Solutions & Hints

Quiz 1

1.(a) Suppose

$$A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

& $$B = \begin{bmatrix} b & b \\ b & b \end{bmatrix}$$

with $a, b \in \mathbb{R}$.

$\Rightarrow A, B \in S$.

Let $AB = O$

$$\Rightarrow \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{Zero elements in}}$$

S.

$\Rightarrow \quad 2ab = 0$

$\Rightarrow$ either $a = 0$ [implying $A = O \in S$]

or $\quad b = 0$ [implying $B = O \in S$]

Thus, the given property holds for S.

(b) Suppose $\begin{bmatrix} p & p \\ p & p \end{bmatrix} \in S$ is the multiplicative id. in S.

$\Rightarrow$ for any $a \in \mathbb{R}$,

we must have

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} p & p \\ p & p \end{bmatrix}$$

$$= \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2ap & 2ap \\ 2ap & 2ap \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\Rightarrow \quad 2ap = a$$

$$\Rightarrow \quad a\left(p - \tfrac{1}{2}\right) = 0$$

either $a = 0$, in which case,

any element multiplied with $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ results in $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$

or $\quad p = \frac{1}{2}$.

$$\Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \in S$$

is the multiplicative identity in S.

(c) Suppose $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ has a multiplicative inverse in S given by

$$\begin{bmatrix} p & p \\ p & p \end{bmatrix}; \text{ i.e.}$$

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} p & p \\ p & p \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2ap & 2ap \\ \ldots \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2ap & 2ap \\ 2ap & 2ap \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Evidently, we need

$$2ap = \frac{1}{2}$$

$$\Rightarrow ap = \frac{1}{4}.$$

$\Rightarrow$ Unless $a = 0$,

we have, $p = \frac{1}{4a}$

i.e. unless we are dealing with $0 \in S$, every other element has a multiplicative inverse in S given by

$$\begin{bmatrix} \frac{1}{4a} & \frac{1}{4a} \end{bmatrix}$$

$$\begin{bmatrix} \frac{}{4a} & \frac{}{4a} \end{bmatrix}.$$

2 (a) Consider $\mathbb{Z}^2 \subseteq \mathbb{R}^2$

so that $\mathbb{Z}^2 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{Z} \right\}$

which is a subset of $\mathbb{R}^2$.

If we add $v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

& $v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, we get

$$v = v_1 + v_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

$$\in \mathbb{Z}^2$$

Also for $v_1 = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^2$,

$$\begin{bmatrix} -x \\ \end{bmatrix} \quad \mathbb{Z}^2$$

$[-y] \in \mathbb{Z}^2$ as well.

Thus $\mathbb{Z}^2$ is closed under addition & additive inverse.

However consider

$\hat{v} = \alpha v$ for $\alpha \notin \mathbb{Z}$ & $\alpha \in \mathbb{R}$.

$\Rightarrow \hat{v} \notin \mathbb{Z}^2$ even if $v \in \mathbb{Z}^2$.

Thus, $\mathbb{Z}^2$ is not a subspace as it is not closed under scalar multiplication.

(b) Consider $(\mathbb{Z}_3, \oplus_3, \odot_3)$

i.e. addition & multiplication modulo 3 over $\{0, 1, 2\}$

modulo 3 over $\{0,1,2\}$.

This is a field since 3 is a prime number.

Further consider $\mathbb{Z}_3^4$

$$:= \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : x,y,z,w \in \mathbb{Z}_3 \right\}$$

with vector addition & scalar multiplication being defined in the usual manner (elementwise modulo addition & multiplication)

Clearly (no need to prove this, but think about why this is "clear") $\mathbb{Z}_3^4$ ... $\mathbb{Z}$ ...

$\mathbb{Z}_3$ over $\mathbb{Z}_3$ is a vector space. Further each of the 4-tuple entries could be either of 0, 1, or 2. Thus, the total number of elements is $3 \times 3 \times 3 \times 3 = 81$.

(c) Consider $S = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

Clearly, $\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \notin S$.

Hence, S is not closed under vector addition.

However $\alpha v = \begin{bmatrix} \alpha x \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 0 \\ \alpha y \end{bmatrix}$

both of which belong to S for $\alpha \in \mathbb{R}$. So S is closed under scalar multiplication.

3. $(x, y) \oplus (a, b) = (x+a, y+b)$

$(x, y) \otimes (a, b) = (ax + 2by, bx + ay)$

Clearly, $(X, \oplus)$ is an Abelian group. For '0' we need

$(x, y) + (a, b) = (x, y)$

$$(x,y)+(a,b)=(x,y)$$

with (a,b) being the '0'.

$$\Rightarrow \left.\begin{array}{r} a+x = x \\ b+y = y \end{array}\right\} \Rightarrow \begin{array}{l} a=0, \\ b=0. \end{array}$$

Thus $\boxed{0 \equiv (0,0)}$

For '1', we need.

$$(x,y) \otimes (a,b) = (x,y)$$

with $(a,b) \equiv 1$.

$$\Rightarrow \begin{array}{r} ax+2by = x \\ bx+ay = y \end{array}$$

$$\Rightarrow \begin{array}{r} (a-1)x+2by = 0 \\ bx+(a-1)y = 0 \end{array}$$

$$\Rightarrow \begin{bmatrix} a-1 & 2b \\ b & a-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

(for any choice of $x, y \in \mathbb{Z}$)

$$\Rightarrow \quad a = 1, \quad b = 0$$

i.e., $\boxed{(1, 0) \equiv 1}$

Now, we need to verify distributivity.

Consider

$$(a_1, b_1) \otimes \big((a_2, b_2) \oplus (a_3, b_3) \big)$$

$$= (a_1, b_1) \otimes (a_2 + a_3, b_2 + b_3)$$

$$= \left(a_1(a_2+a_3) + 2b_1(b_2+b_3),\ a_1(b_2+b_3) + b_1(a_2+a_3)\right)$$

Also,

$$\left((a_1, b_1) \otimes (a_2, b_2)\right) \oplus \left((a_1, b_1) \otimes (a_3, b_3)\right)$$

$$= (a_1a_2 + 2b_1b_2,\ a_1b_2 + a_2b_1) \oplus (a_1a_3 + 2b_1b_3,\ a_1b_3 + b_1a_3)$$

Again, <u>distributivity</u> holds.

It is readily verifiable that associativity also holds (You need to do this)

Finally let

$$(x, y) \otimes (a, b) = \underbrace{(0, 0)}_{\text{"0"}}$$

$$\Rightarrow (ax + 2by,\ bx + ay) = (0, 0)$$

$$\Rightarrow \quad ax + 2by = 0$$
$$bx + ay = 0$$

$$\Rightarrow \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $(x, y) = (0, 0)$, that is one possibility.

Else, $\begin{bmatrix} a & 2b \\ \ldots \end{bmatrix}$

$\begin{bmatrix} b & a \end{bmatrix}$ needs to

have a non-trivial kernel for at least one of $a, b \neq 0$

$\Rightarrow$ $\mathrm{rref}\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\right)$ must contain at most one, non-zero row.

Suppose $a \neq 0$

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2b/a \\ b & a \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2b/a \\ 0 & a - \frac{2b^2}{a} \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{rref}\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\right)$$

$$a^2 - 2b^2 = 0$$

$$\Rightarrow \frac{a}{b} = \pm\sqrt{2}$$

but $a, b \in \mathbb{Z}$. $\Rightarrow \frac{a}{b} \in \mathbb{Q}$

while $\pm\sqrt{2}$ is irrational.

So $a^2 - 2b^2 \neq 0$.

$\Rightarrow \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ has full rank.

unless $a = 0$

Suppose $a = 0$.

Then $\begin{bmatrix} 0 & 2b \\ b & 0 \end{bmatrix}$

also has full rank, unless

$b = 0$

$b = 0.$

So. $(a, b) = (0, 0)$,

unless. $(x, y) = (0, 0)$.

So. $(x, y) \otimes (a, b) = 0$

$\Rightarrow$ either $(x, y) = (0, 0)$

or $(a, b) = (0, 0)$.

Hence $(X, \oplus, \otimes)$ is

an integral domain.

(b) $p^2 = p \otimes p$

$= (p_x, p_y) \otimes (p_x, p_y)$

$= (p_x^2 + 2p_y^2, 2p_x p_y)$

$2 = (1, 0) \oplus (1, 0)$

$$= (2, 0).$$

$$\Rightarrow \quad p_x^2 + 2p_y^2 = 2$$

$$\& \quad 2p_xp_y = 0$$

either $p_x = 0$, $p_y = \pm 1$.

or $p_y = 0$ $p_x = \pm\sqrt{2}$.

But $\pm\sqrt{2} \notin \mathbb{Z}$

$\therefore$ The only solutions are $(0, 1)$ and $(0, -1)$.

4. Suppose $x \in \ker(A)$

$$\Rightarrow \quad Ax = 0 \quad (i)$$

Further, let $x \in im(A)$

$\Rightarrow \exists\, y \in \mathbb{R}^n : Ay = x. \text{(ii)}$

From, (i) & (ii) we need

$$\boxed{A^2 y = 0.}$$

Obviously for $y = 0$, $x = 0$

we have $0 \in ker(A) \cap im(A)$.

The question then boils down to ascertaining if any non-zero y exists such that $x = Ay \neq 0$, but

$A^2 y = 0$.

y

Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

Obviously, $A^2 y = 0$ for any $y \neq 0$, since

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

But choosing $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

we get $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq 0$.

Thus, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \ker(A) \cap \operatorname{im}(A)$.

Hence, the assertion stands disproved by counter-example.

5. $\mathbb{Z}_2$ is a field & hence also an integral domain. So we can factorize $x^2-10x+16$

as $(x-8)(x-2)$ & look for roots of $x-8=0$ &

$x-2=0$

since $(x-8).(x-2)=0$

implies either $x-8=0$

i.e. $\boxed{x=0}$

or $x-2=0$, i.e. $\boxed{x=0}$

However for $\mathbb{Z}_8$, since this is not an integral domain, apart from

$x-8=0$ i.e. $\boxed{x=0}$ &

$x-2=0$. i.e. $\boxed{x=2}$,

we may need to look for other roots as well since

$(x-2)(x-8)=0$ is possible even if $x-2\neq 0$ and $x-8\neq 0$.

We can search over the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ and verify that besides

$x = 0$ and $x = 2$,

we also have

$\boxed{x = 4 \ \& \ x = 6}$ as

roots of the given equation.

6. Let x_N, x_E, x_S & x_W be the flow thru North, East, South & West streets, respectively.

$\therefore \quad x_N + 330 = x_E + 175$

$\Rightarrow \quad x_N - x_E = -155 \longrightarrow (1)$

$$x_E + 570 = x_S + 425$$

$$\Rightarrow x_E - x_S = -145 \longrightarrow (2)$$

$$x_S + 250 = x_W + 300$$

$$\Rightarrow x_S - x_W = 50 \longrightarrow (3)$$

$$x_W + 150 = x_N + 400$$

$$\Rightarrow x_W - x_N = 250 \longrightarrow (4)$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_N \\ x_E \\ x_S \\ x_W \end{bmatrix} = \begin{bmatrix} -155 \\ -145 \\ 50 \\ 250 \end{bmatrix}$$

Performing rref, we get

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -155 \\ 0 & 1 & -1 & 0 & -145 \\ 0 & 0 & 1 & -1 & 50 \end{array}\right.$$

$$\begin{bmatrix} 0 & -1 & 0 & 1 & \vdots & 93 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -155 \\ 0 & 1 & -1 & 0 & -145 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & -1 & 1 & -50 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -155 \\ 0 & 1 & -1 & 0 & -145 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -300 \\ 0 & 1 & -1 & 0 & -145 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -250 \\ 0 & 1 & 0 & -1 & -95 \\ 0 & 0 & 1 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_N = x_W - 250$$

$$x_E = x_W - 95$$

$$x_S = x_W + 50$$

& x_W is the free variable.

Clearly, the solution is non-unique, but it exists.

(b) Clearly adding all the rows, we should get 0 on both sides, implying the inflow = outflow of vehicles [$\because$ there is no accumulation due to "NO PARKING" restrictions].

But with an inc in net inflow by 50 & an inc in net outflow by 30, this does not hold. So, this does no[illegible]

no. solution exists!

(c) Since all streets are one-way, we need the solution in (a) to have all values positive. Clearly, with $x_W = 200$, $x_N = -50$, which is inadmissible.

[This problem illustrates a simple application of linear algebra, in a day-to-day problem].