EE635 : Solution : Assignment 1

(3) Consider $x, y \in \mathbb{Z}_n$

The Product $xy = 0$ implies xy is divisible by $n = pq$ ie. divisible by either p or q respectively from the property given in the hint. (a prime number dividing a product of 2 integers divides atleast one of them). Now when $x^2 = x.x = 0$ must hold, ~~then~~ it has to be divisible by $n = pq$; for this x must be divisible by both p and q, this is possible only when $x = n$ and since $\mathbb{Z}_n = \{0, 1, \dots n-1\} \Rightarrow n \notin \mathbb{Z}_n$

$\therefore x^2 = 0$ has a unique solution in $\mathbb{Z}_n$

(5) Given $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

To obtain P^{-1}. consider $E_1P = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$

$$E_2E_1P = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 7 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$E_3E_2E_1P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$E_4E_3E_2E_1P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $P^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 2 \end{bmatrix}$

Now,

$$P^{-1}AP = \begin{bmatrix} 6 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 24 & 28 \\ 14 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$ verified.

⑦ $A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ obtain $x \in \mathbb{R}^3$ $AX = \lambda X$

Sol:

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$

finding λ:

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & 1 & 0 \\ 0 & 5-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)^3 = 0$$

$\Rightarrow \lambda = 5, 5, 5$

Finding x:

$(A - 5I)x = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix $(A - 5I)$ is in RRE form, with x_1 as free variable

$\therefore x_2 = 0 \quad x_3 = 0 \quad x_1 = k$

$\therefore x = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ will satisfy $Ax = \lambda x$ with $\lambda = 5$

(11) Solution:-

a) Requirement of minerals for each square foot of ground 10 units Phosphrous, 9 units of Potassium and 19 units of Nitrogen.

Let say, x, y, z amount products procured from the brand X, Y, Z respectively.

The problem can be formulated as—

$$2x + y + z = 10$$
$$3x + 3y = 9$$
$$5x + 4y + z = 19$$

i.e.

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 3 & 0 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 19 \end{bmatrix}$$

Now perform elementary row operation to get rref of the following matrix

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 3 & 0 & 9 \\ 5 & 4 & 1 & 19 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 1 & 1 & 0 & 3 \\ 1 & 4/5 & 1/5 & 19/5 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \end{array}\right.$$

$$\left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & -2 \\ 0 & 3/10 & -3/10 & -6/5 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & +1 & 7 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -2 \\ 0 & 3/10 & -3/10 & -\frac{6}{5} \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & +1 & 7 \\ 0 & 1 & -1 & -4 \\ 0 & 3 & -3 & -12 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & +1 & 7 \\ 0 & 1 & -1 & -4 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & +1 & 7 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i.e. the equivalent system can be written as –

$$\begin{bmatrix} 1 & 0 & +1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \quad \begin{bmatrix} 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

let $z = c \in \mathbb{R}$

$y = -4 + c = c - 4$

$x = 7 - c$

Now, for meaning full solution (x, y, z) to be positive integer

i.e. $c \in \mathbb{Z}^+$

so, $c - 4 \geq 0$

$\Rightarrow c \geq 4$

$7 - c \geq 0$

$\Rightarrow 7 \geq c$

then $4 \leq c \leq 7$

so the meaning full solution set

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 7 \end{pmatrix} \right\}$$

(b) The cost of the fertilizer is given by -

$$100x + 600y + 300z$$

for $\begin{pmatrix} 3 \\ \cdot \end{pmatrix}$ the cost is $= 100 \cdot 3 + 600 \cdot 0 + 300 \cdot 4$

$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ $= 1500$ rs.

for $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ the cost is $= 200 + 600 + 1500$
$= 2300$ rs

for $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ the cost is $= 100 + 1200 + 1800$
$= 3100$ rs

for $\begin{pmatrix} 0 \\ 3 \\ 7 \end{pmatrix}$ the cost is $= 1800 + 2100 = 3900$ rs

$\therefore$ least expensive solution is $(3, 0, 4)$

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12 sol^n:

Let, $\underbrace{1+1+1+\cdots+1}_{n} = 0$ and

Let n is a composite number so that
$n = p.q \quad p, q \in \mathbb{Z}, p, q < n$

$$\underbrace{1+1+1+\cdots+1}_{p.q}$$

$$= \overbrace{\underbrace{(1+1+\cdots+1)}_{p \text{ times}} + \cdots + \underbrace{(1+1+\cdots+1)}_{p \text{ times}}}^{q \text{ times}}$$

$$= \underbrace{\alpha + \cdots + \alpha}_{q\text{-time}}$$

where $\alpha = \underbrace{1 + 1 + \cdots + 1}_{p \text{ times}}$ (Please note: It is important to not call this as "p")

If $\alpha = 0$ then the characterstic is $p < n$ which is a contradiction.

<u>Now</u> if $\alpha \neq 0$ then α^{-1} exist

$\because$ $\alpha \in F$ and F is closed under addition.

$$\therefore \quad 0 = \underbrace{\alpha + \alpha + \cdots + \alpha}_{q \text{ times}}$$

$$\Rightarrow \alpha^{-1} \cdot 0 = \alpha^{-1}\alpha + \alpha^{-1}\alpha + \cdots + \alpha^{-1}\alpha$$

$$\Rightarrow 0 = \underbrace{1 + 1 + \cdots + 1}_{q \text{ times}}$$

$\Rightarrow$ the characterstic is $q < n$, a contradiction

Thus, n should be prime number or '0'.

$\boxed{\text{Q.E.D}}$

15 Sol^n:-

Let, a be the additive identity if present then satisfies

$$x \oplus a = x$$
$$\Rightarrow x + a - 1 = x \quad \text{(as per } \oplus \text{ definition)}$$
$$\Rightarrow a = 1$$

Now, check

$$x \oplus 1 = x + 1 - 1 = x + 1 + (-1) = x$$
$$1 \oplus x = 1 + x - 1 = x + 1 + (-1) = x$$

Let 'm' be the multiplicative identity, if present then satisfies

$$x \otimes b = x + b - x \cdot b = x$$
$$\Rightarrow x + (-x) + b - x \cdot b = x + (-x)$$
$$\Rightarrow b(1 - x) = 0$$
$$\Rightarrow b = 0$$

Check

$$x \otimes 0 = x + 0 - x \cdot 0 = 0$$
$$0 \otimes x = 0 + x - (0) \cdot x = x.$$

Since $(\mathbb{Z}, \oplus, \otimes)$ is a commutative ring with identity. (Please Verify!)

For integral Domain check the following

$$x \otimes y = 1$$

$$\Rightarrow x + y - xy = 1$$

$$\Rightarrow x + y - xy - 1 = 0$$

$$\Rightarrow x(1-y) - 1(1-y) = 0$$

$$\Rightarrow (x-1)(1-y) = 0$$

$\Rightarrow$ Either $x-1=0$ or $y-1=0$

$x = 1$ $\Rightarrow y = 1$

both $(x-1)(y-1)=0$

$$\Rightarrow x = 1,\ y = 1$$

Thus $(Z, \oplus, \otimes)$ is an integral Domain

But $(Z, \oplus, \otimes)$ is not a field

(multiplicative inverse doesnot exist)

EE-635 Assignment No-1

$\boxed{Q26}$

Sol^n [a] Not a Vector Space.

e.g $f(n) = \frac{1}{n}$, $g(n) = -n$.

Lets take the linear Combination

$$h(n) := f(n) - g(n)$$
$$= \frac{1}{n} + n$$

& $h(n+1) > h(n)$

hence $h \notin W$

[b] Vector space,

Let $\lim_{n\to\infty} f(n) = 0$ & $\lim_{n\to\infty} g(n) = 0$

Define $h_n := \alpha f_n + \beta g_n$

$$\Rightarrow \lim_{n\to\infty} h(n) = \alpha \lim_{n\to\infty} f(n) + \beta \lim_{n\to\infty} g(n)$$

$$= 0$$

$$\Rightarrow h(n) \in W.$$

[c] vector space. Let $f_n = a_f + (n-1)d_f$, $a_f, d_f \in \mathbb{R}$

& $g_n = a_{f_1} + (n-1)d_{f_1}$ $a_{f_1}, d_{f_1} \in \mathbb{R}$

Now $\alpha f_n + \beta g_n = \alpha(a_f) + \beta a_{f_1} + (n-1)(\alpha d_f + \beta d_{f_1}) \in W.$

[d] Not a Vector Space..

Check for $\alpha f_n + \beta g_n$. with $f_n = a_f \, r_f^{(n-1)}$

& $g_n = a_{f_1} \, r_{f_1}^{(n-1)}$

Q26 (e) It is a Vector Space, $f(n)$ & $g(n) \neq 0$ for finitely mank $n \in \mathbb{N}$.

As for any $f(n)$ & $g(n) \in W$.

$f(n) + g(n) \neq 0$. for finitely many n.

& hence $f(n) + g(n) \in W$.

Q26 (f): Not a Vector Space

Let Consider

$$f(n) = 0 \quad \text{for odd } n$$
$$\neq 0 \quad \text{for even } n.$$

&

$$g(n) \neq 0 \quad \text{for odd } n$$
$$= 0 \quad \text{for even } n.$$

Now $f(n) + g(n) \neq 0$ ~~for~~ $\forall n$.

Hence it is not a vector space.

$\boxed{Q\,28}$

Sol^n

(a) Consider $f_1(x)$ & $f_2(x)$ in V_e

Now $f_1(x) = f_1(-x)$ &

$f_2(x) = f_2(-x)$

Consider $g(x) = (f_1 + \alpha f_2)(x)$

$= f_1(x) + \alpha f_2(x)$

Now $g(-x) = (f_1 + \alpha f_2)(x) \quad \alpha \in \mathbb{R}$

$= f_1(-x) + \alpha f_2(-x)$

$= f_1(x) + \alpha f_2(x)$

$= g(x)$

$\Rightarrow g(-x) = g(x) \Rightarrow g(x) \in V_e$.

Similarly for $h_1(x)$ & $h_2(x) \in V_o$

& $K(x) = (h_1 + \alpha h_2)(x) \quad (\alpha \in \mathbb{R})$

$= h_1(x) + \alpha h_2(x)$

We have $K(-x) = (h_1 + \alpha h_2)(-x)$

$= h_1(-x) + \alpha h_2(-x)$

$= -h_1(x) - \alpha h_2(x)$

$= -K(x)$

$\Rightarrow K(x) \in V_o$.

$\therefore$ Both V_o & V_e are subspace of V.

(b) Consider $f(x) \in V$

Now. $$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

$$= f_1(x) + f_2(x)$$

where $f(x) = f_1(-x)$

$\Rightarrow f(x) \in V_e$

& $f_2(x) = -f_2(-x)$

$f_2(x) \in V_o$

$V = V_e + V_o$

Now Let $f(x) \in V_e \cap V_o$

$\Rightarrow f(x) = f(-x)$ — ①

$[\because f \in V_e)$

& $f(x) = -f(-x)$ $[\because f \in V_o]$ — ②

$2f(x) = 0$ [Adding ① & ②

$\Rightarrow f(x) = 0$

Thus $V_e \cap V_o = \{0_V\}$

Hence $\boxed{V = V_o \oplus V_e}$

Q32

Soln

Given $S := \langle \{r, s\} \rangle$

$P := \langle \{p, r, s\} \rangle$

$Q := \langle \{q, r, s\} \rangle$

$q \in P \Rightarrow \exists\, a, b, c \in \mathbb{R}$

Such that $q = ap + br + cs$

Also $q \notin S \Rightarrow a \neq 0$

i.e. q has a non-zero component along p.

Now Let Us assume $p \notin Q$.

$\Rightarrow$ p can not be repersented as a linear Combination of q, r & s.

However, we Know

$p = \frac{1}{a} \cdot q - br - cs \;, \; a \neq 0$

which is a contradiction.

Hence our assumption is incorrect,

$p \in Q$.

Q. 22) (b) It is obvious that

$$W_1 + W_2 \subseteq \mathbb{F}^{2\times 2}.$$

To show that $W_1 + W_2 = \mathbb{F}^{2\times 2}$, we need to show that

$$\mathbb{F}^{2\times 2} \subseteq W_1 + W_2 .$$

Consider any matrix $M \in \mathbb{F}^{2\times 2}$ such that

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \; m_{ij} \in \mathbb{F} \; \forall i,j.$$

$$\therefore \; M = \begin{bmatrix} m_{11}' + m_{11}'' & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

where $m_{11}' + m_{12}'' = m_{11}$ for some $m_{11}', m_{11}'' \in \mathbb{F}$

$$= \underbrace{\begin{bmatrix} m_{11}' & m_{12} \\ m_{21} & 0 \end{bmatrix}}_{M_1} + \underbrace{\begin{bmatrix} m_{11}'' & 0 \\ 0 & m_{22} \end{bmatrix}}_{M_2}$$

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Clearly, $M_1 \in W_1$ & $M_2 \in W_2$.

$\therefore$ $M \in W_1 + W_2$.

$\therefore$ $\mathbb{F}^{2\times 2} \subseteq W_1 + W_2$.

Hence, $\mathbb{F}^{2\times 2} = W_1 + W_2$.

(a) Suppose $M \in W_1 \cap W_2$.

$\therefore$ $M \in W_1$

and $M \in W_2$.

Hence. $M = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}$ since $M \in W_2$ $(m_{11}, m_{22} \in \mathbb{F})$.

Additionally, $m_{22} = 0$, since $M \in W_1$.

Hence $M = m_{11} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $M \in \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle$. $\Rightarrow W_1 \cap W_2 \subseteq \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle$

Conversely, let $M \in \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle$.

$\therefore M = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ $a \in \mathbb{F}$.

Clearly, $M \in W_1$ & $M \in W_2$

$M \in W_1 \cap W_2$.

Hence, $\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle \subseteq W_1 + W_2$

Therefore, $\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle = W_1 + W_2$.

Q18. Let $\mathbb{F}$ be a field, and $\mathbb{E} \subset \mathbb{F}$ be a sub-field of it. Let the addition operation $+$ and the scalar multiplication operation $\cdot$ be defined as the same as the addition and multiplication operation defined for the elements of filed $\mathbb{F}$. Let, $u, v, w \in \mathbb{F}$ and $\alpha, \beta \in \mathbb{E}$. Note that as $\alpha, \beta \in \mathbb{E}$, it implies $\alpha, \beta \in \mathbb{F}$. Now,

- $u + v = v + u$ (due to the commutative property of field w.r.t addition)
- $(u+v)+w = u+(v+w)$ (due to the associative property of field w.r.t addition)
- $\exists\ 0 \in \mathbb{F}$ such that $v + 0 = v$ (0 is same as the additive identity for the field)
- $\exists\ -v \in \mathbb{F}$ such that $-v + v = 0$ ($-v$ is same as the additive inverse in the field)
- $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$ (due to distributive property of field)
- $(\alpha+\beta)\cdot v = v\cdot(\alpha+\beta) = v\cdot\alpha+v\cdot\beta = \alpha\cdot v+\beta\cdot v$ (using closure w.r.t addition, commutative property of field w.r.t multiplication and distributive property of field.)
- $(\alpha \cdot \beta) \cdot v = \alpha \cdot (\beta \cdot v)$ (using associative property for field w.r.t. multiplication)
- $\exists 1$ such that $1 \cdot v = v$ (1 is the same as the multiplicative identity for the sub-field $\mathbb{E}$.)

Hence, field is a vector space over its sub-fields.

Q17. Sol^n :-

Construct a family of fields such that -

$$\mathbb{Q} \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \mathbb{F}_3 \dots \subset \mathbb{R}$$

Let us say $p_1, p_2, p_3 \dots$ be the increasing sequence of the prime numbers.

Define:

$$F_1 := \mathbb{Q}(\sqrt{p_1})$$ and define the sequence as

$$F_i := F_{i-1}(\sqrt{p_i}) \; \forall i > 1$$

e.g.:

Now think for such

$$F_1 = \mathbb{Q}(\sqrt{2}) = a + b\sqrt{2} \quad a, b \in \mathbb{Q}$$

$$F_2 = \mathbb{Q}(\sqrt{2}, \sqrt{3}) = a + b\sqrt{2} + c\sqrt{3}$$
$a, b, c \in \mathbb{Q}$

First note $\mathbb{F}_i$'s are subset of $\mathbb{R}$

Need to show

$F_{i-1} \subset F_i$ and F_i's are subfield of $\mathbb{R}$

Proof by induction:-

Define $F_0 = \mathbb{Q}$

and $F_i = F_{i-1}(\sqrt{p_i}) \; \forall i \geq 1$

Base case:-

$F_0 = \mathbb{Q} \subset \mathbb{R}$ is a subfield of $\mathbb{R}$

Inductive steps:-

For a given F_{i-1} is a subfield of $\mathbb{R}$ then need to show F_i is a subfield of $\mathbb{R}$ and also $F_{i-1} \subset F_i$

$\because \sqrt{p_i} \notin F_{i-1}$ (why ?)

but $\sqrt{p_i} \in F_i$ (see the example)

$\Rightarrow F_{i-1} \subset F_i$ (It is a subset)

Now to show F_i is a subfield

<u>closure:-</u>

$x, y \in F_i$

$(\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{Q})$

so, $x = \alpha_1 + \beta_1 \sqrt{p_i}$

$y = \alpha_2 + \beta_2 \sqrt{p_i}$

$$x+y = (\alpha_1+\alpha_2) + (\beta_1+\beta_2)\sqrt{p_i}$$

$$= \alpha + \beta\sqrt{p_i} \in F_i$$

$(\alpha, \beta \in \mathbb{Q}$ as $\mathbb{Q}$ is subfield $\mathbb{R})$

<u>multiplication:-</u>

$x, y \in F_i$

$$x = \alpha_1 + \beta_1\sqrt{p_i}$$

$$y = \alpha_2 + \beta_2\sqrt{p_i}$$

$(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Q})$

$$x \cdot y = (\alpha_1\alpha_2 + \beta_1\beta_2 p_i) + (\alpha_1\beta_2 + \beta_1\alpha_2)\sqrt{p_i}$$

$$= \alpha + \beta\sqrt{p_i} \in F_i$$

$(\alpha, \beta \in \mathbb{Q})$ (as $p_i \in \mathbb{Q}$)

why ?

Additive inverse:

$x \in F_i \qquad x = \alpha + \beta\sqrt{p_i}$

$(\alpha, \beta \in \mathbb{Q})$

then $\tilde{x} = -\alpha + (-\beta)\sqrt{p_i}$

s.t $\quad x + \tilde{x} = 0.$

so additive inverse exist $(-\alpha - \beta(\sqrt{p_i}))$

multiplicative inverse:

Let $x \in F_i \qquad x = \alpha + \beta\sqrt{p_i}$

$(\alpha, \beta \in \mathbb{Q})$

$x \cdot \tilde{x} = 1$

$$\Rightarrow \tilde{x} = \frac{1}{\alpha + \beta\sqrt{p_i}}$$

$$= \frac{\alpha - \beta\sqrt{p_i}}{\alpha^2 - \beta^2 p_i}$$

$$= \frac{\alpha}{\alpha^2 - \beta^2 p_i} + \left(- \frac{\beta\sqrt{p_i}}{\alpha^2 - \beta^2 p_i} \right)$$

$$= \gamma_1 + \left(\gamma_2 \sqrt{p_i} \right)$$

$$(\gamma_1, \gamma_2 \in \mathbb{Q})$$

$\therefore$ F_i is a subfield of $\mathbb{R}$

$\therefore$ $\mathbb{Q} \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \cdots \subset \mathbb{R}$

Next

$\mathbb{Q} \subset F \subset \mathbb{C}$

take this

$$F = \mathbb{Q}(\alpha + \beta\sqrt{-p_i})$$

is subfield of $\mathbb{C}$

(please verify as above)

For e.g:

$$F = a + b\sqrt{-2}$$
$$= a + ib\sqrt{2}$$
$$(a, b \in \mathbb{Q})$$