Q 1. (a)

Suppose $u^T v \neq 0$.

Now, $A = u v^T$

$$\Rightarrow Au = u v^T u$$
$$= \underbrace{(v^T u)}_{\lambda} u$$

Hence, λ is an eigenvalue ($\lambda = v^T u = u^T v$) with eigenvector u.

Further, if $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ &

$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with $v_1 \neq 0$, without

loss of generality, then consider

$$w_i = \begin{bmatrix} \frac{v_i}{v_1} \\ \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow i^{\text{th}} \text{ position.}$$

for $i = 2, 3, \dots, n$.

$$\therefore A w_i = \begin{bmatrix} v_1 u & v_2 u & \dots & v_n u \end{bmatrix} w_i$$

$$= 0$$

& also, $\sum \alpha_i w_i = 0$

$$\Rightarrow \begin{bmatrix} \left(\dfrac{-\sum_{i=2}^{n} \alpha_i v_i}{v_1} \right) \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} = 0$$

$\Rightarrow \alpha_i = 0$ for $i = 2, \dots, n$.

$\Rightarrow$ $\{w_2, w_3, \dots w_n\}$ is a lin. ind set.

Also, eigenvector corr to distinct eig values are lin. ind.

$\Rightarrow$ $\{u, w_2, \dots, w_n\}$ is a basis for $\mathbb{R}^n$ comprising eigenvectors.

$\Rightarrow$ A is diagonalizable.

Supp $A = uv^T$ is diag.

Now. A has an eig value of $u^Tv = v^Tu$.

In the prev. part we have seen

that A has at least

that A has ...

$(n-1)$ eigenvalues at 0.

Thus, if $u^T v = 0$, then A has all its eigenvalues zero. So A being diagonalizable implies it is similar to

$$O_{n\times n} \Rightarrow A = O_{n\times n}.$$

Thus for any non-trivial matrix, A, we must have

$$u^T v \neq 0.$$

Alternatively, for the second part, assume $u^T v = 0$ & $u \neq 0, v \neq 0$.

$$\Rightarrow \text{rank}(A) = 1.$$

$$\therefore \dim(\text{Ker}(A)) = n-1.$$

$$\Rightarrow g.m.(0) = n-1$$

$\Rightarrow g.m(0)$

On the other hand,

$$A^2 = uv^Tuv^T = u(v^Tu)v^T$$
$$= (u^Tv)\,uv^T$$
$$= 0$$

$\Rightarrow$ A is a nilpotent matrix.

$\Rightarrow$ $\lambda(A) = 0$ and A cannot have any non zero eigenvalue, due to nilpotence.

$$\Rightarrow \chi_A(x) = x^n$$

$$\Rightarrow a.m(0) = n.$$

Thus, $a.m.(0) > g.m.(0)$

$\Rightarrow$ A is not diagonalizable.

Q 1. (b)

Counter example.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ is not Hermitian.

But

$\lambda_A = 2, -2.$

$$\{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix} \right\}.$$

Observe that $\langle v_1 | v_2 \rangle$

$$= \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= 1 + i^2 = 0$$

$\Rightarrow v_1 \perp v_2.$

[In general, check that for $A^H A = A A^H$ (such matrices are called "normal" matrices) the eigenvectors are an orthogonal set. Clearly, Hermitian matrices satisfy this, but so do skew-Hermitian matrices]. Hence false

Q 1. (c) $Av_i = \lambda_i v_i$ (since (λ_i, v_i) is an eigenpair)

Since A has n-lin ind. eig vectors, it is diagonalizable.

Hence its minimal polynomial is of the form

$$\mu_A(x) = \prod_{i=1}^{r} (x - \lambda_i) \quad r \leq n$$

when $\{\lambda_1, \lambda_2, \cdots \lambda_r\}$ are distinct.

Now, $\chi_A(x) = \prod_{i=1}^{r} (x - \lambda_i)^{k_i}$

where k_i is alg. mult of λ_i.

Thus, $\chi_A(x) = \mu_A(x)$ if and only if $k_i = 1 \; \forall i$ &

$$r = n.$$

In other words, we need to check whether eigenvalues of A are all distinct.

Now,

$$A(w_1 + 2w_2 + \cdots + n w_n)$$
$$= (1^2+1)w_1 + (2^2+2)w_2 + \cdots + (n^2+n)w_n$$

$$\Rightarrow [\lambda_1 - (1^2+1)]w_1 + (2\lambda_2 - [2^2+2])w_2 + \cdots + [n\lambda_n - (n^2+n)]w_n = 0$$

But $\{w_1, w_2, \ldots w_n\}$ is a lin. ind. set. Hence, no non-trivial lin. comb leads to zero

$$\Rightarrow \lambda_i = \frac{i^2+i}{i} = i+1$$

i

Thus, indeed λ_i's are all distinct $\Rightarrow \mu_A(x) = \chi_A(x)$.

True

Q 1. (d) Consider

$$\alpha_0 v + \alpha_1 A v + \dots + \alpha_{m-1} A^{m-1} v = 0. \quad (*)$$

T.S.T. $\alpha_i = 0 \ \forall i = 0, 1, \dots, m-1$.

Multiply $(*)$ by A^{m-1} from the left and noting that if $A^k = 0$ then.

$$A^j = 0 \quad \forall j \geq k,$$

we conclude that,

$$\alpha_0 A^{m-1} v + \alpha_1 \cancelto{0}{A^{m}} v + \dots + \alpha_{m-1} \cancelto{0}{A^{2m-2}} v = 0$$

$$\Rightarrow \alpha_0 A^{m-1} v = 0$$

But $A^{m-1} v \neq 0$.

$\Rightarrow \alpha_0 = 0$

Rewriting (*) we get

$$\alpha_1 A v + \alpha_2 A^2 v + \dots + \alpha_{m-1} A^{m-1} v = 0 \quad (**)$$

Multiplying (**) by A^{m-2} from the left, we get

$$\alpha_1 A^{m-1} v = 0$$

$\Rightarrow \alpha_1 = 0$

Proceeding similarly, it readily follows that

$$\alpha_i = 0 \ \forall i = 0, 1, \ldots, m-1.$$

Hence,

$\{v, Av, \ldots, A^{m-1}v\}$ is linearly independent.

Q2

(a) Sol^n:- $A \in \mathbb{R}^{m \times n}$

SVD of $A = U \Sigma V^T$ (U, V are $\in \mathbb{R}^{m\times m}$, $\mathbb{R}^{n \times n}$ orthogonal)

$\Rightarrow \quad AV = U\Sigma$

$$\Rightarrow \quad A[v_1 \cdots v_n] = [u_1 \cdots u_m] \begin{bmatrix} \sigma_1 & & 0 & \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}_{m \times n}$$

so $\quad Av_1 = u_1 \sigma_1$

$$\frac{\|Av_1\|_2}{\|v_1\|_2} = \frac{\|\sigma_1 u_1\|_2}{\|v_1\|_2} = \sigma_1$$

(u_1 & v_1 are orthonormal)

since. $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_r > 0$

$$\therefore \quad \frac{\|Av_1\|_2}{\|v_1\|_2} \geq \frac{\|Av_2\|_2}{\|v_2\|} \cdots \geq \frac{\|Av_r\|_2}{\|v_r\|_2} > 0$$

Hence $\quad \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \geq \frac{\|Av_1\|_2}{\|v_1\|_2} \geq \sigma_1 \rightarrow (1)$

~~To prove~~

To prove, we need to show that no other ~~other~~ vector scaled more than σ_1.

Let $x \in \mathbb{R}^n$

$\therefore$ x can be represented in terms of right singular ~~eigen~~ vectors of A

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$$\|x\|_2^2 = |c_1|^2 + \dots + |c_n|^2 \quad \left(v_i \text{ are orthonormal} \right)$$

$$Ax = c_1 A v_1 + c_2 A v_2 + \dots + c_n A v_n$$

$$= c_1 \sigma_1 u_1 + c_2 \sigma_2 u_2 + \dots + c_r \sigma_r u_r + 0$$

$$Ax = c_1 \sigma_1 u_1 + c_2 \sigma_2 u_2 + \dots + 0$$

$$\|Ax\|_2^2 = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + \dots + c_r^2 \sigma_r^2 \quad (r = \min(m,n))$$

$$\Rightarrow \|Ax\|_2^2 \leq \sigma_1^2 \left(|c_1|^2 + \dots + |c_r|^2 \right) \quad (\because \sigma_1 \text{ is maximum})$$

$$\Rightarrow \frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \sigma_1^2 \quad \left(\|x\|_2^2 = |c_1|^2 + \dots |c_n|^2 \right)$$

$$\Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \leq \sigma_1 \quad \text{--- } (2)$$

① & ② $\Rightarrow$

$$\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1 \quad \text{Shown.}$$

② b) Sol^n :-

Given $$\|A\|_F^2 = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

$$\Rightarrow \|A\|_F^2 = \text{trace}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \right)$$

$$= \text{trace}(AA^T)$$

Let SVD of $A = U \Sigma V^T$

Now, $$\text{trace}(AA^T) = \text{trace}(U\Sigma V^T V \Sigma U^T)$$
$$= \text{trace}(U\Sigma^2 U^T)$$
$$= \text{trace}(U^T U \Sigma^2)$$
$$= \text{trace}(\Sigma^2)$$

$$\|A\|_F^2 = (\sigma_1^2 + \sigma_2^2 + \cdots \sigma_r^2) \Rightarrow \|A\|_F = (\sigma_1^2 + \sigma_2^2 + \cdots \sigma_r^2)^{1/2}$$

3) a)

$$A = \begin{bmatrix} a & 1 & 0 & & & & \\ 0 & a & 1 & & & & \\ 0 & 0 & a & & & & \\ & & & a & 1 & 0 & \\ & & & 0 & a & 1 & \\ & & & 0 & 0 & a & \\ & & & & & & a \end{bmatrix}$$

$$B = \begin{bmatrix} a & 1 & 0 & & & & \\ 0 & a & 1 & & & & \\ 0 & 0 & a & & & & \\ & & & a & 1 & & \\ & & & 0 & a & & \\ & & & & & a & 1 \\ & & & & & 0 & a \end{bmatrix}$$

$$\mu_A(x) = \mu_B(x) = (x-a)^3$$

$$\chi_A(x) = \chi_B(x) = (x-a)^7$$

3 b) let A be in structure $A = \begin{bmatrix} x & 0 & \cdots & & 0 \\ x & x & 0 & \cdots & 0 \\ x & x & x & 0 \cdots & 0 \\ & & & \ddots & \\ x & x & x & \cdots & x \end{bmatrix}$

$$A^H = \begin{bmatrix} x & x & \cdots & x \\ 0 & x & x \cdots & x \\ 0 & & \ddots & \vdots \\ \vdots & & & \\ 0 & \cdots & & x \end{bmatrix}$$

- consider $AA^H = A^H A$

$$\begin{bmatrix} x & 0 & \cdots & 0 \\ \cancel{x} & x & 0 \cdots & 0 \\ x & & \ddots & \\ \vdots & & & \\ x & & & x \end{bmatrix} \begin{bmatrix} x & x & \cdots & x \\ 0 & x & \cancel{x} \cdots & \cancel{x} \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & & x \end{bmatrix} = \begin{bmatrix} x & x & \cdots & \cancel{x} \\ 0 & x & \cdots & x \\ & & \ddots & \\ & & & x \end{bmatrix} \begin{bmatrix} x & 0 & \cdots & 0 \\ x & x & 0 \cdots & 0 \\ x & & \ddots & \\ \vdots & & & \\ x & & & x \end{bmatrix}$$

Equating the diagonal entries of $(A^H A)$ and (AA^H)

$$(AA^H)_{11} = \|a_{11}\|^2$$

$$(A^H A)_{11} = \|a_{11}\|^2 + \|a_{21}\|^2 + \cdots + \|a_{n1}\|^2$$

In order to satisfy, $(AA^H)_{11} = (A^H A)_{11}$

$\Rightarrow \|a_{21}\|^2 + \cdots + \|a_{n1}\|^2$ each entry has to be zero

$\Rightarrow$ ~~$a_{i1} = 0$~~ ~~$\forall i$~~

$a_{i1} = 0 \quad \forall\, i = 2, \cdots n$

$(AA^H)_{22} = \|a_{21}\|^2 + \|a_{22}\|^2 = \|a_{22}\|^2$ (since a_{21} is zero from previous step)

$(A^H A)_{22} = \|a_{22}\|^2 + \|a_{32}\|^2 + \cdots + \|a_{n2}\|^2$

$\Rightarrow \|a_{32}\|^2 = 0 \cdots \|a_{n2}\|^2 = 0$

$a_{i2} = 0 \quad \forall i = 3, \cdots n$

Proceeding further, we end up with $A^H A$ and AA^H as a diagonal matrix.

Further, A and A^H are individually diagonal matrices too.

4(a) Observe that

$$\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$$

Further, $\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}$ is nonsingular. Hence,

let $T := \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}$

Then $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} T = T \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$

$$\Rightarrow T^{-1} \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} T = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$$

Thus $\chi_{\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}}(x) = \chi_{\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}}(x)$

$$\Rightarrow \begin{vmatrix} xI & 0 \\ -B & xI-BA \end{vmatrix} = \begin{vmatrix} xI-AB & 0 \\ -B & xI \end{vmatrix}$$

$$\Rightarrow x^n \underbrace{\det(xI-BA)}_{\chi_{BA}(x)} = x^n \underbrace{\det(xI-AB)}_{\chi_{AB}(x)}$$

$$\Rightarrow \quad \chi_{BA}(x) = \chi_{AB}(x).$$

4(b) Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be n - (possibly non-distinct) eigenvalues of $A = A^H \in \mathbb{C}^{n \times n}$

Now, we know that

$$A = V^H \Lambda V, \text{ where}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\& \quad V^H V = V V^H = I.$$

$\therefore$ For any $\|x\| = 1$,

consider $x^H A x = x^H V^H \Lambda V x$

$$= (Vx)^H \Lambda (Vx)$$

$$= y^H \Lambda y$$

where $y = Vx \in \mathbb{C}^n$

$$\Rightarrow x = V^H y$$

$$\therefore \min_{\|x\|=1} x^H A x = \min_{\|V^H y\|=1} y^H \Lambda y$$

& $|||^{ly},$ $\max_{\|x\|=1} x^H A x = \max_{\|V^H y\|=1} y^H \Lambda y$

Also, if $\|V^H y\|^2 = 1$

then $y^H \underbrace{V V^H}_{I} y = 1$

$$\Rightarrow y^H y = 1$$

$\Rightarrow \|y\|^2 = 1.$

Thus, we are looking for the max & min of $y^H \Lambda y$ subject to $\|y\| = 1$.

Now $y^H \Lambda y = \sum |y_i|^2 \lambda_i$

where $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{C}^n$

$\therefore \lambda_{min} \sum |y_i|^2 \leq y^H \Lambda y \leq \lambda_{max} \sum |y_i|^2$

As $\sum |y_i|^2 = \|y\|^2 = 1$,

we have,

$$\boxed{\lambda_{min} \leq y^H \Lambda y \leq \lambda_{max}} \quad (*)$$

Choosing $y = [1\ 0\ 0\ 0 \cdots 0]^H$,

we get $y^H \Lambda y = \lambda_{max} = \lambda_1$

& choosing $y = [0\ 0 \cdots 1]^H$,

we get $y^H \Lambda y = \lambda_{min} = \lambda_n$

which leads to equality on either sides of (*) in turn.

Thus, $\lambda_1 = \max\limits_{\|x\|=1} x^H A x = \max\limits_{\|y\|=1} y^H \Lambda y$

& $\lambda_n = \min\limits_{\|x\|=1} x^H A x = \min\limits_{\|y\|=1} y^H \Lambda y$.