Applied Linear Algebra: Problem set-4

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1. In $\mathbb{R}^4$ with standard inner product consider a subspace, $\mathbb{W}$ which is orthogonal to both $v_1 = [1\ 1\ 2\ 4]^T$, and $v_2 = [2\ -1\ -5\ 2]^T$. Construct an orthonorml basis for $\mathbb{W}$. Also obtain an orthonormal basis for span(v_1, v_2) using Gram Schmidt procedure.

2. For an inner product space $\mathbb{V}$, show that two vectors $v_1, v_2 \in \mathbb{V}$ are equal if and only if $\langle v_1|v\rangle = \langle v_2|v\rangle$ for every $v \in \mathbb{V}$.

3. Let $\mathbb{W}$ be a subspace of $\mathbb{R}^2$ given by span$(1,1)$. For the standard inner product, let P be the projection of any vector in R^2 onto $\mathbb{W}$. For $(x,y) \in \mathbb{R}^2$ how can you write $P(x,y) \in \mathbb{R}^2$? What is the matrix representation of P in the standard basis? Obtain the matrix representation of the projection onto $\mathbb{W}^\perp$. Does there exist a basis under which the matrix representation of P is identity (if so, obtain the same, else argue why not)?

4. For an inner product over $\mathbb{C}^{n\times n}$ given by $\langle A|B\rangle = \text{trace}(AB^*)$, let $\mathbb{D}$ represent the subspace of all diagonal matrices in $\mathbb{C}^{n\times n}$. Characterize elements of $\mathbb{D}^\perp$.

5. Considering the standard inner product, the "best approximation" of $f(x) = 5\sin(x)$ over the interval $(-\pi, \pi)$ by a function $g(x) = b\cos(x)$ is obtained by choosing $b = 0$. Prove it or give a counterexample if it is not true.

6. For a finite dimensional inner product space $\mathbb{V}$, show that every linear functional $f : \mathbb{V} \to \mathbb{F}$, can be given by $f(v) = \langle v|\tilde{v}\rangle$ for a unique $\tilde{v} \in \mathbb{V}$. Further, show that this unique vector $\tilde{v} \in (\text{Ker}(f))^\perp$. Hence, show that if P is the orthogonal projection of any vector in $\mathbb{V}$ on $(\text{Ker}(f))^\perp$ then $f(v) = f(Pv)$ for all $v \in \mathbb{V}$.

7. For a finite dimensional inner product space, $\mathbb{V}$, show that every linear functional $f : \mathbb{V} \to \mathbb{F}$, can be given by $f(v) = \langle v|\tilde{v}\rangle$ for a unique $\tilde{v} \in \mathbb{V}$ (This result is called *Riesz Representation Theorem*). Further, show that this unique vector $\tilde{v} \in (\text{Ker}(f))^\perp$. Hence, show that if P is the orthogonal projection of any vector in $\mathbb{V}$ on $(\text{Ker}(f))^\perp$ then $f(v) = f(Pv)$ for all $v \in \mathbb{V}$.

8. Using the result(s) in Question 6, show that for a linear operator $T : \mathbb{V} \to \mathbb{V}$ ($\mathbb{V}$ being a finite dimensional inner product space), show that there exists a unique linear operator T^* on $\mathbb{V}$ such that $\langle Tv_1|v_2\rangle = \langle v_1|T^*v_2\rangle\ \forall v_1, v_2 \in \mathbb{V}$.

9. In a finite dimensional inner product space. $\mathbb{V}$, with an orthonormal ordered basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$, show that a linear operator, $T : \mathbb{V} \to \mathbb{V}$, has a matrix representation, A, in the ordered basis given by $[A]_{ij} = \langle Tv_j|v_i\rangle$. What is the relation between the matrix corresponding to T and the one corresponding to T^* (T^* is as defined in Question 8)?

10. If an operator T on an inner product space, $\mathbb{V}$, has a corresponding unique T^* as defined in Question 8, then T is said to have an ***adjoint*** in $\mathbb{V}$. For two such linear operators, T and R on $\mathbb{V}$, and $\alpha \in \mathbb{F}$, show that (a) $(R+T)^* = R^* + T^*$, (b) $(\alpha T)^* = \bar{\alpha}T^*$, (c) $(TR)^* = R^*T^*$, (d) $(T^*)^* = T$ (e) for an invertible T, $(T^*)^{-1} = (T^{-1})^*$ (f) im(T^*)=$(\text{ker}(T))^\perp$

11. An operator, $T : \mathbb{V} \to \mathbb{V}$, is said to be ***self-adjoint*** if $T = T^*$. Prove the following: (a) If $\mathbb{V}$ is finite dimensional, and $T = T^2$, then $TT^* = T^*T \iff T$ is self adjoint. (b) Composition of two self adjoint operators is also self adjoint if and only if they commute. (c) If $\mathbb{V}$ is a finite dimensional complex inner product space, then T is self-adjoint if and only if $\langle Tv|v\rangle$ is real for every $v \in \mathbb{V}$.

12. Suppose T is an operator on the finite dimensional inner product space $\mathbb{V}$ such that it satisfies the condition $||Tv|| < ||v||\ \forall v \in \mathbb{V}$. Show that $T - \sqrt{2}I$ is invertible, where I is the identity map on $\mathbb{V}$.

13. Verify whether the set of self-adjoint operators on a real inner product space is a subspace.

14. Suppose $\{w_1, w_2, \ldots, w_n\}$ is an orthonormal basis for $\mathbb{V}$ and that there exists another set of vectors, $S = \{s_1, s_2, \ldots, s_n\}$ such that $||s_i - w_i|| < \frac{1}{\sqrt{n}}\ \forall i$. Show that S is a basis of $\mathbb{V}$.

15. Suggest a method for testing the linear independence (or otherwise) of a given set of vectors, belonging to an inner product space, using Gram Schmidt orthogonalization process.

16. Show that for two subspaces $\mathbb{W}_1$ and $\mathbb{W}_2$, we have $(\mathbb{W}_1 + \mathbb{W}_1)^\perp = \mathbb{W}_1^\perp \cap \mathbb{W}_1^\perp$.

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17. Show that for a finite dimensional inner product space, $\mathbb{V}$ over $\mathbb{R}$, given an ordered basis $\mathcal{B}$, there always exists a matrix, $A \in \mathbb{R}^{n\times n}$ such that the inner product can be expressed as $\langle v_1|v_2\rangle = [v_1]_{\mathcal{B}} A [v_2]_{\mathcal{B}}$, where $\dim(\mathbb{V}) = n$ for all $v_1, v_2 \in \mathbb{V}$.

18. Suppose a vector space $\mathbb{V}$ can be expressed as a direct sum of two of its subspaces as $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2$. Let $u_1(\cdot,\cdot) : \mathbb{U}_1 \times \mathbb{U}_1 \to \mathbb{C}$ and $u_2(\cdot,\cdot) : \mathbb{U}_2 \times \mathbb{U}_2 \to \mathbb{C}$ be inner products defined on the two subspaces $\mathbb{U}_1$ and $\mathbb{U}_2$, respectively. Show that there exists a unique inner product $v(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ such that we have $\mathbb{U}_1^{\perp} = \mathbb{U}_2$ and $v(p,q) = u_k(p,q)$, when $p, q \in \mathbb{U}_k,\ \ k \in \{1,2\}$.

19. For a subspace, $\mathbb{W}$, of a finite dimensional inner product space $(\mathbb{V}, \langle\cdot|\cdot\rangle)$, let $\mathbb{W}^{\perp}$ denote the orthogonal complement of $\mathbb{W}$. If we have another subspace, $\mathbb{U}$, of $(\mathbb{V}, \langle\cdot|\cdot\rangle)$ such that $u \in \mathbb{U} \implies \langle u|w^{\perp}\rangle = 0\ \forall w^{\perp} \in \mathbb{W}^{\perp}$, then $\dim(\mathbb{U}^{\perp}) \geq \dim(\mathbb{W}^{\perp})$. Prove it or give a counterexample if it is not true.

20. What is the 'distance' (induced by a norm) between any two orthogonal 'unit vectors', say v_1, v_2, in an inner product space? Justify through proper calculations/reasoning. What can you say about the relation between $v_1 - v_2$ and $v_1 + v_2$?

21. Show that two vectors v_1, v_2 in a complex inner product space are orthogonal if and only if $||\alpha v_1 + \beta v_2||^2 = ||\alpha v_1||^2 + ||\beta v_2||^2$ for all $\alpha, \beta \in \mathbb{C}$.

22. For two vectors v_1, v_2 in a complex inner product space, suppose $\langle v_1|v_2\rangle = 1 - 3i$. Evaluate $||v_1 + iv_2||^2 - ||v_1 - iv_2||^2$.

23. (a) For $v_1, v_2 \in \mathbb{V}$, show that $\langle v_1|v_2\rangle = 0$ if and only if $||v_1|| \leq ||v_1 + \alpha v_2||\ \forall \alpha \in \mathbb{C}$.

    (b) Suppose $\Pi : \mathbb{V} \to \mathbb{V}$ is an idempotent linear operator on the finite dimensional vector space $\mathbb{V}$, and satisfies the condition: $||\Pi v|| \leq ||v||\ \forall v \in \mathbb{V}$. Show that there exists a subspace $\mathbb{U} \subseteq \mathbb{V}$, such that Π is an orthogonal projection from $\mathbb{V}$ onto $\mathbb{U}$.

24. Let $\mathfrak{C}_{\mathbb{R}}[a,b]$ be the vector space of all real valued functions on the interval $[a,b]$ (usual addition and scalar multiplication hold). Consider an inner product defined as $\langle f|g\rangle := \int_a^b f(t)g(t)dt$. Further, let $\mathbb{W}$ be the subspace of all polynomials with real coefficients. What is $\mathbb{W}^{\perp}$? Is $\mathbb{W} = (\mathbb{W}^{\perp})^{\perp}$? Show appropriate steps to justify your answer.

25. With the inner product as defined in Question 24, obtain an orthonormal basis for all polynomials of degree less than or equal to 3, starting from the basis $\{1, x, x^2, x^3\}$, using the Gram Schmidt procedure, over the interval $[0,1]$.

26. For a subspace, $\mathbb{U}$ of a finite dimensional inner product space $\mathbb{V}$, show that $\mathbb{U}^{\perp}$ is isomorphic to $\mathbb{V}/\mathbb{U}$.

27. Prove that for every projection operation, P, on an inner product space, $\mathbb{V}$, $I - P$ is also a projection operation. Further, show that for $P : \mathbb{V} \to \mathbb{W}$, with $\mathbb{W} \subseteq \mathbb{V}$, we have that $\mathrm{im}(I - P) = \mathbb{W}^{\perp}$.

28. (*'Mindless'(?) Interpolation*)
    (a) Argue why the Vandermonde matrix given by

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{bmatrix}$$

    must have full column rank whenever $n \leq m$ and (real-valued) x_i's are all distinct.
    (b) Suppose a set of m points with distinct x_i's are given by $\{(x_1,y_1), (x_2,y_2), \dots, (x_m,y_m)\}$. Show that the unique polynomial of degree $m-1$ passing through all the points is given by $\ell(x) = \sum_{i=1}^{m}\left(y_i \dfrac{\prod_{j\neq i}^{m}(x - x_j)}{\prod_{j\neq i}^{m}(x_i - x_j)}\right)$.
    (c) Suppose a defender kicks a ball from his/her half of the football field and the following table contains the positions of the football (on some suitable vertical plane) at different instants of time, as tracked by a camera:

Down range (x) in m	0	15	30	45	60
Height (y) in m	0	8	15	19	20

    It is required to predict the position where the ball lands on the ground. Obtain this using the interpolation formula derived in part (b).
    (d) Suppose, instead of the formula used in part (c), one knows that the football follows a parabolic trajectory (from experience or from high school physics) of the form $y = ax^2 + bx + c$ and uses this to predict where the ball lands. What would be the answer? Which solution makes more *sense* and why?

"Men pass away, but their deeds abide."
—**[Supposedly, his last words] Augustin-Louis Cauchy**