Tools_for_combining_measurements

September 28, 2021

Contents

1 The inputs

Suppose we have,

- n experimental results, denoted as y_i = $\{y_1, y_2, y_3 \dots\ y_n\}$
- Covariance matrix of the measurements $M_{ij} = \text{cov}(y_i,y_j)$ is a $n \times n$ matrix
- N observables, X_α = $\{X_1,X_2, X_3 \dots\ X_N\}$

So it's obvious $n = \sum_{\alpha=1}^{N} n_\alpha \geq N$

- The link between measurement y_i and the observables X_α is denoted by a $n \times N$ matrix,

amsmath

$$U_{i\alpha} = \begin{cases} 1, & \text{if } y_i \text{ is a measurement of} X_\alpha \\ 0, & \text{if } y_i \text{ is a not measurement of} X_\alpha \end{cases}$$

2 The desired outputs

- **Observable Estimation:** $\hat{x}_\alpha = \lambda_{\alpha i} y_i$ as estimation of the observable X_α
- **Covariance matrix of measured observables:**as cov($\hat{x}_\alpha$,$\hat{x}_\beta$) = $\lambda_{\alpha i} M_{ij} \lambda_{\beta j}$ = $(\lambda M \lambda^T)_{\alpha\beta}$
- The ref. says $\lambda = (U^T M^{-1} U)^{-1} (U^T M^{-1})$, or in index notation $\lambda_{\alpha i} = \sum_{\beta=1}^{N} (U^T M^{-1} U)^{-1}_{\alpha\beta} (U^T M^{-1})_{\beta i}$.
- Puting that in covariance matrix expression we get, cov($\hat{x}_\alpha$,$\hat{x}_\beta$)= $(U^T M^{-1} U)^{-1}_{\alpha\beta}$
- **Decomposition of covariance matrix to statistical and systematics:** Suppose the covariance of measurements can be written as sum of statistical and systematic uncertainty like $M_{ij} = M_{ij}^{\text{stat}} + M_{ij}^{\text{sys}}$. The covariance matrix of observables can also be decomposed as,

$$\text{cov}(\hat{x}_\alpha, \hat{x}_\beta) = (\lambda M^{\text{stat}} \lambda^T)_{\alpha\beta} + (\lambda M^{\text{sys}} \lambda^T)_{\alpha\beta}$$

3 Implementation in python script

```python
import numpy as np
from scipy import stats
from sympy import Matrix
def combine_measurement(y, U, M, stat=None, sys=None):
    """
    Combine measurements and give estimated linear observables with covariance matrix
    @param list Measurements
    @param np.array The link between measurement and observables
    @param np.array Covariance matric
    @param np.array (optional) Statistical component of Covariance matrix
    @param np.array (optional) Systematics component of Covariance matrix
    return:
    """

    n = U.shape[0]              # to find the number of measurement
    if len(U.shape) > 1:
        N = U.shape[1]
    else:
        N = 1
    Minv = np.linalg.inv(M)         # M^-1
    uT_Minv = np.dot(U.T,Minv)      # UT.M^-1
    uT_Minv_u = np.dot(uT_Minv,U)  #UT.M^-1.U

    if len(uT_Minv_u.shape) > 0:
        lambd =  np.dot(np.linalg.inv(uT_Minv_u),uT_Minv) # weight factors calculate
        cov = np.linalg.inv(uT_Minv_u)
    else:
        lambd =  np.multiply(uT_Minv,1/uT_Minv_u)
        cov = 1/uT_Minv_u
```

```
x = np.dot(lambd,y)                                       # Measure observables
                                                          # corrlation matrix

if len(x.shape) == 0:
    print('Measurement:%s+/-%s'%(x,np.sqrt(cov)))
else:
    print('Measurement: X = ')
    display(Matrix(x))
    print('\n Correlation matrix: E = ')
    display(Matrix(cov))
    print('\n sqrt(Correlation matrix): sqrt{E}')
    display(Matrix(np.sqrt(cov)))
    print('\n\n')

# If we need statistical and systematics component individually
if stat is not None or sys is not None:
    stat_cov = np.dot(np.dot(lambd,stat),lambd.T)
    print('\n Correlation matrix: E = ')
    display(Matrix(stat_cov))
    print('\n sqrt(Correlation matrix): sqrt{E}')
    display(Matrix(np.sqrt(stat_cov)))
    print('\n\n')


    sys_cov = np.dot(np.dot(lambd,sys),lambd.T)
    print('\n Correlation matrix: E = ')
    display(Matrix(sys_cov))
    print('\n sqrt(Correlation matrix): sqrt{E}')
    display(Matrix(np.sqrt(sys_cov)))
    print('\n\n')

yprime = np.subtract(y,np.dot(U,x))
chi_2 = np.dot(np.dot(yprime.T,Minv),yprime)
p = stats.chi2.pdf(chi_2 , n-N)
print('Minimum chisqauare for the combination is %s (d.o.f = %s) '+\
      'with p value %s'%(chi_2,n-N, p))
```

4 Example

Measurement of Branching fraction in e and τ decay channel in two different experiment.

- $\hat{\mathcal{B}}^e_A = (10.50 \pm 1)\%$
- $\hat{\mathcal{B}}^e_B = (13.50 \pm 3)\%$
- $\hat{\mathcal{B}}^\tau_A = (9.50 \pm 3)\%$
- $\hat{\mathcal{B}}^\tau_B = (14.00 \pm 3)\%$

So using the same notation, four meaurements, $B_i = \{\hat{\mathcal{B}}^e_A, \hat{\mathcal{B}}^e_B, \hat{\mathcal{B}}^\tau_A, \hat{\mathcal{B}}^\tau_B\}$ and define two observables

$B_\alpha = \{B^e, B^\tau\}$.

The link matrix U =

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

4.1 No correlations

So the measurement matrix will be: y=

$$\begin{bmatrix} 0.105 \\ 0.135 \\ 0.095 \\ 0.14 \end{bmatrix}$$

and its covariance matrix will be: M=

$$\begin{bmatrix} 0.0001 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0009 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0009 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0009 \end{bmatrix}$$

- Now lets use the script to get the output executing,

`combine_measurement(y, U, M)`

The weights:

$$\begin{bmatrix} 0.9 & 0.1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.5 \end{bmatrix}$$

Measurement: X =

$$\begin{bmatrix} 0.108 \\ 0.1175 \end{bmatrix}$$

Covariance matrix: E =

$$\begin{bmatrix} 9.0 \cdot 10^{-5} & 0.0 \\ 0.0 & 0.00045 \end{bmatrix}$$

sqrt(Covariance matrix): sqrt{E}

$$\begin{bmatrix} 0.00948683298050514 & 0.0 \\ 0.0 & 0.0212132034355964 \end{bmatrix}$$

```
Minimum chisqauare for the combination is 2.0250000000000012 (d.o.f = 2)  with p
value 0.1816547846795055
```

Now suppose assuming LFU we are interested in one observables $B_\alpha = \{B^\ell\}$.

```
The link matrix U =
```

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The script gives us,

```
The weights:
```

$$\begin{bmatrix} 0.75 \\ 0.0833333333333333 \\ 0.0833333333333333 \\ 0.0833333333333333 \end{bmatrix}$$

```
Measurement:0.10958333333333332+/-0.008660254037844387
Minimum chisqauare for the combination is 2.192129629629631 (d.o.f = 3)  with p
value 0.19739142442639945
```

4.2 Correlations between measuremens of same observables

Let's assume that a 15% correlation exists between measrements performed by A and B for the same errors given before. So correlatoion will be $\rho\sigma_1\sigma_1 = 0.15 \times 1 \times 3 \times 10^{-4} = 0.45 \times 10^{-4}$

```
The covariance matrix will be: M=
```

$$\begin{bmatrix} 0.0001 & 4.5 \cdot 10^{-5} & 0.0 & 0.0 \\ 4.5 \cdot 10^{-5} & 0.0009 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0009 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0009 \end{bmatrix}$$

- Measurement of $B_\alpha = \{B^e, B^\tau\}$ gives,

```
The weights:
```

$$\begin{bmatrix} 0.93956043956044 & 0.0604395604395604 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.5 \end{bmatrix}$$

```
Measurement: X =
```

$$\begin{bmatrix} 0.106813186813187 \\ 0.1175 \end{bmatrix}$$

```
Covariance matrix: E =
```

$$\begin{bmatrix} 9.66758241758242 \cdot 10^{-5} & 0.0 \\ 0.0 & 0.00045 \end{bmatrix}$$

```
sqrt(Covariance matrix): sqrt{E}
```

$$\begin{bmatrix} 0.00983238649442871 & 0.0 \\ 0.0 & 0.0212132034355964 \end{bmatrix}$$

```
Minimum chisqauare for the combination is 2.1140109890109904 (d.o.f = 2)  with p
value 0.17374741452746106
```

- Measurement of $B_\alpha = \{B^\ell\}$ gives,

```
The weights:
```

$$\begin{bmatrix} 0.773405698778833 \\ 0.0497512437810945 \\ 0.0884215287200362 \\ 0.0884215287200362 \end{bmatrix}$$

```
Measurement:0.10870307553143374+/-0.008920727316089902
Minimum chisqauare for the combination is 2.3229245188200434 (d.o.f = 3)  with p
value 0.19033162900540035
```

4.3 Correlation between measurements of different observables

4.3.1 Positive correlation

Now suppose +99.5% correlation exists between measurements of B^e and B^τ performed by B, while the results of A and B are uncorrelated. In the same experiment there are se

```
The covariance matrix will be: M=
```

$$\begin{bmatrix} 0.0001 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0009 & 0.0 & 0.000896 \\ 0.0 & 0.0 & 0.0009 & 0.0 \\ 0.0 & 0.000896 & 0.0 & 0.0009 \end{bmatrix}$$

- This leads to measurement of $B_\alpha = \{B^e, B^\tau\}$ as,

```
The weights:
```

$$\begin{bmatrix} 0.819491688595084 & 0.180508311404916 & 0.0898530261215584 & -0.089853026121558 \\ 0.808677235094026 & -0.808677235094026 & 0.0974584429754188 & 0.902541557024581 \end{bmatrix}$$

```
Measurement: X =
```

$$\begin{bmatrix} 0.106371863166678 \\ 0.111354053013285 \end{bmatrix}$$

```
Covariance matrix: E =
```

$$\begin{bmatrix} 8.19491688595084 \cdot 10^{-5} & 8.08677235094026 \cdot 10^{-5} \\ 8.08677235094026 \cdot 10^{-5} & 8.77125986778769 \cdot 10^{-5} \end{bmatrix}$$

```
sqrt(Covariance matrix): sqrt{E}
```

$$\begin{bmatrix} 0.00905257802283463 & 0.00899264830344224 \\ 0.00899264830344224 & 0.0093655004499427 \end{bmatrix}$$

```
Minimum chisqauare for the combination is 1.22926160066748 (d.o.f = 2)  with p
value 0.2704202682994562
```

- This leads to measurement of $B_\alpha = \{B^\ell\}$ as,

```
The weights:
```

$$\begin{bmatrix} 0.818016194331984 \\ 0.0455465587044532 \\ 0.0908906882591094 \\ 0.0455465587044532 \end{bmatrix}$$

```
Measurement:0.10705161943319838+/-0.009044424770719166
Minimum chisqauare for the combination is 4.360880566801648 (d.o.f = 3)  with p
value 0.09413345065568042
```

4.3.2 Negetive correlation

In the example above let the correlation between $\hat{\mathcal{B}}^e_A$ and $\hat{\mathcal{B}}^e_A$ be equal to -99.5%.

Negetive correlation can be understood in this case as mis-identification of electron as tau or vice virsa.

```
The covariance matrix will be: M=
```

$$\begin{bmatrix} 0.0001 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0009 & 0.0 & -0.000896 \\ 0.0 & 0.0 & 0.0009 & 0.0 \\ 0.0 & -0.000896 & 0.0 & 0.0009 \end{bmatrix}$$

- This leads to measurement of $B_\alpha = \{B^e, B^\tau\}$ as,

```
The weights:
```

$$\begin{bmatrix} 0.819491688595084 & 0.180508311404916 & -0.0898530261215584 & 0.089853026121558 \\ -0.808677235094026 & 0.808677235094026 & 0.0974584429754188 & 0.902541557024581 \end{bmatrix}$$

```
Measurement: X =
```

$$\begin{bmatrix} 0.114458635517618 \\ 0.159874687118927 \end{bmatrix}$$

```
Covariance matrix: E =
```

$$\begin{bmatrix} 8.19491688595084 \cdot 10^{-5} & -8.08677235094026 \cdot 10^{-5} \\ -8.08677235094026 \cdot 10^{-5} & 8.77125986778769 \cdot 10^{-5} \end{bmatrix}$$

```
sqrt(Covariance matrix): sqrt{E}
```

```
/home/soumen/source/anaconda/lib/python3.7/site-
packages/ipykernel_launcher.py:43: RuntimeWarning: invalid value encountered in
sqrt
```

$$\begin{bmatrix} 0.00905257802283463 & \mathrm{NaN} \\ \mathrm{NaN} & 0.0093655004499427 \end{bmatrix}$$

```
Minimum chisqauare for the combination is 6.081325011231629 (d.o.f = 2)  with p
value 0.02390160455332634
```

- This leads to measurement of $B_\alpha = \{B^\ell\}$ as,

```
The weights:
```

$$\begin{bmatrix} 0.0195652173913041 \\ 0.489130434782609 \\ 0.00217391304347824 \\ 0.489130434782609 \end{bmatrix}$$

```
Measurement:0.13677173913043478+/-0.0013987572123604632
Minimum chisqauare for the combination is 12.305329476130543 (d.o.f = 3)  with p
value 0.002977750761029625
```

4.3.3 Breakdown of error contributions

```
The statictical compoments of covariance matrix is :  MStat
```

$$\begin{bmatrix} 0.0001 & 0.0 & 0.0 & 0.0 \\ 0.0 & 4.0 \cdot 10^{-6} & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0009 & 0.0 \\ 0.0 & 0.0 & 0.0 & 4.0 \cdot 10^{-6} \end{bmatrix}$$

- For $\hat{\mathcal{B}}^e_B$ and $\hat{\mathcal{B}}^\tau_B$ +100% correlated scenario
  - $B_\alpha = \{B^e, B^\tau\}$
  - $B_\alpha = \{B^\ell\}$

```
The systematic compoments of covariance matrix is :  MSys
```

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.000896 & 0.0 & 0.000896 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.000896 & 0.0 & 0.000896 \end{bmatrix}$$

```
The weights:
```

$$\begin{bmatrix} 0.819491688595084 & 0.180508311404916 & 0.0898530261215584 & -0.089853026121558 \\ 0.808677235094026 & -0.808677235094026 & 0.0974584429754188 & 0.902541557024581 \end{bmatrix}$$

```
Measurement: X =
```

$$\begin{bmatrix} 0.106371863166678 \\ 0.111354053013285 \end{bmatrix}$$

```
Covariance matrix: E =
```

$$\begin{bmatrix} 8.19491688595084 \cdot 10^{-5} & 8.08677235094026 \cdot 10^{-5} \\ 8.08677235094026 \cdot 10^{-5} & 8.77125986778769 \cdot 10^{-5} \end{bmatrix}$$

```
sqrt(Covariance matrix): sqrt{E}
```

$$\begin{bmatrix} 0.00905257802283463 & 0.00899264830344224 \\ 0.00899264830344224 & 0.0093655004499427 \end{bmatrix}$$

```
Estat =
```

$$\begin{bmatrix} 7.45854997076814 \cdot 10^{-5} & 7.32433935026436 \cdot 10^{-5} \\ 7.32433935026436 \cdot 10^{-5} & 7.98183808832681 \cdot 10^{-5} \end{bmatrix}$$

```
sqrt(Estat) =
```

$$\begin{bmatrix} 0.00863628969567843 & 0.00855823541991243 \\ 0.00855823541991243 & 0.00893411332384295 \end{bmatrix}$$

```
Esys =
```

$$\begin{bmatrix} 7.36366915182715 \cdot 10^{-6} & 7.62433000675895 \cdot 10^{-6} \\ 7.62433000675896 \cdot 10^{-6} & 7.89421779460871 \cdot 10^{-6} \end{bmatrix}$$

```
sqrt(Esys) =
```

$$\begin{bmatrix} 0.00271360814264461 & 0.00276121893495589 \\ 0.00276121893495589 & 0.00280966506804792 \end{bmatrix}$$

```
Minimum chisqauare for the combination is 1.22926160066748 (d.o.f = 2)  with p
value 0.2704202682994562
```

```
The weights:
```

$$\begin{bmatrix} 0.818016194331984 \\ 0.0455465587044532 \\ 0.0908906882591094 \\ 0.0455465587044532 \end{bmatrix}$$

```
Measurement:0.10705161943319838+/-0.009044424770719166
Measurement:0.10705161943319838+/-0.008623610080587478 +/-0.002726713885098403
Minimum chisqauare for the combination is 4.360880566801648 (d.o.f = 3)  with p
value 0.09413345065568042
```

- For $\hat{\mathcal{B}}_B^e$ and $\hat{\mathcal{B}}_B^\tau$ uncorrelated scenario
  - $B_\alpha = \{B^e, B^\tau\}$
  - $B_\alpha = \{B^\ell\}$

```
The systematic compoments of covariance matrix is :  MSys
```

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.000896 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.000896 \end{bmatrix}$$

```
The weights:
```

$$\begin{bmatrix} 0.9 & 0.1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.5 \end{bmatrix}$$

```
Measurement: X =
```

$$\begin{bmatrix} 0.108 \\ 0.1175 \end{bmatrix}$$

```
 Covariance matrix: E =
```

$$\begin{bmatrix} 9.0 \cdot 10^{-5} & 0.0 \\ 0.0 & 0.00045 \end{bmatrix}$$

```
 sqrt(Covariance matrix): sqrt{E}
```

$$\begin{bmatrix} 0.00948683298050514 & 0.0 \\ 0.0 & 0.0212132034355964 \end{bmatrix}$$

```
Estat =
```

$$\begin{bmatrix} 8.104 \cdot 10^{-5} & 0.0 \\ 0.0 & 0.000226 \end{bmatrix}$$

```
sqrt(Estat) =
```

$$\begin{bmatrix} 0.00900222194794152 & 0.0 \\ 0.0 & 0.0150332963783729 \end{bmatrix}$$

```
Esys =
```

$$\begin{bmatrix} 8.96 \cdot 10^{-6} & 0.0 \\ 0.0 & 0.000224 \end{bmatrix}$$

```
sqrt(Esys) =
```

$$\begin{bmatrix} 0.00299332590941915 & 0.0 \\ 0.0 & 0.0149666295470958 \end{bmatrix}$$

```
Minimum chisqauare for the combination is 2.0250000000000012 (d.o.f = 2)  with p
value 0.1816547846795055
```

```
The weights:
```

$$\begin{bmatrix} 0.75 \\ 0.0833333333333333 \\ 0.0833333333333333 \\ 0.0833333333333333 \end{bmatrix}$$

```
Measurement:0.10958333333333332+/-0.008660254037844387
Measurement:0.10958333333333332+/-0.007909207011803114 +/-0.0035276684147527875
Minimum chisqauare for the combination is 2.192129629629631 (d.o.f = 3)  with p
value 0.19739142442639945
```

- For $\hat{\mathcal{B}}_B^e$ and $\hat{\mathcal{B}}_B^\tau$ -100% scenario
  - $B_\alpha = \{B^e, B^\tau\}$
  - $B_\alpha = \{B^\ell\}$

```
The systematic compoments of covariance matrix is :  MSys
```

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.000896 & 0.0 & -0.000896 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -0.000896 & 0.0 & 0.000896 \end{bmatrix}$$

```
The weights:
```

$$\begin{bmatrix} 0.819491688595084 & 0.180508311404916 & -0.0898530261215584 & 0.089853026121558 \\ -0.808677235094026 & 0.808677235094026 & 0.0974584429754188 & 0.902541557024581 \end{bmatrix}$$

```
Measurement: X =
```

$$\begin{bmatrix} 0.114458635517618 \\ 0.159874687118927 \end{bmatrix}$$

```
 Covariance matrix: E =
```

$$\begin{bmatrix} 8.19491688595084 \cdot 10^{-5} & -8.08677235094026 \cdot 10^{-5} \\ -8.08677235094026 \cdot 10^{-5} & 8.77125986778769 \cdot 10^{-5} \end{bmatrix}$$

```
 sqrt(Covariance matrix): sqrt{E}
```

```
/home/soumen/source/anaconda/lib/python3.7/site-
packages/ipykernel_launcher.py:43: RuntimeWarning: invalid value encountered in
sqrt
```

$$\begin{bmatrix} 0.00905257802283463 & \mathrm{NaN} \\ \mathrm{NaN} & 0.0093655004499427 \end{bmatrix}$$

```
 Estat =
```

$$\begin{bmatrix} 7.45854997076814 \cdot 10^{-5} & -7.32433935026436 \cdot 10^{-5} \\ -7.32433935026436 \cdot 10^{-5} & 7.98183808832681 \cdot 10^{-5} \end{bmatrix}$$

```
 sqrt(Estat) =
```

```
/home/soumen/source/anaconda/lib/python3.7/site-
packages/ipykernel_launcher.py:57: RuntimeWarning: invalid value encountered in
sqrt
```

$$\begin{bmatrix} 0.00863628969567843 & \mathrm{NaN} \\ \mathrm{NaN} & 0.00893411332384295 \end{bmatrix}$$

```
Esys =
```

$$\begin{bmatrix} 7.36366915182715 \cdot 10^{-6} & -7.62433000675895 \cdot 10^{-6} \\ -7.62433000675896 \cdot 10^{-6} & 7.89421779460871 \cdot 10^{-6} \end{bmatrix}$$

```
sqrt(Esys) =
```

```
/home/soumen/source/anaconda/lib/python3.7/site-
packages/ipykernel_launcher.py:65: RuntimeWarning: invalid value encountered in
sqrt
```

$$\begin{bmatrix} 0.00271360814264461 & \mathrm{NaN} \\ \mathrm{NaN} & 0.00280966506804792 \end{bmatrix}$$

```
Minimum chisqauare for the combination is 6.081325011231629 (d.o.f = 2)  with p
value 0.02390160455332634
```

```
The weights:
```

$$\begin{bmatrix} 0.0195652173913041 \\ 0.489130434782609 \\ 0.00217391304347824 \\ 0.489130434782609 \end{bmatrix}$$

```
Measurement:0.13677173913043478+/-0.0013987572123604632
Measurement:0.13677173913043478+/-0.0013987572123604708 +/-0.0
Minimum chisqauare for the combination is 12.305329476130543 (d.o.f = 3)  with p
value 0.002977750761029625
```

- Under the assumption systematic uncertainty is zero,
  - $B_\alpha = \{B^e, B^\tau\}$
  - $B_\alpha = \{B^\ell\}$

```
The statistical compoments of covariance matrix is :  MStat
```

$$\begin{bmatrix} 0.0001 & 0.0 & 0.0 & 0.0 \\ 0.0 & 4.41 \cdot 10^{-6} & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0009 & 0.0 \\ 0.0 & 0.0 & 0.0 & 4.41 \cdot 10^{-6} \end{bmatrix}$$

```
The weights:
```

$$\begin{bmatrix} 0.0422373335887367 & 0.957762666411263 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.00487610707533088 & 0.995123892924669 \end{bmatrix}$$

```
Measurement: X =
```

$$\begin{bmatrix} 0.133732879992338 \\ 0.13978057518161 \end{bmatrix}$$

Covariance matrix: E =

$$\begin{bmatrix} 4.22373335887367 \cdot 10^{-6} & 0.0 \\ 0.0 & 4.38849636779779 \cdot 10^{-6} \end{bmatrix}$$

sqrt(Covariance matrix): sqrt{E}

$$\begin{bmatrix} 0.00205517234286414 & 0.0 \\ 0.0 & 0.00209487383099742 \end{bmatrix}$$

Estat =

$$\begin{bmatrix} 4.22373335887367 \cdot 10^{-6} & 0.0 \\ 0.0 & 4.38849636779779 \cdot 10^{-6} \end{bmatrix}$$

sqrt(Estat) =

$$\begin{bmatrix} 0.00205517234286414 & 0.0 \\ 0.0 & 0.00209487383099742 \end{bmatrix}$$

Esys =

$$\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

sqrt(Esys) =

$$\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

Minimum chisqauare for the combination is 10.858892756781882 (d.o.f = 2) with p value 0.0021927615255231438

The weights:

$$\begin{bmatrix} 0.0215226939970717 \\ 0.488042947779405 \\ 0.00239141044411908 \\ 0.488042947779405 \end{bmatrix}$$

```
Measurement:0.13669887750122012+/-0.0014670614846376325
Measurement:0.13669887750122012+/-0.0014670614846376325 +/-0.0
Minimum chisqauare for the combination is 15.105715967857792 (d.o.f = 3)  with p
value 0.0008134224940035461
```

5 Negetive weights and interpretation

You might have been noticed some of the weights are negetive. This cannot be interpreted properly. We have to take special treatment for those cases. The references below shows proper guide map.

- Valassi, A., Chierici, R.: Information and treatment of unknown correlations in the combination of measurements using the BLUE method. Eur. Phys. J. C 74, 2717 (2014)
- Lista, L.: The bias of the unbiased estimator: a study of the iterative application of the BLUE method. Nucl. Inst. Methods A764, 82–93 (2014) and corr. ibid. A773, 87–96 (2015)

Moreover there are efforts to include these algorithm to ROOT framework. Here is the link of th webpage.