Chapter 3: Linear Maps

Linear Algebra Done Right, by Sheldon Axler

A: The Vector Space of Linear Maps

Problem 1

Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

Proof. ($\Leftarrow$) Suppose $b = c = 0$. Then

$$T(x, y, z) = (2x - 4y + 3z, 6x).$$

Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$. Then

$$\begin{aligned} T((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2), 6(x_1 + x_2)) \\ &= (2x_1 + 2x_2 - 4y_1 - 4y_2 + 3z_1 + 3z_2, 6x_1 + 6x_2) \\ &= (2x_1 - 4y_1 + 3z_1, 6x_1) + (2x_2 - 4y_2 + 3z_2, 6x_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2). \end{aligned}$$

Now, for $\lambda \in \mathbb{F}$ and $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned} T(\lambda(x, y, z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x) - 4(\lambda y) + 3(\lambda z), 6(\lambda x)) \\ &= (\lambda(2x - 4y + 3z), \lambda(6x)) \\ &= \lambda(2x - 4y + 3z, 6x) \\ &= \lambda T(x, y, z), \end{aligned}$$

and thus T is a linear map.

($\Rightarrow$) Supose T is a linear map. Then

$$T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \qquad (\dagger)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$. In particular, by applying the definition of T and comparing first coordinates of both sides of ($\dagger$), we have

$$2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + b = \\ (2x_1 - 4y_1 + 3z_1 + b) + (2x_2 - 4y_2 + 3z_2 + b),$$

and after simplifying, we have $b = 2b$, and hence $b = 0$. Now by applying the definition of T and comparing second coordinates of both sides of (†), we have

$$\begin{aligned} 6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) &= 6x_1 + c(x_1y_1z_1) + 6x_2 + c(x_2y_2z_2) \\ &= 6(x_1 + x_2) + c(x_1y_1z_1 + x_2y_2z_2), \end{aligned}$$

which implies

$$c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) = c(x_1y_1z_1 + x_2y_2z_2).$$

Now suppose $c \neq 0$. Then choosing $(x_1, y_1, z_1) = (x_2, y_2, z_2) = (1, 1, 1)$, the equation above implies $8 = 2$, a contradiction. Thus $c = 0$, completing the proof. □

Problem 3

Suppose $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Proof. Given $x \in \mathbb{F}^n$, we may write

$$x = x_1e_1 + \cdots + x_ne_n,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$. Since T is linear, we have

$$Tx = T(x_1e_1 + \cdots + x_ne_n) = x_1Te_1 + \cdots + x_nTe_n.$$

Now for each $Te_k \in \mathbb{F}^m$, where $k = 1, \ldots, n$, there exist $A_{1,k}, \ldots, A_{m,k} \in \mathbb{F}$ such that

$$\begin{aligned} Te_k &= A_{1,k}e_1 + \cdots + A_{m,k}e_m \\ &= \big(A_{1,k}, \ldots, A_{m,k}\big) \end{aligned}$$

and thus

$$x_kTe_k = \big(A_{1,k}x_k, \ldots, A_{m,k}x_k\big).$$

Therefore, we have

$$\begin{aligned} Tx &= \sum_{k=1}^{n} \big(A_{1,k}x_k, \ldots, A_{m,k}x_k\big) \\ &= \left(\sum_{k=1}^{n} A_{1,k}x_k, \ldots, \sum_{k=1}^{n} A_{m,k}x_k\right), \end{aligned}$$

and thus there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ of the desired form. □

Problem 5

Prove that $\mathcal{L}(V, W)$ is a vector space.

Proof. We check each property in turn.
Commutative: Given $S, T \in \mathcal{L}(V, W)$ and $v \in V$, we have

$$(T + S)(v) = Tv + Sv = Sv + Tv = (S + T)(v)$$

and so addition is commutative.
Associative: Given $R, S, T \in \mathcal{L}(V, W)$ and $v \in V$, we have

$$\begin{aligned} ((R + S) + T)(v) = (R + S)(v) + Tv &= Rv + Sv + Tv \\ &= R + (S + T)(v) = (R + (S + T))(v) \end{aligned}$$

and so addition is associative. And given $a, b \in \mathbb{F}$, we have

$$((ab)T)(v) = (ab)(Tv) = a(b(Tv)) = (a(bT))(v)$$

and so scalar multiplication is associative as well.
Additive identity: Let $0 \in \mathcal{L}(V, W)$ denote the zero map, let $T \in \mathcal{L}(V, W)$, and let $v \in V$. Then

$$(T + 0)(v) = Tv + 0v = Tv + 0 = Tv$$

and so the zero map is the additive identity.
Additive inverse: Let $T \in \mathcal{L}(V, W)$ and define $(-T) \in \mathcal{L}(V, W)$ by $(-T)v = -Tv$. Then

$$(T + (-T))(v) = Tv + (-T)v = Tv - Tv = 0,$$

and so $(-T)$ is the additive inverse for each $T \in \mathcal{L}(V, W)$.
Multiplicative identity: Let $T \in \mathcal{L}(V, W)$. Then

$$(1T)(v) = 1(Tv) = Tv$$

and so the multiplicative identity of $\mathbb{F}$ is the multiplicative identity of scalar multiplication.
Distributive properties: Let $S, T \in \mathcal{L}(V, W)$, $a, b \in \mathbb{F}$, and $v \in V$. Then

$$\begin{aligned} (a(S + T))(v) = a((S + T)(v)) = a(Sv + Tv) &= a(Sv) + a(Tv) \\ &= (aS)(v) + (aT)(v) \end{aligned}$$

and

$$((a + b)T)(v) = (a + b)(Tv) = a(Tv) + b(Tv) = (aT)(v) + (bT)(v)$$

and so the distributive properties hold.

Since all properties of a vector space hold, we see $\mathcal{L}(V, W)$ is in fact a vector space, as desired. $\square$

Problem 7

Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Proof. Since $\dim V = 1$, a basis of V consists of a single vector. So let $w \in V$ be such a basis. Then there exists $\alpha \in \mathbb{F}$ such that $v = \alpha w$ and $\lambda \in \mathbb{F}$ such that $Tw = \lambda w$. It follows

$$Tv = T(\alpha w) = \alpha T w = \alpha \lambda w = \lambda(\alpha w) = \lambda v,$$

as desired. $\square$

Problem 9

Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$ but φ is not linear. (Here $\mathbb{C}$ is thought of as a complex vector space.)

Proof. Define

$$\begin{aligned} \varphi &: \mathbb{C} \to \mathbb{C} \\ x + yi &\mapsto x - yi. \end{aligned}$$

Then for $x_1 + y_1 i, x_2 + y_2 i \in \mathbb{C}$, it follows

$$\begin{aligned} \varphi((x_1 + y_1 i) + (x_2 + y_2 i)) &= \varphi((x_1 + x_2) + (y_1 + y_2)i) \\ &= (x_1 + x_2) - (y_1 + y_2)i \\ &= (x_1 - y_1)i + (x_2 - y_2)i \\ &= \varphi(x_1 + y_1 i) + \varphi(x_2 + y_2 i) \end{aligned}$$

and so φ satisfies the additivity requirement. However, we have

$$\varphi(i \cdot i) = \varphi(-1) = -1$$

and

$$i \cdot \varphi(i) = i(-i) = 1$$

and hence φ fails the homogeneity requirement of a linear map. $\square$

Problem 11

Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Suppose U is a subspace of V and $S \in \mathcal{L}(U, W)$. Let $v_1, \ldots, u_m$ be a basis of U and let $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$ be an extension of this basis to V. For any $z \in V$, there exist $a_1, \ldots, a_n \in \mathbb{F}$ such that $z = \sum_{k=1}^n a_k v_k$, and so we define

$$\begin{aligned} T &: V \to W \\ \sum_{k=1}^n a_k v_k &\mapsto \sum_{k=1}^m a_k S v_k + \sum_{k=m+1}^n a_k v_k. \end{aligned}$$

Since every $v \in V$ has a unique representation as a linear combination of elements of our basis, the map is well-defined. We first show T is a linear map. So suppose $z_1, z_2 \in V$. Then there exist $a_1, \ldots a_n \in \mathbb{F}$ and $b_1, \ldots, b_n \in \mathbb{F}$ such that

$$z_1 = a_1 v_1 + \cdots + a_n v_n \quad \text{and} \quad z_2 = b_1 v_1 + \cdots + b_n v_n.$$

It follows

$$\begin{aligned} T(z_1 + z_2) &= T\left(\sum_{k=1}^n a_k v_k + \sum_{k=1}^n b_k v_k\right) \\ &= T\left(\sum_{k=1}^n (a_k + b_k) v_k\right) \\ &= \sum_{k=1}^m (a_k + b_k) S v_k + \sum_{k=m+1}^n (a_k + b_k) v_k \\ &= \left(\sum_{k=1}^m a_k S v_k + \sum_{k=m+1}^n a_k v_k\right) + \left(\sum_{k=1}^m b_k S v_k + \sum_{k=m+1}^n b_k v_k\right) \\ &= T\left(\sum_{k=1}^n a_k v_k\right) + T\left(\sum_{k=1}^n b_k v_k\right) \\ &= T z_1 + T z_2 \end{aligned}$$

and so T is additive. To see that T is homogeneous, let $\lambda \in \mathbb{F}$ and $z \in V$, so that we may write $z = \sum_{k=1}^{n} a_k v_k$ for some $a_1, \ldots, a_n \in \mathbb{F}$. We have

$$\begin{aligned}
T(\lambda z) &= T\left(\lambda \sum_{k=1}^{n} a_k v_k\right) \\
&= T\left(\sum_{k=1}^{n} (\lambda a_k) v_k\right) \\
&= S\left(\sum_{k=1}^{m} (\lambda a_k) v_k\right) + \sum_{k=m+1}^{n} (\lambda a_k) v_k \\
&= \lambda S\left(\sum_{k=1}^{m} a_k v_k\right) + \lambda \sum_{k=m+1}^{n} a_k v_k \\
&= \lambda\left(S\left(\sum_{k=1}^{m} a_k v_k\right) + \sum_{k=m+1}^{n} \lambda a_k v_k\right) \\
&= \lambda T\left(\sum_{k=1}^{n} a_k v_k\right) \\
&= \lambda T z
\end{aligned}$$

and so T is homogeneous as well hence $T \in \mathcal{L}(V, W)$. Lastly, to see that $T\mid_U = S$, let $u \in U$. Then there exist $a_1, \ldots, a_m \in \mathbb{F}$ such that $u = \sum_{k=1}^{m} a_k v_k$, and hence

$$\begin{aligned}
Tu &= T\left(\sum_{k=1}^{m} a_k v_k\right) \\
&= S\left(\sum_{k=1}^{m} a_k v_k\right) \\
&= Su,
\end{aligned}$$

and so indeed T agrees with S on U, completing the proof. □

Problem 13

Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

Proof. Since $v_1, \ldots, v_m$ is linearly dependent, one of them may be written as a linear combination of the others. Without loss of generality, suppose this is v_m.

Then there exist $a_1, \ldots, a_{m-1} \in \mathbb{F}$ such that

$$v_m = a_1 v_1 + \cdots + a_{m-1} v_{m-1}.$$

Since $W \neq \{0\}$, there exists some nonzero $z \in W$. Define $w_1, \ldots, w_m \in W$ by

$$w_k = \begin{cases} z & \text{if } k = m \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose there exists $T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k$ for $k = 1, \ldots, m$. It follows

$$\begin{aligned} T(0) &= T(v_m - a_1 v_1 - \cdots - a_{m-1} v_{m-1}) \\ &= Tv_m - a_1 T v_1 - \cdots - a_{m-1} T v_{m-1} \\ &= z. \end{aligned}$$

But $z \neq 0$, and thus $T(0) \neq 0$, a contradiction, since linear maps take 0 to 0. Therefore, no such linear map can exist. $\square$

B: Null Spaces and Ranges

Problem 1

Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Proof. Define the map

$$\begin{aligned} T : \mathbb{R}^5 &\to \mathbb{R}^5 \\ (x_1, x_2, x_3, x_4, x_5) &\mapsto (0, 0, 0, x_4, x_5). \end{aligned}$$

First we show T is a linear map. Suppose $x, y \in \mathbb{R}^5$. Then

$$\begin{aligned} T(x + y) &= T((x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5) \\ &= (0, 0, 0, x_4 + y_5, x_5 + y_5) \\ &= (0, 0, 0, x_4, x_5) + (0, 0, 0, y_4, y_5) \\ &= T(x) + T(y). \end{aligned}$$

Next let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} T(\lambda x) &= T(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) \\ &= (0, 0, 0, \lambda x_4, \lambda x_5) \\ &= \lambda(0, 0, 0, x_4, x_5) \\ &= \lambda T(x), \end{aligned}$$

and so T is in fact a linear map. Now notice that

$$\operatorname{null} T = \{(x_1, x_2, x_3, 0, 0) \in \mathbb{R}^5 \mid x_1, x_2, x_3 \in \mathbb{R}\}.$$

This space clearly has as a basis $e_1, e_2, e_3 \in \mathbb{R}^5$ and hence has dimension 3. Now, by the Fundamental Theorem of Linear Maps, we have

$$\dim \mathbb{R}^5 = 3 + \dim \operatorname{range} T$$

and hence $\dim \operatorname{range} T = 2$, as desired. $\square$

Problem 3

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

(b) What property of T corresponds to $v_1, \ldots, v_m$ being linearly independent?

Proof. (a) We claim surjectivity of T corresponds to $v_1, \ldots, v_m$ spanning V. To see this, suppose T is surjective, and let $w \in V$. Then there exists $z \in \mathbb{F}^m$ such that $Tz = w$. This yields

$$z_1 v_1 + \cdots + z_m v_m = w,$$

and hence every $w \in V$ can be expressed as a linear combination of $v_1, \ldots, v_n$. That is, $\operatorname{span}(v_1, \ldots, v_n) = V$.

(b) We claim injectivity of T corresponds to $v_1, \ldots, v_m$ being linearly independent. To see this, suppose T is injective, and let $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Then

$$T(a) = T(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n = 0$$

which is true iff $a = 0$ since T is injective. That is, $a_1 = \cdots = a_n = 0$ and hence $v_1, \ldots, v_n$ is linearly independent. $\square$

Problem 5

Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

Proof. Define

$$T : \mathbb{R}^4 \to \mathbb{R}^4$$
$$(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, 0, 0).$$

Clearly T is a linear map, and we have

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \mid x_3 = x_4 = 0 \in \mathbb{R}\} = \mathbb{R}^2 \times \{0\}^2$$

and

$$\operatorname{range} T = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\} = \mathbb{R}^2 \times \{0\}^2.$$

Hence $\operatorname{range} T = \operatorname{null} T$, as desired. ☐

Problem 7

Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Proof. Let $Z = \{T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\}$, let $v_1, \ldots, v_m$ be a basis of V, where $m \geq 2$, and let $w_1, \ldots, w_n$ be a basis of W, where $n \geq m$. We define $T \in \mathcal{L}(V, W)$ by its behavior on the basis

$$Tv_k := \begin{cases} 0 & \text{if } k = 1 \\ w_2 & \text{if } k = 2 \\ \frac{1}{2} w_k & \text{otherwise} \end{cases}$$

so that clearly T is not injective since $Tv_1 = 0 = T(0)$, and hence $T \in Z$. Similarly, define $S \in \mathcal{L}(V, W)$ by its behavior on the basis

$$Sv_k := \begin{cases} w_1 & \text{if } k = 1 \\ 0 & \text{if } k = 2 \\ \frac{1}{2} w_k & \text{otherwise} \end{cases}$$

and note that S is not injective either since $Sv_2 = 0 = S(0)$, and hence $S \in Z$. However, notice

$$(S + T)(v_k) = w_k \text{ for } k = 1, \ldots, n$$

and hence $\operatorname{null}(S + T) = \{0\}$ since it takes the basis of V to the basis of W, so that $S + T$ is in fact injective. Therefore $S + T \notin Z$, and Z is not closed under addition. Thus Z is not a subspace of $\mathcal{L}(V, W)$. ☐

Problem 9

Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Proof. Suppose $a_1, \ldots, a_n \in \mathbb{F}$ are such that

$$a_1 T v_1 + \cdots + a_n T v_n = 0.$$

Since T is a linear map, it follows

$$T(a_1 v_1 + \cdots + a_n v_n) = 0.$$

But since $\operatorname{null} T = \{0\}$ (by virtue of T being a linear map), this implies $a_1 v_1 + \cdots + a_n v_n = 0$. And since $v_1, \ldots, v_n$ are linearly independent, we must have $a_1 = \cdots = a_n = 0$, which in turn implies $Tv_1, \ldots, Tv_n$ is indeed linearly independent in W. $\square$

Problem 11

Suppose $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \ldots S_n$ makes sense. Prove that $S_1 S_2 \ldots S_n$ is injective.

Proof. For $n \in \mathbb{Z}_{\geq 2}$, let $P(n)$ be the statement: $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \ldots S_n$ makes sense, and the product $S_1 S_2 \ldots S_n$ is injective. We induct on n.

Base case: Suppose $n = 2$, and assume $S_1 \in \mathcal{L}(V_0, V_1)$ and $S_2 \in \mathcal{L}(V_1, V_2)$, so that the product $S_1 S_2$ is defined, and assume that both S_1 and S_2 are injective. Suppose $v_1, v_2 \in V_0$ are such that $v_1 \neq v_2$, and let $w_1 = S_2 v_1$ and $w_2 = S_2 v$. Since S_2 is injective, $w_1 \neq w_2$. And since S_1 is injective, this in turn implies that $S_1(w_1) \neq S_1(w_2)$. In other words, $S_1(S_2(v_1)) \neq S_1(S_2(v_2))$, so that $S_1 S_2$ is injective as well, and hence $P(2)$ is true.

Inductive step: Suppose $P(k)$ is true for some $k \in \mathbb{Z}^+$, and consider the product $(S_1 S_2 \ldots S_k) S_{k+1}$. The term in parentheses is injective by hypothesis, and the product of this term with S_{k+1} is injective by our base case. Thus $P(k+1)$ is true.

By the principle of mathematical induction, the statement $P(n)$ is true for all $n \in \mathbb{Z}_{\geq 2}$, as was to be shown. $\square$

Problem 13

Suppose T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Proof. We claim the list

$$(5, 1, 0, 0), (0, 0, 7, 1)$$

is a basis of null T. This implies

$$\begin{aligned}\dim \operatorname{range} T &= \dim \mathbb{F}^4 - \dim \operatorname{null} T \\ &= 4 - 2 \\ &= 2,\end{aligned}$$

and hence T is surjective (since the only 2-dimensional subspace of $\mathbb{F}^2$ is the space itself). So let's prove our claim that this list is a basis.

Clearly the list is linearly independent. To see that it spans null T, suppose $x = (x_1, x_2, x_3, x_4) \in \operatorname{null} T$, so that $x_1 = 5x_2$ and $x_3 = 7x_4$. We may write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5x_2 \\ x_2 \\ 7x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 7 \\ 1 \end{pmatrix},$$

and indeed x is in the span of our list, so that our list is in fact a basis, completing the proof. $\square$

Problem 15

Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Proof. Suppose such a $T \in \mathcal{L}(\mathbb{F}^5, \mathbb{F}^2)$ did exist. We claim

$$(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$$

is a basis of null T. This implies

$$\begin{aligned}\dim \operatorname{range} T &= \dim \mathbb{F}^5 - \dim \operatorname{null} T \\ &= 5 - 2 \\ &= 3,\end{aligned}$$

which is absurd, since the codomain of T has dimension 2. Hence such a T cannot exist. So, let's prove our claim that this list is a basis.

Clearly $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ is linearly independent. To see that it spans null T, suppose $x = (x_1, \dots, x_5) \in \operatorname{null} T$, so that $x_1 = 3x_2$ and $x_3 = x_4 = x_5$. We may write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

and indeed x is in the span of our list, so that our list is in fact a basis, completing the proof. $\square$

Problem 17

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Proof. ($\Rightarrow$) Suppose $T \in \mathcal{L}(V, W)$ is injective. If $\dim V > \dim W$, Thereom 3.23 tells us that no map from V to W would be injective, a contradiction, and so we must have $\dim V \leq \dim W$.

($\Leftarrow$) Suppose $\dim V \leq \dim W$. Then the inclusion map $\iota : V \to W$ is both a linear map and injective. ☐

Problem 19

Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Proof. ($\Leftarrow$) Suppose $\dim U \geq \dim V - \dim W$. Since U is a subspace of V, there exists a subspace U' of V such that

$$V = U \oplus U'.$$

Let $u_1, \dots u_m$ be a basis for U, let $u'_1, \dots, u'_n$ be a basis for U', and let $w_1, \dots, w_p$ be a basis for W. By hypothesis, we have

$$m \geq (m + n) - p,$$

which implies $p \geq n$. Thus we may define a linear map $T \in \mathcal{L}(V, W)$ by its values on the basis of $V = U \oplus U'$ by taking $Tu_k = 0$ for $k = 1, \dots m$ and $Tu'_j = w_j$ for $j = 1, \dots, n$ (since $p \geq n$, there is a w_j for each u'_j). The map is linear by Theorem 3.5, and its null space is U by construction.

($\Rightarrow$) Suppose U is a subspace of V, $T \in \mathcal{L}(V, W)$, and $\operatorname{null} T = U$. Then, since $\operatorname{range} T$ is a subspace of W, we have $\dim \operatorname{range} T \leq \dim W$. Combining this inequality with the Fundamental Theorem of Linear Maps yields

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W. \end{aligned}$$

Since $\dim \operatorname{null} T = \dim U$, we have the desired inequality. ☐

Problem 21

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

Proof. ($\Rightarrow$) Suppose $T \in \mathcal{L}(V, W)$ is surjective, so that W is necessarily finite-dimensional as well. Let $v_1, \dots, v_m$ be a basis of V and let $n = \dim W$, where $m \geq n$ by surjectivity of T. Note that

$$Tv_1, \dots, Tv_m$$

span W. Thus we may reduce this list to a basis by removing some elements (possibly none, if $n = m$). Suppose this reduced list were $Tv_{i_1}, \dots, Tv_{i_n}$ for some $i_1, \dots, i_n \in \{1, \dots, m\}$. We define $S \in \mathcal{L}(W, V)$ by its behavior on this basis

$$S(Tv_{i_k}) := v_{i_k} \text{ for } k = 1, \dots, n.$$

Suppose $w \in W$. Then there exist $a_1, \dots, a_n \in \mathbb{F}$ such that

$$w = a_1 Tv_{i_1} + \cdots + a_n Tv_{i_n}$$

and thus

$$\begin{aligned}
TS(w) &= TS\left(a_1 Tv_{i_1} + \cdots + a_n Tv_{i_n}\right) \\
&= T\big(S\left(a_1 Tv_{i_1} + \cdots + a_n Tv_{i_n}\right)\big) \\
&= T\big(a_1 S(Tv_{i_1}) + \cdots + a_n S(Tv_{i_n})\big) \\
&= T(a_1 v_{i_1} + \cdots + a_n v_{i_n}) \\
&= a_1 Tv_{i_1} + \cdots + a_n Tv_{i_n} \\
&= w,
\end{aligned}$$

and so TS is the identity map on W.

($\Leftarrow$) Suppose there exists $S \in \mathcal{L}(W, V)$ such that $TS \in \mathcal{L}(W, W)$ is the identity map, and suppose by way of contradiction that T is not surjective, so that $\dim \operatorname{range} TS < \dim W$. By the Fundamental Theorem of Linear Maps, this implies

$$\begin{aligned}
\dim W &= \dim \operatorname{null} TS + \dim \operatorname{range} TS \\
&< \dim \operatorname{null} TS + \dim W
\end{aligned}$$

and hence $\dim \operatorname{null} TS > 0$, a contradiction, since the identity map can only have trivial null space. Thus T is surjective, as desired. □

Problem 23

Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

Proof. We will show that both $\dim \operatorname{range} ST \leq \dim \operatorname{range} S$ and $\dim \operatorname{range} ST \leq \dim \operatorname{range} T$, since this implies the desired inequality.

We first show that $\dim \operatorname{range} ST \leq \dim \operatorname{range} S$. Suppose $w \in \operatorname{range} ST$. Then there exists $u \in U$ such that $ST(u) = w$. But this implies that $w \in \operatorname{range} S$ as well, since $Tu \in S^{-1}(w)$. Thus $\operatorname{range} ST \subseteq \operatorname{range} S$, which implies $\dim \operatorname{range} ST \leq \dim \operatorname{range} S$.

We now show that $\dim \operatorname{range} ST \leq \dim \operatorname{range} T$. Note that if $v \in \operatorname{null} T$, so that $Tv = 0$, then $ST(v) = 0$ (since linear maps take zero to zero). Thus we have $\operatorname{null} T \subseteq \operatorname{null} ST$, which implies $\dim \operatorname{null} T \leq \dim \operatorname{null} ST$. Combining this inequality with the Fundamental Theorem of Linear Maps applied to T yields

$$\dim U \leq \dim \operatorname{null} ST + \dim \operatorname{range} T. \tag{1}$$

Similarly, we have

$$\dim U = \dim \operatorname{null} ST + \dim \operatorname{range} ST. \tag{2}$$

Combining (1) and (2) yields

$$\dim \operatorname{null} ST + \dim \operatorname{range} ST \leq \dim \operatorname{null} ST + \dim \operatorname{range} T$$

and hence $\dim \operatorname{range} ST \leq \dim \operatorname{range} T$, completing the proof. $\square$

Problem 25

Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 \subseteq \operatorname{range} T_2$ if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$.

Proof. ($\Leftarrow$) Suppose there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$, and let $w \in \operatorname{range} T_1$. Then there exists $v \in V$ such that $T_1 v = w$, and hence $T_2 S(v) = w$. But then $w \in \operatorname{range} T_2$ as well, and hence $\operatorname{range} T_1 \subseteq \operatorname{range} T_2$.

($\Rightarrow$) Suppose $\operatorname{range} T_1 \subseteq \operatorname{range} T_2$, and let $v_1, \ldots, v_n$ be a basis of V. Let $w_k = T v_k$ for $k = 1, \ldots, n$. Then there exist $u_1, \ldots, u_n \in V$ such that $T_2 u_k = w_k$ for $k = 1, \ldots, n$ (since $w_k \in \operatorname{range} T_1$ implies $w_k \in \operatorname{range} T_2$). Define $S \in \mathcal{L}(V, V)$ by its behavior on the basis

$$S v_k := u_k \text{ for } k = 1, \ldots, n.$$

It follows that $T_2 S(v_k) = T_2 u_k = w_k = T_1 v_k$. Since $T_2 S$ and T_1 are equal on the basis, they are equal as linear maps, as was to be shown. $\square$

Problem 27

Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.

Proof. Suppose $\deg p = n$, and consider the linear map

$$\begin{aligned} D : \mathcal{P}_{n+1}(\mathbb{R}) &\to \mathcal{P}_n(\mathbb{R}) \\ q &\mapsto 5q'' + 3q'. \end{aligned}$$

If we can show D is surjective, we're done, since this implies that there exists some $q \in \mathcal{P}_{n+1}(\mathbb{R})$ such that $Dq = 5q'' + 3q' = p$. To that end, suppose $r \in \operatorname{null} D$. Then we must have $r'' = 0$ and $r' = 0$, which is true if and only if r is constant. Thus any $\alpha \in \mathbb{R}^\times$ is a basis of $\operatorname{null} D$, and so $\dim \operatorname{null} D = 1$. By the Fundamental Theorem of Linear Maps, we have

$$\dim \operatorname{range} D = \dim \mathcal{P}_{n+1}(\mathbb{R}) - \dim \operatorname{null} D,$$

and hence

$$\dim \operatorname{range} D = (n+2) - 1 = n + 1.$$

Since the only subspace of $\mathcal{P}_n(\mathbb{R})$ with dimension $n+1$ is the space itself, D is surjective, as desired. ☐

Problem 29

Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in $\operatorname{null} \varphi$. Prove that

$$V = \operatorname{null} \varphi \oplus \{au \mid a \in \mathbb{F}\}.$$

Proof. First note that since $u \in V - \operatorname{null} \varphi$, there exists some nonzero $\varphi(u) \in \operatorname{range} \varphi$ and hence $\dim \operatorname{range} \varphi \geq 1$. But since $\operatorname{range} \varphi \subseteq \mathbb{F}$, and $\dim \mathbb{F} = 1$, we must have $\dim \operatorname{range} \varphi = 1$. Thus, letting $n = \dim V$, it follows

$$\begin{aligned} \dim \operatorname{null} \varphi &= \dim V - \dim \operatorname{range} \varphi \\ &= n - 1. \end{aligned}$$

Let $v_1, \ldots, v_{n-1}$ be a basis for $\operatorname{null} \varphi$. We claim $v_1, \ldots, v_{n-1}, u$ is an extension of this basis to a basis of V, which would then imply $V = \operatorname{null} \varphi \oplus \{au \mid a \in \mathbb{F}\}$, as desired.

To show $v_1, \ldots, v_{n-1}, u$ is a basis of V, it suffices to show linearly independence (since it has length $n = \dim V$). So suppose $a_1, \ldots, a_n \in \mathbb{F}$ are such that

$$a_1 v_1 + \cdots + a_{n-1} v_{n-1} + a_n u = 0.$$

We may write

$$a_n u = -a_1 v_1 - \cdots - a_{n-1} v_{n-1},$$

which implies $a_n u \in \operatorname{null} \varphi$. By hypothesis, $u \notin \operatorname{null} \varphi$, and thus we must have $a_n = 0$. But now each of the $a_1, \ldots, a_{n-1}$ must be 0 as well (since $v_1, \ldots, v_{n-1}$ form a basis of $\operatorname{null} \varphi$ and thus are linearly independent). Therefore, $v_1, \ldots, v_{n-1}, u$ is indeed linearly independent, proving our claim. ☐

Problem 31

Give an example of two linear maps T_1 and T_2 from $\mathbb{R}^5$ to $\mathbb{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

Proof. Let $e_1, \ldots, e_5$ be the standard basis of $\mathbb{R}^5$. We define $T_1, T_2 \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^2)$ by their behavior on the basis (using the standard basis for $\mathbb{R}^2$ as well)

$$\begin{aligned} T_1 e_1 &:= e_2 \\ T_1 e_2 &:= e_1 \\ T_1 e_k &:= 0 \text{ for } k = 3, 4, 5 \end{aligned}$$

and

$$\begin{aligned} T_2 e_1 &:= e_1 \\ T_2 e_2 &:= e_2 \\ T_2 e_k &:= 0 \text{ for } k = 3, 4, 5. \end{aligned}$$

Clearly $\operatorname{null} T_1 = \operatorname{null} T_2$. We claim T_2 is not a scalar multiple of T_1. To see this, suppose not. Then there exists $\alpha \in \mathbb{R}$ such that $T_1 = \alpha T_2$. In particular, this implies $T_1 e_1 = \alpha T_2 e_1$. But this is absurd, since $T_1 e_1 = e_2$ and $T_2 e_1 = e_1$, and of course e_1, e_2 is linearly independent. Thus no such α can exist, and T_1, T_2 are as desired. ☐

C: Matrices

Problem 1

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Proof. Let $v_1, \ldots, v_n$ be a basis of V, let $w_1, \ldots, w_m$ be a basis of W, let $r = \dim \operatorname{range} T$, and let $s = \dim \operatorname{null} T$. Then there are s basis vectors of V which map to zero and r basis vectors of V with nontrivial representation as linear combinations of $w_1, \ldots, w_m$. That is, suppose $Tv_k \neq 0$, where $k \in \{1, \ldots, n\}$. Then there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all zero, such that

$$Tv_k = a_1 w_1 + \cdots + a_m w_m.$$

The coefficients form column k of $\mathcal{M}(T)$, and there are r such vectors in the basis of V. Hence there are r columns of $\mathcal{M}(T)$ with at least one nonzero entry, as was to be shown. ☐

Problem 3

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

Proof. Let R be the subspace of V such that

$$V = R \oplus \operatorname{null} T,$$

let $r_1, \ldots, r_m$ be a basis of R (where $m = \dim \operatorname{range} T$), and let $v_1, \ldots, v_n$ be a basis of $\operatorname{null} T$ (where $n = \dim \operatorname{null} T$). Then $r_1, \ldots, r_m, v_1, \ldots, v_n$ is a basis of V. It follows that $Tr_1, \ldots, Tr_m$ is a basis of $\operatorname{range} T$, and hence there is an extension of this list to a basis of W. Suppose $Tr_1, \ldots, Tr_m, w_1, \ldots, w_p$ is such an extension (where $p = \dim W - m$). Then, for $j = 1, \ldots m$, we have

$$Tr_j = \left(\sum_{i=1}^{m} \delta_{i,j} \cdot Tr_t \right) + \left(\sum_{k=1}^{p} 0 \cdot w_k \right),$$

where $\delta_{i,j}$ is the Kronecker delta function. Thus, column j of $\mathcal{M}(T)$ is has an entry of 1 in row j and 0's elsewhere, where j ranges over 1 to $m = \dim \operatorname{range} T$. Since $Tv_1 = \cdots = Tv_n = 0$, the remaining columns of $\mathcal{M}(T)$ are all zero. Thus $\mathcal{M}(T)$ has the desired form. $\square$

Problem 5

Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all the entries in the first row of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row, first column.

Proof. First note that if $\operatorname{range} T \subseteq \operatorname{span}(w_2, \ldots, w_n)$, the first row of $\mathcal{M}(T)$ will be all zeros regardless of choice of basis for V.

So suppose $\operatorname{range} T \not\subseteq \operatorname{span}(w_2, \ldots, w_n)$ and let $u_1 \in V$ be such that $Tu_1 \notin \operatorname{span}(w_2, \ldots, w_n)$. There exist $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$Tu_1 = a_1 w_1 + \cdots + a_n w_n,$$

and notice $a_1 \neq 0$ since $Tu_1 \notin \operatorname{span}(w_2, \ldots, w_n)$. Hence we may define

$$z_1 := \frac{1}{a_1} u_1.$$

It follows

$$Tz_1 = w_1 + \frac{a_2}{a_1} w_2 + \cdots + \frac{a_n}{a_1} w_n. \tag{3}$$

Now extend z_1 to a basis $z_1, \ldots, z_m$ of V. Then for $k = 2, \ldots, m$, there exist $A_{1,k}, \ldots, A_{n,k} \in \mathbb{F}$ such that

$$Tz_k = A_{1,k}w_1 + \cdots + A_{n,k}w_n,$$

and notice

$$\begin{aligned} T(z_k - A_{1,k}z_1) &= Tz_k - A_{1,k}Tz_1 \\ &= \big(A_{1,k}w_1 + \cdots + A_{n,k}w_n\big) - A_{1,k}\left(w_1 + \frac{a_2}{a_1}w_2 + \cdots + \frac{a_n}{a_1}w_n\right) \\ &= \big(A_{2,k} - A_{1,k}\big)\frac{a_2}{a_1}w_2 + \cdots + \big(A_{n,k} - A_{1,k}\big)\frac{a_n}{a_1}w_n. \end{aligned} \tag{4}$$

Now we define a new list in V by

$$v_k := \begin{cases} z_1 & \text{if } k = 1 \\ z_k - A_{1,k}z_1 & \text{otherwise} \end{cases}$$

for $k = 1, \ldots, m$. We claim $v_1, \ldots, v_m$ is a basis. To see this, it suffices to prove the list is linearly independent, since its length equals $\dim V$. So suppose $b_1, \ldots, b_m \in \mathbb{F}$ are such that

$$b_1v_1 + \cdots + b_mv_m = 0.$$

By definition of the v_k, it follows

$$b_1z_1 + b_2(z_2 - A_{1,k}z_1) + \cdots + b_m(z_m - A_{1,k}z_1) = 0.$$

But since $z_1, \ldots, z_m$ is a basis of V, the expression on the LHS above is simply a linear combination of vectors in a basis. Thus we must have $b_1 = \cdots = b_m = 0$, and indeed $v_1, \ldots, v_m$ are linearly independent, as claimed.

Finally, notice (3) tells us the first column of $\mathcal{M}(T, v_k, w_k)$ is all 0's except a 1 in the first entry, and (4) tells us the remaining columns have a 0 in the first entry. Thus $\mathcal{M}(T, v_k, w_k)$ has the desired form, completing the proof. □

Problem 7

Suppose $S, T \in \mathcal{L}(V, W)$. Prove that $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Proof. Let $v_1, \ldots, v_m$ be a basis of V and let $w_1, \ldots, w_n$ be a basis of W. Also, let $A = \mathcal{M}(S)$ and $B = \mathcal{M}(T)$ be the matrices of these linear transformations with respect to these bases. It follows

$$\begin{aligned} (S + T)v_k &= Sv_k + Tv_k \\ &= \big(A_{1,k}w_1 + \cdots + A_{n,k}w_n\big) + \big(B_{1,k}w_1 + \cdots + B_{n,k}w_n\big) \\ &= (A_{1,k} + B_{1,k})w_1 + \cdots + (A_{n,k} + B_{n,k})w_n. \end{aligned}$$

Hence $\mathcal{M}(S+T)_{j,k} = A_{j,k}+B_{j,k}$, and indeed we have $\mathcal{M}(S+T) = \mathcal{M}(S)+\mathcal{M}(T)$, as desired. □

Problem 9

Suppose A is an m-by-n matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n-by-1 matrix. Prove that

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}.$$

Proof. By definition, it follows

$$\begin{aligned}
Ac &= \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,n} \\ A_{2,1} & A_{2,2} & \dots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\
&= \begin{pmatrix} A_{1,1}c_1 + A_{1,2}c_2 + \cdots + A_{1,n}c_n \\ A_{2,1}c_1 + A_{2,2}c_2 + \cdots + A_{2,n}c_n \\ \vdots \\ A_{m,1}c_1 + A_{m,2}c_2 + \cdots + A_{m,n}c_n \end{pmatrix} \\
&= c_1 \begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + c_2 \begin{pmatrix} A_{1,2} \\ A_{2,2} \\ \vdots \\ A_{m,2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{1,n} \\ A_{2,n} \\ \vdots \\ A_{m,n} \end{pmatrix} \\
&= c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n},
\end{aligned}$$

as desired. □

Problem 11

Suppose $a = (a_1, \dots, a_n)$ is a 1-by-n matrix and C is an n-by-p matrix. Prove that

$$aC = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}.$$

Proof. By definition, it follows

$$\begin{aligned}
aC &= (a_1, \dots, a_n) \begin{pmatrix} C_{1,1} & C_{1,2} & \dots & C_{1,p} \\ C_{2,1} & C_{2,2} & \dots & C_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \dots & C_{n,p} \end{pmatrix} \\
&= \left(\sum_{k=1}^{n} a_k C_{k,1}, \sum_{k=1}^{n} a_k C_{k,2}, \dots, \sum_{k=1}^{n} a_k C_{k,p} \right) \\
&= \sum_{k=1}^{n} \big(a_k C_{k,1}, \dots, a_k C_{k,p} \big) \\
&= \sum_{k=1}^{n} a_k \big(C_{k,1}, \dots, C_{k,p} \big) \\
&= \sum_{k=1}^{n} a_k C_{k,\cdot},
\end{aligned}$$

as desired. ☐

Problem 13

Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Prove that $AB+AC$ and $DF+EF$ both make sense and that $A(B+C) = AB+AC$ and $(D+E)F = DF+EF$.

Proof. First note that if $A(B+C)$ makes sense, then the number of columns of A must equal the number of rows of $B+C$. But the sum of two matrices is only defined if their dimensions are equal, and hence the number of rows of both B and C must equal the number of columns of A. Thus $AB+AC$ makes sense. So suppose $A \in \mathbb{F}^{m,n}$ and $B, C \in \mathbb{F}^{n,p}$. It follows

$$\begin{aligned}
\big(A(B+C)\big)_{j,k} &= \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} \\
&= \sum_{r=1}^{n} A_{j,r}(B_{r,k} + C_{r,k}) \\
&= \sum_{r=1}^{n} \big(A_{j,r}B_{r,k} + A_{j,r}C_{r,k} \big) \\
&= \sum_{r=1}^{n} A_{j,r}B_{r,k} + \sum_{r=1}^{n} A_{j,r}C_{r,k} \\
&= (AB)_{j,k} + (AC)_{j,k},
\end{aligned}$$

proving the first distributive property.

Now note that if $(D+E)F$ makes sense, then the number of columns of $D+E$ must equal the number of rows of F. Hence the number of columns of both D and E must equal the number of rows of F, and thus $DF+EF$ makes sense as well. So suppose $D, E \in \mathbb{F}^{m,n}$ and $F \in \mathbb{F}^{n,p}$. It follows

$$\begin{aligned}
\big((D+E)F\big)_{j,k} &= \sum_{r=1}^{n} (D+E)_{j,r} F_{r,k} \\
&= \sum_{r=1}^{n} (D_{j,r} + E_{j,r}) F_{r,k} \\
&= \sum_{r=1}^{n} D_{j,r} F_{r,k} + E_{j,r} F_{r,k} \\
&= \sum_{r=1}^{n} D_{j,r} F_{r,k} + \sum_{r=1}^{n} E_{j,r} F_{r,k} \\
&= (DF)_{j,k} + (EF)_{j,k},
\end{aligned}$$

proving the second distributive property. □

Problem 15

Suppose A is an n-by-n matrix and $1 \leq j, k \leq n$. show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.$$

Proof. For $1 \leq p, k \leq n$, we have

$$(A^2)_{p,k} = \sum_{r=1}^{n} A_{p,r} A_{r,k}.$$

Thus, for $1 \leq j, k \leq n$, it follows

$$\begin{aligned}
(A^3)_{j,k} &= \sum_{p=1}^{n} A_{j,p} (A^2)_{p,k} \\
&= \sum_{p=1}^{n} A_{j,p} \sum_{r=1}^{n} A_{p,r} A_{r,k} \\
&= \sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k},
\end{aligned}$$

as desired. □

D: Invertibility and Isomorphic Vector Spaces

Problem 1

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. For all $u \in U$, we have

$$\begin{aligned}(T^{-1}S^{-1}ST)(u) &= T^{-1}(S^{-1}(S(T(u)))) \\ &= T^{-1}(I(T(u)) \\ &= T^{-1}(T(u)) \\ &= v\end{aligned}$$

and hence $T^{-1}S^{-1}$ is a left inverse of ST. Similarly, for all $w \in W$, we have

$$\begin{aligned}(STT^{-1}S^{-1})(w) &= S(T(T^{-1}(S^{-1}(w)))) \\ &= S(I(S^{-1}(w))) \\ &= S(S^{-1}(w)) \\ &= w\end{aligned}$$

and hence $T^{-1}S^{-1}$ is a right inverse of ST. Therefore, ST is invertible, as desired. □

Problem 3

Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Proof. ($\Leftarrow$) Suppose S is injective, and let W be the subspace of V such that $V = U \oplus W$. Let $u_1, \ldots, u_m$ be a basis of U and let $w_1, \ldots, w_n$ be a basis of W, so that $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis of V. Define $T \in \mathcal{L}(V)$ by its behavior on this basis of V

$$\begin{aligned}Tu_k &:= Su_k \\ Tw_j &:= w_j\end{aligned}$$

for $k = 1, \ldots, m$ and $j = 1, \ldots, n$. Since S is injective, so too is T. And since V is finite-dimensional, this implies that T is invertible, as desired.

($\Rightarrow$) Suppose there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$. Since T is invertible, it is also injective. And since T is injective, so to is $S = T\,|_U$, completing the proof. □

Problem 5

Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$.

Proof. ($\Rightarrow$) Suppose $\operatorname{range} T_1 = \operatorname{range} T_2 := R$, so that $\operatorname{null} T_1 = \operatorname{null} T_2 := N$ as well. Let Q be the unique subspace of V such that

$$V = N \oplus Q,$$

and let $u_1, \ldots, u_m$ be a basis of N and $v_1, \ldots, v_n$ be a basis of Q. We claim there exists a unique $q_k \in Q$ such that $T_2 q_k = T_1 v_k$ for $k = 1, \ldots, n$. To see this, suppose $q_k, q_k' \in Q$ are such that $T_2 q_k = T_2 q_k' = T_1 v_k$. Then $T_2(q_k - q_k') = 0$, and hence $q_k - q_k' \in N$. But since $N \cap Q = \{0\}$, this implies $q_k - q_k' = 0$ and thus $q_k = q_k'$. And so the choice of q_k is indeed unique. We now define $S \in \mathcal{L}(V)$ by its behavior on the basis

$$\begin{aligned} Su_k &= u_k \text{ for } k = 1, \ldots, m \\ Sv_j &= q_j \text{ for } j = 1, \ldots, n. \end{aligned}$$

Let $v \in V$, so that there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n.$$

It follows

$$\begin{aligned} (T_2 S)(v) &= T_2(S(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n)) \\ &= T_2(a_1 S u_1 + \cdots + a_m S u_m + b_1 S v_1 + \cdots + b_n S v_n) \\ &= T_2(a_1 u_1 + \cdots + a_m u_m + b_1 q_1 + \cdots + b_n q_n) \\ &= a_1 T_2 u_1 + \cdots + a_m T_2 u_m + b_1 T_2 q_1 + \cdots + b_n T_2 q_n \\ &= b_1 T_1 v_1 + \cdots + b_n T_1 v_n. \end{aligned}$$

Similarly, we have

$$\begin{aligned} T_1 v &= T_1(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) \\ &= a_1 T_1 u_1 + \cdots + a_m T_1 u_m + b_1 T_1 v_1 + \cdots + b_n T_1 v_n \\ &= b_1 T_1 v_1 + \cdots + b_n T_1 v_n, \end{aligned}$$

and so indeed $T_1 = T_2 S$. To see that S is invertible, it suffices to prove it is injective. So let $v \in V$ be as before, and suppose $Sv = 0$. It follows

$$\begin{aligned} Sv &= S(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) \\ &= (a_1 u_1 + \cdots + a_m u_m) + (b_1 S v_1 + \cdots + b_n S v_n) \\ &= 0. \end{aligned}$$

By the proof of Theorem 3.22, $Sv_1, \ldots, Sv_n$ is a basis of R, and thus the list $u_1, \ldots, u_m, Sv_1, \ldots, Sv_n$ is a basis of V, and each of the a's and b's must be zero. Therefore S is indeed injective, completing the proof in this direction.

($\Leftarrow$) Suppose there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$. If $w \in \operatorname{range} T_1$, then there exists $v \in V$ such that $T_1 v = w$, and hence $(T_2 S)(v) = T_2(S(v)) = w$, so that $w \in \operatorname{range} T_2$ and we have $\operatorname{range} T_1 \subseteq \operatorname{range} T_2$. Conversely, suppose $w' \in \operatorname{range} T_2$, so that there exists $v' \in V$ such that $T_2 v' = w'$. Then, since $T_2 = T_1 S^{-1}$, we have $(T_1 S^{-1})(v') = T_1(S^{-1}(v')) = w'$, so that $w' \in \operatorname{range} T_1$. Thus $\operatorname{range} T_2 \subseteq \operatorname{range} T_1$, and we have shown $\operatorname{range} T_1 = \operatorname{range} T_2$, as desired. $\square$

Problem 7

Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) \mid Tv = 0\}.$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E$?

Proof. (a) First note that the zero map is clearly an element of E, and hence E contains the additive identity of $\mathcal{L}(V, W)$. Now suppose $T_1, T_2 \in E$. Then

$$(T_1 + T_2)(v) = T_1 v + T_2 v = 0$$

and hence $T_1 + T_2 \in E$, so that E is closed under addition. Finally, suppose $T \in E$ and $\lambda \in \mathbb{F}$. Then

$$(\lambda T)(v) = \lambda T v = \lambda 0 = 0,$$

and so E is closed under scalar multiplication as well. Thus E is indeed a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$, and let $\dim V = m$ and $\dim W = n$. Extend v to a basis $v, v_2, \ldots, v_m$ of V, and endow W with any basis. Let $\mathcal{E}$ denote the subspace of $\mathbb{F}^{m,n}$ of matrices whose first column is all zero.

We claim $T \in E$ if and only if $\mathcal{M}(T) \in \mathcal{E}$, so that $\mathcal{M} : E \to \mathcal{E}$ is an isomorphism. Clearly if $T \in E$ (so that $Tv = 0$), then $\mathcal{M}(T)_{\cdot,1}$ is all zero, and hence $T \in \mathcal{E}$. Conversely, suppose $\mathcal{M}(T) \in \mathcal{E}$. It follows

$$\begin{aligned}
\mathcal{M}(Tv) &= \mathcal{M}(T)\mathcal{M}(v) \\
&= \begin{pmatrix} 0 & A_{1,2} & \dots & A_{1,n} \\ 0 & A_{2,2} & \dots & A_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & A_{m,2} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\end{aligned}$$

and thus we must have $Tv = 0$ so that $T \in E$, proving our claim. So indeed $E \cong \mathcal{E}$.

Now note that $\mathcal{E}$ has as a basis the set of all matrices with a single 1 in a column besides the first, and zeros everywhere else. There are $mn - n$ such matrices, and hence $\dim \mathcal{E} = mn - n$. Thus we have $\dim E = mn - n$ as well, as desired. ☐

Problem 9

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof. ($\Leftarrow$) Suppose S and T are both invertible. Then by Problem 1, ST is invertible.

($\Rightarrow$) Suppose ST is invertible. We will show T is injective and S is surjective. Since V is finite-dimensional, this implies that both S and T are invertible. So suppose $v_1, v_2 \in V$ are such that $Tv_1 = Tv_2$. Then $(ST)(v_1) = (ST)(v_2)$, and since ST is invertible (and hence injective), we must have $v_1 = v_2$, so that T is injective. Next, suppose $v \in V$. Since T^{-1} is surjective, there exists $w \in V$ such that $T^{-1}w = v$. And since ST is surjective, there exists $p \in V$ such that $(ST)(p) = w$. It follows that $(STT^{-1})(p) = T^{-1}(w)$, and hence $Sp = v$. Thus S is surjective, completing the proof. ☐

Problem 11

Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Proof. Notice STU is invertible since $STU = I$ and I is invertible. By Problem 9, we have that $(ST)U$ is invertible if and only if ST and U are invertible. By a second application of the result, ST is invertible if and only if S and T are invertible. Thus S, T, and U are all invertible. To see that $T^{-1} = US$, notice

$$\begin{aligned}
US &= (T^{-1}T)US \\
&= T^{-1}(S^{-1}S)TUS \\
&= T^{-1}S^{-1}(STU)S \\
&= T^{-1}S^{-1}S \\
&= T^{-1},
\end{aligned}$$

as desired. $\square$

Problem 13

Suppose V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove that S is injective.

Proof. Since V is finite-dimensional and RST is surjective, RST is also invertible. By Problem 9, we have that $(RS)T$ is invertible if and only if RS and T are invertible. By a second application of the result, RS is invertible if and only if R and S are invertible. Thus R, S, and T are all invertible, and hence injective. In particular, S is injective, as desired. $\square$

Problem 15

Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an m-by-n matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$.

Proof. Endow both $\mathbb{F}^{n,1}$ and $\mathbb{F}^{m,1}$ with the standard basis, and let $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$. Let $A = \mathcal{M}(T)$ with respect to these bases, and note that if $x \in \mathbb{F}^{n,1}$, then $\mathcal{M}(x) = x$ (and similarly if $y \in \mathbb{F}^{m,1}$, then $\mathcal{M}(y) = y$). Hence

$$\begin{aligned}
Tx &= \mathcal{M}(Tx) \\
&= \mathcal{M}(T)\mathcal{M}(x) \\
&= Ax,
\end{aligned}$$

as desired. $\square$

Problem 16

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Proof. ($\Rightarrow$) Suppose $T = \lambda I$ for some $\lambda \in \mathbb{F}$, and let $S \in \mathcal{L}(V)$ be arbitrary. For any $v \in V$, we have $STv = S(\lambda I)v = \lambda Sv$ and $TSv = (\lambda I)Sv = \lambda Sv$, and hence $ST = TS$. Since S was arbitrary, we have the desired result.

($\Leftarrow$) Suppose $ST = TS$ for every $S \in \mathcal{L}(V)$, and let $v \in V$ be arbitrary. Consider the list v, Tv. We claim it is linearly dependent. To see this, suppose not. Then v, Tv can be extended to a basis $v, Tv, u_1, \ldots, u_n$ of V. Define $S \in \mathcal{L}(V)$ by

$$S(\alpha v + \beta Tv + \gamma_1 u_1 + \cdots + \gamma_n u_n) = \beta v,$$

where $\alpha, \beta, \gamma_1, \ldots, \gamma_n$ are the unique coefficients of our basis for the given input vector. In particular, notice $S(Tv) = v$ and $Sv = 0$. Thus $STv = TSv$ implies $v = T(0) = 0$, contradicting our assumption that v, TV is linearly independent. So v, Tv must be linearly dependent, and so for for all $v \in V$ there exists $\lambda_v \in \mathbb{F}$ such that $Tv = \lambda_v v$ (where λ_0 can be any nonzero element of $\mathbb{F}$, since $T0 = 0$). We claim λ_v is independent of the choice of v for $v \in V - \{0\}$, hence $Tv = \lambda v$ for all $v \in V$ (including $v = 0$) and some $\lambda \in \mathbb{F}$, and thus $T = \lambda I$.

So suppose $w, z \in V - \{0\}$ are arbitrary. We want to show $\lambda_w = \lambda_z$. If w and z are linearly dependent, then there exists $\alpha \in \mathbb{F}$ such that $w = \alpha z$. It follows

$$\begin{aligned} \lambda_w w &= Tw \\ &= T(\alpha z) \\ &= \alpha T z \\ &= \alpha \lambda_z z \\ &= \lambda_z(\alpha z) \\ &= \lambda_z w. \end{aligned}$$

Since $w \neq 0$, this implies $\lambda_w = \lambda_z$. Next suppose w and z are linearly independent. Then we have

$$\begin{aligned} \lambda_{w+z}(w+z) &= T(w+z) \\ &= Tw + Tz \\ &= \lambda_w w + \lambda_z z, \end{aligned}$$

and hence

$$(\lambda_{w+z} - \lambda_w)w + (\lambda_{w+z} - \lambda_z)z = 0.$$

Since w and z are assumed to be linearly independent, we have $\lambda_{w+z} = \lambda_w$ and $\lambda_{w+z} = \lambda_z$, and hence again we have $\lambda_w = \lambda_z$, completing the proof. $\square$

Problem 17

Suppose V is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$.

Proof. If $\mathcal{E} = \{0\}$, we're done. So suppose $\mathcal{E} \neq \{0\}$. If $\dim V = n$, then $\mathcal{L}(V) \cong \mathbb{F}^{n,n}$, and so there exists an isomorphic subspace $\mathfrak{E} := \mathcal{M}(\mathcal{E}) \subseteq \mathbb{F}^{n,n}$ with the property that $AB \in \mathfrak{E}$ and $BA \in \mathfrak{E}$ for all $A \in \mathbb{F}^{n,n}$ and all $B \in \mathfrak{E}$. It suffices to show $\mathfrak{E} = \mathbb{F}^{n,n}$.

Define $E^{i,j}$ to be the matrix which is 1 in row i and column j and 0 everywhere else, and let $A \in \mathbb{F}^{n,n}$ be nonzero. Then there exists some $1 \leq j, k \leq n$ such that $A_{j,k} \neq 0$. Notice for $1 \leq i, j, r, s \leq n$, we have $E^{i,j}A \in \mathfrak{E}$, and hence $E^{i,j}AE^{r,s} \in \mathfrak{E}$. This product has the form

$$E^{i,j}AE^{k,\ell} = A_{j,k} \cdot E^{i,\ell}.$$

In other words, $E^{i,j}AE^{k,\ell}$ takes $A_{j,k}$ and puts it in the i^{th} row and ℓ^{th} column of a matrix which is 0 everywhere else. Since $\mathfrak{E}$ is closed under addition, this implies

$$E^{1,j}AE^{k,1} + E^{2,j}AE^{k,2} + \cdots + E^{n,j}AE^{k,n} = A_{j,k} \cdot I \in \mathfrak{E}.$$

But since $\mathfrak{E}$ is closed under scalar multiplication, and $A_{j,k} \neq 0$, we have

$$\left(\frac{1}{A_{j,k}} \cdot A_{j,k}\right) \cdot I = I \in \mathfrak{E}.$$

Since $\mathfrak{E}$ contains I, by our characterization of $\mathfrak{E}$ it must also contain every element of $\mathbb{F}^{n,n}$. Thus $\mathfrak{E} = \mathbb{F}^{n,n}$, and since $\mathfrak{E} \cong \mathcal{E}$, we must have $\mathcal{E} = \mathcal{L}(V)$, as desired. $\square$

Problem 19

Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

(a) Prove that T is surjective.

(b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$.

Proof. (a) Let $q \in \mathcal{P}(\mathbb{R})$, and suppose $\deg q = n$. Let $T_n = T\mid_{\mathcal{P}_n(\mathbb{R})}$, so that T_n is the restriction of T to a linear operator on $\mathcal{P}_n(\mathbb{R})$. Since T is injective, so is T_n. And since T_n is an injective linear operator over a finite-dimensional vector space, T_n is surjective as well. Thus there exists $r \in \mathcal{P}_n(\mathbb{R})$ such that $T_n r = q$, and so we have $Tr = q$ as well. Therefore T is surjective.

(b) We induct on the degree of the restriction maps $T_n \in \mathcal{L}(\mathcal{P}_n(\mathbb{R}))$, each of which is bijective by (a). Let $P(k)$ be the statement: $\deg T_k p = k$ for every nonzero $p \in \mathcal{P}_k(\mathbb{R})$.
Base case: Suppose $p \in \mathcal{P}_0(\mathbb{R})$ is nonzero. Since T_0 is a bijective , $T_0 p = 0$ iff $p = 0$ (the zero polynomial), which is the only polynomial with degree

< 0. Since p is nonzero by hypothesis, we must have $\deg T_0 p = 0$. Hence $P(0)$ is true.
Inductive step: Let $n \in \mathbb{Z}^+$, and suppose $P(k)$ is true for all $0 \leq k < n$. Let $p \in \mathcal{P}_n(\mathbb{R})$ be nonzero. If $\deg T_n p < n$, then for some $k < n$ there exists $q \in \mathcal{P}_k(R)$ and $T_k \in \mathcal{P}(\mathbb{R})$ such that $T_k q = p$ (since T_k is surjective). Hence $Tq = Tp$, a contradiction since $\deg p \neq \deg q$ and T is injective. Thus we must have $\deg T_n p = n$, and $P(n)$ is true.

By the principle of mathematical induction, $P(k)$ is true for all $k \in \mathbb{Z}_{\geq 0}$. Hence $\deg Tp = \deg p$ for all nonzero $p \in \mathcal{P}(\mathbb{R})$, since $Tp = T_k p$ for $k = \deg p$. □

E: Products and Quotients of Vector Spaces

Problem 1

Suppose T is a function from V to W. The **graph** of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W \mid v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Proof. Define $G := \{(v, Tv) \in V \times W \mid v \in V\}$.

($\Rightarrow$) Suppose T is a linear map. Since T is linear, $T0 = 0$, and hence $(0, 0) \in G$, so that G contains the additive identity. Next, let $(v_1, Tv_1), (v_2, Tv_2) \in G$. Then

$$(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2) = (v_1 + v_2, T(v_1 + v_2)) \in G,$$

and G is closed under addition. Lastly, let $\lambda \in \mathbb{F}$ and $(v, Tv) \in G$. It follows

$$\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v)) \in G,$$

and G is closed under scalar multiplication. Thus G is a subspace of $V \times W$.

($\Leftarrow$) Suppose G is a subspace of $V \times W$, and let $(v_1, Tv_1), (v_2, Tv_2) \in G$. Since G is closed under addition, it follows

$$(v_1 + v_2, Tv_1 + Tv_2) \in G,$$

and hence we must have $Tv_1 + Tv_2 = T(v_1 + v_2)$, so that T is additive. And since G is closed under scalar multiplication, for $\lambda \in \mathbb{F}$ and $(v, Tv) \in G$, it follows

$$(\lambda v, \lambda Tv) \in G,$$

and hence we must have $\lambda Tv = T(\lambda v)$, so that T is homogeneous. Therefore, T is a linear map, as desired. □

Problem 3

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Proof. Define the following two subspaces of $\mathcal{P}(\mathbb{R})$

$$\begin{aligned} U_1 &:= \mathcal{P}(\mathbb{R}) \\ U_2 &:= \mathbb{R}, \end{aligned}$$

so that $U_1 \cap U_2 = \mathbb{R}$ and the sum $U_1 + U_2 = \mathcal{P}(\mathbb{R})$ is not direct. Endow $\mathcal{P}(\mathbb{R})$ and $\mathbb{R}$ with their standard bases, and define φ by its behavior on the basis of $U_1 \times U_2$

$$\begin{aligned} \varphi : U_1 \times U_2 &\to U_1 + U_2 \\ \left(X^k, 0\right) &\mapsto X^{k+1} \\ (0, 1) &\mapsto 1. \end{aligned}$$

We claim φ is an isomorphism. To see that φ is injective, suppose

$$\left(a_0 + a_1 X + \cdots + a_m X^m, \alpha\right), \left(b_0 + b_1 X + \cdots + b_n X^n, \beta\right) \in U_1 \times U_2$$

and

$$\left(a_0 + a_1 X + \cdots + a_m X^m, \alpha\right) \neq \left(b_0 + b_1 X + \cdots + b_n X^n, \beta\right).$$

We have

$$\varphi\left(a_0 + a_1 X + \cdots + a_m X^m, \alpha\right) = \alpha + a_0 X + a_1 X^2 + \cdots + a_m X^{m+1} \tag{5}$$

and

$$\varphi\left(b_0 + b_1 X + \cdots + b_n X^n, \beta\right) = \beta + b_0 X + b_1 X^2 + \cdots + b_n X^{n+1}. \tag{6}$$

Since $\alpha \neq \beta$, this implies (5) does not equal (6) and hence φ is injective. To see that φ is surjective, suppose $c_0 + c_1 X + \cdots + c_p X^p \in U_1 + U_2$. Then

$$\varphi\left(c_1 + c_2 X + \cdots + c_p X^{p-1}, c_0\right) = c_0 + c_1 X + \cdots + c_p X^p$$

and φ is indeed surjective.

Since φ an injective and surjective linear map, it is an isomorphism. Thus $U_1 \times U_2 \cong U_1 + U_2$, as was to be shown. □

Problem 5

Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

Proof. Define the projection map π_k for $k = 1, \dots, m$ by

$$\begin{aligned} \pi_k : W_1 \times \cdots \times W_m &\to W_k \\ (w_1, \dots, w_m) &\mapsto w_k. \end{aligned}$$

Clearly π_k is linear. Now define

$$\begin{aligned} \varphi : \mathcal{L}(V, W_1 \times \cdots \times W_m) &\to \mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m) \\ T &\mapsto (\pi_1 T, \dots, \pi_m T). \end{aligned}$$

To see that φ is linear, let $T_1, T_2 \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$. It follows

$$\begin{aligned} \varphi(T_1 + T_2) &= (\pi_1(T_1 + T_2), \dots, \pi_m(T_1 + T_2)) \\ &= (\pi_1 T_1 + \pi_1 T_2, \dots, \pi_m T_1 + \pi_m T_2) \\ &= (\pi_1 T_1, \dots, \pi_m T_1) + (\pi_1 T_2, \dots, \pi_m T_2) \\ &= \varphi(T_1) + \varphi(T_2), \end{aligned}$$

and hence φ is additive. Now for $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$, we have

$$\begin{aligned} \varphi(\lambda T) &= (\pi_1(\lambda T), \dots, \pi_m(\lambda T)) \\ &= (\lambda(\pi_1 T), \dots, \lambda(\pi_m T)) \\ &= \lambda(\pi_1 T, \dots, \pi_m T), \end{aligned}$$

and thus φ is homogenous. Therefore, φ is linear.

We now show φ is an isomorphism. To see that it is injective, suppose $T \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\varphi(T) = 0$. Then

$$(\pi_1 T, \dots, \pi_m T) = (0, \dots, 0)$$

which is true iff T is the zero map. Thus φ is injective. To see that φ is surjective, suppose $(S_1, \dots, S_m) \in \mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$. Define

$$\begin{aligned} S : V &\to W_1 \times \cdots \times W_m \\ v &\mapsto (S_1 v, \dots, S_m v), \end{aligned}$$

so that $\varphi_k S = S_k$ for $k = 1, \dots, m$. Then

$$\begin{aligned} \varphi(S) &= (\pi_1 S, \dots, \pi_m S) \\ &= (S_1, \dots, S_n) \end{aligned}$$

and S is indeed surjective. Therefore, φ is an isomorphism, and we have

$$\mathcal{L}(V, W_1 \times \cdots \times W_m) \cong \mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m),$$

as desired. $\square$

Problem 7

Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Proof. First note that since $v + 0 = v \in v + U$, there exists $w_0 \in W$ such that $v = x + w_0$, and hence $v - x = w_0 \in W$. Similarly, there exists $u_0 \in U$ such that $x - v = u_0 \in U$.

Suppose $u \in U$. Then there exists $w \in W$ such that $v + u = x + w$, and hence

$$u = (x - v) + w = -w_0 + w \in W,$$

and we have $U \subseteq W$. Conversely, suppose $w' \in W$. Then there exists $u' \in U$ such that $x + w' = v + u'$, and hence

$$w' = (v - x) + u' = -u_0 + u' \in U,$$

and we have $W \subseteq U$. Therefore $U = W$, as desired. □

Problem 8

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.

Proof. ($\Rightarrow$) Suppose $A \subseteq V$ is an affine subset of V. Then there exists $x \in V$ and a subspace $U \subseteq V$ such that $A = x + U$. Suppose $v, w \in A$. Then there exist $u_1, u_2 \in U$ such that $v = x + u_1$ and $w = x + u_2$. Thus, for all $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} \lambda v + (1 - \lambda)w &= \lambda(x + u_1) + (1 - \lambda)(x + u_2) \\ &= x + \lambda u_1 + (1 - \lambda)u_2. \end{aligned}$$

Since $\lambda u_1 + (1 - \lambda)u_2 \in U$, this implies $v + (1 - \lambda)w \in x + U = A$, as desired.

($\Leftarrow$) Suppose $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$. Choose $a \in A$ and define

$$U := -a + A.$$

We claim U is a subspace of V. Clearly $0 \in U$ since $a \in A$. Let $x \in U$, so that $x = -a + a_0$ for some $a_0 \in A$, and let $\lambda \in \mathbb{F}$. It follows

$$\lambda a_0 + (1 - \lambda)a \in A \Rightarrow -\lambda a + \lambda a_0 + a \in A \Rightarrow \lambda(-a + a_0) \in -a + A = U,$$

and thus $\lambda x = \lambda(-a + a_0) \in U$, and U is closed under scalar multiplication. Now let $x, y \in U$. Then there exist $a_1, a_2 \in A$ such that $x = -a + a_1$ and $y = -a + a_2$. Notice

$$\frac{1}{2}a_1 + \left(1 - \frac{1}{2}\right)a_2 = \frac{1}{2}a_1 + \frac{1}{2}a_2 \in A,$$

and hence

$$-a + \frac{1}{2}a_1 + \frac{1}{2}a_2 \in U.$$

It follows

$$\begin{aligned} x + y &= -2a + a_1 + a_2 \\ &= 2\left(-a + \frac{1}{2}a_1 + \frac{1}{2}a_2\right) \in U, \end{aligned}$$

using the fact that U has already been shown to be closed under scalar multiplication. Thus U is also closed under addition, and so U is a subspace of V. Now, since $A = a + U$, we have that A is indeed an affine subset of V, as desired. $\square$

Problem 9

Suppose A_1 and A_2 are affine subsets of V. Prove that the intersection $A_1 \cap A_2$ is either an affine subset of V or the empty set.

Proof. If $A_1 \cap A_2 = \emptyset$, we're done, so suppose $A_1 \cap A_2$ is nonempty and let $v \in A_1 \cap A_2$. Then we may write

$$A_1 = v + U_1 \quad \text{and} \quad A_2 = v + U_2$$

for some subspaces $U_1, U_2 \subseteq V$.

We claim $A_1 \cap A_2 = v + (U_1 \cap U_2)$, which is an affine subset of V. To see this, suppose $x \in v + (U_1 \cap U_2)$. Then there exists $u \in U_1 \cap U_2$ such that $x = v + u$. Since $u \in U_1$, we have $x \in v + U_1 = A_1$. And since $u \in U_2$, we have $x \in v + U_2 = A_2$. Thus $x \in A_1 \cap A_2$ and $v + (U_1 \cap U_2) \subseteq A_1 \cap A_2$. Conversely, suppose $y \in A_1 \cap A_2$. Then there exist $u_1 \in U_1$ and $u_2 \in U_2$ such that $y = v + u_1$ and $y = v + u_2$. But this implies $u_1 = u_2$, and hence $u_1 = u_2 \in U_1 \cap U_2$, thus $y \in v + (U_1 \cap U_2)$. Therefore $A_1 \cap A_2 \subseteq v + (U_1 \cap U_2)$, and hence we have $A_1 \cap A_2 = v + (U_1 \cap U_2)$, as claimed. $\square$

Problem 11

Suppose $v_1, \ldots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{F} \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}.$$

(a) Prove that A is an affine subset of V.

(b) Prove that every affine subset of V that contains $v_1, \ldots, v_m$ also contains A.

(c) Prove that $A = v + U$ for some $v \in V$ and some subspace U of V with $\dim U \leq m - 1$.

Proof. (a) Let $v, w \in A$, so that there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ and $\beta_1, \ldots, \beta_m \in \mathbb{F}$ such that

$$v = \alpha_1 v_1 + \cdots + \alpha_m v_m$$
$$w = \beta_1 v_1 + \cdots + \beta_m v_m,$$

where $\sum \alpha_k = 1$ and $\sum \beta_k = 1$. Given $\lambda \in \mathbb{F}$, it follows

$$\begin{aligned} \lambda v + (1-\lambda) w &= \lambda \sum_{k=1}^{m} \alpha_k v_k + (1-\lambda) \sum_{k=1}^{m} \beta_k v_k \\ &= \sum_{k=1}^{m} \left[\lambda \alpha_k + (1-\lambda)\beta_k\right] v_k. \end{aligned}$$

But notice

$$\sum_{k=1}^{m} \left[\lambda \alpha_k + (1-\lambda)\beta_k\right] = \lambda + (1-\lambda) = 1,$$

and hence $\lambda v + (1-\lambda) w \in A$ by the way we defined A. By Problem 8, this implies that A is an affine subset of V, as was to be shown.

(b) We induct on m.
Base case: When $m = 1$, the statement is trivially true, since $A = \{v_1\}$, and hence any affine subset of V that contains v_1 of course contains A.
Inductive step: Let $k \in \mathbb{Z}^+$, and suppose the statement is true for $m = k$. Suppose A' is an affine subset of V that contains $v_1, \ldots, v_{k+1}$, and let $x \in A$. Then there exist $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{F}$ such that $\sum_j \lambda_j = 1$ and

$$x = \lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1}.$$

Now, if $\lambda_{k+1} = 1$, then $x = v_{k+1} \in A'$. Otherwise, we have

$$\frac{\lambda_1}{1-\lambda_{k+1}} + \cdots + \frac{\lambda_k}{1-\lambda_{k+1}} = 1,$$

and hence by our inductive hypothesis, this implies

$$\frac{\lambda_1}{1-\lambda_{k+1}} v_1 + \cdots + \frac{\lambda_k}{1-\lambda_{k+1}} v_k \in A'.$$

By Problem 8, we know

$$(1-\lambda_{k+1}) \left(\frac{\lambda_1}{1-\lambda_{k+1}} v_1 + \cdots + \frac{\lambda_k}{1-\lambda_{k+1}} v_k \right) + \lambda_{k+1} v_{k+1} \in A'.$$

But after simplifying, this tells us

$$\lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1} = x \in A'.$$

Hence $A \subseteq A'$, and the statement is true for $m = k + 1$.

By the principal of mathematical induction, the statement is true for all $m \in \mathbb{Z}^+$. Thus any affine subset of V that contains $v_1, \ldots, v_m$ also contains A, as was to be shown.

(c) Define $U := \operatorname{span}(v_2 - v_1, \ldots, v_m - v_1)$. Let $x \in A$, so that there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ with $\sum_k \lambda_k = 1$ such that

$$x = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Notice

$$\begin{aligned} v_1 + \lambda_2(v_2 - v_1) + \cdots + \lambda_m(v_m - v_1) &= \left(1 - \sum_{k=2}^{m} \lambda_k\right) v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \\ &= \lambda_1 v_1 + \cdots + \lambda_m v_m \\ &= x, \end{aligned}$$

and hence $x \in v_1 + U$, so that $A \subseteq v_1 + U$. Next suppose $y \in v_1 + U$, so that there exist $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{F}$ such that

$$y = v_1 + \alpha_1(v_2 - v_1) + \cdots + \alpha_{m-1}(v_m - v_1).$$

Expanding the RHS yields

$$y = \left(1 - \sum_{k=1}^{m-1} \alpha_k\right) v_1 + \alpha_1 v_2 + \cdots + \alpha_{m-1} v_m.$$

But since

$$\left(1 - \sum_{k=1}^{m-1} \alpha_k\right) + \sum_{k=1}^{m-1} \alpha_k = 1,$$

this implies $y \in A$, and hence $v_1 + U \subseteq A$. Therefore $A = v_1 + U$, and since $\dim U \leq m - 1$, we have the desired result. □

Problem 13

Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V.

Proof. Since

$$\begin{aligned} \dim V &= \dim V/U + \dim U \\ &= m + n, \end{aligned}$$

it suffices to show $v_1, \dots, v_m, u_1, \dots, u_n$ spans V. Suppose $v \in V$. Then there exist $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ such that

$$v + U = \alpha_1(v_1 + U) + \cdots + \alpha_m(v_m + U).$$

But then

$$v + U = (\alpha_1 v_1 + \cdots + \alpha_m v_m) + U$$

and hence

$$v - (\alpha_1 v_1 + \cdots + \alpha_m v_m) \in U.$$

Thus there exist $\beta_1, \dots, \beta_n \in U$ such that

$$v - (\alpha_1 v_1 + \cdots + \alpha_m v_m) = \beta_1 u_1 + \cdots + \beta_n u_n,$$

and we have

$$v = \alpha_1 v_1 + \cdots + \alpha_m v_m + \beta_1 u_1 + \cdots + \beta_n u_n,$$

so that indeed $v_1, \dots, v_m, u_1, \dots, u_n$ spans V. □

Problem 15

Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$. Prove that $\dim V/(\operatorname{null} \varphi) = 1$.

Proof. Since $\varphi \neq 0$, we must have $\dim \operatorname{range} \varphi = 1$, so that $\operatorname{range} \varphi = \mathbb{F}$. Since $V/(\operatorname{null} \varphi) \cong \operatorname{range} \varphi$, this implies $V/(\operatorname{null} \varphi) \cong \mathbb{F}$, and hence $\dim V/(\operatorname{null} \varphi) = 1$, as desired. □

Problem 17

Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that there exists a subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

Proof. Suppose $\dim V/U = n$, and let $v_1 + U, \dots, v_n + U$ be a basis of V/U. Define $W := \operatorname{span}(v_1, \dots, v_n)$. We claim $v_1, \dots, v_n$ must be linearly independent, so that $v_1, \dots, v_n$ is a basis of W. To see this, suppose $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ are such that

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

Then

$$(\alpha_1 v_1 + \cdots + \alpha_n v_n) + U = \alpha_1(v_1 + U) + \cdots + \alpha_n(v_n + U),$$

and hence we must have $\alpha_1 = \cdots = \alpha_n = 0$. Thus $v_1, \dots, v_n$ is indeed linearly independent, as claimed.

We now claim $V = U \oplus W$. To see that $V = U + W$, suppose $v \in V$. Then there exist $\beta_1, \dots, \beta_n \in \mathbb{F}$ such that

$$v + U = \beta_1(v_1 + U) + \cdots + \beta_n(v_n + U).$$

It follows

$$v - \sum_{k=1}^{n} \beta_k v_k \in U,$$

and hence

$$v = \left(v - \sum_{k=1}^{n} \beta_k v_k \right) + \left(\sum_{k=1}^{n} \beta_k v_k \right).$$

Since first term in parentheses is in U and the second term in parentheses is in W, we have $v \in U + W$, and hence $V \subseteq U + W$. Clearly $U + W \subseteq V$, since U and W are each subspaces of V, and hence $V = U + W$. To see that the sum is direct, suppose $w \in U \cap W$. Since $w \in W$, there exist $\lambda_1, \ldots, \lambda_n$ such that $w = \lambda_1 v_1 + \cdots + \lambda_n v_n$, and hence

$$\begin{aligned} w + U &= (\lambda_1 v_1 + \cdots + \lambda_n v_n) + U \\ &= \lambda_1 (v_1 + U) + \cdots + \lambda_n (v_n + U). \end{aligned}$$

Since $w \in U$, we have $w + U = 0 + U$. Thus $\lambda_1 = \cdots = \lambda_n = 0$, which implies $w = 0$. Since $U \cap W = \{0\}$, the sum is indeed direct. Thus $V = U \oplus W$, with $\dim W = n = \dim V/U$, as desired. □

Problem 19

Find a correct statement analogous to 3.78 that is applicable to finite sets, with unions analogous to sums of subspaces and disjoint unions analogous to direct sums.

Theorem. *Suppose $|V| < \infty$ and $U_1, \ldots, U_n \subseteq V$. Then $U_1, \ldots, U_n$ are pairwise disjoint if and only if*

$$|U_1 \cup \cdots \cup U_n| = |U_1| + \cdots + |U_n|.$$

Proof. We induct on n.
Base case: Let $n = 2$. Since $|U_1 \cup U_2| = |U_1| + |U_2| - |U_1 \cap U_2|$, we have $U_1 \cap U_2 = \emptyset$ iff $|U_1 \cup U_2| = |U_1| + |U_2|$.
Inductive hypothesis: Let $k \in \mathbb{Z}_{\geq 2}$, and suppose the statement is true for $n = k$. Let $U_{k+1} \subseteq V$. Then

$$|U_1 \cup \cdots \cup U_{k+1}| = |U_1 \cup \cdots \cup U_k| + |U_{k+1}|$$

iff $U_{k+1} \cap (U_1 \cup \cdots \cup U_k) = \emptyset$ by our base case. Combining this with our inductive hypothesis, we have

$$|U_1 \cup \cdots \cup U_{k+1}| = |U_1| + \cdots + |U_k| + |U_{k+1}|$$

iff $U_1, \ldots, U_{k+1}$ are pairwise disjoint, and the statement is true for $n = k + 1$.

By the principal of mathematical induction, the statement is true for all $n \in \mathbb{Z}_{\geq 2}$. □

F: Duality

Problem 1

Explain why every linear functional is either surjective or the zero map.

Proof. Since $\dim \mathbb{F} = 1$, the only subspaces of $\mathbb{F}$ are $\mathbb{F}$ itself and $\{0\}$. Let V be a vector space (not necessarily finite-dimensional) and suppose $\varphi \in V'$. Since $\operatorname{range} \varphi$ is a subspace of $\mathbb{F}$, it must be either $\mathbb{F}$ itself (in which case φ is surjective) or $\{0\}$ (in which case φ is the zero map). $\square$

Problem 3

Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Proof. Suppose $\dim U = m$ and $\dim V = n$ for some $m, n \in \mathbb{Z}^+$ such that $m < n$. Let $u_1, \ldots, u_m$ be a basis of U. Expand this to a basis $u_1, \ldots, u_m, u_{m+1}, \ldots, u_n$ of V, and let $\varphi_1, \ldots, \varphi_n$ be the corresponding dual basis of V'. For any $u \in U$, there exist $\alpha_1, \ldots, \alpha_m$ such that $u = \alpha_1 u_1 + \cdots + \alpha_m u_m$. Now notice

$$\begin{aligned}
\varphi_{m+1}(u) &= \varphi_{m+1}(\alpha_1 u_1 + \cdots + \alpha_m u_m) \\
&= \alpha_1 \varphi_{m+1}(u_1) + \cdots + \alpha_m \varphi_{m+1}(u_m) \\
&= 0,
\end{aligned}$$

but $\varphi_{m+1}(u_{m+1}) = 1$. Thus $\varphi_{m+1}(u) = 0$ for every $u \in U$ but $\varphi_{m+1} \neq 0$, as desired. $\square$

Problem 5

Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

Proof. For $i = 1, \ldots, m$, let

$$\begin{aligned}
\xi_i : V_i &\to V_1 \times \cdots \times V_m \\
v_i &\mapsto (0, \ldots, v_i, \ldots, 0).
\end{aligned}$$

Now define

$$\begin{aligned}
T : (V_1 \times \cdots \times V_m)' &\to V_1' \times \cdots \times V_m' \\
\varphi &\mapsto (\varphi \circ \xi_1, \ldots, \varphi \circ \xi_m).
\end{aligned}$$

We claim T is an isomorphism. We must show three things: (1) that T is a linear map; (2) that T is injective; and (3) that T is surjective.

To see that T is a linear map, first suppose $\varphi_1, \varphi_2 \in (V_1 \times \cdots \times V_m)'$. It follows

$$\begin{aligned} T(\varphi_1 + \varphi_2) &= \big((\varphi_1 + \varphi_2) \circ \xi_1, \ldots, (\varphi_1 + \varphi_2) \circ \xi_m\big) \\ &= (\varphi_1 \circ \xi_1 + \varphi_2 \circ \xi_1, \ldots, \varphi_1 \circ \xi_m + \varphi_2 \circ \xi_m) \\ &= (\varphi_1 \circ \xi_1, \ldots, \varphi_1 \circ \xi_m) + (\varphi_2 \circ \xi_1, \ldots, \varphi_2 \circ \xi_m) \\ &= T(\varphi_1) + T(\varphi_2), \end{aligned}$$

thus T is additive. To see that it is also homogeneous, suppose $\lambda \in \mathbb{F}$ and $\varphi \in (V_1 \times \cdots \times V_m)'$. We have

$$\begin{aligned} T(\lambda\varphi) &= \big((\lambda\varphi) \circ \xi_1, \ldots, (\lambda\varphi) \circ \xi_m\big) \\ &= \big(\lambda(\varphi \circ \xi_1), \ldots, \lambda(\varphi \circ \xi_m)\big) \\ &= \lambda\,(\varphi \circ \xi_1, \ldots, \varphi \circ \xi_m) \\ &= \lambda T(\varphi), \end{aligned}$$

and thus T is homogeneous as well and therefore it is a linear map.

To see that T is injective, suppose $\varphi, \psi \in (V_1 \times \cdots \times V_m)'$ but $\varphi \neq \psi$. Then there exists some $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ such that $\varphi(v_1, \ldots, v_m) \neq \psi(v_1, \ldots, v_m)$. Since φ and ψ are linear, this means that there exists some index $k \in \{1, \ldots, m\}$ such that $\varphi(0, \ldots, v_k, \ldots, 0) \neq \psi(0, \ldots, v_k, \ldots, 0)$. But then $\varphi \circ \xi_k \neq \psi \circ \xi_k$, and hence $T(\varphi) \neq T(\psi)$, so that T is injective.

To see that T is surjective, suppose $(\varphi_1, \ldots, \varphi_m) \in V_1' \times \cdots \times V_m'$ and define

$$\begin{aligned} \theta : V_1 \times \cdots \times V_m &\to \mathbb{F} \\ (v_1, \ldots, v_m) &\mapsto \sum_{k=1}^{m} \varphi_k(v_k). \end{aligned}$$

We claim $T(\theta) = (\varphi_1, \ldots, \varphi_m)$. To see this, let $k \in \{1, \ldots, m\}$. We will show that the map in the k-th component of $T(\theta)$ is equal to φ_k. Given $v_k \in V_k$, we have

$$\begin{aligned} T(\theta)_k(v_k) &= (\theta \circ \xi_k)(v_k) \\ &= \theta(\xi_k(v_k)) \\ &= \theta(0, \ldots, v_k, \ldots, 0) \\ &= \varphi_1(0) + \cdots + \varphi_k(v_k) + \cdots + \varphi_m(0) \\ &= \varphi_k(v_k), \end{aligned}$$

as desired. Thus $T(\theta) = (\varphi_1, \ldots, \varphi_m)$, and T is indeed surjective. Since T is both injective and surjective, it's an isomorphism. $\square$

Problem 7

Suppose m is a positive integer. Show that the dual basis of the basis $1, \ldots, x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

Proof. For $j = 0, \ldots, m$, we have by direct computation of the j-th derivative

$$\varphi_j\left(x^k\right) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

so that $\varphi_0, \varphi_1, \ldots, \varphi_m$ is indeed the dual basis of $1, \ldots, x^m$. Note the uniqueness of the dual basis follows by uniqueness of a linear map (including the linear functionals in the dual basis) whose values on a basis are specified. $\square$

Problem 9

Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

Proof. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ be such that

$$\psi = \alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n.$$

For $k = 1, \ldots, n$, we have

$$\begin{aligned} \psi(v_k) &= \alpha_1\varphi_1(v_k) + \cdots + \alpha_k\varphi_k(v_k) + \cdots + \alpha_n\varphi_n(v_k) \\ &= \alpha_1 \cdot 0 + \cdots + \alpha_k \cdot 1 + \cdots + \alpha_n \cdot 0 \\ &= \alpha_k. \end{aligned}$$

Thus we have

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n,$$

as desired $\square$

Problem 11

Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbb{F}^m$ and $(d_1, \ldots, d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

Proof. ($\Rightarrow$) Suppose the rank of A is 1. By the assumption that $A \neq 0$, there exists a nonzero entry $A_{i.,j}$ for some $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Thus $\operatorname{span}\{A_{\cdot,1}, \ldots, A_{\cdot,n}\} = \operatorname{span}\{A_{\cdot,j}\}$, and hence there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $A_{\cdot,c} = \alpha_c A_{\cdot,j}$ for $c = 1, \ldots, n$. Expanding out each out these columns, we have

$$A_{r,c} = \alpha_c A_{r,j} \tag{7}$$

for $r = 1, \ldots, m$. Similarly for the rows, we have $\operatorname{span}\{A_{1,\cdot}, \ldots, A_{m,\cdot}\} = \operatorname{span}\{A_{i,\cdot}\}$, and hence there exist $\beta_1, \ldots, \beta_m \in \mathbb{F}$ such that $A_{r',\cdot} = \beta_r A_{i,\cdot}$ for $r' = 1, \ldots, m$. Expanding out each of these rows, we have

$$A_{r',c'} = \beta_{r'} A_{i,c'} \tag{8}$$

for $c' = 1, \ldots, n$. Now by replacing the $A_{r,j}$ term in (7) according to (8), we have $A_{r,j} = \beta_r A_{i,j}$, and hence (7) may be rewritten

$$A_{r,c} = \alpha_c \beta_r A_{i,j},$$

and the result follows by defining $c_r = \beta_r A_{i,j}$ and $d_c = \alpha_c$ for $r = 1, \ldots, m$ and $c = 1, \ldots, n$.

($\Leftarrow$) Suppose there exist $(c_1, \ldots, c_m) \in \mathbb{F}^m$ and $(d_1, \ldots, d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$. Then each of the columns is a scalar multiple of $(d_1, \ldots, d_n)^t \in \mathbb{F}^{n,1}$ and the column rank is 1. Since the rank of a matrix equals its column rank, the rank of A is 1 as well. $\square$

Problem 13

Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbb{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbb{R}^3$.

(a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as a linear combination of ψ_1, ψ_2, ψ_3.

Proof. (a) Endowing $\mathbb{R}^3$ and $\mathbb{R}^2$ with their respective standard basis, we have

$$\begin{aligned}(T'(\varphi_1))(x, y, z) &= (\varphi_1 \circ T)(x, y, z) \\ &= \varphi_1(T(x, y, z)) \\ &= \varphi_1(4x + 5y + 6z, 7x + 8y + 9z) \\ &= 4x + 5y + 6z\end{aligned}$$

and similarly

$$\begin{aligned}(T'(\varphi_2))(x, y, z) &= \varphi_2(4x + 5y + 6z, 7x + 8y + 9z) \\ &= 7x + 8y + 9z.\end{aligned}$$

(b) Notice

$$\begin{aligned}(4\psi_1 + 5\psi_2 + 6\psi_3)(x, y, z) &= 4\psi_1(x, y, z) + 5\psi_2(x, y, z) + 6\psi_3(x, y, z)\\ &= 4x + 5y + 6z\\ &= T'(\varphi_1)(x, y, z)\end{aligned}$$

and

$$\begin{aligned}(7\psi_1 + 8\psi_2 + 9\psi_3)(x, y, z) &= 7\psi_1(x, y, z) + 8\psi_2(x, y, z) + 9\psi_3(x, y, z)\\ &= 7x + 8y + 9z\\ &= T'(\varphi_2)(x, y, z),\end{aligned}$$

as desired. $\square$

Problem 15

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Proof. ($\Rightarrow$) Suppose $T' = 0$. Let $\varphi \in W'$ and $v \in V$ be arbitrary. We have

$$0 = (T'(\varphi))(v) = \varphi(Tv).$$

Since φ is arbitrary, we must have $Tv = 0$. But now since v is arbitrary, this implies $T = 0$ as well.

($\Leftarrow$) Suppose $T = 0$. Again let $\varphi \in W'$ and $v \in V$ be arbitrary. We have

$$(T'(\varphi))(v) = \varphi(Tv) = \varphi(0) = 0,$$

and hence $T' = 0$ as well. $\square$

Problem 17

Suppose $U \subseteq V$. Explain why $U^0 = \{\varphi \in V' \mid U \subseteq \operatorname{null} \varphi\}$.

Proof. It suffices to show that, for arbitrary $\varphi \in V'$, we have $U \subseteq \operatorname{null} \varphi$ if and only if $\varphi(u) = 0$ for all $u \in U$. So suppose $U \subseteq \operatorname{null} \varphi$. Then for all $u \in U$, we have $\varphi(u) = 0$ (simply by definition of $\operatorname{null} \varphi$). Conversely, suppose $\varphi(u) = 0$ for all $u \in U$. Then if $u' \in U$, we must have $u' \in \operatorname{null} \varphi$. That is, $U \subseteq \operatorname{null} \varphi$, completing the proof. $\square$

Problem 19

Suppose V is finite-dimensional and U is a subspace of V. Show that $U = V$ if and only if $U^0 = \{0\}$.

Proof. ($\Rightarrow$) Suppose $U = V$. Then

$$\begin{aligned} U^0 &= \{\varphi \in V' \mid U \subseteq \operatorname{null} \varphi\} \\ &= \{\varphi \in V' \mid V \subseteq \operatorname{null} \varphi\} \\ &= \{0\}, \end{aligned}$$

since only the zero functional can have all of V in its null space.

($\Leftarrow$) Suppose $U^0 = \{0\}$. It follows

$$\begin{aligned} \dim V &= \dim U + \dim U^0 \\ &= \dim U + 0 \\ &= \dim U. \end{aligned}$$

Since the only subspace of V with dimension $\dim V$ is V itself, we have $U = V$, as desired. □

Problem 20

Suppose U and W are subsets of V with $U \subseteq W$. Prove that $W^0 \subseteq U^0$.

Proof. Suppose $\varphi \in W^0$. Then $\varphi(w) = 0$ for all $w \in W$. If $\varphi \notin U^0$, then there exists some $u \in U$ such that $\varphi(u) \neq 0$. But since $U \subseteq W$, $u \in W$. This is absurd, hence we must have $\varphi \in U^0$. Thus $W^0 \subseteq U^0$, as desired. □

Problem 21

Suppose V is finite-dimensional and U and W are subspaces of V with $W^0 \subseteq U^0$. Prove that $U \subseteq W$.

Proof. Suppose not. Then there exists a nonzero vector $u \in U$ such that $u \notin W$. There exists some basis of U containing u. Define $\varphi \in V'$ such that, for any vector v in this basis, we have

$$\varphi(v) = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\varphi \in W^0$, and hence $\varphi \in U^0$. But this implies $\varphi(u) = 0$, a contradiction. □

Problem 22

Suppose U, W are subspaces of V. Show that $(U + W)^0 = U^0 \cap W^0$.

Proof. Since $U \subseteq U+W$ and $W \subseteq U+W$, Problem 20 tells us that $(U+W)^0 \subseteq U^0$ and $(U+W)^0 \subseteq W^0$. Thus $(U+W)^0 \subseteq U^0 \cap W^0$. Conversely, suppose $\varphi \in U^0 \cap W^0$. Let $x \in U+W$. Then there exist $u \in U$ and $w \in W$ such that $x = U + W$. Then

$$\begin{aligned}\varphi(x) &= \varphi(u+w) \\ &= \varphi(u) + \varphi(w) \\ &= 0,\end{aligned}$$

where the second equality follows since $\varphi \in U^0$ and $\varphi \in W^0$ by assumption. Hence $\varphi \in (U+W)^0$ and we have $U^0 + W^0 \subseteq (U+W)^0$. Thus $(U+W)^0 = U^0 \cap W^0$, as desired. ☐

Problem 23

Suppose V is finite-dimensional and U and W are subspaces of V. Prove that $(U \cap W)^0 = U^0 + W^0$.

Proof. Since $U \cap W \subseteq U$ and $U \cap W \subseteq W$, Problem 20 tells us that $U^0 \subseteq (U \cap W)^0$ and $W^0 \subseteq (U \cap W)^0$. Thus $U^0 + W^0 \subseteq (U \cap W)^0$. Now, notice (using Problem 22 to deduce the second equality)

$$\begin{aligned}\dim(U^0 + W^0) &= \dim(U^0) + \dim(W^0) - \dim(U^0 \cap W^0) \\ &= \dim(U^0) + \dim(W^0) - \dim((U+W)^0) \\ &= (\dim V - \dim U) + (\dim V - \dim W) - [\dim V - \dim(U+W)] \\ &= \dim V - \dim U - \dim W + \dim(U+W) \\ &= \dim V - [\dim U + \dim W - \dim(U+W)] \\ &= \dim V - \dim(U \cap W) \\ &= \dim((U \cap W)^0).\end{aligned}$$

Hence we must have $U^0 + W^0 = (U \cap W)^0$, as desired. ☐

Problem 25

Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V \mid \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

Proof. Let $A = \{v \in V \mid \varphi(v) = 0 \text{ for every } \varphi \in U^0\}$. Suppose $u \in U$. Then $\varphi(u) = 0$ for all $\varphi \in U^0$, and hence $u \in A$, showing $U \subseteq A$.

Conversely, suppose $v \in A$ but $v \notin U$. Since $0 \in U$, we must have $v \neq 0$. Thus there exists a basis $u_1, \ldots, u_m, v, v_1, \ldots, v_n$ of V such that $u_1, \ldots, u_m$ is a basis of U. Let $\psi_1, \ldots, \psi_m, \varphi, \varphi_1, \ldots, \varphi_n$ be the dual basis of V', and consider for a moment the functional φ. Clearly we have both $\varphi \in U^0$ and $\varphi(v) = 1$ by construction, but this is a contradiction, since we assumed $v \in A$. Thus $A \subseteq U$, and we conclude $U = A$, as was to be shown. $\square$

Problem 27

Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_5(\mathbb{R}))$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbb{R})$ defined by $\varphi(p) = p(8)$. Prove that $\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbb{R}) \mid p(8) = 0\}$.

Proof. By Theorem 3.107, we know $\operatorname{null} T' = (\operatorname{range} T)^0$, and hence $(\operatorname{range} T)^0 = \{\alpha\varphi \mid \alpha \in \mathbb{R}\}$. It follows by Problem 25

$$\begin{aligned}
\operatorname{range} T &= \{p \in \mathcal{P}_5(\mathbb{R}) \mid \psi(p) = 0 \text{ for all } \psi \in (\operatorname{range} T)^0\} \\
&= \{p \in \mathcal{P}_5(\mathbb{R}) \mid (\alpha\varphi)(p) = 0 \text{ for all } \alpha \in \mathbb{R}\} \\
&= \{p \in \mathcal{P}_5(\mathbb{R}) \mid \varphi(p) = 0\} \\
&= \{p \in \mathcal{P}_5(\mathbb{R}) \mid p(8) = 0\},
\end{aligned}$$

as desired. $\square$

Problem 29

Suppose V and W are finite-dimensional, $T \in \mathcal{L}(V, W)$, and there exists $\varphi \in V'$ such that $\operatorname{range} T' = \operatorname{span}(\varphi)$. Prove that $\operatorname{null} T = \operatorname{null} \varphi$.

Proof. By Theorem 3.107, we know $\operatorname{range} T' = (\operatorname{null} T)^0$, and hence $(\operatorname{null} T)^0 = \{\alpha\varphi \mid \alpha \in \mathbb{R}\}$. It follows by Problem 25

$$\begin{aligned}
\operatorname{null} T &= \{v \in V \mid \psi(v) = 0 \text{ for all } \psi \in (\operatorname{null} T)^0\} \\
&= \{v \in V \mid \alpha\varphi(v) = 0 \text{ for all } \alpha \in \mathbb{F}\} \\
&= \{v \in V \mid \varphi(v) = 0\} \\
&= \operatorname{null} \varphi,
\end{aligned}$$

as desired. $\square$

Problem 31

Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$.

Proof. To prove this, we will first show $V \cong V''$. We will then take an existing basis of V', map it to its dual basis in V'', and then use the inverse of the isomorphism to take this basis of V'' to a basis in V. This basis of V will have the known basis of V' as its dual.

So, for any $v \in V$, define $E_v \in V''$ by $E_v(\varphi) = \varphi(v)$. We claim the map $\hat{\cdot} : V \to V''$ given by $\hat{v} = E_v$ is an isomorphism. To do so, it suffices to show it to be both linear and injective, since $\dim(V'') = \dim((V')') = \dim(V') = \dim(V)$.

We first show $\hat{\cdot}$ is linear. So suppose $u, v \in V$. Then for any $\varphi \in V'$, we have

$$\begin{aligned}
(\widehat{u+v})(\varphi) &= E_{u+v}(\varphi) \\
&= \varphi(u+v) \\
&= \varphi(u) + \varphi(v) \\
&= E_u(\varphi) + E_v(\varphi) \\
&= \hat{u}(\varphi) + \hat{v}(\varphi)
\end{aligned}$$

so that $\hat{\cdot}$ is indeed linear. Next we show it to be homogeneous. So suppose $\lambda \in \mathbb{F}$, and again let $v \in V$. Then for any $\varphi \in V'$, we have

$$\begin{aligned}
(\widehat{\lambda v})(\varphi) &= E_{\lambda v}(\varphi) \\
&= \varphi(\lambda v) \\
&= \lambda \varphi(v) \\
&= \lambda E_v(\varphi) \\
&= \lambda \hat{v},
\end{aligned}$$

so that $\hat{\cdot}$ is homogenous as well. Being both linear and homogenous, it is a linear map.

Next we show $\hat{\cdot}$ is injective. So suppose $\hat{v} = 0$ for some $v \in V$. We want to show $v = 0$. Let $v_1, \ldots, v_n$ be a basis of V. Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Then, for all $\varphi \in V'$, we have

$$\begin{aligned}
\hat{v} = 0 &\implies (\alpha_1 v_1 + \cdots + \alpha_n v_n)^{\wedge} = 0 \\
&\implies \alpha_1 \widehat{v_1} + \cdots + \alpha_n \widehat{v_n} = 0 \\
&\implies (\alpha_1 \widehat{v_1} + \cdots + \alpha_n \widehat{v_n})(\varphi) = 0 \\
&\implies \alpha_1 \widehat{v_1}(\varphi) + \cdots + \alpha_n \widehat{v_n}(\varphi) = 0 \\
&\implies \alpha_1 \varphi(v_1) + \cdots + \alpha_n \varphi_n(v_n) = 0.
\end{aligned}$$

Since this last equation holds for all $\varphi \in V'$, it holds in particular for each element of the dual basis $\varphi_1, \ldots, \varphi_n$. That is, for $k = 1, \ldots, n$, we have

$$\alpha_1 \varphi_k(v_1) + \cdots + \alpha_k \varphi_k(v_k) + \cdots + \alpha_n \varphi_k(v_n) = 0 \implies \alpha_k = 0,$$

and therefore $v = 0 \cdot v_1 + \cdots + 0 \cdot v_n = 0$, as desired. Thus $\hat{\cdot}$ is indeed an isomorphism.

We now prove the main result. Suppose $\varphi_1, \ldots, \varphi_n$ is a basis of V', and let $\Phi_1, \ldots, \Phi_n$ be the dual basis in V''. For each Φ_k, let v_k be the inverse of Φ_k under the isomorphism $\hat{\cdot}$. Since the inverse of an isomorphism is an isomorphism, and isomorphisms take bases to bases, $v_1, \ldots, v_n$ is a basis of V. Let us now check that its dual basis is $\varphi_1, \ldots, \varphi_n$. For $j, k = 1, \ldots, n$, we have

$$\begin{aligned}\varphi_j(v_k) &= \widehat{v_k}(\varphi_j)\\ &= \Phi_k(\varphi_j)\\ &= \begin{cases}1 & \text{if } k = j\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

so indeed there exists a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$, as was to be shown. □

Problem 32

Suppose $T \in \mathcal{L}(V)$ and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent:

(a) T is invertible.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ span $F^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

Proof. We prove the following: $(a) \iff (b) \iff (c) \iff (e) \iff (d)$.

$(a) \iff (b)$. Suppose T is invertible. That is, for any $w \in V$, there exists a unique $x \in V$ such that $w = Tx$. It follows

$$\begin{aligned}\mathcal{M}(w) &= \mathcal{M}(Tx)\\ &= \mathcal{M}(T)\mathcal{M}(x)\\ &= \mathcal{M}(x)_1\mathcal{M}(T)_{\cdot,1} + \cdots + \mathcal{M}(x)_n\mathcal{M}(T)_{\cdot,n}.\end{aligned}$$

That is, every vector in $\mathbb{F}^{1,n}$ can be exhibited as a unique linear combination of the columns of $\mathcal{M}(T)$. This is true if and only if the columns of $\mathcal{M}(T)$ are linearly independent.

$(b) \iff (c)$. Suppose the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$. Since they form a linearly independent list of length $\dim(\mathbb{F}^{n,1})$, they are a basis. But this is true if and only if they span $\mathbb{F}^{n,1}$ as well.

$(c) \iff (e)$. Suppose the columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$, so that the column rank is n. Since the row rank equals the column rank, so too must the rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

$(e) \iff (d)$. Suppose the rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$. Since they form a spanning list of length $\dim(\mathbb{F}^{1,n})$, they are a basis. But this is true if and only if they are linearly independent in $\mathbb{F}^{1,n}$ as well. □

Problem 33

Suppose m and n are positive integers. Prove that the function that takes A to A^t is a linear map from $F^{m,n}$ to $\mathbb{F}^{n,m}$. Furthermore, prove that this linear map is invertible.

Proof. We first show taking the transpose is linear. So suppose $A, B \in \mathbb{F}^{m,n}$ and let $j = 1, \ldots, n$ and $k = 1, \ldots, m$. It follows

$$\begin{aligned}(A+B)^t_{j,k} &= (A+B)_{k,j} \\ &= A_{k,j} + B_{k,j} \\ &= A^t_{j,k} + B^t_{j,k},\end{aligned}$$

so that taking the transpose is additive. Next, let $\lambda \in \mathbb{F}$. It follows

$$\begin{aligned}(\lambda A)^t_{j,k} &= (\lambda A)_{k,j} \\ &= \lambda A_{k,j} \\ &= \lambda A^t_{j,k},\end{aligned}$$

so that taking the transpose is homogenous. Since it is both additive and homogeneous, it is a linear map. To see that taking the transpose is invertible, note that $(A^t)^t = A$, so that the inverse of the transpose is the transpose itself. $\square$

Problem 34

The ***double dual space*** of V, denoted V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Proof. We proved (a) and (c) in Problem 31 (where we defined $\hat{\cdot}$ in precisely the same way as Λ). So it only remains to prove (b). So suppose $v \in V$ and $\varphi \in V'$ are arbitrary. Evaluating $T'' \circ \Lambda$, notice

$$\begin{aligned}((T'' \circ \Lambda)(v))(\varphi) &= (T''(\Lambda v))(\varphi) \\ &= (\Lambda v)(T'\varphi) \\ &= (T'\varphi)(v) \\ &= \varphi(Tv),\end{aligned}$$

where the second and fourth equalities follow by definition of the dual map, and the third equality follows by definition of Λ. And evaluating $\Lambda \circ T$, we have

$$\begin{aligned} ((\Lambda \circ T)(v))(\varphi) &= (\Lambda(Tv))(\varphi) \\ &= \varphi(Tv), \end{aligned}$$

so that the two expressions evaluate to the same thing. Since the choice of both v and φ was arbitrary, we have $T'' \circ \Lambda = \Lambda \circ T$, as desired. □

Problem 35

Show that $(\mathcal{P}(\mathbb{R}))'$ and $\mathbb{R}^\infty$ are isomorphic.

Proof. For any sequence $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathbb{R}^\infty$, let φ_α be the unique linear functional in $(\mathcal{P}(\mathbb{R}))'$ such that $\varphi_\alpha(X^k) = \alpha_k$ for all $k \in \mathbb{Z}^+$ (note that since the list $1, X, X^2, \dots$ is a basis of $\mathcal{P}(\mathbb{R})$, this description of φ_α is sufficient). We claim

$$\begin{aligned} \Phi : \mathbb{R}^\infty &\to (\mathcal{P}(\mathbb{R}))' \\ \alpha &\mapsto \varphi_\alpha \end{aligned}$$

is an isomorphism. There are three things to show: that Φ is a linear map, that it's injective, and that it's surjective.

We first show Φ is linear. Suppose $\alpha, \beta \in \mathbb{R}^\infty$. For any $k \in \mathbb{Z}^+$, it follows

$$\begin{aligned} (\Phi(\alpha + \beta))(X^k) &= \varphi_{\alpha+\beta}(X^k) \\ &= (\alpha + \beta)_k \\ &= \alpha_k + \beta_k \\ &= \varphi_\alpha(X^k) + \varphi_\beta(X^k) \\ &= (\Phi(\alpha))(X^k) + (\Phi(\beta))(X^k), \end{aligned}$$

so that Φ is additive. Next suppose $\lambda \in \mathbb{R}$. Then we have

$$\begin{aligned} \Phi(\lambda\alpha)(X^k) &= \varphi_{\lambda\alpha}(X^k) \\ &= (\lambda\alpha)_k \\ &= \lambda\alpha_k \\ &= \lambda\Phi(\alpha), \end{aligned}$$

so that Φ is homogenous. Being both additive and homogeneous, Φ is indeed linear.

Next, to see that Φ is injective, suppose $\Phi(\alpha) = 0$ for some $\alpha \in \mathbb{R}^\infty$. Then $\varphi_\alpha(X^k) = \alpha_k = 0$ for all $k \in \mathbb{Z}^+$, and hence $\alpha = 0$. Thus Φ is injective.

Lastly, to see that Φ is surjective, suppose $\varphi \in (\mathcal{P}(\mathbb{R}))'$. Define $\alpha_k = \varphi(X^k)$ for all $k \in \mathbb{Z}^+$ and let $\alpha = (\alpha_0, \alpha_1, \dots)$. By construction, we have $(\Phi(\alpha))(X^k) = \alpha_k$ for all $k \in \mathbb{Z}^+$, and hence $\Phi(\alpha) = \varphi_\alpha$. Thus Φ is surjective.

Since Φ is linear, injective, and surjective, it's an isomorphism, as desired. □

Problem 37

Suppose U is a subspace of V. Let $\pi : V \to V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is injective.

(b) Show that $\operatorname{range} \pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

Proof. (a) Let $\varphi \in (V/U)'$, and suppose $\pi'(\varphi) = 0$. Then $(\varphi \circ \pi)(v) = \varphi(v + U) = 0$ for all $v \in V$. This is true only if $\varphi = 0$, and hence π' is indeed injective.

(b) First, suppose $\varphi \in \operatorname{range} \pi'$. Then there exists $\psi \in (V/U)'$ such that $\pi'(\psi) = \varphi$. So for all $u \in U$, we have

$$\begin{aligned}\varphi(u) &= (\pi'(\psi))(u) \\ &= \psi(\pi(u)) \\ &= \psi(u + U) \\ &= \psi(0 + U) \\ &= 0,\end{aligned}$$

and thus $\varphi \in U^0$, showing $\operatorname{range} \pi' \subseteq U^0$. Conversely, suppose $\varphi \in U^0$, so that $\varphi(u) = 0$ for all $u \in U$. Define $\psi \in (V/U)'$ by $\psi(v + U) = \varphi(v)$ for all $v \in V$. Then $(\pi'(\psi))(v) = \psi(\pi(v)) = \psi(v + U) = \varphi(v)$, and so indeed $\varphi \in \operatorname{range} \pi'$, showing $U^0 \subseteq \operatorname{range} \pi'$. Therefore, we have $\operatorname{range} \pi' = U^0$, as desired.

(c) Notice that (b) may be interpreted as saying $\pi' : (V/U)' \to U^0$ is surjective. Since π' was shown to be injective in (a), we conclude π' is an isomorphism from $(V/U)'$ onto U^0, as desired. $\square$