COORDINATE LINEAR MAPS

Coordinate Assignments :

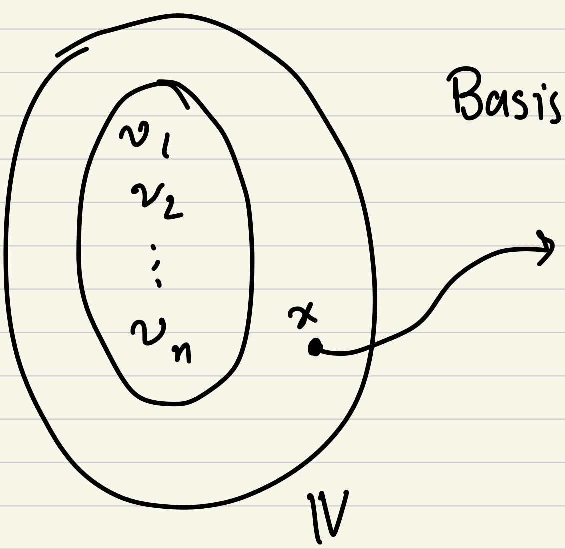
The coordinates of a vector v in vector space $\mathbb{V}$ relative to the basis B will be the scalars that serve to express v as the linear combination of basis vectors.

$$v \in \mathbb{V} \longrightarrow v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where $\forall a_i \in \mathbb{F}$ and $B = \{v_1, v_2, \ldots, v_n\}$

We will be able to say which is the i^{th} coordinate because B is the ordered basis. If B is an arbitrary basis then there is no natural ordering of vectors in B.

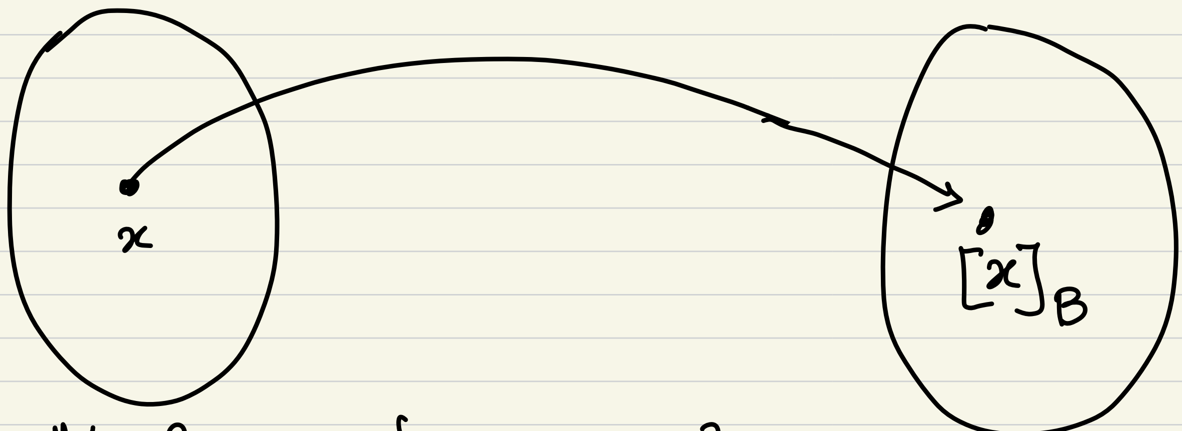
> If $\mathbb{V}$ is a f.d.v.s over $\mathbb{F}$, an ordered basis for $\mathbb{V}$ has a finite sequence (not set) of vectors which is LI spanning set of $\mathbb{V}$.



Basis $B = (v_1, v_2, \ldots, v_n)$

$$x = \sum_{i=1}^{n} a_i v_i$$

The representation of x is unique in terms of OB, B.
Since it is unique, we can collect the scalars a_i and put it in n tuple of scalars in $\mathbb{F}^n$.

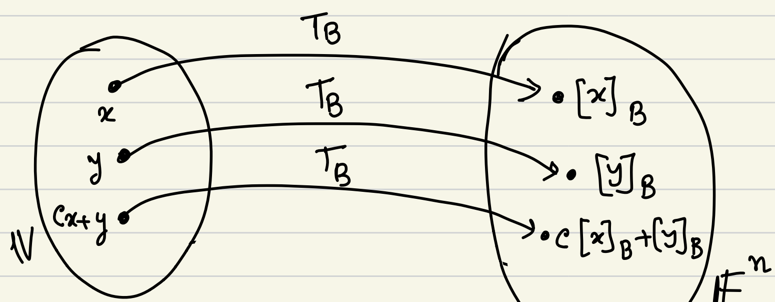


$\mathbb{V}$: Basis $B = \{v_1, v_2, \ldots, v_n\}$ $\qquad$ $\mathbb{F}^n$

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \qquad [x]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$x \longmapsto [x]_B$$

We shall call a_i as the i^{th} coordinate of the vector x relative to the ordered basis B. This order basis B acts like a coordinate system. Where coordinates are relative to system B to which we are working.



$$cx + y = c \cdot \sum a_i v_i + \sum b_i v_i$$
$$= \sum (a_i c + b_i) v_i$$

$$[cx+y]_B = \begin{bmatrix} a_1c_1+b_1 \\ a_2c+b_2 \\ \vdots \\ a_nc+b_n \end{bmatrix} = \begin{bmatrix} a_1c_1 \\ a_2c \\ \vdots \\ a_nc \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$[x]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad [y]_B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Therefore, $[cx+y]_B = c.[x]_B + [y]_B$.

Hence the underlying transformation $T_B : \mathbb{V} \longrightarrow \mathbb{F}^n$ is a linear transformation that takes v from $\mathbb{V}$ and gives $[v]_B$ in $\mathbb{F}^n$ which is coordinates of v relative to B.

Every n tuple in $\mathbb{F}^n$ is the coordinates of some vector v in $\mathbb{F}$ namely,

$$v = \sum_{i=1}^{n} a_i v_i$$

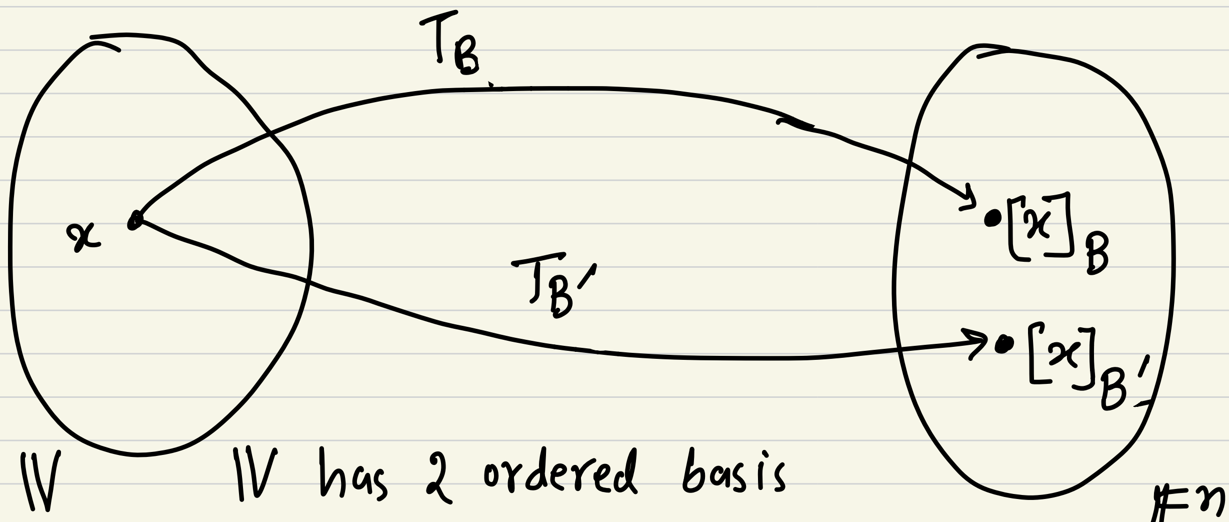
Each ordered basis for $\mathbb{V}$ determines a one to one correspondance

$$v \xmapsto{T_B} (a_1, a_2, \dots, a_n)$$

between the set of all vectors in $\mathbb{F}$ and set of all n-tuples in $\mathbb{F}^n$. Instead of working with $v \in \mathbb{V}$, it is convenient to use the coordinates relative to OB, B.

$$v \in \mathbb{V} \xmapsto{\quad T_B \quad} [v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$$

It is natural to ask now, if we change the coordinate system B in which we are representing the vector v then how will we go from one cordinate n tuple in $\mathbb{F}^n$ to another coordinate n-tuple in $\mathbb{F}^n$?



$\mathbb{V}$ has 2 ordered basis

$$B = \{v_1, v_2, \ldots, v_n\} \quad \forall v_i \in \mathbb{V}$$

$$B' = \{v_1', v_2', \ldots, v_n'\} \quad \forall v_i' \in \mathbb{V}.$$

Suppose we want to represent v_1' in terms of $\{v_i\}_{i=1}^{n}$

$$\left.\begin{aligned} v_1' &= P_{11} v_1 + P_{21} v_2 + \cdots \cdots + P_{n1} v_n \\ v_2' &= P_{12} v_1 + P_{22} v_2 + \cdots \cdots + P_{n2} v_2 \\ &\vdots \\ v_n' &= P_{1n} v_1 + P_{2n} v_2 + \cdots \cdots + P_{nn} v_n \end{aligned}\right\} \forall P_{ij} \in \mathbb{F}$$

Consider a vector $x \in \mathbb{V}$ and x has 2 representation,

$$\left.\begin{aligned} x &= a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \quad \text{—— (i)} \\ x &= a_1' v_1' + a_2' v_2' + \cdots + a_n' v_n' \quad \text{—— (ii)} \end{aligned}\right\} \quad \begin{aligned} &\forall a_i \in \mathbb{F} \\ &\forall a_i' \in \mathbb{F} \end{aligned}$$

Since we now know how to write v_i' in terms of v_i so we substitute the values -

$$\begin{aligned} x = \; & a_1' \cdot \left(P_{11} v_1 + P_{21} v_2 + \cdots + P_{n1} v_n \right) + \\ & a_2' \left(P_{12} v_1 + P_{22} v_2 + \cdots + P_{n2} v_n \right) + \\ & \vdots \\ & a_n' \left(P_{1n} v_1 + P_{2n} v_2 + \cdots + P_{nn} v_n \right) \end{aligned}$$

$$\begin{aligned} = \; & \left(a_1' P_{11} + a_2' P_{12} + \cdots + a_n' P_{1n} \right) v_1 + \\ & \left(a_1' P_{21} + a_2' P_{22} + \cdots + a_n' P_{2n} \right) v_2 + \\ & \vdots \\ & \left(a_1' \cdot P_{n1} + a_2' P_{n2} + \cdots + a_n' P_{nn} \right) v_n \quad \text{—— (iii)} \end{aligned}$$

Equation (i) & (iii) we get -

$$\begin{aligned} a_1 &= a_1' P_{11} + a_2' P_{12} + \cdots + a_n' P_{1n} \\ a_2 &= a_1' P_{21} + a_2' P_{22} + \cdots + a_n' P_{2n} \\ &\vdots \\ a_n &= a_1' P_{n1} + a_2' P_{n2} + \cdots + a_n' P_{nn} \end{aligned}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ & \vdots & & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{bmatrix}$$

$\downarrow$ coordinate representation of x relative to B

$\downarrow$ coordinate representation of x relative to B'

$\Rightarrow \quad [x]_B = P\,[x]_{B'}$ where P is a change of basis matrix.

Closely look at the columns of P.

A basis vector (new), v_1' written in terms of old basis B as:

$$v_1' = P_{11}v_1 + P_{21}v_2 + \cdots + P_{n1}v_n$$

$$\begin{tikzcd} v_1' \arrow[r, mapsto, "T_B"'] \arrow[d, mapsto, "T_{B'}"'] & {[v_1']_B} \arrow[dl, dashed, no head, "\text{relation?}"] \\ {[v_1']_{B'}} & \end{tikzcd}$$

$$v_1' = 1 \cdot v_1' + 0 \cdot v_2' + \cdots + 0 \cdot v_n'$$

$$[v_1']_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$[v_1']_B = \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix}$$

Therefore we can write the old coordinates in terms of the new cordinates as:

$$\underbrace{\begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix}}_{[v_1']_B} = \underbrace{\begin{bmatrix} p_{11} & p_{12} & & p_{1n} \\ p_{21} & p_{22} & & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & & p_{nn} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{[v']_{B'}}$$

$$\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\text{coordinate representation of } x \text{ relative to } B} = \begin{bmatrix} \underbrace{\boxed{\begin{matrix} p_{11} \\ p_{21} \\ \\ p_{n1} \end{matrix}}}_{[v_1']_B} & \underbrace{\boxed{\begin{matrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{matrix}}}_{[v_2']_B} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \underbrace{\boxed{\begin{matrix} p_{1n} \\ p_{2n} \\ \\ p_{nn} \end{matrix}}}_{[v_n']_B} \end{bmatrix} \underbrace{\begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{bmatrix}}_{\text{coordinate representation of } x \text{ relative to } B'}$$

$$
\begin{tikzcd}
v \in V \arrow[r, mapsto, "T_B"] \arrow[d, mapsto, "T_{B'}"'] & \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [v]_B \quad \text{(English)} \arrow[dl, bend left, "P^{-1}[v]_B"] \\
\text{(French)} \quad \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{bmatrix} = [v]_{B'} \arrow[ur, bend left, "P\cdot[v]_{B'}"] &
\end{tikzcd}
$$

$$
\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [v_1']_B & [v_2']_B & \cdots & [v_n']_B \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}
\underbrace{\begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{bmatrix}}_{\text{French}}
=
\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\text{English}}
$$

We need to know how to write each basis of french vocabulary to english.

$$
\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [v_1]_{B'} & [v_2]_{B'} & \cdots & [v_n]_{B'} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}
\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\text{English}}
=
\underbrace{\begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{bmatrix}}_{\text{French.}}
$$

We need to know how to write each english basis of English vocabulary to French.

$$\begin{tikzcd}
x \in V \arrow[r, mapsto, "T_B"] \arrow[d, mapsto, "T_{B'}"'] & {[x]_B} \arrow[dl, shift left, "P^{-1}[x]_B"'] \\
{[x]_{B'}} \arrow[ur, shift left, "P\,[x]_{B'}"'] &
\end{tikzcd}$$

Therefore,

$$[x]_B = P\,[x]_{B'}$$

$$[x]_{B'} = P^{-1}\,[x]_B$$

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [v_1']_B & [v_2']_B & \cdots & [v_n']_B \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

We need to find the matrix P and P^{-1} to change basis from one to another.

Exercise 1: Show that the vectors

$$\alpha_1 = (1, 1, 0, 0), \quad \alpha_2 = (0, 0, 1, 1)$$

$$\alpha_3 = (1, 0, 0, 4), \quad \alpha_4 = (0, 0, 0, 2)$$

form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \underline{OB}$$

$$B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\} \qquad \underline{OB}$$

$$e_1 \in V \xmapsto{T_B} e_1 = 1 \cdot e_1 + 0 \cdot e_2 + \cdots + 0 \cdot e_4$$

$$[e_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow T_{B'}$$

$$e_1 = R_{11} \cdot e_1' + R_{21} \cdot e_2' + R_{31} \cdot e_3' + R_{41} \cdot e_4'$$

$$[e_1]_{B'} = \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \\ R_{41} \end{bmatrix}$$

Our objective is to find $[R_{11}, R_{21}, R_{31}, R_{41}]^t$.

$$R_{11} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + R_{21} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + R_{31} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix} + R_{41} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \\ R_{41} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Similarly for other basis e_2, e_3, e_4 we can write-

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -0.5 & 0.5 \end{bmatrix}$$

$$[e_1]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \quad [e_3]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -0.5 \end{bmatrix}$$

$$[e_2]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad [e_4]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}$$

Exercise 3: Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0).$$

What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?

$$B = \left\{ \begin{array}{ccc} b_1 & b_2 & b_3 \\ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \right\}$$

x is written in standard basis, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

as :

$$x = a \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[x]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{to find } [x]_B = ?$$

$$[x]_B = \begin{bmatrix} & & \\ [e_1]_B & [e_2]_B & [e_3]_B \\ & & \end{bmatrix} [x]_S$$

$$\begin{tikzcd} e_1 \arrow[r, mapsto, "T_S"] \arrow[d, mapsto, "T_B"] & {[e_1]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \\ {} & \end{tikzcd}$$

$$e_1 = R_{11} \cdot b_1 + R_{21} b_2 + R_{31} b_3$$

$$[e_1]_B = \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \end{bmatrix}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \underbrace{\begin{bmatrix} R_{11}\\ R_{21}\\ R_{31}\end{bmatrix}}_{[e_1]_B}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \underbrace{\begin{bmatrix} R_{12}\\ R_{22}\\ R_{32}\end{bmatrix}}_{[e_2]_B}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \underbrace{\begin{bmatrix} R_{13}\\ R_{23}\\ R_{33}\end{bmatrix}}_{[e_3]_B}$$

$$\Rightarrow \begin{pmatrix} R_{11} & R_{12} & R_{13}\\ R_{21} & R_{22} & R_{23}\\ R_{31} & R_{32} & R_{33}\end{pmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^{-1}$$

$$[x]_B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^{-1} [x]_S$$

$$[x]_B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^{-1} \begin{bmatrix} a\\ b\\ c\end{bmatrix} = \begin{bmatrix} 0 & 1 & -1\\ 0 & 1 & 0\\ 1 & -2 & 1\end{bmatrix} \begin{bmatrix} a\\ b\\ c\end{bmatrix}$$

$$[x]_B = \begin{bmatrix} b-c\\ b\\ a-2b+c\end{bmatrix}$$

Remarks: It is not like $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is mapped to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

But $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the LC of vectors b_1, b_2, b_3.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = R_{11} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + R_{21} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + R_{31} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Here $R_{11} = 0$, $R_{21} = 0$, $R_{31} = 1$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

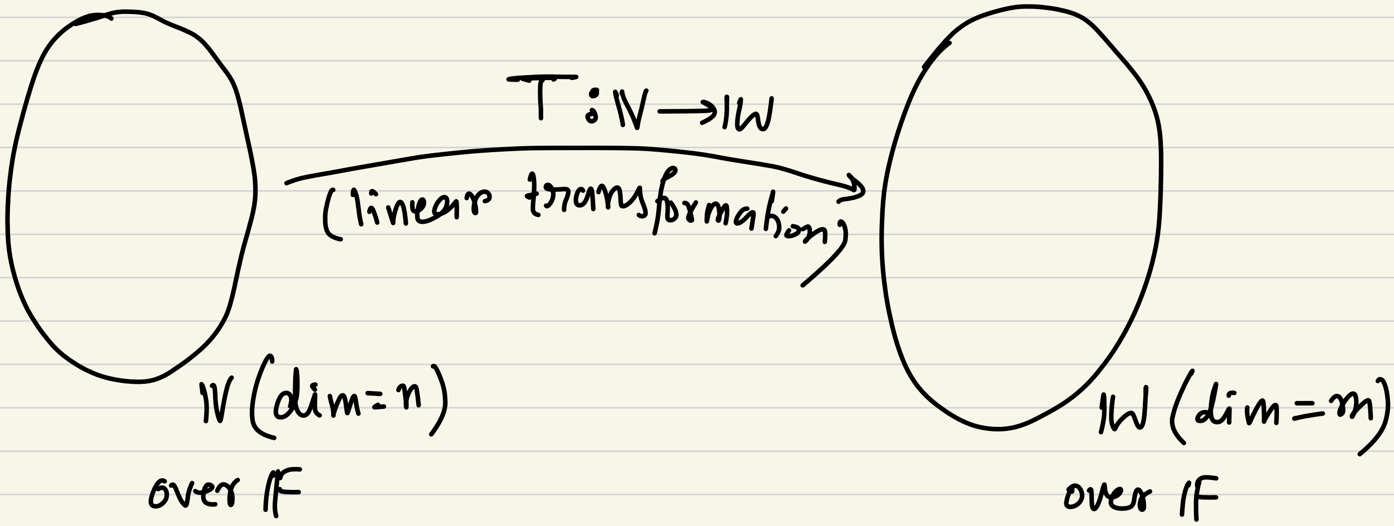
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1 \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Plugging the 1st basis in the equation

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{[e_1]_S} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{[e_1]_B}$$

Do not think that $[e_1]_S$ will map to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $(= b_1)$ which is definitely not the case. The equation gives translation but $[e_1]_S$ is written as a LC of b_i's. It's not like $[e_1]_S$ will be encoded as b_1 in B.

Representation of Transformation by Matrices:



$B_v = \{v_1, v_2, \ldots, v_n\}$ (OB)

$B_W = \{w_1, w_2, \ldots, w_m\}$ (OB)

$$v_1 \in \mathbb{V} \xmapsto{\quad T \quad} Tv_1 \in \mathbb{W}$$

Since Tv_1 belongs to $\mathbb{W}$, it can be represented as a LC of basis of $\mathbb{W}$.

$$Tv_1 = A_{11}w_1 + A_{21}w_2 + \cdots + A_{m1}w_m$$

$$Tv_2 = A_{12}w_1 + A_{22}w_2 + \cdots + A_{m2}w_m$$

$$\vdots$$

$$Tv_n = A_{1n}w_1 + A_{2n}w_2 + \cdots + A_{mn}w_m$$

In terms of coordinates also we can write -

$$\begin{tikzcd}
v_1 \in \mathbb{V} \arrow[r, mapsto, "T"] & Tv_1 \in \mathbb{W}
\end{tikzcd}$$

$$Tv_1 = A_{11}w_1 + \cdots + A_{m1}w_m$$

$$\Big\downarrow T_{B_W}$$

$$[Tv_1]_{B_W} = \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} \in \mathbb{F}^m$$

Therefore -

$$\begin{bmatrix} Tv_1 \\ Tv_2 \\ \vdots \\ Tv_n \end{bmatrix} = \begin{bmatrix} [Tv_1]_{B_W} & [Tv_2]_{B_W} & \cdots & [Tv_n]_{B_W} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

Consider any arbitrary $v \in \mathbb{V}$ and see how is it transformed to $\mathbb{W}$?

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

$$Tv = a_1(Tv_1) + a_2(Tv_2) + \cdots + a_n(Tv_n)$$

Try to represent these in terms of w_i

$$
\begin{array}{ccc}
v \in V & \xmapsto{\quad T \quad} & Tv \\
v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n & & Tv = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m \\
\Big\downarrow T_{B_V} & & \Big\downarrow T_{B_W} \\
[v]_{B_V} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n & \xrightarrow{\quad A \quad} & [Tv]_{B_W} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{F}^m
\end{array}
$$

$$
\begin{aligned}
Tv = {} & a_1 \left(A_{11} w_1 + A_{21} w_2 + \cdots + A_{m1} w_m \right) + \\
& a_2 \left(A_{12} w_1 + A_{22} w_2 + \cdots + A_{m2} w_m \right) + \\
& \vdots \\
& a_n \left(A_{1n} w_1 + A_{2n} w_2 + \cdots + A_{mn} w_m \right) \\
= {} & \left(a_1 A_{11} + a_2 A_{12} + \cdots + a_n A_{1n} \right) w_1 + \\
& \left(a_1 A_{21} + a_2 A_{22} + \cdots + a_n A_{2n} \right) w_2 + \\
& \vdots \\
& \left(a_1 A_{m1} + a_2 A_{m2} + \cdots + a_n A_{mn} \right) w_m
\end{aligned}
$$

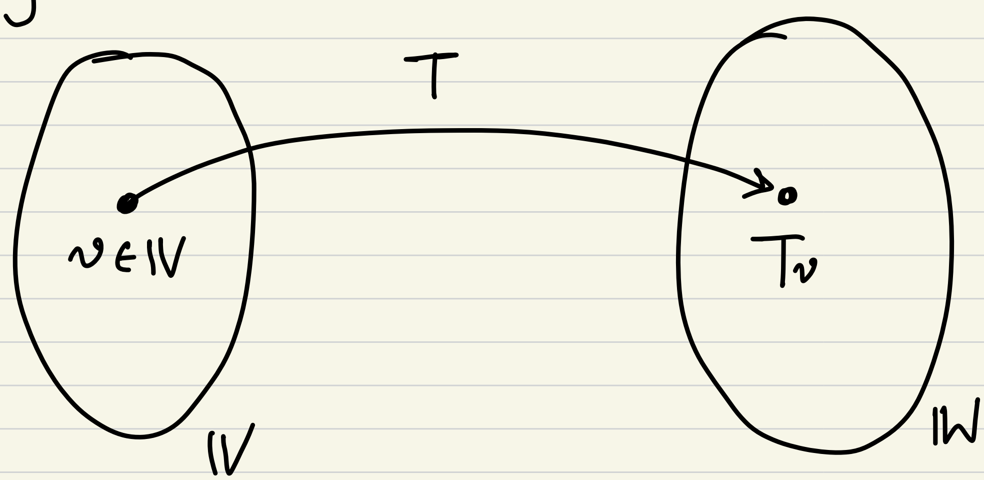
$$= \gamma_1 w_1 + \gamma_2 w_2 + \cdots + \gamma_m w_m$$

$$[Tv]_{B_W} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Therefore :

$$\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{[Tv]_{B_W}} = \underbrace{\begin{array}{cccc} [Tv_1]_{B_W} & [Tv_2]_{B_W} & & [Tv_n]_{B_W} \\ \left[\begin{array}{c} A_{11} \\ A_{21} \\ \\ A_{m1} \end{array}\right. & \begin{array}{c} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \\ \cdots \end{array} & \left.\begin{array}{c} A_{1n} \\ A_{2n} \\ \\ A_{mn} \end{array}\right] \end{array}}_{A} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{[v]_{B_V}}$$

Objective:



We want to represent Tv in terms of B_W or $[Tv]_{B_W}$

$$\begin{tikzcd}
v \in \mathbb{V} \arrow[r, "T"] \arrow[d, "T_{B_V}"'] & Tv \in \mathbb{W} \arrow[d, "T_{B_W}"] \\
[v]_{B_V} \in \mathbb{F}^n \arrow[r, "A"] & [Tv]_{B_W} \in \mathbb{F}^m
\end{tikzcd}$$

Matrix A : It is the matrix of transformation T relative to the pair of ordered bases B_V and B_W.

Let $\mathbb{V}$ be a n dim vector space over $\mathbb{F}$.
Let $\mathbb{W}$ be a m dim vector space over $\mathbb{F}$.
Let B_V be an ordered basis for $\mathbb{V}$.
Let B_W be an ordered basis for $\mathbb{W}$.
Let T be a linear transformation from $\mathbb{V}$ to $\mathbb{W}$.

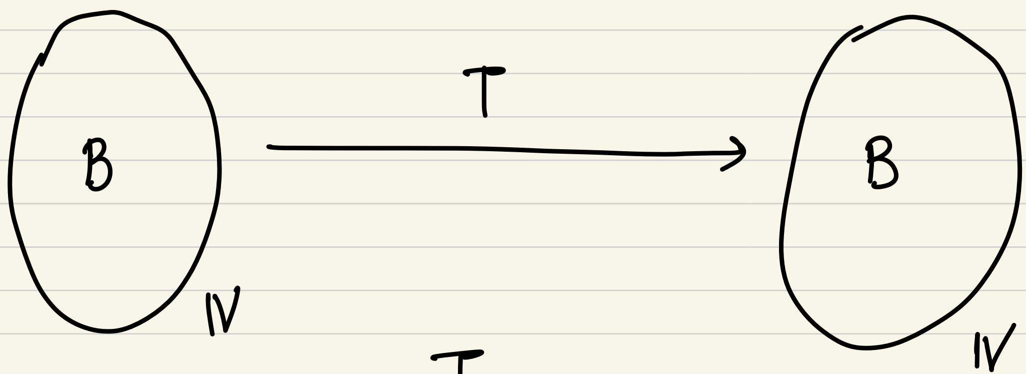
There exist a matrix $A \in \mathbb{F}^{m \times n}$ such that -

$$[Tv]_{B_W} = A\,[v]_{B_V} \quad \text{for every vector } v \in \mathbb{V}.$$

Furthermore, $T \rightarrow A$ is one to one correspondance b/w every linear transformation from $\mathbb{V}$ to $\mathbb{W}$ and every matrix belonging to $\mathbb{F}^{m \times n}$. over $\mathbb{F}$.

We might be interested the representin of linear operators by matrices.

Suppose we use the same ordered basis $B = \{v_i\}_{i=1}^{n}$



$$\begin{tikzcd}
v \in \mathbb{V} \arrow[r, mapsto, "T"] \arrow[d, mapsto, "T_B"] & Tv \in \mathbb{V} \arrow[d, mapsto, "T_B"] \\
{[v]_B} \arrow[r, "A"] & {[Tv]_B}
\end{tikzcd}$$

In this case we shall call A as the matrix of T relative to the ordered basis B.

$$Tv_1 = A_{11}v_1 + A_{21}v_2 + \cdots + A_{n1}v_n$$
$$Tv_2 = A_{12}v_1 + A_{22}v_2 + \cdots + A_{n2}v_n$$
$$\vdots$$
$$Tv_n = A_{1n}v_1 + A_{2n}v_2 + \cdots + A_{nn}v_n$$

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

$$Tv = a_1 (Tv_1) + a_2 (Tv_2) + \cdots + a_n (Tv_n)$$

$$= a_1 (A_{11} v_1 + \cdots + A_{n1} v_n) +$$

$$\vdots$$

$$a_n (A_{1n} v_1 + \cdots + A_{nn} v_n)$$

$$= (a_1 A_{11} + \cdots + a_n A_{1n}) v_1 + \cdots$$

$$(a_1 A_{n1} + \cdots + A_n A_{nn}) v_n$$

$$= \gamma_1 v_1 + \cdots + \gamma_n v_n$$

$$[Tv]_B = \underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}}_{[Tv]_B} = \begin{array}{c} \begin{bmatrix} \boxed{\begin{matrix} A_{11} \\ \vdots \\ A_{n1} \end{matrix}} & \cdots & \boxed{\begin{matrix} A_{1n} \\ \\ A_{nn} \end{matrix}} \end{bmatrix} \\ \begin{matrix} \swarrow \qquad\qquad \swarrow \\ [Tv_1]_B \qquad [Tv_n]_B \end{matrix} \end{array} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{[v]_B}$$

$$\Rightarrow [Tv]_B = A \cdot [v]_B = [T]_B [v]_B$$

This matrix representing T depends upon the ordered basis B and that there is a representing matrix for T in each ordered basis for V. (For transformation from one to another space depends upon 2 OB – B_V, B_W)

So to represent the matrix A we can use $[T]_B$.

$[T]_B$ = Matrix of the linear operator T relative to B.
$= A$

Ex1: Let T be a linear operator on the space $\mathbb{F}^2$ defined as:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

Suppose, $B = \{\varepsilon_1, \varepsilon_2\}$ where $\varepsilon_1, \varepsilon_2$ are standard ordered basis.

How to find $[T]_B$?

$$[T]_B = \begin{bmatrix} [T\varepsilon_1]_B & [T\varepsilon_2]_B \end{bmatrix}$$

$$\varepsilon_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \varepsilon_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$
\begin{tikzcd}
x \in \mathbb{V} \arrow[r, mapsto, "T"] \arrow[d, mapsto, "R_B"] & Tx \in \mathbb{V} \arrow[d, mapsto, "R_B"] \\
{[x]_B} \in \mathbb{F}^2 \arrow[r, "{[T]_B}"] & {[Tx]_B} \in \mathbb{F}^2
\end{tikzcd}
$$

Therefore we can write: (by choosing $x = e_1$ and $x = e_2$)

$$[T]_B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [T\varepsilon_1]_B$$

$$[T]_B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = [T\varepsilon_2]_B$$

$$[T]_B \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow [T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We have shown how to write the transformation in terms of matrix relative to B and B'.

$$\begin{tikzcd}
\mathbb{V} \text{ (dim } n) & & \mathbb{V} \text{ (dim } n) \\[-20pt]
v = \sum a_i v_i \arrow[rr, mapsto, "T"] \arrow[d, mapsto, "R_B"] & & Tv = \sum b_i v_i \arrow[d, mapsto, "R_B"] \\
{[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n} \arrow[rr, "{[T]_B}"] & & {[Tv]_B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{F}^n}
\end{tikzcd}$$

This was one part of the story where we know how to represent v in coordinates relative to B. Now we will apply transformation $[T]_B$ and we will get the coordinates of transformed vector relative to B.

But what if we change the basis from B to B' then how will $[T]_B$ be different from $[T]_{B'}$. This is a valid question since the choice of ordered basis is not unique.

It may be easy to work with one basis but difficult to work with another basis. So we may need to represent the transformation in another basis

$$
\begin{tikzcd}
{[v]_{B'} = \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{bmatrix} \in \mathbb{F}^n} \arrow[r, "{[T]_{B'}}"] & {[Tv]_{B'} = \begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_n' \end{bmatrix} \in \mathbb{F}^n} \\
v = \sum a_i' v_i' \arrow[u, mapsto, "R_{B'}"] \arrow[r, mapsto, "T"] & Tv = \sum b_i' v_i' \arrow[u, mapsto, "R_{B'}"']
\end{tikzcd}
$$

$\mathbb{V}$ (dim n) $\qquad\qquad$ $\mathbb{V}$ (dim n)

$$
\begin{tikzcd}
v = \sum a_i v_i \arrow[r, mapsto, "T"] \arrow[d, mapsto, "R_B"'] & Tv = \sum b_i v_i \arrow[d, mapsto, "R_B"] \\
{[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n} \arrow[r, "{[T]_B}"] & {[Tv]_B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{F}^n}
\end{tikzcd}
$$

$$[Tv]_{B'} = R_{B'} R_B^{-1} [T]_B R_B R_{B'}^{-1} [v]_{B'}$$

$$= [T]_{B'} \cdot [v]_{B'}$$

Therefore, $\quad [T]_{B'} = R_{B'} R_B^{-1} [T]_B R_B R_{B'}^{-1}$

Note the effect of $R_{B'} R_B^{-1}$

$$\begin{tikzcd}
{[v]_{B'}} \arrow[dd, bend left, "P"] \\
v \arrow[u, "R_{B'}"] \arrow[d, "R_B"] \\
{[v]_B}
\end{tikzcd}$$

$$[v]_B = R_B R_{B'}^{-1} [v]_{B'}$$

We know that there exist a change of basis matrix P such that -

$$[v]_B = P [v]_{B'}$$

Therefore, $\boxed{[T]_{B'} = P^{-1} [T]_B P}$

Ex: We have already seen that $[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Now consider $B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. Calculate $[T]_{B'}$

$$\begin{tikzcd}
{[v]_{B'}} \arrow[dd, bend left, "P"] \\
v \in V \arrow[u] \arrow[d] \\
{[v]_B} \arrow[uu, bend left, "Q"]
\end{tikzcd}$$

$$e_1 = 1 \cdot e_1 + 0 \cdot e_2$$

$$e_1 = a \cdot e_1' + b \cdot e_2'$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_1' & e_2' \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e_1' & e_2' \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1' & e_2' \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} e_1' & e_2' \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Therefore,

$$[v]_B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} [v]_{B'} \quad , \quad [v]_{B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} [v]_B$$

Hence $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$$[T]_{B'} = P^{-1} [T]_B \cdot P = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

Exercise 6: Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T(x_1, x_2) = (-x_2, x_1).$$

(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

(b) What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?

$$\begin{tikzcd}
v \in \mathbb{R}^2 \arrow[rr, "T"] \arrow[d, "R_S"] & & Tv \in \mathbb{R}^2 \arrow[d, "R_S"] \\
{[v]_S = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \arrow[rr, "{[T]_S = ?}"] & & {[Tv]_S = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}}
\end{tikzcd}$$

Let's see the action of T on standard basis.

$$[T]_S \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T]_S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad [T]_S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The matrix T in OB will be $[T]_B$.

For that we have to find the change of basis matrix.

$$
\begin{tikzcd}
{[v]_B} \arrow[rr, "{[T]_B}"] \arrow[dd, bend right=60, mapsto, "P"'] & & {[Tv]_B} \\
v \in V \arrow[u, mapsto, "R_B"'] \arrow[rr, "T"] \arrow[d, mapsto, "R_S"] & & Tv \in V \arrow[u, mapsto] \arrow[d, mapsto, "R_S"] \\
{[v]_S} \arrow[rr, "{[T]_S}"] & & {[Tv]_S} \arrow[uu, bend right=60, mapsto, "P^{-1}"']
\end{tikzcd}
$$

Therefore, $[Tv]_B = \underbrace{P^{-1}[T]_S P}\,[v]_B = \underbrace{[T]_B}\,[v]_B$

So we can write, $[T]_B = P^{-1}[T]_S P.$

where P is the change of basis matrix from B to S.

$$
\begin{tikzcd}
v \in V \arrow[r, mapsto, "R_S"] \arrow[d, mapsto, "R_B"'] & {[v]_S} \arrow[dl, shift left, "{P^{-1}[v]_S}"] \\
{[v]_B} \arrow[ur, shift left, "{P.[v]_B}"] &
\end{tikzcd}
$$

let's see the action of basis vectors. $S = \{e_1, e_2\}$

$B = \{e_1', e_2'\}$

$e_1 = 1 \cdot e_1 + 0 \cdot e_2$, $e_2 = 0 \cdot e_1 + 1 \cdot e_2$

$e_1 = a_1 e_1' + a_2 e_2'$ $e_2 = b_1 e_1' + b_2 e_2'$

$$\Rightarrow \quad e_1 = (a_1 e_1' + a_2 e_2') + 0$$

$$e_2 = 0 + (b_1 e_1' + b_2 e_2')$$

$$\Rightarrow \quad \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} e_1' & e_2' \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

$$\Rightarrow \quad I = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} = -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right] = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$$

$[e_1]_B$ $[e_2]_B$

Therefore, $$[v]_B = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} [v]_S = P^{-1}[v]_S$$

$$[v]_S = P \cdot [v]_B$$

So $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$

Therefore, $$[T]_B = P^{-1}[T]_S \; P$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & 2/3 \\ -5/3 & 1/3 \end{bmatrix}$$

<u>Shortcut to find the matrix P</u>

$$\begin{tikzcd} v \in V \arrow[rr, mapsto] \arrow[d] & & {[v]_S} \arrow[dll, bend left, "P^{-1}[v]_S"] \\ {[v]_B} \arrow[urr, "P.[v]_B"] & & \end{tikzcd}$$

We know that,

$$[v]_B = \begin{bmatrix} [e_1]_B & [e_2]_B \end{bmatrix} [v]_S$$

$$[v]_B = P^{-1} \cdot [v]_S$$

Try to find $[e_1]_B$.

$$[e_1]_B = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad ; \; e_1 = a_1 \cdot e_1' + a_2 \cdot e_2'$$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_1' & e_2' \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = [B][e_1]_B$$

Similarly, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [B] \, [e_2]_B$

$$\Rightarrow [B] \begin{bmatrix} [e_1]_B & [e_2]_B \end{bmatrix} = I$$

$$\Rightarrow [B] \cdot P^{-1} = I$$

$\Rightarrow P^{-1} = [B]^{-1}$ (This is true if S = standard basis)

The general rule is: $[B][P]^{-1} = [S]$

Basis where we want to go.

S is any basis

Algebra of Linear Transformation:

Let $\mathbb{V}$ and $\mathbb{W}$ be vector space over field $\mathbb{F}$.

let $T : \mathbb{V} \longrightarrow \mathbb{W}$ (Linear)

let $U : \mathbb{V} \longrightarrow \mathbb{W}$ (linear)

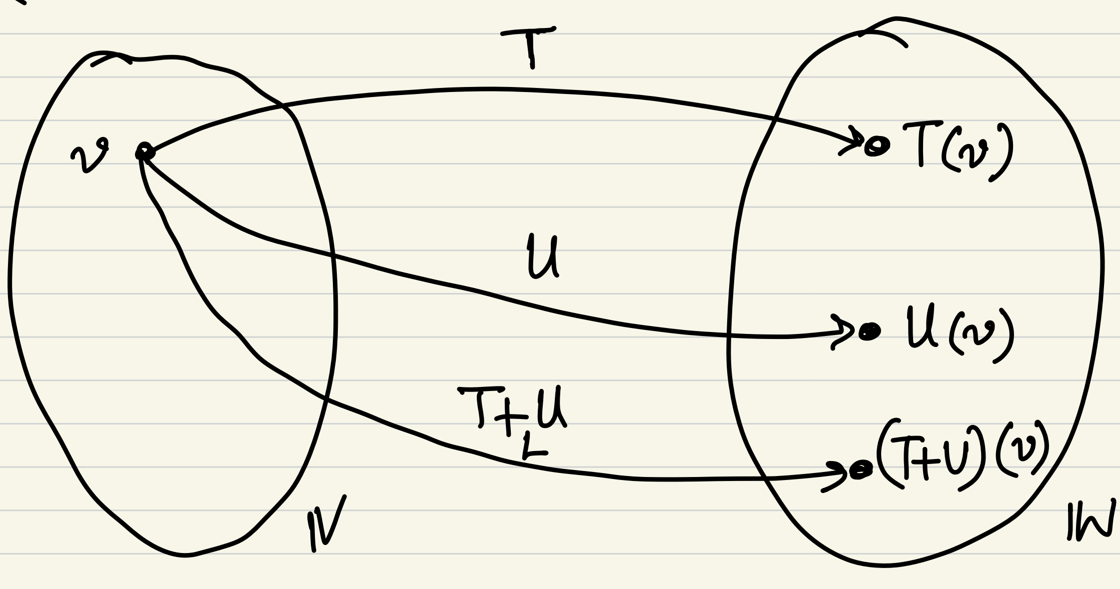
We collect all such linear transformations from $\mathbb{V}$ to $\mathbb{W}$.

$$L(\mathbb{V}, \mathbb{W}) := \{ T \mid T : \mathbb{V} \to \mathbb{W} \text{ and } T \text{ is linear} \}$$

We want to show that $(L(\mathbb{V}, \mathbb{W}), \mathbb{F}, +, \cdot)$ is a vector space with the vector addition $+$ (addition of 2 linear transformation) and scalar multiplication $(\cdot)$ over the field $\mathbb{F}$ on which $\mathbb{V}$ and $\mathbb{W}$ are defined.

The map/function $T+U$ is defined by –

$$(T +_L U)(v) := T(v) +_{\mathbb{W}} U(v) \quad (\forall v \in \mathbb{V})$$



Is $T+U$ a linear transformation?

$\Rightarrow (T+U)(\alpha v_1 + v_2) = \alpha (T+U)(v_1) + (T+U)(v_2)$

$\Rightarrow (T+U)(\alpha v_1) + (T+U)(v_2)$

$\Rightarrow T(\alpha v_1) + U(\alpha v_1) + (T+U)(v_2)$

$\Rightarrow \alpha T(v_1) + \alpha \cdot U(v_1) + (T+U)(v_2)$

$\Rightarrow \alpha \left(T(v_1) + U(v_1) \right) + (T+U)(v_2)$

$\Rightarrow \alpha (T+U)(v_1) + (T+U)(v_2)$

Hence $T+U$ is a linear map from $\mathbb{V}$ to $\mathbb{W}$.

Therefore, the vector addition $+_L$ is closed in $L(\mathbb{V}, \mathbb{W})$.

The map $c \cdot T$ is defined as:

$$\underbrace{(c \cdot T)}_{\text{scalar mult in } L}(v) := \underbrace{c \cdot T(v)}_{\text{scalar mult in } \mathbb{W}} \quad (\forall c \in \mathbb{F}, v \in \mathbb{V})$$

Is $c \cdot T$ a linear map from $\mathbb{V}$ to $\mathbb{W}$?

$\Rightarrow (c \cdot T)(d v_1 + v_2) = d \cdot (c \cdot T)(v_1) + (c \cdot T)(v_2)$

$\Rightarrow c \cdot T(d v_1 + v_2)$

$\Rightarrow c \cdot \left(T(d v_1) + T(v_2) \right)$

$\Rightarrow c \cdot d \cdot T(v_1) + c \cdot T(v_2)$

$\Rightarrow d \cdot c \cdot T(v_1) + c \cdot T(v_2)$

$\Rightarrow d \cdot (c \cdot T)(v_1) + (c \cdot T)(v_2)$

Hence $c \cdot T$ is also a linear map from $\mathbb{V}$ to $\mathbb{W}$ or $c \cdot T$ is closed in $L(\mathbb{V}, \mathbb{W})$.

Therefore with the definition of $+$ and $\cdot$ on the set $L(\mathbb{V}, \mathbb{W})$ is a vector space over field $\mathbb{F}$.

Proof: We need to check $(L(\mathbb{V}, \mathbb{W}), +_L)$ is an Abelian group and it also satisfies the 4 properties of scalar multiplication. Hence -

$$L(\mathbb{V}, \mathbb{W}) := \{ T \mid T : \mathbb{V} \to \mathbb{W} \text{ is linear} \}$$

is a vector space over $\mathbb{F}$.

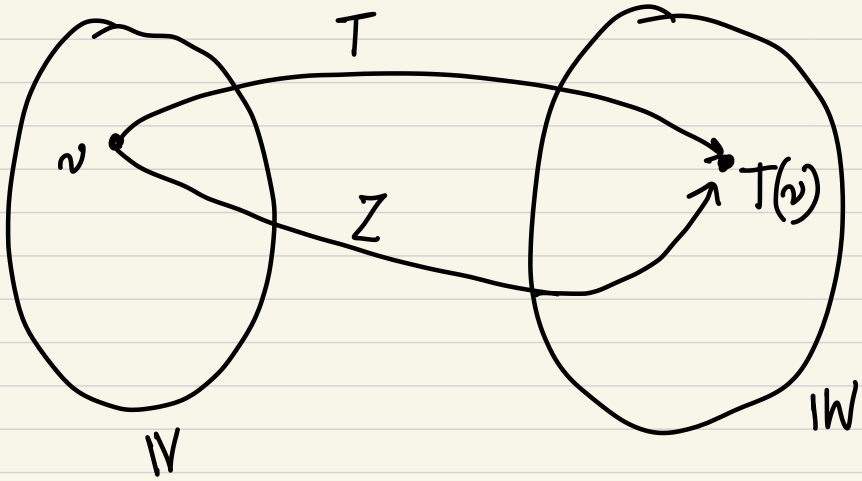
Q. What is the additive identity (or the zero vector) in the vector space $L(\mathbb{V}, \mathbb{W})$?

Let $T \in L(\mathbb{V}, \mathbb{W})$, $Z \in L(\mathbb{V}, \mathbb{W})$

If $(T +_L Z)(v) = T(v) \quad \forall v \in \mathbb{V}$

then Z is additive identity for $L(\mathbb{V}, \mathbb{W})$.

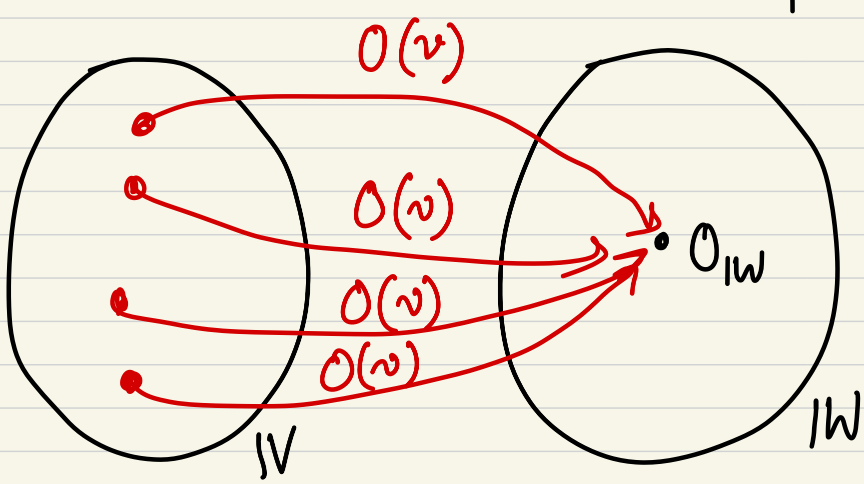
$\Rightarrow T(v) +_{\mathbb{W}} Z(v) = T(v)$



Since $T(v) \in \mathbb{W}$, $Z(v) \in \mathbb{W}$ therefore,

$$T(v) +_{\mathbb{W}} Z(v) = T(v) \text{ iff } Z(v) = 0_{\mathbb{W}}. \ (\forall v)$$

That mean $\forall v \in \mathbb{V}$, $Z(v) = 0_{\mathbb{W}}$. which implies Z is the zero transformation of the space $L(\mathbb{V}, \mathbb{W})$



The zero transformation is defined as: $O : \mathbb{V} \rightarrow \mathbb{W}$ such that, $\forall v \in \mathbb{V}$, $O(v) = 0_{\mathbb{W}}$.

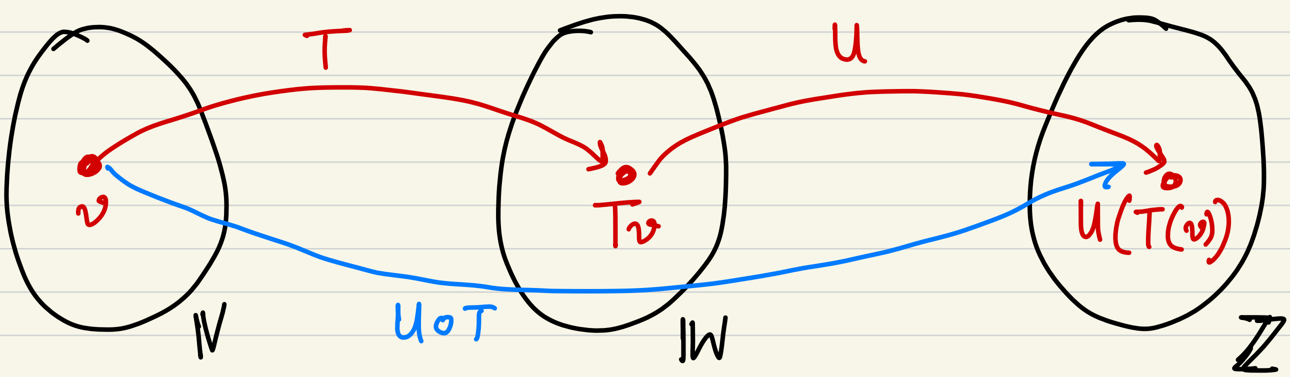
$$\left.\begin{aligned} \dim(\mathbb{V}) &= n \\ \dim(\mathbb{W}) &= m \\ \dim(L(\mathbb{V}, \mathbb{W})) &= mn \end{aligned}\right] \quad \text{proof is non trivial}$$

Suppose $\mathbb{V}, \mathbb{W}, \mathbb{Z}$ are vector spaces over field $\mathbb{F}$.

$$\left.\begin{aligned} \text{let } T &: \mathbb{V} \longrightarrow \mathbb{W}. \\ U &: \mathbb{W} \to \mathbb{Z}. \end{aligned}\right\} \quad \text{Linear transformation.}$$

The composition map, $U \circ T : \mathbb{V} \longrightarrow \mathbb{Z}$ defined as:

$$(U \circ T)(v) := U(T(v))$$



prove that $U \circ T$ is also a linear transformation:

$$\Rightarrow (U \circ T)(\alpha \cdot v_1 + v_2) = \alpha \cdot (U \circ T)(v_1) + (U \circ T)(v_2)$$

$$\Rightarrow U(T(\alpha v_1 + v_2))$$

$$\Rightarrow U(T(\alpha v_1) + T(v_2))$$

$$\Rightarrow U(\alpha T(v_1) + T(v_2))$$

$$\Rightarrow U(\alpha T(v_1)) + U(T(v_2))$$

$$\Rightarrow \alpha \cdot U(T(v_1)) + U(T(v_2))$$

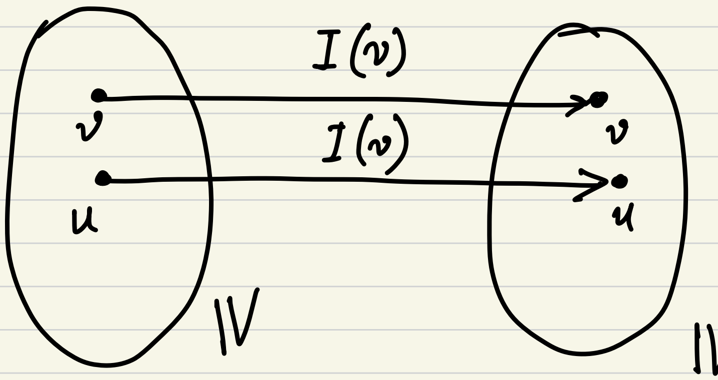
$$\Rightarrow \alpha \cdot (U \circ T)(v_1) + (U \circ T)(v_2)$$

Hence $U \circ T$ is also a linear map from $\mathbb{V}$ to $\mathbb{Z}$.

Invertibility of Transformation:

A identity map $I : \mathbb{V} \to \mathbb{V}$ is defined as:

$$I(v) := v \quad \forall v \in \mathbb{V}.$$



consider a linear operator $T : \mathbb{V} \to \mathbb{V}$ and another linear operator $T^{-1} : \mathbb{V} \to \mathbb{V}$ such that -

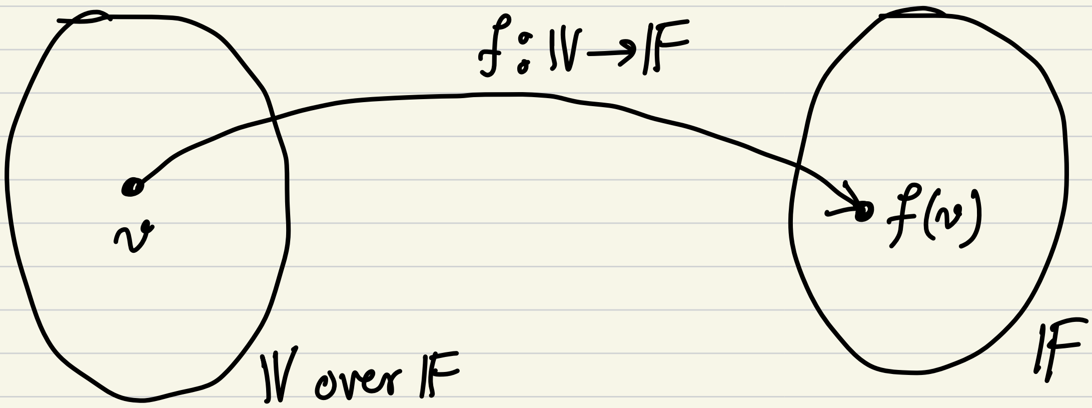
$$(T^{-1} \circ T)(v) = I(v) \quad \text{and} \quad (T \circ T^{-1})(v) = I(v)$$

T is invertible if and only if.

1. T is one to one (Kernel of $\mathbb{V}$ has only $0_{\mathbb{V}}$)
2. T is onto (Image of $\mathbb{V}$ is all of $\mathbb{V}$)

Linear Functional:

If $\mathbb{V}$ is a vector space over field $\mathbb{F}$, a linear transformation 'f' from $\mathbb{V}$ into its scalar field $\mathbb{F}$ is called linear functional on $\mathbb{V}$.



Since f is a linear functional so it must satisfy 2 properties of the linear transformation.

1. $f(v +_{\mathbb{V}} u) = f(v) +_{\mathbb{F}} f(u) \quad \forall u, v \in \mathbb{V}$

2. $f(c \cdot v) = c \times f(v) \quad \forall v \in \mathbb{V}, c \in \mathbb{F}$

Ex1: $f : \mathbb{F}^n$ over $\mathbb{F} \longrightarrow \mathbb{F}$

Consider a standard ordered basis for $\mathbb{F}^n = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ and any vector $x \in \mathbb{F}^n$ can be written in terms of it coordinates,

$$x \in \mathbb{V} \longmapsto [x]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \forall x_i \in \mathbb{F}$$

Define the linear functional f such that,

$$x \in \mathbb{V} \longrightarrow f(x) := a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

$\forall a_i \in \mathbb{F}$. (Scalars are given to us)

Is it linear transformation?

$$f(c \cdot x + y) = c \cdot f(x) + f(y).$$

$$\Rightarrow a_1 \cdot (cx+y)_1 + a_2 (cx+y)_2 + \cdots + a_n (cx+y)_n$$

$$\Rightarrow a_1 (cx_1 + y_1) + a_2 (cx_2 + y_2) + \cdots + a_n (cx_n + y_n)$$

$$\Rightarrow c(a_1x_1 + \cdots + a_nx_n) + (a_1y_1 + \cdots + a_ny_n)$$

$$\Rightarrow c \cdot f(x) + f(y)$$

Every linear functional on $\mathbb{F}^n$ is of the above form for some scalars $a_1, a_2, \ldots, a_n$.

Notice that $f(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$

$$= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} [x]_B$$

consider $x = \varepsilon_1$ (the first standard ordered basis)

so $f(\varepsilon_1) = [a_1 \; a_2 \cdots a_n] \, [1 \; 0 \; 0 \cdots 0]^t = a_1$

similarly $f(\varepsilon_j) = a_j \quad \forall j = 1, 2, \ldots, n$

$$f(x) = f\left(\sum_{j=1}^{n} x_j \varepsilon_j \right)$$

$$= f\left(x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n \right)$$

$$= f(x_1\varepsilon_1) + f(x_2\varepsilon_2) + \cdots + f(x_n\varepsilon_n)$$

$$= x_1 \cdot f(\varepsilon_1) + x_2 f(\varepsilon_2) + \cdots + x_n f(\varepsilon_n)$$

$$= x_1 \cdot a_1 + x_2 \cdot a_2 + \cdots + x_n \cdot a_n$$

Ex2: $A \in \mathbb{F}^{n \times n}$

$$tr(A) = A_{11} + A_{22} + \cdots + A_{nn}$$

Suppose $f : \mathbb{F}^{n \times n}$ over $\mathbb{F} \longrightarrow \mathbb{F}$ such that,

$$f(A) := A_{11} + A_{22} + \cdots + A_{nn} = Tr(A)$$

prove that f is linear functional on $\mathbb{F}^{n \times n}$.

$Tr(c \cdot A + B) = c \cdot Tr(A) + Tr(B)$ (to be proved)

$$= (cA+B)_{11} + (cA+B)_{22} + \cdots + (cA+B)_{nn}$$

$$= (c \cdot A_{11} + B_{11}) + (c \cdot A_{22} + B_{22}) + \cdots + (c\, A_{nn} + B_{nn})$$

$$= c \cdot (A_{11} + A_{22} + \cdots + A_{nn}) + (B_{11} + B_{22} + \cdots + B_{nn})$$

$$= c \cdot Tr(A) + Tr(B).$$

Ex3: let $[a,b]$ be the closed interval on $\mathbb{R}$.
$C([a,b])$ is the space of all real valued continuous function on $[a,b]$.

Define $L(g) := \int_a^b g(t)\, dt. \in \mathbb{R}. \quad \forall g \in C([a,b])$

$L : C[a,b] \longrightarrow \mathbb{R}.$

prove that L is a linear functional.

$$L(c \cdot g + h) = c \cdot L(g) + L(h)$$

$$\Rightarrow \int_a^b (cg+h)(t)\, dt = \int_a (c \cdot g(t) + h(t))\, dt$$

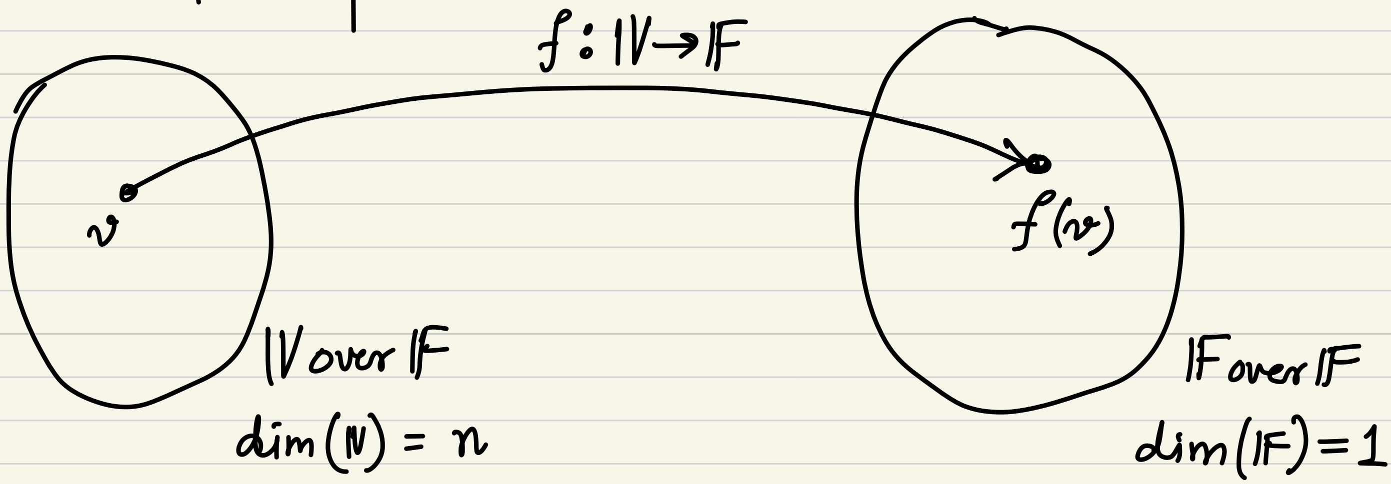
$$\Rightarrow c \int_a^b \cdot g(t)\, dt + \int_a^b h(t)\, dt \Rightarrow c \cdot L(g) + L(h)$$

Dual Space:

If $\mathbb{V}$ is a vector space, the collection of all linear functionals on $\mathbb{V}$ forms a vector space in a natural way. The space is denoted by $L(\mathbb{V}, \mathbb{F})$

$$L(\mathbb{V}, \mathbb{F}) := \{ f \mid f : \mathbb{V} \longrightarrow \mathbb{F} \} \quad \text{over } \mathbb{F}$$

We denote, $V^* = L(\mathbb{V}, \mathbb{F})$ over $\mathbb{F}$ and it is called "dual" space of $\mathbb{V}$.



If $L(\mathbb{V}, \mathbb{F})$ is set of all linear functionals from $\mathbb{V}$ to $\mathbb{F}$ then $L(\mathbb{V}, \mathbb{F})$ is a vector space over $\mathbb{F}$ with the dimension of $\dim(\mathbb{V}) \times \dim(\mathbb{F}) = \dim(V^*)$.

$$V^* = L(\mathbb{V}, \mathbb{F}) \quad , \quad \dim(V^*) = \dim(\mathbb{V})$$

Consider $B = \{v_1, v_2, \ldots, v_n\}$ as a basis for $\mathbb{V}$. We want to construct the basis for dual space $\mathbb{V}^*$.

Consider the following, $\forall f_i : \mathbb{V} \to \mathbb{F}$

$f_1(v_1) = 1$	$f_2(v_1) = 0$		$f_n(v_1) = 0$
$f_1(v_2) = 0$	$f_2(v_2) = 1$	$\cdots$	$f_n(v_2) = 0$
$\vdots$	$\vdots$		$\vdots$
$f_1(v_n) = 0$	$f_2(v_n) = 0$		$f_n(v_n) = 0$

Therefore, $f_i(v_j) = \delta_{ij} \quad \forall\, i,j$ from 1 to n

Consider another set, $B' = \{f_1, f_2, \ldots, f_n\}$.

Is the set B' linearly independent?

$\Rightarrow$ Yes.

$$c_1 \cdot f_1 + c_2 \cdot f_2 + \cdots + c_n \cdot f_n = 0_L$$

$$\Rightarrow c_1 \cdot f_1(v) + c_2 f_2(v) + \cdots + c_n f_n(v) = 0_L(v) = 0_{\mathbb{F}}$$

Suppose $v = v_1$ then $c_1 \cdot f_1(v_1) = c_1 \cdot 1 = 0 \Rightarrow c_1 = 0$

Suppose, $v = v_2$ then $c_2 f_2(v_2) = c_2 \cdot 1 = 0 \Rightarrow c_2 = 0$

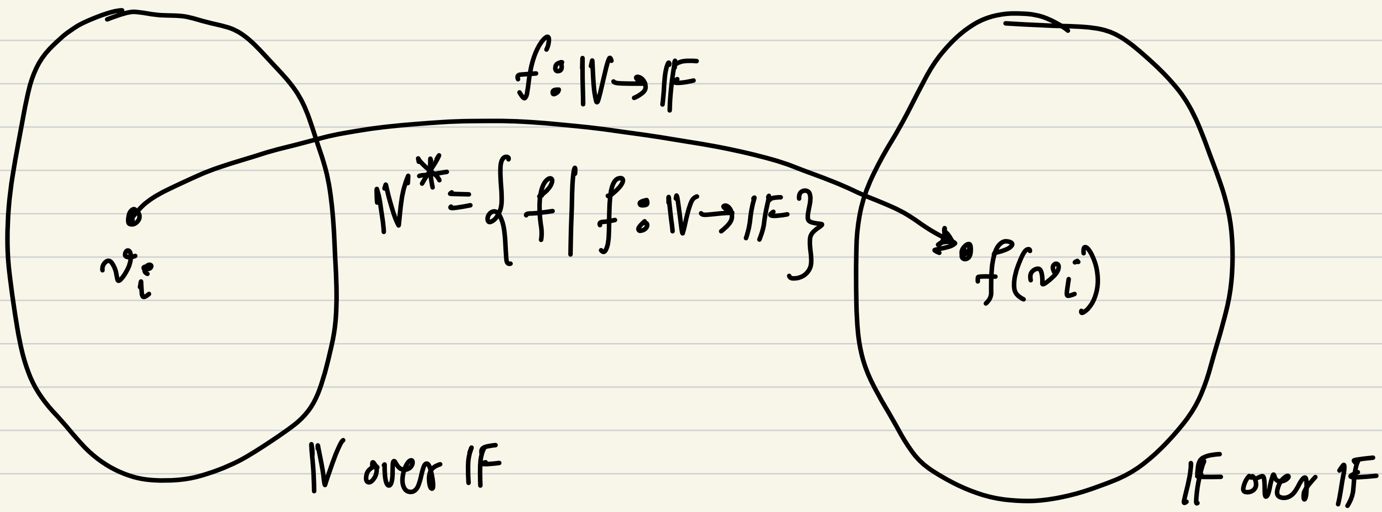
$\vdots$

Suppose, $v = v_n$ then $c_n f_n(v_n) = c_n \cdot 1 = 0 \Rightarrow c_n = 0$

Therefore, for $v = v_i$, $c_i = 0 \;\forall i$ hence the set B' is linearly independent.

Since we already know that $\dim(V^*) = \dim(\mathbb{V}) = n$ therefore, the set B' with $|B'| = n$ will be spanning set of $L(\mathbb{V}, \mathbb{F})$. So,

Basis for dual space; V^* will be: $\{f_1, f_2, \ldots, f_n\}$.
It is also called dual basis of B.



$B = \{v_1, v_2, \ldots, v_n\}$
$B^* = \{f_1, f_2, \ldots, f_n\}$ (dual basis for B)
$f_i(v_j) = \delta_{ij}$

We can construct any linear functional 'f' as a linear combination of f_i (basis of $\mathbb{V}^*$).

$$f = \sum_{i=1}^{n} c_i \cdot f_i \quad \equiv \quad f(v) = \sum_{i=1}^{n} (c_i f_i)(v)$$

$$f(v) = \sum_{i=1}^{n} c_i f_i(v)$$

$$f(v) = c_1 f_1(v) + c_2 f_2(v) + \cdots + c_n f_n(v)$$

Suppose $v = v_1$, $f(v_1) = c_1$

$v = v_2$, $f(v_2) = c_2$

$\vdots$

$v = v_n$, $f(v_n) = c_n$

Therefore,

$$f(v) = f(v_1) \cdot f_1(v) + \cdots + f(v_n) f_n(v)$$

$$f(v) = \sum_{i=1}^{n} f(v_i) f_i(v)$$

$$\Rightarrow \boxed{f = \sum_{i=1}^{n} f(v_i) f_i}$$

where the scalars are given by $f(v_i)$.

Suppose, $v = a_1 \cdot v_1 + a_2 v_2 + \cdots + a_n v_n$

then,

$$f_j(v) = f_j(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = a_j$$

$= a_1$ (if $j=1$)

$= a_2$ (if $j=2$)

$\vdots$

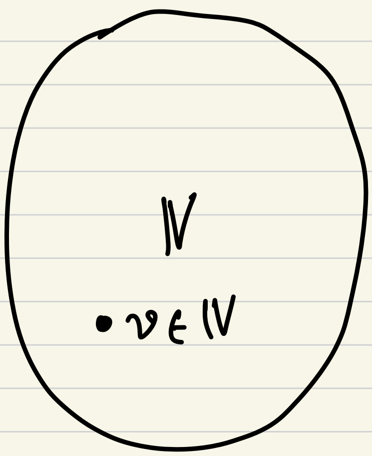
$= a_n$ (if $j=n$)

Therefore, $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$$= f_1(v) v_1 + f_2(v) v_2 + \cdots + f_n(v) v_n$$

$$\boxed{v = \sum_{i=1}^{n} f_i(v)\, v_i}$$

Compare the 2 equations -

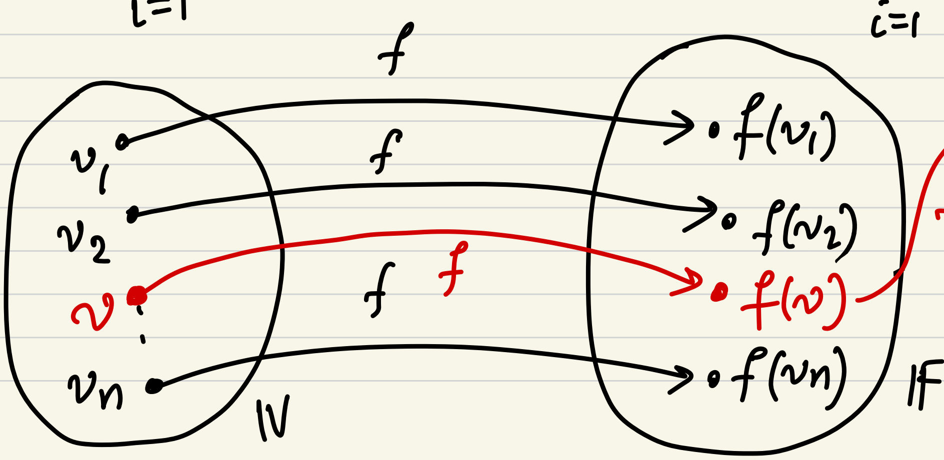


$B = \{v_1, v_2, \ldots, v_n\}$

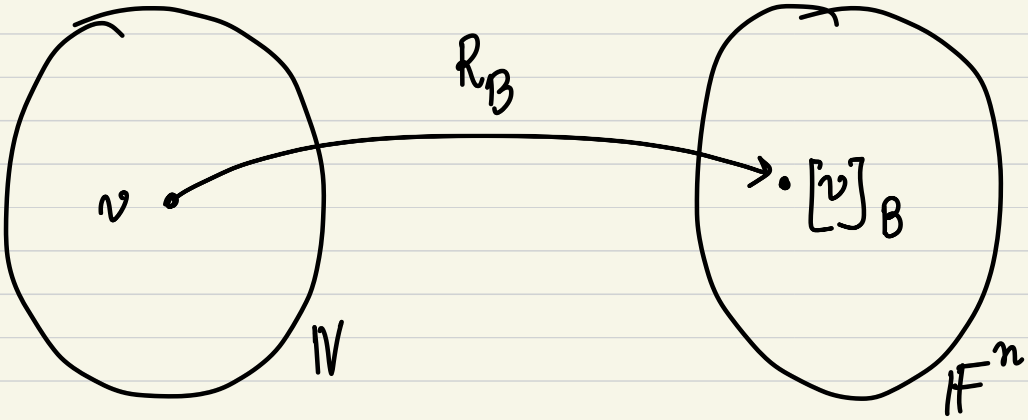
$B' = \{f_1, f_2, \ldots, f_n\}$

$\forall f_i(v_j) = \delta_{ij}$

$$v = \sum_{i=1}^{n} f_i(v)\, v_i \qquad f = \sum_{i=1}^{n} f(v_i)\, f_i$$



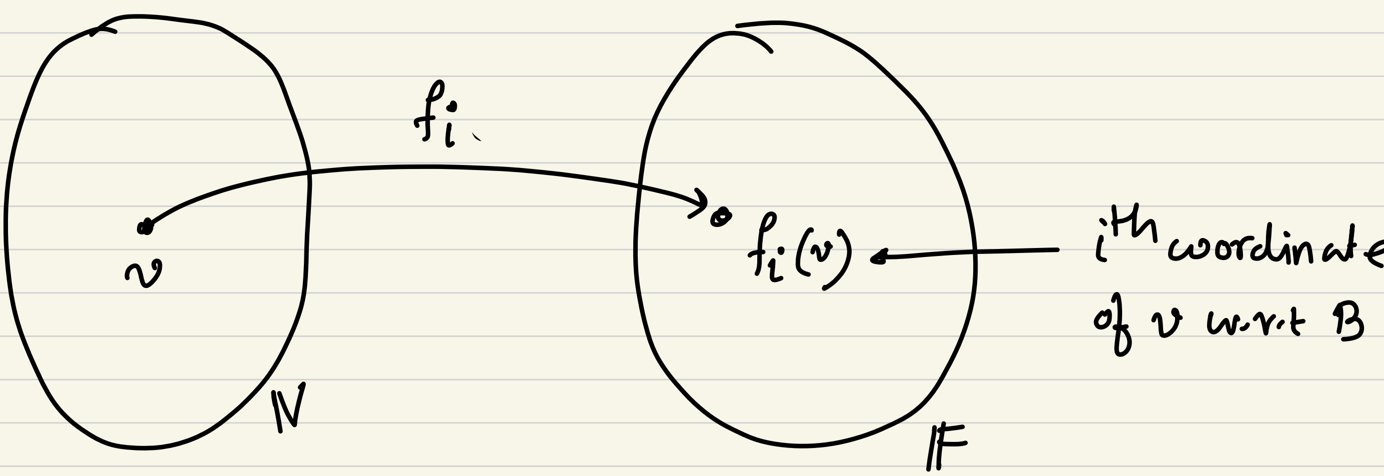
$$f(v) = \sum_{i=1}^{n} f(v_i)\, f_i(v)$$



$$[v]_B = \begin{bmatrix} f_1(v) \\ f_2(v) \\ \vdots \\ f_n(v) \end{bmatrix}$$

i^{th} coordinate of v will be $- f_i(v)$

We can say f_i is the function which assigns each vector v in $\mathbb{V}$ and gives the i^{th} coordinate w.r.t B.



$$[v]_B = \begin{bmatrix} f_1(v) \\ f_2(v) \\ \vdots \\ f_n(v) \end{bmatrix}, \quad [f]_{B'} = \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix}$$

$$f = f(v_1)\cdot f_1 + f(v_2)\cdot f_2 + \cdots + f(v_n)\cdot f_n$$

$$v = f_1(v)\, v_1 + f_2(v)\, v_2 + \cdots + f_n(v)\, v_n$$

$$f(v) = f_1(v)\, f(v_1) + f_2(v)\, f(v_2) + \cdots + f_n(v)\, f(v_n)$$

$$f(v) = f(v_1)\cdot f_1(v) + f(v_2)\, f_2(v) + \cdots + f(v_n)\, f_n(v)$$

Every linear functional on V has the following form:

$$f(v) = f(v_1)\cdot f_1(v) + f(v_2)\cdot f_2(v) + \cdots + f(v_n)\, f_n(v)$$

$$= a_1\cdot f_1(v) + a_2\cdot f_2(v) + \cdots + a_n f_n(v)$$

$$= [a_1 \quad a_2 \quad \cdots \quad a_n]^t\, [v]_B$$

Where $a_i = f(v_i)$

Note that it is same result we obtained $f : \mathbb{F}^n \longrightarrow \mathbb{F}$

$$v \in \mathbb{F}^n \longrightarrow [v]_B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

a linear functional, $f(v) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

Where $a_i = f(\varepsilon_i)$ where ε_i is the basis of $\mathbb{F}^n$.

$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$f(v) = a_1 f(v_1) + a_2 f(v_2) + \cdots + a_n f(v_n) \in \mathbb{F}$

is always a linear functional. Every linear functional $f: \mathbb{V} \to \mathbb{F}$ has the above form.

$$v \longmapsto [v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad f = [f(v_i)]_{i=1}^{n} \cdot [v]_B$$

always linear functional.

$$v_i \longmapsto f(v_i) \ \forall i$$

$$[f(v_1) \ f(v_2) \ \cdots \ f(v_n)]$$

EX1: $\mathbb{V} = \{ P \mid P(x) = a_0 + a_1 x + a_2 x^2, \ P: \mathbb{R} \to \mathbb{R} \}$

Suppose $t_1, t_2, t_3 \in \mathbb{R}$ and let -

$L_1: \mathbb{V} \to \mathbb{R}$ s.t. $L_1(P) = P(t_1)$

$L_2: \mathbb{V} \to \mathbb{R}$ s.t. $L_2(P) = P(t_2)$ $\quad \forall P \in \mathbb{V}$

$L_3: \mathbb{V} \to \mathbb{R}$ s.t $L_3(P) = P(t_3)$

Definitely L_1, L_2, L_3 are linear functionals on $\mathbb{V}$.

Q: Is the set $S = \{L_1, L_2, L_3\}$ linearly independent?

$\Rightarrow$ consider,

$$c_1 L_1 + c_2 L_2 + c_3 L_3 = 0_L \quad (\text{Zero map})$$

$$\Rightarrow c_1 \cdot L_1(P) + c_2 L_2(P) + c_3 L_3(P) = 0_L(P)$$

$$\Rightarrow c_1 \cdot P(t_1) + c_2 \cdot P(t_2) + c_3 \cdot P(t_3) = 0_{\mathbb{F}}$$

$$\Rightarrow c_1 (a_0 + a_1 t_1 + a_2 t_1^2) + c_2 (a_0 + a_1 t_2 + a_2 t_2^2) + c_3 (a_0 + a_1 t_3 + a_2 t_3^2) = 0_{\mathbb{F}}$$

$$\Rightarrow (c_1 + c_2 + c_3) a_0 + (c_1 t_1 + c_2 t_2 + c_3 t_3) a_1 + (c_1 t_1^2 + c_2 t_2^2 + c_3 t_3^2) a_3 = 0_{\mathbb{F}}. \quad \forall t_1, t_2, t_3 \in \mathbb{R}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It can be shown that the matrix is invertible so

$c_1 = 0, c_2 = 0, c_3 = 0$

That means the set S is LI set.

Since $|S| = 3$ therefore S is spanning set for $L(\mathbb{V}, \mathbb{F})$
That means $S = \{L_1, L_2, L_3\}$ is a basis for $L(\mathbb{V}, \mathbb{F})$.

What is the basis of $\mathbb{V}$ to which the set S is dual basis ?

$\Rightarrow$ If $S = \{L_1, L_2, L_3\}$ is a dual basis of B where $B = \{P_1, P_2, P_3\}$ then we know that -

$$L_i(P_j) = \delta_{ij}$$

Therefore, $L_1(P_1) = 1$, $L_2(P_1) = 0$, $L_3(P_1) = 0$
$L_1(P_2) = 0$, $L_2(P_2) = 1$, $L_3(P_2) = 0$
$L_1(P_3) = 0$, $L_2(P_3) = 0$, $L_3(P_3) = 1$

We also know, $L_1(P) = P(t_1)$
$L_2(P) = P(t_2)$
$L_3(P) = P(t_3)$

That means,
$$\left.\begin{aligned} L_1(P_1) &= P_1(t_1) = 1 \\ L_1(P_2) &= P_2(t_1) = 0 \\ L_1(P_3) &= P_3(t_1) = 0 \end{aligned}\right\} P_i(t_j) = \delta_{ij}$$

One can see that, $$P_1(x) = \frac{(x-t_2)(x-t_3)}{(t_1-t_2)(t_1-t_3)}$$

$$P_2(x) = \frac{(x-t_1)(x-t_3)}{(t_2-t_1)(t_2-t_3)}$$

$$P_3(x) = \frac{(x-t_1)(x-t_2)}{(t_3-t_1)(t_3-t_2)}$$

Note that $\left.\begin{aligned} P_1(t_1) &= 1 \\ P_1(t_2) &= 0 \\ P_1(t_3) &= 0 \end{aligned}\right\}$

actually, $P_i(t_j) = \delta_{ij}$

Hence $B = \{P_1, P_2, P_3\}$ is the dual basis of $\{L_1, L_2, L_3\}$

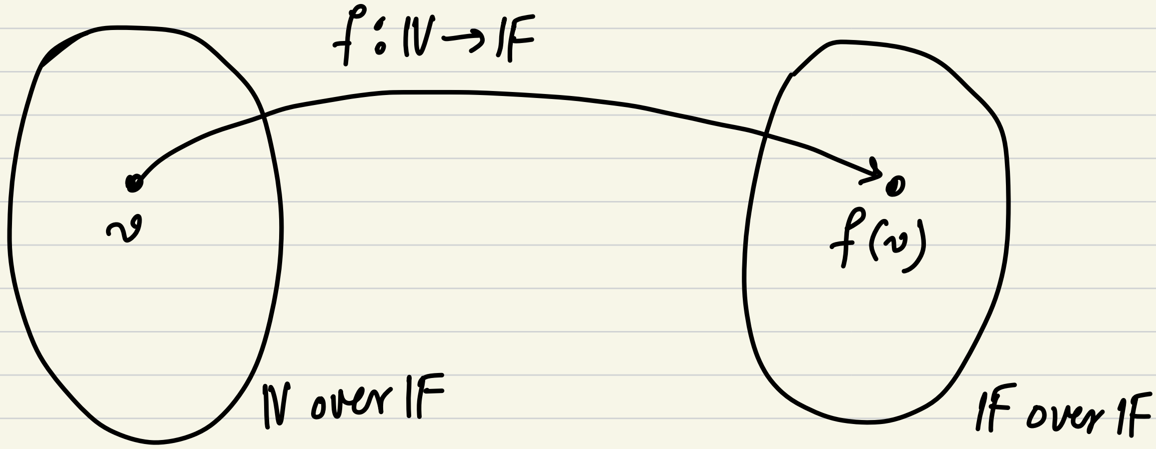
any polynomial $P \in \mathbb{V}$ can be written as: $P = P_1 c_1 + \cdots + P_3 c_3$

$$[P]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{matrix} \xrightarrow{x} L(P_1) \\ \xrightarrow{x} L(P_2) \\ \xrightarrow{x} L(P_3) \end{matrix}$$

$$L(P) = c_1 L(P_1) + \cdots + c_3 L(P_3)$$

linear functional $L: \mathbb{V} \to \mathbb{F}$ is therefore,

$$\begin{aligned} L(P) &= L(P_1)\cdot c_1 + L(P_2)\cdot c_2 + L(P_3)\cdot c_3 \\ &= L(P_1)\cdot P(t_1) + L(P_2)\cdot P(t_2) + L(P_3)\cdot P(t_3) \end{aligned}$$



Suppose 'f' is non zero linear functional on $\mathbb{V}$. for this linear functional, what will be rank & nullity?

$$\text{rank}(f) = \dim(\text{Im}(f))$$
$$\text{nullity}(f) = \dim(\text{Ker}(f))$$

$$\text{Im}(f) = \{ \alpha \mid \exists v \in \mathbb{V},\ f(v) = \alpha \}$$
$$\text{Ker}(f) = \{ v \mid f(v) = 0_{\mathbb{F}} \}$$

Since f is non zero transformation so $\exists v \in \mathbb{V}$, $f(v) = \alpha$ and if $\alpha \in \text{Im}(f)$ then we can say that $c\alpha \in \text{Im}(f)$ since $\mathbb{F}$ over $\mathbb{F}$ is of dimension 1 hence the scalar α will span the entire space $\mathbb{F}$ over $\mathbb{F}$.

Hence, $\text{Im}(f) = \mathbb{F}$.

$$\dim(\text{Im}(f)) = \dim(\mathbb{F}) = 1.$$

From rank-nullity theorem,

$$\dim(\mathrm{Im}(f)) + \dim(\mathrm{Ker}(f)) = \dim(\mathbb{V})$$

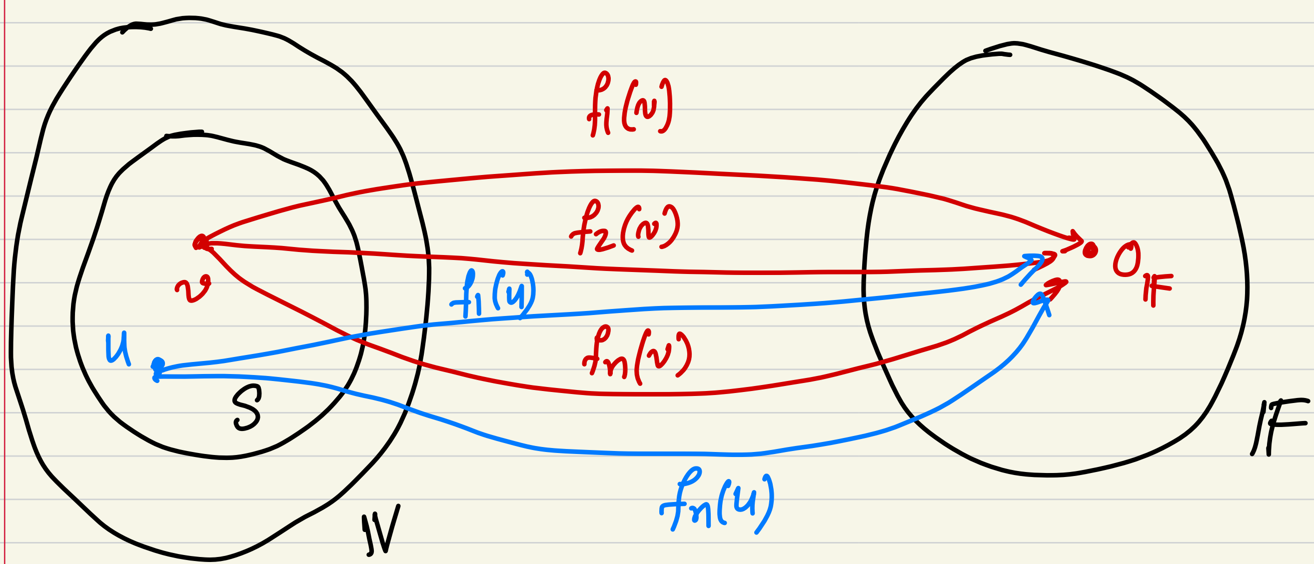
$$\Rightarrow \mathrm{rank}(f) + \mathrm{nullity}(f) = \dim(\mathbb{V}) = n$$

$$\Rightarrow \mathrm{nullity}(f) = n-1.$$

The subspace $\mathrm{Ker}(f)$ is called "hyperspace" of $\mathbb{V}$.

Annihilator :

If $\mathbb{V}$ is a vector space over $\mathbb{F}$ and S is subset of $\mathbb{V}$ the annihilator of S is the set S^0 of linear functionals 'f' on $\mathbb{V}$ such that $f(v)$ is 0 for every $v \in S$.



$$S^0 = \{ f \mid \forall v \in S,\ f(v) = 0_{\mathbb{F}},\ f : \mathbb{V} \to \mathbb{F} \}$$

It is very obvious that $f \in \mathbb{V}^*$ (the dual space of $\mathbb{V}$ which is the collection of all such linear functionals).

Claim: S^0 is the subspace of $\mathbb{V}^*$. But it really doesn't matter whether S is subspace of $\mathbb{V}$ or not.

If $S = \mathbb{V}$ itself then -

$$S^0 = \{ f \in \mathbb{V}^* \mid \forall v \in S, \; f(v) = 0_{\mathbb{F}} \}$$

Note that only zero functional $O(v) = 0_{\mathbb{F}} \; \forall v \in \mathbb{V}$ so S^0 contains only O (zero functional).

$$\boxed{\text{If } S = \mathbb{V} \text{ then } S^0 = \{ 0_{\mathbb{V}^*} \}}$$

If $S = \{0_{\mathbb{V}}\}$ then,

$$S^0 = \{ f \in \mathbb{V}^* \mid \forall v \in 0_{\mathbb{V}}, \; f(0_{\mathbb{V}}) = 0_{\mathbb{F}} \}$$

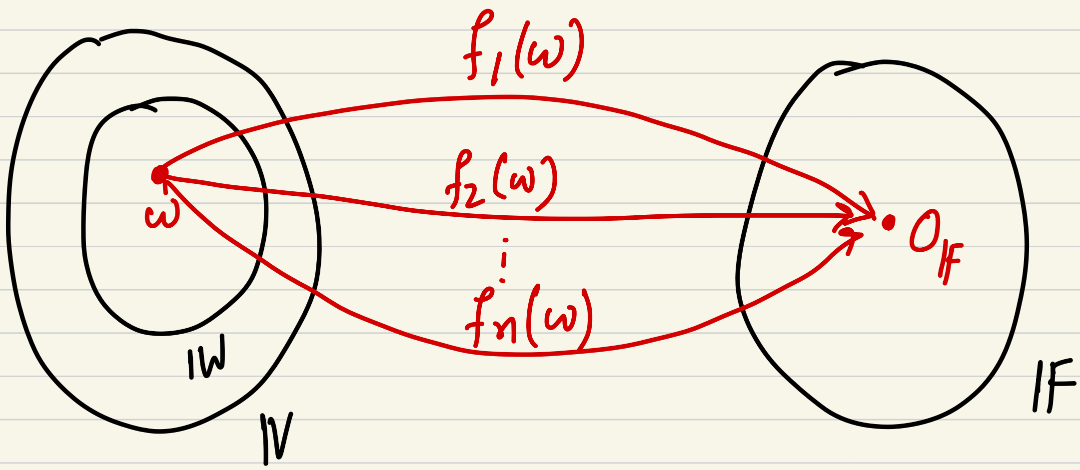
We take any functional $f \in \mathbb{V}^*$, it is always true that $f(0_{\mathbb{V}}) = 0_{\mathbb{F}}$ to be a valid linear functional. Therefore the $S^0 = \mathbb{V}^*$ itself.

$$\boxed{\text{If } S = \{0_{\mathbb{V}}\} \text{ then } S^0 = \mathbb{V}^*.}$$

Combining these 2 facts we can state a theorem.

Consider $\mathbb{V}$ is a f.d.v.s over $\mathbb{F}$. let $\mathbb{W}$ be a subspace of $\mathbb{V}$. (This is subspace and not subset).

$$\dim(\mathbb{W}) + \dim(\mathbb{W}^0) = \dim(\mathbb{V}).$$



$$\mathbb{W}^0 = \{ f \in \mathbb{V}^* \mid \forall w \in \mathbb{W} ,\ f(w) = 0_{\mathbb{F}} \}$$

$\mathbb{W}^0$ is a subspace of $\mathbb{V}^*$.

$\mathbb{W}$ is a subspace of $\mathbb{V}$.

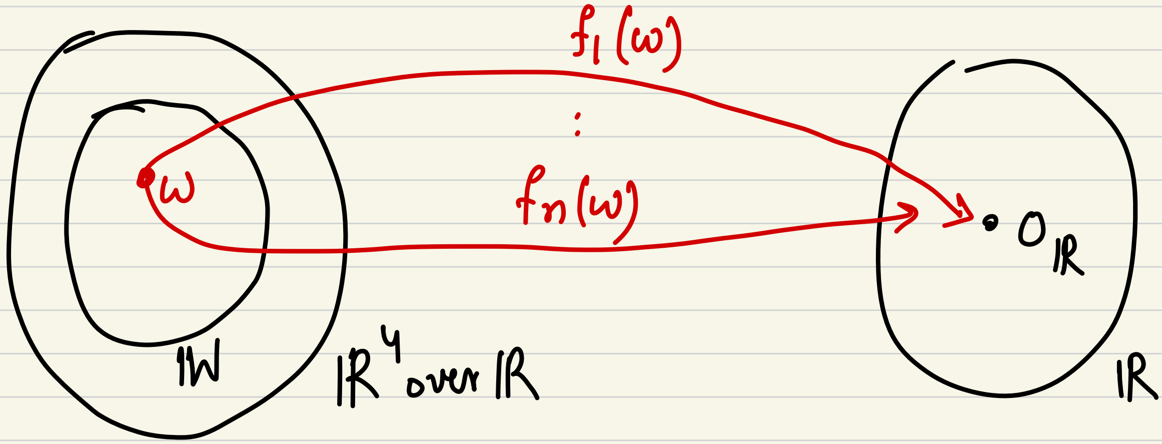
$$\boxed{\dim(\mathbb{W}) + \dim(\mathbb{W}^0) = \dim(\mathbb{V})}$$

We say $\mathbb{W}^0$ is the annihilator of subspace $\mathbb{W}$ or subspace $\mathbb{W}$ is annihilated by $\mathbb{W}^0$. Whatever linear functional you choose from the dual space $\mathbb{V}^*$ of $\mathbb{V}$, all the vectors inside $\mathbb{W}$ will be mapped to $0_{\mathbb{F}}$. That's why the name annihilator.

Ex 1: Consider the vector space $\mathbb{V} = \mathbb{R}^4$ over $\mathbb{R}$.
The 3 linear functionals are given below.

$$f_1(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 2x_3 + x_4$$
$$f_2(x_1, x_2, x_3, x_4) = 2x_2 + x_4$$
$$f_3(x_1, x_2, x_3, x_4) = -2x_1 \div 4x_3 + 3x_4$$

We are seeking the subspace of $\mathbb{R}^4$ annihilated by the linear functions $\{f_1, f_2, f_3\}$.



Consider the annihilator, $\mathbb{W}^0$ of $\mathbb{W}$ contains only 3 linear functionals f_1, f_2, f_3 so $\mathbb{W}^0 = \{f_1, f_2, f_3\}$

$\Rightarrow \forall w \in \mathbb{W}, \quad f_1(\omega) = 0_{\mathbb{R}}$

$\forall w \in \mathbb{W}, \quad f_2(\omega) = 0_{\mathbb{R}}$

$\forall w \in \mathbb{W}, \quad f_3(\omega) = 0_{\mathbb{R}}$

That means if we want to find the $\mathbb{W}$ which is annihilated by $\mathbb{W}^{\circ}$ then we have to solve for the inverse equations:

$$\left.\begin{array}{l} f_1(w) = 0 \\ f_2(w) = 0 \\ f_3(w) = 0 \end{array}\right\} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for $w = [x_1 \ x_2 \ x_3 \ x_4]^t$ such that the equation holds and the solution w will be the null space of the matrix A which is nothing the subspace $\mathbb{W}$ that we are looking for.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} \text{only } x_3 \text{ is free variable.} \\ x_4 = 0 \\ x_2 = 0 \\ x_1 = -2x_3 = -2 \end{array}\right\} \quad w = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \times c$$

Therefore, $\mathbb{W} = \left\{ (-2c, 0, c, 0) \mid \forall c \in \mathbb{R} \right\}$

Ex 2: let $\mathbb{V} = \mathbb{R}^5$ over $\mathbb{R}$,

a subspace $\mathbb{W}$ of $\mathbb{V}$ is given by the basis –

$$\left\{ v_1, v_2, v_3, v_4 \right\} \qquad \alpha_i \in \mathbb{R}^5.$$

$$\left.\begin{aligned} v_1 &= (2,-2,3,4,-1)^t \\ v_2 &= (0,0,-1,-2,3)^t \\ v_3 &= (-1,1,2,5,2)^t \\ v_4 &= (1,-1,2,3,0)^t \end{aligned}\right]$$

any linear functional can be written in the form.

$$f(v) = c_1x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$[v]_B = (x_1, x_2, \ldots, x_n)^t$$

Find out the annihilator of $\mathbb{W}$.

$$\mathbb{W}^0 = \left\{ f \in \mathbb{V}^* \,\middle|\, \forall w \in \mathbb{W},\ f(w) = 0_{\mathbb{F}} \right\}$$

$$f(v_1) = 0 \implies c_1 \cdot 2 - 2c_2 + 3c_3 + 4c_4 - c_5 = 0$$

$$f(v_2) = 0 \implies c_1 \cdot 0 + c_2 \cdot 0 - 1 \cdot c_3 - 2 \cdot c_4 + 3 \cdot c_5 = 0$$

$$f(v_3) = 0 \implies -c_1 \cdot 1 + c_2 \cdot 1 + 2 \cdot c_3 + 5 \cdot c_4 + 2 \cdot c_5 = 0$$

$$f(v_4) = 0 \implies 1 \cdot c_1 + c_2 \cdot (-1) + 2 \cdot c_3 + 3c_4 + 0 \cdot c_5 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 3 & 4 & -1 \\ 0 & 0 & -1 & -2 & 3 \\ -1 & 1 & 2 & 5 & 2 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

c_2, c_4 are free variables.

$c_5 = 0$

$c_3 \cdot 1 + 2 \cdot c_4 = 0$

$c_1 \cdot 1 - c_2 - c_4 = 0$

choose $c_4 = 1$, $\quad c_3 \cdot 1 = -2 \Rightarrow c_3 = -2$

$\qquad c_2 = 0$, $\quad c_1 = c_4 = 1$.

choose $c_4 = 0$, $\quad c_1 = c_2 = 1$

$\qquad c_2 = 1$, $\quad c_3 = -2 \times 0 = 0$

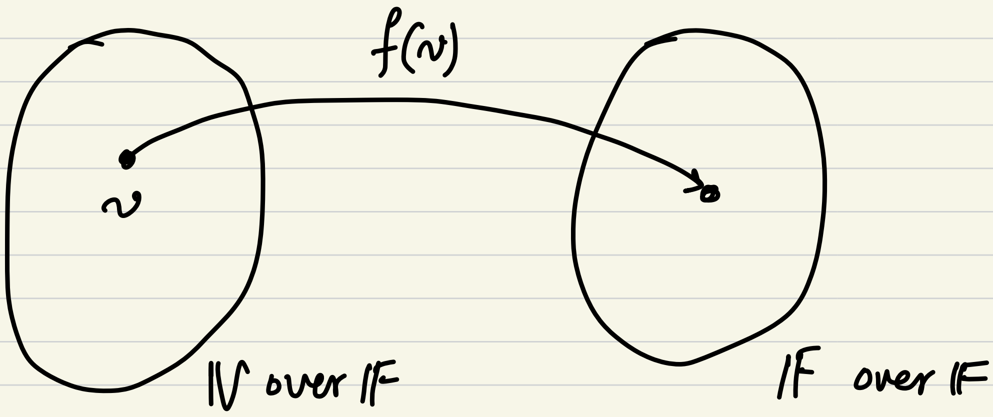
Therefore,
$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Therefore the linear functional is written as:

$f_1(v) = x_1 + x_2$, $\quad f_2(v) = x_1 - 2x_3 + x_4$

$\mathbb{W}^0 = \{ f \in \mathbb{V}^* \mid f = c \cdot f_1 + d \cdot f_2 \ \forall c, d \in \mathbb{R} \}$.

Double Dual:

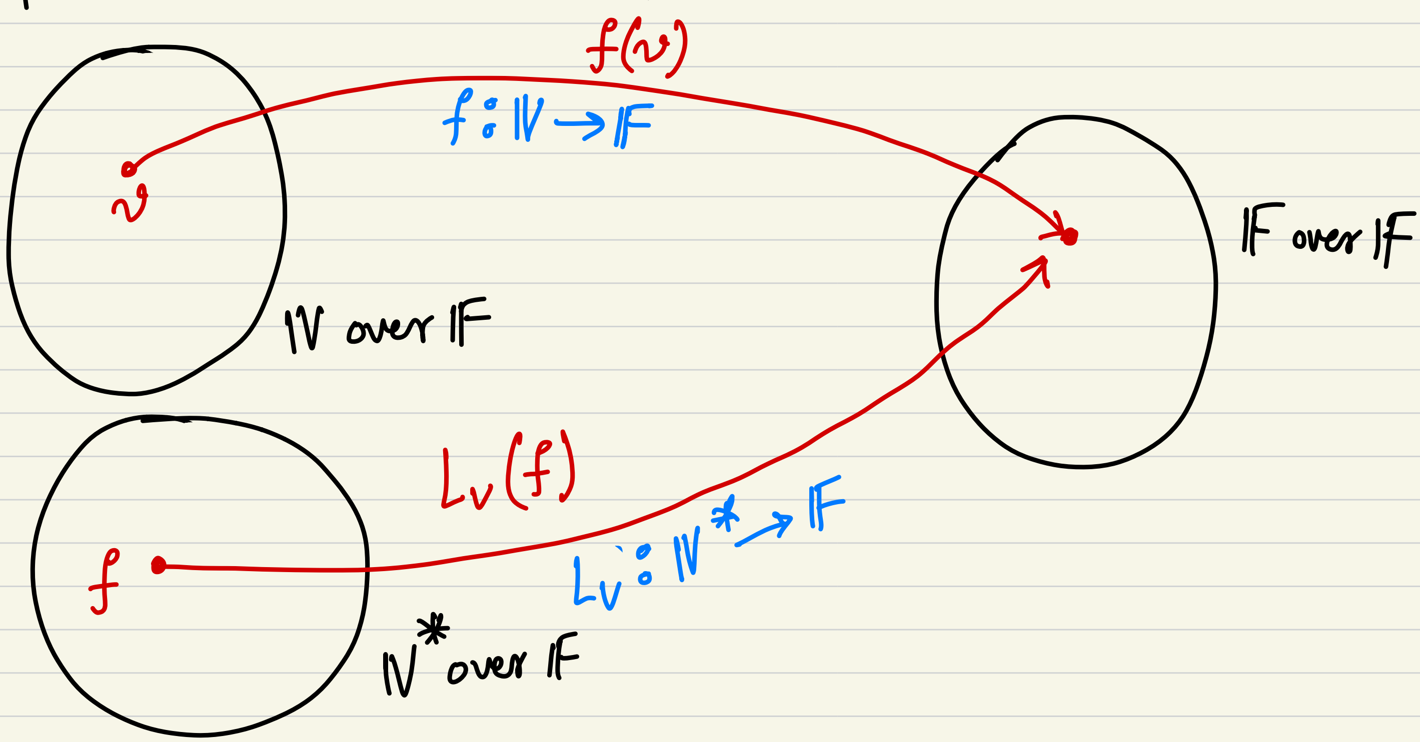


$$V^* = \{ f \mid f : V \to F \}$$

$B = \{ v_1, v_2, \ldots, v_n \}$, basis for V.

$B^* = \{ f_1, f_2, \ldots, f_n \}$, basis for V^* s.t $f_i(v_j) = \delta_{ij}$

Now given V^* and B^*, we can construct the dual of V^* and dual basis of B^*.



Suppose a vector $v \in \mathbb{V}$. We can construct a linear functional on $\mathbb{V}$ by $f: \mathbb{V} \to \mathbb{F}$ and we have seen that $f \in \mathbb{V}^*$. but $f(v) \in \mathbb{F}$.

Because of v in $\mathbb{V}$, there exist a linear functional on $\mathbb{V}^*$ such that $L_v(f) := f(v)$ where $L_v : \mathbb{V}^* \longrightarrow \mathbb{F}$.

The claim is is linear transformation. from $\mathbb{V}^*$ to $\mathbb{F}$.

$$L_v(c \cdot f_1 + f_2) = c L_v(f_1) + L_v(f_2)$$

$$\Rightarrow (c \cdot f_1 + f_2)(v)$$

$$\Rightarrow c \cdot f_1(v) + f_2(v)$$

$$\Rightarrow c \cdot L_v(f_1) + L_v(f_2)$$

Hence L_v is a linear transformation or linear functional on V^*. L_v is called double dual of $\mathbb{V}$.

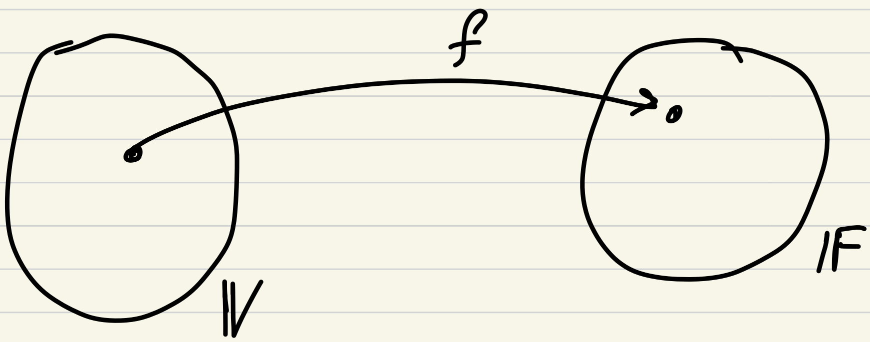
Double dual, $V^{**} = \{ L_v \mid L_v(f) = f(v), v \in \mathbb{V} \}$

Consider $\mathbb{V}$ as vector space over $\mathbb{F}$.
For each $v \in \mathbb{V}$ define $L_v(f) := f(v)$, $f \in \mathbb{V}^*$

$$\dim(V^{**}) = \dim(V^*) = \dim(\mathbb{V}).$$

Every linear functional is either surjective or zero map.

$\Rightarrow$



$f: \mathbb{V} \to \mathbb{F}$ is surjective iff $\mathrm{Im}(f) = \mathbb{F}$ or

$\dim(\mathrm{Im}(f)) = \dim(\mathbb{F}) = 1$

Since $\mathbb{F}$ over $\mathbb{F}$ has only 2 subspaces possible

$\rightarrow$ $\mathbb{F}$ itself with $\dim = 1$

$\rightarrow$ $\{0\}$ with $\dim = 0$

Since $\mathrm{Im}(f)$ is a subspace of $\mathbb{F}$ and that subspace can be

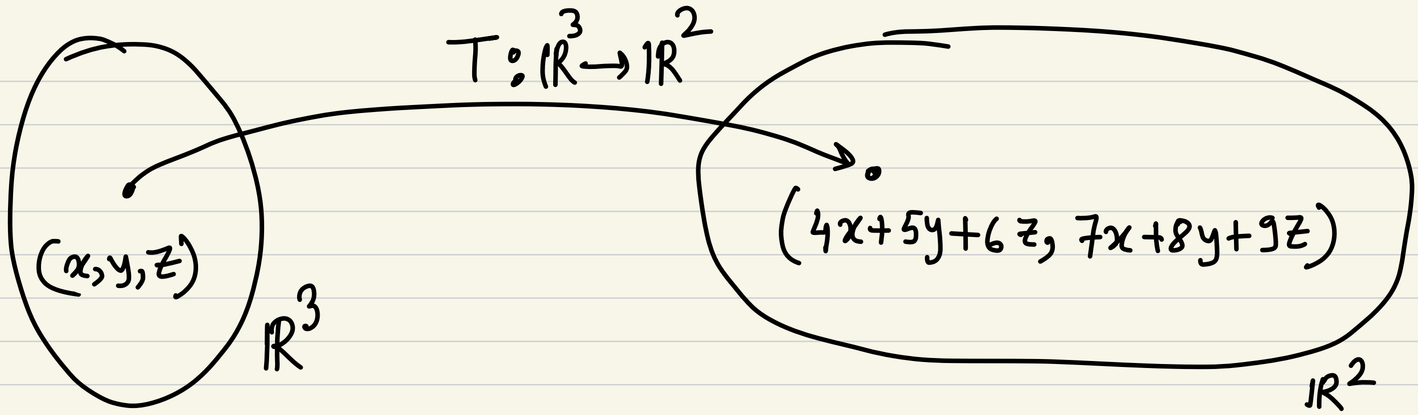
$\rightarrow$ $\mathbb{F}$. in that case f is surjective as $\mathrm{Im}(f) = \mathbb{F}$.

$\rightarrow$ $\{0_{\mathbb{F}}\}$. in that case f is zero map: $0(v) = 0_{\mathbb{F}}$.

13 Define $T: \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbf{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbf{R}^3$.

(a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

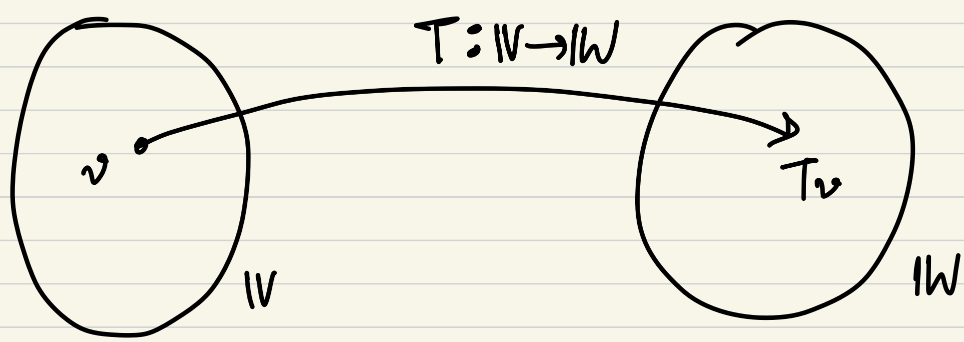
(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

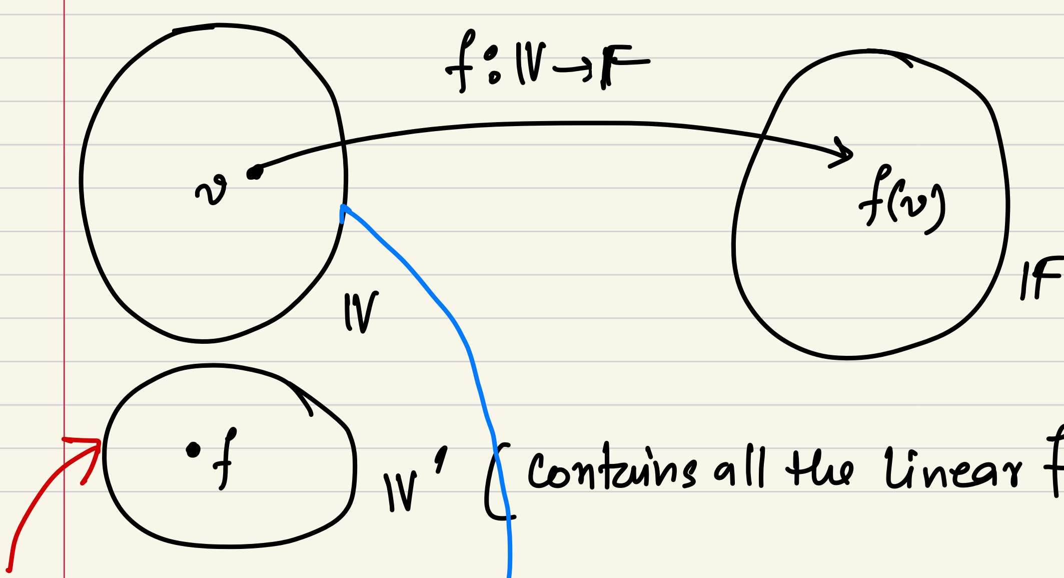


$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$S_1' = \{ \psi_1, \psi_2, \psi_3 \} \qquad S_2' = \{ \varphi_1, \varphi_2 \}$$

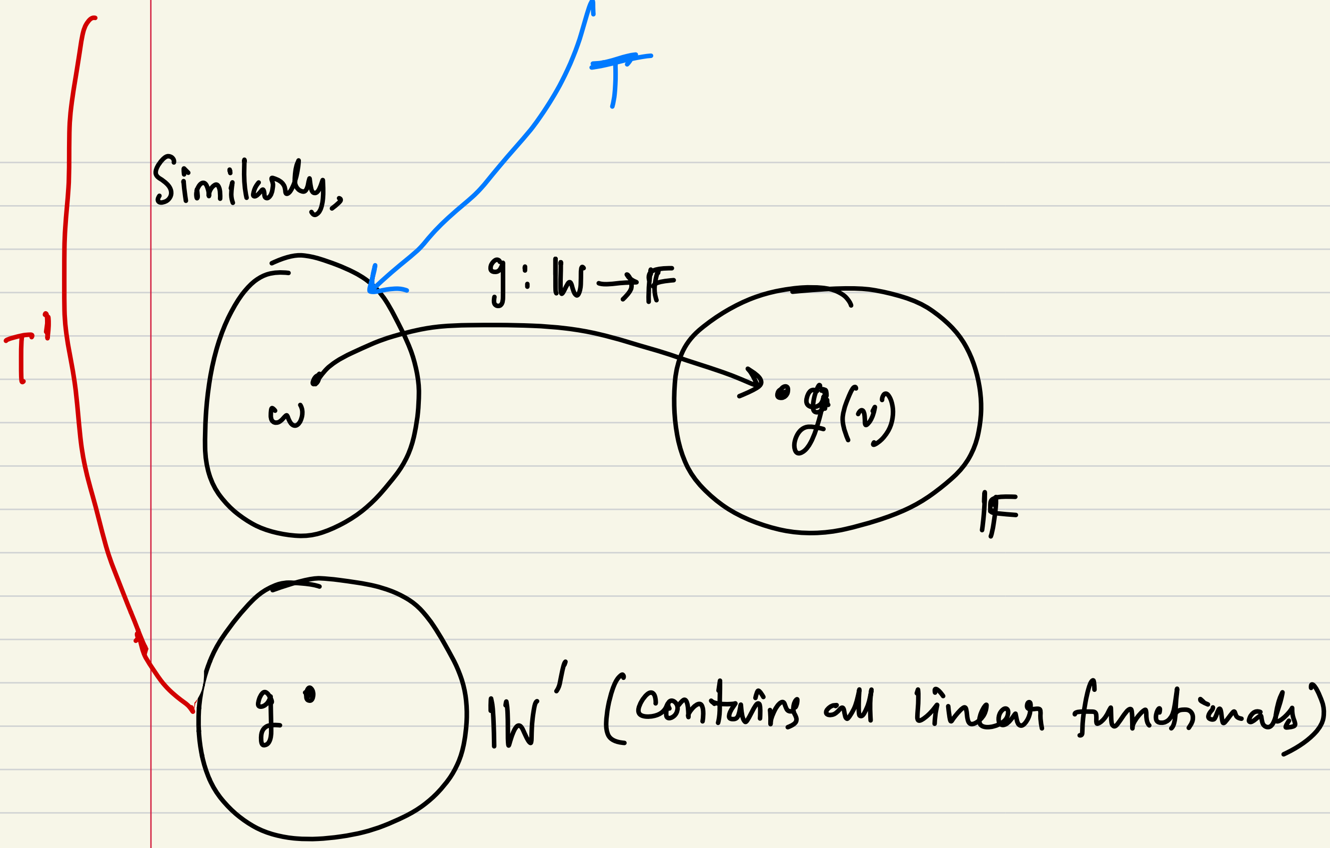
If $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ then the dual map of T is the linear map $T' \in \mathcal{L}(\mathbb{W}', \mathbb{V}')$.





(contains all the linear functionals)

Similarly,



The map $T' : \mathbb{W}' \longrightarrow \mathbb{V}'$ is called dual map of T.

T' is defined as:

for $g \in \mathbb{W}'$, $T'(g) = f = g \circ T$

$$\underbrace{T'(g)}_{f}(v) = g \circ T(v) = g\big(\underbrace{T(v)}_{w}\big) = g(w) \in \mathbb{F}$$

$f \rightarrow f(v) \in \mathbb{F}$.

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "T"] & \mathbb{W} \\
v \arrow[d, "f"] & Tv \arrow[d, "\varphi_1"] \\
\mathbb{F} & \mathbb{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{V}' & \mathbb{W}' \arrow[l, "T'"'] \\
f = T'(\varphi_1) & \varphi_1
\end{tikzcd}$$

$$f(v) = \big(T'(\varphi_1)\big)(v) = \varphi_1(Tv)$$

$$(T'(\varphi_1))(v) = \varphi_1(T(v))$$

$$\Rightarrow T'(\varphi_1)(x,y,z) = \varphi_1(T(x,y,z))$$

$$= \varphi_1(4x+5y+6z,\ 7x+8y+9z)$$

Dual basis of $\begin{bmatrix}1\\0\end{bmatrix}$ is φ_1 where $\varphi_1(v_j) = 1 \quad j=1$

$= 0$ otherwise

$$T'(\varphi_1)(x,y,z) = (4x+5y+6z)\cdot \underbrace{c_1}_{\substack{\varphi_1(v_1) \\ =1}} + (7x+8y+9z)\cdot \underbrace{c_2}_{\substack{\varphi_1(v_2) \\ =0}}$$

$$= 4x+5y+6z.$$

Similarly, $T'(\varphi_2)(x,y,z) = 7x+8y+9z.$

$T'(\varphi_1)$, $T'(\varphi_2)$ as LC of ψ_1, ψ_2, ψ_3.

$\psi_1\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 1$

$\psi_2\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 1$

$\psi_3\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 1$

$$(4\psi_1 + 5\psi_2 + 6\psi_3)(x,y,z)$$

$$= 4\psi_1(x,y,z) + 5\psi_2(x,y,z) + 6\psi_3(\ldots)$$

$$= 4x + 5y + 6z.$$

$$= T'(\varphi_1)(x,y,z)$$

Dual Map:

$T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$

$T' \in \mathcal{L}(\mathbb{W}', \mathbb{V}')$

T' is called the dual map of T.

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "T"] & \mathbb{W} \\
v \arrow[r, mapsto] \arrow[d, "T'(\varphi) \in \mathbb{V}'"'] \arrow[dr, dotted, "\varphi \circ T"] & Tv \arrow[d, "\varphi \in \mathbb{W}'"] \\
\mathbb{F} & \mathbb{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{W}' \arrow[r, "T'"] & \mathbb{V}' \\
\varphi \arrow[r, mapsto] & T'(\varphi)
\end{tikzcd}$$

$$\big(T'(\varphi)\big)(v) \quad = \quad \varphi(T(v))$$

Therefore T' is defined as:

$$\boxed{\begin{aligned} T'(\varphi) &:= \varphi \circ T \\ \Rightarrow \big(T'(\varphi)\big)(v) &= \varphi(T(v)) \end{aligned}}$$

$T'(\varphi)$ spits out another linear functional which belongs to $\mathbb{V}'$ (dual space of $\mathbb{V}$).

① For $S, T \in \mathcal{L}(V, W)$, the following things hold.

(i) $(S+T)' = S' + T'$

(ii) $(\lambda T)' = \lambda T'$

$$\left.\begin{array}{l} T \in \mathcal{L}(U, V) \\ S \in \mathcal{L}(V, W) \end{array}\right\} \quad (ST)' = T'S'$$

<u>proof:</u>

$(S+T)'$ is the dual map of $S+T$.

$(S+T)' \in \mathcal{L}(W', V')$ and $S, T \in \mathcal{L}(V, W)$

Suppose $\varphi \in W'$ and $v \in V$

$$\Rightarrow \left((S+T)'(\varphi)\right)(v) := \varphi\left((S+T)(v)\right)$$

$$\Rightarrow \varphi(Sv + Tv)$$

$$\Rightarrow \varphi(Sv) + \varphi(Tv)$$

$$\Rightarrow \varphi(S(v)) + \varphi(T(v))$$

$$\Rightarrow \varphi \circ S(v) + \varphi \circ T(v)$$

$$\Rightarrow \left(S'(\varphi)\right)(v) + \left(T'(\varphi)\right)(v)$$

$$\Rightarrow \Big(S'(\varphi) + T'(\varphi) \Big)(v)$$

$$\Rightarrow \Big((S'+T')(\varphi) \Big)(v)$$

$$\Rightarrow \Big((S+T)'(\varphi) \Big)(v)$$

$$(S+T)' = S'+T'$$

(iii) $\left.\begin{array}{l} T \in L(U,V) \\ S \in L(V,W) \end{array}\right\} \quad (S.T)' = T'.S'$

$$\begin{array}{ccccc} U & \xrightarrow{T} & V & \xrightarrow{S} & W \\ u & \longmapsto & Tu & \longmapsto & S(Tu) \end{array}$$

$$\longmapsto_{S\circ T} ST(u)$$

$$\left\{\begin{array}{l} W' \xrightarrow{(ST)'} U' \\ \varphi \longmapsto (ST)'(\varphi) \\ W' \xrightarrow{S'} V' \xrightarrow{T'} U' \\ \varphi \longmapsto S'(\varphi) \longmapsto T'(S'(\varphi)) \end{array}\right.$$

We can say that, $(ST)'(\varphi) = T'S'(\varphi)$

$$\Rightarrow (ST)' = T'S'$$

Examples of Dual map:

① Consider derivative operator on polynomial space.

$D \in L(\mathbb{R}[x], \mathbb{R}[x])$ defined as,

$$D(p(x)) := \frac{d}{dx}(p(x)) = p'(x) \qquad \forall p \in \mathbb{R}[x]$$

$$\Rightarrow D(p) = p'$$

It is given that $\varphi \in \mathbb{R}[x]^*$ (dual space of $\mathbb{R}[x]$)

$$\varphi(f) := \int_0^1 f(x)\,dx \qquad \text{where } f \in \mathbb{R}[x]$$

$$\begin{tikzcd} \mathbb{R}[x] \arrow[r, "\varphi"] & \mathbb{R} \\ f \arrow[r, mapsto] & \varphi(f) \end{tikzcd}$$

For the transformation D, we have to find its dual map?

$$\begin{tikzcd} \mathbb{R}[x] \arrow[r, "D"] & \mathbb{R}[x] \\ f \arrow[r, mapsto] \arrow[d, mapsto, "D^*(\varphi) \in \mathbb{R}[x]^*"'] & D(f) \arrow[d, mapsto, "\varphi \in \mathbb{R}[x]^*"] \\ (D^*(\varphi))(f) \in \mathbb{F} & \varphi(D(f)) \in \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}[x]^* \arrow[r, "D^*"] & \mathbb{R}[x]^* \\ \varphi \arrow[r, mapsto] & D^*(\varphi) \end{tikzcd}$$

$$(D^*(\varphi))(f) \in \mathbb{F} \quad = \quad \varphi(D(f)) \in \mathbb{R}$$

We can write that, $D^*(\varphi) = \varphi \circ D$

$\Rightarrow \left(D^*(\varphi)\right)(f) = \varphi\left(D(f)\right)$

$\Rightarrow \left(D^*(\varphi)\right)(f(x)) = \varphi\left(D(f(x))\right)$

$\Rightarrow \left(D^*(\varphi)\right)(f(x)) = \varphi\left(\dfrac{df(x)}{dx}\right)$

Choose $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$\varphi\left(\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right)\right)$

$= \varphi\left(a_1 + 2a_2 x + 3a_3 x^2 + \cdots\right)$

$= \varphi(a_1) + \varphi(2a_2 x) + \varphi(3a_3 x^2) + \cdots$

$= \int_0^1 a_1\,dx + \int_0^1 2a_2 x\,dx + \int_0^1 3a_3 x^2\,dx + \cdots$

$= \left[a_1 x\right]_0^1 + \left[a_2 x^2\right]_0^1 + \left[a_3 x^3\right]_0^1 + \cdots$

$= \left[a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right]_0^1 = a_1 + a_2 + a_3 + \cdots$

$= (a_0 + a_1 + a_2 + \cdots) - a_0 = f(1) - f(0)$

$$\boxed{\Rightarrow \left(D^*(\varphi)\right)(f(x)) = f(1) - f(0)}$$

So we can say $D^*(\varphi)$ is a linear functional on $\mathbb{R}[x]$ spitting out scalars in $\mathbb{R}$.

$$\begin{aligned} \mathbb{R}[x] &\xrightarrow{\quad D^*(\varphi) \quad} \mathbb{F} \\ f &\longmapsto f(1) - f(0) \end{aligned}$$

Hence $D^*(\varphi) \in \mathbb{R}[x]^*$ (Dual space of polynomials)

<u>Ex2:</u> Suppose φ is a linear functional from $\mathbb{R}[x]$ to $\mathbb{R}$. defined as: $\varphi(f) := f(3)$ and $\varphi \in \mathbb{R}[x]^*$.

If $D : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ and $D(f) := \frac{d}{dx} f(x)$ then obtain $D^* : R[x]^* \longrightarrow R[x]^*$ as $D^*(\varphi)$.

$$\begin{aligned} R[x] &\xrightarrow{\quad D \quad} R[x] \\ f &\longmapsto D(f) \end{aligned} \qquad \begin{aligned} R[x]^* &\xrightarrow{\quad D^* \quad} R[x]^* \\ \varphi &\longmapsto D^*(\varphi) \end{aligned}$$

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto, "D^*(\varphi) \in \mathbb{R}[x]^*"] & D(f) \arrow[d, mapsto, "\varphi \in \mathbb{R}[x]^*"] \\ (D^*(\varphi))(f) \in \mathbb{F} = & \varphi(D(f)) \in \mathbb{F} \end{tikzcd}$$

$\Rightarrow D^*(\varphi)(f) = \varphi(D(f))$

$$\Rightarrow D^*(\varphi)(f) = D(f)(3)$$

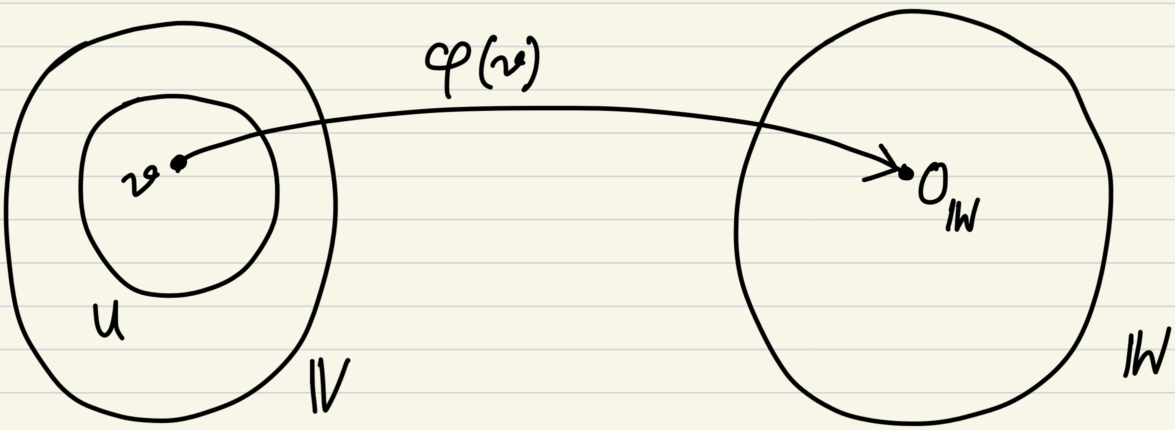
$$\boxed{\Rightarrow D^*(\varphi)(f) = f'(3)}$$

Kernel and Image of Dual map:

Linear map: $T \in L(\mathbb{V}, \mathbb{W})$

Dual map: $T' \in L(\mathbb{W}', \mathbb{V}')$

How can we describe $\ker(T')$ and $\mathrm{Im}(T')$ in terms of $\ker(T)$ and $\mathrm{Im}(T)$?



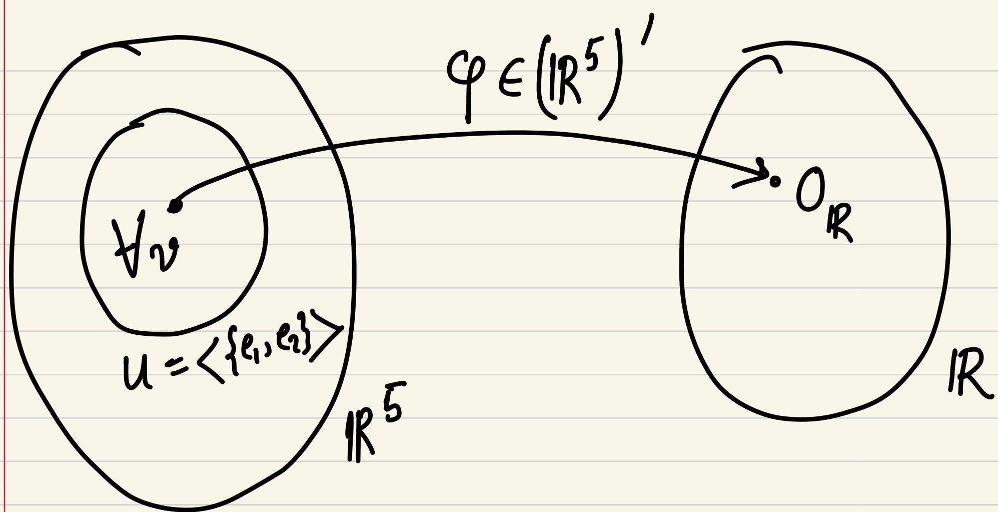
$$U^0 := \{\varphi \in \mathbb{V}' \mid \forall v \in U, \ \varphi(v) = 0_{\mathbb{W}}\}$$

U^0 is the collection of all those linear functionals which takes all the vectors from a set U and spits to $0_{\mathbb{W}}$ in $\mathbb{W}$.

3.104 Example Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbf{R}^5$, and let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbf{R}^5)'$. Suppose

$$U = \operatorname{span}(e_1, e_2) = \{(x_1, x_2, 0, 0, 0) \in \mathbf{R}^5 : x_1, x_2 \in \mathbf{R}\}.$$

Show that $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.



$$B = \{e_1, e_2, e_3, e_4, e_5\}$$

$$B' = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\} \text{ s.t. } \varphi_i(e_j) = \delta_{ij}$$

The set, $U = \langle \{e_1, e_2\} \rangle$

$$= \{(x_1, x_2, 0, 0, 0) \mid \forall x_1, x_2 \in \mathbb{R}\}$$

The anhiliator of U is given by:

$$U^\circ := \{\varphi \in (\mathbb{R}^5)' \mid \forall v \in U, \ \varphi(v) = 0_{\mathbb{R}}\}$$

We have to prove that $U^\circ = \langle \{\varphi_3, \varphi_4, \varphi_5\} \rangle$

(i) If $\varphi \in U^\circ$ then $\varphi \in \langle \{\varphi_3, \varphi_4, \varphi_5\} \rangle$

Assume $\varphi \in U^\circ$

$\Rightarrow \varphi(v) = 0_{\mathbb{R}} \ \forall v \in U$

Consider $v = (x_1, x_2, 0, 0, 0)$

$\varphi(x_1, x_2, 0, 0, 0) = 0_{\mathbb{R}}$.

We know how the basis of dual space is selected.

$\Rightarrow \varphi_i(e_j) = \delta_{ij}$

$\Rightarrow \varphi_1(e_1) = 1, \varphi_2(e_2) = 0, \dots, \varphi_5(e_5) = 0$

Express the $\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n \in U^0$

$\varphi(e_1) = c_1, \varphi(e_2) = c_2, \dots, \varphi(e_n) = c_n$

$\Rightarrow \varphi = \varphi(e_1)\cdot\varphi_1 + \varphi(e_2)\cdot\varphi_2 + \dots + \varphi(e_n)\cdot\varphi_n$

Since $e_1, e_2 \in U$ therefore, $\varphi(e_1) = 0_{\mathbb{R}}, \varphi(e_2) = 0_{\mathbb{R}}$ as φ is the linear functional in the annihilator U^0.

Therefore, $\varphi = \varphi(e_3)\cdot\varphi_3 + \dots + \varphi(e_5)\cdot\varphi_5$

$\Rightarrow \varphi \in \langle\{\varphi_3, \varphi_4, \varphi_5\}\rangle$

Therefore we can say that $\varphi \in U^0 \longrightarrow \varphi \in \langle\{\varphi_3, \varphi_4, \varphi_5\}\rangle$

Now consider, $\varphi \in \langle\{\varphi_3, \varphi_4, \varphi_5\}\rangle$

$\Rightarrow \varphi = c_3\cdot\varphi_3 + c_4\varphi_4 + c_5\cdot\varphi_5$.

$\Rightarrow \varphi(x_1, x_2, 0, 0, 0) = c_3\cdot\varphi_3(x_1, x_2, 0, 0, 0) + \dots$

$= c_3\varphi_3(x_1\cdot e_1 + x_2\cdot e_2) + \dots$

$= c_3 x_1 \varphi_3(e_1) + c_3\cdot x_2\cdot\varphi_3(e_2) + \dots$

$= 0 + 0 + 0 + \dots = 0_{\mathbb{R}}$.

$\Rightarrow \underline{\varphi \in U^0}$.

The annhilator of U represented by U^0 which is the Collection of Linear functionals from V' so U^0 is a subspace of V'.

<u>proof</u>: $U^0 := \{ f \in V' \mid \forall v \in U \text{ s.t. } f(v) = 0_{\mathbb{F}} \}$

consider f and $g \in U^0$.

Construct $h = c.f + g$

$$(c.f+g)(v) = (c.f)(v) + g(v)$$
$$= c.f(v) + g(v)$$
$$= 0 + 0 \qquad [\because f, g \in U^0]$$
$$= 0$$

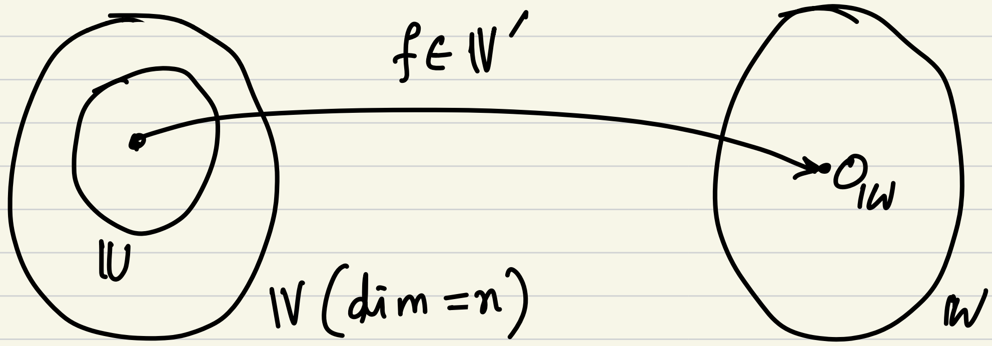
Therefore $c.f+g$ is also in U^0 hence U^0 is subspace of V'

We know $\dim(V) = \dim(V')$

$\dim(U^0) \leq \dim(V')$

claim: $\dim(U^0) + \dim(U) = \dim(V')$

proof:



Suppose $\{v_1, v_2, \cdots, v_K\}$ is the basis of $\mathbb{U}$.
Extend this basis to construct the basis of $\mathbb{V}$.

Basis for $\mathbb{V} = \{v_1, v_2, \cdots, v_K, v_{K+1}, v_{K+2}, \cdots, v_n\}$
Basis for $\mathbb{V}' = \{f_1, f_2, f_3, \cdots, f_n\}$

The claim: $\{f_{K+1}, f_{K+2}, \cdots, f_n\}$ is the basis of $\mathbb{U}^0$.

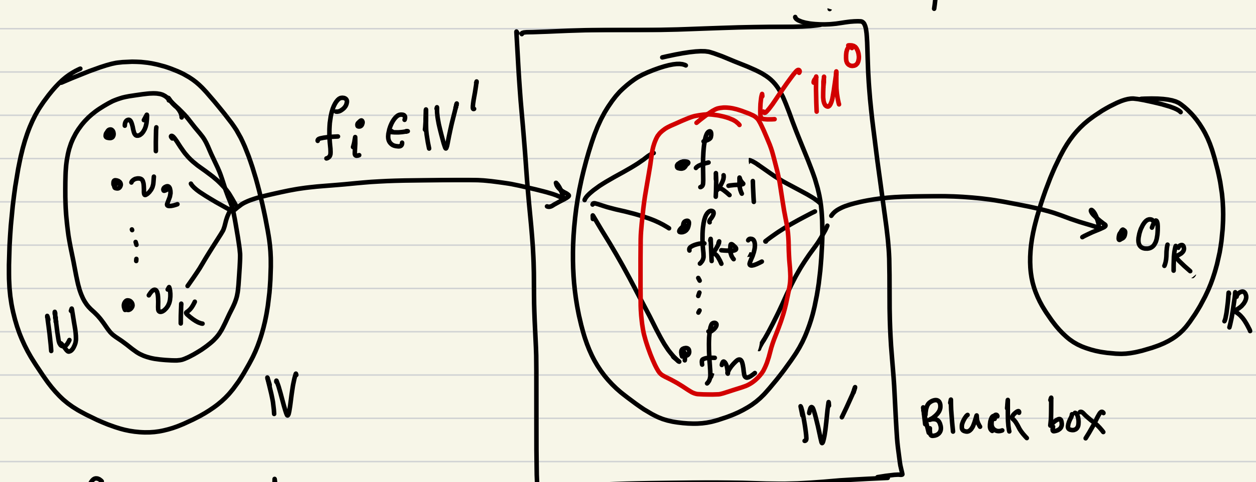
By definition: $f_i(v_j) = \delta_{ij}$

Definitely $\{f_i\}_{i=K+1}^{n}$ are LI. We have to show that they span $\mathbb{U}^0$.

$$f = c_1 f_{K+1} + c_2 f_{K+2} + \cdots + c_{K-n} f_n$$

$$f(v) = f(v_{K+1}) \cdot f_{K+1} + \cdots + f(v_n) f_n$$

Suppose $f \in \mathbb{U}^0$ then $f(v_{K+1}) = f(v_{K+2}) = \cdots = f(v_n) = 0$
That means $f(v) = 0$ hence the set spans $\mathbb{U}^0$.



$$U^0 = \{f \in \mathbb{V}' \mid \forall v \in \mathbb{U},\ f(v) = 0_{\mathbb{R}}\}.$$

Basis for $\mathbb{U}^0 = \{f_{K+1}, f_{K+2}, \dots, f_n\} \longrightarrow \dim(\mathbb{U}^0) = n-K$

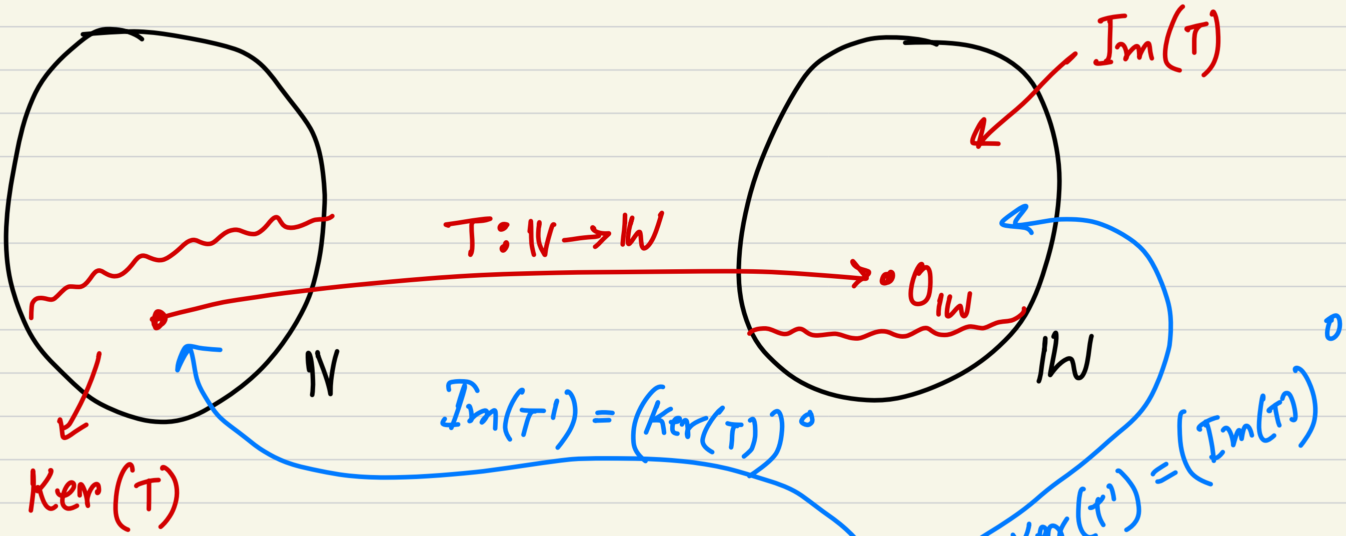
Basis for $\mathbb{V}' = \{f_1, f_2, \dots, f_n\} \longrightarrow \dim(\mathbb{V}') = n$

Basis for $\mathbb{U} = \{v_1, v_2, \dots, v_K\} \longrightarrow \dim(\mathbb{U}) = K.$

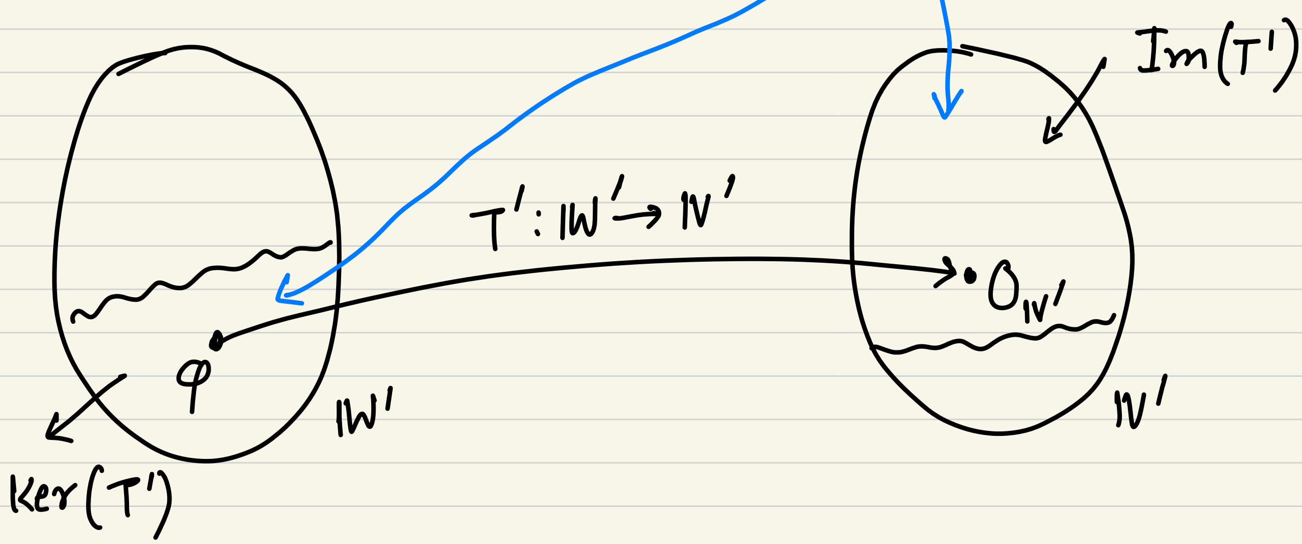
$$\dim(\mathbb{U}) + \dim(\mathbb{U}^0) = \dim(\mathbb{V}') = \dim(\mathbb{V})$$

Suppose $\mathbb{V}$ and $\mathbb{W}$ are f.d.v.s.

$T \in L(\mathbb{V}, \mathbb{W})$.



claim: $\operatorname{Ker}(T') = (\operatorname{Im}(T))^0$.



prove that, $\varphi \in \mathrm{Ker}(T') \longrightarrow \varphi \in (\mathrm{Im}(T))^0$

assume $\varphi \in \mathrm{Ker}(T')$

$\Rightarrow T'(\varphi) = \varphi \circ T = 0_{V'}$ (zero functional in V')

This should be true for all $v \in V$.

$\Rightarrow (T'(\varphi))(v) = \varphi(T(v)) = 0_{\mathbb{F}}$

$\mathrm{Im}(T) = \{ w \in W \mid \exists v \in V \text{ s.t } T(v) = w \}$

$(\mathrm{Im}(T))^0 = \{ \varphi \in W' \mid \forall w \in \mathrm{Im}(T) \text{ s.t. } \varphi(w) = 0_{\mathbb{F}} \}$

$= \{ \varphi \in W' \mid \forall w \in \mathrm{Im}(T) \text{ s.t. } \varphi(T(v)) = 0_{\mathbb{F}} \}$

That means $(T'(\varphi))(v) = \varphi(T(v)) = 0_{\mathbb{F}}$

$\Rightarrow \varphi \in (\mathrm{Im}(T))^0$. from the above definition.

prove that $\varphi \in (\mathrm{Im}(T))^0 \longrightarrow \varphi \in \mathrm{Ker}(T')$

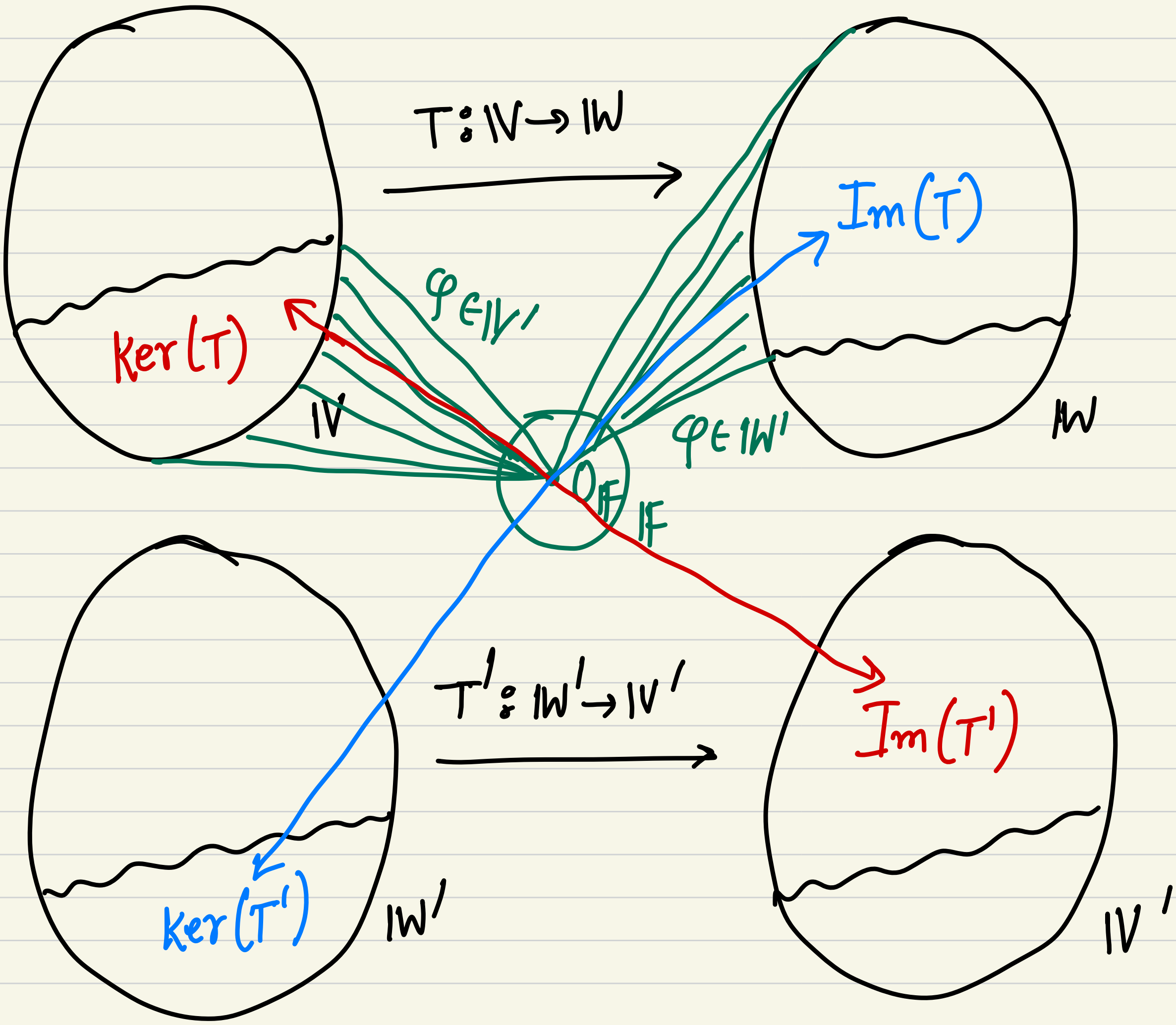
assume $\varphi \in (\mathrm{Im}(T))^0$

$\Rightarrow \varphi(T(v)) = 0_{\mathbb{F}} \quad \forall v \in V.$

$\Rightarrow (T'(\varphi))(v) = 0_{\mathbb{F}} \quad \forall v \in V.$

$\Rightarrow T'(\varphi) = 0_{V'} \Rightarrow \varphi \in \mathrm{Ker}(T')$

$$\dim(\mathrm{Ker}(T')) = \dim(\mathrm{Ker}(T)) + \dim(\mathbb{W}) - \dim(\mathbb{V})$$



$\mathrm{Ker}(T') = (\mathrm{Im}(T))^0$ $\qquad$ $\mathrm{Im}(T') = (\mathrm{Ker}\,T)^0$

We know, $\dim(\mathrm{Im}(T)) + \dim((\mathrm{Im}(T))^0) = \dim(\mathbb{W})$

$\dim(\mathrm{Ker}(T)) + \dim((\mathrm{Ker}(T))^0) = \dim(\mathbb{V})$

We know, $\dim(\mathrm{Ker}(T')) = \dim((\mathrm{Im}(T))^0)$

$= \dim(\mathbb{W}) - \dim(\mathrm{Im}(T))$

We know, $\dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)) = \dim(\mathbb{V})$.

Therefore, $\dim(\operatorname{Ker}(T')) = \dim(\mathbb{W}) - \dim(\mathbb{V}) + \dim(\operatorname{Ker}(T))$

(Hence proved)

Claim: $\operatorname{Im}(T') = (\operatorname{Ker} T)^0$

prove that : $\varphi \in \operatorname{Im}(T') \longrightarrow \varphi \in (\operatorname{Ker} T)^0$

assume that $\varphi \in \operatorname{Im}(T')$ and $\operatorname{Im}(T') \subseteq \mathbb{V}'$

$\Rightarrow \exists\, \psi \in \mathbb{W}'$ such that, $\varphi = T'(\psi)$

$\Rightarrow \varphi = \psi \circ T$

$\Rightarrow \varphi(v) = (\psi \circ T)(v) = \psi(T(v))$

If $v \in \operatorname{Ker}(T)$ then, $\psi(T(v)) = \psi(0_{\mathbb{W}}) = 0_{\mathbb{F}}$

Therefore, $(\operatorname{Ker} T)^0 = \{ \varphi \in \mathbb{V}' \mid \forall v \in \operatorname{Ker}(T),\ \varphi(v) = 0_{\mathbb{F}} \}$

$\Rightarrow \varphi(v) = \psi(T(v)) = 0_{\mathbb{F}} \quad \forall v \in \operatorname{Ker}(T)$

$\Rightarrow \varphi \in (\operatorname{Ker}(T))^0$

Similarly assume $\varphi \in (\operatorname{Ker} T)^0 \longrightarrow \varphi \in \operatorname{Im}(T')$

$$(\operatorname{Ker}(T))^0 = \{ \varphi \in \mathbb{V}' \mid \forall v \in \operatorname{Ker}(T) \text{ s.t } \varphi(v) = 0_{\mathbb{F}} \}$$

$\varphi \in (\operatorname{Ker} T)^0 \Rightarrow \varphi(v) = 0_{\mathbb{F}} \quad \forall v \in \operatorname{Ker}(T).$

If $v \in \mathrm{Ker}(T) \Rightarrow Tv = 0_{\mathbb{W}}$ for all $v \in \mathrm{Ker}(T)$.

$$\mathrm{Im}(T') = \left\{ \varphi \in \mathbb{V}' \;\middle|\; \exists \psi \in \mathbb{W}' \text{ s.t. } T'(\psi) = \phi \right\}$$

$\varphi(v) = 0_{\mathbb{F}}$

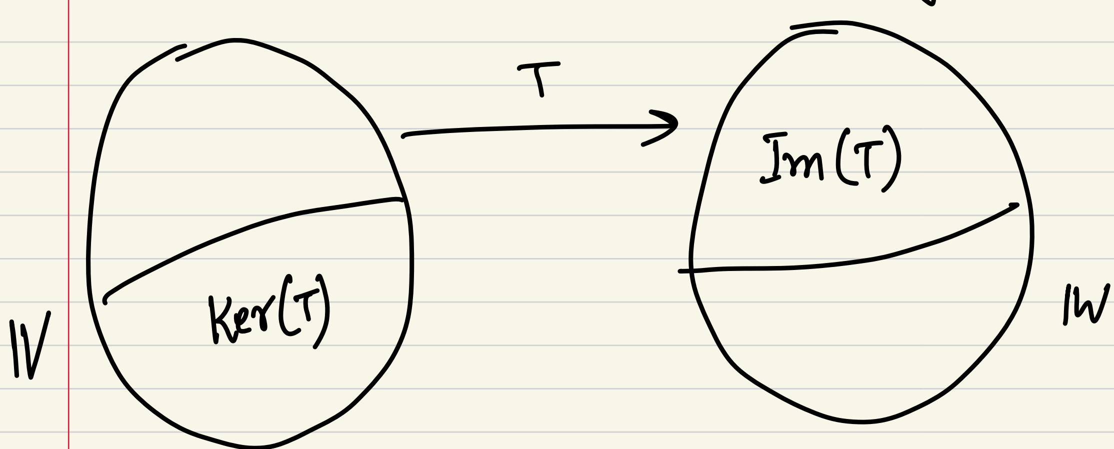
$\Rightarrow \varphi(v) = \psi(0_{\mathbb{W}}) \quad (\exists \psi \in \mathbb{W}')$

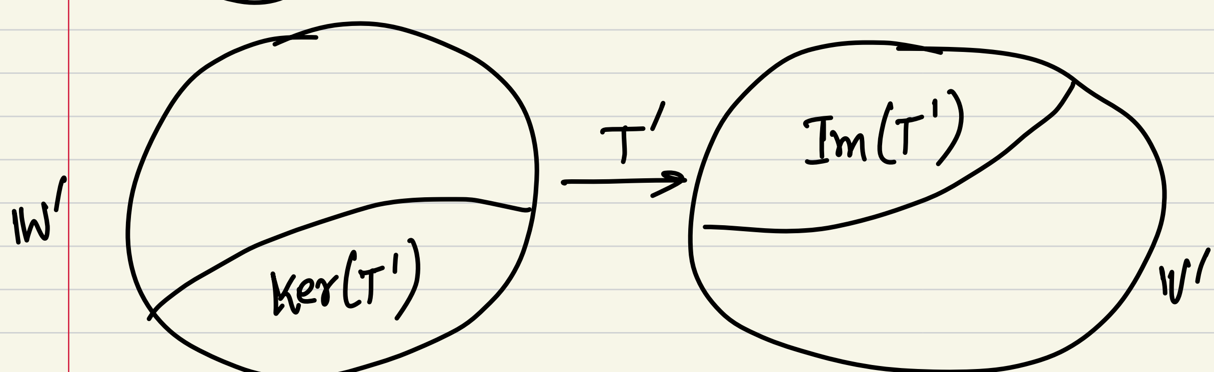
$\Rightarrow \psi(T(v)) = \phi(v)$

$\Rightarrow T'(\psi) = \phi$

$\Rightarrow \phi \in \mathrm{Im}(T')$

Claim: T is surjective if and only if T' is injective.





T is surjective $\longrightarrow T'$ is injective.

Assume T is surjective.

$\Rightarrow \text{Im}(T) = \mathbb{W}$.

We know, $\text{Ker}(T') = \left(\text{Im}(T)\right)^0 = \mathbb{W}^0$

We also know that if U is full set of $\mathbb{W}$ then.

$U^0 = W^0 = \{0_{\mathbb{W}'}\}$ (zero functional)

Therefore, $\text{Ker}(T') = \mathbb{W}^0 = \{0_{\mathbb{W}'}\}$

If $\text{Ker}(T')$ is $\{0_{\mathbb{W}'}\}$ then T' is injective.

So proved.

T' is injective $\longrightarrow T$ is surjective.

Assume T' is injective

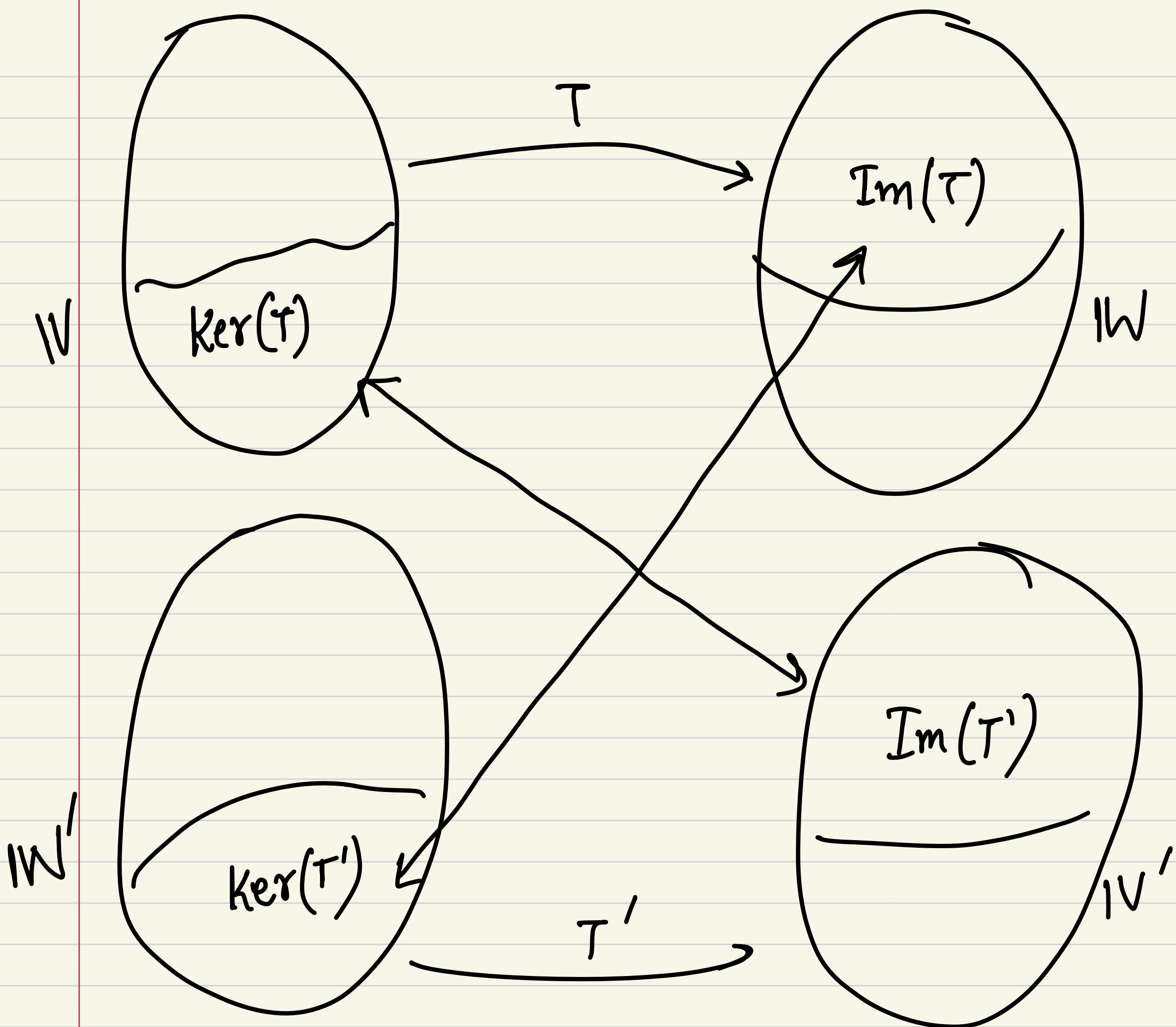
$\Rightarrow \text{Ker}(T') = \{0_{\mathbb{W}'}\}$.

$\Rightarrow \left(\text{Im}(T)\right)^0 = \{0_{\mathbb{W}'}\}$

$\Rightarrow \text{Im}(T) = \mathbb{W}$

Therefore, T is surjective.

Claim: T is injective $\longleftrightarrow T'$ is surjective.



assume T is injective.

$\Rightarrow \operatorname{Ker}(T) = \{0_V\}$.

$$\underbrace{\cancel{\dim(\operatorname{Ker}(T))}}_{0} + \dim((\operatorname{Ker}(T))^0) = \dim(V) = \dim(V')$$

$$\Rightarrow \dim((\operatorname{Ker}(T))^0) = \dim(V) = \dim(V')$$

2 subspace has same dim so they are same.

$\Rightarrow (\operatorname{Ker}(T))^0 = V'$

$\Rightarrow \operatorname{Im}(T') = V' \Rightarrow T'$ is surjective.

Assume T' is surjective.

$\Rightarrow Im(T') = V'$

$\Rightarrow (Ker(T))^0 = V'$

$$\underbrace{\dim(Ker(T))}_{=0} + \underbrace{\dim((Ker(T))^0)}_{=\dim(V')} = \dim(V') = \dim(V)$$

Therefore, $\dim(Ker(T)) = 0$

$\Rightarrow Ker(T) = \{0_V\}$

$\Rightarrow T$ is injection.

Products of Vector spaces:

Suppose $V_1, V_2, \ldots, V_m$ are vector spaces over $\mathbb{F}$.
The product of vector spaces are defined by –

$$P = V_1 \times V_2 \times \cdots \times V_m := \{(v_1, v_2, \ldots, v_m) \mid \forall v_i \in V_i\}$$

The addition operation is defined as:

$$+_P : P \times P \longrightarrow P.$$

$$(u_1, u_2, \ldots, u_m) +_P (v_1, v_2, \ldots, v_m) := (u_1 \underset{+_{V_1}}{+} v_1, \ldots, u_m \underset{+_{V_m}}{+} v_m)$$

Scalar multiplication operation is defined as:

$$\cdot_{\mathbb{P}} : \mathbb{F} \times \mathbb{P} \longrightarrow \mathbb{P}$$

$$\underbrace{c}_{\text{in } \mathbb{P}} \cdot (v_1, v_2, \ldots, v_m) := (\underbrace{cv_1}_{\mathbb{V}_1}, \underbrace{cv_2}_{\mathbb{V}_2}, \ldots, \underbrace{cv_m}_{\mathbb{V}_m})$$

products of vector spaces is a vector space itself with the defined addition & scalar multiplication rule above

$(\mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m, \mathbb{F}, +_p, \cdot_p)$ is a vector space.

Additive identity for this vector space 0_p will be -

$$(v_1, v_2, \ldots, v_m) +_p (u_1, u_2, \ldots, u_m) = (v_1, v_2, \ldots, v_m)$$

$$\Rightarrow (v_1 + u_1, v_2 + u_2, \ldots, v_m + u_m) = (v_1, v_2, \ldots, v_m)$$

$$\Rightarrow u_1 = 0_{V_1}, u_2 = 0_{V_2}, \ldots, u = 0_{V_m}.$$

So the additive identity, $0_p = (0_{V_1}, 0_{V_2}, \ldots, 0_{V_m})$

the additive inverse $= (-v_1, -v_2, \ldots, -v_m)$

<u>Ex1:</u> Is $\mathbb{R}^2 \times \mathbb{R}^3$ equal to $\mathbb{R}^5$?

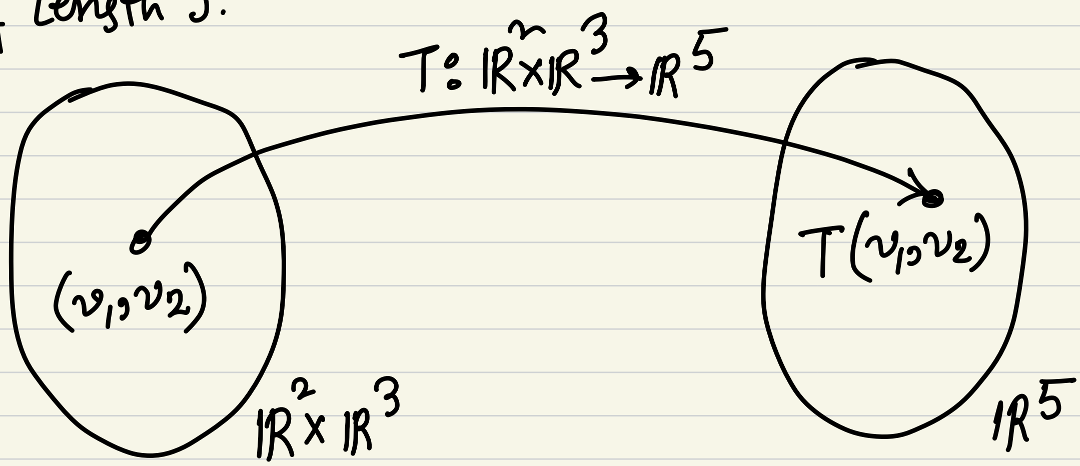
$$\mathbb{R}^2 = \{ (x_1, x_2) \mid \forall x_i \in \mathbb{R} \}$$

$$\mathbb{R}^3 = \{ (x_3, x_4, x_5) \mid \forall x_i \in \mathbb{R} \}$$

$$\mathbb{R}^2 \times \mathbb{R}^3 = \{ ((x_1, x_2), (x_3, x_4, x_5)) \mid (x_1, x_2) \in \mathbb{R}^2, (x_3, x_4, x_5) \in \mathbb{R}^3 \}$$

$$\mathbb{R}^5 = \{ (x_1, x_2, x_3, x_4, x_5) \mid \forall x_i \in \mathbb{R} \}$$

Definitely these 2 objects are not same. Elements of $\mathbb{R}^2 \times \mathbb{R}^3$ is a list of length 2. Elements of $\mathbb{R}^5$ is a list of length 5.



The linear transformation, T that takes a vector (v_1, v_2) from $\mathbb{R}^2 \times \mathbb{R}^3$ and sends to $T(v_1, v_2)$ in $\mathbb{R}^5$ is clearly one to one and onto so it is isomorphism between $\mathbb{R}^2 \times \mathbb{R}^3$ and $\mathbb{R}^5$. Therefore these 2 vector spaces are clearly isomorphic but they are not same.

Dimension of product space:

$$\dim(V_1 \times V_2 \times \cdots \times V_m) = \dim(V_1) + \dim(V_2) + \cdots + \dim(V_m)$$

<u>proof</u>: Choose basis for V_j.

Basis of $\mathbb{V}_1$: $v_1^1, v_2^1, v_3^1, \ldots, v_{n_1}^1$

Basis of $\mathbb{V}_2$: $v_1^2, v_2^2, v_3^2, \ldots, v_{n_2}^2$

$\vdots$

Basis of $\mathbb{V}_m$: $v_1^m, v_2^m, v_3^m, \ldots, v_{n_m}^m$

Consider the set :

$$S = \Big\{ (v_1^1, 0_{V_2}, \ldots, 0_{V_m}), (v_2^1, 0_{V_2}, \ldots, 0_{V_m}), \ldots,$$
$$(v_{n_1}^1, 0_{V_2}, \ldots, 0_{V_m}), (0_{V_1}, v_1^2, \ldots, 0_{V_m}),$$
$$(0_{V_1}, v_2^2, \ldots, 0_{V_m}), \ldots, (0_{V_1}, v_{n_2}^2, \ldots, 0_{V_m}),$$
$$\vdots$$
$$(0_{V_1}, 0_{V_2}, \ldots, v_1^m), (0_{V_1}, 0_{V_2}, \ldots, v_2^m), \ldots,$$
$$(0_{V_1}, 0_{V_2}, \ldots, v_{n_m}^m) \Big\}$$

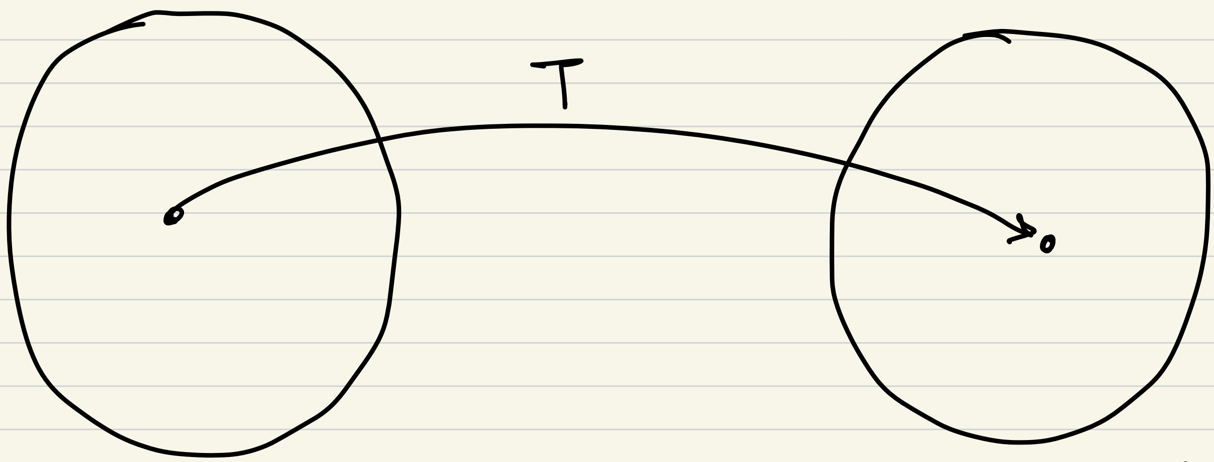
Claim: S is linearly independent set and S spans $\mathbb{P}$.
Hence S is basis for $\mathbb{P}$.

Therefore, $\dim(\mathbb{P}) = \sum_{i=1}^{m} \dim(\mathbb{V}_i)$

$U_1, U_2, \ldots, U_m$ are subspaces of $\mathbb{V}$.

$$\dim(U_1 \oplus U_2 \oplus \cdots \oplus U_m) = \dim(U_1) + \cdots + \dim(U_m)$$

Consider a linear map:



$V_1 \times V_2 \times \cdots \times V_m$ $\qquad$ $V_1 + V_2 + \cdots + V_m$

$V_1, V_2, \ldots, V_m$ are all subspaces of $\mathbb{V}$.

T is injective iff $V_1 + V_2 + \cdots + V_m = V_1 \oplus V_2 \oplus \cdots \oplus V_m$

otherwise $\ker(T) \neq \{0_{\mathbb{P}}\}$.

Clearly the map T is surjective therefore,

$$\dim(V_1 \oplus V_2 \oplus \cdots \oplus V_m) = \dim(V_1 \times V_2 \times \cdots \times V_m)$$

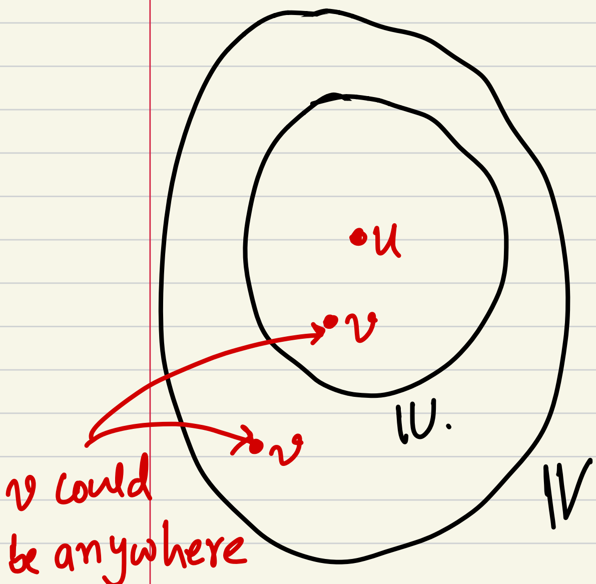
and the 2 spaces are isomorphic to each other.

Quotient of Vector spaces:

We need to define the sum of vector and vector space.

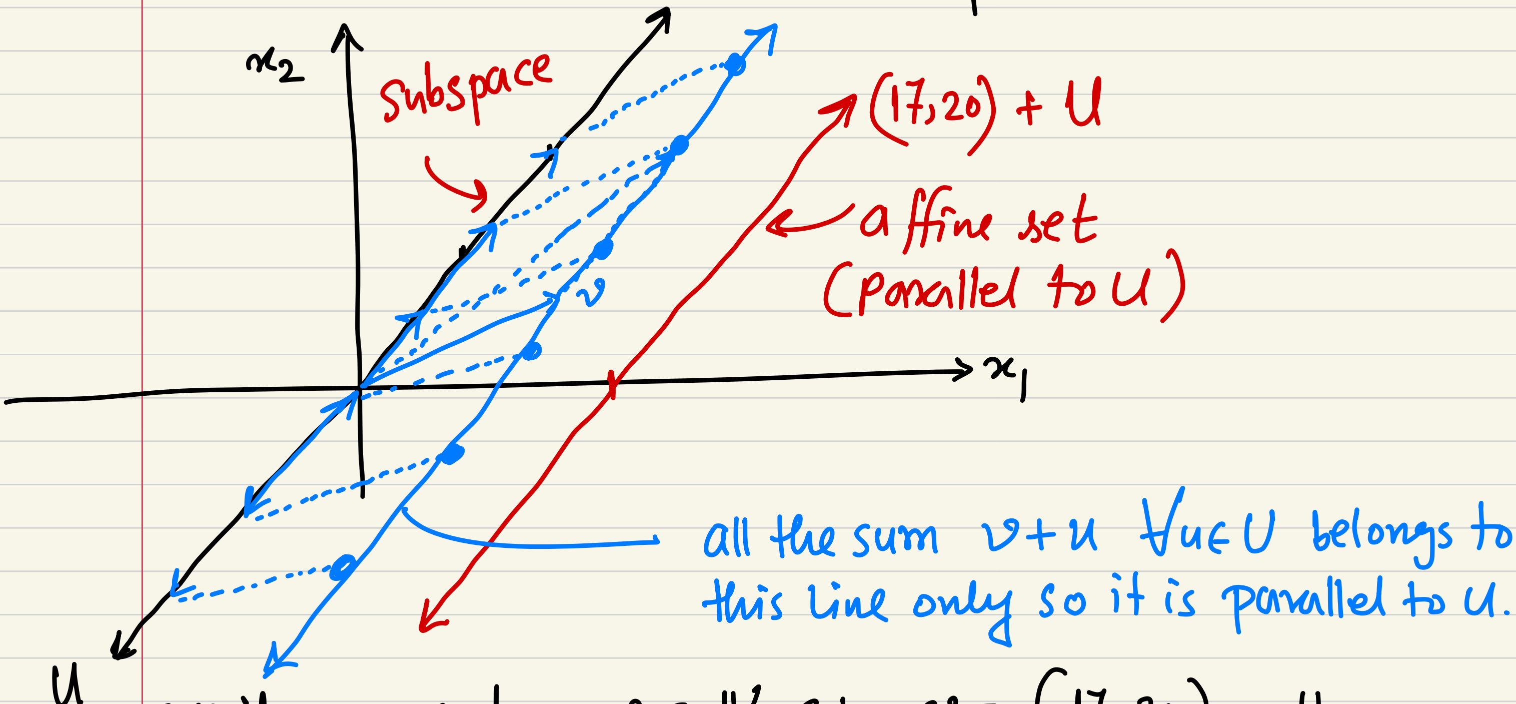
Suppose $\mathbb{V}$ is a vector space. $\mathbb{U}$ is a subspace of $\mathbb{V}$.

Consider any vector $v \in \mathbb{V}$ and we will define the sum with the subspace $\mathbb{U}$.



$$v + U := \{ v +_v u \mid \forall u \in U \}$$

Suppose $U = \{ (x, 2x) \in \mathbb{R}^2 \mid \forall x \in \mathbb{R} \}$



consider a vector $v \in \mathbb{V}$ s.t. $v = (17, 20)$. Hence,

$$(17,20) +_v U := \{ (17,20) +_v u \mid \forall u \in U \}$$

$$= \{ (17 + x, 20 + 2x) \mid \forall x \in \mathbb{R} \}$$

Affine Subset :

An affine subset of $\mathbb{V}$ is a subset of $\mathbb{V}$ of the form

$v + U \subseteq \mathbb{V}$ and $v + U := \{ v + u \mid \forall u \in \mathbb{U} \}$

Here v & U are free to choose. Depending on the choice of v & U, we will get different affine sets.
For a $v \in \mathbb{V}$ and $U \subseteq \mathbb{V}$ (subspace), the affine set $v + U$ is called parallel to U.

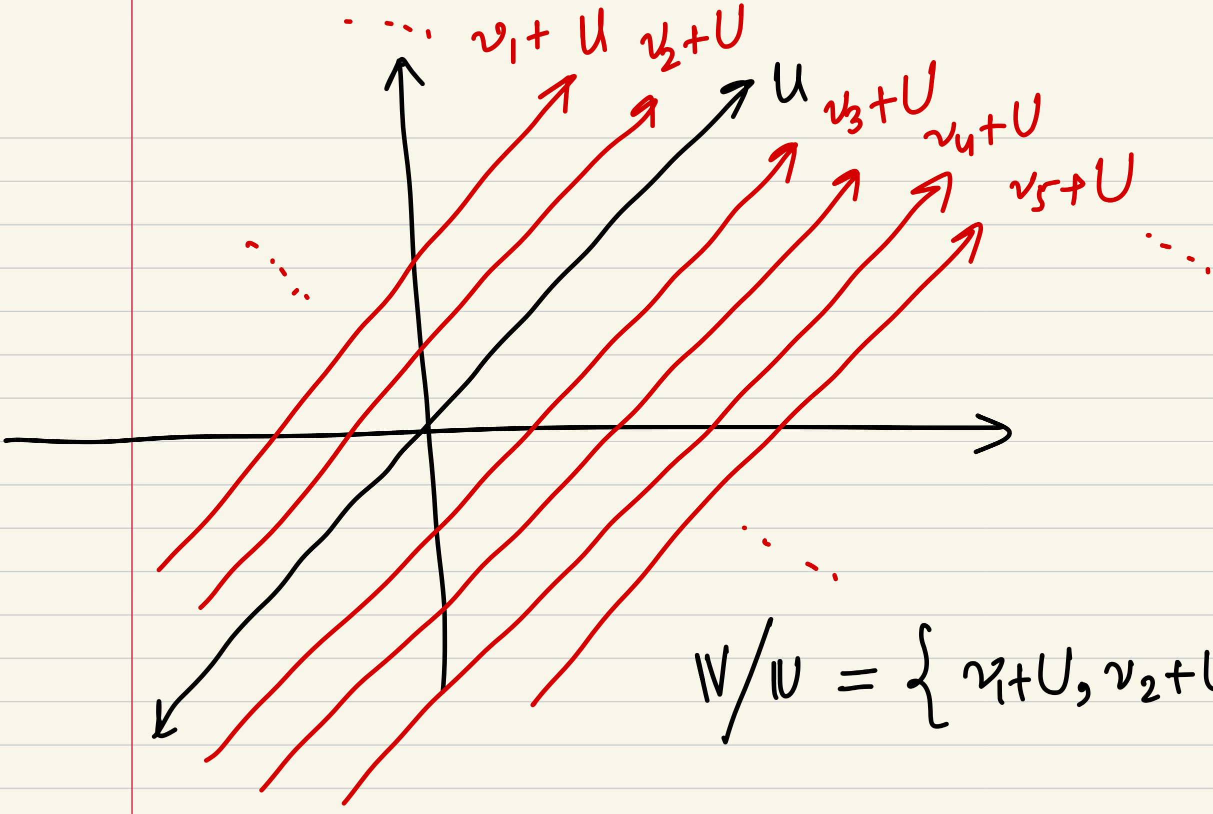
Quotient Space :

Suppose U is a subspace of $\mathbb{V}$. Then the quotient space $\mathbb{V}/\mathbb{U}$ is the set of all affine subsets of $\mathbb{V}$. parallel to $\mathbb{U}$.

$$\mathbb{V}/\mathbb{U} := \{ v + U \mid \forall v \in \mathbb{V} \}$$

Remember $\mathbb{V}$ is set of vectors.
$\mathbb{V}/\mathbb{U}$ is set of sets of vectors because $v + U$ itself is a set of vectors.

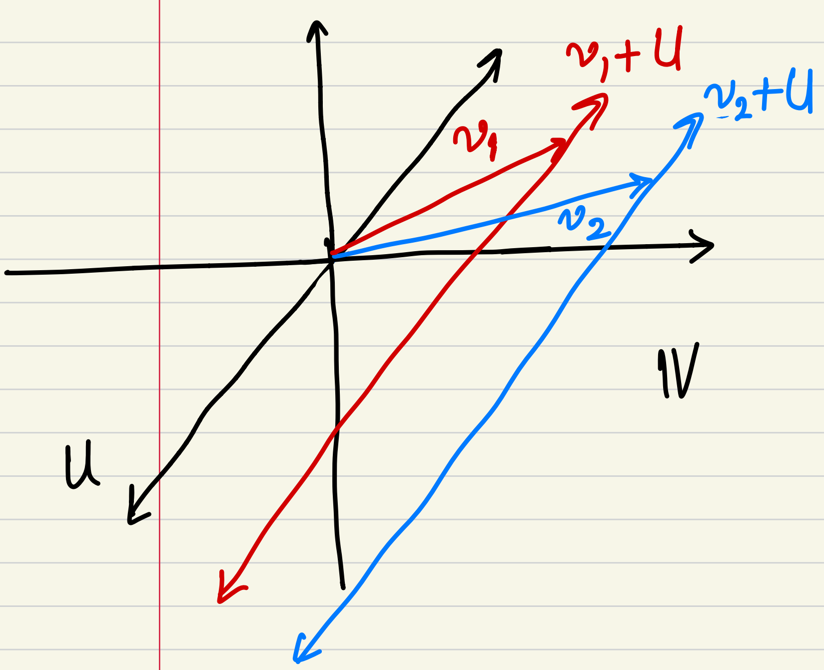
Quotient space of $\mathbb{V}$, quotiented by $\mathbb{U}$ contains all the parallel of $\mathbb{U}$ or all the affine sets possible for $\mathbb{U}$.



$$\mathbb{V}/\mathbb{U} = \{ v_1+U, v_2+U, \dots \}$$

We will inverstigate whether $\mathbb{V}/\mathbb{U}$ is a vector space or not over field $\mathbb{F}$?

Claim: Two affine set parallel to U are either equal or completely disjoint.



$U \subseteq \mathbb{V}$ (subspace of $\mathbb{V}$)

prove that -

$v_1+\mathbb{U} = v_2+\mathbb{U}$ iff

$v_1 - v_2 \in \mathbb{U}$

Let $\mathbb{V}$ be a vector space and $\mathbb{U}$ be a subspace of $\mathbb{V}$.
For $v_1, v_2 \in \mathbb{V}$, $\quad v_1 + \mathbb{U} = v_2 + \mathbb{U}$ iff $v_1 - v_2 \in \mathbb{U}$

Proof: $\quad v_1 - v_2 \in \mathbb{U} \longrightarrow v_1 + \mathbb{U} = v_2 + \mathbb{U}$

assume that $\quad u = v_1 - v_2 \in \mathbb{U}$.

$$\Rightarrow v_1 = v_2 + u$$

consider an element in $v_1 + \mathbb{U}$ say $v_1 + u_1$, $u_1 \in \mathbb{U}$.

$$\Rightarrow v_1 + u_1 = (v_2 + u) + u_1 = v_2 + (u_1 + u)$$

$$\Rightarrow \underbrace{v_1 + u_1}_{\in v_1 + \mathbb{U}} = \underbrace{v_2 + \hat{u}_1}_{\in v_2 + \mathbb{U}} \quad \text{say } u_1 + u = \hat{u}_1 \in \mathbb{U}.$$

$$\Rightarrow v_1 + \mathbb{U} \subseteq v_2 + \mathbb{U}$$

consider an element in $v_2 + \mathbb{U}$ say $v_2 + u_2$, $u_2 \in \mathbb{U}$.

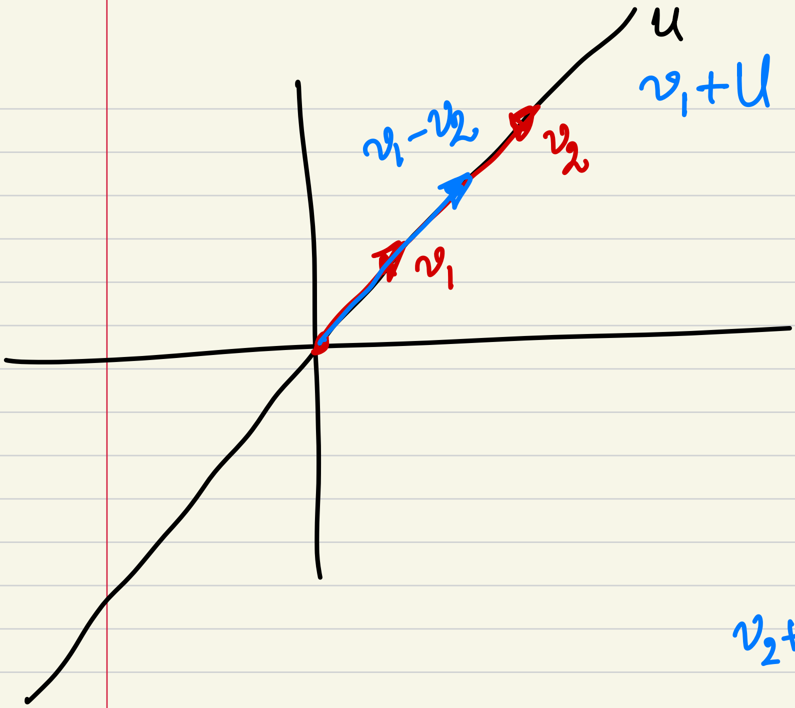
$$\Rightarrow v_2 + u_2 = v_1 - u + u_2 = v_1 + (u_2 - u)$$

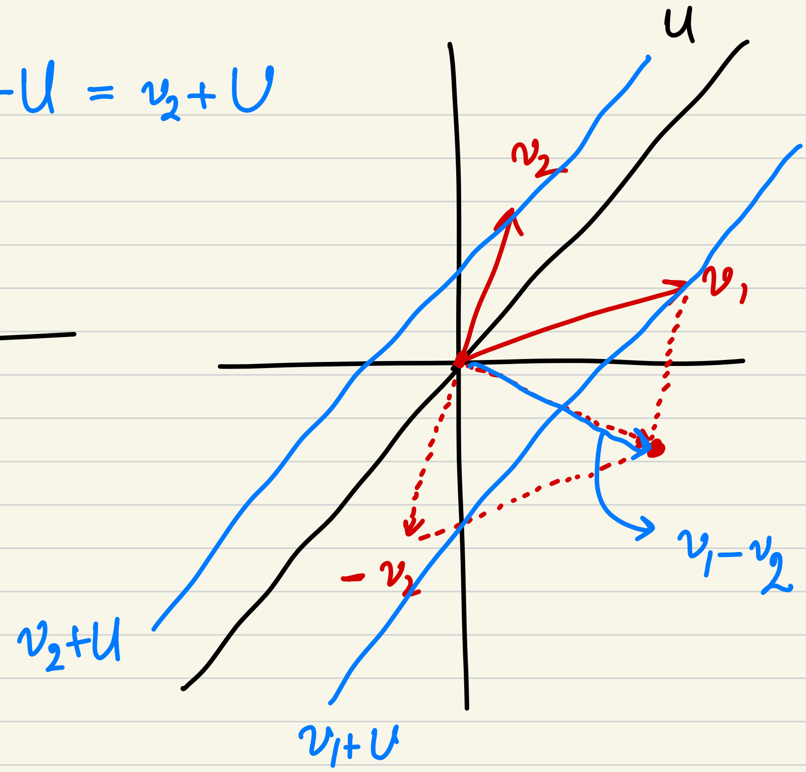
$$\Rightarrow \underbrace{v_2 + u_2}_{\in v_2 + \mathbb{U}} = \underbrace{v_1 + \hat{u}_2}_{\in v_1 + \mathbb{U}.} \quad \text{say } \hat{u}_2 = u_2 - u$$

$$\Rightarrow v_2 + \mathbb{U} \subseteq v_1 + \mathbb{U}.$$

Hence $\quad v_1 + \mathbb{U} = v_2 + \mathbb{U}$.

This essentially says that -





Notice $v_1 - v_2 \notin U \rightarrow v_1 + U \neq v_2 + U$ or

$$(v_1 + U) \cap (v_2 + U) = \emptyset$$

prove that: $v_1 + U = v_2 + U \longrightarrow v_1 - v_2 \in U$.

assume that $v_1 + U = v_2 + U$,

Take 2 arbitrary elements-

$v_1 + u_1 \in v_1 + U$ where $u_1 \in U$.

$v_2 + u_2 \in v_2 + U$ where $u_2 \in U$.

such that, $v_1 + u_1 = v_2 + u_2$

$$\Rightarrow v_1 - v_2 = u_2 - u_1 = \hat{u} \in U.$$

Hence $v_1 - v_2 \in U$. (Hence proved)

Therefore 2 affine sets are either same or disjoint.

$$(v_1 + \mathbb{U}) \cap (v_2 + \mathbb{U}) \neq \varphi \longleftrightarrow v_1 + \mathbb{U} = v_2 + \mathbb{U}.$$

vector addition on $\mathbb{V}/\mathbb{U}$. Suppose, $Q = \mathbb{V}/\mathbb{U}$.

$$+_Q : \mathbb{V}/\mathbb{U} \times \mathbb{V}/\mathbb{U} \longrightarrow \mathbb{V}/\mathbb{U}.$$

$$(v_1 + \mathbb{U}) +_Q (v_2 + \mathbb{U}) := (v_1 +_V v_2) + \mathbb{U}$$

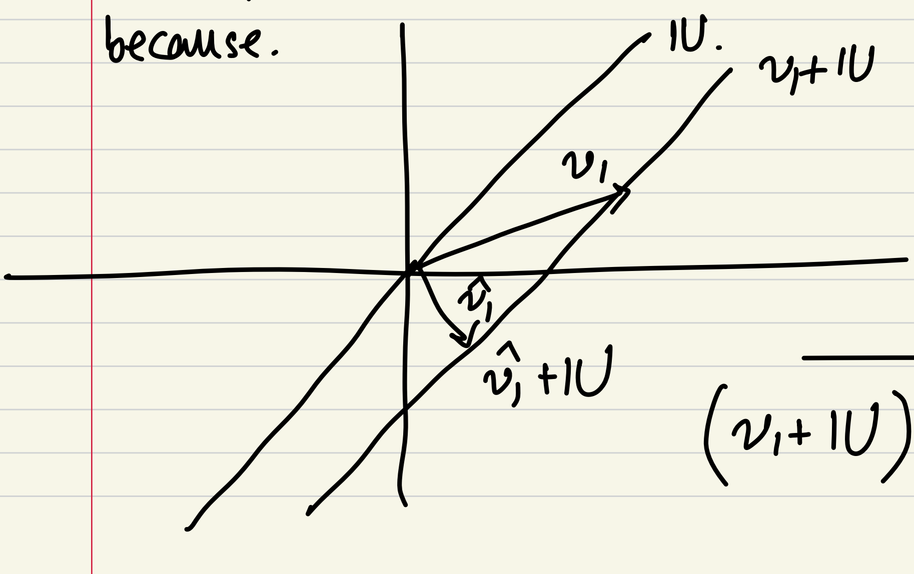
Scalar multiplication on $\mathbb{V}/\mathbb{U}$. Suppose, $Q = \mathbb{V}/\mathbb{U}$.

$$\cdot_Q : \mathbb{F} \times \mathbb{V}/\mathbb{U} \longrightarrow \mathbb{V}/\mathbb{U}.$$

$$c \underset{Q}{\cdot} (v + \mathbb{U}) := (c \underset{\mathbb{V}}{\cdot} v) + \mathbb{U}.$$

Claim: Suppose $\mathbb{U}$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$. Then $(\mathbb{V}/\mathbb{U}, \mathbb{F}, +_Q, \cdot_Q)$ is a vector space.

The representation $v_1 + \mathbb{U}$ and $v_2 + \mathbb{U}$ is not unique. because.



$$v_1 + \mathbb{U} = \hat{v}_1 + \mathbb{U}$$

$$v_2 + \mathbb{U} = \hat{v}_2 + \mathbb{U}$$

$$(v_1 + \mathbb{U}) +_Q (v_2 + \mathbb{U}) = (v_1 + v_2) + \mathbb{U}$$
$$= (\hat{v}_1 + \hat{v}_2) + \mathbb{U}$$

We need to prove that even we change the representation,

$(v_1+v_2) + \mathbb{U} = (\hat{v}_1 + \hat{v}_2) + \mathbb{U}$.

Suppose, $v_1 + \mathbb{U} = \hat{v}_1 + \mathbb{U}$.

$v_2 + \mathbb{U} = \hat{v}_2 + \mathbb{U}$.

$v_1 - \hat{v}_1 \in \mathbb{U}$ iff $v_1 + \mathbb{U} = \hat{v}_1 + \mathbb{U}$.

$v_2 - \hat{v}_2 \in \mathbb{U}$ iff $v_2 + \mathbb{U} = \hat{v}_2 + \mathbb{U}$.

Since $\mathbb{U}$ is a subspace so $(v_1 - \hat{v}_1) + (v_2 - \hat{v}_2) \in \mathbb{U}$.

$\Rightarrow (v_1 + v_2) - (\hat{v}_1 + \hat{v}_2) \in \mathbb{U}$.

iff $(v_1+v_2) + \mathbb{U} = (\hat{v}_1 + \hat{v}_2) + \mathbb{U}$ (hence proved)

Similarly, $c \cdot (v + \mathbb{U}) = cv + \mathbb{U}$

$= c\hat{v} + \mathbb{U}$

We need to prove that $cv + \mathbb{U} = c\hat{v} + \mathbb{U}$.

Suppose, $v + \mathbb{U} = \hat{v} + \mathbb{U}$.

$v - \hat{v} \in \mathbb{U}$ iff $v + \mathbb{U} = \hat{v} + \mathbb{U}$

$\Rightarrow \underbrace{c(v - \hat{v})}_{\text{Closed in } \mathbb{U} \text{ under scalar multiplication}} \in \mathbb{U}$ iff $cv + \mathbb{U} = c\hat{v} + \mathbb{U}$. (proved)

Therefore 2 definitions makes sense.

Additive identity of $Q = \mathbb{V}/\mathbb{U}$. denoted by 0_Q.

$$\left(v + \mathbb{U}\right) +_Q \left(x + \mathbb{U}\right) = \left(v + \mathbb{U}\right)$$

$$\Rightarrow (v+x) + \mathbb{U} = \left(v + \mathbb{U}\right)$$

Definitely, x must be $0_{\mathbb{V}}$.

So the $0_Q = 0 + \mathbb{U} = \mathbb{U}$ itself. (affine set)

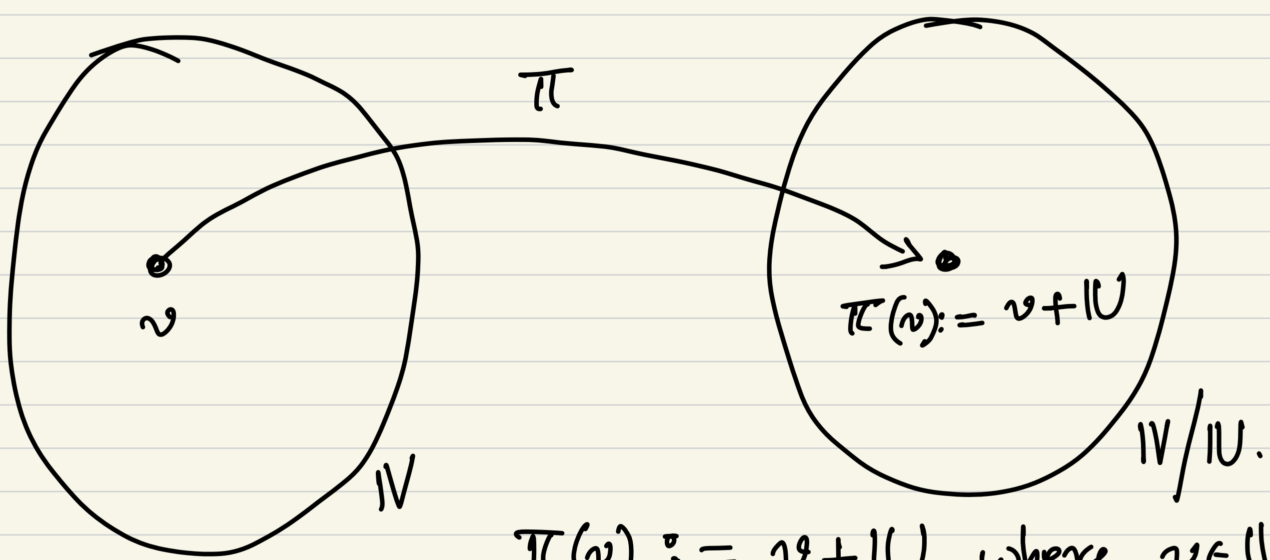
Therefore every quotient subspace must contain $\mathbb{U}$.

Additive inverse of Q will be, $(-v) + \mathbb{U}$. (affine set)

Quotient Map:

Suppose $\mathbb{U}$ is the subspace of $\mathbb{V}$. Quotient map π is defined as:

$$\pi : \mathbb{V} \rightarrow \mathbb{V}/\mathbb{U}.$$



$$\pi(v) := v + \mathbb{U} \text{ where } v \in \mathbb{V}.$$

Prove that π is linear map.

$\pi(c \cdot v_1 + v_2) = c \cdot \pi(v_1) + \pi(v_2)$

$\Rightarrow (\underline{cv_1} + \underline{v_2}) + \mathbb{U}$

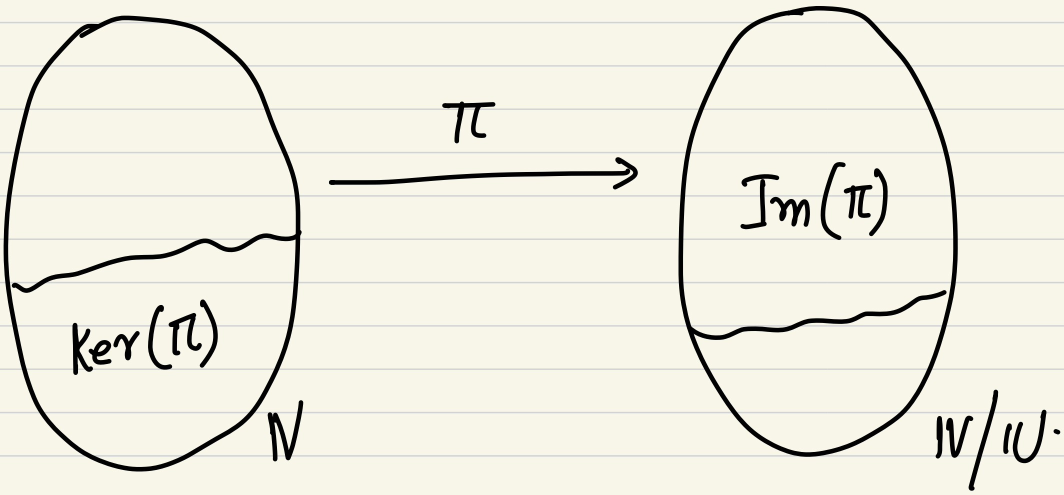
$\Rightarrow (\underline{cv_1 + \mathbb{U}}) + (\underline{v_2 + \mathbb{U}})$ [By def of vector addition on $\mathbb{V}/\mathbb{U}$]

$\Rightarrow c \cdot (v_1 + \mathbb{U}) + (v_2 + \mathbb{U})$ [By def of scalar mult on $\mathbb{V}/\mathbb{U}$]

$\Rightarrow c \cdot \pi(v_1) + \pi(v_2)$.

(Hence proved)

Dimension of the quotient space:



The way map π is defined $\pi(v) = v + \mathbb{U}$ so it is guarented that for all $v + \mathbb{U}$, there will be a pre image in $\mathbb{V}$ so π is surjective hence, $Im(\pi) = \mathbb{V}/\mathbb{U}$.

$Ker(\pi)$ means $\pi(v) = 0_{\mathbb{V}/\mathbb{U}} = \mathbb{U}$ for those v sitting in $Ker(\pi)$. Definitely if $v \in \mathbb{U}$ then v will be mapped to $v + \mathbb{U}$ $(v \in \mathbb{U})$.

$$v + \mathbb{U} := \{ v + u \mid v, u \in \mathbb{U} \} = \{ \hat{u} \mid u \in \mathbb{U} \} = \mathbb{U}.$$

Hence $\ker(\pi) = \mathbb{U}$.

Apply rank-nullity theorem –

$$\dim(\ker(\pi)) + \dim(\operatorname{Im}(\pi)) = \dim(\mathbb{V})$$

$$\Rightarrow \dim(\mathbb{U}) + \dim(\mathbb{V}/\mathbb{U}) = \dim(\mathbb{V})$$

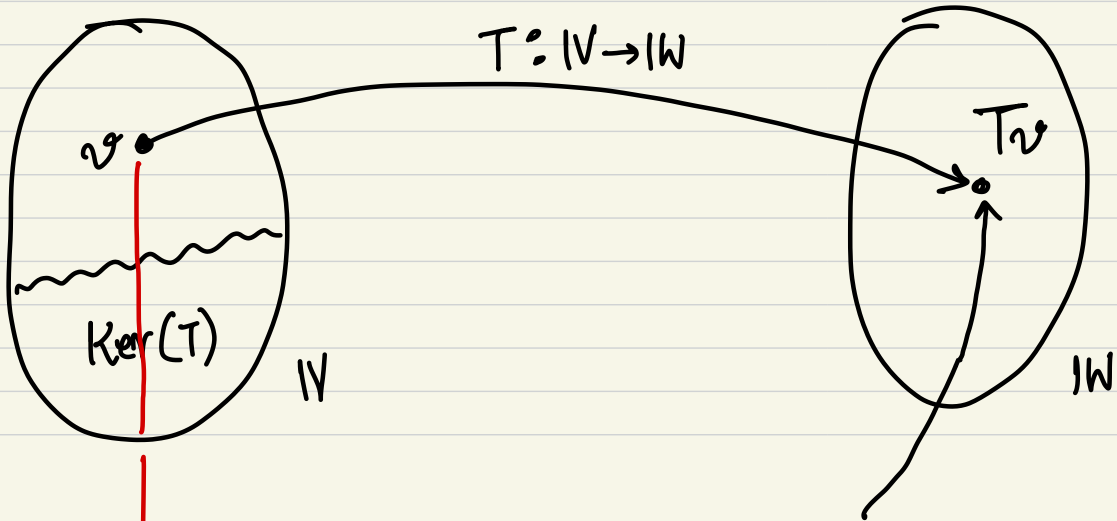
$$\Rightarrow \boxed{\dim(\mathbb{V}/\mathbb{U}) = \dim(\mathbb{V}) - \dim(\mathbb{U})}$$

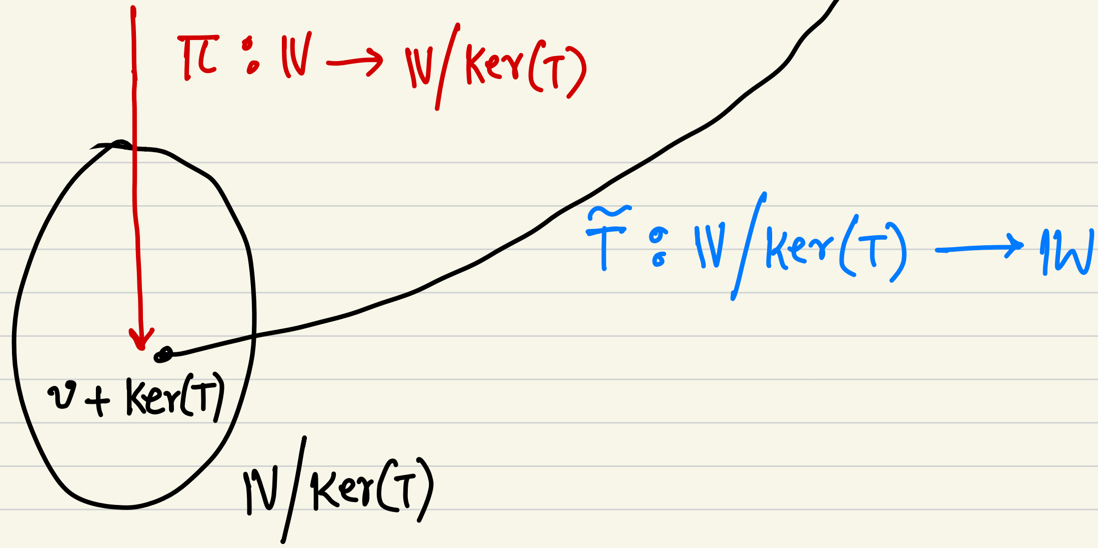
Induced Map:

Each linear map T on $\mathbb{V}$ induces a linear map $\tilde{T}$ on $\mathbb{V}/\ker(T)$.

Suppose, $T \in L(\mathbb{V}, \mathbb{W})$

$\tilde{T} \in L(\mathbb{V}/\ker(T), \mathbb{W})$





$$\tilde{T}\big(v + \mathrm{Ker}(T)\big) := T(v)$$

Suppose, $u, v \in \mathbb{V}$ such that $u + \mathrm{Ker}(T) = v + \mathrm{Ker}(T)$

Therefore, $u - v \in \mathrm{Ker}(T)$

$\Rightarrow T(u-v) = 0_{\mathbb{W}}$.

$\Rightarrow Tu = Tv$

This indeed makes sense as $\tilde{T}\big(u + \mathrm{Ker}(T)\big) := Tu$

$\tilde{T}\big(v + \mathrm{Ker}(T)\big) := Tv$

If $u + \mathrm{Ker}(T) = v + \mathrm{Ker}(T)$ then $Tu = Tv$. (so it is a valid map).

Is it a linear map?

$\Rightarrow \tilde{T}\Big(\underline{c} \cdot \underline{(v + \mathrm{Ker}(T))} + (u + \mathrm{Ker}(T))\Big) =$

$c \cdot \tilde{T}\big(v + \mathrm{Ker}(T)\big) + \tilde{T}\big(u + \mathrm{Ker}(T)\big)$

$\Rightarrow \tilde{T}\big(\underline{c \cdot v + \mathrm{Ker}(T)} +_Q \underline{u + \mathrm{Ker}(T)}\big)$ [$\because$ scalar mult on $\mathbb{V}/\mathrm{Ker}(T)$]

$\Rightarrow \tilde{T}\left(cv + u + \mathrm{Ker}(T)\right)$ [$\because$ Addition on $\mathbb{V}/\mathrm{Ker}(T)$]

$\Rightarrow T(cv+u)$

$\Rightarrow c\cdot T(v) + T(u)$ [$\because$ T is linear]

$\Rightarrow c.\tilde{T}\left(v + \mathrm{Ker}(T)\right) + \tilde{T}\left(u + \mathrm{Ker}(T)\right)$ [$\because$ By def $\tilde{T}$]

Hence $\tilde{T}$ is a linear map.

Claim: $\tilde{T}$ is injective.

We have to prove that $\mathrm{Ker}(\tilde{T}) = \{0_{\mathbb{V}/\mathrm{Ker}(T)}\}$

But $0_{\mathbb{V}/\mathrm{Ker}(T)} = \mathrm{Ker}(T)$.

So we have to show that $\mathrm{Ker}(\tilde{T}) = \mathrm{Ker}(T)$

Suppose $v \in \mathbb{V}$ and $\tilde{T}\left(v + \mathrm{Ker}(T)\right) = 0_{\mathbb{W}}$.

By def: $\tilde{T}\left(v + \mathrm{Ker}(T)\right) = Tv$

So $Tv = 0_{\mathbb{W}}$

$\Rightarrow v \in \mathrm{Ker}(T)$.

We know that $v \in \mathrm{Ker}(T)$, $0_{\mathbb{V}} \in \mathrm{Ker}(T)$ so

$v - 0_{\mathbb{V}} \in \mathrm{Ker}(T)$

$\Rightarrow v + \mathrm{Ker}(T) = 0_{\mathbb{V}} + \mathrm{Ker}(T) = \mathrm{Ker}(T) = 0_{\mathbb{V}/\mathrm{Ker}(T)}$

Therefore, $\tilde{T}\left(0_{\mathbb{V}/\mathrm{Ker}(T)}\right) = 0_{\mathbb{W}}$.

$\Rightarrow \mathrm{Ker}(\tilde{T}) = 0_{\mathbb{V}/\mathrm{Ker}(T)} = \mathrm{Ker}(T)$ (Hence proved)

Hence $\tilde{T}$ is an injective map.

claim: $\tilde{T}$ is an surjective map. b/w $V/\text{Ker}(T)$ and $\text{Im}(T)$.

clearly, $\tilde{T}(v + \text{Ker}(T)) := Tv$

Now $Tv \in \text{Im}(T)$ so all the objects inside Tv must have a preimage in $V/\text{Ker}(T)$ so it is a surjection b/w $V/\text{Ker}(T)$ and $\text{Im}(T)$.

$\Rightarrow \text{Im}(\tilde{T}) = \text{Im}(T)$ clear from the definition (the way $\tilde{T}$ is defined).

We have already proved $\tilde{T}$ is injection and $\tilde{T}$ is surjection b/w $V/\text{Ker}(T)$ and $\text{Im}(T)$ hence $\tilde{T}$ is bijection b/w $V/\text{Ker}(T)$ and $\text{Im}(T)$.

$V/\text{Ker}(T)$ is isomorphic to $\text{Im}(T)$.

$$\Rightarrow \dim(V/\text{Ker}(T)) = \dim(\text{Im}(T))$$

$$\Rightarrow \dim(V) - \dim(\text{Ker}(T)) = \dim(\text{Im}(T))$$

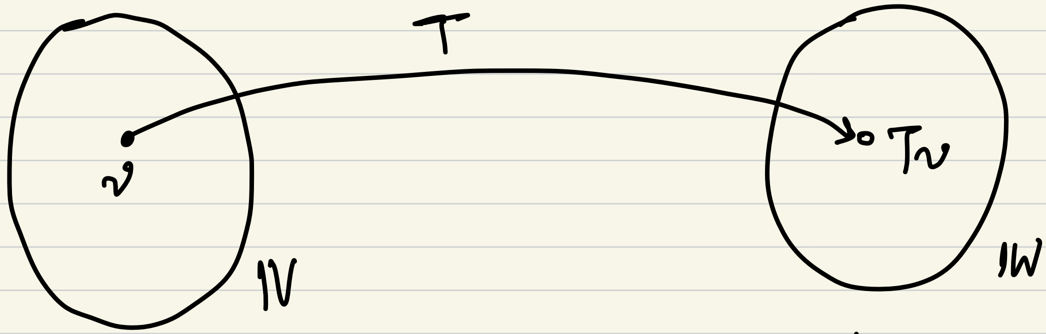
$$\Rightarrow \dim(V) = \dim(\text{Im}(T)) + \dim(\text{Ker}(T))$$

(rank-nullity theorem).

1 Suppose T is a function from V to W. The ***graph*** of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.



$$\text{graph}(T) = \{ (v, Tv) \in V \times W \mid v \in V \}$$

Prove : T is linear $\longrightarrow$ graph(T) is subspace of $V \times W$.

assume T is linear.

$$T(c \cdot v + u) = c \cdot T(v) + T(u) \quad \forall u, v \in V, \; c \in F$$

consider $P_1 = (v, Tv) \in \text{graph}(T)$

$P_2 = (u, Tu) \in \text{graph}(T)$

Construct $P = \alpha \cdot P_1 + P_2$. $\exists \alpha \in F$

$$P = \alpha \cdot (v, Tv) + (u, Tu)$$

$$= (\alpha \cdot v, \alpha Tv) + (u, Tu)$$ [$\because$ As per rule of "$\cdot$" on $V \times W$]

$= (\alpha v + u, \ \alpha Tv + Tu)$ [$\because$ As per rule of "+" on $\mathbb{V} \times \mathbb{W}$]

$= (\alpha v + u, \ T(\alpha v + u))$ [$\because$ T is linear]

$= (\hat{p}, T\hat{p}) \in$ graph(T) [$\because \hat{p} = \alpha v + u \in \mathbb{V}$]

Therefore, graph(T) is subspace of $\mathbb{V} \times \mathbb{W}$.

prove: graph(T) is subspace of $\mathbb{V} \times \mathbb{W} \longrightarrow$ T is linear.

assume: graph(T) is subspace of $\mathbb{V} \times \mathbb{W}$.

We need to prove that $T(\alpha u + v) = \alpha T(u) + T(v)$.

Start with $T(\alpha u + v)$

Because graph(T) is subspace of $\mathbb{V} \times \mathbb{W}$ so,

$(u, Tu) \in$ graph(T)

$(v, Tv) \in$ graph(T)

$\alpha \cdot (u, Tu) + (v, Tv) \in$ graph(T)

$\Rightarrow (\alpha u, \alpha Tu) + (v, Tv)$

$\Rightarrow (\alpha u + v, \ \alpha Tu + Tv) \in$ graph(T).

It is possible iff $\alpha u + v \in \mathbb{V}$ and

$\alpha Tu + Tv \in \mathbb{W}$.

But we also know that, $\left(\alpha u+v,\ T(\alpha u+v)\right) \in \text{graph}(T)$
So matching the 2 form we get -

$$\alpha \cdot Tu + Tv = T(\alpha u + v) \quad \text{so } T \text{ is } \underline{\text{linear}}.$$

7 Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

$v \in \mathbb{V}$. | $\mathbb{U}, \mathbb{W}$ are subspaces of $\mathbb{V}$.
$x \in \mathbb{V}$. | $v + \mathbb{U} = x + \mathbb{W}$.

$$v + \mathbb{U} := \{ v + u \mid \forall u \in \mathbb{U} \}$$
$$x + \mathbb{W} := \{ x + w \mid \forall w \in \mathbb{W} \}$$

Since they are equal so, $v + u = x + w \quad \forall u, w$.
$\Rightarrow u = (x - v) + w$

We know that, $v + 0_V = v \in v + \mathbb{U}$ therefore,
$v + 0_V = x + w_0 \quad \exists w_0 \in \mathbb{W}$.

Therefore, $v = x + w_0$
We also know that, $x + 0_V = x \in x + \mathbb{W}$ therefore,
$v + u_0 = x + 0_V \quad \exists u_0 \in \mathbb{U}$.

Therefore, $x = v + u_0$.

We already have, $u = (x-v) + w = -w_0 + w \in \mathbb{W}$

Since $u \in \mathbb{U} \longrightarrow u \in \mathbb{W}$ so $\mathbb{U} \subseteq \mathbb{W}$.

We also have, $w = (v-x) + u = -u_0 + u \in \mathbb{U}$.

Since $w \in \mathbb{W} \longrightarrow w \in \mathbb{U}$ so $\mathbb{W} \subseteq \mathbb{U}$.

therefore, $\mathbb{W} = \mathbb{U}$. (Hence proved).

9 Suppose A_1 and A_2 are affine subsets of V. Prove that the intersection $A_1 \cap A_2$ is either an affine subset of V or the empty set.

$$A_1 = v + \mathbb{U}_1 = \{ v + u_1 \mid u_1 \in \mathbb{U}_1, \mathbb{U}_1 \subseteq \mathbb{V} \}$$

$$A_2 = v + \mathbb{U}_2 = \{ v + u_2 \mid u_2 \in \mathbb{U}_2, \mathbb{U}_2 \subseteq \mathbb{V} \}$$

$$A_1 \cap A_2 = \{ v + u \mid u \in \mathbb{U}_1 \cap \mathbb{U}_2, v \in \mathbb{V}, \mathbb{U} \cap \mathbb{W} \subseteq \mathbb{V} \}$$

The claim is that. $A_1 \cap A_2 = v + \mathbb{U}_1 \cap \mathbb{U}_2$.

proof: suppose, $x \in v + \mathbb{U}_1 \cap \mathbb{U}_2$

$\Rightarrow \exists u \in \mathbb{U}_1 \cap \mathbb{U}_2$ s.t. $v + u = x$

Since $u \in \mathbb{U}_1 \cap \mathbb{U}_2 \Rightarrow u \in \mathbb{U}_1$

$\Rightarrow x \in v + \mathbb{U}_1 \Rightarrow x \in A_1$

Similarly $x \in A_2$ so $x \in A_1 \cap A_2$

$\Rightarrow v + \mathbb{U}_1 \cap \mathbb{U}_2 \subseteq A_1 \cap A_2$.

Suppose $y \in A_1 \cap A_2$.

$\Rightarrow y \in A_1$ and $y \in A_2$.

If $y \in A_1$ then $\exists\, u_1 \in U_1$ s.t. $v + u_1 = y$

If $y \in A_2$ then $\exists\, u_2 \in U_2$ s.t $v + u_2 = y$

$\Rightarrow v + u_1 = v + u_2$

$\Rightarrow u_1 = u_2 = u$ (say)

$$\left.\begin{array}{l} \text{Since } u_1 \in U_1 \text{ so } u \in U_1 \\ \phantom{\text{Since }} u_2 \in U_2 \text{ so } u \in U_2 \end{array}\right\} u \in U_1 \cap U_2.$$

Therefore, $\exists\, u \in U_1 \cap U_2$ s.t $v + u = y$

$\Rightarrow y \in v + U_1 \cap U_2$.

$\Rightarrow A_1 \cap A_2 \subseteq v + U_1 \cap U_2$.

Hence $A_1 \cap A_2 = v + U_1 \cap U_2$

Clearly if $U_1 \cap U_2 = \phi$ and $A_1 = v + U_1$
$A_2 = u + U_2$ and $v = u = 0_V$ then $A_1 \cap A_2 = \emptyset$ as no 2 vectors will be common in A_1 and A_2. So $A_1 \cap A_2 = \phi$.

8 Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1 - \lambda) w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.

prove: A is affine subset of $V \longrightarrow \lambda v + (1-\lambda)w \in A$

assume: A is affine set.

$\Rightarrow \exists x \in V, \exists U \subseteq V$ s.t. $x + U = \{x + u \mid \forall u \in U\}$

We need to show that $\lambda v + (1-\lambda) w \in A$.

$\Rightarrow \lambda v + (1-\lambda) w = x + u$

Given that $v, w \in A$ that means, $v = x + u_1$
$w = x + u_2$

$\Rightarrow \lambda(x+u_1) + (1-\lambda)(x+u_2)$

$\Rightarrow \cancel{\lambda x} + \lambda u_1 + x + u_2 - \cancel{\lambda x} - \lambda u_2$

$\Rightarrow x + (u_2 + \lambda u_1 - \lambda u_2)$

We know that $u_1, u_2 \in U$ and if we consider U as subspace then $u_2 + \lambda u_1 - \lambda u_2 \in U$ that mean consider $u_2 + \lambda u_1 - \lambda u_2 = u$ so,

$\Rightarrow x + u$ (which is in the form of $x + U$)

Therefore, $\lambda v + (1-\lambda) w \in A$.

prove that: $\lambda v + (1-\lambda)w \in A \longrightarrow A$ is affine set.

assume that: $\lambda v + (1-\lambda) w \in A$. $\forall v, w \in A$.

We have to prove that A is affine set that means $A = a + \mathbb{U}$ for some $a \in \mathbb{V}$ and some set $\mathbb{U} \subseteq \mathbb{V}$.

We have seen that if we choose the set $\mathbb{U}$ as subspace then the above prove worked so here also we will take $\mathbb{U}$ as subspace. So we need to show $\mathbb{U}$ exists.

claim: $\mathbb{U} = -a + A$ is subspace of $\mathbb{V}$. with $a \in A$

If we can Show $\mathbb{U}$ is a subspace of $\mathbb{V}$ then $A = a + \mathbb{U}$. is an affine set.

Suppose, $x = -a + a_1 \in A, \mathbb{U}$

$y = -a + a_2 \in A, \mathbb{U}$

look at $x + y$.

$\Rightarrow -2a + a_1 + a_2$

$\Rightarrow \left(-a + \frac{1}{2}a_1 + \frac{1}{2}a_2\right) \times 2$

$\Rightarrow 2 \cdot \left(-a + \frac{1}{2}a_1 + \frac{1}{2}a_2\right)$

$\Rightarrow 2(\underbrace{-a}_{\in A} + \underbrace{a_0}_{\in A}) \Rightarrow 2(\underbrace{p}_{\in A}) \in A$ so it is closed under addition.

$\lambda \cdot v + (1-\lambda) w \in A$

Choose $v = a_1$, $w = a_2$,

$\lambda = \frac{1}{2}$

so $\frac{1}{2}a_1 + \frac{1}{2}a_2 \in A$.

look at $x = -a + a_0$ for some $a_0 \in A$

$\lambda a_0 + (1-\lambda) a \in A$

$\Rightarrow \underbrace{-a}_{\in A} + \lambda \underbrace{(a_0 - a)}_{\in A} \in A$

$\Rightarrow -a + \lambda \cdot x \in A$

$\hookrightarrow x \in A.$

So $\lambda x \in A$ hence it is closed under scalar mult.

Therefore, $\mathbb{U}$ is a subspace.

13 Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V.

$\mathbb{U}$ is subspace of $\mathbb{V}$.

Basis for $\mathbb{V}/\mathbb{U} = \{ v_1 + \mathbb{U}, v_2 + \mathbb{U}, \ldots, v_m + \mathbb{U} \}$

Basis for $\mathbb{U} = \{ u_1, u_2, \ldots, u_n \}$

$\mathbb{V}/\mathbb{U} = \{ v + \mathbb{U} \mid \forall v \in \mathbb{V} \}$

$\dim(\mathbb{V}/\mathbb{U}) = \dim(\mathbb{V}) - \dim(\mathbb{U}) = (m+n) - (n)$

$= m.$

Given, $B = \{ v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n \}.$

Prove B is LI set.

consider the LC of set of vectors in B giving rise to $0_{\mathbb{V}}$

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n = 0_{\mathbb{V}}$$

Since $\{v_1, v_2, \ldots, v_m\}$ is LI set so

$$\Rightarrow \quad \sum \alpha_i v_i = 0_{\mathbb{V}} \longrightarrow \forall \alpha_i = 0$$

Since $\{u_1, u_2, \ldots, u_n\}$ is LI set so

$$\Rightarrow \quad \sum \beta_i u_i = 0_{\mathbb{V}} \longrightarrow \forall \beta_i = 0$$

Therefore,

$$(\alpha_1 v_1 + \cdots + \alpha_m v_m) + (\beta_1 u_1 + \cdots + \beta_n u_n) = 0_{\mathbb{V}}$$

$$\Rightarrow \quad 0_{\mathbb{V}} + 0_{\mathbb{V}} = 0_{\mathbb{V}}$$

$$\Rightarrow \forall \alpha_i = 0, \ \forall \beta_i = 0$$

$\Leftrightarrow$ B is LI set.

Now prove that B is spanning set of $\mathbb{V}$.

Consider, $v \in \mathbb{V}$.

We have to show that v can be written as LC of vectors in B.

Consider, $v + \mathbb{U} = \{ v + u \mid \forall u \in \mathbb{U} \}$

Since $v + \mathbb{U} \in \mathbb{V}/\mathbb{U}$ therefore we can write it in terms of basis.

$$v + \mathbb{U} = \alpha_1 (v_1 + \mathbb{U}) + \alpha_2 (v_2 + \mathbb{U}) + \cdots + \alpha_m (v_m + \mathbb{U})$$

$$\Rightarrow v + \mathbb{U} = (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) + \mathbb{U} \quad (\text{By def}^{+})$$

$$\Rightarrow v - (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) \in \mathbb{U}.$$

Since this belongs to $\mathbb{U}$ so it can be represented as LC of basis vectors of $\mathbb{U}$.

$$v - (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m) = \beta_1 u_1 + \cdots + \beta_n u_n$$

$$\Rightarrow v = \alpha_1 v_1 + \cdots + \alpha_m v_m + \beta_1 u_1 + \cdots + \beta_n u_n$$

$$= \langle \{ v_1, v_2, \ldots, v_m, u_1, \ldots, u_n \} \rangle$$

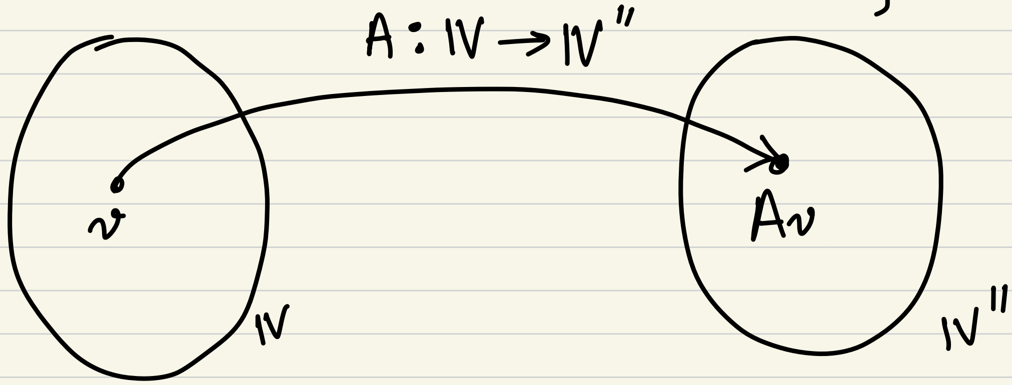
$$= \langle B \rangle$$

Therefore B is generating set for $\mathbb{V}$. So B is a basis for $\mathbb{V}$.

12. The *double dual* of a vector space, $\mathbb{V}$, denoted by $\mathbb{V}''$, is the dual space of $\mathbb{V}'$ by definition. Define $\Lambda : \mathbb{V} \to \mathbb{V}''$ as $(\Lambda v)(\varphi) = \varphi(v)$, where $v \in \mathbb{V}$ and $\varphi \in \mathbb{V}'$.
(a) Prove that $\Lambda : \mathbb{V} \to \mathbb{V}''$ is a linear map.
(b) Prove that for $\tau \in \mathcal{L}(\mathbb{V})$, we have $\tau'' \circ \Lambda = \Lambda \circ \tau$, where $\tau'' = (\tau')'$.
(c) Prove that if $\mathbb{V}$ is finite dimensional, then Λ is an isomorphism between $\mathbb{V}$ and $\mathbb{V}''$.

$\mathbb{V}' = \{ f \mid f : \mathbb{V} \to \mathbb{F} \}$ (set of all linear functionals defined on $\mathbb{V}$)

$\mathbb{V}'' = \{ g \mid g : \mathbb{V}' \to \mathbb{F} \}$ (set of all linear functionals defined on $\mathbb{V}'$)



$(Av)(\varphi) := \varphi(v)$ where $v \in \mathbb{V}$ and $\varphi \in \mathbb{V}'$

(i) A is linear: $A(cv_1 + v_2) = cA(v_1) + A(v_2)$.

$$\Rightarrow \quad A(cv_1+v_2)(\varphi) = \varphi(cv_1+v_2)$$

$$= \varphi(c \cdot v_1) + \varphi(v_2)$$

$$= c \cdot \varphi(v_1) + \varphi(v_2)$$

$$= c\, A(v_1)(\varphi) + A(v_2)(\varphi)$$

$\Rightarrow \quad A(cv_1+v_2) = c \cdot A(v_1) + A(v_2)$. Hence proved.

(ii) $T'' \circ A = (T')' \circ A$

$$
\begin{tikzcd}
V \arrow[r, "T"] & V \\
v \arrow[r, mapsto] \arrow[d, mapsto, "A"] & T(v) \arrow[d, mapsto, "A"] \\
V'' \ A(v) \arrow[d, mapsto, "(T')'"] & V'' \ A(T(v)) \arrow[dl, leftrightarrow] \\
V'' \ (T')'(A(v)) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
V' \arrow[r, "g \in V''"] & F \\
\varphi \arrow[r, mapsto] & g(\varphi)
\end{tikzcd}
$$

We know that:

$$T'(\varphi) = \varphi \circ T$$

$$(T')'(A(v)) = A(T(v))$$

consider one $\varphi \in V'$ to evaluate it.

$$
\begin{aligned}
(T')'(A(v))(\varphi) &= (T')'(Av)(\varphi) \\
&= (Av \circ T')(\varphi) \qquad [\because T'(\varphi) = \varphi \circ T] \\
&= Av(T'(\varphi)) \\
&= Av(\varphi \circ T) \qquad [\because T'(\varphi) = \varphi \circ T] \\
&= (\varphi \circ T)(v) \qquad [\because (Av)(\varphi) = \varphi(v)] \\
&= \varphi(T(v))
\end{aligned}
$$

Now, $A(T(v))$ is in $\mathbb{W}''$ so evaluate it.

$$A(T(v))(\varphi) = \varphi(T(v))$$

Hence proved.

(iii) Since $\dim(\mathbb{V}) = \dim(V'')$ so we just have to prove that A is injective map.

for some $v \in \mathbb{V}$, A is injective iff $(Av = 0_{V''} \rightarrow v = 0)$ $\forall v \in$

Suppose $v \neq 0$

Consider $\{v_1, v_2, \ldots, v_n\}$ as basis of $\mathbb{V}$.

$$\Rightarrow v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n \qquad \forall \alpha_i \in \mathbb{F}$$

$$\Rightarrow \phi(v) = \alpha_1 \phi(v_1) + \alpha_2 \varphi(v_2) + \cdots + \alpha_n \varphi(v_n)$$

Where φ is linear functional from $\mathbb{V}$ to $\mathbb{F}$. so $\varphi \in \mathbb{V}'$

This equation holds for $\forall \varphi \in \mathbb{V}'$

consider $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ as basis for $\mathbb{V}'$

Now pass $\varphi_1, \varphi_2, \ldots, \varphi_n$ to the equation.

$$\varphi_1(v) = \alpha_1 \varphi_1(v_1) + 0 + 0 + \cdots + 0 = \alpha_1$$

$$\varphi_2(v) = 0 + \alpha_2 \varphi_2(v_2) + 0 + \cdots + 0 = \alpha_2$$

$$\vdots$$

$$\varphi_n(v) = \alpha_n.$$

Therefore,

$$\varphi(v) = \varphi_1(v)\,\varphi(v_1) + \varphi_2(v)\cdot\varphi(v_2) + \cdots + \varphi_n(v)\,\varphi(v_K)$$

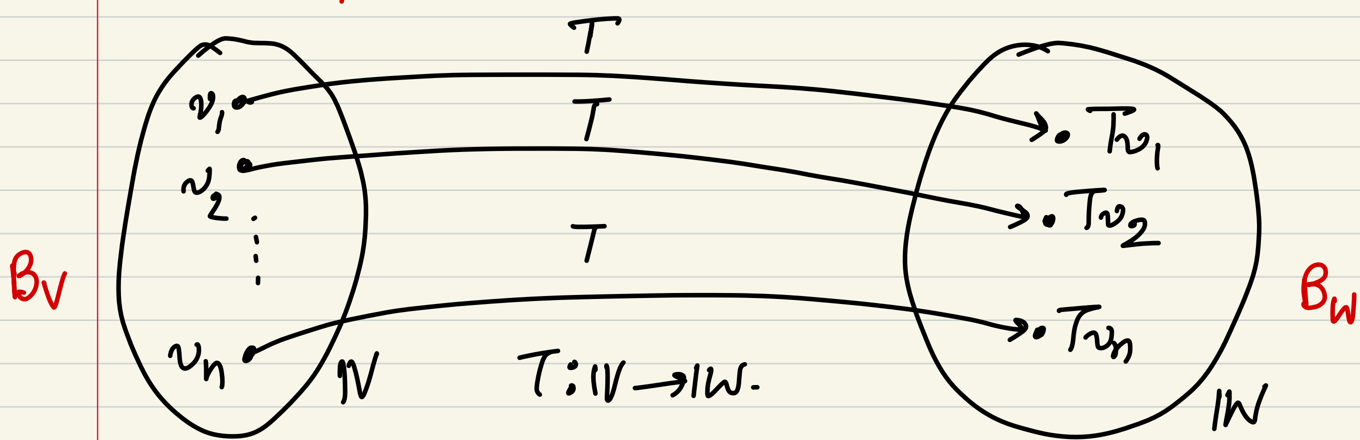
$\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ are LI set therefore,

$$\varphi_1(v)\cdot\varphi(v_1) + \cdots + \varphi_n(v)\,\varphi(v_K) = 0_{\mathbb{F}}$$

$$\Rightarrow \varphi(v_i) = 0 \quad \forall v_i$$

That means $\varphi(v) = 0_{\mathbb{F}}$

Matrix representation of a linear map:



The basis for $\mathbb{V}$: $\{v_1, v_2, \ldots, v_n\}$

The basis for $\mathbb{W}$: $\{w_1, w_2, \ldots, w_m\}$

The transformed vectors in $\mathbb{W}$ space can be written in terms of basis of $\mathbb{W}$.

$$Tv_1 = A_{11}w_1 + A_{21}w_2 + \cdots + A_{m1}w_m$$

$$Tv_2 = A_{12}w_1 + A_{22}w_2 + \cdots + A_{m2}w_m$$

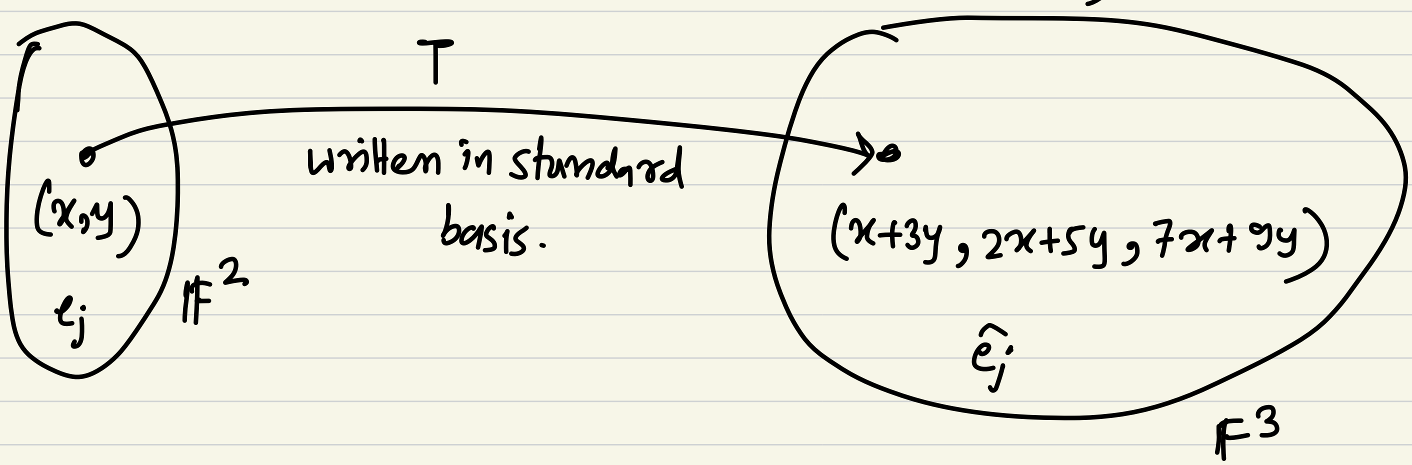
$$\vdots$$

$$Tv_n = A_{1n}w_1 + A_{2n}w_2 + \cdots + A_{mn}w_m$$

$$\begin{bmatrix} [Tv_1]_{B_V} & [Tv_2]_{B_V} & \cdots & [Tv_n]_{B_V} \end{bmatrix} = \begin{bmatrix} [w_1]_{B_W} & [w_2]_{B_W} & \cdots & [w_m]_{B_W} \end{bmatrix} \begin{array}{cccc} [Tv_1]_{B_W} & & & [Tv_n]_{B_W} \\ \left[\begin{matrix} A_{11} & A_{12} & \cdots & A_{m1} \\ A_{21} & A_{22} & \cdots & A_{m2} \\ & \vdots & & \\ A_{m1} & A_{2n} & \cdots & A_{mn} \end{matrix}\right] \\ & [Tv_2]_{B_W} & & \underline{m \times n} \end{array}$$

Suppose, $\quad T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$

$$T(x,y) := (x+3y,\ 2x+5y,\ 7x+9y)$$



$$T(1,0) = (1,2,7) = A_{11} \cdot \hat{e}_1 + A_{12}\hat{e}_2 + A_{13}\hat{e}_3$$

$$T(0,1) = (3,5,9) = A_{21}\hat{e}_1 + A_{22}\hat{e}_2 + A_{23}\hat{e}_3$$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = A_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + A_{12}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + A_{13}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} = A_{21}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + A_{22}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + A_{23}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{22} \\ A_{23} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} \qquad T(x,y) = \begin{bmatrix} x+3y \\ 2x+5y \\ 7x+9y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} \qquad T(x,y) = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{(x,y)}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_{11} & A_{21} \\ A_{22} & A_{22} \\ A_{13} & A_{23} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 2 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 3 \\ 5 \\ 9 \end{matrix}} \end{bmatrix} \qquad \begin{matrix} [T(v_1)]_{B_W} \\ [T(v_2)]_{B_W} \end{matrix}$$

Therefore the transformation is given as:

$$\begin{tikzcd} V \; (\dim n) \arrow[r, "T"] & W \; (\dim m) \\ v \arrow[r, mapsto] \arrow[d, mapsto] & Tv \arrow[d, mapsto] \\ {[v]_{B_W}} \arrow[r, mapsto, "{[A]^{B_V}_{B_W}}"] & {[v]_{B_V}} \end{tikzcd}$$

$$[v]_{B_V} = [A]^{B_V}_{B_W} [v]_{B_W}$$

Ex2: $D \in L\left(P_3(\mathbb{R}),\, P_2(\mathbb{R})\right)$, $Dp = p'$

Standard bases of $P_3(\mathbb{R})$: $\{1, x, x^2, x^3\}$

Standard bases of $P_2(\mathbb{R})$: $\{1, x, x^2\}$

$$D(1) = 0 = (0)\cdot 1 + (0)\cdot x + (0)\cdot x^2 = [0]_{S_W}$$
$$D(x) = 1 = (1)\cdot 1 + (0)\cdot x + (0)\cdot x^2 = [1]_{S_W}$$
$$D(x^2) = 2x = (0)\cdot 1 + (2)\cdot x + (0)\cdot x^2 = [2x]_{S_W}$$
$$D(x^3) = 3x^2 = (0)\cdot 1 + (0)\cdot x + (3)\cdot x^2 = [3x^2]_{S_W}$$

Therefore the matrix of D will be -

$$\begin{bmatrix} 0 & 1 & 2x & 3x^2 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Therefore the matrix $[D]_S$ will be

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ [0]_{S_W} & [1]_{S_W} & [2x]_{S_W} & [3x^2]_{S_W} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$[Tv]_S = [A]\,[v]_S$$

Suppose, $D \in \mathcal{L}(P_3(R), P_2(R))$, $Dp = p'$

Given matrix,

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

find $P_3(R)$ and $P_2(R)$ basis such that it is possible.

let $B = \{p_1, p_2, p_3, p_4\}$ (to be found)

$B' = \{q_1, q_2, q_3\}$ (to be found).

$$D(p_1) = p_1' = a_{11}q_1 + a_{21}q_2 + a_{31}q_3 \Rightarrow [0]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$D(p_2) = p_2' = a_{12}q_1 + a_{22}q_2 + a_{32}q_3 \Rightarrow [1]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

$$D(p_3) = p_3' = a_{13}q_1 + a_{23}q_2 + a_{33}q_3 \Rightarrow [2x]_{B'} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$D(p_4) = p_4' = a_{14}q_1 + a_{24}q_2 + a_{34}q_3 \Rightarrow [3x^2]_{B'} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$$

$$\begin{bmatrix} p_1' & p_2' & p_3' & p_4' \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \Bigg[\underbrace{[0]_{B'}, [1]_{B'}, [2x]_{B'}, [3x^2]_{B'}}_{A} \Bigg]$$

Since A is given to us:

$$\begin{bmatrix} [0]_{B'} & [1]_{B'} & [2x]_{B'} & [3x^2]_{B'} \end{bmatrix}_{3\times 4} \underset{4\times 1}{[v]_B} = \underset{3\times 1}{[v]_{B'}}$$

$$\Rightarrow [\,p_1' \; p_2' \; p_3' \; p_4'\,] = [\,q_1 \; q_2 \; q_3\,] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow [\,p_1' \; p_2' \; p_3' \; p_4'\,] = [\,q_1 \; q_2 \; q_3 \; 0\,]$$

$$\Rightarrow p_4' = 0 \Rightarrow p_4 = \int 0 . dx + 1 = 1$$

$$\Rightarrow p_3' = q_3 \Rightarrow p_3 = \int q_3 \, dx$$

$$\Rightarrow p_2' = q_2 \Rightarrow p_2 = \int q_2 \, dx$$

$$\Rightarrow p_1' = q_1 \Rightarrow p_1 = \int q_1 \, dx$$

Suppose, $q_1 = 1 \Rightarrow p_1 = x$

$q_2 = x \Rightarrow p_2 = x^2/2$

$q_3 = x^2 \Rightarrow p_3 = x^3/3$

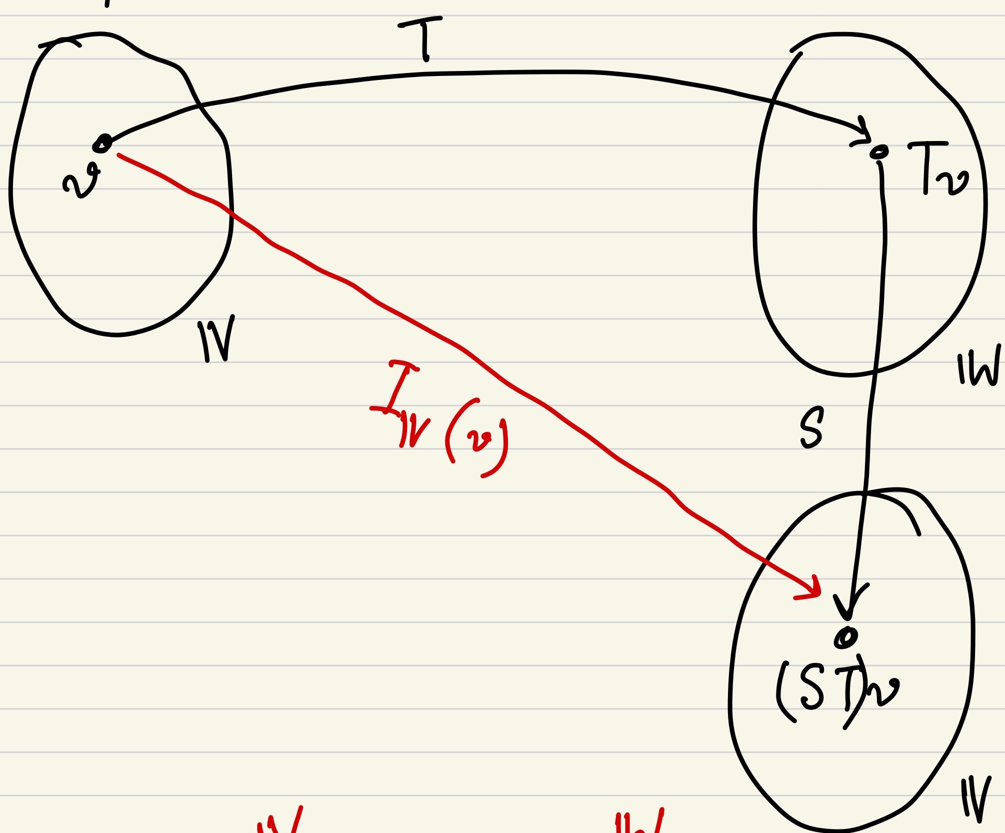
OB (q_1, q_2, q_3)

$= \underline{(1, x, x^2)}$

Therefore $\underline{\text{OB}}$ $(p_1, p_2, p_3, p_4) = \left(x, \frac{x^2}{2}, \frac{x^3}{3}, 1\right)$

Invertibility :

A linear map, $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ is called invertible if there exists another linear map, $S \in \mathcal{L}(\mathbb{W}, \mathbb{V})$ such that ST equals to identity map on $\mathbb{V}$ and TS equals to identity map on $\mathbb{W}$.



$$
\begin{tikzcd}
\mathbb{V} & \mathbb{W} \\
v \arrow[r, mapsto, "T"] \arrow[d, "I_V"] & Tv \arrow[d, "T^{-1}"] \\
I_V(v) = v \arrow[r, phantom, "\equiv"] & T^{-1}(Tv) = (T^{-1}T)(v) = v
\end{tikzcd}
$$

$\Rightarrow \boxed{I_V = T^{-1} \circ T}$

$$
\begin{tikzcd}
\overset{\mathbb{W}}{w} \arrow[rr, mapsto, "T^{-1}"] \arrow[d, mapsto, "I_W"] & & \overset{\mathbb{W}}{T^{-1}(w)} \arrow[d, mapsto, "T"] \\
\overset{\mathbb{W}}{I_W(w) = w} \arrow[rr, phantom, "="] & & \overset{\mathbb{W}}{(TT^{-1})(w) = w}
\end{tikzcd}
$$

$\Rightarrow \boxed{I_W = T \circ T^{-1}}$

<u>Invertibility is equivalent to bijection:</u>

$T \in L(\mathbb{V}, \mathbb{W})$. T is invertible iff T is bijective.

assume : T is invertible.

Show that: T is injective.

Suppose, $u, v \in \mathbb{V}$ such that $Tu = Tv \longrightarrow u = v$

$\Rightarrow T^{-1}Tu = T^{-1}Tv$ (since T is intertible)

$\Rightarrow I_{\mathbb{V}} \cdot u = I_V \cdot v$

$\Rightarrow u = v$ (So it is injective).

Show that: T is surjective.

let $w \in \mathbb{W}$ then $T^{-1}(w) = v$ ($\exists\, v \in \mathbb{V}$.)

$\Rightarrow TT^{-1}(w) = Tv \Rightarrow w = Tv$

Which means $\exists v \in \mathbb{V}$ s.t $Tv = w$ so every $w \in \mathbb{W}$ has a preimage in $\mathbb{V}$ so T is surjective.

Hence T is invertible $\longrightarrow$ T is bijective.

Left inverse:

injection: $n \leq m$

$$\begin{tikzcd} \overset{(n)}{\mathbb{V}} \arrow[r, "\varphi"] & \overset{(m)}{\mathbb{W}} \\ v \arrow[r, mapsto] \arrow[d, mapsto, "I_{\mathbb{V}}"] & \varphi_v \arrow[d, mapsto, "\psi"] \\ I_{\mathbb{V}}(v) \arrow[r, phantom, "="] & (\psi \circ \varphi)(v) \\ & \mathbb{V} \end{tikzcd}$$

$I_{\mathbb{V}} = \psi \circ \phi$

(Def. of left inverse of φ)

$I_{V} = \phi^{-1} \circ \phi.$

ϕ is injection $\longleftrightarrow$ ϕ^{-1} exists (left inverse)

Right inverse:

surjection: $n \geqslant m$

$$\begin{tikzcd} \mathbb{W} \arrow[r, "\psi"] & \mathbb{V} \\ w \arrow[r, mapsto] \arrow[d, mapsto, "I_{\mathbb{W}}"] & \psi w \arrow[d, mapsto, "\phi"] \\ I_{\mathbb{W}}(w) \arrow[r, phantom, "="] & \phi \circ \psi(w) \\ \mathbb{W} & \mathbb{W} \end{tikzcd}$$

$I_{\mathbb{W}} = \phi \circ \psi$

(Def of right inverse of φ)

$I_{\mathbb{W}} = \phi \circ \phi^{-1}$

φ is surjection $\longleftrightarrow$ φ^{-1} exists (right inverse)

When $\dim(V) = \dim(W) = n$ then.

ϕ is injective $\longrightarrow$ φ is bijection

φ is surjective $\longrightarrow$ φ is bijective.

So φ^{-1} exists if and only if φ is bijection.

Ex: Suppose $T \in L(V, V)$. , T^{-1} exists
$S \in L(V, W)$. , S^{-1} exists.

prove that, $ST \in L(V, W)$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

$$
\begin{tikzcd}
U \arrow[r, "T"] & V \arrow[r, "S"] & W \\
u \arrow[r, mapsto] \arrow[d, "I_u"'] & Tu \arrow[r, mapsto] & STu \arrow[dl, "S^{-1}"] \\
I_u(u) & \underset{V}{S^{-1}STu} \arrow[dl, "T^{-1}"'] & \\
\underset{U}{T^{-1}S^{-1}STu} & &
\end{tikzcd}
$$

$$I_u = \underbrace{(T^{-1}S^{-1})}_{\text{exists.}}(ST)$$

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{ST} & \mathbb{W} \\ u & \longmapsto & STu \\ \Big\downarrow I_u & & \Big\downarrow (ST)^{-1} \\ I_u u & & (ST)^{-1}(ST)u \\ & & \mathbb{U}. \end{array}$$

So, $I_u = \underbrace{(ST)^{-1}}_{\text{must exist}}(ST)$

as $\underline{T^{-1}S^{-1} \text{ exists}}$

Since $(T^{-1}S^{-1})(ST) = (ST)^{-1}(ST)$

$\Rightarrow T^{-1}S^{-1} = (ST)^{-1}$. (Left inverse exists)

$T^{-1}S^{-1}$ is the left inverse of ST.

Similarly, $T^{-1}S^{-1}$ is also the right inverse of ST.

$$\begin{array}{ccc} \mathbb{W} & \xrightarrow{(ST)^{-1}} & \mathbb{U} \\ w & \longmapsto & (ST)^{-1}w \\ \Big\downarrow I_W & & \Big\downarrow (ST) \\ I_W(w) & = & (ST)(ST)^{-1}w \\ & & \mathbb{W} \end{array}$$

$I_W = (ST)(ST)^{-1}$

$$
\begin{tikzcd}
\mathbb{W} \arrow[r, "S^{-1}"] & \mathbb{V}. \arrow[r, "T^{-1}"] & \mathbb{U}. \\
w \arrow[r, mapsto] \arrow[d, "I_W"'] & S^{-1}w \arrow[r, mapsto] & T^{-1}S^{-1}w \arrow[dl, "T"'] \\
I_W(w) & TT^{-1}S^{-1}w \ (\mathbb{V}.) \arrow[dl, "S"'] & \\
\parallel \quad STT^{-1}S^{-1}w \ (\mathbb{W}) & &
\end{tikzcd}
$$

$$I_W = ST\underbrace{T^{-1}S^{-1}}_{\text{exists}}$$

$$\Rightarrow \quad I_W = ST(ST)^{-1}$$

Therefore right inverse of ST exists

So $(ST)^{-1} = T^{-1}S^{-1}$. (proved).

Prove that: $S, T \in L(V,V)$.

ST is invertible $\longleftrightarrow$ S, T are invertible.

S, T are invertible $\longrightarrow$ ST is invertible,

$(ST)^{-1} = T^{-1} \circ S^{-1}$ (So it is done)

We have to prove, ST is invertible $\longrightarrow$ S, T are invertible.

ST is invertible means, ST is surjective, injective.