Stochastic Processes, Brownian Motion, and Time Series

1. Formal Definitions

- A **stochastic process** is a collection of random variables $X_t\_t \in \mathcal{T}$ indexed by time or space, representing the evolution of a system with inherent randomness.
- A process $X_t\_t \in \mathbb{Z}$ is **weakly stationary** (or covariance stationary) if:
    1. $\mathbb{E}[X_t] = \mu$ (mean is constant over time),
    2. $\mathrm{Var}(X_t) < \infty$ (finite, constant variance),
    3. $\mathrm{Cov}(X_t, X\_t + h) = \gamma(h)$ depends only on lag h, not on t.

---

2. Brownian Motion

A **Brownian motion** (or **Wiener process**) $B_t\_t \geq 0$ satisfies:

1. **Initial Condition**:

$$B_0 = 0 \quad \text{(almost surely)}$$

2. **Independent Increments**: For $0 \leq t_0 < t_1 < \cdots < t_n$, the increments:

$$B_{t_1} - B_{t_0},\ B_{t_2} - B_{t_1},\ \ldots,\ B_{t_n} - B_{t_{n-1}}$$

   are independent.

3. **Stationary Increments**: For $s < t$,

$$B_t - B_s \sim \mathcal{N}(0, t - s)$$

4. **Continuity**: Paths $t \mapsto B_t(\omega)$ are continuous almost surely.

   Summary Notation:

$$B_t \sim \mathcal{N}(0, t), \quad \mathbb{E}[B_t] = 0, \quad \mathrm{Var}(B_t) = t$$

   The sample paths are continuous but almost surely nowhere differentiable.

---

3. Example: Black-Scholes Price Dynamics

Consider the Black-Scholes stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

Here, the return dS_t follows:

$$dS_t \sim \mathcal{N}(\mu\, dt, \sigma^2\, dt)$$

This implies that **log-returns** over fixed intervals (e.g. 1 hour) are normally distributed:

Time (T)	Price (S_T)	Change (dS_T)
t_0	1.20	–
$2t_0$	1.32	0.12
$3t_0$	1.94	0.62
$4t_0$	1.50	–0.44
$5t_0$	1.76	0.26

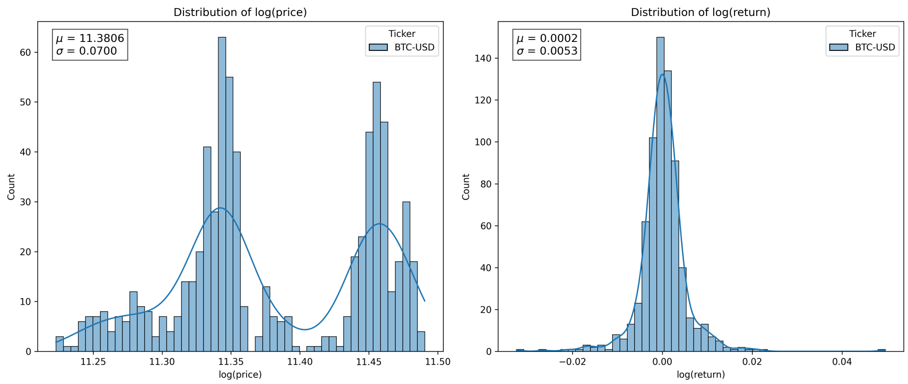
Visualization:

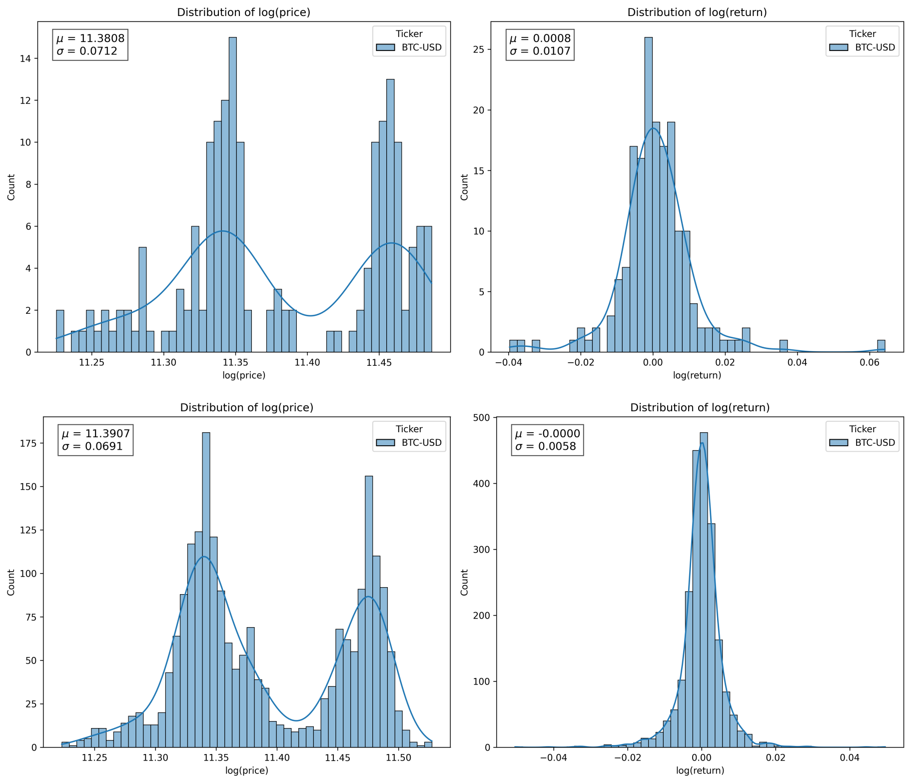
We observe log-price and log-return distributions for:

- Upper: $T = 30$ days, $\Delta t = 1$ hour
- Middle: $T = 30$ days, $\Delta t = 4$ hours
- Lower: $T = 90$ days, $\Delta t = 1$ hour

Observations:

- Larger total period (T) yields better approximation of the true (population) distribution.
- Doubling Δt (timeframe) roughly doubles the return volatility.





---

4. Stationarity of Log-Returns

- Log-return $d \log S_t$ follows $\mathcal{N}(\mu t_0, \sigma^2 t_0)$ for fixed timeframe $\Delta t = t_0$.
- Mean and variance **do not** change with absolute time T, making the process **covariance stationary**.
- Returns at each Δt are **i.i.d.**, so volatility scales with $\sqrt{\Delta t}$, not T.

  Illustration:
- Suppose a price process S_t has log-values:

$$\log S_t = \{f(t_0), f(2t_0), f(3t_0), f(4t_0), \ldots, f(T)\},$$

$$\text{where } f(t) \sim \mathcal{N}(M_t, \Sigma_t^2)$$

$$E[\log S_t] = \log S_0 + (\mu - \sigma^2/2)t = M_t$$

$$\mathrm{Var}[\log S_t] = \sigma^2 t = \Sigma_t^2$$

This means that the **mean and variance depend on both** the timeframe $\Delta t = t_0$ and the **total time** T.

- In contrast, the log-returns (modeled as dW_t) are:

$$\log\left(\frac{S_t}{S_{t-t_0}}\right) = \{f(t_0), f(t_0), f(t_0), f(t_0), \dots\}$$

where each term is i.i.d. $\sim \mathcal{N}(\mu t_0, \sigma^2 t_0)$—independent of T.

This shows:

- **Log-prices** evolve over time and are **non-stationary** (variance increases with T),
- **Log-returns** are **stationary**—their distribution depends on Δt, but **not** on T.

Is log-return an $I(0)$ process?

- Yes, by definition. A stationary process is an $I(0)$ process.

What ARIMA model fits log-return?

- Log-return under Black-Scholes is **white noise**, modeled by **ARIMA(0,0,0)**.

---

5. Discrete vs Continuous Processes

Discrete Model	Continuous Counterpart
Random Walk: $X_t = X\_t - 1 + \epsilon_t$	Brownian motion: W_t
White Noise: $Y_t = dX_t = \epsilon_t$	$Y_t = dW_t$ (stationary increments)

Random Walk vs First Difference:

- Random walk accumulates white noise:

$$X_t = \sum_{i=1}^{t} \epsilon_i, \quad \mathrm{Var}(X_t) = t\sigma^2$$

$\rightarrow$ **Not stationary**

- First difference:

$$Y_t = X_t - X_{t-1} = \epsilon_t$$

$\rightarrow$ **Stationary**

---

6. Brownian Motion vs Log-Price

- Brownian motion: $\text{Var}(W_t) = t \rightarrow$ **not stationary**
- Log-price also follows a non-stationary path:

$$\mathbb{E}[\log S_t] = \log S_0 + (\mu - \sigma^2/2)t\text{Var}[\log S_t] = \sigma^2 t$$

So, log-prices evolve with time. But **log-returns**, defined over fixed Δt, are i.i.d. and **stationary**.

---

7. Cointegration and Statistical Arbitrage

- Let X_t, Y_t be two $I(1)$ processes.
- If there exists a linear combination:

$$S_t = Y_t - \alpha - \beta X_t$$

such that S_t is **stationary**, then X_t and Y_t are **cointegrated**.

Is the spread S_t affected by the timeframe Δt?

- **No.**
- Here, the $I(0)$ process is **constructed** via linear combination, not by differencing.
- Therefore, its distribution is independent of Δt (unlike log-return, which depends on Δt).

Here's a polished, elegant, and LaTeX-consistent version of your draft with clearer structure, smoother phrasing, and enhanced formatting:

Random Walk vs AR(1) Process

We consider the first-order autoregressive (AR(1)) model:

$$X_t = \phi X_{t-1} + \epsilon_t$$

where ϵ_t is white noise with mean zero and variance σ^2. The explicit solution is:

$$X_t = \sum_{k=0}^{t-1} \phi^k \epsilon_{t-k}$$

Stationary Case: $|\phi| < 1$

- The process becomes **weakly stationary** (i.e., ARIMA(1,0,0)) when the characteristic root satisfies $|\phi| < 1$, or equivalently, $1/\phi > 1$.
- For a stationary process, the mean and variance are time-independent. Taking variance on both sides:

$$\mathrm{Var}(X_t) = \phi^2 \mathrm{Var}(X_{t-1}) + \mathrm{Var}(\epsilon_t) \Rightarrow \mathrm{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$$

Unit Root Case: $\phi = 1$

- This corresponds to a **random walk** (i.e., ARIMA(1,1,0)):

$$X_t = \sum_{i=1}^{t} \epsilon_i$$

- The variance grows linearly with time: $\mathrm{Var}(X_t) = t\sigma^2$, and the process is **non-stationary**.

Explosive Case: $|\phi| > 1$

- The process becomes **non-stationary and explosive**:

$$X_t = \phi^t X_0 + \sum_{k=0}^{t-1} \phi^k \epsilon_{t-k}$$

- Both mean and variance diverge exponentially over time.

White Noise vs Other Stationary Processes

White Noise

- A **white noise** process ϵ_t is stationary with zero autocorrelation:

$$\mathrm{Cov}(\epsilon_t, \epsilon_s) = \sigma^2 \delta_{t,s}$$

- Each ϵ_t is independent and identically distributed (i.i.d.).

AR(1) Process

- The AR(1) process is also stationary for $|\phi| < 1$, but exhibits **non-zero autocorrelation**:

$$\mathrm{Cov}(X_t, X_{t-s}) = \frac{\sigma^2 \phi^s}{1 - \phi^2}$$

- The autocorrelation decays **geometrically** with lag s, a key distinguishing feature from white noise.