4) Suppose a symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\mathbb{U}$ is an A-invariant subspace. Show that $\mathbb{U}^{T}$ must also be A-invariant.

$\mathbb{U}^{\perp} = \{ v \in \mathbb{V} \mid \langle u | v \rangle = 0 \ \forall\, u \in \mathbb{U} \}$.

& $A = A^{T}$

Since $\mathbb{U}$ is invariant under A.

if $u_1 \in \mathbb{U}$ then $\exists\ u_1' \in \mathbb{U}$ such that

$$A u_1' = u_1$$

consider $w_1 \in \mathbb{U}^{\perp}$ then we know that

$$\langle u_1 | w_1 \rangle = 0$$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow w_1^{T} u_1 = 0 \quad [u_1 = A u_1']$

$\Rightarrow w_1^{T} (A u_1') = 0 \quad [\text{Since } A = A^{T}]$

$\Rightarrow (A w_1)^{T} u_1' = 0$

$\exists\ y \in V$ such that $A w_1 = y$.

$$y^{T} u_1' = 0 \Rightarrow y \in \mathbb{U}^{\perp}$$

$A w_1 \in \mathbb{U}^{\perp}$ & $w_1 \in \mathbb{U}^{\perp}$

$\therefore\ \mathbb{U}^{\perp}$ is also invariant under A

Q6

a) Suppose $v \in \operatorname{Im}[B \; AB \; A^2B \; \text{——} \; A^{n-1}B]$

$\Rightarrow v = [B \; AB \; \text{——} \; A^{n-1}B]x$

Consider $Av = [AB \; A^2B \; \text{——} \; A^nB]x$.

Now the characteristics polynomial of A is of degree 'n' & A satisfies its own characteristics equation.

Hence, if $\chi_A(s)$ is the characteristics polynomial of A, then $\chi_A(A) = 0$. But $\deg(\chi_A(s)) = n$, & it is monic.

Hence any power of A greater than $n-1$, can be written as a linear combination of $\{I, A, A^2 \text{——} A^{n-1}\}$

$$\Rightarrow A^nB = \left(\sum_{i=0}^{n-1} d_i A^i\right)B \qquad [\because A^0 = I]$$

$$\Rightarrow Av = [AB \; A^2B \; \text{——} \; A^nB]x = \left[AB \; A^2B \; \ldots \; \sum_{i=0}^{n-1} d_i A^i B\right]x$$

$x \in \mathbb{R}^{m \cdot n}$

$\therefore$ Av is expressible as a linear combination of the columns of $B, AB, \ldots, A^{n-1}B$.

Hence, $Av \in \text{im}[B, AB, \ldots, A^{n-1}B]$. Thus, $\mathcal{C}$ is invariant.

b) Let $v \in \Theta$ ie $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0$ —— (*)

Consider Av $\therefore$ $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} Av = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} v$

from (*) we know that $Cv = 0$

$CAv = 0$

$\vdots$

$CA^{n-1}v = 0$.

$\therefore \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ CA^n v \end{bmatrix}$.

Now, $A^n = \sum_{i=0}^{n-1} d_i A^i \Rightarrow CA^n v = C\sum_{i=0} d_i A^i v = 0$.

$\therefore \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} v = 0$. Hence, Θ is also A-invariant.

not all

Q7

Since W is A-invariant & $W = \langle w_1, w_2, \ldots w_k \rangle$

$$\Rightarrow Aw_i \in W \Rightarrow Aw_i = \sum_{j=1}^{k} a_{ij} w_j$$

Now, $W = [w_1, w_2, \ldots w_k]$

$$\Rightarrow AW = [Aw_1, Aw_2, \ldots Aw_k] = \left[\sum_{j=1}^{k} a_{1j} w_j \; \sum_{j=1}^{k} a_{2j} w_j \ldots \sum_{j=1}^{k} a_{kj} w_j\right]$$

$$\Rightarrow AW = [w_1, w_2 \ldots w_k] \begin{bmatrix} d_{11} & \ldots & d_{k1} \\ d_{12} & & d_{k2} \\ \vdots & & \vdots \\ d_{1k} & & d_{kk} \end{bmatrix} = WS$$

where $S = \begin{bmatrix} d_{11} & \ldots & d_{k1} \\ d_{12} & & d_{k2} \\ \vdots & & \vdots \\ d_{1k} & & d_{kk} \end{bmatrix}$

let v be an eigenvector of S with eigenvalue 'λ'.

$\Rightarrow Sv = \lambda v \Rightarrow WSv = \lambda Wv$ (premultiplying by W)

$\Rightarrow AWv = \lambda Wv$ $\quad [\because WS = AW]$

$\Rightarrow Ay = \lambda y$ where $Wv = y$

So, if λ is an eigenvalue of S with eigenvector v, then λ is also an eigenvalue of A with eigenvector

Wv, unless $v \in Ker(W)$.

Now $\exists$ atmost $k-r$ linearly independent vectors v_i s.t.

$Wv_i = 0$. So, $Ker(W)$ can atmost accommodate $(k-r)$

eigenvectors of S. The remaining $k-(k-r) = r = rank(w)$

eigenvalues of S must have their eigenvectors s.t.

$Wv \neq 0$ & $Sv = \lambda v$. Thus atleast 'r' eigenvalues

must be shared between A & S.

13)

(a) Consider A & A^H together with their characteristic polynomials given by:

$$\chi_A(x) = \det(A - xI)$$

& $$\chi_{A^H}(y) = \det(A^H - yI)$$

Now

$$\chi_{A^H}(y) = \overline{\overline{\det(A^H - yI)}}$$

$$= \overline{\det(\overline{A^H} - y^* I)}$$

$$= \overline{\det(A^T - y^* I)}$$

$$= \overline{\det(A^T - y^* I^T)}$$

$$= \chi_{A^T}(y^*).$$

But $\chi_A(z) = \chi_{A^T}(z)$, since $I^T = I$ & determinant is invariant under transposition.

$$\Rightarrow \quad \chi_{A^H}(y) = \chi_A(y^*).$$

Hence for $z \in \mathbb{C}$: $\chi_A(z) = 0$,

we have $\chi_{A^H}(z^*) = 0$.

(b) Suppose

$$Av = \lambda v \quad \text{--- (1)}$$

$$\& \quad A^H w = \mu w, \quad \text{--- (2)}$$

when $v \neq 0$, $w \neq 0$ &

$$\lambda \neq \mu^*.$$

from (1)

$$\Rightarrow \quad v^* A = \lambda v^* \quad \text{—} ③$$

Now, $\langle w | v \rangle = v^* w$ — ④

Also, from ③, we get

$$\underbrace{v^* A^H w}_{} = \lambda^* v^* w \quad [\text{by post mult. with } w]$$

$$\Rightarrow v^* \mu w = \lambda^* v^* w \quad [\text{using } ②$$

$$\Rightarrow (\mu - \lambda^*) v^* w = 0$$

$$\Rightarrow (\mu - \lambda^*) \langle w | v \rangle = 0 \quad (\text{from } ④)$$

$$\Rightarrow \langle w | v \rangle = 0 \quad [\because \mu^* \neq \lambda]$$

$\Rightarrow w$ is $\perp$ to v.

(c) Consider $J \in \mathbb{C}^{n \times n}$ as the Jordan canonical form of A (even if $A \in \mathbb{R}^{n \times n}$), we can always consider it to be a

matrix in $\mathbb{C}^{n\times n}$ & consider its
<u>complex</u> JCF).

Since $a.m.(\lambda) = 1$,
we have

$$J_1 = \begin{bmatrix} \lambda & \\ 0 & * \end{bmatrix}$$

for some $V = [v_1 \ldots v_n]$
where $Av_1 = \lambda v_1$ &.

V is non-singular, with its columns containing the (generalized) eigenvectors of A.

v_1 is clearly the right eigenvector of A corresp. to λ.

For left (generalized) eigenvectors, we have similarly.

$$\underbrace{\begin{bmatrix} w_1^* \\ w_2^* \\ \vdots \\ w_n^* \end{bmatrix}}_{W} A = \underbrace{\begin{bmatrix} \lambda & \text{---} \\ 0 & Q \end{bmatrix}}_{J_2} \underbrace{\begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix}}_{W},$$

where w_1^* is the left eig vector of A w.r.t. λ.

Suppose w_1 & v_1 are orthogonal,

i.e. $\langle v_1 | w_1 \rangle = 0$

$$\Rightarrow w_1^* v_1 = 0.$$

Hence,

$$WA = \begin{bmatrix} \lambda & \text{---} \\ 0 & Q \end{bmatrix} W$$

$$WAv_1 = \begin{bmatrix} \lambda & \text{---} \\ 0 & Q \end{bmatrix} Wv_1$$

$$\Rightarrow \lambda \begin{bmatrix} 0 \\ w_2^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix} = \begin{bmatrix} \lambda & \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 \\ w_2^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix}$$

comparing the last $(n-1)$ terms in the vectors on both sides, we get

$$\lambda \underbrace{\begin{bmatrix} w_2^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix}}_{p} = Q \underbrace{\begin{bmatrix} w_2^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix}}_{p}$$

$\Rightarrow$ λ is an eigenvalue of Q. [Note that v_1 cannot be orthogonal to w_i for

all $i = 2, 3, \ldots n$, since
then we would have
$v_1 = 0$, a
contradiction. Hence
$p \neq 0$]

But $\chi_A(x) = \chi_{J_2}(x)$
$$= (x-\lambda) \cdot \chi_{\mathcal{B}}(x).$$

Now if λ is an eigenvalue of $\mathcal{B}$, then
$$\chi_{\mathcal{B}}(x) = (x-\lambda)\, q(x)$$
for some $q(x) \in \mathbb{R}[x]$.

Hence, $\chi_A(x) = (x-\lambda)^2 \cdot q(x)$.

$\Rightarrow$ a.m.$(\lambda) = 2 > 1$, a
contradiction. Thus

contradiction. Thus

$$\langle v_1 | w_1 \rangle \neq 0.$$

20) Which of the following are Ideals and Why / why not?

a) {all polynomials in $\mathbb{C}[x]$ having constant term equal to zero}.

$\mathfrak{a} := \{ p(x) \in \mathbb{C}[x] , p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x \}$

Let $p, q \in \mathfrak{a}$

clearly $p+q = (a_n + b_n) x^n + \cdots + (a_1 + b_1) x \in \mathfrak{a}$

now, Consider $g(x) = c_m x^k + \cdots + c_1 x + c_0 \in \mathbb{C}[x]$

$pg = c_m x^k (p(x)) + \cdots + c_0 (p(x))$

notice $p(x) g(x)$ is also a polynomial with zero constant co-efficient.

$\therefore p(x) g(x) \in \mathfrak{a}$

$\therefore \mathfrak{a}$ is an ideal.

b) $\mathfrak{a} := \{ p(x) \mid p(x)$ has only even degree terms$\}$.

if p & $q \in \mathfrak{a}$, it is clear that $p+q \in \mathfrak{a}$

now consider

$g = a_0 + a_1 x + \cdots + a_k x^k \in \mathbb{C}[x]$

& $p(x) = b_2 x^2 + b_4 x^4 + \cdots + b_{2k} x^{2k}$

$p(x) g(x) = b_2 x^2 (g(x)) + \cdots + b_{2k} x^{2k} (g(x))$

$p(x) g(x)$ clearly has terms ~~which are~~ with odd degree. i.e $p(x) g(x) \notin \mathfrak{a}$

$\therefore$ not an ideal.

classmate

Date

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c) $\{0, 2, 4\} \subseteq \mathbb{Z}_6$

$0+2 = 2 \bmod 6 = 2$ $\therefore p, q \in a$

$2+4 = 6 \bmod 6 = 0$ & $p+q \in a$

Similarly when $p \in \{0, 1, 2, 3, 4, 5\}$
it can be verified that
every $2p \in \{0, 2, 4\}$ & $4p \{0, 2, 0\}$ &
$0p = 0 \in \{0, 2, 4\}$
$\therefore \{0, 2, 4\}$ is an ideal.

d) $a := \{p(x) \in \mathbb{Z}[x] \mid p(x)$ has even co-efficients$\}$

Let $p(x) = \sum_{i=0}^{k} a_i x^i \in a$ & $q(x) = \sum_{i=0}^{m} b_i x^i$

as a_i & b_i are even $\forall i$ we observe that
& $p+q \in a$ since $a_i + b_i$ is even $\forall i$

Let $g(x) = c_n x^n + \cdots + c_1 x + c_0 \in \mathbb{Z}[x]$
$c_i \; \forall i$ are not necessarily all odd or even

$p(x)\, g(x) = c_n x^n (p(x)) + \cdots + c_0 (p(x))$
every co-efficient of $g(x)$ is multiplied by an even number resulting in all even co-efficients & their addition (for the terms with same degree) results in an even number
$\therefore pg \in a \Rightarrow a$ is an ideal.

e) $\mathbb{R} \subseteq \mathbb{R}[x]$

not an ideal, consider $x \in \mathbb{R}[x]$

& $p \in \mathbb{R}$, $px \notin \mathbb{R}$.

Q22)

Given

$$\langle v, Av, \ldots A^{n-1}v \rangle = \mathbb{C}^n$$

$$\Rightarrow \quad \mathcal{B} = \{v, Av, \ldots, A^{n-1}v\}$$

is a basis for $\mathbb{C}^n$

(set of n vectors spanning $\mathbb{C}^n$ must be lin. ind. hence, a basis)

Now,

$$[A]_{\mathcal{B}} = \left[[Av]_{\mathcal{B}} \; [A\,Av]_{\mathcal{B}} \ldots [A.A^{n-1}v]_{\mathcal{B}} \right]$$

since we can have an ordered basis

$$\{v_1, v_2, \ldots v_n\}$$

where $v_i = A^{i-1}v$.

Thus, $Av_i = v_{i+1}$

$$\Rightarrow [Av_i]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \leftarrow (i+1)^{\text{th}} \text{ position.}$$

for $i = 1, 2, \ldots n-1$,

from Cayley Hamilton theorem, we know that

$$\chi_A(A) = 0$$

$$\Rightarrow A^n = -\sum_{i=0}^{n-1} \alpha_i A^i$$

[since $\chi_A(x)$ is monic, $\alpha_n = 1$]

$$\Rightarrow A \cdot v_n$$

$$= A \cdot A^{n-1} v$$

$$= A^n v$$

$$= -\sum_{i=0}^{n-1} \alpha_i A^i v$$

$$\sum_{i=0}$$

$$= -\sum_{i=0}^{n-1} \alpha_i v_{i+1}$$

$$\Rightarrow [Av_n]_{\mathcal{B}} = \begin{bmatrix} -\alpha_0 \\ -\alpha_1 \\ \vdots \\ -\alpha_{n-1} \end{bmatrix}$$

$$\Rightarrow [A]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & & 0 & -\alpha_0 \\ 1 & 0 & & 0 & -\alpha_1 \\ 0 & 1 & & \vdots & \vdots \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & & 1 & -\alpha_{n-1} \end{bmatrix}$$

Since $\{v, Av, \ldots, A^{n-1}v\}$

is a linearly ind. set,

there does not exist

$\{\alpha_0, \alpha_1, \ldots \alpha_{n-1}\}$;

$\sum \alpha_i A^i v = 0$, unless

$\alpha_i = 0 \quad \forall i = 0, 1, \ldots, n-1$

minimal polynomial of A w.r.t. v must be of degree at least equal to n.

Also, $\mu_A(x)$ must be a multiple of this polynomial.

Thus $\deg(\mu_A(x)) = n$.

and we know that

$$\mu_A(x) \mid \chi_A(x).$$

so $\mu_A(x) = \chi_A(x)$

Q.12) Given $A \in \mathbb{R}^{m \times n}$ and SVD of A is given by $A = U \begin{bmatrix} \Sigma_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} V^T$.

$$A^{+} = V \begin{bmatrix} \Sigma_{r \times r}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

(a) Let A is invertible. $\Rightarrow A \in \mathbb{R}^{n \times n}$

and $A^{+} = V \Sigma_{n \times n}^{-1} U^T$

with $V \in \mathbb{R}^{n \times n}$ and $U \in \mathbb{R}^{n \times n}$.

Now $A^{-1} = (U \Sigma_{n \times n} V^T)^{-1}$

$$= (V^T)^{-1} \Sigma_{n \times n}^{-1} U^{-1}$$

$$= V \Sigma_{n \times n}^{-1} U^T = A^{+}$$

(b) $$(A^{+})^{+} = \left(V \begin{bmatrix} \Sigma_{r \times r}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T \right)^{+}$$

$$= U \begin{bmatrix} (\Sigma_{r \times r}^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$= U \begin{bmatrix} \Sigma_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} V^T = A$$

(c) $$(A^{+})^T = \left(V \begin{bmatrix} \Sigma_{r\times r}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T\right)^T$$

$$= U \begin{bmatrix} (\Sigma_{r\times r}^{-1})^T & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$(A^T)^{+} = \left(\left(U \begin{bmatrix} (\Sigma_{r\times r}^{-1}) & 0 \\ 0 & 0 \end{bmatrix} V^T\right)^T\right)^{+}$$

$$= \left(V \begin{bmatrix} (\Sigma_{r\times r}^{-1})^T & 0 \\ 0 & 0 \end{bmatrix} U^T\right)^{+}$$

$$= U \begin{bmatrix} \Sigma_{r\times r}^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^T = (A^{+})^T$$

(d) Let A has full column rank.

Then $A^T A = V \Sigma_{n\times n}^{2} V^T \in \mathbb{R}^{n\times n}$

Now, $$(A^T A)^{-1} A = V^T \Sigma_{n\times n}^{-2} V V^T \begin{bmatrix} \Sigma_{n\times n}^{1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= V^T \begin{bmatrix} \Sigma_{n\times n}^{-2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_{n\times n} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= V^T \begin{bmatrix} \Sigma_{n\times n}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= A^{+}$$

Similarly when A has full row rank,

$$AA^T = U\Sigma^2_{m\times m}U^T \in \mathbb{R}^{m\times m}$$

& $$A^T(AA^T)^{-1} = V\begin{bmatrix}\Sigma_{m\times m} & 0\\ 0 & 0\end{bmatrix}U^TU\Sigma^{-2}_{m\times m}U^T$$

$$= V\begin{bmatrix}\Sigma_{m\times m} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Sigma^{-2}_{m\times m} & 0\\ 0 & 0\end{bmatrix}U^T$$

$$= V\begin{bmatrix}\Sigma^{-1}_{m\times m} & 0\\ 0 & 0\end{bmatrix}U^T = A^+$$

(e) $$(A^TA)^+ = \left(V\begin{bmatrix}\Sigma^2_{r\times r} & 0\\ 0 & 0\end{bmatrix}V^T\right)^+$$

$$= V\begin{bmatrix}\Sigma^{-2}_{r\times r} & 0\\ 0 & 0\end{bmatrix}V^T$$

& $$A^+(A^T)^+ = V\begin{bmatrix}\Sigma^{-1}_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^T\left(V\begin{bmatrix}\Sigma^{-1}_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^T\right)^+$$

$$= V\begin{bmatrix}\Sigma^{-1}_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^TU\begin{bmatrix}\Sigma^{-1}_{r\times r} & 0\\ 0 & 0\end{bmatrix}V$$

$$= V\begin{bmatrix}\Sigma^{-2}_{r\times r} & 0\\ 0 & 0\end{bmatrix}V = (A^TA)^+$$

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Q 24

Is $Z[x]$ a PID ? justify

for $Z[x]$ to be a PID, every ideal generated must be a principle ideal (Ideal generated by a single element).

Consider the ideal $a = \langle 3, x^2 \rangle$

i.e. $h(x) = 3p(x) + x^2 g(x)$, $h(x) \in a$: $p(x), g(x) \in Z[x]$.

If $Z[x]$ is a PID then $a = \langle 3, x^2 \rangle = \langle h(x) \rangle$

i.e

$3 = h(x) \, f_1(x)$ — (I) , $f_1(x) \in Z[x]$

$x^2 = h(x) \, f_2(x)$ — (II) , $f_2(x) \in Z[x]$

~~from (I) & (II)~~

from (I) it is clear that $h(x) \, f_1(x)$ has to be a constant polynomial.

$\therefore h(x) = \{\pm 1, \pm 3\}$

Suppose $a = \langle 3, x^2 \rangle = \langle 1 \rangle$

$\exists \; r_1(x)$ & $r_2(x)$, such that $[r_1, r_2 \in Z[x]]$

$$3r_1(x) + x^2 r_2(x) = 1$$

the term $x^2 r_2(x)$ has no ~~de~~. constant term thus the only Constant term avaliable is that $r_1(x)$

Suppose $r_1(0)$ is the Constant term then by Comprasion of Poly Co-efficient , we have .

$3r_1(0) = 1$. this only possible only if $r_1(0) = 1/3$

$r_1(x) \notin \mathbb{Z}[x]$

$$\langle 3, x^2 \rangle \neq \langle 1 \rangle$$

Suppose $\langle 3, x^2 \rangle = \langle 3 \rangle$ [i.e. $h(x) = \pm 3$]

then $\exists\, r(x)$ such that

$$x^2 = 3\, r(x)$$

which again implies $r(x) \notin \mathbb{Z}[x]$

$\therefore \langle 3, x^2 \rangle$ is not a principle ideal

$\Rightarrow \mathbb{Z}[x]$ is not PID.

Q27

Given two diagonalizable matrices A, B.

⟨1⟩ $AB = BA$ ⟨2⟩ $\exists\, S \in \mathbb{R}^{n \times n}$ s.t. $S^{-1}AS$ & $S^{-1}BS$ are both diagonal.

⟨2⟩ ⇒ ⟨1⟩ let $S^{-1}AS = D_1$ & $S^{-1}BS = D_2$ are diagonal matrices

$$S^{-1}ASS^{-1}BS = D_1 D_2$$

{diagonal matrices commutes}.

$$S^{-1}ABS = D_2 D_1 = S^{-1}BS\,S^{-1}AS.$$

$$S^{-1}ABS = S^{-1}BAS \Rightarrow AB = BA.$$

⟨1⟩ ⇒ ⟨2⟩ Given $AB = BA$, A & B are diagonalizable

$\exists\, Q$ s.t. $Q^{-1}AQ = D_A = \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \lambda_2 I_{n_2} & \\ & & \ddots \\ & & & \lambda_r I_{n_r} \end{bmatrix}$

is given by ${}^{n}C_{k}$.

Hence, the total number of invariant subspaces is given by

$$\underbrace{{}^{n}C_{0}}_{\text{for } 0} + {}^{n}C_{1} + \cdots \underbrace{{}^{n}C_{n}}_{\text{for } V} = (1+1)^{n} = 2^{n}.$$

TST: Consider matrix $L = \begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{bmatrix}$ L_i's are block matrix.

Then L is diagonalizable iff each L_i is diagonalizable.

($\Leftarrow$) let $L_1, L_2, \ldots L_k$ be diagonalizable then $\exists$ invertible matrix Q_i s.t. $Q_i^{-1} L_i Q_i$ is diagonal matrix. let $Q := Q_1 \oplus \cdots \oplus Q_k$.

Then clearly, we have $D = Q^{-1} L Q$ is a diagonal matrix.

Hence, L is diagonalizable.

($\Rightarrow$) let L is diagonalizable then $\exists\, Q$ s.t. $D = Q^{-1} L Q$

where $D = \text{diag}(\lambda_1, \lambda_2 - \lambda_k)$ is a diagonal matrix.

Then $LQ = QD$, we have

$$\begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{bmatrix} \begin{bmatrix} x_1^j \\ | \\ | \\ x_k^j \end{bmatrix} = \lambda_j \begin{bmatrix} x_1^j \\ | \\ | \\ x_k^j \end{bmatrix}$$

where $v_j = [x_1^j \text{ --- } x_k^j]^T$ j^{th} column of Q & x_i^j is a vector

s.t $L_i x_i^j = \lambda_j x_i^j$ for $i = 1, - k$ & L_i is $n_i \times n_i$ matrix.

Since Q is invertible, defining invertible submatrix

$$Y := [x_i^{m_i} \cdots x_i^{m_{n_i}}] \quad \therefore \; Y \in \mathbb{R}^{n_i \times n_i}$$

So, $L_i Y = Y \Lambda$ where $\Lambda = \text{diag}(\lambda_{m_1} \text{ --- } \lambda_{m_{n_i}})$

which is diagonal matrix. Hence $L_i = Y \Lambda Y^{-1}$. Proved.

Using above result, $\exists$ nonsingular $X_i \in M_{n_i}$ s.t $X_{n_i}^{-1} D_B^{n_i} X_{n_i}$ for $i=1,\dots r$

is a diagonal matrix.

Consider $X := diag(X_{n_1}, X_{n_2}, \dots X_{n_r})$ (a block diagonal matrix)

$\Rightarrow$ $\Lambda = X^{-1} D_B X$ & $\Gamma = X^{-1} D_A X$ are diagonal.

Therefore $\exists$ invertible $S := QX$ s.t $\Lambda = S^{-1}BS$ & $\Gamma = S^{-1}AS$

are diagonal matrices. Hence proved.

$\boxed{Q2B}$

$$A = \begin{bmatrix} -\pi/2 & \pi/2 \\ \pi/2 & -\pi/2 \end{bmatrix},$$

$$\cos(A) = P^{-1} \cos(D)\, P$$

$$sI - A = \begin{bmatrix} s+\pi/2 & -\pi/2 \\ -\pi/2 & s+\pi/2 \end{bmatrix}$$

$$\det(sI-A) = 0 \Rightarrow (s+\pi/2)^2 - (\pi/2)^2 = 0$$

$$s(s+\pi) = 0$$

$$s = 0, -\pi; \text{ (eigen values)}$$

eig vector corresponding to $\lambda = 0$

$$\begin{bmatrix} \pi/2 & -\pi/2 \\ -\pi/2 & \pi/2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eig vector corresponding to $\lambda = -\pi$

$$\begin{bmatrix} -\pi/2 & -\pi/2 \\ -\pi/2 & -\pi/2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = -x_2 \; ; v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\cos A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos 0 & 0 \\ 0 & \cos(-\pi) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{\cos(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$\boxed{Q29}$

Given $A = -A^T$

Let $Q = e^A$

Now $QQ^T = e^A e^{A^T}$

$= e^{-A^T} e^{A^T}$

$= e^{-A^T + A^T}$

$= e^{O_{n \times n}}$

$= I$

where I is the Identity matrix

hence Q is orthogonal.

(32)

Consider the function,

$$F(t) = e^{At+Bt} - e^{At} e^{Bt}$$

Differentiating both sides wrt ~~to~~ t, we get

$$F'(t) = (A+B) e^{(A+B)t} - A e^{At} e^{Bt} - e^{At} e^{Bt} . B$$

$$= A e^{(A+B)t} - A e^{At} e^{Bt} + B e^{(A+B)t} - \cancel{B} e^{At} B e^{Bt}$$

$$= A F(t) + B F(t)$$

$\left[\because BA = AB \right.$
$A^k B = B A^k$
$\left. \forall k \right]$

$$\Rightarrow F'(t) = (A+B) F(t)$$

write $F(t) = [f_1(t), \dots f_n(t)]$, where $f_i(t) \in \mathbb{R}^n$

$$\Rightarrow \dot{f}_i(t) = (A+B) f_i(t) \quad \forall i = 1, 2, \dots, n$$

Consider $f_i(t) = e^{(A+B)t} f_i(0)$

and observe that $\dot{f}_i(t) = (A+B) f_i(t)$ is satisfied for this choice of $f_i(t)$

$$\Rightarrow F(t) = e^{(A+B)t} F(0)$$

Now, $F(0) = e^{A \cdot 0 + B \cdot 0} - e^{A \cdot 0} e^{B \cdot 0}$

$$= I - I$$

$$= 0$$

$$\Rightarrow F(t) = 0, \forall t \Rightarrow \cancel{e^{(A+B)t}} = \cancel{e^{A}} \; \cancel{\forall t}$$

$$\Rightarrow e^{(A+B)t} = e^{At} e^{Bt}$$

Solution of other part is not unique. Take any $A, B : AB \neq BA$ and verify

(34) suppose we have

$$Av = \lambda_M v \to (*)$$

where λ_M is the largest eigenvalue of A

Equating component-wise on both sides of (*), we get

$$\sum_{j=1}^{n} a_{ij} v_j = \lambda_M v_i \quad \text{for } i = 1, \ldots, n$$

suppose $\max_i v_i = v_k$

Note that v_k is always positive, unless $v_i < 0$ for all i.

In case $v_i < 0 \; \forall i$, ~~anyway~~ we may just choose $-v$ as the eigenvector.

Further, $\sum_{j=1}^{n} a_{ij} = \text{RowSum}_i(A)$

Now,

$$\lambda_M v_k = \sum_{j=1}^{n} a_{kj} v_j \leq \sum_{j=1}^{n} a_{kj} v_k \quad \left[\because a_{kj} v_j < a_{kj} v_k \; \forall j \text{ as } a_{kj} > 0 \text{ by definition}\right]$$

$$\Rightarrow \lambda_M v_k \leq v_k \sum_{j=1}^{n} a_{kj}$$

$$\Rightarrow \lambda_M \leq \sum_{j=1}^{n} a_{kj} = \text{RowSum}_k(A)$$

$$\leq \max_i \text{RowSum}_i(A)$$

Since $A^T = A$, we have an orthonormal set of eigenvectors that diagonalize A.

consider any $x \in \mathbb{R}^n$, represented as

$$x = \sum_{i=1}^{n} c_i v_i$$

where $\{v_1, v_2, \ldots v_n\}$ is the orthonormal set of eigenvectors of A.

Observe that,

$$\langle x | Ax \rangle = x^T A x = \left(\sum c_i v_i\right)^T A \left(\sum c_i v_i\right)$$

$$= \left(\sum c_i v_i\right)^T \sum c_i A v_i$$

$$= \sum c_i v_i^T \left[\sum c_i \lambda_i v_i\right]$$

$$= \sum c_i^2 \lambda_i v_i^T v_i$$

$$= \sum c_i^2 \lambda_i$$

Notice that $\|x\|^2 = \langle x | x \rangle = \langle \sum c_i v_i | \sum c_i v_i \rangle$

$$= \sum_{i=1}^{n} c_i^2$$

Now, $$\max_x \frac{\langle x | Ax \rangle}{\langle x | x \rangle}$$

$$= \max_{\{c_1, c_2, \ldots c_n\}} \frac{\sum c_i^2 \lambda_i}{\sum c_i^2}$$

$$\leq \lambda_M \frac{\sum c_i^2}{\sum c_i^2} = \lambda_M$$

$$\therefore \lambda_M \geq \max_{x \neq 0} \frac{\langle x | Ax \rangle}{\langle x | x \rangle}$$

$\because \lambda_M \geq \dfrac{\langle x | Ax \rangle}{\langle x | x \rangle}$ for any choice of non-trivial vector, x

we choose $x = [1 \; 1 \; \ldots \; 1]^T$

$$\Rightarrow \lambda_M \geq \frac{\sum_{i=1}^{n} \text{RowSum}_i(A)}{n}$$

Now,
$$\sum_{i=1}^{n} RowSum_i(A) \geq n \left[\min_i RowSum_i(A)\right]$$

$$\Rightarrow \frac{\sum_{i=1}^{n} RowSum_i(A)}{n} \geq \min_i RowSum_i(A)$$

$$\Rightarrow \lambda_M \geq \min RowSum_i(A)$$

$$\Rightarrow \min RowSum_i(A) \leq \lambda_M \leq \max_i RowSum_i(A)$$

(35) $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 3 & 6 & 8 \end{bmatrix}$

consider a basis for $\mathbb{R}^3$ as $\{e_1, e_2, e_3\}$; $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ae_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad A^2e_1 = \begin{bmatrix} 22 \\ 31 \\ 39 \end{bmatrix} \quad A^3e_1 = \begin{bmatrix} 310 \\ 441 \\ 564 \end{bmatrix}$$

$$\therefore \alpha_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 22 \\ 31 \\ 39 \end{bmatrix} = \begin{bmatrix} 310 \\ 441 \\ 564 \end{bmatrix}$$

$$\alpha_0 = 5, \quad \alpha_1 = 19, \quad \alpha_2 = 13$$

$$\boxed{\mu_{e_1} = s^3 - 13s^2 - 19s - 5}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ae_2 = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \quad A^2e_2 = \begin{bmatrix} 45 \\ 64 \\ 81 \end{bmatrix} \quad A^3e_2 = \begin{bmatrix} 642 \\ 913 \\ 1167 \end{bmatrix}$$

$$\alpha_0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} + \alpha_2 \begin{bmatrix} 45 \\ 64 \\ 81 \end{bmatrix} = \begin{bmatrix} 642 \\ 913 \\ 1167 \end{bmatrix}$$

$$\Rightarrow \alpha_0 = 5 \quad \alpha_1 = 19 \quad \alpha_2 = 13$$

$$\boxed{\mu_{e_2} = s^3 - 13s^2 - 19s - 5}$$

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad Ae_3 = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \quad A^2 e_3 = \begin{bmatrix} 66 \\ 94 \\ 121 \end{bmatrix} \quad A^3 e_3 = \begin{bmatrix} 953 \\ 1355 \\ 1730 \end{bmatrix}$$

$$\alpha_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} + \alpha_2 \begin{bmatrix} 66 \\ 94 \\ 121 \end{bmatrix} = \begin{bmatrix} 953 \\ 1355 \\ 1730 \end{bmatrix}$$

$\Rightarrow \quad \alpha_0 = 5, \quad \alpha_1 = 19, \quad \alpha_2 = 13$

$$\Rightarrow \boxed{M e_3 = s^3 - 13s^2 - 19s - 5}$$

$$M = \operatorname{lcm}(M e_1, M e_2, M e_3)$$

$$\boxed{M = s^3 - 13s^2 - 19s - 5}$$

Now, eigenvalues are roots of both $X(s)$ and $M(s)$.

solving for $M(s) = 0$, we get by inspection that

$S = -1$ is a root

Now,

$$\begin{array}{r|l} & s^2 - 14s - 5 \\ \hline s+1 & s^3 - 13s^2 - 19s - 5 \\ & s^3 + s^2 \\ \hline & -14s^2 - 19s - 5 \\ & -14s^2 - 14s \\ \hline & -5s - 5 \\ & -5s - 5 \\ \hline & 0 \end{array}$$

the other two eigenvalues are roots of $s^2 - 14s - 5 = 0$.

$$\therefore \quad \lambda = \{-1, \; 7 \pm \sqrt{54}\}$$

(40) Given $\phi : V \to V$ with $\dim(V) = n < \infty$

T.S.T. $\operatorname{Ker}(\phi^n) \cap \operatorname{Im}(\phi^n) = \{0\}$

Suppose $v \in \operatorname{Ker}(\phi^n) \cap \operatorname{Im}(\phi^n)$. Then $\phi^n v = 0$

Also, $\exists u \in V$ s.t $v = \phi^n u$,

so, $\phi^n v = \phi^{2n} u = 0$

But $\operatorname{Ker} \phi^n = \operatorname{Ker} \phi^{n+1} = \dots$ Null spaces stop growing

$\Rightarrow \phi^n u = 0$

$\Rightarrow v = \phi^n u = 0$

$\Rightarrow \operatorname{Ker}(\phi^n) \cap \operatorname{Im}(\phi^n) = \{0\}$

Thus $\operatorname{Ker}(\phi^n) + \operatorname{Im}(\phi^n)$ is direct sum.

Also, $\dim\left[(\operatorname{Ker}(\phi^n)) \oplus (\operatorname{Im}(\phi^n))\right] = \dim \operatorname{Ker}(\phi^n) + \dim \operatorname{Im}(\phi^n)$

Now by fundamental theorem of linear maps,

$\dim \operatorname{ker}(\phi^n) + \dim \operatorname{Im}(\phi^n) = \dim V$

thus, $\operatorname{Ker}(\phi^n) \oplus \operatorname{Im}(\phi^n) = V$.

Assignment 5

Q7. We already have.

$$AW = WS.$$

$$\Rightarrow A^2 W = AWS = WS.S = WS^2$$

$$\Rightarrow A^l W = WS^l \ \forall l$$

$$\therefore \underbrace{\chi_A(A)}_{0 \text{ (identically)}} W = W \chi_A(S)$$

where $\chi_A(A)$ is the char poly of A.

$$\Rightarrow W.\chi_A(S) = 0$$

$$\left(\chi_A(S)\right)^T W^T = 0$$

$$\Rightarrow \chi_A(S^T).W^T = 0$$

[since $f(S)^T = f(S^T)$ for any $S \in \mathbb{R}^{K \times K}$ & $f(\cdot)$ being a polynomial]

(Check this!).

$\Rightarrow \chi_A(S^T)$ maps at least the 'γ' independent columns of W^T to zero.

so that entire W^T is mapped to 0

$$\Rightarrow (S^T-\lambda_1 I)(S^T-\lambda_2 I)\dots(S^T-\lambda_n I).$$
$$W^T = 0$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A.

$\Rightarrow$ The r lin. ind columns of W must belong to

$$\operatorname{Ker}(S^T - \lambda_i I) \text{ for}$$

some value of i or the other. These λ_i's are therefore eigenvalues of S^T with the corresp columns of W^T being the eigenvectors. Thus, distinct or not, we have at least 'r' set of eqns of the form

$$(S^T - \lambda_i I)\bar{w}_i = 0$$

where w_i is some i-th column of W^T.

Such λ_i's, being r in number, are common eigenvalues of both A & S^T.

S & S^T have same eigenvalues, so these λ_i's are also eigenvalues of S.