# Applied Linear Algebra

***(Course Code: EE 635)***

Department of Electrical Engineering
Indian Institute of Technology Bombay

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| Instructor: Dwaipayan Mukherjee |
| Date: 30th August, 2023 |

Time: **1.25 Hours**

## Quiz-1

Total Points: 40

## Instructions

- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

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1. Suppose $S$ is a set of matrices of the form $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$, where $a \in \mathbb{Q}$ (set of rational numbers). It is known that an integral domain has the property that *product of two non-zero elements is non-zero.*

   (a) Does the aforesaid property hold for $S$? Justify your answer.

   (b) Determine the multiplicative identity of $S$, if it exists.

   (c) Does every element is $S$ have a multiplicative inverse? Obtain the same for those elements that may have such an inverse.

   [2+1.5+1.5]

2. Provide an example of each of the following:

   (a) A non-empty subset of $\mathbb{R}^2$, say $\mathbb{V} \subseteq \mathbb{R}^2$, which is closed under vector addition and under taking additive inverse, but is not a subspace of $\mathbb{R}^2$.

   (b) A vector space that has exactly 81 elements.

   (c) A non-empty subset of $\mathbb{R}^2$, say $\mathbb{U} \subseteq \mathbb{R}^2$, which is closed under scalar multiplication, but is not a subspace of $\mathbb{R}^2$.

   [$2 \times 3$]

3. Suppose $X = \mathbb{Z}^2 := \{(x, y) : x, y \in \mathbb{Z}\}$. Define addition, $\oplus$, and multiplication, $\otimes$ in the following manner:
   $(x, y) \oplus (a, b) = (x + a, y + b)$, and
   $(x, y) \otimes (a, b) = (ax + 2by, bx + ay)$.

(a) Assert through proper justification whether $(X, \oplus, \otimes)$ is an integral domain. Hence, obtain the '0' and '1' of this integral domain (or even if it is not an integral domain).

(b) Solve for an element $p = (p_x, p_y) \in X$ such that $p^2 = 2$ (if such a $p$ exists). [Hint: For the '1' obtained in (a), define $n := \underbrace{1 \oplus 1 \oplus \ldots \oplus 1}_{n \text{ times}}$]

[4+5]

4. Prove or disprove the following assertion: For a matrix $A \in \mathbb{R}^{n\times n}$, it follows that $\ker(A) \cap \mathrm{im}(A) = 0_n$.

[6]

5. Obtain all solutions of $x^2 - 10x + 16 = 0$ over $\mathbb{Z}_2$, and $\mathbb{Z}_8$. [4]

6. (*Traffic flow*) Consider the traffic flow depicted in the map shown in Fig. 1 over a

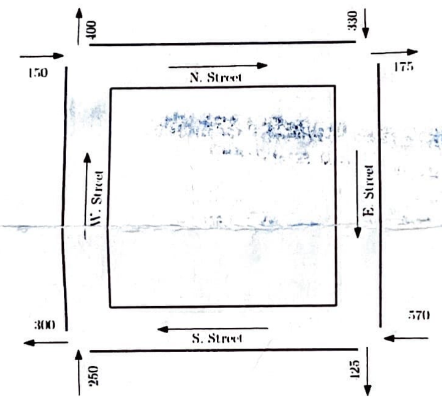


Figure 1: Traffic flow over a 1-hour period

one-hour open window while there is no flow of traffic during the remainder of the day. It is known that the streets are one-way, there are no residences in the map, and the entire area is a 'No Parking' zone.

(a) Does there exist a solution to the problem of determining the number of vehicles passing through each of the four main roads? Justify your answer mathematically. If it is solvable, is the solution unique? In any case outline possible solution(s).

(b) Suppose the number of vehicles entering the north-east junction were increased by 50 and the number of vehicles leaving through the south-west junction increased by 30. Can you solve for the problem in (a)? Justify.

(c) For the problem in (a), does a solution exist if the total number of vehicles through W. Street was 200 during the one-hour period? Justify.

[5+3+2]



End of the question paper.

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| Time: **1.25 Hours** | **Applied Linear Algebra**<br>***(Course Code: EE 635)***<br>Department of Electrical Engineering<br>Indian Institute of Technology Bombay<br>**Mid-semester Examination** | Instructor: Dwaipayan Mukherjee<br>Date: 18th September, 2023<br>Total Points: 50 |

## Instructions

- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

1. Prove or disprove (possibly through some counter-example) the following:

   (a) For two subspaces $\mathbb{U}$ and $\mathbb{W}$ of the vector space $\mathbb{V}$, if every vector belonging to $\mathbb{V}$ either belong to $\mathbb{U}$ or to $\mathbb{W}$ (or both), then either $\mathbb{V} = \mathbb{U}$, or $\mathbb{V} = \mathbb{W}$ (or both).

   (b) Every subfield of the complex numbers must contain every rational number.

   (c) Suppose $v_1, v_2 \in \mathbb{V}$ (a vector space). Further, suppose $\mathbb{U}$ and $\mathbb{W}$ are subspaces of $\mathbb{V}$. If $v_1 + \mathbb{U} = v_2 + \mathbb{W}$, then $\mathbb{U} = \mathbb{W}$.

   (d) If $\mathbb{U}_1, \mathbb{U}_2$, and $\mathbb{W}$ are subspaces of the vector space $\mathbb{V}$ such that $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{W}$ and $\mathbb{V} = \mathbb{U}_2 \oplus \mathbb{W}$, then $\mathbb{U}_1 = \mathbb{U}_2$.

   [3 × 4]

2. Give an example of each of the following

   (a) A mapping $T : \mathbb{R}^2 \to \mathbb{R}$ such that $T(\alpha v) = \alpha T(v)$ for $\alpha \in \mathbb{R}, v \in \mathbb{R}^2$, but $T$ is not linear.

   (b) A mapping $T : \mathbb{C} \to \mathbb{C}$ such that $T(v_1 + v_2) = T(v_1) + T(v_2)$ for $v_1, v_2 \in \mathbb{C}$, but $T$ is not linear.

   [3 × 2]

3. For a linear transformation $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{U})$, show that

   (a) $\dim(\text{im}(\varphi'))=\dim(\text{im}(\varphi))$, and

   (b) $\text{im}(\varphi')=(\ker(\varphi))^0$

[4+5]

4. For vector spaces $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$, of which $\mathbb{U}$, and $\mathbb{V}$ are finite dimensional, consider two linear transformations $\tau : \mathbb{U} \to \mathbb{V}$ and $\psi : \mathbb{V} \to \mathbb{W}$. Suppose the composition of the two transformations is given by $\psi \circ \tau$. Establish that $\dim(\mathrm{Ker}(\psi \circ \tau)) \leq \dim(\mathrm{Ker}(\tau)) + \dim(\mathrm{Ker}(\psi))$. When does equality hold? What can you say about the inequality when $\tau$ is surjective?

[4+1+2]

5. Let $\mathbb{V} = \mathbb{R}[x]_2$, the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$, that have degree 2 or less, given by $p(x) = c_2x^2 + c_1x + c_0$. Define $f_1(p) = \int_0^1 p(x)dx$, $f_2(p) = \int_0^2 p(x)dx$, and $f_3(p) = \int_0^{-1} p(x)dx$. Show that $\{f_1, f_2, f_3\}$ is a basis for $\mathbb{V}'$. Obtain the basis for $\mathbb{V}$ to which $\{f_1, f_2, f_3\}$ is a dual.

[3+3]

6. (a) Suppose $\mathbb{U} \subseteq \mathbb{V}$ and $\{v_1 + \mathbb{U}, v_2 + \mathbb{U}, \ldots, v_m + \mathbb{U}\}$ is a basis for $\mathbb{V}/\mathbb{U}$, while $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{U}$. Show that $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{V}$.

(b) Verify the first isomorphism theorem (you are not allowed to use it/invoke it at any point!) for the case when $\mathbb{V}$ is the space of all polynomials in $x$ over the real field, of degree less than or equal to $n$, $\mathbb{W} = \mathbb{V}$, and the linear map $T : \mathbb{V} \to \mathbb{W}$ is the derivative map. Then provide an interpretation for what each element in $\mathbb{V}/\mathrm{Ker}(T)$ represents. In other words, you are required to define the induced map corresponding to the derivative map and show that this induced map is linear, injective, has an image identical with the original derivative map, and finally, $\mathbb{V}/\mathrm{Ker}(T) \cong \mathrm{im}(T)$. Thereafter, you need to describe what each element in $\mathbb{V}/\mathrm{Ker}(T)$ represents.

[5+5]

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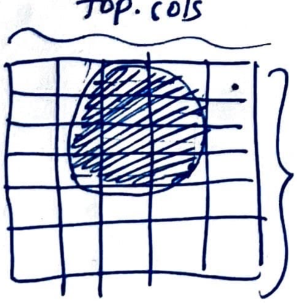
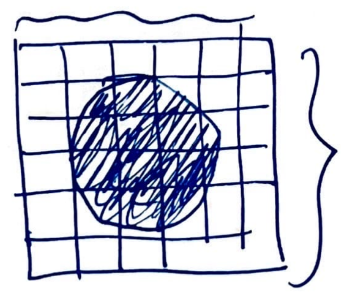


top.rows



$H_s$ = side.rows

top

Side.

$L_S^1$: # of food pixel in 1st row

$L_S^k$: # of food pixel in kth row

foreground pixel

background pixel.

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| | **Applied Linear Algebra**<br>***(Course Code: EE 635)***<br>Department of Electrical Engineering<br>Indian Institute of Technology Bombay | Instructor: Dwaipayan Mukherjee<br>Date: 18th October, 2023 |

Time: **1 hour and 15 minutes** **Quiz-2** Total Points: 40

## Instructions

- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

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1. Prove or disprove (possibly through some counter-example) the following:

   (a) For two subspaces, $\mathbb{W}_1$ and $\mathbb{W}_2$, of an inner product space, we have $(\mathbb{W}_1 + \mathbb{W}_2)^\perp = \mathbb{W}_1^\perp \cap \mathbb{W}_2^\perp$.

   (b) For an inner product space $\mathbb{V}$, two vectors $v_1, v_2 \in \mathbb{V}$ are equal if and only if $\langle v_1|v\rangle = \langle v_2|v\rangle$ for every $v \in \mathbb{V}$.

   (c) If $v_1, v_2 \in \mathbb{V}$ (an inner product space) are orthogonal, then the vectors $v_1 + v_2$ and $v_1 - v_2$ cannot be orthogonal.

   (d) If $\Xi_1$ and $\Xi_2$ are non-empty subsets of an inner product space $\mathcal{V}$ such that $\Xi_1 \subseteq \Xi_2$, then $\Xi_1^\perp \subseteq \Xi_2^\perp$.

   (e) Let $\{v_1, v_2, \ldots, v_m\}$ be a linearly independent set of vectors in a real inner product space $\mathbb{V}$. There exist exactly $2^m$ orthonormal sets of vectors of the form $\{w_1, w_2, \ldots, w_m\}$ in $\mathbb{V}$ such that span($\{v_1, v_2, \ldots, v_i\}$)=span($\{w_1, w_2, \ldots, w_i\}$) for all $i \in \{1, 2, \ldots, m\}$.

   [2.5× 5]

2. (a) For $v_1, v_2 \in \mathbb{V}$ (an inner product space), show that $\langle v_1|v_2\rangle = 0$ if and only if $||v_1|| \leq ||v_1 + \alpha v_2|| \; \forall \alpha \in \mathbb{C}$.

   (b) Suppose $\Pi : \mathbb{V} \to \mathbb{V}$ is an idempotent linear operator on the finite dimensional vector space $\mathbb{V}$, and satisfies the condition: $||\Pi v|| \leq ||v|| \; \forall v \in \mathbb{V}$. Show that there exists a subspace $\mathbb{U} \subseteq \mathbb{V}$, such that $\Pi$ is an orthogonal projection from $\mathbb{V}$ onto $\mathbb{U}$.

   [3.5+4]

3. (a) Show that a linear operator, $\phi$, on an inner product space, $\mathbb{V}$ (over $\mathbb{R}$ or $\mathbb{C}$), is identically the 'zero' operator if and only if $\langle \phi(v_1)|v_2\rangle = 0$ for all $v_1, v_2 \in \mathbb{V}$.

(b) Suppose $v_1$ and $v_2$ belong to an inner product space, $\mathbb{V}$ (over $\mathbb{C}$), and $\alpha, \beta \in \mathbb{C}$, while $\phi : \mathbb{V} \to \mathbb{V}$ is a linear operator. Show that $\alpha\bar{\beta}\langle\phi(v_1)|v_2\rangle + \bar{\alpha}\beta\langle\phi(v_2)|v_1\rangle = \langle\phi(\alpha v_1 + \beta v_2)|\alpha v_1 + \beta v_2\rangle - |\alpha|^2\langle\phi(v_1)|v_1\rangle - |\beta|^2\langle\phi(v_2)|v_2\rangle$.

(c) Show that for an inner product space, $\mathbb{V}$ over $\mathbb{C}$, a necessary and sufficient condition for a linear operator, $\phi$, to be the 'zero' operator is that $\langle\phi(v)|v\rangle = 0$ for all $v \in \mathbb{V}$

(d) Provide an example of a non-zero linear operator, $\phi$, on an inner product space, $\mathbb{V}$ over $\mathbb{R}$, such that $\langle\phi(v)|v\rangle = 0$ for all $v \in \mathbb{V}$.

(e) Show that for an inner product space, $\mathbb{V}$ over $\mathbb{R}$, a necessary and sufficient condition for a linear, self-adjoint operator, $\phi$, to be the 'zero' operator is that $\langle\phi(v)|v\rangle = 0$ for all $v \in \mathbb{V}$

[2+2+5+2+5]

4. Suppose $\mathcal{P} : \mathbb{V} \to \mathbb{U}$ is an orthogonal projection onto a subspace $\mathbb{U}$ of $\mathbb{V}$, where $\mathbb{V}$ is a finite dimensional inner product space. Prove that $\mathcal{P}$ is self-adjoint. [4]

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| | **Applied Linear Algebra**<br>***(Course Code: EE 635)***<br>Department of Electrical Engineering<br>Indian Institute of Technology Bombay | Instructor: Dwaipayan Mukherjee<br>Date: 5th November, 2023 |
| Time: **1 hour and 15 minutes** | **Quiz-3** | Total Points: 40 |

## Instructions

- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

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1. Prove or disprove (possibly through some counter-example) the following:

   (a) For the Jordan form representation of a nilpotent matrix, $N \in \mathbb{C}^{n\times n}$, the number of Jordan blocks of size $i \times i$ or greater is equal to $\text{rank}(N^{i-1})-\text{rank}(N^i)$.

   (b) If $A \in \mathbb{C}^{n\times n}$ has a set of $n$ orthonormal eigenvectors, it must be a Hermitian matrix.

   (c) For $A \in \mathbb{R}^{n\times n}$, suppose $\{w_1, w_2, \ldots, w_n\}$ is a set of $n$ linearly independent eigenvectors and for $w = \sum_{k=1}^{k=n} kw_k$, we have $Aw = \sum_{k=1}^{k=n}(k^2+k)w_k$. Then the minimal polynomial, $\mu_A(x)$, and characteristic polynomial, $\chi_A(x)$, of $A$ are identical.

   (d) For a linear operator $A : \mathbb{V} \to \mathbb{V}$ on a finite dimensional vector space, a positive integer $m$, and a vector $v \in \mathbb{V}$, suppose $A^{m-1}v \neq 0$, but $A^m v = 0$. Then $\{v, Av, A^2v, \ldots, A^{m-1}v\}$ must be linearly independent.

[3× 4]

2. (a) The *spectral* norm of a matrix, $A \in \mathbb{R}^{m\times n}$, is defined as $||A||_2 := \max_{x\neq 0} \frac{||Ax||_2}{||x||_2}$, where $x \in \mathbb{R}^n$ and the vector norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ result from the conventional inner products on those spaces. If the non-zero singular values of $A$ are given by $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ ($r \leq \min(m,n)$), then prove that $||A||_2 = \sigma_1$.

   (b) The *Frobenius* norm of a matrix $A \in \mathbb{R}^{m\times n}$ is given by $||A||_F := \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}$. If the non-zero singular values of $A$ are given by $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ ($r \leq \min(m,n)$), then prove that $||A||_F = (\sigma_1^2+\sigma_2^2+\ldots\sigma_r^2)^{\frac{1}{2}}$.

[5+5]

3. Provide an example for each of the following or explain clearly why no such example can be provided, if such is the case.

   (a) Two matrices $A, B \in \mathbb{R}^{7\times 7}$ are such that $\chi_A(x) = \chi_B(x)$, $\mu_A(x) = \mu_B(x)$, and $A$ and $B$ have the same eigenvalues with same algebraic and geometric multiplicities for each eigenvalue, but there exists no $T \in \mathbb{R}^{7\times 7}$ such that $B = T^{-1}AT$.

   (b) A lower/upper triangular matrix, $A \in \mathbb{C}^{n\times n}$, satisfying $A^H A = AA^H$, which is not a diagonal matrix.

[4 × 2]

4. (a) For matrices $A, B \in \mathbb{C}^{n\times n}$, show that $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ represent the same operator under different choices of bases. Hence, show that $\chi_{AB}(x) = \chi_{BA}(x)$.

   (b) (*Rayleigh quotient*) Suppose the eigenvalues of a Hermitian matrix, $A \in \mathbb{C}^{n\times n}$, are given by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Show that $\lambda_1 = \max_{||x||_2=1} x^H Ax$ and $\lambda_n = \min_{||x||_2=1} x^H Ax$.

[5+5]

| Applied Linear Algebra | Instructor: Dwaipayan Mukherjee |
|---|---|
| *(Course Code: EE 635)* | Date: 24th November, 2023 |

Department of Electrical Engineering
Indian Institute of Technology Bombay

Time: **3.00 Hours** **End-semester Examination** Total Points: 90

### Instructions

$\begin{bmatrix} 0.01 & -0.02 \\ -0.02 & 0.01 \end{bmatrix}$ $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

1. Prove or disprove (possibly through some counter-example) the following:

   0 (a) Components of the eigenvector for a matrix $A \in \mathbb{C}^{n\times n}$ vary continuously with the entries of $A$.

   4 (b) If $(\lambda_k, v_k)$ is an eigenvalue-eigenvector pair for $A \in \mathbb{R}^{n\times n}$, then $\dfrac{v_k}{\lambda_k - \lambda} = (A - \lambda I)^{-1} v_k$, where $\lambda$ is not an eigenvalue of $A$.

   0 (c) The non-zero matrix $A = uv^T$ for $u, v \in \mathbb{R}^n$ is diagonalizable if and only if $v^T u \neq 0$.

   3 (d) For $A \in \mathbb{R}^{n\times n}$, $\det(e^A) = e^{\text{trace}(A)}$.

   0 (e) If $A \in \mathbb{C}^{n\times n}$ is a diagonalizable matrix with characteristic polynomial $\chi_A(x) = (x-1)^{k_1}(x+1)^{k_2}x^{k_3}$, then the rank of $A$ can be increased by adding or subtracting the identity matrix to it.

   3 (f) Suppose $A \in \mathbb{R}^{n\times n}$ has a characteristic polynomial given by $\chi_A(x) = (x-\lambda_1)^{k_1}(x-\lambda_2)^{k_2}$ and $\text{rank}(A - \lambda_1 I) = n - k_1$. Then $A$ must be diagonalizable.

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

[3 × 6]

2. A common way of encoding messages is to associate distinct positive integers with each letter of the alphabet. For instance, a very basic encoding would be to choose $A \to 1, B \to 2, C \to 3, \ldots, Z \to 26$. However, this can be easily cracked if it falls into the hands of an adversary. To add a level of security, the messenger and receiver both agree upon a predetermined invertible matrix, say $A \in \mathbb{R}^{n\times n}$. Thereafter, the sender splits up the message (without spaces!) into chunks of $n$ consecutive letters in the message leading to vectors in $\mathbb{R}^n$. If the total number of letters is not a multiple of $n$, the remaining spaces are filled by a place-holder number such as 27. Thereafter,

18

the matrix acts on each of the vectors and yields transformed vectors as the output, which are then sequentially transmitted to the receiver. The receiver then decodes this message by passing each of these transformed vectors it received through $A^{-1}$. Then the message is read by mapping the numbers to their corresponding letters.

Example: Suppose the message is 'I SEE' and the matrix is $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ (with $n = 2$). Then the sender, using an encoder machine, first converts the message into $\{9, 19, 5, 5\}$, splits it up into two vectors $v_1 = [9\ 19]^T$, $v_2 = [5\ 5]^T$, operates on $v_1$ and $v_2$ using $A$ to get $y_1 = Av_1 = [-10\ 37]^T$ and $y_2 = Av_2 = [0\ 15]^T$, and then transmits $y_1$ and $y_2$ sequentially to the receiver. The receiver then recovers $v_1 = A^{-1}y_1$ and $v_2 = A^{-1}y_2$ using a decoder, which then reads the message by replacing the numbers with suitable letters.

Suppose it is known that an adversary uses this form of coding with $n = 3$ but the matrix is unknown to you. However, your spies have managed to steal a model of this encoder machine used by the sender, and you can now experiment with this machine (without ripping it open!) to find out what this matrix is. You decide to send the message 'LET ME TRY' (without spaces, of course, and using 27 as the place-holder), and obtain a series of numbers at the output.

$(1-\lambda)(2-\lambda)(3-\lambda)$

(a) Determine the matrix $A \in \mathbb{R}^{3\times 3}$ if the encoder gives an output sequence $\{17, -8, 37, 18, -7, 38, 43, -9, 70\}$ for the chosen message. Give an example of a message (not necessarily meaningful words), using which you will certainly not be able to determine the matrix $A$.

(b) Suppose your friend playfully decides to send you a message using this encoder, which you read as the sequence of numbers given by $\{21, 0, 30, 16, -1, 28, 23, 4, 28, 19, 0, 20, 19, 7, 24, 20, 1, 21\}$ at the output of the encoder. What is your friend trying to tell you? [9 + 9]

3. For each of the following short answers, show appropriate calculations/steps/justifications.

3 (a) If $u, v, w$ are orthonormal vectors in an inner product space, $\mathbb{V}$, then what is the value of $||u+v||^2 + ||v+w||^2 + ||w+u||^2$?

$(3-\lambda)\left[(1-\lambda)^2 - 4\right] = 0$

3 (b) For $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \in \mathbb{Z}_5$, what are the eigenvalues and their corresponding algebraic multiplicities?

5 mod 5 = 0

$-1$, 3, 3   4   mod 5   2, 2, 4.   $\boxed{\lambda = 3}$

$(1-\lambda)^2 = 4 = 2^2$

$1-\lambda = \pm 2$

$\lambda = 3$

$\lambda = -1$

3 (c) Which of the following matrices, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, is/are diagonalizable over $\mathbb{Z}_2$?

3 (d) For a diagonalizable matrix, $A \in \mathbb{C}^{9\times 9}$, with characteristic polynomial given by $\chi(x) = (x-1)^3(x+1)^2(x+2)^4$, what is the rank of $A^T + I$?

$R_2 \to R_2 \times 3 - R_1$   $3+3+4 = 6+4 = 10$ mod 5 $= 0$

0 (e) What is the value of $t$ so that the set $\left\{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ t \end{bmatrix}\right\}$ is linearly dependent in $\mathbb{R}^3$?

$\times 3$

3 2 5
0 $-11$ 13
0 7 $(3t-20)$

$\frac{6}{5} = (5\times 1) + 1$

$\frac{7}{5} = (5\times 1) + 2$

$\frac{-1}{5} = (5\times -1) + 4 = 4$

$\frac{4}{5} = 5\times 1 - 1$

$-1 = 5\times(-1) + 4$

0 1 $-1$

3 2 5
0 $-11$ 22
0 0 $(3t-20)\times 11 + 7\times 22 = 0$

$3t = 44 + 20$

$= 26$

$t =$

0 (f) Suppose $A$ is a self-adjoint operator on a finite dimensional inner product space over $\mathbb{C}$ and has only 2 and 3 (multiplicities unknown) as its eigenvalues. What is the minimal polynomial of $A$?

[3 × 6]

4. (*Competition vs cooperation*) Consider evolution of variables in continuous time for this problem

18 (a) Consider two species, say $\mathcal{P}$ and $\mathcal{Q}$, in an environment competing for the same resources. Thus, each species' population increases in proportion to its current population and decreases in proportion to that of its competitor. Suppose the constants of proportionality for both $\mathcal{P}$ and $\mathcal{Q}$ are 0.02 (for increase) and $-0.01$ (for decrease), respectively, while their initial populations are 10,000 and 20,000, respectively. Obtain the expressions for the population evolution of the two species. Does any of the two species go extinct within some finite time? If so, which one?

(b) Suppose two symbiotic species, say $\mathcal{R}$ and $\mathcal{S}$, inhabit the same environment so that each species' population increases in proportion to the current population of its symbiotic neighbour, while it decreases in proportion to its own current population. Further, suppose the two constants of proportionality (for both increase and decrease, respectively) are 0.01 and $-0.01$ for both the species. If the initial populations of $\mathcal{R}$ and $\mathcal{S}$ are 10.000 and 20,000, respectively, obtain the expressions for the population evolution of the two species. What can you say about the long run trend of either population?

[9+9]

5. 0 (a) For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, establish that $\text{rank}(AB) = \text{rank}(B) - \dim(\text{Ker}(A) \cap \text{Im}(B))$.

0 (b) For the above matrices, show that $\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB)$.

3 (c) Establish that $\text{Im}(C^TC) = \text{Im}(C^T)$ and $\text{Ker}(C^TC) = \text{Ker}(C)$ for $C \in \mathbb{R}^{m\times n}$.

[6 × 3]

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