Stochastic Processes, Brownian Motion, and Time Series

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1. Formal Definitions

- A **stochastic process** is a collection of random variables ${X_t}_{t \in \mathcal{T}}$ indexed by time or space, representing the evolution of a system with inherent randomness.

- A process ${X_t}_{t \in \mathbb{Z}}$ is **weakly stationary** (or covariance stationary) if:

    1. $\mathbb{E}[X_t] = \mu$ (mean is constant over time),
    2. $\text{Var}(X_t) < \infty$ (finite, constant variance),
    3. $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ depends only on lag h, not on t.

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2. Brownian Motion

A **Brownian motion** (or **Wiener process**) ${B_t}_{t \geq 0}$ satisfies:

1. **Initial Condition**:

    $$ B_0 = 0 \quad \text{(almost surely)} $$

2. **Independent Increments**: For $0 \leq t_0 < t_1 < \dots < t_n$, the increments:

    $$ B_{t_1} - B_{t_0},\ B_{t_2} - B_{t_1},\ \dots,\ B_{t_n} - B_{t_{n-1}} $$

    are independent.

3. **Stationary Increments**: For $s < t$,

    $$ B_t - B_s \sim \mathcal{N}(0, t - s) $$

4. **Continuity**: Paths $t \mapsto B_t(\omega)$ are continuous almost surely.

> **Summary Notation**:
>
> $$ B_t \sim \mathcal{N}(0, t), \quad \mathbb{E}[B_t] = 0, \quad \text{Var}(B_t) = t $$
>
> The sample paths are continuous but almost surely nowhere differentiable.

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3. Example: Black-Scholes Price Dynamics

Consider the Black-Scholes stochastic differential equation:

$$ \frac{d S_t} {S_t} = \mu, dt + \sigma, dW_t $$

Here, the return dS_t follows:

$$ dS_t \sim \mathcal{N}(\mu, dt, \sigma^2, dt) $$

This implies that **log-returns** over fixed intervals (e.g. 1 hour) are normally distributed:

Time (T)	**Price (S_T)**	**Change (dS_T)**
t_0	1.20	–
$2t_0$	1.32	0.12
$3t_0$	1.94	0.62
$4t_0$	1.50	–0.44
$5t_0$	1.76	0.26

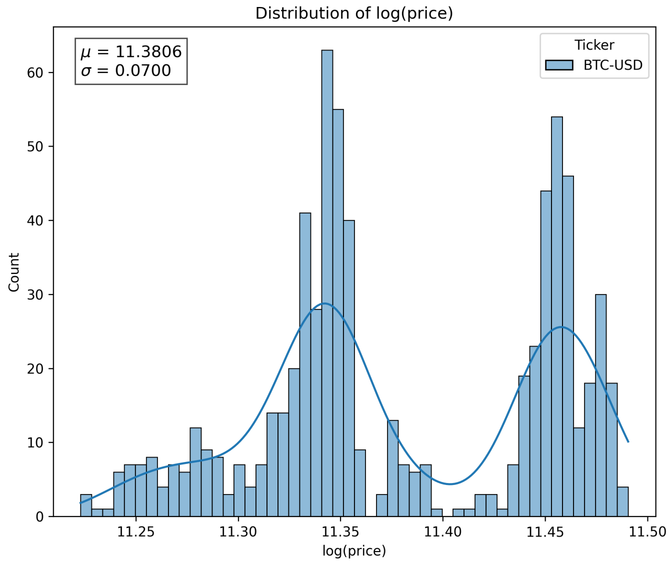
Visualization:

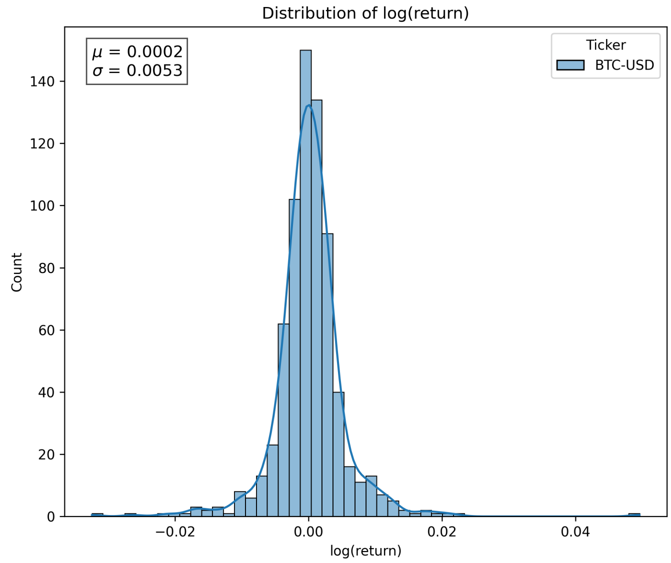
We observe log-price and log-return distributions for:

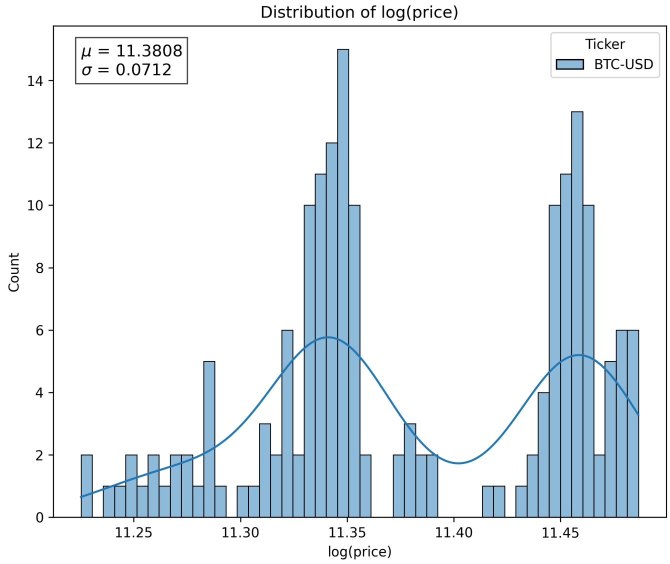
- Upper: $T = 30$ days, $\Delta t = 1$ hour
- Middle: $T = 30$ days, $\Delta t = 4$ hours
- Lower: $T = 90$ days, $\Delta t = 1$ hour

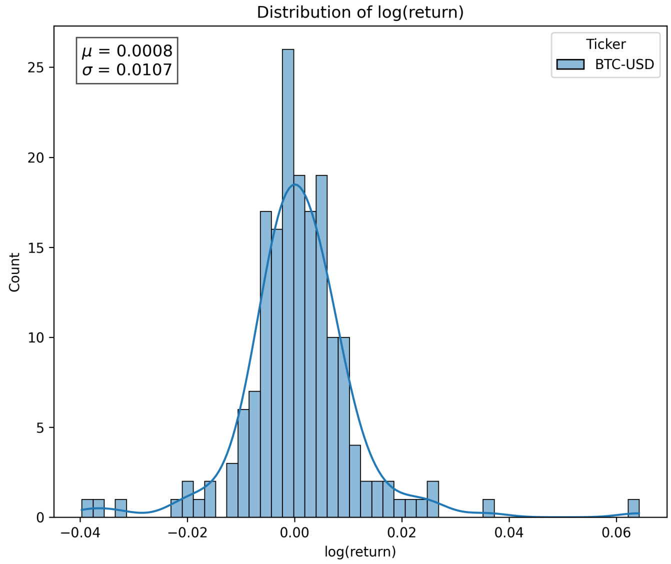
Observations:

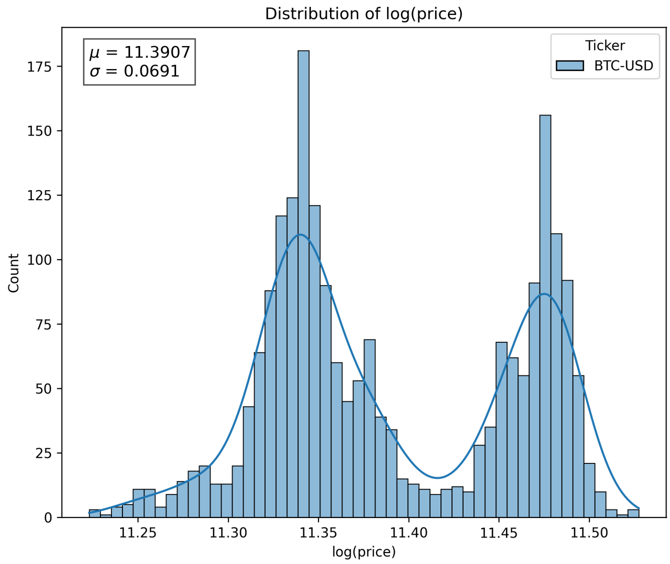
- Larger total period (T) yields better approximation of the true (population) distribution.
- Doubling Δt (timeframe) roughly doubles the return volatility.

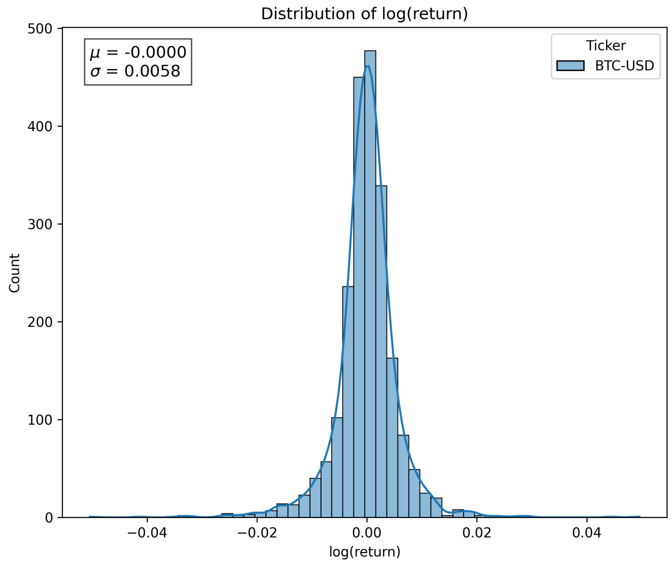












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4. Stationarity of Log-Returns

- Log-return $d \log S_t$ follows $\mathcal{N}(\mu t_0, \sigma^2 t_0)$ for fixed timeframe $\Delta t = t_0$.

- Mean and variance **do not** change with absolute time T, making the process **covariance stationary**.

- Returns at each Δt are **i.i.d.**, so volatility scales with $\sqrt{\Delta t}$, not T.

  Illustration:

- Suppose a price process S_t has log-values:

  $$ \log S_t = {f(t_0), f(2t_0), f(3t_0), f(4t_0), \dots, f(T)}, \quad $$

  $$ \text{where } f(t) \sim \mathcal{N}(\mu_t \log S_0 + (\mu -\sigma^2/2) t , \sigma_t^2=\sigma^2 t) $$

  $$E[\log S_t] = \log S_0 + (\mu -\sigma^2/2) t = \Mu_t$$ $$\text{Var}[\log S_t] = \sigma^2 t = \Sigma_t$$

  This means that the **mean and variance depend on both** the timeframe $\Delta t = t_0$ and the **total time** T.

- In contrast, the log-returns (modeled as dW_t) are:

  $$ \log\left(\frac{S_t}{S_{t - t_0}}\right) = {f(t_0), f(t_0), f(t_0), f(t_0), \dots} $$

  where each term is i.i.d. $\sim \mathcal{N}(\mu t_0, \sigma^2 t_0)$—independent of T.

This shows:

- **Log-prices** evolve over time and are **non-stationary** (variance increases with T),
- **Log-returns** are **stationary**—their distribution depends on Δt, but **not** on T.

4. Stationarity of Log-Returns

- Log-return $d \log S_t$ follows $\mathcal{N}(\mu t_0, \sigma^2 t_0)$ for time frame $\Delta t = t_0$.
- The **mean** and **variance** are time-independent, hence the process is **covariance stationary**.
- Even though returns scale with Δt, they do **not** depend on absolute time T.
- The scaling of volatility follows: $\sigma_{\Delta t} \propto \sqrt{\Delta t}$.

> **Clarification**:
>
> - Stationarity implies that properties do not evolve with time.
> - Timeframe Δt determines variance, not the dataset duration T.

Is log-return an $I(0)$ process?

- Yes, by definition. A stationary process is an $I(0)$ process.

What ARIMA model fits log-return?

- Log-return under Black-Scholes is **white noise**, modeled by **ARIMA(0,0,0)**.

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5. Discrete vs Continuous Processes

Discrete Model	Continuous Counterpart
Random Walk: $X_t = X_{t-1} + \epsilon_t$	Brownian motion: W_t
White Noise: $Y_t = dX_t = \epsilon_t$	$Y_t = dW_t$ (stationary increments)

Random Walk vs First Difference:

- Random walk accumulates white noise:

  $$ X_t = \sum_{i=1}^t \epsilon_i, \quad \text{Var}(X_t) = t\sigma^2 $$

  → **Not stationary**

- First difference:

  $$ Y_t = X_t - X_{t-1} = \epsilon_t $$

  → **Stationary**

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6. Brownian Motion vs Log-Price

- Brownian motion: $\text{Var}(W_t) = t$ → **not stationary**

- Log-price also follows a non-stationary path:

  $$ \mathbb{E}[\log S_t] = \log S_0 + (\mu - \sigma^2/2)t \ \text{Var}[\log S_t] = \sigma^2 t $$

So, log-prices evolve with time. But **log-returns**, defined over fixed Δt, are i.i.d. and **stationary**.

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7. Cointegration and Statistical Arbitrage

- Let X_t, Y_t be two $I(1)$ processes.

- If there exists a linear combination:

  $$ S_t = Y_t - \alpha - \beta X_t $$

  such that S_t is **stationary**, then X_t and Y_t are **cointegrated**.

Is the spread S_t affected by the timeframe Δt?

- **No.**
- Here, the $I(0)$ process is **constructed** via linear combination, not by differencing.
- Therefore, its distribution is independent of Δt (unlike log-return, which depends on Δt).