Chapter 6: Inner Product Spaces

Linear Algebra Done Right, by Sheldon Axler

A: Inner Products and Norms

Problem 1

Show that the function that takes $\big((x_1, x_2), (y_1, y_2)\big) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbb{R}^2$.

Proof. Suppose it were. First notice

$$\begin{aligned}\langle (1,1) + (-1,-1), (1,1) \rangle &= \langle (0,0), (1,1) \rangle \\ &= |0 \cdot 1| + |0 \cdot 1| \\ &= 0.\end{aligned}$$

Next, since inner products are additive in the first slot, we also have

$$\begin{aligned}\langle (1,1) + (-1,-1), (1,1) \rangle &= \langle (1,1), (1,1) \rangle + \langle (-1,-1), (1,1) \rangle \\ &= |1 \cdot 1| + |1 \cdot 1| + |(-1) \cdot 1| + |(-1) \cdot 1| \\ &= 4.\end{aligned}$$

But this implies $0 = 4$, a contradiction. Hence we must conclude that the function does not in fact define an inner product. □

Problem 3

Suppose $\mathbb{F} = \mathbb{R}$ and $V \neq \{0\}$. Replace the positivity condition (which states that $\langle v, v \rangle \geq 0$ for all $v \in V$) in the definition of an inner product (6.3) with the condition that $\langle v, v \rangle > 0$ for some $v \in V$. Show that this change in the definition does not change the set of functions from $V \times V$ to $\mathbb{R}$ that are inner products on V.

Proof. Let V be a nontrivial vector space over $\mathbb{R}$, let A denote the set of functions $V \times V \to \mathbb{R}$ that are inner products on V in the standard definition, and let B denote the set of functions $V \times V \to \mathbb{R}$ under the modified definition. We will show $A = B$.

Suppose $\langle \cdot, \cdot \rangle_1 \in A$. Since $V \neq \{0\}$, there exists $v \in V - \{0\}$. Then $\langle v, v \rangle_1 > 0$, and so $\langle \cdot, \cdot \rangle_1 \in B$. Thus $A \subseteq B$.

Conversely, suppose $\langle \cdot, \cdot \rangle_2 \in B$. Then there exists some $v' \in V$ such that

$\langle v', v' \rangle_2 > 0$. Suppose by way of contradiction there exists $u \in V$ is such that $\langle u, u \rangle_2 < 0$. Define $w = \alpha u + (1-\alpha)v'$ for $\alpha \in \mathbb{R}$. It follows

$$\begin{aligned}\langle w, w \rangle_2 &= \langle \alpha u + (1-\alpha)v', \alpha u + (1-\alpha)v' \rangle_2 \\ &= \langle \alpha u, \alpha u \rangle_2 + 2\langle \alpha u, (1-\alpha)v' \rangle_2 + \langle (1-\alpha)v', (1-\alpha)v' \rangle_2 \\ &= \alpha^2 \langle u, u \rangle_2 + 2\alpha(1-\alpha)\langle u, v' \rangle_2 + (1-\alpha)^2 \langle v', v' \rangle_2.\end{aligned}$$

Notice the final expression is a polynomial in the indeterminate α, call it p. Since $p(0) = \langle v', v' \rangle_2 > 0$ and $p(1) = \langle u, u \rangle_2 < 0$, by Bolzano's theorem there exists $\alpha_0 \in (0,1)$ such that $p(\alpha_0) = 0$. That is, if $w = \alpha_0 u + (1-\alpha_0)v'$, then $\langle w, w \rangle_2 = 0$. In particular, notice $\alpha_0 \neq 0$, for otherwise $w = v'$, a contradiction since $\langle v', v' \rangle_2 > 0$. Now, since $\langle w, w \rangle_2 = 0$ iff $w = 0$ (by the definiteness condition of an inner product), it follows

$$u = \frac{\alpha_0 - 1}{\alpha_0} v.$$

Letting $t = \frac{\alpha_0 - 1}{\alpha_0}$, we now have

$$\begin{aligned}\langle u, u \rangle_2 &= \langle tv', tv' \rangle_2 \\ &= t^2 \langle v', v' \rangle_2 \\ &> 0,\end{aligned}$$

where the inequality follows since $t \in (-1, 0)$ and $\langle v', v' \rangle_2 > 0$. This contradicts our assumption that $\langle u, u \rangle_2 < 0$, and so we have $\langle \cdot, \cdot \rangle_2 \in A$. Therefore, $B \subseteq A$. Since we've already shown $A \subseteq B$, this implies $A = B$, as desired. □

Problem 5

Let V be finite-dimensional. Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Let $v \in \operatorname{null}(T - \sqrt{2}I)$, and suppose by way of contradiction that $v \neq 0$. Then

$$\begin{aligned}Tv - \sqrt{2}v = 0 &\implies Tv = \sqrt{2}v \\ &\implies \|\sqrt{2}v\| \leq \|v\| \\ &\implies \sqrt{2} \cdot \|v\| \leq \|v\| \\ &\implies \sqrt{2} \leq 1,\end{aligned}$$

a contradiction. Hence $v = 0$ and $\operatorname{null}(T - \sqrt{2}I) = \{0\}$, so that $T - \sqrt{2}I$ is injective. Since V is finite-dimensional, this implies $T - \sqrt{2}I$ is invertible, as desired. □

Problem 7

Suppose $u, v \in V$. Prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbb{R}$ if and only if $\|u\| = \|v\|$.

Proof. ($\Rightarrow$) Suppose $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbb{R}$. Then this equation holds when $a = 1$ and $b = 0$. But then we must have $\|u\| = \|v\|$, as desired.

($\Leftarrow$) Conversely, suppose $\|u\| = \|v\|$. Let $a, b \in \mathbb{R}$ be arbitrary, and notice

$$\begin{aligned}
\|au + bv\| &= \langle au + bv, au + bv\rangle \\
&= \langle au, au\rangle + \langle au, bv\rangle + \langle bv, au\rangle + \langle bv, bv\rangle \\
&= a^2\|u\|^2 + ab\left(\langle u, v\rangle + \langle v, u\rangle\right) + b^2\|v\|^2. \qquad (1)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|bu + av\| &= \langle bu + av, bu + av\rangle \\
&= \langle bu, bu\rangle + \langle bu, av\rangle + \langle av, bu\rangle + \langle av, av\rangle \\
&= b^2\|u\|^2 + ab\left(\langle u, v\rangle + \langle v, u\rangle\right) + a^2\|v\|^2. \qquad (2)
\end{aligned}$$

Since $\|u\| = \|v\|$, (1) equals (2), and hence $\|au + bv\| = \|bu + av\|$. Since a, b were arbitrary, the result follows. $\square$

Problem 9

Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|v\|^2} \leq 1 - |\langle u, v\rangle|.$$

Proof. By the Cauchy-Schwarz Inequality, we have $|\langle u, v\rangle| \leq \|u\|\|v\|$. Since $\|u\| \leq 1$ and $\|v\| \leq 1$, this implies

$$0 \leq 1 - \|u\|\|v\| \leq 1 - |\langle u, v\rangle|,$$

and hence it's enough to show

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|v\|^2} \leq 1 - \|u\|\|v\|.$$

Squaring both sides, it suffices to prove

$$\left(1 - \|u\|^2\right)\left(1 - \|v\|^2\right) \leq \left(1 - \|u\|\|v\|\right)^2. \qquad (3)$$

Notice

$$\begin{aligned}
\left(1 - \|u\|\|v\|\right)^2 - \left(1 - \|u\|^2\right)\left(1 - \|v\|^2\right) &= \|u\|^2 - 2\|u\|\|v\| + \|v\|^2 \\
&= \left(\|u\| - \|v\|\right)^2 \\
&\geq 0,
\end{aligned}$$

and hence inequality (3) holds, completing the proof. $\square$

Problem 11

Prove that

$$16 \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

for all positive numbers a, b, c, d.

Proof. Define

$$x = \left(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right) \quad \text{and} \quad y = \left(\sqrt{\frac{1}{a}}, \sqrt{\frac{1}{b}}, \sqrt{\frac{1}{c}}, \sqrt{\frac{1}{d}}\right).$$

Then the Cauchy-Schwarz Inequality implies

$$\begin{aligned}(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) &\geq \left(\sqrt{a}\sqrt{\frac{1}{a}}+\sqrt{b}\sqrt{\frac{1}{b}}+\sqrt{c}\sqrt{\frac{1}{c}}+\sqrt{d}\sqrt{\frac{1}{d}}\right)^2 \\ &= (1+1+1+1)^2 \\ &= 16,\end{aligned}$$

as desired. $\square$

Problem 13

Suppose u, v are nonzero vectors in $\mathbb{R}^2$. Prove that

$$\langle u, v\rangle = \|u\|\|v\|\cos\theta,$$

where θ is the angle between u and v (thinking of u and v as arrows with initial point at the origin).

Proof. Let A denote the line segment from the origin to u, let B denote the line segment from the origin to v, and let C denote the line segment from v to u. Then A has length $\|u\|$, B has length $\|v\|$ and C has length $\|u-v\|$. Letting θ denote the angle between A and B, by the Law of Cosines we have

$$C^2 = A^2 + B^2 - 2BC\cos\theta,$$

or equivalently

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta.$$

It follows

$$\begin{aligned}2\|u\|\|v\|\cos\theta &= \|u\|^2 + \|v\|^2 - \|u-v\|^2 \\ &= \langle u, u\rangle + \langle v, v\rangle - \langle u-v, u-v\rangle \\ &= \langle u, u\rangle + \langle v, v\rangle - \big(\langle u, u\rangle - 2\langle u, v\rangle + \langle v, v\rangle\big) \\ &= 2\langle u, v\rangle.\end{aligned}$$

Dividing both sides by 2 gives the desired result. $\square$

Problem 15

Prove that
$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} j{a_j}^2\right)\left(\sum_{j=1}^{n} \frac{{b_j}^2}{j}\right)$$
for all real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$.

Proof. Let
$$u = \left(a_1, \sqrt{2}a_2, \dots, \sqrt{n}a_n\right) \quad \text{and} \quad v = \left(b_1, \frac{1}{\sqrt{2}}b_2, \dots, \frac{1}{\sqrt{n}}b_n\right).$$
Since $\langle u, v\rangle = \sum_{k=1}^{n} a_k b_k$, the Cauchy-Schwarz Inequality yields
$$\begin{aligned}(a_1b_1 + \cdots + a_nb_n)^2 &\leq \|u\|^2\|v\|^2 \\ &= \left({a_1}^2 + 2{a_2}^2 + \cdots + n{a_n}^2\right)\left({b_1}^2 + \frac{{b_2}^2}{2} + \cdots + \frac{{b_n}^2}{n}\right),\end{aligned}$$
as desired. □

Problem 17

Prove or disprove: there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by
$$\|(x, y)\| = \max\{|x|, |y|\}$$
for all $(x, y) \in \mathbb{R}^2$.

Proof. Suppose such an inner product existed. Then by the Parallelogram Equality, it follows
$$\|(1,0) + (0,1)\|^2 + \|(1,0) - (0,1)\|^2 = 2\left(\|(1,0)\|^2 + \|(0,1)\|^2\right).$$
After simplification, this implies $2 = 4$, a contradiction. Hence no such inner product exists. □

Problem 19

Suppose V is a real inner product space. Prove that
$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$
for all $u, v \in V$.

Proof. Suppose V is a real inner product space and let $u, v \in V$. It follows

$$\begin{aligned}
\frac{\|u+v\|^2 - \|u-v\|^2}{4} &= \frac{\left(\|u\|^2 + 2\langle u, v\rangle + \|v\|^2\right) - \left(\|u\|^2 - 2\langle u, v\rangle + \|v\|^2\right)}{4} \\
&= \frac{4\langle u, v\rangle}{4} \\
&= \langle u, v\rangle,
\end{aligned}$$

as desired. ∎

Problem 20

Suppose V is a complex inner product space. Prove that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$$

for all $u, v \in V$.

Proof. Notice we have

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2
\end{aligned}$$

and

$$\begin{aligned}
-\|u-v\|^2 &= -\langle u-v, u-v\rangle \\
&= -\|u\|^2 + \langle u, v\rangle + \langle v, u\rangle - \|v\|^2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|u+iv\|^2 i &= i\left(\langle u+iv, u+iv\rangle\right) \\
&= i\left(\|u\|^2 + \langle u, iv\rangle + \langle iv, u\rangle + \langle iv, iv\rangle\right) \\
&= i\left(\|u\|^2 - i\langle u, v\rangle + i\langle v, u\rangle + \|v\|^2\right) \\
&= i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle + i\|v\|^2
\end{aligned}$$

and

$$\begin{aligned}
-\|u-iv\|^2 i &= -i\left(\langle u-iv, u-iv\rangle\right) \\
&= -i\left(\|u\|^2 - \langle u, iv\rangle - \langle iv, u\rangle + \langle iv, iv\rangle\right) \\
&= -i\left(\|u\|^2 + i\langle u, v\rangle - i\langle v, u\rangle + \|v\|^2\right) \\
&= -i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle - i\|v\|^2.
\end{aligned}$$

Thus it follows

$$\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i = 4\langle u, v\rangle.$$

Dividing both sides by 4 yields the desired result. $\square$

Problem 23

Suppose $V_1, \ldots, V_m$ are inner product spaces. Show that the equation

$$\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m)\rangle = \langle u_1, v_1\rangle + \cdots + \langle u_m, v_m\rangle$$

defines an inner product on $V_1 \times \cdots \times V_m$.

Proof. We prove that this definition satisfies each property of an inner product in turn.
Positivity: Let $(v_1, \ldots, v_m) \in V_1 \times \ldots V_m$. Since $\langle v_k, v_k\rangle$ is an inner product on V_k for $k = 1, \ldots, m$, we have $\langle v_k, v_k\rangle \geq 0$. Thus

$$\langle (v_1, \ldots, v_m), (v_1, \ldots, v_m)\rangle = \langle v_1, v_1\rangle + \cdots + \langle v_m, v_m\rangle \geq 0.$$

Definiteness: First suppose $\langle (v_1, \ldots, v_m), (v_1, \ldots, v_m)\rangle = 0$ for $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$. Then

$$\langle v_1, v_1\rangle + \cdots + \langle v_m, v_m\rangle = 0.$$

By positivity of each inner product on V_k (for $k = 1, \ldots, m$), we must have $\langle v_k, v_k\rangle \geq 0$. Thus the equation above holds only if $\langle v_k, v_k\rangle = 0$ for each k, which is true iff $v_k = 0$ (by definiteness of the inner product on V_k). Hence $(v_1, \ldots, v_m) = (0, \ldots, 0)$. Conversely, suppose $(v_1, \ldots, v_m) = (0, \ldots, 0)$. Then

$$\begin{aligned}\langle (v_1, \ldots, v_m), (v_1, \ldots, v_m)\rangle &= \langle v_1, v_1\rangle + \cdots + \langle v_m, v_m\rangle \\ &= \langle 0, 0\rangle + \cdots + \langle 0, 0\rangle \\ &= 0 + \cdots + 0 \\ &= 0,\end{aligned}$$

where the third equality follows from definiteness of the inner product on each V_k, respectively.
Additivity in first slot: Let

$$(u_1, \ldots, u_m), (v_1, \ldots, v_m), (w_1, \ldots, w_m) \in V_1 \times \cdots \times V_m.$$

It follows

$$\begin{aligned}&\langle (u_1, \ldots, u_m) + (v_1, \ldots, v_m)), (w_1, \ldots, w_m)\rangle \\ &\qquad = \langle (u_1 + v_1, \ldots, u_m + v_m), (w_1, \ldots, w_m)\rangle \\ &\qquad = \langle u_1 + v_1, w_1\rangle + \cdots + \langle u_m + v_m, w_m\rangle \\ &\qquad = \langle u_1, w_1\rangle + \langle v_1, w_1\rangle + \cdots + \langle u_m, w_m\rangle + \langle v_m, w_m\rangle \\ &\qquad = \langle (u_1, \ldots, u_m), (w_1, \ldots, w_m)\rangle + \langle (v_1, \ldots, v_m), (w_1, \ldots, w_m)\rangle,\end{aligned}$$

where the third equality follows from additivity in the first slot of each inner product on V_k, respectively.

Homogeneity in the first slot: Let $\lambda \in \mathbb{F}$ and

$$(u_1, \dots, u_m), (v_1, \dots, v_m) \in V_1 \times \cdots \times V_m.$$

It follows

$$\begin{aligned}
\langle \lambda(u_1, \dots, u_m), (v_1, \dots, v_m) \rangle &= \langle (\lambda u_1, \dots, \lambda u_m), (v_1, \dots, v_m) \rangle \\
&= \langle \lambda u_1, v_1 \rangle + \cdots + \langle \lambda u_m, v_m \rangle \\
&= \lambda \langle u_1, v_1 \rangle + \cdots + \lambda \langle u_m, v_m \rangle \\
&= \lambda(\langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle) \\
&= \lambda \langle (u_1, \dots, u_m), (v_1, \dots, v_m) \rangle,
\end{aligned}$$

where the third equality follows from homogeneity in the first slot of each inner product on V_k, respectively.

Conjugate symmetry: Again let

$$(u_1, \dots, u_m), (v_1, \dots, v_m) \in V_1 \times \cdots \times V_m.$$

It follows

$$\begin{aligned}
\langle (u_1, \dots, u_m), (v_1, \dots, v_m) \rangle &= \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle \\
&= \overline{\langle v_1, u_1 \rangle} + \cdots + \overline{\langle v_m, u_m \rangle} \\
&= \overline{\langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle} \\
&= \overline{\langle (v_1, \dots, v_m), (u_1, \dots, u_m) \rangle},
\end{aligned}$$

where the second equality follows from conjugate symmetry of each inner product on V_k, respectively. ☐

Problem 24

Suppose $S \in \mathcal{L}(V)$ is an injective operator on V. Define $\langle \cdot, \cdot \rangle_1$ by

$$\langle u, v \rangle_1 = \langle Su, Sv \rangle$$

for $u, v \in V$. Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on V.

Proof. We prove that this definition satisfies each property of an inner product in turn.

Positivity: Let $v \in V$. Then $\langle v, v \rangle_1 = \langle Sv, Sv \rangle \geq 0$.

Definiteness: Suppose $\langle v, v \rangle = 0$ for some $v \in V$. This is true iff $\langle Sv, Sv \rangle = 0$ (by definition) which is true iff $Sv = 0$ (by definiteness of $\langle \cdot, \cdot \rangle$), which is true iff

$v = 0$ (since S is injective).

Additivity in first slot: Let $u, v, w \in V$. Then

$$\begin{aligned}
\langle u+v, w\rangle_1 &= \langle S(u+v), Sw\rangle \\
&= \langle Su + Sv, Sw\rangle \\
&= \langle Su, Sw\rangle + \langle Sv, Sw\rangle \\
&= \langle u, w\rangle_1 + \langle v, w\rangle_1.
\end{aligned}$$

Homogeneity in first slot: Let $\lambda \in \mathbb{F}$ and $u, v \in V$. Then

$$\begin{aligned}
\langle \lambda u, v\rangle_1 &= \langle S(\lambda u), Sv\rangle \\
&= \langle \lambda Su, Sv\rangle \\
&= \lambda\langle Su, Sv\rangle \\
&= \lambda\langle u, v\rangle_1.
\end{aligned}$$

Conjugate symmetry Let $u, v \in V$. Then

$$\begin{aligned}
\langle u, v\rangle_1 &= \langle Su, Sv\rangle \\
&= \overline{\langle Sv, Su\rangle} \\
&= \overline{\langle v, u\rangle_1}.
\end{aligned}$$

$\square$

Problem 25

Suppose $S \in \mathcal{L}(V)$ is not injective. Define $\langle \cdot, \cdot\rangle_1$ as in the exercise above. Explain why $\langle \cdot, \cdot\rangle_1$ is not an inner product on V.

Proof. If S is not injective, then $\langle \cdot, \cdot\rangle_1$ fails the definiteness requirement in the definition of an inner product. In particular, there exists $v \neq 0$ such that $Sv = 0$. Hence $\langle v, v\rangle_1 = \langle Sv, Sv\rangle = 0$ for a nonzero v. $\square$

Problem 27

Suppose $u, v, w \in V$. Prove that

$$\left\| w - \frac{1}{2}(u+v) \right\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}.$$

Proof. We have

$$\begin{aligned}
\left\| w - \frac{1}{2}(u+v) \right\|^2 &= \left\| \left(\frac{w-u}{2} \right) + \left(\frac{w-v}{2} \right) \right\|^2 \\
&= 2\left\| \frac{w-u}{2} \right\|^2 + 2\left\| \frac{w-v}{2} \right\|^2 - \left\| \left(\frac{w-u}{2} \right) - \left(\frac{w-v}{2} \right) \right\|^2 \\
&= \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \left\| \frac{-u+v}{2} \right\|^2 \\
&= \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4},
\end{aligned}$$

where the second equality follows by the Parallelogram Equality. □

The next problem requires some extra work to prove. We first include a definition and prove a theorem.

Definition. *Suppose* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are norms on vector space* V*. We say* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are* equivalent *if there exist* $0 < C_1 \leq C_2$ *such that*

$$C_1\|v\|_1 \leq \|v\|_2 \leq C_2\|v\|_1$$

for all $v \in V$*.*

Theorem. *Any two norms on a finite-dimensional vector space are equivalent.*

Proof. Let V be finite-dimensional with basis $e_1, \ldots, e_n$. It suffices to prove that every norm on V is equivalent to the ℓ_1-style norm $\|\cdot\|_1$ defined by

$$\|v\|_1 = |\alpha_1| + \cdots + |\alpha_n|$$

for all $v = \alpha_1 e_1 + \cdots + \alpha_n e_n \in V$.

Let $\|\cdot\|$ be a norm on V. We wish to show $C_1\|v\|_1 \leq \|v\| \leq C_2\|v\|_1$ for all $v \in V$ and some choice of C_1, C_2. Since this is trivially true for $v = 0$, we need only consider $v \neq 0$, in which case we have

$$C_1 \leq \|u\| \leq C_2, \tag{*}$$

where $u = v/\|v\|_1$. Thus it suffices to consider only vectors $v \in V$ such that $\|v\|_1 = 1$.

We will now show that $\|\cdot\|$ is continuous under $\|\cdot\|_1$ and apply the Extreme Value Theorem to deduce the desired result. So let $\epsilon > 0$ and define $M = \max\{\|e_1\|, \ldots, \|e_n\|\}$ and

$$\delta = \frac{\epsilon}{M}.$$

It follows that if $u, v \in V$ are such that $\|u - v\|_1 < \delta$, then

$$\begin{aligned}
\big| \|u\| - \|v\| \big| &\leq \|u - v\| \\
&\leq M\|u - v\|_1 \\
&\leq M\delta \\
&= \epsilon,
\end{aligned}$$

and $\|\cdot\|$ is indeed continuous under the topology induced by $\|\cdot\|_1$. Let $\mathcal{S} = \{u \in V \mid \|u\|_1 = 1\}$ (the unit sphere with respect to $\|\cdot\|_1$). Since $\mathcal{S}$ is compact and $\|\cdot\|$ is continuous on it, by the Extreme Value Theorem we may define

$$C_1 = \min_{u \in \mathcal{S}} \|u\| \quad \text{and} \quad C_2 = \max_{u \in \mathcal{S}} \|u\| .$$

But now C_1 and C_2 satisfy (*), completing the proof. □

Problem 29

For $u, v \in V$, define $d(u, v) = \|u - v\|$.

(a) Show that d is a metric on V.

(b) Show that if V is finite-dimensional, then d is a complete metric on V (meaning that every Cauchy sequence converges).

(c) Show that every finite-dimensional subspace of V is a closed subset of V (with respect to the metric d).

Proof. (a) We show that d satisfies each property of the definition of a metric in turn.

Identity of indiscernibles: Let $u, v \in V$. It follows

$$\begin{aligned} d(u, v) = 0 &\iff \sqrt{\langle u - v, u - v \rangle} = 0 \\ &\iff \langle u - v, u - v, = \rangle 0 \\ &\iff u - v = 0 \\ &\iff u = v. \end{aligned}$$

Symmetry: Let $u, v \in V$. We have

$$\begin{aligned} d(u, v) &= \|u - v\| \\ &= \big\|(-1)(u - v)\big\| \\ &= \|v - u\| \\ &= d(v, u). \end{aligned}$$

Triangle inequality: Let $u, v, w \in V$. Notice

$$\begin{aligned} d(u, v) + d(v, w) &= \|u - v\| + \|v - w\| \\ &\le \big\|(u - v) + (v - w)\big\| \\ &= \|u, w\| \\ &= d(u, w). \end{aligned}$$

(b) Suppose V is a p-dimensional vector space with basis $e_1, \ldots, e_p$. Assume $\{v_k\}_{k=1}^{\infty}$ is Cauchy. Then for $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$\|v_m - v_n\| < \epsilon$ whenever $m, n > N$. Given any v_i in our Cauchy sequence, we adopt the notation that $\alpha_{i,1}, \ldots, \alpha_{i,p} \in \mathbb{F}$ are always defined such that

$$v_i = \alpha_{i,1} e_1 + \cdots + \alpha_{i,p} e_p.$$

By our previous theorem, $\|\cdot\|$ is equivalent to $\|\cdot\|_1$ (where $\|\cdot\|_1$ is defined in that theorem's proof). Thus there exists some $c > 0$ such that, whenever $m, n > N$, we have

$$c\|v_m - v_n\|_1 \leq \|v_m - v_n\| < \epsilon,$$

and hence

$$c\left(\sum_{i=1}^{p} \left|\alpha_{m,i} - \alpha_{n,i}\right|\right) < \epsilon.$$

This implies that $\{\alpha_{k,i}\}_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$ for each $i = 1, \ldots, p$. Since $\mathbb{R}$ is complete, these sequences converge. So let $\alpha_i = \lim_{k\to\infty} \alpha_{k,i}$ for each i, and define $v = \alpha_1 e_1 + \cdots + \alpha_p e_p$. It follows

$$\begin{aligned} \left\|v_j - v\right\| &= \left\|(\alpha_{j,1} - \beta_1)e_1 + \cdots + (\alpha_{j,p} - \beta_p)e_p\right\| \\ &\leq \left|\alpha_{j,1} - \alpha_1\right|\|e_1\| + \cdots + \left|\alpha_{j,p} - \alpha_p\right|\left\|e_p\right\|. \end{aligned}$$

Since $\alpha_{j,i} \to \alpha_i$ for $i = 1, \ldots, p$, the RHS can be made arbitrarily small by choosing sufficiently large $M \in \mathbb{Z}^+$ and considering $j > M$. Thus $\{v_k\}_{k=1}^{\infty}$ converges to v, and V is indeed complete with respect to $\|\cdot\|$.

(c) Suppose U is a finite-dimensional subspace of V, and suppose $\{u_k\}_{k=1}^{\infty} \subseteq U$ is Cauchy. By (b), $\lim_{k\to\infty} u_k \in U$, hence U contains all its limit points. Thus U is closed. □

Problem 31

Use inner products to prove Apollonius's Identity: In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$

Proof. Consider a triangle formed by vectors $v, w \in \mathbb{R}^2$ and the origin such that $\|w\| = a$, $\|v\| = c$, and $\|w - v\| = b$. The identity follows by applying Problem 27 with $u = 0$. □

B: Orthonormal Bases

Problem 1

(a) Suppose $\theta \in \mathbb{R}$. Show that $(\cos\theta, \sin\theta)$, $(-\sin\theta, \cos\theta)$ and $(\cos\theta, \sin\theta)$, $(\sin\theta, -\cos\theta)$ are orthonormal bases of $\mathbb{R}^2$.

(b) Show that each orthonormal basis of $\mathbb{R}^2$ is of the form given by one of the two possibilities of part (a).

Proof. (a) Notice

$$\langle(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\rangle = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

and

$$\langle(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)\rangle = \sin\theta\cos\theta - \sin\theta\cos\theta = 0,$$

hence both lists are orthonormal. Clearly the three distinct vectors contained in the two lists all have norm 1 (following from the identity $\cos^2\theta + \sin^2\theta = 1$). Since both lists have length 2, by Theorem 6.28 both lists are orthonormal bases.

(b) Suppose e_1, e_2 is an orthonormal basis of $\mathbb{R}^2$. Since $\|e_1\| = \|e_2\| = 1$, there exist $\theta, \varphi \in [0, 2\pi)$ such that

$$e_1 = (\cos\theta, \sin\theta) \quad \text{and} \quad e_2 = (\cos\varphi, \sin\varphi).$$

Next, since $\langle e_1, e_2\rangle = 0$, we have

$$\cos\theta\cos\varphi + \sin\theta\sin\varphi = 0.$$

Since $\cos\theta\cos\varphi = \frac{1}{2}(\cos(\theta+\varphi) + \cos(\theta-\varphi))$ and $\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$, the above implies

$$\cos(\theta - \varphi) = 0$$

and thus $\varphi = \theta + \frac{3\pi}{2} - n\pi$, for $n \in \mathbb{Z}$. Since $\theta, \varphi \in [0, 2\pi)$, this implies $\varphi = \theta \pm \frac{\pi}{2}$. If $\varphi = \theta + \frac{\pi}{2}$, then

$$\begin{aligned} e_2 &= \left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right) \\ &= (-\sin\theta, \cos\theta), \end{aligned}$$

and if $\varphi = \theta - \frac{\pi}{2}$, then

$$\begin{aligned} e_2 &= \left(\cos\left(\theta - \frac{\pi}{2}\right), \sin\left(\theta - \frac{\pi}{2}\right)\right) \\ &= (\sin\theta, -\cos\theta). \end{aligned}$$

Thus all orthonormal bases of $\mathbb{R}^2$ have one of the two forms from (a). □

Problem 3

Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ has an upper-triangular matrix with respect to the basis $(1,0,0)$, $(1,1,1)$, $(1,1,2)$. Find an orthonormal basis of $\mathbb{R}^3$ (use the usual inner product on $\mathbb{R}^3$) with respect to which T has an upper-triangular matrix.

Proof. Let $v_1 = (1,0,0), v_2 = (1,1,1)$, and $v_3 = (1,1,2)$. By the proof of 6.37, T has an upper-triangular matrix with respect to the the basis e_1, e_2, e_3 generated by applying the Gram-Schmidt Procedure to v_1, v_2, v_3. Since $\|v_1\| = 1$, $e_1 = v_1$. Next, we have

$$\begin{aligned}
e_2 &= \frac{v_2 - \langle v_2, e_1\rangle e_1}{\big\|v_2 - \langle v_2, e_1\rangle e_1\big\|} \\
&= \frac{(1,1,1) - \langle (1,1,1),(1,0,0)\rangle (1,0,0)}{\big\|(1,1,1) - \langle (1,1,1),(1,0,0)\rangle (1,0,0)\big\|} \\
&= \frac{(1,1,1) - (1,0,0)}{\big\|(1,1,1) - (1,0,0)\big\|} \\
&= \frac{(0,1,1)}{\big\|(0,1,1)\big\|} \\
&= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)
\end{aligned}$$

and

$$\begin{aligned}
e_3 &= \frac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\big\|v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\big\|} \\
&= \frac{(1,1,2) - \langle (1,1,2),(1,0,0)\rangle (1,0,0) - \left\langle (1,1,2), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}{\left\|(1,1,2) - \langle (1,1,2),(1,0,0)\rangle (1,0,0) - \left\langle (1,1,2), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\|} \\
&= \frac{(1,1,2) - (1,0,0) - \frac{3}{\sqrt{2}}\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}{\left\|(1,1,2) - (1,0,0) - \frac{3}{\sqrt{2}}\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\|} \\
&= \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\left\|\left(0, -\frac{1}{2}, \frac{1}{2}\right)\right\|} \\
&= \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),
\end{aligned}$$

and we're done. ☐

Problem 4

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $\mathcal{C}[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Proof. First we show that all vectors in the list have norm 1. Notice

$$\begin{aligned}
\left\| \frac{1}{\sqrt{2\pi}} \right\| &= \sqrt{\int_{-\pi}^{\pi} \frac{1}{2\pi} dx} \\
&= \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} dx} \\
&= 1.
\end{aligned}$$

And for $k \in \mathbb{Z}^+$, we have

$$\begin{aligned}
\left\| \frac{\cos(kx)}{\sqrt{\pi}} \right\| &= \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx)^2 dx} \\
&= \sqrt{\frac{1}{\pi} \left[\frac{\sin(2kx)}{4k} + \frac{x}{2} \right]_{-\pi}^{\pi}} \\
&= \sqrt{\frac{1}{\pi} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right]} \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\left\| \frac{\sin(kx)}{\sqrt{\pi}} \right\| &= \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kx)^2 dx} \\
&= \sqrt{\frac{1}{\pi} \left[\frac{x}{2} - \frac{\cos(2kx)}{4k} \right]_{-\pi}^{\pi}} \\
&= \sqrt{\frac{1}{\pi} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right]} \\
&= 1,
\end{aligned}$$

so indeed all vectors have norm 1. Now we show them to be pairwise orthogonal. Suppose $j, k \in \mathbb{Z}$ are such that $j \neq k$. It follows from basic calculus

$$\begin{aligned}\left\langle \frac{\sin(jx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(jx)\sin(kx)dx \\ &= \frac{1}{\pi}\left[\frac{k\sin(jx)\cos(kx) + j\cos(jx)\sin(kx)}{j^2 - k^2}\right]_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\left\langle \frac{\sin(jx)}{\sqrt{\pi}}, \frac{\cos(kx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(jx)\cos(kx)dx \\ &= -\frac{1}{\pi}\left[\frac{k\sin(jx)\sin(kx) + j\cos(jx)\cos(kx)}{j^2 - k^2}\right]_{-\pi}^{\pi} \\ &= -\frac{1}{\pi}\left[\left(\frac{j\cos(j\pi)\cos(k\pi)}{j^2 - k^2}\right) - \left(\frac{j\cos(-j\pi)\cos(-k\pi)}{j^2 - k^2}\right)\right] \\ &= 0,\end{aligned}$$

$$\begin{aligned}\left\langle \frac{\cos(jx)}{\sqrt{\pi}}, \frac{\cos(kx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(jx)\cos(kx)dx \\ &= \frac{1}{\pi}\left[\frac{j\sin(jx)\cos(kx) - k\cos(jx)\sin(kx)}{j^2 - k^2}\right]_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\left\langle \frac{\sin(jx)}{\sqrt{\pi}}, \frac{\cos(jx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(jx)\cos(jx)dx \\ &= \left[-\frac{\cos^2(jx)}{2j}\right]_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\cos(jx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} \cos(jx)dx \\ &= \frac{1}{\sqrt{2}\pi}\left[\frac{\sin(jx)}{j}\right]_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\sin(jx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin(jx)dx \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{\cos(jx)}{j} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{\cos(j\pi) - \cos(-j\pi)}{j} \right] \\
&= 0.
\end{aligned}$$

Thus the list is indeed an orthonormal list in $\mathcal{C}[-\pi, \pi]$. □

Problem 5

On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\, dx.$$

Apply the Gram-Schmidt Procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

Proof. First notice $\|1\| = 1$, hence $e_1 = 1$. Next notice

$$\begin{aligned}
v_2 - \langle v_1, e_1 \rangle e_1 &= x - \langle x, 1 \rangle \\
&= x - \int_0^1 x\, dx \\
&= x - \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\left\| x - \frac{1}{2} \right\| &= \sqrt{\left\langle x - \frac{1}{2}, x - \frac{1}{2} \right\rangle} \\
&= \sqrt{\int_0^1 \left(x - \frac{1}{2} \right) \left(x - \frac{1}{2} \right) dx} \\
&= \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{4} \right) dx} \\
&= \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} \\
&= \frac{1}{2\sqrt{3}},
\end{aligned}$$

and therefore we have

$$e_2 = 2\sqrt{3}\left(x - \frac{1}{2}\right).$$

To compute e_3, first notice

$$\begin{aligned}
v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2 &= x^2 - \int_0^1 x^2\,dx - \left[2\sqrt{3}\int_0^1 x^2\left(x - \frac{1}{2}\right)\,dx\right] e_2 \\
&= x^2 - \frac{1}{3} - \left[2\sqrt{3}\int_0^1 \left(x^3 - \frac{x^2}{2}\right)\,dx\right]\left[2\sqrt{3}\left(x - \frac{1}{2}\right)\right] \\
&= x^2 - \frac{1}{3} - 12\left(\frac{1}{4} - \frac{1}{6}\right)\left(x - \frac{1}{2}\right) \\
&= x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) \\
&= x^2 - x + \frac{1}{6}
\end{aligned}$$

and

$$\begin{aligned}
\left\|x^2 - x + \frac{1}{6}\right\| &= \sqrt{\left\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6}\right\rangle} \\
&= \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6}\right)\left(x^2 - x + \frac{1}{6}\right)\,dx} \\
&= \sqrt{\int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{x}{3} + \frac{1}{36}\right)\,dx} \\
&= \sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} \\
&= \frac{1}{\sqrt{180}} \\
&= \frac{1}{6\sqrt{5}}.
\end{aligned}$$

Thus

$$e_3 = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right),$$

and we're done. $\square$

Problem 7

Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Proof. Consider the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$ on $\mathcal{P}_2(\mathbb{R})$. Define $\varphi \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $\varphi(p) = p\left(\frac{1}{2}\right)$ and let e_1, e_2, e_3 be the orthonormal basis found in Problem 5. By the Riesz Representation Theorem, there exists $q \in \mathcal{P}_2(\mathbb{R})$ such that $\varphi(p) = \langle p, q\rangle$ for all $p \in \mathcal{P}_2(\mathbb{R})$. That is, such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)\,dx.$$

Equation 6.43 in the proof of the Riesz Representation Theorem fashions a way to find q. In particular, we have

$$\begin{aligned}
q(x) &= \overline{\varphi(e_1)}\,e_1 + \overline{\varphi(e_2)}\,e_2 + \overline{\varphi(e_3)}\,e_3 \\
&= e_1 + 2\sqrt{3}\left(\frac{1}{2} - \frac{1}{2}\right)e_2 + 6\sqrt{5}\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{6}\right)e_3 \\
&= 1 + 6\sqrt{5}\left(\frac{-1}{12}\right)\left[6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right] \\
&= -15(x^2 - x) - \frac{3}{2},
\end{aligned}$$

as desired. ☐

Problem 9

What happens if the Gram-Schmidt Procedure is applied to a list of vectors that is not linearly independent?

Proof. Suppose $v_1, \ldots, v_m$ are linearly dependent. Let j be the smallest integer in $\{1, \ldots, m\}$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$. Then $v_1, \ldots, v_{j-1}$ are linearly independent. The first $j - 1$ steps of the Gram-Schmidt Procedure will produce an orthonormal list $e_1, \ldots, e_{j-1}$. At step j, however, notice

$$v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1} = v_j - v_j = 0,$$

and we are left trying to assign e_j to $\frac{0}{0}$, which is undefined. Thus the procedure cannot be applied to a linearly dependent list. ☐

Problem 11

Suppose $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on V such that $\langle v,w\rangle_1 = 0$ if and only if $\langle v,w\rangle_2 = 0$. Prove that there is a positive number c such that $\langle v,w\rangle_1 = c\langle v,w\rangle_2$ for every $v,w \in V$.

Proof. Let $v,w \in V$ be arbitrary. By hypothesis, if v and w are orthogonal relative to one of the inner products, they're orthogonal relative to the other. Hence any choice of $c \in \mathbb{R}$ would satisfy $\langle v,w\rangle_1 = c\langle v,w\rangle_2$. So suppose v and w are not orthogonal relative to either inner product. Then both v and w must be nonzero (by Theorem 6.7, parts b and c, respectively). Thus $\langle v,v\rangle_1$, $\langle w,w\rangle_1$, $\langle v,v\rangle_2$, and $\langle w,w\rangle_2$ are all nonzero as well. It now follows

$$\begin{aligned}
0 &= \langle v,w\rangle_1 - \frac{\langle v,w\rangle_1}{\langle v,v\rangle_1}\langle v,v\rangle_1 \\
&= \langle v,w\rangle_1 - \left\langle v, \overline{\left(\frac{\langle v,w\rangle_1}{\langle v,v\rangle_1}\right)} v \right\rangle_1 \\
&= \left\langle v, w - \overline{\left(\frac{\langle v,w\rangle_1}{\langle v,v\rangle_1}\right)} v \right\rangle_1 \\
&= \left\langle v, w - \overline{\left(\frac{\langle v,w\rangle_1}{\langle v,v\rangle_1}\right)} v \right\rangle_2 \\
&= \langle v,w\rangle_2 - \left\langle v, \overline{\left(\frac{\langle v,w\rangle_1}{\langle v,v\rangle_1}\right)} v \right\rangle_2 \\
&= \langle v,w\rangle_2 - \frac{\langle v,w\rangle_1}{\langle v,v\rangle_1}\langle v,v\rangle_2 \\
&= \langle v,w\rangle_2 - \frac{\langle v,v\rangle_2}{\langle v,v\rangle_1}\langle v,w\rangle_1,
\end{aligned}$$

where the fifth equality follows by our hypothesis. Thus

$$\langle v,w\rangle_1 = \frac{\|v\|_1{}^2}{\|v\|_2{}^2}\langle v,w\rangle_2. \tag{4}$$

By a similar computation, notice

$$\begin{aligned}
0 &= \langle v, w \rangle_1 - \frac{\langle v, w \rangle_1}{\langle w, w \rangle_1} \langle w, w \rangle_1 \\
&= \langle v, w \rangle_1 - \left\langle \frac{\langle v, w \rangle_1}{\langle w, w \rangle_1} w, w \right\rangle_1 \\
&= \left\langle v - \frac{\langle v, w \rangle_1}{\langle w, w \rangle_1} w, w \right\rangle_1 \\
&= \left\langle v - \frac{\langle v, w \rangle_1}{\langle w, w \rangle_1} w, w \right\rangle_2 \\
&= \langle v, w \rangle_2 - \left\langle \frac{\langle v, w \rangle_1}{\langle w, w \rangle_2} w, w \right\rangle_2 \\
&= \langle v, w \rangle_2 - \frac{\langle v, w \rangle_1}{\langle w, w \rangle_2} \langle w, w \rangle_2 \\
&= \langle v, w \rangle_2 - \frac{\langle w, w \rangle_2}{\langle w, w \rangle_1} \langle v, w \rangle_1,
\end{aligned}$$

and thus

$$\langle v, w \rangle_1 = \frac{\|w\|_1^{\,2}}{\|w\|_2^{\,2}} \langle v, w \rangle_2 \tag{5}$$

as well. By combining Equations (4) and (5), we conclude

$$\frac{\langle v, v \rangle_1}{\langle v, v \rangle_2} = \frac{\langle w, w \rangle_1}{\langle w, w \rangle_2}.$$

Since v and w were arbitrary nonzero vectors in V, choosing $c = \|u\|_1^{\,2} / \|u\|_2^{\,2}$ for any $u \neq 0$ guarantees $\langle v, w \rangle_1 = c \langle v, w \rangle_2$ for every $v, w \in V$, as was to be shown. □

Problem 13

Suppose $v_1, \ldots, v_m$ is a linearly independent list in V. Show that there exists $w \in V$ such that $\langle w, v_j \rangle > 0$ for all $j \in \{1, \ldots, m\}$.

Proof. Let $W = \operatorname{span}(v_1, \ldots, v_m)$. Given $v \in W$, let $a_1, \ldots, a_m \in \mathbb{F}$ be such that $v = a_1 v_1 + \cdots + a_m v_m$. Define $\varphi \in \mathcal{L}(W)$ by

$$\varphi(v) = a_1 + \cdots + a_m.$$

By the Riesz Representation Theorem, there exists $w \in W$ such that $\varphi(v) = \langle v, w \rangle$ for all $v \in W$. But then $\varphi(v_j) = 1$ for $j \in \{1, \ldots, m\}$, and indeed such a $w \in V$ exists. □

Problem 15

Suppose $C_{\mathbb{R}}([-1,1])$ is the vector space of continuous real-valued functions on the interval $[-1,1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

for $f, g \in C_{\mathbb{R}}([-1,1])$. Let φ be the linear functional on $C_{\mathbb{R}}([-1,1])$ defined by $\varphi(f) = f(0)$. Show that there does not exist $g \in C_{\mathbb{R}}([-1,1])$ such that

$$\varphi(f) = \langle f, g\rangle$$

for every $f \in C_{\mathbb{R}}([-1,1])$.

Proof. Suppose not. Then there exists $g \in C_{\mathbb{R}}([-1,1])$ such that

$$\varphi(f) = \langle f, g\rangle$$

for every $f \in C_{\mathbb{R}}([-1,1])$. Choose $f(x) = x^2 g(x)$. Then $f(0) = 0$, and hence

$$\int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} [xg(x)]^2 dx = 0.$$

Now, let $h(x) = xg(x)$. Since h is continuous on $[-1,1]$, there exists an interval $[a,b] \subseteq [-1,1]$ such that $h(x) \neq 0$ for all $x \in [a,b]$. By the Extreme Value Theorem, $h(x)^2$ has a minimum at some $m \in [a,b]$. Thus $h(m)^2 > 0$, and we now conclude

$$0 = \int_{-1}^{1} h(x)^2 dx = \int_{a}^{b} h(x)^2 dx \geq (b-a)h(m)^2 > 0,$$

which is absurd. Thus it must be that no such g exists. □

Problem 17

For $u \in V$, let Φ_u denote the linear functional on V defined by

$$(\Phi_u)(v) = \langle v, u\rangle$$

for $v \in V$.

(a) Show that if $\mathbb{F} = \mathbb{R}$, then Φ is a linear map from V to V'.

(b) Show that if $\mathbb{F} = \mathbb{C}$ and $V \neq \{0\}$, then Φ is not a linear map.

(c) Show that Φ is injective.

(d) Suppose $\mathbb{F} = \mathbb{R}$ and V is finite-dimensional. Use parts (a) and (c) and a dimension-counting argument (but without using 6.42) to show that Φ is an isomorphism from V to V'.

Proof. (a) Suppose $\mathbb{F} = \mathbb{R}$. Let $u, u' \in V$ and $\alpha \in \mathbb{R}$. Then, for all $v \in V$, we have

$$\Phi_{u+u'}(v) = \langle v, u + u' \rangle = \langle v, u \rangle + \langle v, u' \rangle = \Phi_u(v) + \Phi_{u'}(v)$$

and

$$\Phi_{\alpha u}(v) = \langle v, \alpha u \rangle = \overline{\alpha} \langle v, u \rangle = \alpha \langle v, u \rangle = \alpha \Phi_u(v).$$

Thus Φ is indeed a linear map.

(b) Suppose $\mathbb{F} = \mathbb{C}$ and $V \neq \{0\}$. Let $u \in V$. Then, given $v \in V$, we have

$$\Phi_{iu}(v) = \langle v, iu \rangle = \bar{i} \langle v, u \rangle,$$

whereas

$$i\Phi_u(v) = i\langle v, u \rangle.$$

Thus $\Phi_{iu} \neq i\Phi_u$, and indeed Φ is not a linear map, since is is not homogeneous.

(c) Suppose $u, u' \in V$ are such that $\Phi_u = \Phi_{u'}$. Then, for all $v \in V$, we have

$$\begin{aligned} &\Phi_u(v) = \Phi_{u'}(v) \\ \implies &\langle v, u \rangle = \langle v, u' \rangle \\ \implies &\langle v, u \rangle - \langle v, u' \rangle = 0 \\ \implies &\langle v, u - u' \rangle = 0. \end{aligned}$$

In particular, choosing $v = u - u'$, the above implies $\langle u - u', u - u' \rangle = 0$, which is true iff $u - u' = 0$. Thus we conclude $u = u'$, so that Φ is indeed injective.

(d) Suppose $\mathbb{F} = \mathbb{R}$ and $\dim V = n$. Notice that since $\Phi : V \hookrightarrow V'$, we have

$$\dim V = \dim \operatorname{null} \Phi + \dim \operatorname{range} \Phi = \dim \operatorname{range} \Phi.$$

Thus Φ is surjective as well, and we have $V \cong V'$, as was to be shown. □

C: Orthogonal Complements and Minimization Problems

Problem 1

Suppose $v_1, \ldots, v_m \in V$. Prove that

$$\{v_1, \ldots, v_m\}^\perp = (\operatorname{span}(v_1, \ldots, v_m))^\perp.$$

Proof. Suppose $v \in \{v_1, \ldots, v_m\}^\perp$. Then $\langle v, v_k \rangle = 0$ for $k = 1, \ldots, m$. Let $u \in \operatorname{span}(v_1, \ldots, v_m)$ be arbitrary. We want to show $\langle v, u \rangle = 0$, since this implies $v \in (\operatorname{span}(v_1, \ldots, v_m))^\perp$ and hence $\{v_1, \ldots, v_m\}^\perp \subseteq (\operatorname{span}(v_1, \ldots, v_m))^\perp$. To see this, notice

$$\begin{aligned}
\langle v, u \rangle &= \langle v, \alpha_1 v_1 + \cdots + \alpha_m v_m \rangle \\
&= \alpha_1 \langle v, v_1 \rangle + \cdots + \alpha_m \langle v, v_m \rangle \\
&= 0,
\end{aligned}$$

as desired. Next suppose $v' \in (\operatorname{span}(v_1, \ldots, v_m))^\perp$. Since $v_1, \ldots, v_m$ are all clearly elements of $\operatorname{span}(v_1, \ldots, v_m)$, clearly $v' \in \{v_1, \ldots, v_m\}^\perp$, and thus $(\operatorname{span}(v_1, \ldots, v_m))^\perp \subseteq \{v_1, \ldots, v_m\}^\perp$. Therefore we conclude $\{v_1, \ldots, v_m\}^\perp = (\operatorname{span}(v_1, \ldots, v_m))^\perp$. □

Problem 3

Suppose U is a subspace of V with basis $u_1, \ldots, u_m$ and

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V. Prove that if the Gram-Schmidt Procedure is applied to the basis of V above, producing a list $e_1, \ldots, e_m, f_1, \ldots, f_n$, then $e_1, \ldots, e_m$ is an orthonormal basis of U and $f_1, \ldots, f_n$ is an orthonormal basis of $U^\perp$.

Proof. By 6.31, $\operatorname{span}(u_1, \ldots, u_m) = \operatorname{span}(e_1, \ldots, e_m)$. Since $e_1, \ldots, e_m$ is an orthonormal list by construction (and linearly independent by 6.26), $e_1, \ldots, e_m$ is indeed an orthonormal basis of U. Next, since each of f_i is orthogonal to each e_j, so too is each f_i orthogonal to any element of U. Thus $f_k \in U^\perp$ for $k = 1, \ldots, n$. Now, since $\dim U^\perp = \dim V - \dim U = n$ by 6.50, we conclude $f_1, \ldots, f_n$ is an orthonormal list of length $\dim U^\perp$ and hence an orthonormal basis of $U^\perp$. □

Problem 5

Suppose V is finite-dimensional and U is a subspace of V. Show that $P_{U^\perp} = I - P_U$, where I is the identity operator on V.

Proof. For $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. It follows

$$\begin{aligned}
P_{U^\perp}(v) &= w \\
&= (u + w) - u \\
&= Iv - P_U v,
\end{aligned}$$

and therefore $P_{U^\perp} = I - P_U$, as desired. □

Problem 7

Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$. Prove that there exists a subspace U of V such that $P = P_U$.

Proof. By Problem 4 of Chapter 5B, we know $V = \operatorname{null} P \oplus \operatorname{range} P$. Let $v \in V$. Then there exist $u \in \operatorname{null} P$ and $w \in \operatorname{range} P$ such that $v = u + w$ and hence

$$\begin{aligned} Pv &= P(u + w) \\ &= Pu + Pw \\ &= Pw. \end{aligned}$$

Let $U = \operatorname{range} P$ and notice that $\operatorname{null} P \subseteq \operatorname{null} P_U = U^\perp$ by 6.55e. Then $Pv = Pw = P_U(v)$, and so U is the desired subpace. □

Problem 9

Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

Proof. (⇐) Suppose $P_U T P_U = T P_U$ and let $u \in U$. It follows

$$TP_u(u) = P_U T P_U(v)$$

and thus

$$Tu = P_U T u.$$

Since $\operatorname{range} P_U = U$ by 6.55d, this implies $Tu \in U$. Thus U is indeed invariant under T.

(⇒) Now suppose U is invariant under T and let $v \in V$. Since $P_U(v) \in U$, it follows that $TP_U(v) \in U$. And thus $P_U T P_U(v) = T P_U(v)$, as desired. □

Problem 11

In $\mathbb{R}^4$, let

$$U = \operatorname{span}\big((1, 1, 0, 0), (1, 1, 1, 2)\big).$$

Find $u \in U$ such that $\big\|u - (1, 2, 3, 4)\big\|$ is as small as possible.

Proof. We first apply the Gram-Schmidt Procedure to $v_1 = (1,1,0,0)$ and $v_2 = (1,1,1,2)$. This yields

$$\begin{aligned} e_1 &= \frac{v_1}{\|v_1\|} \\ &= \frac{(1,1,0,0)}{\big\|(1,1,0,0)\big\|} \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) \end{aligned}$$

and

$$\begin{aligned} e_2 &= \frac{v_2 - \langle v_2, e_1\rangle e_1}{\big\|v_2 - \langle v_2, e_1\rangle e_1\big\|} \\ &= \frac{(1,1,1,2) - \left\langle (1,1,1,2), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)\right\rangle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)}{\left\|(1,1,1,2) - \left\langle (1,1,1,2), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)\right\rangle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)\right\|} \\ &= \frac{(1,1,1,2) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)}{\left\|(1,1,1,2) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)\right\|} \\ &= \frac{(0,0,1,2)}{\big\|(0,0,1,2)\big\|} \\ &= \left(0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right). \end{aligned}$$

Now, with our orthonormal basis e_1, e_2, it follows by 6.55(i) and 6.56 that $\big\|u - (1,2,3,4)\big\|$ is minimized by the vector

$$\begin{aligned} u &= P_U(1,2,3,4) \\ &= \langle (1,2,3,4), e_1\rangle e_1 + \langle (1,2,3,4), e_2\rangle e_2 \\ &= \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) + \frac{11}{\sqrt{2}}\left(0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= \left(\frac{3}{2}, \frac{3}{2}, 0, 0\right) + \left(0, 0, \frac{11}{5}, \frac{22}{5}\right) \\ &= \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5}\right), \end{aligned}$$

completing the proof. ∎

Problem 13

Find $p \in \mathcal{P}_5(\mathbb{R})$ that makes

$$\int_{-\pi}^{\pi} \left|\sin x - p(x)\right|^2 dx$$

as small as possible.

Proof. Let $\mathcal{C}_{\mathbb{R}}[-\pi, \pi]$ denote the real inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx,$$

and let U denote the subspace of $\mathcal{C}_{\mathbb{R}}[-\pi, \pi]$ consisting of the polynomials with real coefficients and degree at most 5. In this inner product space, observe that

$$\left\|\sin x - p(x)\right\| = \sqrt{\int_{-\pi}^{\pi} \left(\sin x - p(x)\right)^2 dx} = \sqrt{\int_{-\pi}^{\pi} \left|\sin x - p(x)\right|^2 dx}.$$

Notice also that $\sqrt{\int_{-\pi}^{\pi} \left|\sin x - p(x)\right|^2 dx}$ is minimized if and only if $\int_{-\pi}^{\pi} \left|\sin x - p(x)\right|^2 dx$ is minimized. Thus by 6.56, we may conclude $p(x) = P_U(\sin x)$ minimizes $\int_{-\pi}^{\pi} \left|\sin x - p(x)\right|^2 dx$. To compute $P_U(\sin x)$, we first find an orthonormal basis of $\mathcal{C}_{\mathbb{R}}[-\pi, \pi]$ by applying the Gram-Schmidt Procedure to the basis $1, x, x^2, x^3, x^4, x^5$ of U. A lengthy computation yields the orthonormal basis

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{2\pi}} \\
e_2 &= \frac{\sqrt{\frac{3}{2}}x}{x^{3/2}} \\
e_3 &= -\frac{\sqrt{\frac{5}{2}}\left(\pi^2 - 3x^2\right)}{2\pi^{5/2}} \\
e_4 &= -\frac{\sqrt{\frac{7}{2}}\left(3\pi^2 x - 5x^3\right)}{2\pi^{7/2}} \\
e_5 &= \frac{3\left(3\pi^4 - 30\pi^2 x^2 + 35x^4\right)}{8\sqrt{2}\pi^{9/2}} \\
e_6 &= -\frac{\sqrt{\frac{11}{2}}\left(15\pi^4 x - 70\pi^2 x^3 + 63x^5\right)}{8\pi^{11/2}}.
\end{aligned}$$

Now we compute $P_U(\sin x)$ using 6.55(i), yielding

$$P_U(\sin x) = \frac{105\left(1485 - 153\pi^2 + \pi^4\right)}{8\pi^6}x - \frac{315\left(1155 - 125\pi^2 + \pi^4\right)}{4\pi^8}x^3 + \frac{693\left(945 - 105\pi^2 + \pi^4\right)}{8\pi^{10}}x^5,$$

which is the desired polynomial. $\square$