EIGENVALUE & EIGENVECTOR

Invariant Subspace:

Suppose, $T \in L(V,V)$. A subspace $\mathbb{U}$ of $\mathbb{V}$ is called invariant under T if and only if

$$u \in \mathbb{U} \longrightarrow Tu \in \mathbb{U}.$$

That means $\forall u \in \mathbb{U}$, the action of T must be closed under $\mathbb{U}$.

Suppose, $T \in L(V,V)$ then $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are the T invariant subspaces. Definitely $\{0_V\}$ and $\mathbb{V}$ are the trivial T-invariant subspaces.

Take any $v \in \mathbb{V}$ with $v \neq 0_{\mathbb{V}}$

Let $\mathbb{U}$ be equal to the set of scalar multiples of v.

$$\mathbb{U} := \left\{ \lambda v \;\middle|\; \lambda \in \mathbb{F} \right\} = \langle \{v\} \rangle$$

Definitely $\mathbb{U}$ is a subspace of dimension 1. If $\mathbb{U}$ is invariant subspace under an operator $T \in L(V,V)$. then $Tv \in \mathbb{U}$. Therefore the vector Tv may be written as scalar multiple of v.

$$Tv = \lambda v \quad \text{where } \lambda \in \mathbb{F}.$$

Similarly, if $T v = \lambda v$ for some $\lambda \in \mathbb{F}$ then $\langle \{v\} \rangle$ is a 1 dimensional subspace of $\mathbb{V}$ invariant under T.

Eigenvalue - Eigenvector :

Suppose, $T \in L(V, V)$. A number $\lambda \in \mathbb{F}$ is called an eigenvalue of T if there exists $v \in \mathbb{V}$ such that $v \neq 0_{\mathbb{V}}$ and $T v = \lambda v$.

$\mathbb{V}$ has a 1 dimensional invariant subspace under T if and only if T has an eigenvalue.

In the definition $v \neq 0_{\mathbb{V}}$ because if $v = 0_{\mathbb{V}}$ then any λ will satisfy the equation.

$$T v = \lambda v$$
$$\Rightarrow (T - \lambda I) v = 0_{V}$$

That means $v \in \operatorname{Ker}(T - \lambda I)$ and $v \neq 0_{\mathbb{V}}$. So $T - \lambda I$ has non trivial kernel. Hence the matrix $T - \lambda I$ is not injective and it is not invertible either. By rank nullity theorem, it can be shown that $T - \lambda I$ is not surjective either.

Suppose $T \in L(V, V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in \mathbb{V}$ is called an "eigenvector" of T corresponding to eigenvalue λ if $v \neq 0_{\mathbb{V}}$ and $Tv = \lambda v$.

$$Tv = \lambda v \longleftrightarrow (T - \lambda I)\, v = 0_{\mathbb{V}} \longleftrightarrow v \in \operatorname{Ker}(T - \lambda I)$$

Linearly independent eigenvectors:

Let $T \in L(V, V)$. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, v_2, \ldots, v_m$ are corresponding eigenvectors of T. Then $\{v_1, v_2, \ldots, v_m\}$ is LI set.

Proof: Suppose $\{v_1, v_2, \ldots, v_m\}$ are LD set.
$\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_m$.

Suppose there are k - linearly independent set of vectors $\{v_1, v_2, \ldots, v_k\}$ where $k < m$.

Consider a vector v_{k+1} which can be written as the linear combination of $\{v_1, v_2, \ldots, v_k\}$. because $v_{k+1}, v_{k+2}, \ldots, v_m$ can be reduced to $0_{\mathbb{V}}$ after performing row operations implying that these vectors stays in the span of $\{v_1, v_2, \ldots, v_k\}$.

$$v_{k+1} = a_1 \cdot v_1 + a_2 v_2 + \cdots + a_k v_k \qquad \text{—— (i)}$$

$$\Rightarrow T v_{k+1} = T(a_1 v_1) + T(a_2 v_2) + \cdots + T(a_k v_k)$$

$$\Rightarrow \lambda_{k+1} \cdot v_{k+1} = a_1 \cdot \lambda_1 v_1 + a_2 \cdot \lambda_2 v_2 + \cdots + a_k \lambda_k v_k$$

Multiply by λ_{k+1} in both side of eqn (i)

$$\Rightarrow \lambda_{k+1} \cdot v_{k+1} = a_1 \cdot \lambda_{k+1} \cdot v_1 + a_2 \cdot \lambda_{k+1} \cdot v_2 + \cdots + a_k \lambda_{k+1} \cdot v_k$$

Subtracting we get -

$$\Rightarrow 0_V = a_1(\lambda_1 - \lambda_{k+1}) \cdot v_1 + \cdots + a_k(\lambda_k - \lambda_{k+1}) v_k$$

Since $\{v_1, v_2, \ldots, v_k\}$ are LI set so it implies.

$$\left.\begin{aligned} a_1(\lambda_1 - \lambda_{k+1}) &= 0 \quad \Rightarrow a_1 = 0 \\ a_2(\lambda_2 - \lambda_{k+1}) &= 0 \quad \Rightarrow a_2 = 0 \\ &\vdots \\ a_k(\lambda_k - \lambda_{k+1}) &= 0 \quad \Rightarrow a_k = 0 \end{aligned}\right\} \text{Because all } \lambda_i\text{'s are distinct.}$$

Therefore, $v_{k+1} = 0_V$ which is a contradiction because v_{k+1} must be non zero to be an eigenvector. Therefore, all the eigenvectors corresponding to the distinct eigenvalues must be linearly independent.

Suppose $\mathbb{V}$ is f.d.v.s with dimension 'n'. Then an operator $T \in L(V, V)$ can't have more than n distinct eigenvalues.

$\Rightarrow$ Suppose T has m distinct eigenvalues. So $\{v_1, v_2, \ldots, v_m\}$ is the LI set of eigenvectors. In n dimensional space, we can't have more than n LI set of vectors so $m \leq \dim(\mathbb{V})$.

Polynomials applied to operators:

If a linear operator $T \in L(V, V)$ then TT makes sense and is also in $L(V, V)$. We usually write TT as T^2.

Suppose, $T \in L(V, V)$ and m is a positive integer.

$$T^m := \underbrace{T \cdot T \cdot T \cdots T}_{(m \text{ times})}$$

$$T^0 := I \quad (\text{identity map from } \mathbb{V} \text{ to } \mathbb{V})$$

If T is invertible operator then,

$$T^{-m} := \underbrace{(T^{-1}) \cdot (T^{-1}) \cdots (T^{-1})}_{m \text{ times}} = (T^{-1})^m$$

Suppose, $T \in L(V, V)$ and $p(x)$ is a polynomial where $p(x) \in \mathbb{F}[x]$.

$$p(x) := a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad ; \forall a_i \in \mathbb{F}.$$

We define an operator $p(T)$ as follows:

$$p(T) := a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n \in L(V, V)$$

Suppose,

$$p(T) v = \left(a_0 + a_1 T + \cdots + a_n T^n\right)(v)$$
$$= a_0 I v + a_1 T v + \cdots + a_n T^n v$$

If $p, q \in \mathbb{F}[x]$ then $(pq)(x) := p(x) \cdot q(x)$

Similarly. for a $T \in L(V, V)$

$$(pq)(T) = p(T) \cdot q(T)$$

$$(pq)(x) = (qp)(x) = q(x) \cdot p(x)$$

$$(pq)(T) = q(T) \cdot p(T)$$

Eigenspace:

Suppose $T \in L(V, V)$ and $\lambda \in \mathbb{F}$. The eigenspace of T corresponding to λ is defined as $\operatorname{Ker}(T - \lambda I)$.

In other words the eigenspace contains set of all eigenvectors of T corresponding to eigenvalue λ.

Suppose, $\mathbb{V}$ is f.d.v.s and $T \in L(V, V)$. Also $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of T. Then the eigenspace of T corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_m$ are direct sum.

$$\operatorname{Ker}(T - \lambda_1 I) \oplus \operatorname{Ker}(T - \lambda_2 I) \oplus \cdots \oplus \operatorname{Ker}(T - \lambda_m I)$$

Furthermore,

$$\dim(\operatorname{ker}(T - \lambda_1 I)) + \cdots + \dim(\operatorname{Ker}(T - \lambda_m I)) \leq \dim(\mathbb{V})$$

<u>proof:</u>

consider, $u_1 \in \operatorname{Ker}(T - \lambda_1 I)$ is eigenvector c.t. λ_1

$u_2 \in \operatorname{Ker}(T - \lambda_2 I)$ is eigenvector c.t. λ_2

$\vdots \qquad \vdots$

$u_m \in \operatorname{Ker}(T - \lambda_m I)$ is eigenvector c.t. λ_m

Since $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m$ so $\{u_1, u_2, \ldots, u_m\}$ is a linearly independent set.

consider a vector $u \in \operatorname{Ker}(T - \lambda_1 I) + \cdots + \operatorname{Ker}(T - \lambda_m I)$

So $u = u_1 + u_2 + \cdots + u_m$

We know that $\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_m$ are subspace of $\mathbb{V}$. Then $\mathbb{U}_1 + \mathbb{U}_2 + \cdots + \mathbb{U}_m$ is a direct sum if and only if the only way to write $0_{\mathbb{V}}$ as a sum $u_1 + u_2 + \cdots + u_m$ where $\forall u_j \in \mathbb{U}_j$ and each $u_j = 0_{\mathbb{V}}$.

consider, $u_1 + u_2 + \cdots + u_m = 0_{\mathbb{V}}$

Because $\{u_1, u_2, \ldots, u_m\}$ is LI set so some non trivial LC gives rise to $0_{\mathbb{V}}$ is possible if and only if all u_i are individually $0_{\mathbb{V}}$.

Hence $u_1 = u_2 = u_3 = \cdots = u_m = 0_{\mathbb{V}}$.

Since the only way to write $0_{\mathbb{V}}$ is when all $u_i = 0_{\mathbb{V}}$ that means the sum is direct sum. (proved)

Diagonalizable:

An operator $T \in L(V, V)$ is called diagonalizable if the operator has a diagonal matrix with respect to some basis of $\mathbb{V}$.

Suppose, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of T.

T is diagonalizable iff
$\mathbb{V}$ has a basis consisting of eigenvectors of T.

<u>proof</u>: Suppose, $T v_1 = \lambda_1 v_1$

$$T v_2 = \lambda_2 v_2$$
$$\vdots$$
$$T v_n = \lambda_n v_n$$

$$\Rightarrow \begin{bmatrix} T v_1 & T v_2 & \cdots & T v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$$

$$\Rightarrow T \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow T = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1}}_{P^{-1}}$$

$P =$ Change of basis matrix so in the basis $\{v_1, \ldots v_n\}$, the matrix T is represented as a diagonal matrix Λ.

T is diagonalizable iff
$\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \cdots \oplus \mathbb{U}_n$ each $\mathbb{U}_i$ is T invariant subspace of $\mathbb{V}$.

proof: $\mathbb{V}$ has basis $\{v_1, v_2, \dots, v_n\}$.

let $\mathbb{U}_j = \langle \{v_j\} \rangle$

Obviously each $\mathbb{U}_j$ is 1 dimensional subspace of $\mathbb{V}$.
Each $\mathbb{U}_j$ is also invariant under T because v_j are eigenvectors of T.

Take a vector, $$v = \underbrace{c_1 v_1}_{\in \mathbb{U}_1} + \underbrace{c_2 v_2}_{\in \mathbb{U}_2} + \cdots + \underbrace{c_n v_n}_{\in \mathbb{U}_n}$$

$$= u_1 + u_2 + \cdots + u_n$$

So $\mathbb{V} = \mathbb{U}_1 + \mathbb{U}_2 + \cdots + \mathbb{U}_n$.
But is it direct sum?

$$v = u_1 + u_2 + \cdots + u_n$$
$$v = g_1 + g_2 + \cdots + g_n$$

$$0_{\mathbb{V}} = (u_1 - g_1) + (u_2 - g_2) + \cdots + (u_n - g_n)$$

$$0_{\mathbb{V}} = w_1 + w_2 + \cdots + w_n \qquad \forall w_i \in \mathbb{U}_i$$

$\{w_i\}_{i=1}^{n}$ is also a basis of $\mathbb{V}$ because, it is just a scalar multiple of $\{v_i\}_{i=1}^{n}$. So the only choice of $w_i = 0_{\mathbb{V}}$ $\forall i$ to make the equation work. So $u_i = g_i$ $\forall i$.

T is diagonalizable iff

$$\mathbb{V} = \ker(T-\lambda_1 I) \oplus \ker(T-\lambda_2 I) \oplus \cdots \oplus \ker(T-\lambda_m I)$$

$$\dim(\mathbb{V}) = \sum \dim(\ker(T-\lambda_i I))$$

If T has n distinct eigenvalues then T is surely diagonalizable.

Null spaces of Powers of an operator:

Suppose, $T \in \mathcal{L}(V, V)$. Then

$$\{0_{\mathbb{V}}\} = \ker(T^0) \subset \ker(T^1) \subset \ker(T^2) \cdots$$

Since $T^0 = I$ so $Iv = 0_{\mathbb{V}} \longleftrightarrow v = 0_{\mathbb{V}}$ hence $\ker(T^0) = \{0_{\mathbb{V}}\}$.

We need to prove that $\ker(T^K) \subset \ker(T^{K+1})$

Suppose, $x \in \ker(T^K)$

$$\Rightarrow T^K x = 0_{\mathbb{V}}.$$

$$\Rightarrow T.T^K.x = T.0_{\mathbb{V}} = 0_{\mathbb{V}}$$ (any operator when it takes $0_{\mathbb{V}}$, gives $0_{\mathbb{V}}$ back)

$$\Rightarrow T^{K+1}.x = 0_{\mathbb{V}}.$$

$$\Rightarrow x \in \ker(T^{K+1})$$

We have to prove that $\exists v$ s.t. $v \in Ker(T^{K+1}) \longrightarrow v \notin Ker(T^K)$.

$\Rightarrow T^{K+1}(v) = 0_V$.

$\Rightarrow T.(T^K.v) = 0_V$.

$\Rightarrow T.u = 0_V$. , $u \in Ker(T)$.

If $u = 0_V$ then $T^K.v = 0_V$ so $v \in Ker(T^K)$.
But if $u \neq 0_V$ then $T^K v = u$ so $v \notin Ker(T^K)$.

Since $T: V \to V$ so if T is not injective operator then $v \notin Ker(T^K)$ so it means it is a proper subset.

Hence, $Ker(T^K) \subset Ker(T^{K+1})$

Suppose, $Ker(T^m) = Ker(T^{m+1})$. Then,

$Ker(T^m) = Ker(T^{m+1}) = Ker(T^{m+2}) = Ker(T^{m+3}) = \cdots$

We need to prove that $Ker(T^{m+K}) = Ker(T^{m+K+1})$

We already Know, $Ker(T^{m+K}) \subset Ker(T^{m+K+1})$.

Suppose, $x \in Ker(T^{m+K+1})$

$\Rightarrow T^{m+K+1}.x = 0_V$.

$$\Rightarrow T^{m+1} \cdot (T^K x) = 0_V.$$

$$\Rightarrow T^K x \in \ker(T^{m+1})$$

Since $\ker(T^{m+1}) = \ker(T^m)$ hence $T^K x \in \ker(T^m)$.

$$\Rightarrow T^m \cdot (T^K \cdot x) = 0_V$$

$$\Rightarrow T^{m+K} \cdot x = 0_V.$$

$$\Rightarrow x \in \ker(T^{m+K}).$$

Therefore, $\ker(T^{m+1}) = \ker(T^{m+2}) = \cdots$

But this proposition is true if $\ker(T^m) = \ker(T^{m+1})$.
What gives the guarentee that these 2 kernels are same?
for what m, the 2 kernels will be same?

$T: V \to V$ (dim n).
When $m = n$ then, $\ker(T^n) = \ker(T^{n+1}) = \cdots$
<u>proof</u>: Suppose, it is not true.
We have already seen that,

$$\ker(T^0) \subsetneq \ker(T^1) \subsetneq \ker(T^2) \subsetneq \cdots$$

So it is proper subset. If $\ker(T^m) = \ker(T^{m+1})$
then only the equality condition in the subset holds.

Since $\ker(T)$ is a subspace of dim say 1

$\ker(T^2)$ is a subspace of dim say 2 (at least)

$\vdots$

$\ker(T^n)$ is a subspace of dim say n

$\ker(T^{n+1})$ is a subspace of dim say $n+1$

Which is a contradiction as dim of subspace can't be more than $n+1$.

Therefore, $\ker(T^n) = \ker(T^{n+1})$.

So if $n = \dim(V)$ then, $V = \ker(T^n) \oplus \operatorname{im}(T^n)$

<u>proof</u>:

Suppose, $v \in \ker(T^n) \cap \operatorname{Im}(T^n)$

$\Rightarrow T^n v = 0$

and $\exists u$ s.t. $T^n u = v$

$\Rightarrow T^{2n} u = T^n v = 0$

$\Rightarrow u \in \ker(T^{2n})$

and $\ker(T^n) = \ker(T^{2n})$

So $u \in \ker(T^n)$.

$\Rightarrow T^n \cdot u = 0 \Rightarrow v = 0_V$ hence the intersection contains only 0_V vector and nothing else.

Rank-nullity theorem: $T^n : \mathbb{V} \to \mathbb{V}$.

$$\dim(\operatorname{Ker}(T^n)) + \dim(\operatorname{Im}(T^n)) = \dim(\mathbb{V}).$$

$$\Rightarrow \dim(\operatorname{Ker}(T^n) \oplus \operatorname{Im}(T^n)) = \dim(\mathbb{V}).$$

$$\Rightarrow \operatorname{Ker}(T^n) \oplus \operatorname{Im}(T^n) = \mathbb{V}.$$

Generalized Eigenvectors:

Some operators do not have enough eigenvectors that leads to a good description. Describe an operator by decomposing its domain into invariant subspaces.

Suppose, $T \in L(V, V)$

We want to decompose, $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \cdots \oplus \mathbb{U}_m$.
Where each of the subspaces $\mathbb{U}_j$ is an invariant subspace of $\mathbb{V}$ under T.

Suppose, $\mathbb{V}$ has a basis $v_1, v_2, \ldots, v_n$ consisting of eigenvectors of T.

Let $\mathbb{U}_j = \langle \{v_j\} \rangle$

$\dim(\mathbb{U}_j) = 1$. and $\mathbb{U}_j$ is T-invariant subspace.
because, $Tu_j = \lambda_j u_j \in \mathbb{U}_j$ itself.

any vector $v \in \mathbb{V}$ can be written as -

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$
$$= u_1 + u_2 + \cdots + u_n \quad [\because u_j = c_j v_j]$$

$\Rightarrow v \in \mathbb{U}_1 + \mathbb{U}_2 + \cdots + \mathbb{U}_n.$

$$\dim(\mathbb{V}) = \dim(\mathbb{U}_1 + \mathbb{U}_2 + \cdots + \mathbb{U}_n)$$
$$= \sum \dim(\mathbb{U}_i) \quad = n.$$

Hence $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \cdots \oplus \mathbb{U}_n.$

We also know that $v_j \in \operatorname{Ker}(T - \lambda_i I)$
let $\operatorname{Ker}(T - \lambda_i I) = E(\lambda_i, T).$

Suppose there are m distinct eigenvalues We know that eigenvectors corresponding to distinct eigenvalues are linearly independent.

Suppose, $\quad u \in E(\lambda_1, T) + E(\lambda_2, T) + \cdots + E(\lambda_m, T)$

$\Rightarrow u = u_1 + u_2 + \cdots + u_m. \quad (u_i \in \operatorname{Ker}(T - \lambda_i I))$

Now $0_{\mathbb{V}} = u_1 + u_2 + \cdots + u_m$

$\Rightarrow \forall u_i = 0_{\mathbb{V}}$ because all u_i are LI set of vectors

$\Rightarrow u = 0_{\mathbb{V}}.$ so the sum is direct sum.

Hence, $\mathbb{V} = \ker(T-\lambda_1 I) \oplus \ker(T-\lambda_2 I) \oplus \cdots$
$\cdots \oplus \ker(T-\lambda_m I)$.

$$\dim(\mathbb{V}) = \sum_{i=1}^{m} \dim\left(\ker(T-\lambda_i I)\right)$$

In case T is diagonalizable, all the above statements are equivalent.

T has a basis consisting of eigenvectors of T

$\Leftrightarrow \mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \cdots \oplus \mathbb{U}_n$ ($\mathbb{U}_i$ are T invar. subsp.)

$\Leftrightarrow \mathbb{V} = \ker(T-\lambda_1 I) \oplus \cdots \oplus \ker(T-\lambda_m I)$

$\Leftrightarrow \dim(\mathbb{V}) = \sum \dim(\ker(T-\lambda_i I))$.

We seek to find a directsum decomposition made of T-invariant subspaces,

$$\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \cdots \oplus \mathbb{U}_m.$$

It is possible if and only if $\mathbb{V}$ has a basis consisting of eigenvectors of T. Again this happens if and only if $\mathbb{V}$ has an eigenspace decomposition.

$$\mathbb{V} = \ker(T-\lambda_1 I) \oplus \ker(T-\lambda_2 I) \oplus \cdots \oplus \ker(T-\lambda_m I).$$

If the operator T has the following property that,
$TT^* = T^*T$ (T^* is adjoint of T which is just conjugate transpose in any orth. normal basis).

Called "normal operator". Ofcourse if $T = T^*$ (self adjoint / Hermitian) is normal operator.

If T is normal operator then T has enough eigen-values to form a basis for $\mathbb{V}$ over $\mathbb{C}$. Similarly if T is self adjoint operator in $\mathbb{V}$ over $\mathbb{R}$ then T has enough eigenvalues to form a basis for $\mathbb{V}$. To be proved later.

For more general operators T (which is not normal) the decomposition of $\mathbb{V}$ will not hold true because the eigenvectors do not form basis for $\mathbb{V}$ as they do not have enough eigenvectors. In this case we use generalized eigenvectors.

Suppose, $T \in L(V, V)$ and λ is an eigenvalue of T. A vector $v \in \mathbb{V}$ is called a "generalized eigenvector" of T corresponding to λ if and only if $v \neq 0_{\mathbb{V}}$ and

$$(T - \lambda_i I)^j v = 0 \quad \text{for some positive integer } j$$

$T \in L(V, V)$ and $\lambda \in \mathbb{F}$.

The generalized eigenspace of T corresponding to eigenvalue λ is denoted by $G(\lambda, T)$.

$G(\lambda, T) := \{$ Set of all generalized eigenvectors of T corresponding to λ along with $0_{\mathbb{V}}\}$.

The eigenvector of T is obviously a generalized eigenvector of T (Take $j = 1$). Each eigenspace is contained in the generalized eigenspace.

$$E(\lambda, T) \subset G(\lambda, T)$$

Description of Generalized Eigenspace:

Suppose, $T \in L(V, V)$. and $\lambda \in \mathbb{F}$.

$$G(\lambda, T) = \operatorname{Ker}\left((T - \lambda I)^n \right) \quad \text{where } n = \dim(\mathbb{V}).$$

proof: Although in the definition of generalized eigenvectors, j is allowed to be arbitrary integer (of course $j = 1$ means v is eigenvector) but if we increase j and search for other v so that $(T - \lambda I)^j v = 0$ then those v will be generalized eigenvectors & this theorem says every generalized eigenvector will satisfy the equation, $(T-\lambda I)^n v = 0$

Basically, we have to prove that,

$$G(\lambda, T) = \operatorname{Ker}\left((T-\lambda I)^n\right)$$

Suppose, $v \in \operatorname{Ker}\left((T-\lambda I)^n\right)$

$\Rightarrow (T-\lambda I)^n v = 0$

$\Rightarrow v \in G(\lambda, T)$

Suppose, $v \in G(\lambda, T)$

$\Rightarrow \exists j$ s.t. $(T-\lambda I)^j v = 0$

$\Rightarrow v \in \operatorname{Ker}\left((T-\lambda I)^j\right)$

We know that,

$$\operatorname{Ker}\left((T-\lambda I)^j\right) \subset \operatorname{Ker}\left((T-\lambda I)^{j+1}\right) \subset \cdots\cdots$$

if $\operatorname{Ker}\left((T-\lambda I)^j\right) = \operatorname{Ker}\left((T-\lambda I)^{j+1}\right)$ then,

$$\operatorname{Ker}\left((T-\lambda I)^j\right) = \operatorname{Ker}\left((T-\lambda I)^{j+1}\right) = \operatorname{Ker}\left((T-\lambda I)^{j+2}\right) = \cdots$$

More specifically,

$$\operatorname{Ker}\left((T-\lambda I)^n\right) = \operatorname{Ker}\left((T-\lambda I)^{n+1}\right) = \cdots\cdots$$

Therefore, $v \in \ker\left((T-\lambda I)^j\right)$

$\Rightarrow v \in \ker\left((T-\lambda I)^{j+1}\right)$

$\vdots$

$\Rightarrow v \in \ker\left((T-\lambda I)^n\right)$

Therefore, $G(\lambda, T) = \ker\left((T-\lambda I)^n\right)$.

<u>Example:</u>

$T(z_1, z_2, z_3) = (4z_2, 0, 5z_3)$

Eigenvalues of T will be, $\lambda_1 = 0$, $\lambda_2 = 5$

$E(0,T) = (z_1, 0, 0)$, $z_1 \in \mathbb{C}$

$E(5,T) = (0, 0, z_3)$, $z_3 \in \mathbb{C}$.

Note there is not enough eigenvectors to span $V = \mathbb{C}^3$

Therefore we look for generalized eigenvectors.

$(T-\lambda I)^n v = 0$

$\Rightarrow (T-5I)^3 (z_1, z_2, z_3) = (-125z_1 + 300z_2, -125z_2, 0)$

$\Rightarrow (T-0I)^3 (z_1, z_2, z_3) = (0, 0, 125z_3)$.

$G(0,T) = (z_1, z_2, 0)$, $G(5,T) = (0, 0, z_3)$.

Note that, $\mathbb{V} = G(0,T) \oplus G(5,T)$. So the generalized eigenvectors span the domain of T.

Note that $G(\lambda, T)$ is also a subspace of $\mathbb{V}$.

<u>Theorem:</u> Let $T \in L(V,V)$. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, v_2, \ldots, v_m$ are corresponding generalized eigenvectors. Then $v_1, v_2, \ldots, v_m$ must be linearly independent.

<u>proof:</u> let K be the largest positive integer such that $(T-\lambda_1 I)^K v_1 \neq 0$. and $(T-\lambda_1 I)^{K+1} v_1 = 0$

let, $w = (T-\lambda_1 I)^K v_1$

$$\Rightarrow (T-\lambda_1 I) \cdot w = (T-\lambda_1 I)^{K+1} v_1 = 0$$

$$\Rightarrow Tw = \lambda_1 w$$

$$\Rightarrow Tw - \lambda I w = \lambda_1 w - \lambda w$$

$$\Rightarrow (T-\lambda I) w = (\lambda_1 - \lambda) w \quad \text{for every } \lambda \in \mathbb{F}.$$

$$\Rightarrow (T-\lambda I)^n w = (\lambda_1 - \lambda)^n \cdot w$$

Consider, $a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0_{\mathbb{V}}$.

$$\Rightarrow a_1\left[(T-\lambda_1 I)^K (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n\right] v_1 +$$

$$a_2\left[(T-\lambda_1 I)^K (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n\right] v_2 +$$

$$\vdots$$

$$a_m\left[(T-\lambda_1 I)^K (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n\right] v_m = 0$$

$$\Rightarrow a_1\left[(T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n \cdot w\right]$$

$$\Rightarrow a_1\left[\underbrace{(\lambda_1-\lambda_2)^n}_{\neq 0} \cdot \underbrace{(\lambda_1-\lambda_3)^n}_{\neq 0} \cdots \underbrace{(\lambda_1-\lambda_m)^n}_{\neq 0}\right]$$

$$\Rightarrow a_1 = 0$$

Similarly all the $a_i = 0$ meaning the generalized eigenvectors are linearly independent.

Properties of Generalized Eigenvector :

If $A : V \to V$ of dimension n then the generalized eigenvector of A corresponding to eigenvalue λ is defined as:

$$(A-\lambda I)^K \cdot x = 0 \quad \text{where } K > 0 \text{ and } x \neq 0$$

Here x is the generalized eigenvector. We can also say $x \in \operatorname{Ker}(A - \lambda I)^K$.

Ex:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} .$$

$$\chi_A(x) = (3-\lambda)(1-\lambda)^2$$

$$\lambda_1 = 1, \quad \lambda_2 = 3$$

Eigenvector $v_1 = (1,2,2)$ for $\lambda_1 = 3$

$v_2 = (1,0,0)$ for $\lambda_2 = 1$

Since the matrix fall short of eigenvectors, we can look for generalized eigenvector. (v_3).

$$(A - 3.I) \cdot x = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 = -x_2$$

$$-2x_2 = -2x_3$$

$$\Rightarrow x_2 = x_3 = 2$$

$$x = (x_1, x_2, x_3) = (1,2,2).$$

So No need to increase the power. since AM=GM

$$(A - 1 \cdot I)x = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0, \; x_3 = 0, \; x_1 = 1$ (any).

So $x = (1, 0, 0)$

$(A - 1.I)^2 . x = 0$ can be solved by $(A - \lambda I) v_r = v_{r-1}$

$\Rightarrow x = (0, 1, 0)$. (Generalized eigenvector).

How many powers of $(A - \lambda I)$ do we need to compute in order to find all the generalized eigenvectors?

$\Rightarrow$ If the AM of λ is K.

$\Rightarrow \dim\left(\operatorname{Ker}(A - \lambda I)^n\right) = K$ then we don't have to go beyond K powers.

So $v \in \operatorname{Ker}(A - \lambda I)^K$ and $\dim\left(\operatorname{Ker}(A - \lambda I)^K\right) = K$.

So we have to take at most K powers.

Nilpotent Operators:

An operator is called nilpotent if some power of it equals to 0. We never need to use a power higher than the $\dim(V) = n$.

Suppose, $N \in L(V, V)$ is nilpotent operator.

We show that $N^n = 0$.

proof: First we show that the nilpotent operator has only 1 eigenvalue that is 0.

N is nilpotent

$\Rightarrow \exists K$ s.t. $N^K = 0$

Consider, $Nx = \lambda x$ where (λ, x) is eigenpair

$\Rightarrow N^2 x = \lambda . Nx = \lambda . \lambda x = \lambda^2 x$

$\vdots$

$\Rightarrow N^K x = \lambda^K x = 0.x = 0_V.$

Since $x \neq 0$ so the only possibility is that $\lambda^K = 0$

$\Rightarrow \lambda . \lambda . \lambda \dots . \lambda = 0$

$\Rightarrow \lambda = 0$

Now N has only eigenvalue as 0.

Therefore, we find the generalized eigenvector for 0

$G(0, N) = \operatorname{Ker}\left((N - 0.I)^n \right)$

So if v is an generalized eigenvector then,

$(N - 0.I)^n . v = 0_V \Rightarrow N^n . v = 0_V. \Rightarrow v \in \operatorname{Ker}(N^n)$

We know that $\operatorname{Ker}(N^n) = \operatorname{Ker}(N^{n+1}) = \operatorname{Ker}(N^{n+2}) = \cdots$

$v \in \operatorname{Ker}(N^n) \longrightarrow v \in \operatorname{Ker}(N^{2n})$.

$\Rightarrow N^{2n} \cdot v = 0_{\mathbb{V}}$

$\Rightarrow N^n (N^n \cdot v) = 0_{\mathbb{V}}$.

$\Rightarrow N^n \cdot u = 0_{\mathbb{V}}$. where $u = N^n \cdot v$ so $u \in \operatorname{Im}(N^n)$

But we know that $N^n \cdot v = 0_{\mathbb{V}}$ so, $u = 0_{\mathbb{V}}$.

Hence $\operatorname{Im}(N^n) = \{0_{\mathbb{V}}\}$.

$\dim(\mathbb{V}) = \dim(\operatorname{Ker}(N^n)) = n$

$\Rightarrow \operatorname{Ker}(N^n) = \mathbb{V}$.

That means for any $v \in \mathbb{V}$

$\Rightarrow v \in \operatorname{Ker}(N^n)$

$\Rightarrow N^n \cdot v = 0_{\mathbb{V}}$. $\forall v \in \mathbb{V}$.

$\Rightarrow N^n = 0_{L(V,V)}$ (zero operator by definition).

For a nilpotent operator, $N : \mathbb{V} \to \mathbb{V}$ we have -

$\mathbb{V} = \operatorname{Ker}(N^n)$, $\{0_{\mathbb{V}}\} = \operatorname{Im}(N^n)$, $N^n = 0_{L(V,V)}$

Decomposition of Domain using Generalized Eigenvectors:

Suppose, $T \in L(V, V)$.

$\ker(T)$ is T- invariant subspace.

<u>proof:</u> $v \in \ker(T)$.

$$\Rightarrow T.v = 0_V.$$

$$\Rightarrow T.(Tv) = T.0_V = 0_V$$

$$\Rightarrow T.x = 0_V.$$

$$\Rightarrow x \in \ker(T)$$

$$\Rightarrow Tv \in \ker(T) \quad \text{So } \ker(T) \text{ is } T \text{ invariant.}$$

$\operatorname{Im}(T)$ is T invariant subspace.

<u>proof:</u> $v \in \operatorname{Im}(T)$.

$$\Rightarrow \exists u \text{ st. } T.u = v$$

$$\Rightarrow T(T(u)) = T(v)$$

$$\Rightarrow T\tilde{u} = Tv$$

$$\Rightarrow T\tilde{u} \in \operatorname{Im}(T)$$

Since $T\tilde{u} = Tv$ so $Tv \in \operatorname{Im}(T)$.

Hence $\operatorname{Im}(T)$ is T invariant subspace.

Suppose, $T \in L(V, V)$. and $P(x) \in \mathbb{F}[x]$.
Then $\operatorname{Ker}(P(T))$ and $\operatorname{Im}(P(T))$ is T invariant.

<u>proof</u>: Suppose, $v \in \ker(P(T))$.

$$\Rightarrow P(T)v = 0_V.$$

Consider, $(P(T))(Tv)$

suppose $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$$\Rightarrow P(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n.$$

$$\Rightarrow P(T).T = a_0 T + a_1 T^2 + a_2 T^3 + \cdots + a_n T^{n+1}$$

$$\Rightarrow T.P(T) = a_0.T + a_1 T^2 + a_2 T^3 + \cdots + a_n T^{n+1}$$

Therefore, $P(T).T = T.P(T)$

In general, $P(T).q(T) = q(T).P(T)$. , $\forall p, q \in \mathbb{F}[x]$.

Therefore, $P(T).(T.v)$

$$\left.\begin{aligned} &= P(T).T(v) \\ &= T.P(T)(v) \\ &= T.\big(P(T)(v)\big) \\ &= T.0_V = 0_V. \end{aligned}\right\}$$

therefore, $T(v) \in \operatorname{Ker}(P(T))$

Hence $\operatorname{Ker}(P(T))$ is T-invariant subspace.

Suppose, $v \in \mathrm{im}(P(T))$.

$\Rightarrow P(T).u = v \quad \exists u \in V$.

$\Rightarrow Tv = T.(P(T).u) = P(T).(Tu) = P(T).w$.

$\exists w$ s.t. $P(T).w = Tv \Rightarrow Tv \in \mathrm{Im}(P(T))$.

Hence $\mathrm{Im}(P(T))$ is T invariant subspace.

<u>Theorem:</u> Suppose $T \in L(V, V)$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct eigenvalues of T.
Then $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T)$.

<u>Restriction operator</u> :

Suppose, $T \in L(V, V)$. Let U be a subspace of V.
U is an invariant subspace under T.

The restriction operator $T|_U \in L(U, U)$ is defined as:

$$T|_U(u) := T.u \quad \text{for all } u \in U.$$

Since U is a T-invariant subspace, we can think $T|_U$ as the operator restricted to U only. So its domain is restricted to U.

$G(\lambda_i, T) := \operatorname{Ker}\left((T - \lambda_i I)^n\right)$ is T invariant

proof: Suppose, $P(z) = (z - \lambda_i)^n \in \mathbb{F}[x]$.

So $P(T) = (T - \lambda_i I)^n$.

We have seen that $\operatorname{Ker}(P(T))$ is T invariant.

$\Rightarrow \operatorname{Ker}\left((T - \lambda_i I)^n\right)$ is T invariant.

$(T - \lambda_i I)\Big|_{G(\lambda_i, T)}$ is nilpotent operator.

proof: Suppose, $G(\lambda_i, T) = \operatorname{Ker}\left((T - \lambda_i I)^n\right)$.

consider an operator $X : G(\lambda_i, T) \to G(\lambda_i, T)$ such that X is restricted to $\mathbb{U}_i$ only. $\dim(\mathbb{U}_i) = d_i$

$$(T - \lambda_i I)\Big|_{G(\lambda_i, T)} x := (T - \lambda_i I) \cdot x$$

We don't need till n rather we need till $\dim(\mathbb{U}_i)$

We know that, $(T - \lambda_i I)^{d_i}\Big|_{G(\lambda_i, T)} x = 0_V$ because x is the generalized Eigenvector.

Hence, $(T - \lambda_i I)$ is the nilpotent operator but restricted to only the $\operatorname{Ker}\left((T - \lambda_i I)^n\right)$ or $G(\lambda_i, T)$.

Since we have seen that for general operators T,

$$\mathbb{V} \neq E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_m, T)$$

But, $\mathbb{V} = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T)$.

So we can pick the basis from $\text{Ker}\left((T-\lambda_i I)^n\right)$ and extend it to form the basis for $\mathbb{V}$.

Suppose $T \in \mathcal{L}(V, V)$. The multiplicity of an eigenvalue λ of T is defined as the dimension of the generalized eigenspace.

algebraic multiplicity $= \dim\left(\text{Ker}(T-\lambda I)^n\right)$.

In general AM, it is the number of times an eigenvalue λ repeats in the characteristic polynomials.

$$\mathbb{V} = \text{Ker}(T-\lambda_1 I)^n \oplus \text{Ker}(T-\lambda_2 I)^n \oplus \cdots \oplus \text{Ker}(T-\lambda_m I)^n$$

$$\Rightarrow \dim(\mathbb{V}) = \sum_{i=1}^{m} \dim\left(\text{Ker}(T-\lambda_i I)^n\right)$$

$$\Rightarrow \boxed{n = \sum_{i=1}^{m} AM(\lambda_i)}$$

Geometric multiplicity of λ is defined as -

$$GM(\lambda) := \dim(\ker(T - \lambda I)) = \dim(E(\lambda, T))$$

Block Diagonal matrix :

$$\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix}$$

where $A_1, A_2, \ldots, A_m$ are the square matrices lying in the diagonal.

Suppose $T \in L(V, V)$. let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct eigenvalues of T with multiplicities $d_1, d_2, \ldots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix as follows.

$$[T]_B = \begin{bmatrix} [A_1]_{d_1 \times d_1} & & & 0 \\ & [A_2]_{d_2 \times d_2} & & \\ & & \ddots & \\ 0 & & & [A_m]_{d_m \times d_m} \end{bmatrix}$$

$$[A_i]_{d_i \times d_i} = \begin{bmatrix} \lambda_i & & & * \\ & \lambda_i & & \\ & & \lambda_i \ddots & \\ 0 & & & \lambda_i \end{bmatrix}$$

upper triangular matrix of this form.

We know that each $(T-\lambda_i I)\Big|_{G(\lambda_i, T)}$ is a nilpotent operator on the space $G(\lambda_i, T)$.

Because, $\operatorname{Ker}(T-\lambda_i I)^n$ is the T invariant subspace.

So $G(\lambda_i, T)$ contains only generalized eigenvectors and it is T invariant subspace.

Consider, $x \in G(\lambda_i, T)$.

$\Rightarrow x \in \operatorname{Ker}(T-\lambda_i I)^n$

$\Rightarrow (T-\lambda_i I)^n \cdot x = 0_V$.

Define, $(T-\lambda_i I)^n\Big|_{G(\lambda_i, T)} x = (T-\lambda_i I)^n \cdot x = 0_V$.

$$\Rightarrow (T-\lambda_i I)\Big|_{G(\lambda_i, T)} = 0_{L(G(\lambda_i, T),\, G(\lambda_i, T))}$$

$\Rightarrow (T-\lambda_i I)\Big|_{G(\lambda_i, T)}$ is a nilpotent operator when its domain is restricted to general eigenspace.

Now for each i, choose a basis of $G(\lambda_i, T)$. which is of dimension d_i means B_i will have d_i vectors.

$$\left[(T-\lambda_i I)\Big|_{G(\lambda_i, T)} \right]_{B_i} = \left[N_i \right]_B \qquad N_i : G(\lambda_i, T) \to G(\lambda_i, T)$$

$$\boxed{\operatorname{Ker}(N_i^{d_i}) = G(\lambda_i, T).}$$

Suppose, N is a nilpotent operator, $N : V \to V$ and we know that for any nilpotent operator, we have,

$V = \operatorname{Ker}(N^n)$

First choose a basis for $\operatorname{Ker}(N)$.

Extend the basis to $\operatorname{Ker}(N^2) \supset \operatorname{Ker}(N)$

Extend the basis to $\operatorname{Ker}(N^3) \supset \operatorname{Ker}(N^2)$

$\vdots$

Extend the basis to $\operatorname{Ker}(N^n)$ and this basis becomes the basis for $V = \operatorname{Ker}(N^n)$.

Consider representing matrix of N with respect to this basis.

Suppose $\operatorname{Ker}(N)$ has dimension f_1 so,

$B_1 = \{v_1, v_2, \ldots, v_{f_1}\}$.

Suppose $\operatorname{Ker}(N^2)$ has dimension $f_1 + f_2$ so,

$B_2 = \{v_1, v_2, \ldots, v_{f_1}, v_{f_1+1}, v_{f_1+2}, \ldots, v_{f_1+f_2}\}$.

$\vdots$

Suppose $\ker(N^n)$ has dimension n so,

$$B = \{v_1, v_2, \ldots, v_{f_1}, v_{f_1+1}, \ldots, v_{f_1+f_2}, \ldots \\ \ldots, v_{n-1}, v_n\}.$$

$$[v_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, v_{f_1+1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow (f_1+1)^{th} \text{ position}$$

$$[v_{f_1+f_2+1}]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow (f_1+f_2+1)^{th} \text{ position}, \quad [v_n]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} N \cdot [v_1]_B &= 0_N. \\ N[v_2]_B &= 0_N \\ &\vdots \\ N[v_{f_1}]_B &= 0_N \end{aligned}\right\} \text{ Because } \{v_1, v_2, \ldots, v_{f_1}\} \text{ is the basis for } \ker(N).$$

Therefore, the first f_1 columns are all 0.

$$[N]_B = \left[\, N[v_1]_B \;\; N[v_2]_B \;\cdots\; N[v_{f_1}]_B \;\cdots\cdots \,\right]$$

$$= \left[\begin{array}{cccc|ccc} 0 & 0 & & 0 & & & \\ 0 & 0 & & 0 & & & \\ 0 & 0 & & 0 & & & \\ \vdots & \vdots & \cdots & \vdots & N[v_{f_1+1}]_B & \cdots & N[v_n]_B \\ \vdots & \vdots & & \vdots & & & \\ 0 & 0 & & 0 & & & \end{array}\right]_{n\times n}$$

$$\underbrace{\hphantom{0 \quad 0 \quad \cdots \quad 0}}_{\text{first } f_1 \text{ columns}}$$

$v_{f_1+1} \in \mathrm{Ker}(N^2)$.

$\Rightarrow N^2 v_{f_1+1} = 0_V$.

$\Rightarrow N.\left(N.v_{f_1+1}\right) = 0_N$.

Therefore, $N.v_{f_1+1} \in \mathrm{Ker}(N)$.

So Nv_{f_1+1} can be represented as linear combination of basis vectors chosen for $\mathrm{Ker}(N)$.

$$N[v_{f_1+1}]_B = \underbrace{c_1 \cdot v_1 + c_2 v_2 + \cdots + c_{f_1} \cdot v_{f_1}}_{\substack{\text{all non zero entries are} \\ \text{above the } f_1 \text{ rows.}}} + 0 \cdot v_{f_1+1} + \cdots + 0 \cdot v_n$$

$$\begin{array}{cc} & \overbrace{\hphantom{0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0}}^{f_1 \text{ columns}} \\ [N]_B = \begin{array}{c} \\ \\ f_1 \text{ rows} \\ \\ \\ \\ \\ \\ \end{array} & \left[\begin{array}{cccccc|ccccc} 0 & 0 & 0 & & 0 & 0 & c_{11} & c_{12} & & c_{1f_2} & \\ 0 & 0 & 0 & & 0 & 0 & c_{21} & c_{22} & & c_{2f_2} & \\ 0 & 0 & 0 & & 0 & 0 & c_{31} & c_{32} & & c_{3f_2} & \\ 0 & 0 & 0 & & 0 & 0 & c_{41} & c_{42} & \cdots & c_{4f_2} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & & 0 & 0 & c_{f_11} & c_{f_12} & & c_{f_1f_2} & \\ \hline 0 & 0 & 0 & & 0 & 0 & 0 & 0 & & 0 & \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 & & 0 & \end{array}\right] \\ & \hphantom{0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad} \underbrace{\hphantom{c_{11} \quad c_{12} \quad \cdots \quad c_{1f_2}}}_{f_2 \text{ columns.}} \end{array}$$

Now $v_{f_1+f_2+1} \in \operatorname{Ker}(N^3)$

$\Rightarrow N^3 v_{f_1+f_2+1} = 0_N$

$\Rightarrow N^2 \cdot (N . v_{f_1+f_2+1}) = 0_V$

$\Rightarrow N . v_{f_1+f_2+1} \in \operatorname{Ker}(N^2)$.

So $N . v_{f_1+f_2+1} = c_1 \cdot v_1 + c_2 v_2 + \cdots + c_k \cdot v_{f_1+f_2}$

(linear combination of previous basis vectors).

Therefore, this time, the zeros will be below $f_1 + f_2$ rows so for f_1+f_2 rows, there will be non zeros.

$$
[N]_B = \begin{array}{c}
\begin{matrix} \overbrace{\qquad\qquad\qquad}^{f_1} & \overbrace{\qquad\qquad\qquad}^{f_2} & \overbrace{\qquad\qquad\qquad}^{n-f_1-f_2} \end{matrix} \\
\left[\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & \times & \times & \times & \times & \times & \times \\
0 & 0 & 0 & \times & \times & \times & \times & \times & \times \\
0 & 0 \;\cdots & 0 & \times & \times \;\cdots & \times & \times & \times \;\cdots & \times \\
0 & 0 & 0 & \times & \times & \times & \times & \times & \times \\
\hline
\vdots & \vdots & \vdots & 0 & 0 & 0 & \times & \times & \times \\
 & & & 0 & 0 & 0 & \times & \times & \times \\
 & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & 0 & 0 & 0 & \times & \times & \times \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 \\
 & & & & & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]_{n\times n}
\end{array}
\begin{array}{l} \left.\begin{matrix}\\ \\ \\ \\ \end{matrix}\right\} f_1 \\ \left.\begin{matrix}\\ \\ \\ \\ \end{matrix}\right\} f_2 \\ \left.\begin{matrix}\\ \\ \\ \\ \end{matrix}\right\} n-f_1-f_2 \end{array}
$$

Therefore, if we notice then $[N]_B$ has a special structure that it is upper triangular matrix with all diagonal elements are 0.

$$(T-\lambda_i I)\Big|_{G(\lambda_i, T)} \quad \text{is a nilpotent operator}$$

So if we choose the basis of $\mathbb{V}$ as,

$$B = B_{\text{Ker}(T-\lambda_1 I)^n} \cup B_{\text{Ker}(T-\lambda_2 I)^n} \cup \cdots \cup B_{\text{Ker}(T-\lambda_m I)^n}$$

Then on this basis we have –

$$[T]_B = \left[(T-\lambda_i I)\Big|_{G(\lambda_i, T)}\right]_B + \left[\lambda_i I\Big|_{G(\lambda_i, T)}\right]_B .$$

$$\left[T - \lambda_i I \Big|_{G(\lambda_i, T)} \right]_B = \begin{bmatrix} 0 & & & & & * \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & 0 & & & & & 0 \end{bmatrix}_{n \times n}$$

$\uparrow$

Nilpotent operator

$$\left[\lambda_i I \Big|_{G(\lambda_i, T)} \right]_B = \begin{bmatrix} \lambda_1 & 0 & & 0 & 0 & & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 & & 0 \\ 0 & 0 & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \lambda_1 & 0 & & \vdots \\ \vdots & \vdots & & 0 & \vdots & & \vdots \\ 0 & 0 & & 0 & 0 & & 0 \end{bmatrix}$$

$$\underbrace{\hphantom{\lambda_1 \quad 0 \quad \cdots \quad 0}}_{d_i \text{ columns}}$$

Since $\lambda_i I$ is restricted to $G(\lambda_i, T)$ so for the first d_i columns (dim. of $G(\lambda_i, T)$) the contribution is reflected and for rest of the basis, it is all 0.

Because of restriction operation & direct sum of subspaces.

$$[T]_B = \begin{array}{c} \left[\begin{array}{c|c|c} \begin{matrix} \lambda_1 & & * \\ & \lambda_1 & \\ & & \ddots \; \lambda_1 \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} \lambda_2 & & * \\ & \lambda_2 & \\ & & \ddots \; \lambda_2 \end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix} \lambda_m & & * \\ & \lambda_m & \\ & & \ddots \; \lambda_m \end{matrix} \end{array}\right] \begin{matrix} \}\, d_1 \\ \\ \}\, d_2 \\ \\ \}\, d_m \end{matrix} \\ \begin{matrix} \quad d_1 & \qquad\qquad d_2 & \qquad\qquad d_m \end{matrix} \end{array}$$

$$[T]_B = \begin{bmatrix} [A_1]_{d_1 \times d_1} & 0 & 0 \\ 0 & [A_2]_{d_2 \times d_2} & 0 \\ & & \ddots \\ 0 & 0 & [A_m]_{d_m \times d_m} \end{bmatrix}_{n \times n}.$$

__Example:__

$$T = \begin{bmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$ with respect to standard basis.

Eigenvalues of T are $6, 6, 7$.

Generalized eigenvectors will be-

$$G(6,T) = \langle \{(1,0,0), (0,1,0)\} \rangle = \text{Ker}\,(T - 6I)^3$$

$$G(7,T) = \langle \{(10,2,1)\} \rangle = \text{Ker}\,(T - 7I)^3.$$

If you find,

$$E(6,T) = \langle \{(1,0,0)\} \rangle$$

$$E(7,T) = \langle \{(10,2,1)\} \rangle$$

Note the matrix T has not sufficient eigenvector to diagonalize it.

Also note that when $\dim\left(\operatorname{Ker}(T-\lambda I)^n\right) = 1$ or $AM(\lambda) = 1$ then we see that $G(\lambda, T) = E(\lambda, T)$.
or $AM(\lambda) = GM(\lambda)$.

A quick check $GM(\lambda) \leq AM(\lambda)$

$$\Rightarrow \dim\left(\operatorname{Ker}(T-\lambda I)\right) \leq \dim\left(\operatorname{Ker}(T-\lambda I)^n\right)$$

Obviously

$$\operatorname{Ker}(T-\lambda I) \subseteq \operatorname{Ker}(T-\lambda I)^2 \subseteq \cdots \subseteq \operatorname{Ker}(T-\lambda I)^n.$$

$$\Rightarrow \dim\left(\operatorname{Ker}(T-\lambda I)\right) \leqslant \dim\left(\operatorname{Ker}(T-\lambda I)^n\right)$$

$$\Rightarrow GM(\lambda) \leqslant AM(\lambda)$$

(The equality is possible iff $\operatorname{Ker}(T-\lambda I)^m = \operatorname{Ker}(T-\lambda I)^{m+1}$)

So we have, $\mathbb{C}^3 = G(6, T) \oplus G(7, T)$

Basis for $\mathbb{C}^3$ is, $B = \{(1,0,0), (0,1,0), (10,2,1)\}$.

Matrix of T w.r.t to basis B will be –

$$[T]_B = \left[\begin{array}{cc:c} 6 & 3 & 0 \\ 0 & 6 & 0 \\ \hdashline 0 & 0 & 7 \end{array}\right]$$

Block diagonalizable matrix.

$$[T]_S = \begin{bmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_B = P^{-1} [T]_S \, P$$

$$= \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

When T is not diagonalizable (not enough eigen-vectors) then using the generalized eigenvectors, we have shown that T can be at least block diagonalized. Using Jordan form, even more 0's can be acheived.

Jordan Form:

Suppose $N \in L(\mathbb{F}^4)$ be a nilpotent operator.

$$[N]_S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We can form a basis for $\mathbb{F}^4$ as - $\{N^3v, N^2v, Nv, v\}$ where $v = (1,0,0,0)$.

Then the matrix N w.r.t B will be -

$$[N]_B = P^{-1}[N]_S P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose, $N \in \mathcal{L}(\mathbb{F}^6)$ and the operator is defined by.

$$N(z_1, z_2, \dots, z_6) := (0, z_1, z_2, 0, z_4, 0)$$

But in this case, there doesn't exist a $v \in \mathbb{F}^6$ such that $\{N^5v, N^4v, N^3v, N^2v, N^1v, v\}$ is a basis for $\mathbb{F}^6$.

However if we take $v_1 = (1,0,0,0,0,0)$, $v_2 = (0,0,0,1,0,0)$ and $v_3 = (0,0,0,0,0,1)$ then

$\{\underline{N^2v_1, Nv_1, v_1}, \underline{Nv_2, v_2}, \underline{v_3}\}$ forms a basis for $\mathbb{F}^6$.

Basically, we have to come up with a vectors such that we get a basis for $\mathbb{V}$ where a nilpotent operator can have a nice Jordan form.

$$[N]_B = \left[\begin{array}{ccc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \boxed{1} &
\end{array}\right]$$

Every nilpotent operator $N \in L(V)$ behaves similarly. Specifically there is a finite collection of vectors $v_1, v_2, \dots, v_n \in V$. Such that there is a basis of V consisting of the vectors in the form $N^k v_j$.

Suppose $N \in L(V)$ is a nilpotent operator. Then there exist vectors $v_1, v_2, \dots, v_k \in V$ and non negative integers $m_1, m_2, \dots, m_k$ such that -

$$\{ N^{m_1} v_1, N^{m_1 - 1} v_1, \dots, N v_1, v_1,$$
$$N^{m_2} v_2, N^{m_2 - 1} v_2, \dots, N v_2, v_2,$$
$$\vdots$$
$$N^{m_k} v_k, N^{m_n - 1} v_k, \dots, N v_k, v_k \}$$ is a basis for V.

moreover, $N^{m_1+1} v_1 = N^{m_2+1} v_2 = \cdots = N^{m_n+1} v_K = 0_V$.

Suppose $T \in L(V, V)$. A basis of V is called Jordan basis for T if with respect to that basis T has a block diagonal matrix.

$$[T]_B = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix}_{n \times n}, \quad A_j = \begin{bmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ 0 & & & \lambda_j \end{bmatrix}_{d_j \times d_j}$$

<u>proof</u>: Consider a nilpotent operator $N \in L(V, V)$.
consider vectors $\{v_1, v_2, \ldots, v_K\}$.

$$B = \{ \boxed{N^{m_1} \cdot v_1}, N^{m_1 - 1} v_1, \ldots, N v_1, v_1,$$
$$\boxed{N^{m_2} v_2}, N^{m_2 - 1} v_2, \ldots, N v_2, v_2,$$
$$\vdots$$
$$\boxed{N^{m_K} v_K}, N^{m_K - 1} v_K, \ldots, N v_K, v_K \}$$

Note that $N \cdot (N^{m_j} \cdot v_j) = N^{m_j + 1} \cdot v_j = 0$

So N sends the first vector in the list always to 0 and other vector to its previous vector.

$$\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{m_1+1} & \overbrace{\qquad\qquad\qquad}^{m_2+1} \end{array} \\
\begin{array}{c} m_1+1 \\ \\ \\ m_2+1 \end{array} &
\left[\begin{array}{ccccc|ccccc|c}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \\
0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \cdots & 0 & \\
\vdots & \vdots & \vdots & & 1 & 0 & \vdots & \vdots & & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & & & & 0 & \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & \\
\vdots & \vdots & \vdots & & \vdots & 0 & 0 & 1 & \cdots & 0 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & 0 & \\
 & & & & & \vdots & \vdots & \vdots & & 1 & \ddots \\
 & & \vdots & & & 0 & 0 & 0 & \cdots & 0 & \\
 & & & & & \vdots & \vdots & \vdots & & \vdots &
\end{array}\right]
\end{array}$$

Therefore, each of the A_i will have the above form. With of course the λ_i must be added at the diagonal. Collect all the Jordan basis for each A_i and then combine it to get basis for V.

Jordan Form of an Operator:

A Jordan block of size K is a $K \times K$ matrix of the form -

$$\begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{bmatrix}$$

A square matrix is said to be in Jordan form if it is block diagonal and each block is a Jordan block.

Any operator $T : V \to V$ can be represented by a matrix in Jordan form. This matrix is unique up to the rearrangement of the order of Jordan blocks and is called Jordan form of T. The basis of V that puts the matrix in Jordan form is called Jordan basis.

If v is a non-zero generalized eigenvector of T corresponding to eigenvalue λ, then there exists a smallest positive integer K such that,

$$(T - \lambda I)^{K} \cdot v = 0_{V}.$$

$\{v, (T-\lambda I)v, (T-\lambda I)^2 v, \ldots, (T-\lambda I)^{K-1} v\}$ is LI set.

The above set is called the "Jordan Chain" corresponding to 'v'.

Note that for each Jordan chain, the final vector $(T-\lambda I)^{K-1} v$ in the chain is the ordinary eigenvector because -

$$(T-\lambda I)^{K} v = 0$$

$$\Rightarrow (T-\lambda I)\underbrace{\left((T-\lambda I)^{K-1} \cdot v\right)}_{w \neq 0} = 0$$

$$\Rightarrow (T-\lambda I) w = 0$$

$\Rightarrow$ w is an eigenvector of T so $(T-\lambda I)^{k-1} v$ is the ordinary eigenvector.

A Jordan basis is exactly a basis of V which is composed of Jordan chains. Such a basis exists for Nilpotent operators. Each Jordan block in the Jordan form of T corresponds to exactly one such Jordan Chain.

$\{ (T-\lambda I)^{K-1} v, (T-\lambda I)^{K-2} v, \ldots, (T-\lambda I) v, v \}$ is a Jordan block.

Generalized Eigenspaces:

A vector v is called generalized eigenvector of A corresponding to an eigenvalue λ if -

$$(A-\lambda I)^K v = 0_V \quad \text{for some } K \geq 1$$

The set of all generalized eigenvectors $G(\lambda, A)$ will be defined as:

$$G(\lambda, A) := \ker (A-\lambda I)^1 \cup \ker (A-\lambda I)^2 \cup \cdots$$

$$= \bigcup_{K \geq 1} \ker\left((A-\lambda I)^K\right)$$

We know that the Kernel sequence is an increasing sequence :

$$\ker (A-\lambda I) \subset \ker (A-\lambda I)^2 \subset \ker (A-\lambda I)^3 \subset \cdots$$

But we know that the sequence will eventually stabilize & stops growing further so then we can write -

$$\ker (A-\lambda I)^K = \ker (A-\lambda I)^{K+1} \qquad \forall K > K_\lambda$$

We also know that K_λ is at most $\dim (V) = n$.

But the sequence may get stabilized before n.
We also know that,

for some 'K', $\quad \ker(A-\lambda I)^K = \ker(A-\lambda I)^{K+1} \longrightarrow$

$$\ker(A-\lambda I)^{K+r} = \ker(A-\lambda I)^{K+r+1}$$

Suppose the number $d = d(\lambda)$ on which the sequence $\ker(A-\lambda I)^K$ stabilizes i.e. the number d such that,

$$\ker(A-\lambda I)^{d-1} \subsetneq \ker(A-\lambda I)^d = \ker(A-\lambda I)^{d+1}$$

is called the "index"/"depth" of the eigenvalue λ.

Generalized eigenspace,

$$G_\lambda = \bigcup_{K \geq 1} \ker(A-\lambda I)^K$$

For all v (generalized eigenvector) $\in G_\lambda$,

$$(A-\lambda I)^K . v = 0_V$$

If we take $v \in \ker(A-\lambda I)^{d(\lambda)}$ then for sure all the $v \in G_\lambda$ must be included as we have seen

$$\ker(A-\lambda I)^1 \subset \ker(A-\lambda I)^2 \subset \cdots \subset \ker(A-\lambda I)^{d(\lambda)} = \ker(A-\lambda I)^{d(\lambda)+1} = \cdots$$

Therefore $G_\lambda := \ker(A-\lambda I)^{d(\lambda)}$ (This will include all the generalized eigenvectors).

$\max d(\lambda) = n$ (after this we don't need it)

$$\boxed{\text{Since } \forall v \in G_\lambda \text{ therefore, } (A-\lambda I)^{d(\lambda)} \cdot v = 0_{\mathbb{V}}.}$$

(i) $\ker(A-\lambda I)^{d(\lambda)}$ is an invariant subspace of $\mathbb{V}$.

(ii) We can restrict the operator A on the subspace G_λ only.

$$A\big|_{G_\lambda} : \ker(A-\lambda I)^{d(\lambda)} \longrightarrow \ker(A-\lambda I)^{d(\lambda)}$$

$\Rightarrow A\big|_{G_\lambda} v := Av$ where $v \in \ker(A-\lambda I)^{d(\lambda)}$

Therefore, $\left(A\big|_{G_\lambda} - \lambda I\big|_{G_\lambda}\right)^{d(\lambda)} v = 0_{\mathbb{V}}$.

$$\Rightarrow \left.(A-\lambda I)^{d(\lambda)}\right|_{G_\lambda} = 0_{\mathbb{V}}.$$

Spectrum $\sigma(A)$ of an operator 'A' is the set of all eigenvalues of A (not counting multiplicities).

<u>Theorem</u>: let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigen values of A. The corresponding generalized eigenspaces are $G_{\lambda_1}, G_{\lambda_2}, \ldots, G_{\lambda_r}$. Then we can write,

$$\mathbb{V} = G_{\lambda_1} \oplus G_{\lambda_2} \oplus \cdots \oplus G_{\lambda_r}.$$

<u>proof</u>: The operator $N_K := A - \lambda_K \cdot I$ is the nilpotent operator.

$$\left.\left(A - \lambda_K \cdot I\right)^{d(\lambda)}\right|_{G_{\lambda_K}} v = 0_{\mathbb{V}} \text{ for all } v \in G_{\lambda_K}$$

$$\Rightarrow \left.\left(A - \lambda_K \cdot I\right)^{d(\lambda)}\right|_{G_{\lambda_K}} = 0 \text{ (operator)}$$

$$\Rightarrow \left.N_K^{d(\lambda)}\right|_{G_{\lambda_K}} = 0$$

$$\left.N_K\right|_{G_{\lambda_K}} : \operatorname{Ker}(A - \lambda_K I)^{d(\lambda)} \longrightarrow \operatorname{Ker}(A - \lambda_K I)^{d(\lambda)}$$

So $N_K \big|_{G_{\lambda_K}}$ is the nilpotent operator.

operator A can be written in block diagonal form diag $(A_1, A_2, \dots, A_r)$ where each block size will be $d(\lambda_K) \times d(\lambda_K)$ because of the basis that we will select from each G_{λ_K} space.

Characteristic polynomial,

$$P(z) = \prod_{K=1}^{r} (z - \lambda_K)^{m_K}$$