Introducing spatial availability, a singly-constrained competitive-access accessibility measure

Abstract

Accessibility measures are widely used in transportation, urban and healthcare planning, among other applications. These measures are weighted sums of the opportunities that can be reached given the cost of movement and are estimates of the potential for spatial interaction. These unconstrained measures are useful in understanding spatial structure, but they do not properly account for competition due to multiple counting of opportunities. This leads to interpretability issues, as noted in recent research on balanced floating catchment areas (BFCA) and competitive (doubly- or singly- constrained) measures of accessibility. In this paper, we respond to this limitation by proposing a new measure of *spatial availability* which is calculated by imposing a single constraint on conventional gravity-based accessibility. This constraint ensures that the marginals at the destination are met and thus the number of opportunities are preserved across the analysis. Through examples, we detail the formulation of the proposed measure. Further, we use data from the 2016 Transportation Tomorrow Survey of the Greater Golden Horseshoe area in southern Ontario, Canada, to contrast how the conventional accessibility measure tends to overestimate and underestimate the number of jobs *available* to workers. We conclude with some discussion of the possible uses of spatial availability and argue that, compared to conventional measures of accessibility, it can offer a more interpretable measure of opportunity access. All data and code used in this research are openly available which should facilitate testing by other researchers in their case studies.

1. Introduction

Accessibility analysis is employed in transportation, geography, public health, and many other areas, particularly as mobility-based planning is de-emphasized in favor of access-oriented planning (Deboosere et al., 2018; Handy, 2020; Proffitt et al., 2017; Yan, 2021). The concept of accessibility derives its appeal from combining the spatial distribution of opportunities and the cost of reaching them (Handy and Niemeier, 1997; Hansen, 1959).

Numerous methods for calculating accessibility have been proposed that can be broadly organized into infrastructure-, place-, person-, and utility-based measures (Geurs and van Wee, 2004). Of these, the place-based family of measures is arguably the most common, capturing the number of opportunities reachable from an origin using the transportation network. One common type of accessibility measure is based on the gravity model of spatial interaction; since being first developed by Hansen (1959), it has been widely adopted in many forms (Arranz-López et al., 2019; Cervero et al., 2002; Geurs and van Wee, 2004; Handy and Niemeier, 1997; Levinson, 1998; Paez, 2004). Next to gravity-based approaches, another common type of accessibility measurement are cumulative-opportunity aproaches in which access to opportunities are evaluated by using travel distance or time threshold to opportunities (Vickerman, 1974; Wachs and Kumagai, 1973). Though gravity-based measures are argued to be more theoretically sound (Siddiq and D. Taylor, 2021), results from the more simple-to-understand cumulative-opportunity measures can correlate with gravity-based accessibility when travel thresholds are correctly selected (El-Geneidy and Levinson, 2006; Higgins, 2019; Kwan, 1998; Santana Palacios and El-geneidy, 2022; Wang et al., 2021). Place-based accessibility analysis offers a powerful tool to study the intersection between urban structure and transportation infrastructure - however, the interpretability of the output values can be challenging (Geurs and van Wee, 2004; Miller, 2018). A key issue is that accessibility measures are sensitive to the number of opportunities in a region (e.g., a large city has more jobs than a smaller city), and therefore raw outputs are not easily compared across study areas (Allen and Farber, 2019).

The meaning of conventional gravity-based and cumulative-opportunity accessibility raw outputs are unclear as they measure the *intensity of the possibility of interaction* (as defined by Hansen, 1959) so categorical membership to high- through low- accessibility to opportunities scores are assigned to the spatial unit of analysis. For instance, in the study of accessibility to employment by public transit in Metropolitan São Paulo Metropolitan Region, Boisjoly et al. (2017) find that for every 1% increase in cumulative-opportunity accessibility score the probability of being in informal job sector decreases by 3% . However, this study does not include the raw values as it can be assumed that primary use in visualization is to identify region-relative hot- and cold- spots and not the values themselves.

As a different example, in the work of Bocarejo S. and Oviedo H. (2012), they include the raw gravity-based accessibility to employment values for each community in Bogota. A similar interpretability issue arises: what does it mean for the Zona Franca community and El Rincón community to have 35,704 employment accessibility and 148,238 employment accessibility (i.e., the number of potential job opportunities) respectively? Once these values are normalized per capita, they represent 0.87 and 0.86 accessibility per capita but, but as discussed from the perspective of demand in Paez et al. (2019), opportunities are multiple-counted per capita so normalizing per capita does not consistently address this limitation. Though uses of gravity-based and cumulative-opportunity accessibility measures in the work of Boisjoly et al. (2017) and Bocarejo S. and Oviedo H. (2012) represent urban structure and can help inform policy, the interpretation of the raw magnitudes of the measures are unclear and may even be inaccurate when applied to rival opportunities, like employment opportunities (Merlin and Hu, 2017).

Put another way, conventional measures of accessibility do not capture the *competition* for opportunities but instead quantify access as if every person can take every opportunity given their mobility (Kelobonye et al., 2020; Paez et al., 2019). These conventional approaches (i.e., non-competitive) are not necessarily problematic if the opportunity of interest is non-rival, that is, if use by one unit of population does not preclude use by another. For instance, national parks with abundant space are seldom used to full capacity, so the presence of some population does not exclude use by others. However, inconsistent opportunity-adjustments can be more acute when opportunities are *rival*. For example, the work of Merlin and Hu (2017) found using competitive gravity-based accessibility more accurately reflects empirical access to employment opportunities in Los Angelos than non-competitive methods. Though these rival-type opportunities can still be modelled by conventional accessibility (e.g., jobs in Boisjoly et al. (2017) and Bocarejo S. and Oviedo H. (2012)), the output values ultimately do not reflect the *spatial availability* of the opportunity.

There has been academic efforts to develop accessibility approaches which consider competition. The first approach was introduced by Weibull (1976) in which the distance decay of the supply of employment and the demand for employment (by workers) was formulated under axiomatic assumptions. This approach was then applied by Joseph and Bantock (1984) in the context of healthcare, to quantify the availability of general practioners in Canada. More recently, Shen (1998) re-created Weibull (1976) formula and deconstructed it to consider accessibility for different modes. These advances were reformatted and popularized as the family of two-step floating catchment area (2SFCA) methods (Luo and Wang, 2003) that have found widespread adoption for calculating competitive accessibility to a variety of opportunities such as healthcare, education, and food access (B. Y. Chen et al., 2020; Chen, 2019; Z. Chen et al., 2020; Yang et al., 2006; Ye et al., 2018). The 2SFCA method is mathematically equivalent to the works of Shen (1998) and Weibull (1976).

Another approach which researchers have used to consider competition in accessibility has been the imposition of constraints on the gravity model to ensure potential interaction equals observed totals (i.e., population and/or opportunities). Based on Wilson's (1971) description of the gravity-based modelled interaction, researchers can incorporate constraints to ensure that the modeled origin-destination flows match some known quantities in the data inputs. In this way, models can be **singly-constrained** to match the row- or column-marginals (i.e., the population-trips produced or attracted, respectively), **doubly-constrained** to match both marginals, or **unconstrained** if neither population-trips produced or attracted are preserved. For instance, in Shen (1998) work, he provides a proof that demonstrates that the total sum of opportunities (not population) in the network considered is preserved; this proof which demonstrates that Shen-style accessibility is singly-constrained, is the basis of *spatial availability* and is discussed within this paper. Next, Allen and Farber (2019) employed a doubly-constrained gravity-based accessibility model as both the population and employment opportunity counts are preserved within the accessibility outputs and thus can be easily compared across regions. And as previously mentioned, Bocarejo S. and Oviedo H. (2012) developed a detailed gravity-based accessibility model to quantify the total accessibility to employment for each study area community; this model can be interpreted as unconstrained since the total number of opportunities (or population) are not preserved within the accessibility outputs.

Using Wilson (1971)'s spatial interaction model classifications of doubly-, singly- or unconstrained may be a useful to better interpreting *how* opportunities are counted in place-based measured and *how* competition is considered. In this way, the aim of this paper is three-fold:

- First, we aim to demonstrate that Shen-style (and thus Weibull (1976)) accessibility and the popular 2SFCA method are *singly-constrained* from the perspective of opportunities allocated (i.e., population-trips produced).
- Second, we introduce a new measure, *spatial availability*, as a more interpretable version of Shen-style accessibility, in which the opportunities in the system are preserved and proportionally allocated to each origin.
- Third, we contrast *spatial availability* with conventional gravity-based accessibility and demonstrate how it may be more intuitive for policy planners.

The contribution of this paper is thus to introduce and present spatial availability alongside and in context with doubly-, unconstrained, and other singly-constrained measures. Using spatial availability, we aim to demonstrate that this singly-constrained measure proportionally allocates (based on travel cost and population size) all opportunities to the opportunity-seeking population. By allocating opportunities in a proportional way in a single step, this measure considers competition and avoids the issues that result from multiple allocation of the same opportunities in *unconstrained* accessibility analysis (i.e., Hansen-style accessibility). Spatial availability returns the value of available opportunities per origin and can be effectively normalized as a rate of available opportunities per opportunity-seeking population. The normalized rate is equivalent to Shen-style accessibility thus can be used as a benchmark value to compare *available* opportunity rates both inter- and intra-regionally as well as be used in the context of opportunity provision assessment. This novel approach and clarifying work comes at a time when the quantity and resolution of data is exponentially increasing and the need to operationalize accessibility methods in city-planning objectives is urgent.

This paper is split into four main parts. The first part describes a re-newed interpretation of how congestion and/or competition are treated in existing measures of singly-constrained, doubly-constrained, and unconstrained accessibility. The second part introduces spatial availability and uses a synthetic example to compare it to the established measures discussed. In the third part, we then calculate, compare, and contrast the spatial availability, unconstrained accessibility (i.e., Hansen-style accessibility), and doubly-constrained accessibility for 2016 employment data in the city of Toronto, Canada (Transportation Tomorrow Survey (TTS)). The motivation of this part is to demonstrate how constraints on accessibility distributes opportunities and thus impacts interpretability. Finally, we conclude by remarking on the conceptual limits of unconstrained accessibility analysis, the computational burden of doubly-constrained accessibility analysis and outline the advantages of the spatial availability measure and the breadth of potential uses from the perspective of opportunity-provision planning.

In the spirit of openness of research in the spatial sciences (Brunsdon and Comber, 2021; Páez, 2021) this paper has a companion open data product (Arribas-Bel et al., 2021), and all code will be available for replicability and reproducibility purposes. We call for researchers to examine our contribution and to re-examine the use of unconstrained, singly-constrained, and doubly-constrained accessibility measures based on the qualities of the opportunities and the populations in question.

2. Re-framing accesssibility measures interpretations

We first describe unconstrained (Hansen, 1959), singly-constrained (Luo and Wang, 2003; Shen, 1998), and doubly-constrained (Horner, 2004; Paez et al., 2019), accessibility measures. Then present the synthetic example introduced in Shen (1998) (slightly modified so the population is greater than the job opportunities). Then we introduce the proposed spatial availability measure, calculate the spatial availability values as well as the established accessibility measures for the synthetic example. We discuss how the interpretation of the resulting values from the perspective of opportunity-provision and the impact that opportunity-constraints have on interpretation. The synthetic example is presented in Figure 1, the results are summarized in **Table XX**, and the detailed solutions are in the **Appendix** for all measures.

2.1. Unconstrained accessibility

Accessibility analysis stems from the foundational works of Harris (1954) and Hansen (1959). From their seminal efforts, many accessibility measures (excluding utility-based measures) have been derived, particularly after the influential work of Wilson (1971) on the gravity model. The model follows the formulation shown in Equation (1) and the solved synthetic example is in **Table XX - left**.

$$A_i = \sum_{j=1}^{J} O_j \cdot f(c_{ij}) \tag{1}$$

where:

- A is accessibility.
- i is a set of origin locations.
- j is a set of destination locations.
- O_j is the number of opportunities at location j; $\sum_j O_j$ is the total supply of opportunities in the study region.
- c_{ij} is a measure of the cost of moving between i and j.
- $f(\cdot)$ is an impedance function of c_{ij}; it can take the form of any monotonically decreasing function chosen based on positive or normative criteria (Paez et al., 2012).

As formally defined, accessibility A_i is the weighted sum of opportunities that can be reached from location i, given the cost of travel c_{ij}. Summing the opportunities in the neighborhood of i, as determined by the impedance function $f(\cdot)$, provides estimates of the number of opportunities that can be reached from i at a certain cost. The type of accessibility can be modified depending on the impedance function; for example, the measure could be cumulative opportunities (if $f(\cdot)$ is a binary or indicator function e.g., El-Geneidy et al., 2016; Geurs and van Wee, 2004; Qi et al., 2018; Rosik et al., 2021) or a gravity measure using an impedance function modeled after any monotonically decreasing function (e.g., Gaussian, inverse power, negative exponential, or log-normal, among others, see, *inter alia*, Kwan, 1998; Li et al., 2020; Reggiani et al., 2011; Vale and Pereira, 2017). In practice, the accessibility measures derived from many cumulative and gravity formulations tend to be highly correlated with one another (Higgins, 2019; Kwan, 1998; Santana Palacios and El-geneidy, 2022).

Gravity-based accessibility has been shown to be an excellent indicator of the intersection between urban structure and transportation infrastructure (Kwan, 1998; Reggiani et al., 2011; Shi et al., 2020). However, beyond enabling comparisons of relative values they are not highly interpretable on their own (Miller, 2018). To address this interpretability issue, previous research has aimed to index and normalize values on a per demand-population basis (e.g., Barboza et al., 2021; Pereira et al., 2019; Wang et al., 2021). However, as recent research on accessibility discusses (Allen and Farber, 2019; Kelobonye et al., 2020; Merlin and Hu, 2017; Paez et al., 2019), these steps do not truly adequately consider competition. In effect, when calculating A_i, every opportunity enters the weighted sum once for every origin i that can reach it. Neglecting to constrain opportunity and population counts (i.e., doubly-constrained) or just opportunity or just population counts (i.e., single-constraint) in addition to obscuring the interpretability of accessibility can bias the estimated landscape of opportunity, as we will discuss later.

2.2. *Singly-constrained accessibility*

Past research has considered competition by introducing demand-side modifications to the Hansen-style accessibility. These modifications consider spatially-distributed impacts of competition from the perspective of opportunity-seekers. For example, the influential work of Shen (1998) (and Weibull (1976); Joseph and Bantock (1984)) divides conventional accessibility by the travel-cost adjusted population seeking the opportunities in a given region. These works were reformatted (remaining mathematically equivalent) and popularized by the 2-step floating catchment approach (2SFCA) introduced by Luo and Wang (2003) which are widely used today.

The formulation of the 2SFCA approach is shown in step 1 (Equation (2)) where the provider-to-population ratio (PPR) R_j is calculated for each opportunity and then allocated to populations based on travel cost $f(\cdot)$ in step 2 (Equation (3)).

$$R_j = \frac{O_j}{\sum_i P_i \cdot f(c_{ij})} \tag{2}$$

$$A_i = \sum_j R_j \cdot f(c_{ij}) \tag{3}$$

where:

- A is accessibility.
- i is a set of origin locations.
- j is a set of destination locations.
- O_j is the number of opportunities at location j;
- P_i is the population at location i; $\sum_j R_j$ is the total supply of opportunities in the study region.
- R_j is the provider-to-population (PPR) ratio at location j;
- c_{ij} is a measure of the cost of moving between i and j;
- $f(\cdot)$ is an impedance function of c_{ij}.

It should be noted that both methods used in Shen (1998) and Luo and Wang (2003) are also equivalent in the sense that the resulting values are an artifact of the constrained opportunities (see proof in Shen (1998)). Each resulting value, when multiplied by the population at that origin and summed for the full study region, results in the total number of opportunities in the full study region. This is to say, that the total number of opportunities are constrained and fully preserved.

2.3. *Doubly-constrained accessibility*

As shown by Paez et al. (2019), singly-constraining (referred to as congestion) does not necessarily mean competition, as the same population is claimed by multiple service centers and the same level of service is given to multiple populations. Allen and Farber (2019) also discusses the need to balance both the demand-side (opportunity-seekers) and supply-side (opportunities) Hence, population and opportunity constraints have been introduced to the Hansen-style accessibility to consider spatially-distributed impacts of competition from both opportunity-seekers and between opportunities. These modifications take the form of inverse balancing factors and have been used in works such as (Horner, 2004) and (Allen and Farber, 2019) from the perspective of employment opportunities. These modifications, in an alternative way, have also been taken by Paez et al. (2019) in their balanced 2-step floating catchment approach (B2SFCA) within the health-care context.

For the method used by Allen and Farber (2019), the inverse balancing factor for each origin is calculated through an iterative procedure until convergence. The iterations seek to match the opportunities to the population in the region where both are weighted by the travel cost (impedance function). It requires that the total population and opportunities are equal in the region but the mean accessibilities from previous iterations can be included to standardize the imbalance. The solved synthetic example is in **Table XX - middle**, and the formulation of this method is as follows in Equation (4) and (5).

$$A_i = \frac{\bar{A}^o}{\bar{A}^c} \sum_{j=1}^{J} \frac{O_j f(c_{ij})}{B_j} \tag{4}$$

$$B_i = \sum_{i=1}^{I} \frac{P_i f(c_{ij})}{A_i} \tag{5}$$

where: - B_i is the balancing factor; other variables defined in the gravity model.

Though the unmodified Shen-style accessibility and 2SFCA are equivalent, the inverse balanced modified accessibility (used by Allen and Farber (2019)) is not equivalent to the B2SFCA of Paez et al. (2019). The B2SFCA advances the 2SFCA by incorporating a demand-side balancing factor to the PPR at each origin at both steps. The method results in a consistent number of opportunities being assigned.

In the B2SFCA, the PPR R_j can be interpreted as the total number of opportunities (i.e., jobs) accessible to the total *proportionally* travel-cost adjusted population at each opportunity center. The PPR R_j is then allocated proportionally based on travel cost, to each population center yielding A_i. For this reason, the sum of all A_i adds up to the same value as the sum of all R_j. Since PPR and the subsequent A_i are proportionally allocated based travel costs, it should be noted that A_i no longer considers *potential* interaction as how it was defined in the gravity model (Hansen, 1959) and instead represent the allocation of PPR, based on travel time, to each population. This measure introduces some consistency in how the PPR is calculated (compared to the 2SFCA), but is still lacking interpretability in the resulting values.

The solved synthetic example is in **Table XX - right**, and the formulation of this method is as follows in Equation (6) and (7).

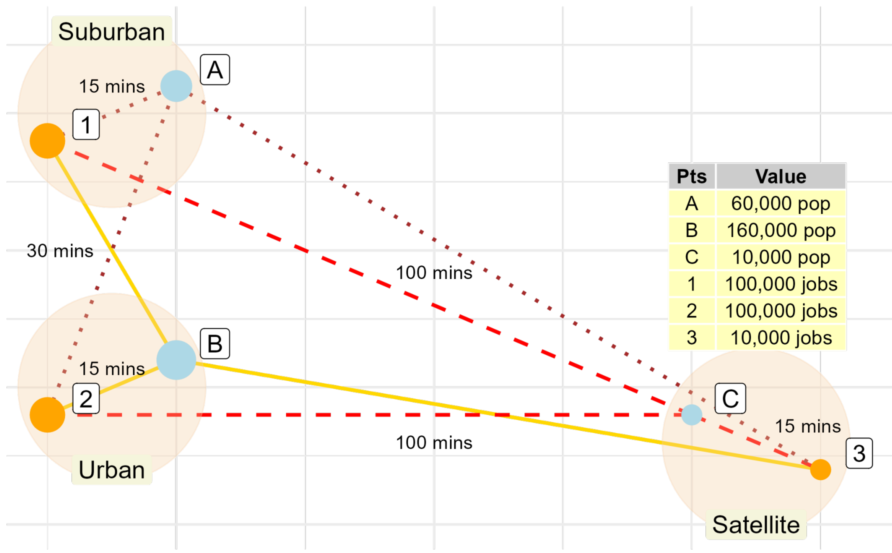


Figure 1: Shen (1998) synthetic example with locations of employment centers, population centers, number of employment opportunities and population at centers, and travel time between population centers to employment centers between and to Urban, Suburban, and Satellite Town regions.

$$R_j = \frac{O_j}{\sum_i P_i \frac{f(c_{ij})}{\sum_j f(c_{ij})}} \tag{6}$$

$$A_i = \sum_j R_j \frac{f(c_{ij})}{\sum_j f(c_{ij})} \tag{7}$$

3. Introducing spatial availability and a synthetic example

Here we introduce the spatial availability model formulation using the synthetic example modified from Shen (1998) (Figure 1).

We define spatial availability V_i as the number of opportunities O that are proportionally allocated based on population and cost of travel, for all origins i to all destinations j.

This idea is reflected in Equation (8), where F_i^p is a population-based allocation factor that grants a larger share of the existing opportunities to larger centers, and F_{ij}^c is a transportation cost-based allocation factor that grants a larger share of the existing opportunities to closer centers. This is in line with the tradition of gravity modeling, and proposed framework distinguishes between opportunities at a destination and demand for opportunities at the origin.

$$V_i = O_j \frac{F_i^p \cdot F_{ij}^c}{\sum_{i=1}^K F_i^p \cdot F_{ij}^c} \tag{8}$$

The terms in Equation 8 are as follows:

- V_i is the spatial availability of opportunities in j to origin i.
- i is a set of origin locations in the region K.
- j is a set of destination locations in the region K.
- O_j is the number of opportunities at location j in the region K.
- F_i^p is a proportional allocation factor of the population in i.
- F_{ij}^c is a proportional allocation factor of travel cost for i; it is a product of a monotonically decreasing (i.e., impedance) function associated with the cost of travel between i and j.

Notice that, unlike A_i in Equation (1), the population in the region enters the calculation of V_i. It is important to detail the role of the two proportional allocations factors in the formulation of spatial availability. We begin by considering the population allocation factor F_i^p followed by the role of the travel cost allocation factor F_{ij}^c; then we show how both allocation factors combine in the final general form of spatial availability V_i. The calculation of spatial availability is introduced with a step-by-step example for the three population centers (A, B, C) in the role of demand (i.e., the number of individuals in the labor market who 'demand' employment) and three employment centers (1, 2, 3) in the role of opportunities.

3.1. Population and travel cost allocation factors

We begin with allocation based on demand by population; consider an employment center j with O_j jobs. In the general case where there are K population centers in the region, we define the following factor:

$$F_i^p = \frac{P_i^\alpha}{\sum_{i=1}^K P_i^\alpha} \tag{9}$$

The population allocation factor F_i^p corresponds to the proportion of the population in origin i relative to the population in the region. On the right hand side of the equation, the numerator P_i is the population at origin i that is eligible for and demands jobs j. The summation in the denominator is over $i = 1, \cdots, K$, the population at origins i in the region. To modulate the effect of demand by population in this factor we include an empirical parameter α (i.e., $\alpha < 1$ places greater weight on smaller centers relative to larger ones while $\alpha > 1$ achieves the opposite effect). This population allocation factor F_i^p can now be used to proportionally allocate a share of the jobs at j to origins.

More broadly, since the factor F_i^p is a proportion, when it is summed over $i = 1, \cdots, K$ it always equals to 1 (i.e., $\sum_i^K F_i^p = 1$). This is notable since the share of jobs at each destination j allocated to (i.e., available to) each origin, based on population, is equal to $V_i^p = O_j \cdot F_i^p$. Since the sum of F_i^p is equal to

1, it follows that $\sum_{i=1}^{I} V_i = O_j$. In other words, the number of opportunities (jobs) across the region is preserved. The result is a proportional allocation of jobs to origins based on the size of their populations.

For simplicity, assume that $\alpha = 1$. The population allocation factors F_i^p is as follows in Equation (10).

$$\begin{aligned} F_1^p &= \frac{P_1^\alpha}{P_1^\alpha + P_2^\alpha + P_3^\alpha} = \frac{260}{260+255+495} = 0.257 \\ F_2^p &= \frac{P_2^\alpha}{P_1^\alpha + P_2^\alpha + P_3^\alpha} = \frac{255}{260+255+495} = 0.252 \\ F_3^p &= \frac{P_3^\alpha}{P_1^\alpha + P_2^\alpha + P_3^\alpha} = \frac{495}{260+255+495} = 0.490 \end{aligned} \tag{10}$$

These F_i^p values can be used to find a *partial* spatial availability in which jobs are allocated proportionally to population; this partial spatial availability V_i^p for each population center is calculated as follows in Equation (11).

$$\begin{aligned} V_1^p &= O_1 \cdot F_1^p + O_2 \cdot F_1^p = 750 \cdot 0.257 + 220 \cdot 0.257 = 249.29 \\ V_2^p &= O_1 \cdot F_2^p + O_2 \cdot F_2^p = 750 \cdot 0.252 + 220 \cdot 0.252 = 244.44 \\ V_3^p &= O_1 \cdot F_3^p + O_2 \cdot F_3^p = 750 \cdot 0.490 + 220 \cdot 0.490 = 475.30 \end{aligned} \tag{11}$$

When using only the proportional allocation factor F_i^p to calculate spatial availability (differentiated here by being defined as V_i^p instead of V_i), proportionally more jobs are allocated to the bigger population center (i.e., 2 times more jobs as it is 2 times larger in population). We can also see that the sum of spatial availability for all population centers equals the total number of opportunities.

Clearly, using only the proportional allocation factor F_i^p to calculate spatial availability does not account for how far population centers are from employment centers. It is the task of the second allocation factor F_{ij}^c to account for the friction of distance, as seen in Equation (12).

$$F_{ij}^c = \frac{f(c_{ij})}{\sum_{i=1}^{K} f(c_{ij})} \tag{12}$$

Travel cost allocation factor F_{ij}^c serves to proportionally allocate more jobs to closer locations through an impedance function. c_{ij} is the cost (e.g., the distance, travel time, etc.) to reach employment center j from i and $f(\cdot)$ is an impedance function that depends on cost (c_{ij}).

To continue with the example, assume that the impedance function is a exponential function with $\beta = -0.00015$ and the distance from population centers to employment centers is as shown in TABLE XX. β modulates the steepness of the impedance effect and is empirically determined in the case of positive accessibility, or set by the analyst to meet a preset condition in the case of normative accessibility (Paez et al., 2012). The proportional allocation factor F_i^p for all population centers is defined in Equation (13).

$$
\begin{aligned}
F^c_{1,1} &= \frac{\exp(\beta * 2548.1)}{\exp(\beta * 2548.1)+\exp(\beta * 1314.1)+\exp(\beta * 2170.2)} = 0.109\\
F^c_{2,1} &= \frac{\exp(\beta * 1314.1)}{\exp(\beta * 2548.1)+\exp(\beta * 1314.1)+\exp(\beta * 2170.2)} = 0.697\\
F^c_{3,1} &= \frac{\exp(\beta * 2170.2)}{\exp(\beta * 2548.1)+\exp(\beta * 1314.1)+\exp(\beta * 2170.2)} = 0.193\\
F^c_{1,2} &= \frac{\exp(\beta * 5419.1)}{\exp(\beta * 5419.1)+\exp(\beta * 2170.2)+\exp(\beta * 1790.1)} = 0.004\\
F^c_{2,2} &= \frac{\exp(\beta * 4762.6)}{\exp(\beta * 5419.1)+\exp(\beta * 2170.2)+\exp(\beta * 1790.1)} = 0.011\\
F^c_{3,2} &= \frac{\exp(\beta * 1790.1)}{\exp(\beta * 5419.1)+\exp(\beta * 2170.2)+\exp(\beta * 1790.1)} = 0.984
\end{aligned} \tag{13}
$$

We can see, for instance, that the proportional allocation factor for P_2 is largest for E_1 since the cost (i.e., distance) to E_1 is lowest. For E_2, P_3 has the largest proportional allocation factor similarly because it is in the closest proximity. Using the travel cost proportional allocation factors F^c_{ij} as defined in Equation (13), we can calculate the spatial availability of jobs for each population center based only on F^c_{ij} and the jobs available at each employment center, as shown in Equation (14).

$$
\begin{aligned}
V^c_{1,1} &= E_1 \cdot F^c_{1,1} = 750 \times 0.109 = 81.75\\
V^c_{2,1} &= E_1 \cdot F^c_{2,1} = 750 \times 0.697 = 522.75\\
V^c_{3,1} &= E_1 \cdot F^c_{3,1} = 750 \times 0.193 = 144.75\\
V^c_{1,2} &= E_2 \cdot F^c_{1,2} = 220 \times 0.004 = 0.88\\
V^c_{2,2} &= E_2 \cdot F^c_{2,2} = 220 \times 0.011 = 2.42\\
V^c_{3,2} &= E_2 \cdot F^c_{3,2} = 220 \times 0.984 = 216.48
\end{aligned} \tag{14}
$$

For instance, spatial availability defined by F^c_{ij} only (i.e., V^c_i) allocates a largest share of jobs from E_1 to P_2 since it is the closest. However, as previously discussed, P_2 has a relatively small population, so $V^p_{2,1}$ is actually the smallest value of any population center for E_1. It is necessary to combine both population and travel cost factors to better reflect demand; these two components are in line with how demand is conventionally modelled in accessibility calculations which are re-scaled on a per demand-population basis or also consider competition (e.g., Allen and Farber, 2019; Barboza et al., 2021; Yang et al., 2006). Fortunately, since both F^c_{ij} and F^p_i preserve the total number of opportunities as they independently sum to 1, they can be combined multiplicatively to calculate the proposed spatial availability V_i which considers demand to be based on both population and travel cost.

3.2. Putting spatial availability together

We can combine the proportional allocation factors by population F^p_i and travel cost F^c_{ij} and calculate spatial availability V_i as introduced in Equation (8) and repeated below:

$$
V_i = O_j \frac{F^p_i \cdot F^c_{ij}}{\sum_{i=1}^{K} F^p_i \cdot F^c_{ij}}
$$

The resulting spatial availability V_i is calculated for all population centers is calculated in Equation (15).

$$
\begin{aligned}
V_{1,1} &= O_1 \cdot \tfrac{F^p_{1,1} \cdot F^c_{1,1}}{F^p_{1,1} \cdot F^c_{1,1} + F^p_{2,1} \cdot F^c_{2,1} + F^p_{3,1} \cdot F^c_{3,1}} = 750 \cdot \tfrac{0.26 \cdot 0.109}{0.26 \cdot 0.109 + 0.25 \cdot 0.697 + 0.49 \cdot 0.193} = 70.45 \\
V_{2,1} &= O_1 \cdot \tfrac{F^p_{2,1} \cdot F^c_{2,1}}{F^p_{1,1} \cdot F^c_{1,1} + F^p_{2,1} \cdot F^c_{2,1} + F^p_{3,1} \cdot F^c_{3,1}} = 750 \cdot \tfrac{0.25 \cdot 0.697}{0.26 \cdot 0.109 + 0.25 \cdot 0.697 + 0.49 \cdot 0.193} = 441.72 \\
V_{3,1} &= O_1 \cdot \tfrac{F^p_{3,1} \cdot F^c_{3,1}}{F^p_{1,1} \cdot F^c_{1,1} + F^p_{2,1} \cdot F^c_{2,1} + F^p_{3,1} \cdot F^c_{3,1}} = 750 \cdot \tfrac{0.49 \cdot 0.193}{0.26 \cdot 0.109 + 0.25 \cdot 0.697 + 0.49 \cdot 0.193} = 237.83 \\
V_{1,2} &= O_2 \cdot \tfrac{F^p_{1,2} \cdot F^c_{1,2}}{F^p_{1,2} \cdot F^c_{1,2} + F^p_{2,2} \cdot F^c_{2,2} + F^p_{3,2} \cdot F^c_{3,2}} = 220 \cdot \tfrac{0.26 \cdot 0.004}{0.26 \cdot 0.004 + 0.25 \cdot 0.011 + 0.49 \cdot 0.984} = 0.46 \\
V_{2,2} &= O_2 \cdot \tfrac{F^p_{2,2} \cdot F^c_{2,2}}{F^p_{1,2} \cdot F^c_{1,2} + F^p_{2,2} \cdot F^c_{2,2} + F^p_{3,2} \cdot F^c_{3,2}} = 220 \cdot \tfrac{0.25 \cdot 0.011}{0.26 \cdot 0.004 + 0.25 \cdot 0.011 + 0.49 \cdot 0.984} = 1.26 \\
V_{3,2} &= O_2 \cdot \tfrac{F^p_{1,2} \cdot F^c_{1,2}}{F^p_{1,2} \cdot F^c_{1,2} + F^p_{2,2} \cdot F^c_{2,2} + F^p_{3,2} \cdot F^c_{3,2}} = 220 \cdot \tfrac{0.49 \cdot 0.984}{0.26 \cdot 0.004 + 0.25 \cdot 0.011 + 0.49 \cdot 0.984} = 218.28
\end{aligned}
\tag{15}
$$

Aggregating by population center gives the following values:

$$
\begin{aligned}
V_1 &= 70.45 + 0.46 = 70.91 \\
V_2 &= 441.72 + 1.26 = 442.98 \\
V_3 &= 237.83 + 218.28 = 456.11
\end{aligned}
\tag{16}
$$

Considering both population and cost allocation factors in V_i, the jobs at $E1$ that are allocated to all population centers are still preserved (i.e., $V_{1,1} + V_{2,1} + V_{3,1} = O_1$). Additionally, the sum of jobs at $E2$ are also all preserved (i.e., $V_{1,2} + V_{2,2} + V_{3,2} = O_2$). Thus the sum of V_i equals the sum of opportunities (i.e.,) Notice that V_i, allocates a number of jobs to P_1, P_2, and P_3 is between the values allocated in V_i^p and V_i^c.

When comparing V_i to the singly-constrained gravity model (see Wilson (1971)), V_i is the result of constraining A_i to match one of the marginals in the origin-destination table, the known total of opportunities. Since the sum of opportunities is preserved in the procedures above, it is possible to calculate an interpretable measure of spatial availability per capita (lower-case v_i) as shown in Equation (17).

$$
v_i = \frac{V_i}{P_i}
\tag{17}
$$

To complete the illustrative example, the per capita spatial availability of jobs is calculated in Equation (18).

$$
\begin{aligned}
v_1 &= \tfrac{V_{1,1} + V_{1,2}}{P_1} = \tfrac{70.91}{260} = 0.272 \\
v_2 &= \tfrac{V_{2,1} + V_{2,2}}{P_2} = \tfrac{442.98}{255} = 1.737 \\
v_3 &= \tfrac{V_{3,1} + V_{3,2}}{P_3} = \tfrac{456.11}{495} = 0.921
\end{aligned}
\tag{18}
$$

We can see that since P_2 is closest to $E1$, is similarly spaced out from $P1$ and $P2$, and is a smaller population center thus having less competition, P_2 benefits with a higher spatial availability of jobs per job-seeking population. We can also compare these values to the overall ratio of jobs-to-population in this region of two job center and three population centers is $\frac{750+220}{260+255+495} = 0.96$ jobs per person.

3.3. Discusion on established accessibility measures and spatial availability

Table 1 contains the output from all the measures reviewed above.

Table 1: Summary description of synthetic example

Origin	**Dest.**	**TT**	**Pop.**	**Jobs**	**A_i**	**S_i**	**BFCA_i**
	1	15		100000			
A	2	30	60000	100000	27292.18	1.17	1.17
	3	100		10000			
	1	30		100000			
B	2	15	160000	100000	27292.18	0.81	0.81
	3	100		10000			
	1	100		100000			
C	2	100	10000	100000	2240.38	1.00	1.00
	3	15		10000			

For instance, from Figure 1, A has equal accessibility (27,292 potential jobs) as B despite A having a three times smaller population. On the other hand, the isolated satellite town of C has low accessibility (2240 potential jobs) but it is still better than *zero* and it has a small population. It is difficult to determine which accessibility values indicate excellent, good, or only fair accessibility access? What does it *mean* for a location to have accessibility to so many jobs?

To address this interpretability issue, previous research has aimed to index and normalize values on a per demand-population basis (e.g., Barboza et al., 2021; Pereira et al., 2019; Wang et al., 2021). However, as recent research on accessibility discusses (Allen and Farber, 2019; Kelobonye et al., 2020; Merlin and Hu, 2017; Paez et al., 2019), these steps do not truly address competition. In effect, when calculating A_i, every opportunity enters the weighted sum once for every origin i that can reach it. Put another way, if the Suburban region A increases its population and all employment centers and other population centers are kept steady, A will have just as high an accessibility score. There is a lurking assumption in this process that all opportunities are potentially available to anyone from any origin $i = 1, \cdots, n$ who can reach them: in other words, opportunities are assumed to be non-rival and inexhaustible. This multiplication of the opportunities means that competition is not really present, and A_i does not consider that neighbouring population centers are seeking the same exhaustible opportunities. Neglecting to constrain opportunity counts (i.e., single-constraint) in addition to obscuring the interpretability of accessibility can also bias the estimated landscape of opportunity, as we will discuss later on in the paper.

We see that the PPR R_j for each employment center can be interpreted as the total number of jobs accessible to the total travel-cost adjusted population. This step recognizes that not all opportunities can be distributed to the entire population evenly since not *all* opportunities can be reached by *all* population centers. It is assumed that all population and employment centers are in the same catchment.

In step two, S_i values represent the travel-cost adjusted PPR for each population center. Put another way, here S_A, S_B, and S_C values represent the number of jobs accessible to each population center after being travel-cost adjusted from both the opportunities-perspective and population-perspective. The value could theoretically be on a scale of 0 to the maximum total number of PPR in the catchment (i.e., $f(c_{ij}) = 0$ to $f(c_{ij}) = 1$); in this case that value is 4.683 jobs per person.

4. Empirical example of Toronto

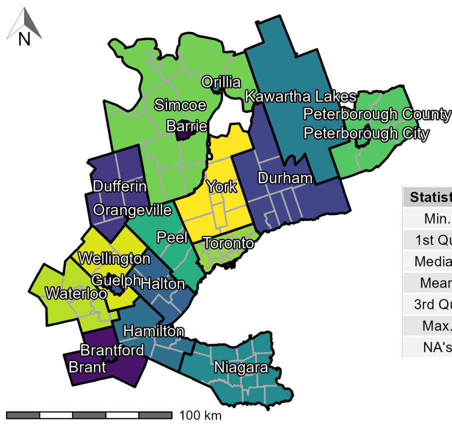
In this section we use population and employment data from the Golden Horseshoe Area (GGH). This is the largest metropolitan region in Canada and includes the cities of Toronto and Hamilton. We calculate gravity accessibility, XXX, and the proposed spatial availability for Toronto after introducing the data used and calibrating an impedance function.

4.1. Data

Population and employment data are drawn from the 2016 Transportation Tomorrow Survey (TTS). This survey collects representative urban travel information from 20 municipalities contained within the GGH area in the southern part of Ontario, Canada (see Figure 2) (Data Management Group, 2018). The data set includes Traffic Analysis Zones (TAZ) (n=3,764), the number of jobs (n=3,081,885) and workers (n=3,446,957) at each origin and destination. The TTS data is based on a representative sample of between 3% to 5% of households in the GGH and is weighted to reflect the population covering the study area has a whole (Data Management Group, 2018).

To generate the travel cost for these trips, travel times between origins and destinations are calculated for car travel using the R package {r5r} (Rafael H. M. Pereira et al., 2021) with a street network retrieved from OpenStreetMap for the GGH area. A the 3 hr travel time threshold was selected as it captures 99% of population-employment pairs (see the travel times summarized in Figure 2). This method does not account for traffic congestion or modal split, which can be estimated through other means (e.g., Allen and Farber, 2021; Higgins et al., 2021). For simplicity, we carry on with the assumption that all trips are taken by car in uncongested travel conditions.

All data and data preparation steps are documented and can be freely explored in the companion open data product {TTS2016R}.



Statistic	Trips	TT	Workers	Jobs
Min.	1	0	0	0
1st Qu.	14	13	25	64
Median	22	20	464	244
Mean	33	23	916	819
3rd Qu.	38	30	1378	700
Max.	1129	179	8491	41821
NA's	NA	3507	NA	NA

Figure 2: TTS 2016 study area (GGH, Ontario, Canada) along with the descriptive statistics of the trips, calculated origin-destination car travel time (TT), workers per TAZ, and jobs per TAZ. Contains 20 regions (black boundaries) and sub-regions (dark gray boundaries).

4.2. Calibration of an impedance function

In the synthetic example introduced in a preceding section, a negative exponential function with an arbitrary parameter was used. For the empirical example, we calibrate an impedance function on the trip length distribution (TLD) of commute trips. Briefly, a TLD represents the proportion of trips that are taken at a specific travel cost (e.g., travel time); this distribution is commonly used to derive impedance functions in accessibility research (Batista et al., 2019; Horbachov and Svichynskyi, 2018).

The empirical and theoretical TLD for this data set are represented in the top-left panel of Figure 3. Maximum likelihood estimation and the Nelder-Mead method for direct optimization available within the {fitdistrplus} package (Delignette-Muller and Dutang, 2015) were used. Based on goodness-of-fit criteria and diagnostics seen in Figure 3, the gamma distribution was selected (also see Figure **??** in Appendix XX).

```
[1] 3069541
```

The gamma distribution takes the following general form where the estimated 'shape' is $\alpha = 2.019$, the estimated 'rate' is $\beta = 0.094$, and $\Gamma(\alpha)$ is defined in Equation (19).

$$
\begin{aligned}
&f(x,\alpha,\beta) = \tfrac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \quad \text{for } 0 \leq x \leq \infty \\
&\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx
\end{aligned}
\tag{19}
$$

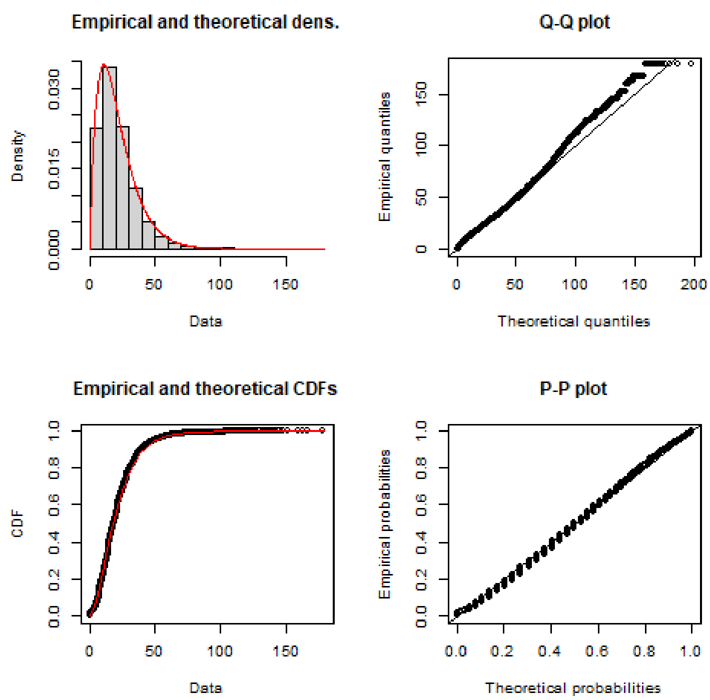


Figure 3: Car trip length distribution and calibrated gamma distribution impedance function (red line) with associated Q-Q and P-P plots. Based on TTS 2016.

4.3. Measuring access to jobs in Toronto

Toronto is the largest city in the GGH and represents a significant subset of workers and jobs in the GGH; 31% of workers in the GGH travel to jobs in Toronto and 40% of jobs are located within Toronto.

To enhance the interpretability, spatial availability can be normalized to provide more meaningful insight into how many jobs are *available* on average for each TAZ. This normalization, shown in Figure **??**, demonstrates which TAZ have above (reds) and below (blue) the average available jobs per worker in the GGH (1.17). Similar to the spatial availability plot of the GGH jobs in Figure **??**, we can see that many average or above average jobs per worker TAZ (whites and reds) are present in southern Peel and Halton (south-west of Toronto), Waterloo and Brantford (even more south-west of Toronto), and Hamilton and Niagara (south of Toronto), however, the distribution is uneven and many TAZ within these areas do have below average values (blues).

Interestingly, when considering *competitive* job access, many areas outside of Toronto have similar jobs per worker values as TAZ in Toronto. This is contrary to the notion that since Toronto has high job access it has a significant density of employment opportunities in the GGH. Not all jobs in Toronto are *available* since Toronto has a high density of *competition* in addition to density of jobs opportunities. For instance, urban centers outside of Toronto such as those found in Brantford, Guelph, southern Peel, Halton, and Niagara have TAZ which are far above the the TTS average jobs per worker and higher than TAZ within Toronto. High job access is not seen in the accessibility plot which suggests that these less densely populated urban centers may have sufficient employment opportunities for their populations; this finding is obscured when only considering the accessibility measure for job access as will be later discussed.

It is also worth noting that there is almost two times more jobs per worker in the GGH jobs spatial availability results than the GGH Toronto spatial availability results. This suggests that all GGH people who work in the city of Toronto, on average, face more competition for jobs than all GGH people who work anywhere in the GGH .

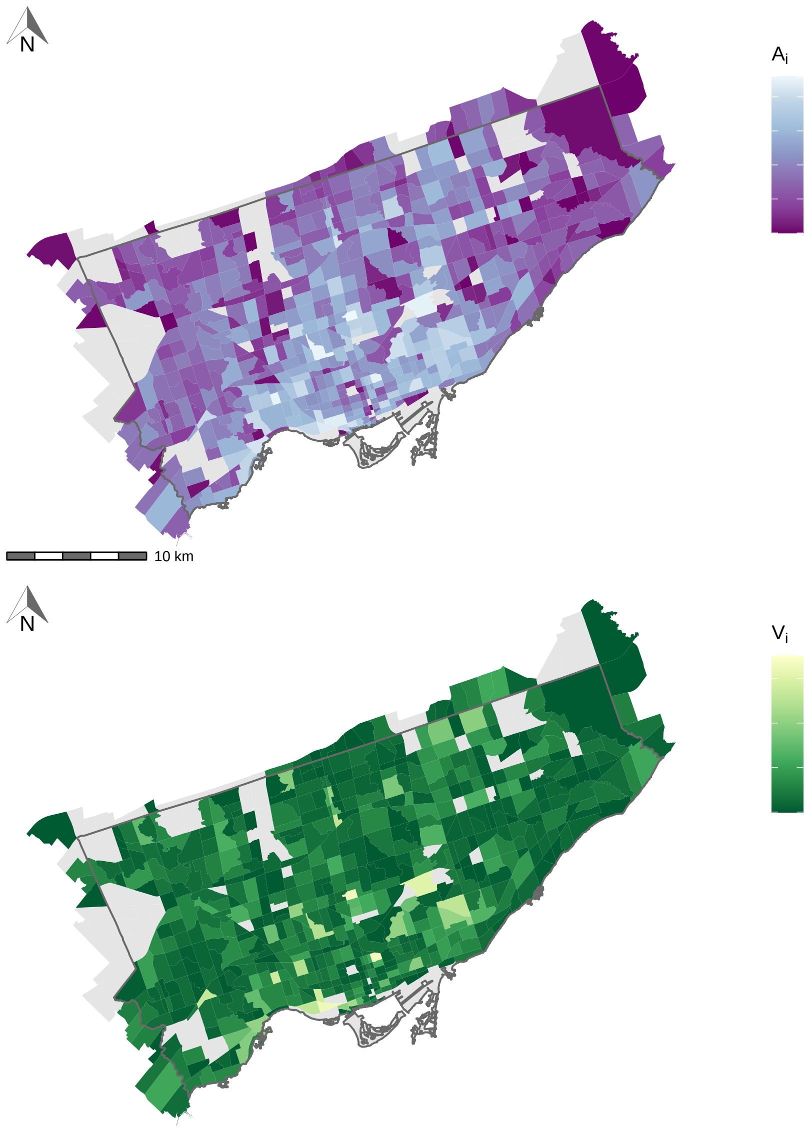


Figure 4: Calculated accessibility (top) and spatial availability (bottom) of employment from origins in destinations and origins in Toronto. Greyed out TAZ represent null accessibility and spatial availability values.

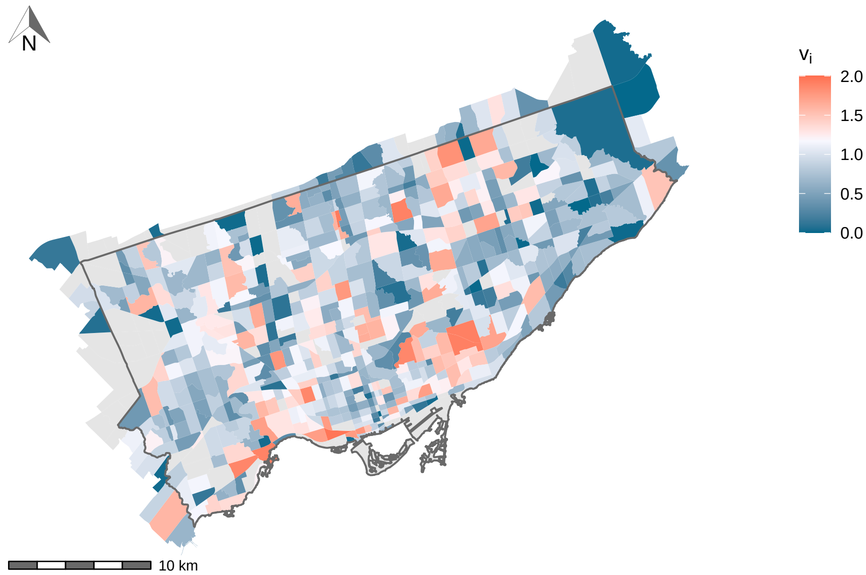


Figure 5: Spatial availability per worker, from origins to job opportunities in Toronto.

5. Discussion and Conclusions

6. Appendix A: Step-by-step accessibility calculations for synthetic example

Details for the synthetic example:
and:

$$\beta = -0.1 inf(c_{ij}) = exp(\beta * tt_{ij})$$

Table 2: Summary description of synthetic example

Origin	Destination	Travel Time	Population	Jobs
A	1	15	60000	100000
A	2	30	60000	100000
A	3	100	60000	10000
B	1	30	160000	100000
B	2	15	160000	100000
B	3	100	160000	10000
C	1	100	10000	100000
C	2	100	10000	100000
C	3	15	10000	10000

6.1. Conventional gravity accessibiliy

$$A_i = \sum_{j=1}^{J} O_j \cdot f(c_{ij})$$

Solved in one step:

$$\sum_{j=1}^{J} O_j = E1 + E2 = 750 + 220 = 970 jobs$$

$$\begin{aligned}
A_A &= 100000 \cdot \exp(\beta * 15) + 100000 \cdot \exp(\beta * 30) + 10000 \cdot \exp(\beta * 100) = 27292 \\
A_B &= 100000 \cdot \exp(\beta * 30) + 100000 \cdot \exp(\beta * 15) + 10000 \cdot \exp(\beta * 100) = 27292 \\
A_C &= 100000 \cdot \exp(\beta * 100) + 100000 \cdot \exp(\beta * 100) + 10000 \cdot \exp(\beta * 100) = 2240
\end{aligned} \tag{20}$$

A_A, A_B, and A_C values represent the number of travel-cost adjusted opportunities accessible to each population. Specifically, only a proportion of opportunities are allocated to population centers based on their travel cost value (higher the travel cost lower the number of opportunities). The population is not considered in this measure and the allocation of opportunities is not constrained, it is only adjusted based on the weight of the travel cost. With our negative exponential distance decay, accessibility can be as high as 210,000 (the total number of opportunities in the region) and as low as essentially 0.

However, in many instances being close to opportunities doesn't necessarily mean much practically to an individual nor can this scale of 0 to the maximum number of total opportunities in the region be operationalized by decision-makers. However, correlates have been found so it is a strong indicator of urban structure, but practically what does it mean for an individual to live in a population center of $A_C = 2{,}240$ potential job opportunities? On a scale of 0 to 210,000 ($f(c_{ij}) = 0$ to $f(c_{ij}) = 1$), this value is low but of the three population centers it is around average. However, A_B also has the largest population of all population centers. It has a population that is three times the population center of A_A but an accessibility value that is equal to A_A's accessibility value. Does it make sense that in both population centers, the accessibility is equal or should it be adjusted based on population? Adjusting based on population is not equivalent as both centers have different travel costs to a different magnitude of opportunities. From this perspective, competitive measures such as the FCA were introduced with the most recently popularized 2SFCA is discussed as follows.

6.2. 2 step floating catchment approach (2SFCA)

Step one:

$$
\begin{aligned}
R_j &= \frac{O_j}{\sum_i P_i \cdot f(c_{ij})} \\
R_1 &= \frac{100000}{60000 \cdot \exp(\beta * 15) + 160000 \cdot \exp(\beta * 30) + 10000 \cdot \exp(\beta * 100)} = 4.683 \\
R_2 &= \frac{100000}{60000 \cdot \exp(\beta * 30) + 160000 \cdot \exp(\beta * 15) + 10000 \cdot \exp(\beta * 100)} = 2.584 \\
R_3 &= \frac{10000}{60000 \cdot \exp(\beta * 100) + 160000 \cdot \exp(\beta * 100) + 10000 \cdot \exp(\beta * 15)} = 4.462
\end{aligned} \tag{21}
$$

Step two:

$$
\begin{aligned}
S_i &= \sum_j R_j \cdot f(c_{ij}) \\
S_A &= 4.683 \cdot \exp(\beta * 15) + 2.584 \cdot \exp(\beta * 30) + 4.462 \cdot \exp(\beta * 100) = 1.174 \\
S_B &= 4.683 \cdot \exp(\beta * 30) + 2.584 \cdot \exp(\beta * 15) + 4.462 \cdot \exp(\beta * 100) = 0.810 \\
S_C &= 4.683 \cdot \exp(\beta * 100) + 2.584 \cdot \exp(\beta * 100) + 4.462 \cdot \exp(\beta * 15) = 0.996
\end{aligned} \tag{22}
$$

We see that the PPR R_j for each employment center can be interpreted as the total number of jobs accessible to the total travel-cost adjusted population. This step recognizes that not all opportunities can be distributed to the entire population evenly since not *all* opportunities can be reached by *all* population centers. It is assumed that all population and employment centers are in the same catchment.

In step two, S_i values represent the travel-cost adjusted PPR for each population center. Put another way, here S_A, S_B, and S_C values represent the number of jobs accessible to each population center after being travel-cost adjusted from both the opportunities-perspective and population-perspective. The value could theoretically be on a scale of 0 to the maximum total number of PPR in the catchment (i.e., $f(c_{ij}) = 0$ to $f(c_{ij}) = 1$); in this case that value is 4.683 jobs per person.

It may seem that this method locates opportunities and population in an unconstrained manner, but it is in fact constrained from the opportunities perspective. See the proof below on how 2SFCA (and Shen (1998) method) cancels out an equals Spatial Availability.

6.2.1. Proof: Spatial availability cancels out into Shen-style accessibility

Population allocation factor:

$F_{ij}^p = \frac{P_{i \in r}^\alpha}{\sum_i^K P_{i \in r}^\alpha}$

$F_A^p = \frac{P_A^\alpha}{P_A^\alpha + P_B^\alpha + P_C^\alpha}$

Cost allocation factor:

$F_{ij}^c = \frac{f(c_{ij})}{\sum_{i=A}^K f(c_{ij})}$

$F_{A1}^c = \frac{f(c_{A1})}{f(c_{A1}) + f(c_{B1}) + f(c_{C1})}$ $F_{B1}^c = \frac{f(c_{A2})}{f(c_{A2}) + f(c_{B2}) + f(c_{C2})}$ $F_{C1}^c = \frac{f(c_{A3})}{f(c_{A3}) + f(c_{B3}) + f(c_{C3})}$

Now let's put it together with P, and see how the denominators end up cancelling out:

$v_i = \sum_j \frac{O_j}{P_{i \in r}^\alpha} \frac{\frac{P_{i \in r}^\alpha}{\sum_i^K P_{i \in r}^\alpha} \cdot \frac{f(c_{ij})}{\sum_i^K f(c_{ij})}}{\sum_i^K \frac{P_{i \in r}^\alpha}{\sum_i^K P_{i \in r}^\alpha} \cdot \frac{f(c_{ij})}{\sum_i^K f(c_{ij})}}$

$$v_A = \frac{O_1}{P_A^\alpha}\left(\frac{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A1})}{f(c_{A1})+f(c_{B1})+f(c_{C1})}}{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A1})}{f(c_{A1})+f(c_{B1})+f(c_{C1})}+\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{B1})}{f(c_{A1})+f(c_{B1})+f(c_{C1})}+\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{C1})}{f(c_{A1})+f(c_{B1})+f(c_{C1})}}\right)+$$

$$\frac{O_2}{P_A^\alpha}\left(\frac{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A2})}{f(c_{A2})+f(c_{B2})+f(c_{C2})}}{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A2})}{f(c_{A2})+f(c_{B2})+f(c_{C2})}+\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{B2})}{f(c_{A2})+f(c_{B2})+f(c_{C2})}+\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{C2})}{f(c_{A2})+f(c_{B2})+f(c_{C2})}}\right)+$$

$$\frac{O_3}{P_A^\alpha}\left(\frac{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A3})}{f(c_{A3})+f(c_{B3})+f(c_{C3})}}{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A3})}{f(c_{A3})+f(c_{B3})+f(c_{C3})}+\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{B3})}{f(c_{A3})+f(c_{B3})+f(c_{C3})}+\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{C3})}{f(c_{A3})+f(c_{B3})+f(c_{C3})}}\right)$$

First, notice how the denominator on the denominator is the same across the summation? Let's simplify it:

$$v_A = \frac{O_1}{P_A^\alpha}\left(\frac{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A1})}{f(c_{A1})+f(c_{B1})+f(c_{C1})}}{\frac{P_A^\alpha\cdot f(c_{A1})+P_A^\alpha\cdot f(c_{B1})+P_A^\alpha\cdot f(c_{C1})}{(P_A^\alpha+P_B^\alpha+P_C^\alpha)\cdot(f(c_{A1})+f(c_{B1})+f(c_{C1}))}}\right)+\frac{O_2}{P_A^\alpha}\left(\frac{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A2})}{f(c_{A2})+f(c_{B2})+f(c_{C2})}}{\frac{P_A^\alpha\cdot f(c_{A2})+P_A^\alpha\cdot f(c_{B2})+P_A^\alpha\cdot f(c_{C2})}{(P_A^\alpha+P_B^\alpha+P_C^\alpha)\cdot(f(c_{A2})+f(c_{B2})+f(c_{C2}))}}\right)+$$

$$\frac{O_3}{P_A^\alpha}\left(\frac{\frac{P_A^\alpha}{P_A^\alpha+P_B^\alpha+P_C^\alpha}\cdot\frac{f(c_{A3})}{f(c_{A3})+f(c_{B3})+f(c_{C3})}}{\frac{P_A^\alpha\cdot f(c_{A3})+P_A^\alpha\cdot f(c_{B3})+P_A^\alpha\cdot f(c_{C3})}{(P_A^\alpha+P_B^\alpha+P_C^\alpha)\cdot(f(c_{A3})+f(c_{B3})+f(c_{C3}))}}\right)$$

See how the denominator of the denominator is the same as the denominator of the numerator's denominator for each J (J=1, J=2, and J=3)? Let's cancel those out and simplify:

$$v_A = \frac{O_1}{P_A^\alpha}\left(\frac{P_A^\alpha\cdot f(c_{A1})}{P_A^\alpha\cdot f(c_{A1})+P_A^\alpha\cdot f(c_{B1})+P_A^\alpha\cdot f(c_{C1})}+\frac{O_2}{P_A^\alpha}\frac{P_A^\alpha\cdot f(c_{A2})}{P_A^\alpha\cdot f(c_{A2})+P_A^\alpha\cdot f(c_{B2})+P_A^\alpha\cdot f(c_{C2})}+\right.$$

$$\left.\frac{O_3}{P_A^\alpha}\frac{P_A^\alpha\cdot f(c_{A3})}{P_A^\alpha\cdot f(c_{A3})+P_A^\alpha\cdot f(c_{B3})+P_A^\alpha\cdot f(c_{C3})}\right)$$

Next, see how we can cancel out the P_A^α? Let's do that.

$$v_A = O_1\left(\frac{f(c_{A1})}{P_A^\alpha\cdot f(c_{A1})+P_B^\alpha\cdot f(c_{B1})+P_C^\alpha\cdot f(c_{C1})}+O_2\frac{f(c_{A2})}{P_A^\alpha\cdot f(c_{A2})+P_B^\alpha\cdot f(c_{B2})+P_C^\alpha\cdot f(c_{C2})}+\right.$$

$$\left.O_3\frac{f(c_{A3})}{P_A^\alpha\cdot f(c_{A3})+P_B^\alpha\cdot f(c_{B3})+P_C^\alpha\cdot f(c_{C3})}\right)$$

6.3. balanced 2 step floating catchment approach (B2SFCA)

As discussed by Paez et al. (2019), in the 2SFCA, the PPR calculation in the first step and allocation of PPR to origins in the second step is not *proportional* to the total population seeking opportunities. Though the 'potential' for interaction is being consistently allocated in these two steps, when looking to decipher the meaning of the measure from the perspective of allocation, the resulting values are difficult to interpret. This issue of interpretability has been attempted to be remedied by adjusting the population and opportunities in both steps by a *proportional* travel cost in the B2SFCA as follows.

Step one:

$$
\begin{aligned}
&R_j = \frac{O_j}{\sum_i P_i \frac{f(c_{ij})}{\sum_j f(c_{ij})}} \\
&R_1 = \frac{100000}{60000 \frac{\exp(\beta*15)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)} + 160000 \frac{\exp(\beta*30)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)} + 10000 \frac{\exp(\beta*100)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)}} \\
&R_1 = 1.278 \\
&R_2 = \frac{100000}{60000 \frac{\exp(\beta*30)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)} + 160000 \frac{\exp(\beta*15)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)} + 10000 \frac{\exp(\beta*100)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)}} \\
&R_2 = 0.706 \\
&R_3 = \frac{10000}{60000 \frac{\exp(\beta*100)}{\exp(\beta*15)+\exp(\beta*100)+\exp(\beta*100)} + 160000 \frac{\exp(\beta*15)}{\exp(\beta*15)+\exp(\beta*100)+\exp(\beta*100)} + 10000 \frac{\exp(\beta*15)}{\exp(\beta*15)+\exp(\beta*100)+\exp(\beta*100)}} \\
&R_3 = 0.995
\end{aligned}
\tag{23}
$$

Step two:

$$
\begin{aligned}
&A_i = \sum_j R_j \frac{f(c_{ij})}{\sum_j f(c_{ij})} \\
&A_A = 1.278 \frac{\exp(\beta*15)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)} + 0.706 \frac{\exp(\beta*30)}{(\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100))} + 0.995 \frac{\exp(\beta*100)}{(\exp(\beta*15)+\exp(\beta*30)+\mathrm{exp}} \\
&A_B = 1.278 \frac{\exp(\beta*30)}{\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100)} + 0.706 \frac{\exp(\beta*15)}{(\exp(\beta*15)+\exp(\beta*30)+\exp(\beta*100))} + 0.995 \frac{\exp(\beta*100)}{(\exp(\beta*15)+\exp(\beta*30)+\mathrm{exp}} \\
&A_C = 1.278 \frac{\exp(\beta*100)}{\exp(\beta*15)+\exp(\beta*100)+\exp(\beta*100)} + 0.706 \frac{\exp(\beta*100)}{(\exp(\beta*15)+\exp(\beta*100)+\exp(\beta*100))} + 0.995 \frac{\exp(\beta*15}{(\exp(\beta*15)+\exp(\beta*100)+} \\
&A_A = 1.173 \\
&A_B = 0.810 \\
&A_C = 0.996
\end{aligned}
\tag{24}
$$

In the B2SFCA, the PPR R_j for each employment center can be interpreted as the total number of jobs accessible to the total population after being *proportionally* adjusted to the travel cost. The PPR R_j is then allocated, proportionally based on travel cost, to each employment center. For this reason, the sum of all A_i adds up to 2.98, the same value as the sum of all R_j.

6.4. Inverse Balancing accessibility, doubly-constrained

This measure results in a opportunities per person metric, however, it constraints opportunities and population from both sides, estimating accessibility iteratively. The formulation requires the number of opportunities equals the population so they propose the iterative estimates are standardized such that opportunities equals population and the disbalanced is carried through as a factor. The formulation for the synthetic example would take the following form:

$$
\begin{aligned}
A_i &= \frac{\bar{A}^o}{\bar{A}^c} \sum_{j=1}^{J} \frac{O_j f(c_{ij})}{L_j} \\
L_i &= \sum_{i=1}^{I} \frac{P_i f(c_{ij})}{A_i}
\end{aligned}
\tag{25}
$$

Iteration 1:

$$
\begin{aligned}
A_{P1} &= (1) * \tfrac{750 \exp(\beta * 2548.1) + 220 * \exp(\beta * 5419.1)}{1} = 16.47 \\
A_{P2} &= (1) * \tfrac{750 \exp(\beta * 1314.1) + 220 * \exp(\beta * 4762.6)}{1} = 104.65 \\
A_{P3} &= (1) * \tfrac{750 \exp(\beta * 2170.2) + 220 * \exp(\beta * 1790.1)}{1} = 43.93 \\
L_{E1} &= \tfrac{260 \exp(\beta * 2548.1)}{16.47} + \tfrac{255 * \exp(\beta * 1314.1)}{104.65} + \tfrac{495 * \exp(\beta * 2170.2)}{43.93} = 1.12 \\
L_{E2} &= \tfrac{260 \exp(\beta * 5419.1)}{16.47} + \tfrac{255 * \exp(\beta * 4762.6)}{104.65} + \tfrac{495 * \exp(\beta * 1790.1)}{43.93} = 0.78
\end{aligned}
\tag{26}
$$

We can complete 8 more iterations until we reach convergence at the second decimal level. I skip writing them out but the final A_i values appear like:

$$
\begin{aligned}
A_{P1} &= 14.56 \\
A_{P2} &= 92.35 \\
A_{P3} &= 47.05
\end{aligned}
\tag{27}
$$

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