Q6) Consider a linear functional $f : \mathbb{V} \to \mathbb{F}$ such that

$$f(\alpha \tilde{v}_1 + \tilde{v}_2) = \alpha f(\tilde{v}_1) + f(\tilde{v}_2)$$

$$\forall\ \tilde{v}_1, \tilde{v}_2 \in \mathbb{V}$$

$$\& \ \alpha \in \mathbb{F}.$$

$f(\cdot)$ can be characterized by its action on some basis of $\mathbb{V}$, say $\mathcal{B} := \{v_1, v_2, \dots, v_n\}$.

Let $f(v_i) = \beta_i \in \mathbb{F}$.

$$\Rightarrow f(v) = f\left(\sum \alpha_i v_i\right)$$

$$= \sum \alpha_i f(v_i)$$

$$\sum \alpha_i f(v_i)$$

$$= \sum \alpha_i \beta_i$$

$$= [\alpha_1 \ \alpha_2 \cdots \alpha_n] \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$= \langle [v]_B \mid [\tilde{v}]_B \rangle$$

where $\tilde{v} = \overline{\beta_1} v_1 + \overline{\beta_2} v_2 + \cdots + \overline{\beta_n} v_n$.

where, $\overline{\beta_i} = \text{conj}(\beta_i)$

This implies

$$f(v) = \langle v / \tilde{v} \rangle$$

where, $\tilde{v} = \sum \overline{f(v_i)} . v_i$.

Q8) $T : V \to V$. (linear)

$\langle Tv \mid w \rangle$

$v_1 \longmapsto \langle Tv_1 | v_2 \rangle$ is a

linear functional acting on $v_1 \in V$,

since $\langle T(\alpha u_1 + u_2) | v_2 \rangle$

$$= \langle \alpha Tu_1 + Tu_2 | v_2 \rangle$$

$$= \alpha \langle Tu_1 | v_2 \rangle + \langle Tu_2 | v_2 \rangle .$$

Now, by qn 5, $\exists$ a unique

$\tilde{v}_2$: $v_1 \longmapsto \langle Tv_1 | v_2 \rangle$

is the same as an inner product

with a unique vector, $v_1 \longmapsto \langle v_1 | \tilde{v}_2 \rangle$.

[$\because$ every lin. functional can be linked with a unique vector so that it is described as an inner product with that vector] $\rightarrow$ see qn 5 & its soln

$\Rightarrow \langle Tv_1 | v_2 \rangle = \langle v_1 | \tilde{v}_2 \rangle.$

Describe $T^*: v_2 \mapsto \tilde{v}_2$.

(i.e. a mapping that carries v_2 to $\tilde{v}_2$, the unique vector with whom the inner product gives us the functional).

Clearly, then $\langle Tv_1 | v_2 \rangle = \langle v_1 | T^* v_2 \rangle$.

It remains to be shown that T^* is linear.

consider

$\langle v_1 | T^*(c w_1 + w_2) \rangle$

$= \langle Tv_1 | c w_1 + w_2 \rangle$

$= \langle Tv_1 | c w_1 \rangle + \langle Tv_1 | w_2 \rangle$

$$= \bar{c}\langle Tv_1 | w_1\rangle + \langle Tv_1 | w_2\rangle$$

$$= \bar{c}\langle v_1 | T^* w_1\rangle + \langle v_1 | T^* w_2\rangle$$

$$= \langle v_1 | cT^* w_1\rangle + \langle v_1 | T^* w_2\rangle$$

$$= \langle v_1 | cT^* w_1 + T^* w_2\rangle$$

$$\Rightarrow T^*(cw_1 + w_2) = cT^* w_1 + T^* w_2$$

Hence, T^* is linear.

T^* can be uniquely determined by studying the effect of the functional $\langle Tv | v_2\rangle$ on a basis set for $v \in V$.

Q7) Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V.

$\therefore$ Any $v \in V$ can be expressed as $v = \sum_{i=1}^{n} \alpha_i v_i$

where $\alpha_i = \langle v / v_i \rangle$.

$\therefore$ $v = \sum_{i=1}^{n} \langle v / v_i \rangle v_i$

Now, consider a functional

$$f : V \to \mathbb{F}$$

$$\therefore \quad f(v) = f\left(\sum \langle v | v_i \rangle v_i\right)$$

$$= \sum_{i=1}^{n} f(\langle v | v_i \rangle v_i)$$

$$= \sum_{i=1}^{n} \langle v | v_i \rangle . f(v_i).$$

$$= \sum_{i=1}^{n} \langle v | \overline{f(v_i)} . v_i \rangle$$

$$= \left\langle v \middle| \sum \overline{f(v_i)} . v_i \right\rangle$$

Let $\tilde{v} = \sum \overline{f(v_i)} . v_i$

$$\therefore \quad f(v) = \langle v | \tilde{v} \rangle.$$

Suppose $\tilde{v}$ is not unique.

Then let $f(v) = \langle v | \hat{v} \rangle$

where $\tilde{v} \neq \hat{v}$ & $\tilde{v}, \hat{v} \in V$.

$\Rightarrow f(v) - f(v) = \langle v | \tilde{v} \rangle - \langle v | \hat{v} \rangle \quad \forall v \in V$

$\Rightarrow \quad 0 = \langle v | \tilde{v} - \hat{v} \rangle$

choose $v = \tilde{v} - \hat{v}$

$\Rightarrow \quad \| \tilde{v} - \hat{v} \| = 0$

$\Rightarrow \quad \tilde{v} - \hat{v} = 0$, which is a contradiction!

Consider $v \in \ker(f)$.

$\Rightarrow \quad f(v) = 0$

$\Rightarrow \quad \langle v | \tilde{v} \rangle = 0$

$\Rightarrow \quad \tilde{v}$ is $\perp$r to v.

Since $\tilde{v}$ is $\perp$r to every

$v \in \text{Ker}(f)$,

$\therefore \tilde{v} \in \text{Ker}(f)^{\perp}$.

Suppose $P : V \longrightarrow \text{Ker}(f)^{\perp}$.

Now $V = \text{Ker}(f) \oplus \text{Ker}(f)^{\perp}$.

[$\because$ V is finite dim. it can always be written as $V = U \oplus U^{\perp}$ for $U \subseteq V$ (why?)]

$\therefore$ any $v \in V$ can be expressed as

$$v = v_{\parallel} + v_{\perp}$$

where $v_{\parallel} \in \text{Ker}(f)$ &

$v_{\perp} \in \text{Ker}(f)^{\perp}$

$\therefore f(v) = f(v_{\parallel}) + f(v_{\perp})$

$$= f(v_{\perp})$$

Now, $P(v) = v_{\perp}$. [By defn]

$\therefore \; f(P(v)) = f(v_{\perp})$

Hence, $f(v) = f(Pv) \; \forall v \in V$.

Q10

(a) Consider $\langle (R+T)v_1 | v_2 \rangle$

$$= \langle v_1 | (R+T)^+ v_2 \rangle$$

Now, $\langle (R+T)v_1 | v_2 \rangle$

$$= \langle Rv_1 + Tv_1 | v_2 \rangle$$

$$= \langle Rv_1 | v_2 \rangle + \langle Tv_1 | v_2 \rangle$$

$$= \langle v_1 | R^* v_2 \rangle + \langle v_1 | T^* v_2 \rangle$$

$$= \langle v_1 | (R^* + T^*) v_2 \rangle$$

$$\Rightarrow \langle v_1 | (R+T)^* v_2 \rangle = \langle v_1 | (R^* + T^*) v_2 \rangle$$

$$\Rightarrow (R+T)^* = R^* + T^*.$$

(b) $\langle \alpha T v_1 | v_2 \rangle = \alpha . \langle T v_1 | v_2 \rangle$

$$= \alpha . \langle v_1 | T^* v_2 \rangle$$

$$= \langle v_1 | \bar{\alpha} T^* v_2 \rangle$$

Now $\langle (\alpha T) v_1 | v_2 \rangle = \langle v_1 | (\alpha T)^* v_2 \rangle$

$$\therefore \quad (\alpha T)^* = \bar{\alpha} T^* .$$

(c) $$\langle TR v_1 | v_2 \rangle = \langle v_1 | (TR)^* v_2 \rangle$$

Further,

$$\langle TR v_1 | v_2 \rangle = \langle R v_1 | T^* v_2 \rangle$$

$$= \langle v_1 | R^* T^* v_2 \rangle$$

$$\Rightarrow (TR)^* = R^* T^*$$

(d) $$\langle T^* v_1 | v_2 \rangle = \langle v_1 | (T^*)^* v_2 \rangle$$

Also,

$$\langle T^* v_1 | v_2 \rangle = \overline{\langle v_2 | T^* v_1 \rangle}$$

$$= \overline{\langle T v_2 | v_1 \rangle}$$

$$= \langle v_1 | T v_2 \rangle$$

$$\Rightarrow \langle v_1 | T v_2 \rangle = \langle v_1 | (T^*)^* v_2 \rangle$$

$$\Rightarrow T = (T^*)^*$$

(e) $\langle T^{-1} v_1 | v_2 \rangle = \langle v_1 | (T^{-1})^* v_2 \rangle$

Also, $\langle T^{-1} v_1 | T^* (T^*)^{-1} v_2 \rangle$

$$= \langle T T^{-1} v_1 | (T^*)^{-1} v_2 \rangle$$

$$= \langle v_1 | (T^*)^{-1} v_2 \rangle$$

$$\Rightarrow (T^{-1})^* = (T^*)^{-1}$$

(f) Let $v \in (\ker T)$

$\Rightarrow \langle Tv | \tilde{v} \rangle = 0 \ \forall \tilde{v} \in V$

$\Rightarrow \langle v | T^* \tilde{v} \rangle = 0 \ \forall \tilde{v} \in V$ $[\because Tv = 0]$

$\Rightarrow v \in [\operatorname{im}(T^*)]^\perp$

$\Rightarrow (\ker(T)) \subseteq (\operatorname{im}(T^*))^\perp$ — 1

Let $v \in (\operatorname{im}(T^*))^\perp$

$$\langle v | T^* z\rangle = 0 \quad \forall z \in V$$

$$\Rightarrow \langle Tv | z\rangle = 0 \quad \forall z \in V$$

$$\Rightarrow \quad Tv = 0 \quad [\because \text{Putting } z = Tv, \text{ we get } \|Tv\|^2 = 0]$$

$$\Rightarrow v \in \ker(T).$$

$$\Rightarrow (\operatorname{im}(T^*))^\perp \subseteq \ker(T) \quad \text{—— } (2)$$

$$\Rightarrow \boxed{(\operatorname{im}(T^*))^\perp = \ker(T)}$$

Q11

(a) Suppose T is self-adjoint.

i.e. $T = T^*$.

then $T.T^* = T.T. = T^2$

and $T^*T = T.T = T^2$

$$\Rightarrow T^*T = TT^*.$$

Next, suppose $T^*T = TT^*$

& $T = T^2$.

Now

Now,
$$\langle Tv|v\rangle$$
$$= \langle T^2 v|v\rangle$$
$$= \langle Tv|T^* v\rangle$$
$$= \langle v|T^{*2} v\rangle$$

Also,
$$\langle Tv|v\rangle = \langle v|T^* v\rangle$$

$$\Rightarrow \langle v|T^* v\rangle = \langle v|T^{*2} v\rangle \quad \forall v$$

$$\Rightarrow T^* = T^{*2}$$

Consider now $\|(T^*T - T)v\|^2$

$$= \langle (T^*T - T)v|(T^*T - T)v\rangle$$

$$= \langle v|(T^*T - T)^*(T^*T - T)v\rangle$$

$$= \langle v|(T^*T - T^*)(T^*T - T)v\rangle$$

$$= \langle v|(T^*TT^*T - T^*T^2 - T^{*2}T + T^*T)v\rangle$$

$$= \langle v|(T^*T^*TT - T^*T - T^*T + T^*T)v\rangle$$

$(\because TT^* = T^*T)$

$$= \langle v | (T^{*2}T^2 - 2T^*T + T^*T) v \rangle$$

$$= \langle v | (T^*T - T^*T) v \rangle$$

$$= 0$$

$$\Rightarrow \boxed{T^*T = T} \longrightarrow (1)$$

Illy, consider

$$\| (TT^* - T^*) v \|^2$$

$$= \langle (TT^* - T^*) v | (TT^* - T^*) v \rangle$$

$$= \langle v | (TT^* - T)(TT^* - T^*) v \rangle$$

$$= \langle v | (TT^*TT^* - TT^{*2} - T^2T^* + TT^*) v \rangle$$

$$= \langle v | (TTT^*T^* - TT^* - TT^* + TT^*) v \rangle$$

$$= \langle v | (T^2T^{*2} - TT^*) v \rangle$$

$= \langle v | (TT^* - T^*T) v \rangle$

$= 0$

$\Rightarrow \boxed{TT^* = T^*} \longrightarrow (2)$

From (1) & (2),

$$T = \underbrace{T^*T = TT^*}_{\text{given}} = T^*$$

$\Rightarrow T = T^*$

Hence T is self adjoint.

[General statement: For an idempotent operator,

Normal $\Longleftrightarrow$ self adjoint]

(b) Suppose $RS = SR$ [commutativity]

& $S = S^*$, $R = R^*$ [self adj]

Now $(RS)^* = S^* R^*$

$= SR$ [due to self adjoint]

$= RS$ [$\because$ they commute]

$\Rightarrow (RS)^* = (RS)$

so RS is also self-adjoint.

Suppose $(RS)^* = RS$. [composition is self adj]

$\Rightarrow S^* R^* = RS$ [$\because (RS)^* = S^* R^*$]

$\Rightarrow SR = RS$ [$\because S^* = S$ & $R^* = R$ due to self adj.]

$\therefore$ S & R commute.

(c) Suppose $T = T^*$ (self adjoint)

Now

$$\langle Tv|v\rangle = \langle v|Tv\rangle$$

& $$\langle Tv|v\rangle = \langle v|T^* v\rangle$$

$$= \langle v|Tv\rangle$$

$$(\because T = T^*).$$

$$\Rightarrow \langle v|Tv\rangle = \overline{\langle v|Tv\rangle}$$

$$(= \langle Tv|v\rangle)$$

$\Rightarrow \langle v|Tv\rangle$ is real.

$\Rightarrow \langle Tv|v\rangle$ is real $\forall v \in V$.

Suppose $\langle Tv|v\rangle$ is real $\forall v \in V$

$$\Rightarrow \langle Tv|v\rangle = \overline{\langle Tv|v\rangle}$$

$$= \langle v|Tv\rangle \rightarrow ①$$

But, $\langle Tv|v\rangle = \langle v|T^* v\rangle. \rightarrow ②$

From (i) & (2),

$$\langle v|Tv\rangle = \langle v|T^{+}v\rangle \quad \forall v \in V.$$

$$\Rightarrow T = T^{+} \quad (\text{self adjoint}).$$

QED.

$\boxed{Q14}$ we need to show that S is linearly independent set and we will be done.

Since V is an ~~infinitely~~ n-dim vector space & or any linearly independent set in V, containing n vectors is a basis of V.

Suppose S is not line independent

$\therefore \exists \{\alpha_i\}_{i=1,2\ldots n}$

not all zero, such that

$$\sum_{i=1}^{n} \alpha_i s_i = 0$$

Now, $\sum_{i=1}^{n} \alpha_i (s_i - w_i) = -\sum_{i=1}^{n} \alpha_i w_i$

$\Rightarrow \|\sum \alpha_i (s_i - w_i)\|^2 = \|\sum \alpha_i w_i\|^2$

$= \langle \sum \alpha_i w_i | \sum \alpha_i w_i \rangle$

$= \sum \alpha_i \bar{\alpha_i} \langle w_i | w_i \rangle \quad [\because \langle w_i | w_j \rangle = 0 \text{ for } i \neq j]$

$= \sum_{i=1}^{n} |\alpha_i|^2$ ——— $(*)$

On the other hand $\|\sum \alpha_i (s_i - w_i)\|^2$

$= \langle \sum \alpha_i (s_i - w_i) | \sum \alpha_i (s_i - w_i) \rangle$

$\leq |\langle \sum \alpha_i (s_i - w_i) | \sum \alpha_i (s_i - w_i) \rangle|$

$$\leq \sum_j \sum_i \left| \langle \alpha_i (x_i - w_i) | \alpha_j (x_j - w_j) \rangle \right|$$

(Δ inequality).

$$\leq \sum_j \sum_i |\alpha_i| |\alpha_j| \left| \langle (x_i - w_i) | (x_j - w_j) \rangle \right|$$

$$\leq \sum_j \sum_i |\alpha_i| |\alpha_j| \, \|x_i - w_i\| \, \|x_j - w_j\|$$

[Cauchy Schwarz inequality]

$$< \frac{1}{n} \sum_j \sum_i |\alpha_i| |\alpha_j| \quad \left[\because \|x_i - w_i\| < \frac{1}{n} \ \forall i \right]$$

$$= \frac{1}{n} \left(\sum_i |\alpha_i| \right)^2 \quad \text{(Key step.)} \ !$$

$\circledast$

Now Consider

$$\begin{pmatrix} |\alpha_1| \\ |\alpha_2| \\ \vdots \\ |\alpha_n| \end{pmatrix} \in \mathbb{F}^n \quad \& \quad \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{F}^n$$

with $v_2^T v_1 = \langle v_1 | v_2 \rangle$

being the inner product

Then by C-S inequality

we get $\sum |\alpha_i| = \left\langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$

$$\leq \sqrt{|\alpha_1|^2 + \dots + |\alpha_n|^2} \cdot \sqrt{1^2 + 1^2 + \dots 1^2}$$

$$= \sqrt{n} \cdot \sqrt{\sum |\alpha_i|^2}$$

$$\Rightarrow \left(\sum_i |\alpha_i| \right)^2 \leq n \sum |\alpha_i|^2$$

$$\Rightarrow \left(\text{from } (**) \right)$$

$$\left\| \sum \alpha_i (s_i - w_i) \right\|^2 < \frac{1}{n} \left(\sum |\alpha_i| \right)^2 \leq \sum |\alpha_i|^2$$

from $(*)$ on the other hand we have

$$\| \alpha_i (s_i - w_i) \|^2 = \sum |\alpha_i|^2$$

we have

$$\sum |\alpha_i|^2 < \sum |\alpha_i|^2$$

a contradiction

Thus S must be a linearly independent set

Q 18

Suppose $\langle \cdot | \cdot \rangle_{\mathbb{V}}$ represent an inner product in $\mathbb{V}$. that satisfies the required condition

for any $f \in \mathbb{V}$ we may express f as $f = v_1 + v_2$

where $v_1 \in \mathbb{V}_1$ & $v_2 \in \mathbb{V}_2$

$\therefore$ we get

$\langle f | \tilde{f} \rangle_{\mathbb{V}}$ for $f, \tilde{f} \in \mathbb{V}$

given by $\langle f | \tilde{f} \rangle_{\mathbb{V}} = \langle v_1 + v_2 | \tilde{v}_1 + \tilde{v}_2 \rangle_{\mathbb{V}}$

where $v_1, \tilde{v}_1 \in \mathbb{V}_1$ &

$v_2, \tilde{v}_2 \in \mathbb{V}_2$

$\Rightarrow \langle f | \tilde{f} \rangle_{\mathbb{V}} = \langle v_1 | \tilde{v}_1 + \tilde{v}_2 \rangle_{\mathbb{V}} + \langle v_2 | \tilde{v}_1 + \tilde{v}_2 \rangle_{\mathbb{V}}$

$= \langle v_1 | \tilde{v}_1 \rangle_{\mathbb{V}} + \langle v_1 | \tilde{v}_2 \rangle_{\mathbb{V}} + \langle v_2 | \tilde{v}_1 \rangle_{\mathbb{V}} + \langle v_2 | \tilde{v}_2 \rangle_{\mathbb{V}}$

But according to the condition $V_1^{\perp} = V_2$, we must have

$\langle v_1 | \tilde{v}_2 \rangle_{\mathbb{V}} = 0$ & $\langle v_2 | \tilde{v}_1 \rangle_{\mathbb{V}} = 0$

$\therefore \langle f | \tilde{f} \rangle_{\mathbb{V}} = \langle v_1 | \tilde{v}_1 \rangle_{\mathbb{V}} + \langle v_2 | \tilde{v}_2 \rangle_{\mathbb{V}}$

further to meet the second Condition; check that

for $f = v_1 \in \mathbb{U}_1$

& $\tilde{f} = \tilde{v}_1 \in \mathbb{U}_1$

We have $\langle f | \tilde{f} \rangle_{\mathbb{V}} = \langle v_1 | \tilde{v}_1 \rangle_{\mathbb{V}} = u_1 (v_1 , \tilde{v}_1 \rangle$

Similarly for $f = v_2 \in \mathbb{U}_2$ &

$\tilde{f} = \tilde{v}_2 \in \mathbb{U}_2 \quad \Rightarrow \langle f | \tilde{f} \rangle_{\mathbb{V}} = \langle v_2 | \tilde{v}_2 \rangle_{\mathbb{V}}$

$= u_2 (v_2 , \tilde{v}_2 \rangle$

Hence, since any vector in $\mathbb{V}$ say f or $\tilde{f}$ can be uniquely written as $f = v_1 + v_2$ & $\tilde{f} = \tilde{v}_1 + \tilde{v}_2$

where $u_1, \tilde{u}_1 \in \mathbb{U}_1$ & $v_2, \tilde{v}_2 \in \mathbb{U}_2$

We have done the above arguments.

$\langle f | \tilde{f} \rangle_{\mathbb{V}} = u_1 (v_1 , \tilde{v}_1 \rangle + u_2 (v_2 , \tilde{v}_2 \rangle$

$\Rightarrow v (f , \tilde{f} \rangle = u_1 (v_1 , \tilde{v}_1 \rangle + u_2 (v_2 , \tilde{v}_2 \rangle$

Note that uniqueness follows from the way we construct the inner product

To verify if $v(\cdot,\cdot)$ is an inner product check that,

① $v(f, f) \geq 0$ & 0 only for $f = 0$

⑪ $v(f_a + f_b, f) = u_1 (f_{a_1} + f_{b_1} , f_1) + u_2 (f_{a_2} + f_{b_2} , f_2)$

where $f_a = f_{a_1} + f_{a_2}$
$(\in V_1)$ $(\in V_2)$

$f_b = f_{b_1} + f_{b_2}$
$(\in V_1)$ $(\in V_2)$

& $f = f_1 + f_2$
$\in V_1$ $\in V_2$

(iii) $v(\alpha f, \tilde{f}) = u_1(\alpha f_1, \tilde{f}_1) + u_2(\alpha f_2, \tilde{f}_2)$

$f_1, \tilde{f}_1 \in V_1$ & $f_2, \tilde{f}_2 \in V_2$,

$\alpha \in \mathbb{C}$ & $f = f_1 + f_2$

$\tilde{f} = \tilde{f}_1 + \tilde{f}_2$

Similarly (iv) $v(f_1, \tilde{f}_1) = u_1(f_1, \tilde{f}_1) + u_2(f_2, \tilde{f}_2)$

$\Rightarrow$ $v(\tilde{f}_1, f) = u_1(\tilde{f}_1, f_1) + u_2(\tilde{f}_2, f_2)$

$= \overline{u_1(f_1, \tilde{f}_1)} + \overline{u_2(f_2, \tilde{f}_2)}$

$= \overline{u_1(f_1, \tilde{f}_1) + u_2(f_2, \tilde{f}_2)}$

$= \overline{v(f_1, \tilde{f}_1)}$

Hence $v(\cdot,\cdot)$ is a legitimate inner product on V.

$\boxed{Q_{21}}$ Suppose $v_1 \perp v_2$

$$\Rightarrow \|\alpha v_1+\beta v_2\|^2 = \langle \alpha v_1+\beta v_2 | \alpha v_1+\beta v_2\rangle$$

$$= \langle \alpha v_1 | \alpha v_1+\beta v_2\rangle + \langle \beta v_2 | \alpha v_1+\beta v_2\rangle$$

$$= \alpha\bar{\alpha}\langle v_1|v_1\rangle + \alpha\bar{\beta}\langle v_1|v_2\rangle + \beta\bar{\alpha}\langle v_2|v_1\rangle + \beta\bar{\beta}\langle v_2|v_2\rangle$$

$$= |\alpha|^2\langle v_1|v_1\rangle + |\beta|^2\langle v_2|v_2\rangle$$

$$= \|\alpha v_1\|^2 + \|\beta v_2\|^2 \quad \forall \alpha,\beta \in \mathbb{F}$$

Next Suppose

$$\|\alpha v_1+\beta v_2\|^2 = \|\alpha v_1\|^2 + \|\beta v_2\|^2$$

$$= \alpha\bar{\beta}\langle v_1|v_2\rangle + \beta\bar{\alpha}\langle v_2|v_1\rangle = 0$$

from previous Calculation

Now $\alpha\bar{\beta}\langle v_1|v_2\rangle = \beta\bar{\alpha}\langle v_2|v_1\rangle$

$$\Rightarrow \alpha\bar{\beta}\langle v_1|v_2\rangle + \overline{\alpha\bar{\beta}\langle v_1|v_2\rangle} = 0$$

$$\mathrm{Re}\left(\alpha\bar{\beta}\langle v_1|v_2\rangle\right) = 0$$

Let $\langle v_1|v_2\rangle = a+ib$
$a,b \in \mathbb{R}$

$$\Rightarrow \mathrm{Re}\left(\alpha\bar{\beta}(a+ib)\right) = 0$$

Let $\alpha\bar{\beta} = c+id$, $c,d \in \mathbb{R}$

$$\Rightarrow \mathrm{Re}((c+id)(a+ib)) = 0$$

$$\Rightarrow ac-bd = 0 \quad \forall c,d \in \mathbb{R}$$

Let $c=1,\ d=0$

$\Rightarrow a=0$

Similarly let $c=0,\ d=1$

$\Rightarrow b=0$

$a+ib=0$

$\Rightarrow \langle v_1|v_2\rangle=0$

$\Rightarrow v_1 \perp v_2$

$\boxed{Q\,24}$ Let $a_0+a_1x+a_2x^2+\cdots \in W$

Consider $f \in W^{\perp}$

$$\Rightarrow \int_a^b f(x)\cdot[a_0+a_1x+a_2x^2+\cdots]\,dx=0$$

$$\Rightarrow \int_a^b f(x)\,a_0\,dx+\int_a^b f(x)\cdot a_1x\,dx+\cdots=0$$

$\forall\ a_0,a_1,a_2,\cdots \in \mathbb{R}$

Let any one of $a_0, a_1, \cdots$ be 1. While the rest ~~of~~ are zero. at a time.

$\Rightarrow \int_a^b x^n\cdot f(x)=0$ for $n\in\mathbb{N}$ $[\mathbb{N}=\{0,1,2\cdots\}]$

this implies that $f(x)=0,\ \forall x$

Clearly $\mathbb{W}^{\perp} = \{0_f\}$

$\Rightarrow (\mathbb{W}^{\perp})^{\perp} = \mathbb{V}$ [$\because$ every f^n is orthogonal to zero f^n]

But $\mathbb{W} \subseteq \mathbb{V}$

$\Rightarrow \mathbb{W} \subset (\mathbb{W}^{\perp})^{\perp}$ & $(\mathbb{W}) \neq (\mathbb{W}^{\perp})^{\perp}$

[Note f^n that are zero "almost everywhere" are also treated as the "zero f^n"]

(25) $\mathcal{B} = \{1, x, x^2, x^3\}$

Let $\tilde{w}_1 = 1$

$$\|\tilde{w}_1\|^2 = \int_0^1 1\,dx = 1$$

$$w_1 = \tilde{w}_1$$

$$\tilde{w}_2 = x - \left(\int_0^1 x\,dx\right) \cdot \frac{1}{1^2} = \left(x - \frac{1}{2}\right)$$

$$\|\tilde{w}_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_0^1$$

$$= \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3\right] = \frac{1}{12}$$

$$\Rightarrow w_2 = 2\sqrt{3}\left(x - \frac{1}{2}\right) = 2\sqrt{3}x - \sqrt{3} = \sqrt{3}(2x-1)$$

Let $\tilde{w}_3 = x^2 - \int_0^1 x^2 \cdot 1\,dx - \int_0^1 x^2 \cdot \sqrt{3}(2x-1)\,dx \cdot \sqrt{3}(2x-1)$

$$= x^2 - \frac{1}{3} - \left(\frac{2\sqrt{3}}{4} - \frac{\sqrt{3}}{3}\right)\sqrt{3}(2x-1)$$

$$= x^2 - \frac{1}{3} - \frac{3}{2}(2x-1) + \cancel{2x+}1(2x-1)$$

$$= x^2 - x + \frac{1}{6}$$

$$\|\tilde{w}_3\|^2 = \int_0^1 \left(x^2 - x - \frac{1}{6}\right)^2 dx$$

$$= \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 + \frac{x^2}{3} - \frac{x}{3}\right) dx$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{2}{4} + \frac{1}{9} - \frac{1}{6}$$

$$= \frac{1}{180}$$

$\Rightarrow \|\tilde{W}_3\| = \dfrac{1}{6\sqrt{5}}$

$W_3 = \sqrt{5}\,(6x^2 - 6x + 1)$.

Similarly, compute $\tilde{w}_4$ and then w_4

$\langle 1, x, x^2, x^3 \rangle = \langle 1, \sqrt{3}(2x-1); \sqrt{5}(6x^2-6x+1), \sqrt{7}(20x^3-30x^2+12x-1) \rangle$

(28)

a) Consider the Vandermonde matrix to have rank deficiency

$\Rightarrow \exists\ \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, not all zero :

$\alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} = 0$ is satisfied for

m distinct values of x

But, by the fundamental theorem of algebra, an $(n-1)^{th}$ degree polynomial over a field can have at most $(n-1)$ distinct roots.

So, $m \leq n-1 < n \Rightarrow m < n$

But we have, $n \geq m$ which is a contradiction.

Hence, the given Vandermonde matrix must be of full column rank.

b) In this case, $n-1 = m-1$.

$\Rightarrow n = m$, so from (a)

V is a square matrix and full column rank (hence also full row rank and invertible)

Thus, $\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{m-1} \end{bmatrix}$ has a unique solution for $V \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{m-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$

given by $\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{m-1} \end{bmatrix} = V^{-1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$

The only thing that needs to be proved is that

$$l(x_i) = y_i.$$

By the given expression, $l(x)$ is a polynomial of degree $m-1$

Check that $$l(x_i) = \frac{y_i \prod_{j \neq i}^{m} (x_i - x_j)}{\prod_{j \neq i}^{m} (x_i - x_j)}$$

Since all the other terms in the sum contain $(x_i - x_i) = 0$ in the numerator

$$\Rightarrow l(x_i) = y_i \text{ . Hence proved.}$$

c) By the above formula,

$$y = l(x) = -8 \cdot \frac{(x-30)(x-45)(x-60)\, x}{15 \cdot 30 \cdot 45 \cdot 15}$$

$$+ 15x \frac{(x-15)(x-45)(x-60)}{30 \cdot 15 \cdot 15 \cdot 30}$$

$$- 19x \frac{(x-15)(x-30)(x-60)}{45 \cdot 30 \cdot 15 \cdot 15}$$

$$+ \frac{20x(x-15)(x-30)(x-45)}{60 \cdot 45 \cdot 30 \cdot 15}$$

is the trajectory of the ball in the given vertical plane.

For the landing on the ground, we need

$$y=l(x)=0$$

$x=0, y=0$ is the point where the defender kicks the ball, hence inadmissible as a solution.

$$\Rightarrow \frac{-8}{154}\frac{(x-30)(x-45)(x-60)}{6}+\frac{15}{154}\frac{(x-15)(x-45)(x-60)}{4}$$
$$-\frac{19}{154}\frac{(x-15)(x-30)(x-60)}{6}+\frac{20}{154}\frac{(x-15)(x-30)(x-45)}{24}=0$$

$$\Rightarrow \frac{-4}{3}(x-30)(x-45)(x-60)+\frac{15}{4}(x-15)(x-45)(x-60)$$
$$-\frac{19}{6}(x-15)(x-30)(x-60)+\frac{5}{6}(x-15)(x-30)(x-45)=0$$

$$\Rightarrow \left(\frac{-4}{3}+\frac{15}{4}-\frac{19}{6}+\frac{5}{6}\right)x^3+\left(\frac{4}{3}.135-\frac{15}{4}.120+\frac{19}{6}.105-\frac{5}{6}.90\right)x^2$$
$$+15^2\left\{\frac{-4}{3}(6+8+12)+\frac{15}{4}(3+4+12)-\frac{19}{6}(2+4+8)\right.$$
$$\left.+\frac{5}{6}(2+3+6)\right\}x$$
$$+15^3\left(\frac{4}{3}.24-\frac{15}{4}.12+\frac{19}{6}.8-\frac{5}{6}.6\right)=0$$

$$\Rightarrow \frac{1}{12}x^3-\frac{25}{2}x^2+\frac{1275}{4}x+24750=0$$

$$\Rightarrow x^3-150x^2+\cancel{38622}\,3825x+297000=0$$

roots of this equation: $-31.278, 90.639 \pm 35.77j$

$\Rightarrow$ No real positive roots, implying that the solution is not good ("Overfitting" case of Lagrange Interpolation)

d) Here we deal with an overdetermined system, So that V is a tall matrix. so there is a hope of getting exact matching at data points.

Let us set it up as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 15 & 15^2 \\ 1 & 30 & 30^2 \\ 1 & 45 & 45^2 \\ 1 & 60 & 60^2 \end{bmatrix}}_{V} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 15 \\ 19 \\ 20 \end{bmatrix}$$

Since by (a), V is full column rank, it has a left inverse given by

$V^+ = (V^T V)^{-1} V^T$. This is also a projection matrix onto the image of V

Thus,

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = V^+ \begin{bmatrix} 0 \\ 8 \\ 15 \\ 19 \\ 20 \end{bmatrix}$$

will give us the "best" solution

$$V^+ = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 15 & 30 & 45 & 60 \\ 0 & 225 & 900 & 2025 & 3600 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 15 & 225 \\ 1 & 30 & 900 \\ 1 & 45 & 2025 \\ 1 & 60 & 3600 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 15 & 30 & 45 & 60 \\ 0 & 225 & 900 & 2025 & 3600 \end{bmatrix}$$

$$V^{+} = \begin{bmatrix} 5 & 150 & 6750 \\ 150 & 6750 & 337500 \\ 6750 & 337500 & 17912500 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 15 & 30 & 45 & 60 \\ 0 & 225 & 900 & 2025 & 3600 \end{bmatrix}$$

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} -0.2286 \\ 0.6636 \\ -0.0054 \end{bmatrix}$$

$$\Rightarrow \quad y = -0.0054x^2 + 0.6636x - 0.2286$$

$$\Rightarrow \quad y = 0 \quad \Rightarrow \quad x = 0.3453,\ 122.66$$

So the ball lands roughly 122 m away from where the ~~deflector~~ defender kicked it.

Note that though, this is a "best fit" instead of exact matching, as in (c) above, this "best fit" curve is much more realistic and obeys the law of physics.