# Applied Linear Algebra: Problem set-3

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[Note: $\mathcal{L}(\mathbb{V}, \mathbb{W})$ denotes the vector space of all linear transformations from the vector space $\mathbb{V}$ to the vector space $\mathbb{W}$.]

1. For a linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$, given by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$, ascertain if $T$ is invertible. If so, then obtain a definition for $T^{-1}$ such as the one for $T$. Prove that $(T^2 - I)(T - 3I) = 0$.

2. For a linear operator $T : \mathbb{V} \to \mathbb{V}$, show that the following holds: $T(T(v)) = 0 \implies T(v) = 0, v \in \mathbb{V}$ is equivalent to $\text{Im}(T) \cap \text{Ker}(T) = \{0\} \subseteq \mathbb{V}$.

3. For a linear map $T : \mathbb{V} \to \mathbb{F}$, ($\mathbb{V}$ finite dimensional) show that if $u \notin \text{Ker}(T)$, then any $v \in \mathbb{V}$ can be expressed uniquely as $v = w_1 + w_2$ with $w_1 \in \text{span}(u)$ and $w_2 \in \text{Ker}(T)$.

4. Obtain a linear map, $\varphi$ from $\mathbb{F}^5$ to $\mathbb{F}^2$ such that its kernel is given by $\text{Ker}(\varphi) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = \alpha x_2; x_3 = x_4 = x_5\}$ $(\alpha \in \mathbb{F})$.

5. Over a finite dimensional vector space, $\mathbb{V}$, suppose we have two linear operators $\psi$ and $\tau$. Prove that the composition of the two operators, $\psi \circ \tau$, is invertible if and only if both $\psi$ and $\tau$ are invertible.

6. Suppose $\mathcal{C}^k(\mathbb{R})$ is the vector space of all $k$-times continuously differentiable functions on $\mathbb{R}$, $k \in \{0, 1, 2, \ldots\}$. Consider $\mathcal{L}_1 : \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ and $\mathcal{L}_2 : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ be given by $\mathcal{L}_1(f(x)) = \frac{d}{dx}(f(x)) + \alpha f(x),\ \alpha \in \mathbb{R}$, and $\mathcal{L}_2(g(x)) = \frac{d^2}{dx^2}(g(x)) + \omega^2 g(x),\ \omega \in \mathbb{R}$. What are the dimensions of the kernels of $\mathcal{L}_1$ and $\mathcal{L}_2$? Write down one possible choice of basis for each operator.

7. Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces. Show that the map which takes $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ to $\varphi' \in \mathcal{L}(\mathbb{W}', \mathbb{V}')$ is an isomorphism of $\mathcal{L}(\mathbb{V}, \mathbb{W})$ onto $\mathcal{L}(\mathbb{W}', \mathbb{V}')$.

8. Let $\mathbb{U}$ and $\mathbb{W}$ be subspaces of a finite dimensional vector space $\mathbb{V}$. Show that $(\mathbb{U} \cap \mathbb{W})^0 = \mathbb{U}^0 + \mathbb{W}^0$.

9. Consider the vector space of all polynomials over the real field denoted by $\mathbb{R}[x]$ and define $\varphi \in \mathbb{R}[x]'$ as $\varphi(f) := \int_0^1 f(x)dx$. Suppose $\mathcal{D}$ is the differentiation operator in $\mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$. Obtain $\mathcal{D}'(\varphi)$ and demonstrate that it belongs to $\mathbb{R}[x]'$.

10. Let $\mathbb{V}$ be a finite dimensional vector space and $\Gamma$ be a subspace of $\mathbb{V}'$. Prove that $\Gamma = \{v \in \mathbb{V} : \varphi(v) = 0\ \forall\ \varphi \in \Gamma\}^0$.

11. Suppose $\mathbb{V}$ is a finite dimensional vector space and $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ is a linearly independent set in $\mathbb{V}'$. Show that $\dim(\cap_{i=1}^m \text{Ker}(\varphi_i)) = \dim(\mathbb{V}) - m$.

12. The *double dual* of a vector space, $\mathbb{V}$, denoted by $\mathbb{V}''$, is the dual space of $\mathbb{V}'$ by definition. Define $\Lambda : \mathbb{V} \to \mathbb{V}''$ as $(\Lambda v)(\varphi) = \varphi(v)$, where $v \in \mathbb{V}$ and $\varphi \in \mathbb{V}'$.
(a) Prove that $\Lambda : \mathbb{V} \to \mathbb{V}''$ is a linear map.
(b) Prove that for $\tau \in \mathcal{L}(\mathbb{V})$, we have $\tau'' \circ \Lambda = \Lambda \circ \tau$, where $\tau'' = (\tau')'$.
(c) Prove that if $\mathbb{V}$ is finite dimensional, then $\Lambda$ is an isomorphism between $\mathbb{V}$ and $\mathbb{V}''$.

13. If $f : \mathbb{V} \to \mathbb{W}$, the *graph* of $f$ is defined as a subset of $\mathbb{V} \times \mathbb{W}$, defined by $\mathcal{G}(f) := \{(x, f(x)) \in \mathbb{V} \times \mathbb{W} : x \in \mathbb{V}\}$. Prove that $f$ is a linear map, if and only if $\mathcal{G}(f)$ is a subspace of $\mathbb{V} \times \mathbb{W}$.

14. For vector spaces $\mathbb{V}_i$, $i = 1, 2, \ldots, m$, suppose $\mathcal{L}(\mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m, \mathbb{W})$ is the space of all linear functions from $\mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m$ to $\mathbb{W}$, while $\mathcal{L}(\mathbb{V}_i, \mathbb{W})$ is the space of all linear functions from $\mathbb{V}_i$ to $\mathbb{W}$. Show that $\mathcal{L}(\mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m, \mathbb{W})$ is isomorphic to $\mathcal{L}(\mathbb{V}_1, \mathbb{W}) \times \mathcal{L}(\mathbb{V}_2, \mathbb{W}) \times \ldots \times \mathcal{L}(\mathbb{V}_m, \mathbb{W})$.

15. Suppose $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional vector spaces. Show that for a linear transformation, $\tau \in \mathcal{L}(\mathbb{V}, \mathbb{W})$, and its dual map, $\tau' \in \mathcal{L}(\mathbb{W}', \mathbb{V}')$, one has the following:
(a) $\tau$ is surjective $\iff$ $\tau'$ is injective, and
(b) $\tau$ is injective $\iff$ $\tau'$ is surjective.

16. For vector spaces $\mathbb{U}, \mathbb{V}, \mathbb{W}$, such that $\mathbb{U}$ and $\mathbb{V}$ are subspaces of $\mathbb{W}$ show that for $w_1, w_2 \in \mathbb{W}$, if we have $w_1 + \mathbb{U} = w_2 + \mathbb{V}$, then $\mathbb{U} = \mathbb{V}$.

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17. For two affine subsets of a vector space $\mathbb{V}$, say $A_1$ and $A_2$, prove that $A_1 \cap A_2$ is either an affine subset of $\mathbb{V}$ or the empty set.

18. Suppose $\mathbb{U} \subseteq \mathbb{V}$ and $\{v_1 + \mathbb{U}, v_2 + \mathbb{U}, \ldots, v_m + \mathbb{U}\}$ is a basis for $\mathbb{V}/\mathbb{U}$, while $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{U}$. Show that $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{V}$.

19. For finite dimensional vector spaces $\mathbb{U} \subseteq \mathbb{V}$, show that there exists a subspace $\mathbb{W}$ of $\mathbb{V}$ such that dim($\mathbb{W}$) =dim($\mathbb{V}/\mathbb{U}$) and $\mathbb{V} = \mathbb{U} \bigoplus \mathbb{W}$. Further, show that $\mathbb{V}$ is isomorphic to $\mathbb{U} \times \mathbb{V}/\mathbb{U}$.

20. For a non-trivial linear functional $f : \mathbb{V} \to \mathbb{F}$, show that $\dim(\mathbb{V}/\ker(f)) = 1$.

21. Show that for vector spaces $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_m$, we may claim that $\mathbb{V}_1 + \mathbb{V}_2 + \ldots + \mathbb{V}_m$ is a direct sum if and only if the unique representation of 0 as a sum of the form $\sum_{i=1}^{m} v_i$, where $v_i \in \mathbb{V}_i$ is given by $v_i = 0 \; \forall i$.

22. Suppose $\varphi \in \mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$ is an injective linear operator on $\mathbb{R}[x]$ such that $\deg(\varphi(f)) \leq \deg(f)$ for any non-zero polynomial, $f$, in $\mathbb{R}[x]$. Prove that $\varphi$ is a surjection and that $\deg(\varphi(f)) = \deg(f)$ for every non-zero polynomial $f$. [**Hint**: One may consider the properties of an operator *restricted* to a subset/subspace of its domain.]

23. Define a linear map $\tau : \mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m \to \mathbb{V}_1 + \mathbb{V}_2 \ldots + \mathbb{V}_m$ as $\tau(v_1, v_2, \ldots, v_n) = v_1 + v_2 + \ldots + v_n$. Show that $\tau$ is injective if and only if $\mathbb{V}_1 + \mathbb{V}_2 \ldots + \mathbb{V}_m$ is a direct sum.

24. State whether the following assertions are true or false and provide brief justifications in support of/against them.
(a) For two operators $\varphi, \psi \in \mathcal{L}(\mathbb{V}, \mathbb{V})$ ($\mathbb{V}$ is a vector space), such that $\mathrm{im}(\varphi) \subset \ker(\psi)$, $(\varphi\psi)^2 = 0$.
(b) There cannot exist a linear operator, $\varphi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, such that $\mathrm{im}(\varphi) = \ker(\varphi)$ for $n = 2m - 1$, where $m \in \mathbb{N}$.
(c) If $\varphi_1, \varphi_2 \in \mathbb{V}'$, where $\mathbb{V}$ is a vector space over field $\mathbb{F}$, such that $\ker(\varphi_1) = \ker(\varphi_2)$, then there exists $\alpha \in \mathbb{F}$ so that $\varphi_1 = \alpha\varphi_2$.
(d) Suppose $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{L}(\mathbb{V}, \mathbb{V})$ are operators on the finite dimensional vector space $\mathbb{V}$ such that $\varphi_1\varphi_2\varphi_3 = \text{identity}_{\mathbb{V}}$. Then $\varphi_2$ is invertible and $\varphi_2^{-1} = \varphi_3\varphi_1$.
(e) Consider the differentiation operator, $\mathcal{D}$, on the vector space of polynomials over the real field, $\mathbb{R}[x]$ (i.e., $\mathcal{D} \in \mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$). For any $\lambda \in \mathbb{R}$, $\ker(\mathcal{D} - \lambda I)$ is a non-trivial subspace of $\mathbb{R}[x]$.

25. Let $\mathbb{R}[x]$ be the vector space of polynomials over $\mathbb{R}$ and consider subspaces of $\mathbb{R}[x]$ given by $\mathbb{W}_1 = \{f(x) \in \mathbb{R}[x] : f(0) = f(10)\}$ and $\mathbb{W}_2 = \{f(x) \in \mathbb{R}[x] : f(0) = f(10) = 0\}$. Then $\mathbb{R}[x]/\mathbb{W}_1$ is isomorphic to $\mathbb{R}[x]/\mathbb{W}_2$. Prove if it is true or give a counterexample to disprove it.

26. (a) Verify the first isomorphism theorem (you are not allowed to use it/invoke it at any point!) for the case when $\mathbb{V}$ is the space of all polynomials in $x$ over the real field, of degree less than or equal to $n$, $\mathbb{W} = \mathbb{V}$, and the linear map $T : \mathbb{V} \to \mathbb{W}$ is the derivative map. Then provide an interpretation for what each element in $\mathbb{V}/\mathrm{Ker}(T)$ represents. In other words, you are required to define the induced map corresponding to the derivative map and show that this induced map is linear, injective, has an image identical with the original derivative map, and finally, $\mathbb{V}/\mathrm{Ker}(T) \cong \mathrm{im}(T)$. Thereafter, you need to describe what each element in $\mathbb{V}/\mathrm{Ker}(T)$ represents.
(b) Define a linear map $\tau : \mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m \to \mathbb{V}_1 + \mathbb{V}_2 \ldots + \mathbb{V}_m$ as $\tau(v_1, v_2, \ldots, v_n) = v_1 + v_2 + \ldots + v_n$. Show that $\tau$ is injective if and only if $\mathbb{V}_1 + \mathbb{V}_2 \ldots + \mathbb{V}_m$ is a direct sum.

*"A consequence of the existence of general system properties is the appearance of structural similarities or isomorphisms in different fields... an exponential law of growth applies to certain bacterial cells, to populations of bacteria, of animals or humans, and to the progress of scientific research measured by the number of publications in genetics or science in general."*

**—in 'General System Theory' (1968) Ludwig von Bertalanffy.**