03035291	4.3399040	6
Sum	—	7	5	8	—

As mentioned, accessibility values are typically interpreted as a summary of the proportionally allocated opportunities at each i. Hence, in interpreting the doubly constrained accessibility from Table 10, V_i^D values (i.e., the sum of values at all three j destinations for each origin i) would be 4.0018381, 9.9963656, and 6.0017963 physician-capacity accessible for Zones 1, 2 and 3 respectively. This approximately equal to the number of population at each of these zones. Conversely, the market potential M_j^D interpretation of these values would be 7, 5, and 8 people accessible from Zones 1, 2 and 3 respectively, equal to the number of *opportunities* (physician-capacities accessible) at each of these zones. Notice, the mass weight at the origin equals the mass weight of the destination: this is precisely the function of the double constraint. In other words, V_i^D is the number of accessed opportunities and M_j^D is the number of population accessed. For the other two travel behaviour, identical V_i^D and M_j^D values are calculated, following the same logic. Hence, the usefulness of the doubly constrained measure lies in the interpretation as V_{ij}^D values.

For instance, differences in V_{ij}^D values between travel behaviour scenarios are notable. These values can be directly compared to discuss mass and distance decay impacts. Examining Zone 2 (the bedroom community), Table 11 demonstrates these i to j access values for this more relatively remote, higher-populated and lower-opportunity rich zone. It can be observed that the number of intra-zonal opportunities proportionally allocated decreases as the assumed distance decay decreases e.g., from 4.9593248 to 2.6672837 out of the ~10 opportunities allocated to Zone 2 (a population of 10). Following the intuition discussed in the singly constrained opportunity case, as decay decreases, the mass effect of the population (at origin) and opportunity (at destination) is more evident: zonal opportunities are supplied and zonal populations demand at the weights assigned to these zones, with minimal decay adjustment, reflected by the proportional allocation factors κ_{ij}^D.

Table 11: Doubly constrained opportunity accessibility for all travel behaviour scenarios for Zone 2 in the modified simple system.

			$V_{\{ij\}}^D$		
			$f_1\ (c_{ij}) = 1/{c_{ij}}^3$	$f_2\ (c_{ij}) = 1/{c_{ij}}^2$	$f_3\ (c_{ij}) = 1/{c_{ij}}^{0.1}$
Dest.	Population at 2 (units: *people in 10,000s*)	Opportunities (units: *capacity in 10,000s*)	units: *physician-capacity in 10,000s*	units: *physician-capacity in 10,000s*	units: *physician-capacity in 10,000s*
1	10.000	7.000	2.133	2.272	3.411
2	10.000	5.000	4.959	4.766	2.667
3	10.000	8.000	2.904	2.958	3.919

Recall, accessibility is defined as "the potential for interaction" and is traditionally presented as a summary zonal measure. In the doubly constrained case, we force the zonal population and zonal opportunities to match one-to-one, hence providing a zonal summary is no longer relevant: the sum of V_{ij}^D for all is ends up being equal to the population at the zone. However, if what interests readers is the "potential for interaction" in the case population and opportunities to match one-to-one, reframing the investigation may be needed. In this sense, it would be examining "interaction" (much less room for potential) through the values of V_{ij}^D e.g., how many opportunities are being allocated to an origin from a destination (or in a transposed sense for M_{ji}^D). In this sense, the doubly constrained case can be thought of as an estimate of *realized* accessibility or access: reflecting spatial interaction. It is also formulaically identical to the doubly constrained spatial interaction model, but with specific interpretations of the origin and destination weights as 'population' and 'opportunity', respectively.

As Wilson explicitly noted, origin and destination weights defined in the spatial interaction model *can* be defined using any unit. Accessibility, however, is often presented and understood as a zonal summary of *potential* for interaction between origins and destinations that contain inherently different units. Arriving at the doubly constrained case through the unconstrained, total, and singly constrained cases make the connection between the *potential* for spatial interaction (accessibility) and *realized* potential for spatial interaction more interpretable. Namely, by demonstrating that these members of the accessibility family all derive from the same root and can be derived from Wilson's original formulation, this more clearly demonstrates what *potential* is, within the context of spatial interaction modelling.

Namely, *potential* depends on the framing of the masses at the origin and destination, and how similar they are in their units. The appropriate constraint should be decided based on the the input data and their similarity. In increasing unit similarity from the perspective of opportunity-constraint: the total-constraint can be used if population (origin mass) is not known, the single constraint can be used if population is known and matters to *potential* but it does not match the capacity of opportunities, and lastly, the double constraint is appropriate if population is known, matters and population matches the capacity of opportunities. These constraint measures can reflect the opportunities accessible at each zone, but reflects the assumptions embedded in the input data about the potential to spatially interact.

Conclusions

In this work, we examined the historical and mathematical commonalities between spatial interaction models and place-based accessibility measures. As accessibility research evolved largely influenced by Hansen (1959), researchers in the field neglected the proportionality constant that was originally present in gravity-based models, and is still present in spatial interaction modeling. Firstly, we have demonstrated that by reintroducing such constraints and defining associated balancing factors and proportional allocation factors, we can derive a unifying family of accessibility measures that reintroduces units to the resulting values. These values become more intuitive for the purpose of analysis and comparison. Secondly, the constraining constants, depending on the case (i.e., total, singly- or doubly- constrained), restrict the degree of *potential*, linking accessibility (the potential for spatial interaction) with access (spatial interaction) on the same continuum based on the constraint used. Lastly, we discussed how popular measures such as the one used in Hansen (1959) and the 2SFCA link into the family of accessibility measures. We summarise the family of accessibility measures, as follows.

We first place the popular Hansen-type accessibility measure (Hansen 1959) within this family of measures as an "unconstrained" case, demonstrating that resulting values cannot be directly compared across different travel scenarios without ad-hoc adjustments. We then show how applying a total constraint balances the units and produces a statistically averaged solution that converges to the regional average for each zone as the decay effect decreases. In other words, the total-constraint model could be a more interpretable alternative for the unconstrained case if population-competition is not relevant and one is interested in capturing the maximum *potential*; specifically, if there is a fixed number of opportunities in the region, and if it makes sense to assume that people accessing proximate opportunities leave fewer for others, *without* considering the population size at the origins.

We then introduce the singly constrained case, which *does* takes into account the population size at the origin in the allocation of opportunities (unlike the total constraint). It is also mathematically equivalent to the spatial availability introduced in Soukhov et al. (2023). In this case, all accessibility values are fixed to sum to a known zonal opportunity-size value (implicitly, the regional total of opportunities), but they are not required to sum to any population-based values at the zone or regional level. The singly constrained model could be useful if regional competition is a factor and if the acknowledgment that only a finite number of opportunities can be allocated from each destination (with those allocations distributed based on origin population size) is suitable. We also introduce an 'accessible' PPR (e.g., opportunities per capita), calculated by dividing each accessibility value by the zonal population. To clarify, this per capita expression of the singly constrained case is equivalent to the 2SFCA (Luo and Wang 2003; Q. Shen 1998b), hence linking this literature back to spatial interaction principles.

Lastly, the doubly constrained case is introduced. In this case, the sums must equal both the regional total and ensure that no zone allocates more opportunities than it has available. Specifically, accessibility values for each i-j pair must be a proportion of he zonal opportunity and population values simultaneously. For example, the accessibility at zone 1 must equal the sum of opportunities from zones 1, 2, and 3, as well as the sum of the population at zone 1. Satisfying the double constraint means the opportunities and population data must match one-to-one, so working with the accessibility i-j pair values should be of interest. In this sense, the research question should be concerned with 'access' (how many opportunities accessed from j at i based on given zonal opportunities and populations) instead of potential spatial interaction (e.g., typically expressed as a zonal summary measure of how many opportunities one could reach (out of a regional total and/or zonal-allocation).

In summary, building on Wilson (1971)'s foundational work, this paper proposed a unified framework for accessibility that is able to account for competition. By reintroducing Wilson's proportionality constant, the proposed family of constrained accessibility measures restores measurement units to accessibility estimates. This enhancement provides a more interpretable, consistent, and theoretically grounded basis for accessibility analysis, which could help advance the adoption of accessibility-oriented planning.

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