Stochastic process, Brownian motion, timeseries and all

Some formal defintion

- A **stochastic process** is a collection of random variables indexed by time or space (or more generally, by some index set), representing the evolution of some random quantity over that index.
- A stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ is **weakly/covariant stationary** if:
  1. $\mathbb{E}[X_t] = \mu$, a constant (mean is time-invariant),
  2. $\mathrm{Var}(X_t) = \mathbb{E}[(X_t - \mu)^2] < \infty$ (finite, constant variance),
  3. $\mathrm{Cov}(X_t, X_{t+h}) = \gamma(h)$ depends only on the lag h, not on time t.
- **Definition of Brownian Motion**

  A **Brownian motion** (also called **Wiener process**) is a continuous-time stochastic process $\{B_t\}_{t\geq 0}$ that satisfies the following properties:

  1. **Initial condition**:

  $$B_0 = 0 \quad \text{(almost surely)}$$

  2. **Independent increments**: For any $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$, the increments

  $$B_{t_1} - B_{t_0},\ B_{t_2} - B_{t_1},\ \ldots,\ B_{t_n} - B_{t_{n-1}}$$

  are independent random variables.

  3. **Stationary increments**: For $s < t$, the increment $B_t - B_s$ has a **normal distribution**:

  $$B_t - B_s \sim \mathcal{N}(0, t-s)$$

  4. **Continuity**: The sample paths $t \mapsto B_t(\omega)$ are **almost surely continuous** functions of t.

Notation:

A standard Brownian motion is often written as:

$$B_t \sim \mathcal{N}(0,t), \quad \text{with } B_0 = 0$$

- $\mathbb{E}[B_t] = 0$
- $\mathrm{Var}(B_t) = t$
- B_t has nowhere differentiable paths (almost surely)

Example

Consider a Black-Scholes price below,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

The distribution of dS_t is a normal distribution $\mathcal{N}(\mu dt, \sigma^2 dt)$ as the first component gives deterministic drift and the second component is a random noise with zero mean, of variance $\sigma^2 dt$. Now let's try to digrace this. What does it mean in real life.

- Suppose you have a price (time) series $S_t = \{1.20, 1.32, 1.94, 1.50, 1.76, \}$. The equation is saying if we plot the $d \log S_T$ **then that will be a normal distrbution**.

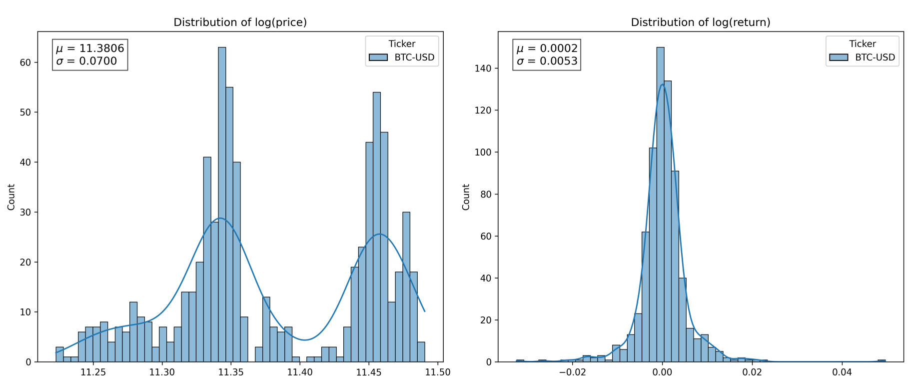
T	S_T	dS_T
t_0	1.20	-
$2t_0$	1.32	0.12
$3t_0$	1.94	0.62
$4t_0$	1.50	-0.44
$5t_0$	1.76	0.26

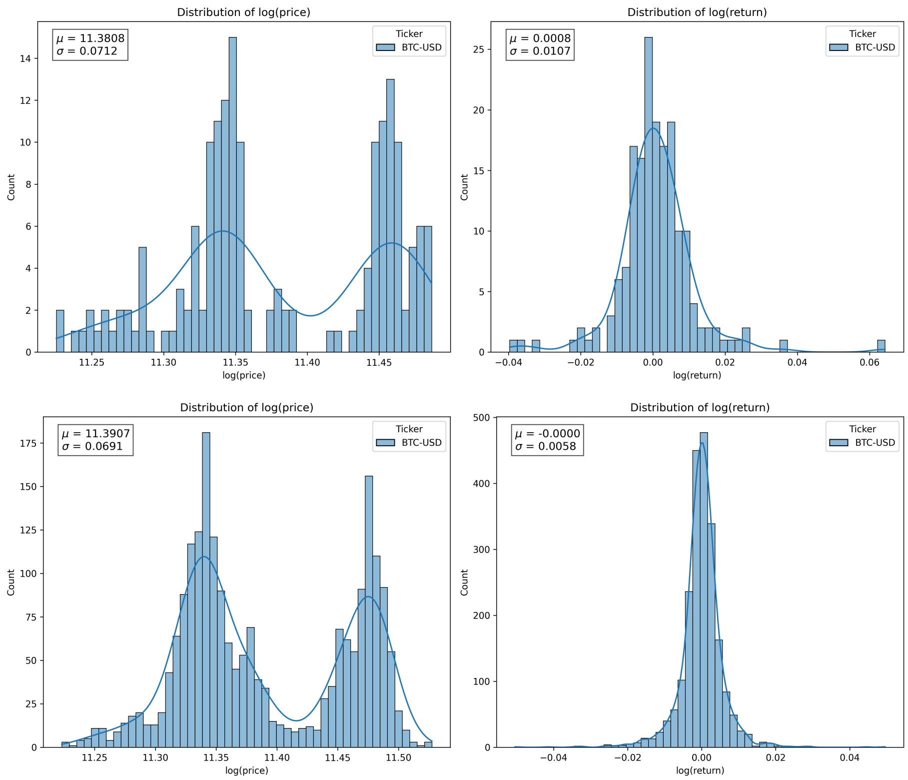
Below I have plotted the log(price) and log(return) for

- Upper ($T = 30\,\text{day}$, $\Delta t = 1\,\text{hr}$)
- Middle ($T = 30\,\text{day}$, $\Delta t = 4\,\text{hr}$)
- Lower ($T = 90\,\text{day}$, $\Delta t = 1\,\text{hr}$)

Notice,

- The third plot has higher T, which makes the sampling distribution more like population distribution.
- The sigma for $\Delta t = 4\,\text{hr}$ is 2x that of $\Delta t = 1\,\text{hr}$





Now what is the mean and sigma of this distribution?

- Here $dt \to t_0$ and hence normal distribution: $\mathcal{N}(\mu t_0, \sigma^2 t_0)$

Is the $d \log S_T$ or log-return is (covariant) stationary process?

- The mean and sigma of the distribution is time independent. Since each time-step is independent (property of brownian motion) the auto-covariance is also zero. Hence it is a covariant stationary process.
- There may arise confustion that they are depending on t_0 then how it is time independent?
  - We are looking for a log(return) of a stock price for a fixed time frame say $t_0 = 1\,\text{hr}$, now with this $t_0 = 1\,\text{hr}$ timeframe data we can make distribution with 1 year or 20 year of data, the normal distribution will resembles population distribution ($T = 1\,\text{yr} \to T = 20\,\text{yr} \equiv$ sampling distribution $\to$ Population distribution)
- Then what about the claim volatality scales with $\sim \sqrt{t}$?
  - To avoid any confustion we can make it $\sim \sqrt{\Delta t}$, and with Δt we mean the timeframe, not absolute time (total duration of data T = 20 year say)

  - When we check log-return of $\Delta t = 1\,\mathrm{yr}$ timeframe, and plot distribution of multiple (successive / simulated) years (T) then the distribution will also will not change (of course if other condition does not change) over the year and will make the sampling distribution towards the Population distribution.
  - But the distribution $\sigma_{1\,\mathrm{yr}} = \sqrt{252}\sigma_{1\,\mathrm{day}}$, so the log(return) is stationary process and mean/width/auto-covariance does not change over time but they depend on the timeframe overwhich these are calculated, 1hr / 1 day/ 1 yr.
- Is the log(return) a $I(0)$ process?
  - By defintion stationary process itself is $I(0)$ process.
- How we can model log(return) with discrete timeseries model in ARIMA?
  - The log (return) of a Black-Scholes price is a white noise ARIMA(0,0,0).

Discrete vs continuous

Discrete	Continuous
Random walk, ARIMA(1, 1, 0), $X_t = X_{t-1} + \epsilon_t$	Brownian motion W_t
White noise (Stationary) ARIMA(0, 0, 0), $Y_t = dX_t = \epsilon_t$	Stationary increment of Brownian motion $Y_t = dW_t$

Random Walk and its first difference

- Random walk $X_t = \sum_{i=1}^{t} \epsilon_i$ is total sum of all the steps of white noise process. So at time t it bags all the stationary increment which makes it dependent on t, $\mathrm{Var}(X_t) = t\mathrm{Var}(\epsilon_0) = t\sigma^2$. Since this depend on time, it's not-stationary process.
- However Y_t does not bag all the increment, but only increment of that particular time, $\mathrm{Var}(Y_t) = \sigma^2$

Brownian motion and its first difference

- According to defition normalized brownian motion's variance $\mathrm{Var}(W_t) = t$. The behaviour is same as discrete one. Since this depend on time, it's not-stationary process.

- As an example we can consider price series, and if we are to make a histogram of the log(price) series it will not give you a normal distribution.

$$E[\log S_t] = \log S_0 + (\mu - \sigma^2/2)t = M_t$$

$$\mathrm{Var}[\log S_t] = \sigma^2 t = \Sigma_t$$

- Suppose $\mathcal{N}(M_{t_0}, \Sigma_{t_0}) = f(t_0)$ is a random variable then So for a price series, the mean and variance depends on $\Delta t = t_0$ and T

$$\log S_t = \{f(t_0), f(2t_0), f(3t_0), f(4t_0), f(5t_0), f(6t_0)...f(T)\}$$

- But for dW_t, all are i.i.d. and hence it will give fixed variance, does not depend on T

$$\log(S_t/S_0) = \{f(t_0), f(t_0), f(t_0), f(t_0), f(t_0), f(t_0)...f(t_0)\}$$

Cointegration and statistical arbitrage spread

Suppose two $I(1)$ series X_t, Y_t are cointegrated, hence there exist a linear combination of these two series $Y_t = \alpha + \beta X_t + S_t$, where S_t is an $I(0)$ process.

If we consider the distribution of S_t series for a timeframe $\Delta t = 1\,\text{hr}$ vs $\Delta t = 4\,\text{hr}$, the width of latter will be 2x that of former?

- **NO**
- Because in earlier case we made $I(1)$ series out of $I(0)$ series by taking the first difference. That caused the difference timeframe Δt, as important parameter, deciding the width of the $I(0)$ series.
- But here Δt has no role, by construction the spread is an $I(0)$ process.

Random walk vs AR(1)

$$X_t = \phi X_{t-1} + \epsilon_t$$

$$X_t = \sum_{k=0}^{t-1} \phi^k \epsilon_{t-k}$$

- Become stationary process (ARIMA(1,0,0)) charecteristic root $1/\phi > 1 \Rightarrow \phi < 1$.We can use earlier expression to calculate mean and variance but for stationary process, we know mean and variance is independent of time, which imply $\text{Var}(X_t) = \text{Var}(X_{t-1})$. So $\text{Var}(X_t) = \phi^2 \text{Var}(X_{t-1}) + \text{Var}(\epsilon_t) \Rightarrow \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$
- For $\phi = 1$ it is a random walk (ARIMA(1, 1, 0)). $X_t = \sum_{i=1}^{t} \epsilon_t$
- For $\phi > 1$ it is a non-stationary process. $X_t = \phi^t + \epsilon_t$, both mean and sigma diverge with time.

White noise vs other stationary process

- White noise are those stationary process whose auto-correlation is zero, that is $\text{Cov}(\epsilon_t, \epsilon_s) = \sigma^2 \delta_{t,s}$.
- Where as for simple AR(1) process $X_t = \phi X_{t-1} + \epsilon_t$, $\text{Cov}(X_t, X_{t-s}) = \frac{\sigma^2 \phi^s}{1-\phi^2}$