Q1. Sol^n :-

Given, $T(x_1, x_2\, x_3) = (3x_1,\ x_1 - x_2\ \ 2x_1 + x_2 + x_3)$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longrightarrow Tx$$

From given map, the matrix representation can be given as –

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}$$

Now, T is invertible matrix transformation is invertible.

i.e

$$\left[\begin{array}{ccc:ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\Downarrow \; \begin{array}{l} R_1 \to R_1/3 \\ R_2 \to 3R_2 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & -3 & 0 & -1 & 3 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\Downarrow \; \begin{array}{l} R_2 \to -R_2/3 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & -1 & 0 \\ 0 & 1 & 1 & -2/3 & 0 & 1 \end{array}\right]$$

$$\Big\Downarrow \; R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

RRE of T is identity $\Rightarrow$ It is invertible

and
$$T^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$TT^{-1} = I_3$$

$\therefore$ Inverse transformation as : $x \to T^{-1}x$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T^{-1}(x_1, x_2, x_3) = \left(\frac{x_1}{3}, \frac{x_1}{3} - x_2, -x_1 + x_2 + x_3\right)$$

To show that

$$(- 2T)(T - 3T) = 0$$

Find $T^2 - I = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 2 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}$$

$$T - 3I = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$\therefore (T^2 - I)(T - 3I) = \begin{bmatrix} 8 & 0 & 0 \\ 2 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

QED

Q.4 Sol^n :-

For the map φ, applying Rank-Nullity theorem —

$$\dim(\mathbb{F}^5) = \dim(\ker \varphi) + \dim(\operatorname{im} \varphi)$$

$$\Rightarrow \dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi)) = 5 \longrightarrow (1)$$

Given, $\ker(\varphi) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = \alpha x_2 ; x_3 = x_4 = x_5\}$ $(\alpha \in \mathbb{F})$

So $\ker(\varphi) = \left\langle \begin{pmatrix} \alpha \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

$\therefore \dim(\ker(\varphi)) = 2$

$\therefore \dim(\operatorname{im}(\varphi)) = 3$ [from (i)]

$\Rightarrow \dim(\operatorname{im}(\varphi)) = 3$ [from (1)]

But $\operatorname{im}(\varphi) \subseteq \mathbb{F}^2$ &

$\dim(\mathbb{F}^2) = 2$

Hence, Such map doesnot exist.

Q6. Sol^n:-

Let $f(x) \in \ker(L_1)$

So, $L_1(f(x)) = 0$

$\Rightarrow \frac{d}{dx} f(x) + \alpha f(x) = 0$

$$\Rightarrow \frac{df(x)}{f(x)} = -\alpha\, dx$$

$$\Rightarrow \ln f(x) = -\alpha x + c_0$$

Let at $f(0) = C$ $\quad c_0 = \ln C$; $C \in \mathbb{R}$

so $\quad f(x) = C e^{-\alpha x}$

$\therefore$ A basis for $\ker(\alpha_1) = \{ e^{-\alpha x} \}$

again, let, $f(x) \in \ker(\alpha_2)$

$$\therefore \quad \alpha_2(f(x)) = 0$$

$$\Rightarrow \frac{d^2}{dx^2}(f(x)) + \omega^2 f(x) = 0$$

$$\Rightarrow f''(x) + \omega^2 f(x) = 0$$

$$f(x) = C_1 \sin(\omega x) + c_2 \cos(\omega x)$$

$\Rightarrow$

A basis for ker (L_2) is

$$\{\sin \omega x \quad \cos \omega x\}$$

$$\dim(\ker(L_1)) = 1$$

$$\dim(\ker(L_2)) = 2$$

Q7.

We have $\varphi \in \mathcal{L}(\mathbb{V},\mathbb{W})$ and $\varphi^{'} \in \mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'})$. let the dimension of $\mathbb{V} = m$ and dimension of $\mathbb{W} = n$. We have dimension of $\mathcal{L}(\mathbb{V},\mathbb{W}) = mn$. Also, the dimension of $\mathbb{V}^{'} =$ dimension of $\mathbb{V} = m$ and dimension of $\mathbb{W}^{'}$ =dimension of $\mathbb{W} = n$. Hence, dimension of $\mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'}) = mn$. Consider an ordered basis for $\mathcal{L}(\mathbb{V},\mathbb{W})$ and $\mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'})$ to be $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Let S be a linear map that takes φ to $\varphi^{'}$, i.e., $S : \mathcal{L}(\mathbb{V},\mathbb{W}) \to \mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'})$ and $S(\varphi_i) = \varphi_i^{'}$, $\forall i$ where $\varphi_i \in \mathcal{B}_1$, and $\varphi_i^{'} \in \mathcal{B}_2$. Now, let $\varphi \in \mathcal{L}(\mathbb{V},\mathbb{W})$ such that $S(\varphi) = 0 \in \mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'}) \implies \exists \alpha_i$, suchthat $S(\sum_i \alpha_i\varphi_i) = 0 \implies \sum_i \alpha_i S(\varphi_i) = 0 \implies \sum_i \alpha_i \varphi_i^{'} = 0$. As $\{\varphi_i\} = \mathcal{B}_2$ is an independent set, $\alpha_i = 0,\ \forall i \implies \varphi = 0$, and hence the map S is one to one. Let an arbitrary $\varphi^{'} \in \mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'})$. Hence, $\varphi^{'} = \sum_i \beta_i\varphi_i^{'} = \sum_i \beta_i S(\varphi_i) = \sum_i S(\beta_i\varphi_i) = S(\varphi)$ with $\varphi = \sum_i \beta_i\varphi_i$. Hence, for all $\varphi_i \in \mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'})$ we have an $\varphi \in \mathcal{L}(\mathbb{V},\mathbb{W})$ such that $S(\varphi) = \varphi^{'}$. Hence, S is onto, and S is an isomorphism of $\mathcal{L}(\mathbb{V},\mathbb{W})$ onto $\mathcal{L}(\mathbb{W}^{'},\mathbb{V}^{'})$.

Q12.

a) Given $\wedge : V \to V^{"}$
Consider $v_1, v_2 \in V$. So, $\wedge(\alpha v_1 + v_2)\varphi$ (for $\alpha \in \mathbb{F}$)
$= \varphi(\alpha v_1 + v_2)$ (using definition of $\wedge$)
$= \varphi(\alpha v_1) + \varphi(v_2) = \alpha\varphi(v_1) + \varphi(v_2) = \alpha \wedge (v_1) + \wedge(v_2)$.
Hence, $\wedge$ is a linear map.

b) Consider an arbitrary $v \in \mathbb{V}$ & $\varphi \in \mathbb{V}$‘.
Now $((\tau'' o \wedge)(v))(\varphi) = (\tau''(\wedge v))(\varphi)$
$= (\wedge v)(\tau'\varphi)$ (using definition of dual)
$= (\tau'\varphi)(v)$ (using $\wedge$ definition)
$= \varphi(\tau v)$ (again using definition of dual).
Also, $((\wedge o \tau)(v))(\varphi) = (\wedge(\tau v))(\varphi) = \varphi(\tau v)$ which same as obtain above for any arbitrary v and φ. Hence, $\tau'' o \wedge = \wedge o \tau$.

c) To show that: if $\mathbb{V}$ is finite dimensional, $\wedge$ is an isomorphism of $\mathbb{V}$ onto $\mathbb{V}''$.
First notice that $v \neq 0$ then $(\wedge v)\varphi = \varphi(v) \neq 0$ $\{\because$ if $v \in Null(\wedge) \implies \varphi(v) = 0\ \forall \varphi in \mathbb{V}'$ but $\implies v = 0\}$
Thus, $\wedge$ is non-singular. Also, $dim(\mathbb{V}'') = dim(\mathbb{V}') = dim(\mathbb{V})$. Hence the transformation is invertible and therefore it is isomorphism.

8) Show that $U^\circ + W^\circ \subseteq (U \cap W)^\circ$

Let $g \in U^\circ + W^\circ \Rightarrow g_1 \in U^\circ$ and $g_2 \in W^\circ$ such that

$$g = g_1 + g_2$$

consider any $x \in U \cap W$

$$\Rightarrow g(x) = g_1(x) + g_2(x) = 0$$

$$\Rightarrow g \in (U \cap W)^\circ$$

$$\therefore \quad U^\circ + W^\circ \subseteq (U \cap W)^\circ$$

Since $U^\circ + W^\circ \subseteq (U \cap W)^\circ$

$$\dim (U^\circ + W^\circ) \leq \dim (U \cap W)^\circ \longrightarrow ①$$

But $\dim (U^\circ + W^\circ) = \dim U^\circ + \dim W^\circ - \dim (U^\circ \cap W^\circ)$

we have $\dim U^\circ + \dim U = \dim V$

$$\dim W^\circ + \dim W = \dim V$$

$\therefore \dim (\cancel{W}U^\circ + W^\circ) = 2 \dim V - \dim U - \dim W - \dim (U^\circ \cap W^\circ) \longrightarrow ②$

consider $f \in (U \cup W)^\circ$

ie every vector $v = u_1 + w_1$ is killed by f

ie $f(v) = f(u_1) + f(w_1) = 0 \Rightarrow f(u_1) = 0$ and $f(w_1) = 0$

$\Rightarrow f \in U^\circ$ and $f \in W^\circ$

$$\Rightarrow f \in (U^\circ \cap W^\circ)$$

$$\therefore \quad (U + W)^\circ \subseteq U^\circ \cap W^\circ$$

$\Rightarrow \dim (U + W)^\circ \leq \dim (U^\circ \cap W^\circ)$ utilizing things in eqn ②,

we have

$$\dim (U^\circ + W^\circ) \geq 2 \dim V - \dim U - \dim W - \dim (U + W)^\circ$$

we have $\dim (U + W)^\circ + \dim (U + W) = \dim V$

$\therefore \dim(U^\circ + W^\circ) \geq \dim V - \dim V - \dim W + \dim(V+W)$

we have

$$\dim U + \dim W = \dim(U+W) + \dim(U\cap W)$$

$\therefore \dim(U^\circ + W^\circ) \geq \dim V - \dim(U\cap W)$

since $\dim(U\cup W) + \dim(U\cup W)^\circ = \dim V$

$$\Rightarrow \dim(U^\circ + W^\circ) \geq \dim(U\cap W^\circ) \rightarrow (3)$$

from (1) and (3), we have,

$$U^\circ + W^\circ = (U\cup W)^\circ$$

Q. 14)

Consider $\Phi \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$

Let S denote a map from

$$\mathcal{L}(V_1 \times \cdots \times V_m, W) \longrightarrow \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$$

such that.

$$S(\Phi) = (\varphi_1, \varphi_2, \ldots, \varphi_m),$$

where $\varphi_i \in \mathcal{L}(V_i, W)$.

& for any.

$v = (0, 0, \ldots, v_i, 0, \ldots 0) \in V_1 \times V_2 \times \cdots \times V_m$,

we have. $\varphi_i(v_i) = \Phi(v) = w \in W$.

"Obviously", due to linearity of

ϕ, ψ_i is a linear map as well.

Now consider

$$S(\phi_1) + \alpha S(\phi_2)$$

$$= (\psi_{11}, \psi_{12}, \dots, \psi_{1m})$$

$$+ (\alpha\psi_{21}, \alpha\psi_{22}, \dots, \alpha\psi_{2m})$$

Now, for any $v_i \in V_i$

consider $\psi_{1i}(v_i) + \alpha\psi_{2i}(v_i)$

$$= \phi_1(v) + \alpha\phi_2(v)$$

where, $v = (0, 0, \dots, v_i, 0, \dots 0)$

as before.

But, $\phi_1(v) + \alpha\phi_2(v)$

$$= (\phi_1 + \alpha \phi_2)(v).$$

$$\therefore \quad S(\phi_1 + \alpha \phi_2)$$

$$= (\psi_{11}, \psi_{12}, \ldots, \psi_{1m})$$

$$+ (\alpha \psi_{21}, \alpha \psi_{22}, \ldots, \alpha \psi_{2m})$$

$$= S(\phi_1) + \alpha S(\phi_2)$$

Hence, S is also linear. $\longrightarrow$ (1)

Now, for any.

$$(\psi_1, \psi_2, \ldots, \psi_m) \in \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$$

$$\exists \ \phi \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$$

naturally given by the following:

$\forall v = (v_1, v_2, \ldots, v_m)$, $v_i \in \mathbb{V}_i$
& $v \in \mathbb{V}_1 \times \cdots \times \mathbb{V}_m$

we have.

$$\Phi(v) \longrightarrow (\psi_1(v_1), \ldots, \psi_m(v_m))$$

i.e. $\exists\, \Phi \in \mathcal{L}(\mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_m, \mathbb{W})$ for $(\psi_1, \psi_2, \ldots, \psi_m)$ such that $S(\Phi) = (\psi_1, \psi_2, \ldots, \psi_m)$.

<u>Hence, S is onto.</u> $\longrightarrow$ 2

Suppose $S(\Phi) = (0, 0, \ldots, 0)$

$$\in \mathcal{L}(\mathbb{V}_1, \mathbb{W}) \times \cdots \times \mathcal{L}(\mathbb{V}_m, \mathbb{W}).$$

Then ...

Thus, ϕ is such that for any $v \in \mathbb{V}_1 \times \mathbb{V}_2 \times \dots \times \mathbb{V}_m$, we have.

$$\psi_i = 0 \ \forall \ i.$$

But. $\psi_i(v_i) = \phi(v)$

if $v = (0, \dots, v_i, 0, \dots, 0)$.

$$\Rightarrow \phi(v) = 0 \ \forall \ v \in \mathbb{V}_1 \times \dots \times \mathbb{V}_m.$$

$\Rightarrow$ ϕ is also the identical zero function, i.e.,

$$\phi = 0_{\mathcal{L}(\mathbb{V}_1 \times \mathbb{V}_2 \times \dots \times \mathbb{V}_m, \mathbb{W})}.$$

Hence, S is injective. $\rightarrow$ (3).

Combining (1), (2) & (3), we have that S is a linear bijection. Thus,

$\mathcal{L}(V_1 \times V_2 \times \cdots \times V_m, W)$ is

isomorphic with

$$\mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W) \times \cdots \times \mathcal{L}(V_m, W).$$

Q. 16)

$$U \subseteq W \;\&\; V \subseteq W.$$

Given $w_1 + U = w_2 + V$
for $w_1, w_2 \in W$.

Consider $p \in w_1 + U = w_2 + V$.

$\Rightarrow p = w_1 + u$ for $u \in U$.

Illy, $p = w_2 + v$ for $v \in V$.

since, $w_1 + U = w_2 + V$.

$\Rightarrow w_1 + u = w_2 + v.$

$\Rightarrow w_1 + u - w_2 = v \in \mathbb{V}.$

$\Rightarrow (w_1 + u) - w_2 \in \mathbb{V}$

$\Rightarrow (w_1 + u) + \mathbb{V} = w_2 + \mathbb{V}.$

$= w_1 + \mathbb{U}$ (given)

$\Rightarrow w_1 + u + \mathbb{V} = w_1 + \mathbb{U}. \longrightarrow 1$

$\therefore$ a vector in $w_1 + u + \mathbb{V}$ is given by $w_1 + u + \tilde{v}$ (for any $\tilde{v} \in \mathbb{V}$)

any vector in $w_1 + \mathbb{U}$ is given by $w_1 + \tilde{u}$

$\therefore$ any vector in $w_1 + \mathbb{U}$, or $w_1 + u + \mathbb{V}$ (equal sets as per 1)

can be written as

$w_1 + \tilde{v} + u = w_1 + \tilde{u}$

$\Rightarrow \tilde{v} = \tilde{u} - u \in \mathbb{U}.$

$\Rightarrow \tilde{v} \in \mathbb{U}$

$\Rightarrow v \in U.$

$$\Rightarrow V \subseteq U.$$

Carrying out similar arguments as above with U & V reversed, we conclude that

$$U \subseteq V.$$

Thus $U = V$.

Consider a basis $\{v_1 + \mathbb{U}, \ldots, v_m + \mathbb{U}\}$ of

Q19)

$\mathbb{V}/\mathbb{U}$. By previous exercise,

$\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

Extend $\{v_1, v_2, \ldots, v_m\}$ to

$\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$

which is a basis for $\mathbb{V}$,

where $\dim(\mathbb{V}) = m+n$.

Again, by previous exercise,

$u_i \in \mathbb{U}$ $\forall$ $i = 1, 2, \ldots, n$.

Thus,

$$v = \sum_{i=1}^{n} \alpha_i u_i + \sum_{j=1}^{m} \beta_j v_j$$

for any $v \in \mathbb{V}$,

where $\{\alpha_i, \beta_j\}$ are uniquely determined, since

$$\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$$

is a basis for V.

$$\Rightarrow v = u + w$$

where $u = \sum_{i=1}^{n} \alpha_i u_i \in U$

& $w = \sum_{j=1}^{m} \beta_j v_j \in \langle v_1, v_2, \ldots, v_m \rangle$.

Let $\langle v_1, v_2, \ldots, v_m \rangle := W$.

Therefore

$$v = u + w$$

where $u \in U$ & $w \in W$, in addition to the fact that the representation is unique.

This implies

$$V = U \oplus W.$$

for $W = \langle v_1, v_2, \dots, v_m \rangle$.

Consider a mapping

$$\phi : V \longrightarrow U \times V/U$$

given by,

$$\phi(v) = \left(v - \sum \beta_j v_j ,\ v + U \right)$$

where, as before,

$$v = \underbrace{\sum \alpha_i u_i}_{\in U} + \underbrace{\sum \beta_j v_j}_{\in \langle v_1, v_2, \dots}.$$

$\ldots, v_n]$

Consider any $(\tilde{u}, \tilde{v} + U)$

$\in U \times V/U.$

[i.e. $\tilde{u} \in U$, $\tilde{v} \in V$]

Thus, $\tilde{u} = \sum_{i=1}^{n} \gamma_i u_i$

& $\tilde{v} = \sum_{i=1}^{n} \tilde{\alpha}_i u_i + \sum_{j=1}^{m} \tilde{\beta}_j v_j$

Let $v = \sum_{i=1}^{n} \gamma_i u_i + \sum_{j=1}^{m} \tilde{\beta}_j v_j$

$\therefore \phi(v) = (v - \sum \tilde{\beta}_j v_j,$

$\ldots + U)$

$v + U)$

$$= \left(\sum_{i=1}^{n} \gamma_i u_i,\ v + U\right)$$

$$= (\tilde{u},\ v + U)$$

Now $v - \tilde{v} = \sum(\gamma_i - \alpha_i) u_i$
$\in U.$

$\Rightarrow v + U = \tilde{v} + U.$

Hence,

$$\phi(v) = (\tilde{u},\ \tilde{v} + U).$$

Thus, $\exists v : \phi(v) = (\tilde{u}, \tilde{v} + U)$

$\forall\ (\tilde{u}, \tilde{v} + U) \in U \times V/U.$

Therefore, ϕ is surjective.

Consider

$$\phi(v) = (0, 0+U) = 0_{U \times V/U}$$

for some $v \in V$.

This means that if

$$v = \sum_{i=1}^{n} \bar{\alpha}_i u_i + \sum_{j=1}^{m} \bar{\beta}_j v_j$$

Then $\phi(v) = \left(\sum \bar{\alpha}_i u_i, v+U\right)$.

$= (0, 0+U)$

But $\{u_1, u_2, \dots u_n\}$ is a linearly ind set.

$\therefore \sum \bar{\alpha}_i u_i = 0$

$\Rightarrow \bar{\alpha}_i = 0 \ \forall i$.

$$\therefore v = \sum_{j=1} \beta_j v_j$$

Further, $v + U = 0 + U$

$$\Rightarrow v \in U$$

$$\Rightarrow \sum_{j=1}^{m} \beta_j v_j \in U.$$

But $\sum \beta_j v_j \in \langle v_1, \dots, v_m \rangle$

& $\langle v_1, v_2, \dots, v_m \rangle \cap U$

$= \{0\}$ [$\because V = U \oplus \langle v_1, \dots, v_m \rangle$]

$$\Rightarrow \sum \beta_j v_j = 0$$

$\Rightarrow \beta_j = 0 \ \forall j$ since $\{v_1, \dots, v_m\}$ is a lin. ind. set.

$$\Rightarrow v = 0 \in V.$$

Hence. $\operatorname{Ker}(\phi) = 0$.

Therefore, ϕ is also an injective map.

Hence. $\mathbb{V}$ is isomorphic to $\mathbb{U} \times \mathbb{V}/\mathbb{U}$.

$\boxed{Q_{22}}$

Consider $f(x) \in \mathbb{R}[x]$

Suppose $\deg(f) = n$

Then, $f \in \mathbb{R}[x]$

Consider the restriction of ϕ, say ϕ_n in $\mathcal{L}[\mathbb{R}[x]_n, \mathbb{R}[x]_n]$

Such that $\phi_n(f) = \phi(f)$

for any $f \in \mathbb{R}[x]_n$

$[\because \deg(\phi_n(f)) \leq \deg(f)$,

hence $\phi_n(f) \in \mathbb{R}[x]_n]$

Now ϕ_n in a linear operator on a finite dim vector space $\mathbb{R}[x]_n$.

and is also injective, since ϕ is injective

Thus, ϕ_n must also be surjective but, this means that for any $f \in \mathbb{R}[x]$, since $\exists\, n \in \mathbb{Z}_+$ such that

$\deg(f) \leq n$, hence $f \in \mathbb{R}[x]_n$ for some n.

Consequently, $\exists\, g \in \mathbb{R}[x]_n$: $\phi_n(g) = f$

$\Rightarrow \phi(g) = f\ [\because \phi_n = \phi$ for $g \in \mathbb{R}[x]_n]$

$\Rightarrow \exists\, g \in \mathbb{R}[x]$: $\phi(g) = f$ and so, ϕ is a surjective.

Suppose for some polynomial, f (non-zero) of degree k,

we have $\deg(\phi(f)) < k$

Let $g = \phi(f)$

Now, Consider ~~ϕ_{k-1}~~

$$\phi_{k-1} \in \mathcal{L}\left[\mathbb{R}[x]_{k-1}, \mathbb{R}[x]_{k-1}\right]$$

Such that

$$\phi_{k-1}(\cdot) = \phi(\cdot)$$

$$\text{for } f \in \mathbb{R}[x]_{k-1}$$

already we have seen that ϕ_{k-1} is a Surjection

(& bijection) over $\mathbb{R}[x]_{k-1}$.

Thus for any $p(x) \in \mathbb{R}[x]_{k-1}$, we must have a

unique $k(x) \in \mathbb{R}[x]_{k-1}$: $\phi_{k-1}(k(x)) \in \mathbb{R}[x]_{k-1}$

Hence we must have $h(x) \in \mathbb{R}[x]_{k-1}$: $\phi_{k-1}(h(x)) = g(x)$

But since $\phi_{k-1} = \phi$ on $\mathbb{R}[x]_{k-1}$

$$\Rightarrow \boxed{\phi(h(x)) = g(x)}$$

On the other hand we have

$$\phi(f(x)) = g(x)$$

Since ϕ is injective, we must have

$$f(x) = h(x)$$

But $\deg(f(x)) = K$ &

$\deg(h(x)) < K$

hence the Contradiction!

Thus $\deg(\phi(f)) = \deg(f) \quad \forall f \in \mathbb{R}[x]$

Q24

(a) True, Consider $v \in V$.

Now, $\phi(\psi(v)) \in \operatorname{im}\phi \subset \ker(\psi)$

$\therefore \psi(\phi(\psi(v))) = \phi(0) = 0$

[$\because$ for a linear transformation $\phi(0) = 0$]

(b) True, Suppose $\operatorname{Im}(\phi) = \ker(\phi)$

$\Rightarrow \dim(\operatorname{Im}(\phi)) = \dim(\ker(\phi))$

But RNT dictates that

$$\dim(V) = \dim(\operatorname{Im}\phi) + \dim(\ker(\phi))$$
$$= 2\dim(\ker(\phi))$$

$$\Rightarrow \underbrace{2m-1}_{\text{odd integer}} = \underbrace{2\cdot\dim(\ker(\phi))}_{\text{even integer}}$$

Hence impossible!

(c) True. for linear functionals that are not zero,

$$\dim(V) = \dim(Im(\phi)) + \dim(Ker(\phi))$$

$$\Rightarrow \dim(Ker(\phi)) = \dim(V) - 1$$

(I) Suppose $\phi_1 = \phi_2 = 0$

Then obviously $\phi_1 = \alpha\phi_2$ for any $\alpha \in \mathbb{F}$

(II) Suppose $\phi_1 \neq 0$, & $\phi_2 \neq 0$

(It can not be case that one of them is zero while the other exist since their Ker will not be identical].

Let $\{v_1, v_2 \dots v_{n-1}\}$ be basis for $Ker(\phi_1) = Ker(\phi_2)$ where $\dim(V) = n$

extend them to $\{v_1, v_2 \dots v_n\}$ so as to get a basis for V.

Now, for any $v \in V$.

we have

$$v = \sum \alpha_i v_i$$

$$\Rightarrow \phi_1(v) = \alpha_n \phi_1(v_n)$$

$$\& \ \phi_2(v) = \alpha_n \phi_2(v_n)$$

Since $\phi_1 \neq 0$ & $\phi_2 \neq 0$

$\therefore \phi_1(v_n) \neq 0$ & $\phi_2(v_n) \neq 0$

$$\Rightarrow \phi_1(v) = \alpha_n \frac{\phi_1(v_n)}{\phi_2(v_n)} \cdot \phi_2(v_n)$$

$$= \frac{\phi_1(v_n)}{\phi_2(v_n)} \cdot (\alpha \cdot \phi_2(v_n))$$

$$\frac{\phi_1(v_n)}{\phi_2(v_n)} \phi_2(v)$$

$$\Rightarrow \phi_1(v) = \alpha \phi_2(v)$$

(d) True. $\phi_1, \phi_2, \phi_3 = id_V$

by def

$$\Rightarrow \phi_1^{-1} = \phi_2 \phi_3$$

$$\& \ \phi_3^{-1} = \phi_1 \phi_2$$

$$\phi_2 \phi_3 \phi_1 = id_V$$

$$\phi_3 \phi_1 \phi_2 = id_V$$

$$\phi_2^{-1} = \phi_3 \phi_1$$

(e) false.

Suppose $(D-\lambda I)(f(x)) = 0$

$\Rightarrow f'(x) - \lambda f(x) = 0$

for $\lambda = 0$, we have

$f(x) = c$ in $\ker(D-\lambda I)$

This is a non trivial

Subspace of $\mathbb{R}[x]$

for any other λ, $(\neq 0)$

Let $f'(x) = \lambda f(x)$

for $f(x) \in \mathbb{R}[x]$

$\Rightarrow f(x) = 0$

Hence, $\ker(D-\lambda I) = 0_{\mathbb{R}[x]}$

Q25

Consider $\phi : \mathbb{R}[x] \to \mathbb{R}$,

a functional such that

$$\phi(f) = f(10) - f(0)$$

we know, from first isomorphism thm

$$\mathbb{R}[x] / \mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$$

Now $\mathrm{Ker}(\phi) = W_1$

& $\mathrm{Im}(\phi) = \mathbb{R}$

$$\boxed{\mathbb{R}[x] / W_1 \cong \mathbb{R}}$$

Next Let $\psi : \mathbb{R}[x] \to \mathbb{R}^2$

be such that

$$\psi(f) = \begin{pmatrix} f(0) \\ f(10) \end{pmatrix}$$

$\mathrm{Ker}(\psi) = W_2$

& $\mathrm{Im}(\psi) = \mathbb{R}^2$

$$\Rightarrow \mathbb{R}[x] / W_2 \cong \mathbb{R}^2$$

Thus

$$\mathbb{R} \not\cong \mathbb{R}^2,$$

the statement is <u>false</u>