# EE 635: Applied Linear Algebra
## Assignment 3

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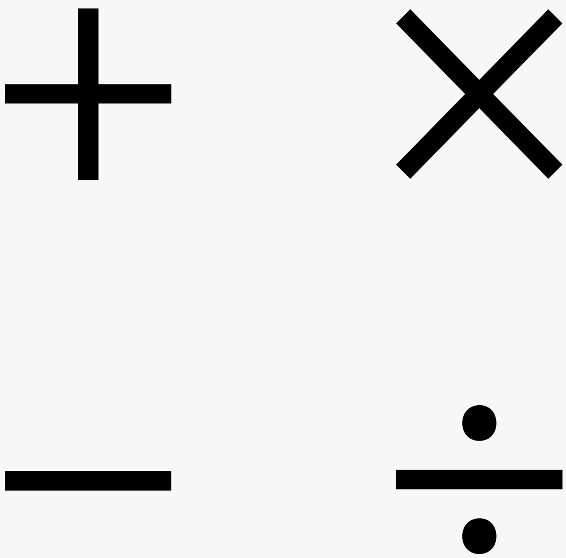


1. For a linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$, given by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$, ascertain if $T$ is invertible. If so, then obtain a definition for $T^{-1}$ such as the one for $T$. Prove that $(T^2 - I)(T - 3I) = 0$.

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$
$$= (y_1, y_2, y_3) \quad \text{(say)}.$$

Since both $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ are in standard basis, it is easy to work with matrix of $T$.

$$\left.\begin{aligned} y_1 &= 3x_1 \\ y_2 &= x_1 - x_2 \\ y_3 &= 2x_1 + x_2 + x_3 \end{aligned}\right\} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Therefore, $$T = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$T$ will be invertible if $T$ is injective and surjective linear map.

Check for kernel:

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $3x_1 = 0 \Rightarrow x_1 = 0$

$x_1 - x_2 = 0 \Rightarrow x_2 = 0$

$2x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = 0.$

Since we get $\operatorname{Ker}(T) = \{(0,0,0)\}$. Therefore $T$ is injective.

Check for image:

rank-nullity theorem:

$$\dim(\operatorname{Im}(T)) + \dim(\operatorname{Ker}(T)) = \dim(\mathbb{R}^3)$$

$$\Rightarrow \dim(\operatorname{Im}(T)) = \dim(\mathbb{R}^3) - \dim(\operatorname{Ker}(T))$$

$$= 3 - 0$$

$$= 3$$

Since $\dim(\operatorname{Im}(T)) = \dim(\mathbb{R}^3) = 3$ so it means, the only possible subspace that has the same dimension with the underlying vector space is the vector space itself. Hence $\operatorname{Im}(T) = \mathbb{R}^3$.

Since we see $\operatorname{im}(T) = \mathbb{R}^3$ so $T$ must be surjective. Since $T$ is both injective & surjective so $T$ is invertible.

Since $T$ is invertible so $T \cdot T^{-1} = T^{-1}T = I$.

Suppose,

$$T^{-1} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving the systems of equation:

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \Rightarrow y_{11} = 1/3 \\ \Rightarrow y_{21} = 1/3 \\ \Rightarrow y_{31} = -1 \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{matrix} \Rightarrow y_{12} = 0 \\ \Rightarrow y_{22} = -1 \\ \Rightarrow y_{32} = +1 \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{matrix} \Rightarrow y_{13} = 0 \\ \Rightarrow y_{23} = 0 \\ \Rightarrow y_{33} = 1. \end{matrix}$$

Therefore the $T^{-1}$ will be-

$$T^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ -1 & +1 & 1 \end{bmatrix}$$

$$(x_1, x_2, x_3) \underset{T^{-1}}{\overset{T}{\rightleftarrows}} (y_1, y_2, y_3)$$

$$T^{-1}(y_1, y_2, y_3) = \left( \frac{1}{3}y_1, \frac{1}{3}y_1 - y_2, -y_1 + y_2 + y_3 \right)$$

We need to prove that,

$$(T^2 - I)(T - 3I) = 0$$

$$\Rightarrow \left( \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) (T - 3I)$$

$$\Rightarrow \left( \begin{bmatrix} 9 & 0 & 0 \\ 2 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} 8 & 0 & 0 \\ 2 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Hence proved)

4. Obtain a linear map, $\varphi$ from $\mathbb{F}^5$ to $\mathbb{F}^2$ such that its kernel is given by $\text{Ker}(\varphi) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = \alpha x_2; x_3 = x_4 = x_5\}$ $(\alpha \in \mathbb{F})$.

$$\mathbb{F}^5 \xrightarrow{\varphi} \mathbb{F}^2$$

$$\text{Ker}(\varphi) = \left\{ (x_1, x_2, \ldots, x_5) \mid x_1 = \alpha x_2,\ x_3 = x_4 = x_5 \right\}$$

$$= \left\{ (\alpha x_2, x_2, x_3, x_3, x_3) \mid \forall x_2, x_3, \alpha \in \mathbb{F} \right\}$$

$$= \left\{ x_2 \cdot (\alpha, 1, 0, 0, 0) + x_3 \cdot (0, 0, 1, 1, 1) \right\}$$

$$= \langle \{ (\alpha, 1, 0, 0, 0), (0, 0, 1, 1, 1) \} \rangle$$

$\{(\alpha, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}$ is LI set

$\Rightarrow c_1 \cdot (\alpha, 1, 0, 0, 0) + c_2 \cdot (0, 0, 1, 1, 1) = 0$

$$\left.\begin{array}{l} \Rightarrow c_1 \alpha_1 = 0 \quad \Rightarrow c_1 = 0 \\ \Rightarrow c_2 = 0 \end{array}\right\} \text{ so it is LI set.}$$

Since $\{(\alpha, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}$ is spanning set of $\text{Ker}(\varphi)$ and it is LI set so it is a basis for $\text{Ker}(\varphi)$

So $\dim(\text{Ker}(\varphi)) = 2$.

From rank nullity theorem:

$$\dim(\mathrm{Ker}(\varphi)) + \dim(\mathrm{Im}(\varphi)) = \dim(\mathbb{F}^5)$$

$$\Rightarrow 2 + \dim(\mathrm{Im}(\varphi)) = 5$$

$$\Rightarrow \dim(\mathrm{Im}(\varphi)) = 3 \quad \text{——— (i)}$$

However, $\mathrm{Im}(\varphi)$ is subspace of $\mathbb{F}^2$ so,

$$\dim(\mathrm{Im}(\varphi)) \leq 2. \quad \text{——— (ii)}$$

Two results in eqn (i) & (ii) are contradictory. Therefore, such a linear map with the given kernel is not possible.

6. Suppose $\mathcal{C}^k(\mathbb{R})$ is the vector space of all $k$-times continuously differentiable functions on $\mathbb{R}$, $k \in \{0, 1, 2, \ldots\}$. Consider $\mathcal{L}_1 : \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ and $\mathcal{L}_2 : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ be given by $\mathcal{L}_1(f(x)) = \frac{d}{dx}(f(x)) + \alpha f(x)$, $\alpha \in \mathbb{R}$, and $\mathcal{L}_2(g(x)) = \frac{d^2}{dx^2}(g(x)) + \omega^2 g(x)$, $\omega \in \mathbb{R}$. What are the dimensions of the kernels of $\mathcal{L}_1$ and $\mathcal{L}_2$? Write down one possible choice of basis for each operator.

$$C^K(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R},\ f(x) \text{ is } K \text{ times cont-diff.} \}$$

$$L_1 : C^1(\mathbb{R}) \to C^0(\mathbb{R})$$

$$L_1(f(x)) = \frac{d}{dx}(f(x)) + \alpha \cdot f(x)$$

$\mathrm{Ker}(L_1)$ is defined as:

$$\mathrm{Ker}(L_1) = \{ f(x) \in C^1(\mathbb{R}) \mid L_1(f(x)) = 0 \}$$

Consider, $L_1(f(x)) = 0$

$$\Rightarrow \frac{d}{dx}(f(x)) + \alpha f(x) = 0$$

$$\Rightarrow \frac{d}{dx}(f(x)) = -\alpha f(x)$$

$$\Rightarrow \frac{d(f(x))}{f(x)} = -\alpha \cdot dx$$

Integrating both sides, we get -

$$\Rightarrow \int \frac{d(f(x))}{f(x)} = \int -\alpha dx + c \qquad (c \in \mathbb{R})$$

$$\Rightarrow \ln|f(x)| = -\alpha x + c$$

$$\Rightarrow |f(x)| = e^{-\alpha x + c} = e^{c} \cdot e^{-\alpha x}$$

Since $e^c$ is just a constant, we can rewrite as $a > 0$

$$\Rightarrow |f(x)| = a \cdot e^{-\alpha x} \qquad (a > 0)$$

Since $f(x)$ can take negative values so we write -

$$\Rightarrow f(x) = a \cdot e^{-\alpha x} \quad (a \in \mathbb{R})$$

We can see that $f(x)$ is scalar multiple of $e^{-\alpha x}$.

$$\mathrm{Ker}(L_1) = \{ f(x) \in C^1(\mathbb{R}) \mid L_1(f(x)) = 0 \}$$

$$= \{ a \cdot e^{-\alpha x} \mid a \in \mathbb{R} \}$$

$$= \langle \{ e^{-\alpha x} \} \rangle$$

Note that $\{e^{-\alpha x}\}$ is clearly LI and it spans $\mathrm{Ker}(L_1)$ that means it is basis for $\mathrm{Ker}(L_1)$.

Basis for $\mathrm{Ker}(L_1) = \{ e^{-\alpha x} \}$

dimension for $\mathrm{Ker}(L_1) = 1$.

$L_2 : C^2(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$

$$L_2(g(x)) = \frac{d^2}{dx^2}(g(x)) + \omega^2 g(x)$$

$$\mathrm{Ker}(L_2) = \{ g(x) \in C^2(\mathbb{R}) \mid L_2(g(x)) = 0 \}$$

Consider, $L_2(g(x)) = 0$

$$\Rightarrow \frac{d^2}{dx^2}(g(x)) + \omega^2 g(x) = 0$$

Consider $g(x) = y$.

$$\Rightarrow \frac{d^2y}{dx^2} + \omega^2 y = 0 \quad (\omega \neq 0)$$

Assume the solution, $y = e^{rx}$ where $r$ is a constant

$$\frac{d^2y}{dx^2} = r^2 \cdot e^{rx}$$

$$\Rightarrow r^2 \cdot e^{rx} + \omega^2 \cdot e^{rx} = 0$$

$$\Rightarrow e^{rx}(r^2 + \omega^2) = 0$$

Since $e^{rx} \neq 0$ therefore, $r^2 + \omega^2 = 0$

$$\Rightarrow r = \pm\sqrt{-\omega^2} = \pm i\omega \quad \text{where } i = \sqrt{-1}$$

Therefore, solution, $y_1 = e^{i\omega x}$ or $y_2 = e^{-i\omega x}$

We can write the general solution as LC of these 2 solutions.

$$y = C \cdot y_1 + D \cdot y_2 = C \cdot e^{i\omega x} + D \cdot e^{-i\omega x}$$

$$= C \cdot \left(\cos(\omega x) + i \sin(\omega x)\right) + D\left(\cos(\omega x) - i \sin(\omega x)\right)$$

$$= (C + D) \cdot \cos(\omega x) + (Ci - Di) \sin(\omega x)$$

$$= A \cos(\omega x) + B \sin(\omega x).$$

Suppose, $C = \alpha + \beta i$, $D = \alpha - \beta i$ $(\alpha, \beta \in \mathbb{R})$

$$C + D = \alpha + \beta$$

$$i(C - D) = 2\beta i \times i = -2\beta \quad [\because i^2 = -1]$$

Hence, $A = C+D = \alpha + \beta \longrightarrow A \in \mathbb{R}$

$B = i(C-D) = -2\beta \longrightarrow B \in \mathbb{R}$.

Therefore, $g(x) = A\cos(\omega x) + B\sin(\omega x)$, $A, B \in \mathbb{R}$.
We can see that $g(x)$ is LC of $\cos(\omega x)$ and $\sin(\omega x)$.

But if $\omega = 0$ then $\dfrac{d^2 g(x)}{dx^2} = 0$

Integrating 2 times, $g(x) = Ax + B \quad (A, B \in \mathbb{R})$.

Hence $\text{Ker}(L_2) = \{ g(x) \in C^2(\mathbb{R}) \mid L_2(g(x)) = 0 \}$

$= \{ A.\cos(\omega x) + B.\cos(\omega x) \mid \omega \neq 0, A, B \in \mathbb{R} \}$

$= \{ A.x + B \mid \omega = 0, A, B \in \mathbb{R} \}$

Therefore, $\text{Ker}(L_2) = \langle \{ \cos(\omega x), \sin(\omega x) \} \rangle \quad (\omega \neq 0)$

$= \langle \{ 1, x \} \rangle \quad (\omega = 0)$

$\{ \cos\omega x, \sin\omega x \}$ is a spanning set for $\text{Ker}(L_2)$.
but is it LI ?

$\Rightarrow \quad c_1 \cos(\omega x) + c_2 \sin(\omega x) = 0$

Suppose, $c_1 = r\sin\theta$ , $c_2 = r\cos\theta$

$\Rightarrow r = \sqrt{c_1^2 + c_2^2}$ , $\theta = \tan^{-1}\left(\frac{c_1}{c_2}\right)$

Therefore, $r\sin\theta\cos(wx) + r\cos\theta\sin(wx) = 0$

$\Rightarrow r.\sin(\theta + wx) = 0$

Since it is true for all $x$ so the only way it is possible is $r = 0$.

$c_1 = r\sin\theta = 0 \times \sin\theta = 0$
$c_2 = r\cos\theta = 0 \times \cos\theta = 0$

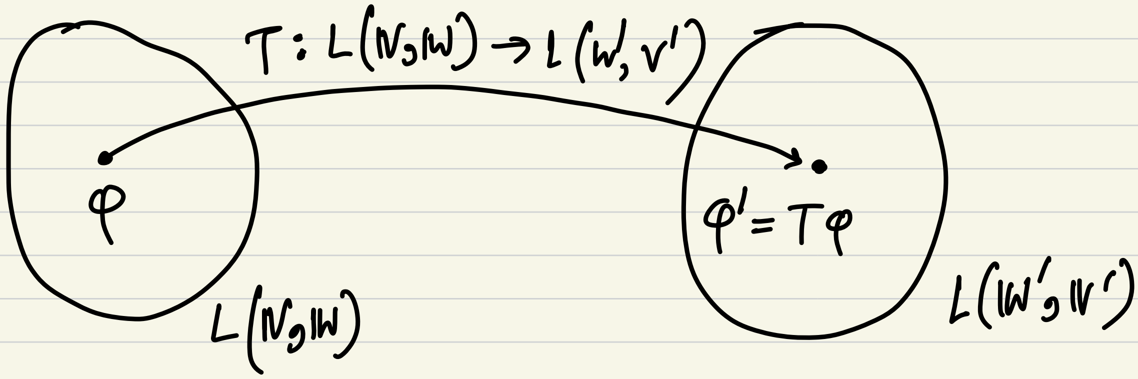
Hence $\{\cos(wx), \sin(wx)\}$ are LI set. Hence it is basis for $Ker(L_2)$.
It is trivial to show that $\{1, x\}$ is LI set since it is standard polynomial bases vectors.

Therefore, $Ker(L_2) = \langle\{\cos wx, \sin wx\}\rangle \quad w \neq 0$
$= \langle\{1, x\}\rangle \quad w = 0$

Basis for $Ker(L_2) = \{\cos(wx), \sin(wx)\} \; (w \neq 0)$
$= \{1, x\} \; (w = 0)$

Dimension for $Ker(L_2) = 2$.

7. Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces. Show that the map which takes $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ to $\varphi' \in \mathcal{L}(\mathbb{W}', \mathbb{V}')$ is an isomorphism of $\mathcal{L}(\mathbb{V}, \mathbb{W})$ onto $\mathcal{L}(\mathbb{W}', \mathbb{V}')$.




Check linearity:

$$T(c\varphi_1 + \varphi_2) = c \cdot T(\varphi_1) + T(\varphi_2) \qquad (c \in \mathbb{F})$$

$$\Rightarrow (c\varphi_1 + \varphi_2)' = c \cdot \varphi_1' + \varphi_2' \quad \text{(to be proved)}$$

For any linear functional $g$ in $\mathbb{W}'$ s.t. $g : \mathbb{W} \to \mathbb{F}$,

$$\Rightarrow (c\varphi_1 + \varphi_2)'(g) = g \circ (c\varphi_1 + \varphi_2)$$

For any vector $v$ in $\mathbb{V}$ we get-

$$\left(g \circ (c\varphi_1 + \varphi_2)\right)(v)$$

$$= g\left((c\varphi_1 + \varphi_2)(v)\right)$$

$$= g\left(c\,\varphi_1(v) + \varphi_2(v)\right) \quad [\because \varphi_1, \varphi_2 \text{ are linear}]$$

$$= g(c \cdot \varphi_1(v)) + g(\varphi_2(v)) \quad [\because g \text{ is linear}]$$

$$= c \cdot g(\varphi_1(v)) + g(\varphi_2(v))$$

$= c\cdot(g\circ\phi_1)(v) + (g\circ\phi_2)(v)$

$= \left(c(g\circ\varphi_1) + g\circ\varphi_2\right)(v)$

$= \left(c\cdot \varphi_1'(g) + \varphi_2'(g)\right)(v)$

Therefore,

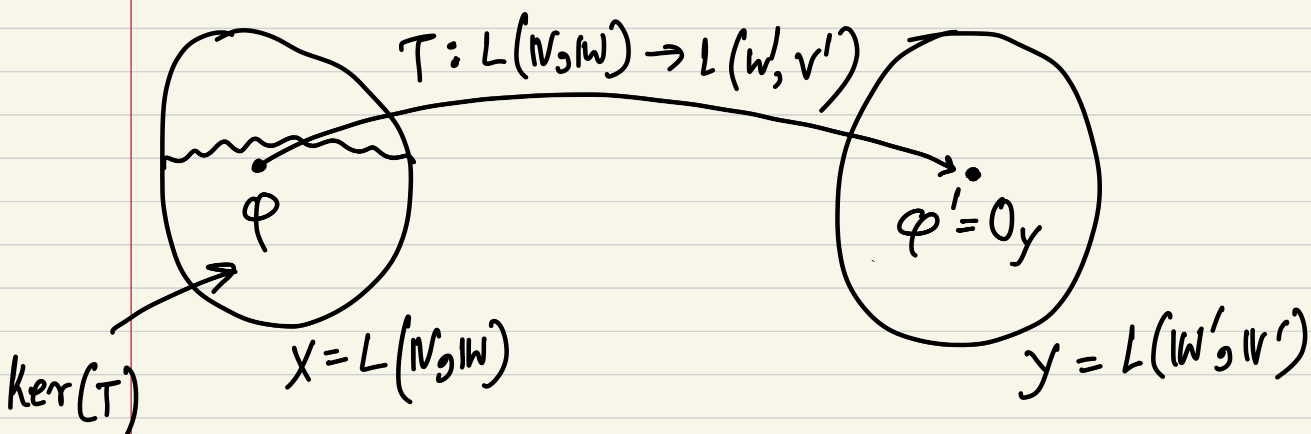
$(c\varphi_1+\varphi_2)'(g) = c\cdot\varphi_1'(g) + \varphi_2'(g)$

$\Rightarrow (c\varphi_1+\varphi_2)'(g) = (c\varphi_1'+\varphi_2')(g)$

$\Rightarrow (c\varphi_1+\varphi_2)' = c\varphi_1'+\varphi_2'$ (proved)

Therefore, T is a linear transformation.

Check injectivity:




We have to prove that $\ker(T) = \{0_Y\}$.

$$\operatorname{Ker}(T) = \left\{ \varphi \in \mathcal{L}(\mathbb{V}, \mathbb{W}) \;\middle|\; T(\varphi) = 0_y \right\}$$

Now $0_y$ is defined as: $\quad 0_y : \mathbb{W}' \longrightarrow \mathbb{V}'$

such that, $0_y(g) = 0_{\mathbb{V}'}$ for $\forall g \in \mathbb{W}'$. $\quad$ —— (i)

$0_{\mathbb{V}'}$ is defined as: $\quad 0_{\mathbb{V}'} : \mathbb{V} \to \mathbb{F}$

such that, $0_{\mathbb{V}'}(v) = 0_{\mathbb{F}}$ for $\forall v \in \mathbb{V}$. $\quad$ —— (ii)

We need to prove that, $\operatorname{Ker}(T) = \{0_x\}$ to prove injectivity where, $0_x$ is defined as: $0_x : \mathbb{V} \longrightarrow \mathbb{W}$.

such that, $0_x(v) = 0_{\mathbb{W}}$ for $\forall v \in \mathbb{V}$. $\quad$ —— (iii)

$$\mathbb{V} \xrightarrow{\varphi} \mathbb{W} \qquad\qquad \mathbb{W}' \xrightarrow{\varphi'} \mathbb{V}'$$

$$v \longmapsto \varphi(v) \qquad\qquad g \longmapsto \varphi'(g)$$

$$\begin{array}{ccc} v & \longmapsto & \varphi(v) \\ \Big\downarrow{\scriptstyle \varphi'(g) \in \mathbb{V}'} & & \Big\downarrow{\scriptstyle g \in \mathbb{W}'} \\ \varphi'(g)(v) \in \mathbb{F} & = & g(\varphi(v)) \in \mathbb{F} \end{array}$$

Suppose, $\varphi \in \operatorname{Ker}(T)$.

$\Rightarrow T(\varphi) = 0_y$

$\Rightarrow (T(\varphi))(g) = 0_y(g) = 0_{\mathbb{V}'}$ $\quad (\forall g \in \mathbb{W}')$ —(from (i))

$\Rightarrow \varphi'(g) = 0_{\mathbb{V}'}$ $\quad (\because T(\varphi) = \varphi')$

$\Rightarrow g \circ \varphi = 0_{\mathbb{V}'}$ $\quad (\because \varphi'(g) = g \circ \varphi)$

$\Rightarrow (g \circ \varphi)(v) = 0_{\mathbb{V}'}(v) \quad (\forall v \in \mathbb{V})$

$\Rightarrow g(\varphi(v)) = 0_{\mathbb{F}}.$ $\quad$ ( from def$^{n}$ (ii)) $(\forall g \in \mathbb{W}')$

$\Rightarrow \varphi(v) = 0_{\mathbb{W}'}$ $\quad$ ($\because \varphi(v) \in \mathbb{W}$ and $0_{\mathbb{W}'}(w \in \mathbb{W}) := 0_{\mathbb{F}}$)

$\Rightarrow \varphi = 0_X$ ( by def$^{n}$ (iii)) $\quad$ (Since it is true for all $g$ so $\varphi(v) = 0_{\mathbb{W}'}$)

Therefore, $\varphi \in \text{Ker}(T) \longrightarrow \varphi = 0_X$

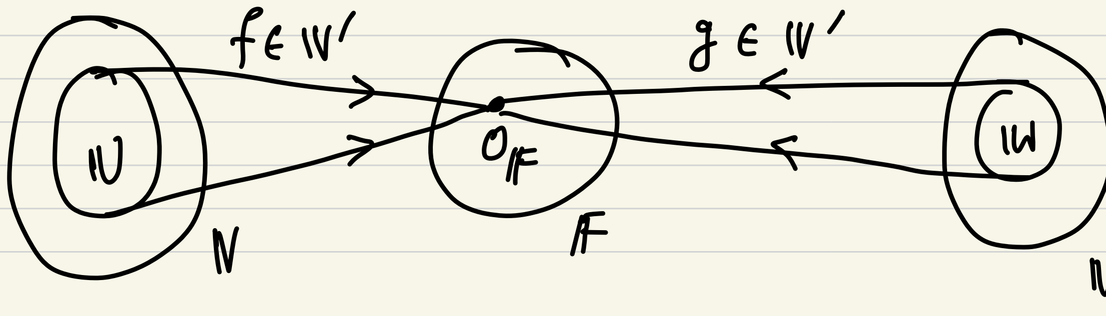
Hence, $\text{Ker}(T) = \{0_X\}$ so $T$ is injective.

Let's find out dimension of $L(\mathbb{V}, \mathbb{W}) = \dim(\mathbb{V}) \times \dim(\mathbb{W})$

$$\dim(L(\mathbb{W}', \mathbb{V}')) = \dim(\mathbb{W}') \times \dim(\mathbb{V}')$$
$$= \dim(\mathbb{W}) \times \dim(\mathbb{V}) \quad (\because \dim(\mathbb{V}') = \dim(\mathbb{V}))$$

Therefore the dimension of 2 vector spaces are same. and we have already proved injectivity of $T$. Hence $T: L(V, W) \longrightarrow L(W', V')$ must be surjective also since the $\dim(L(V,W)) = \dim(L(W', V'))$. Therefore, $T$ is isomorphism of $L(V,W)$ onto $L(W', V')$.

8. Let $\mathbb{U}$ and $\mathbb{W}$ be subspaces of a finite dimensional vector space $\mathbb{V}$. Show that $(\mathbb{U} \cap \mathbb{W})^0 = \mathbb{U}^0 + \mathbb{W}^0$.



$\mathbb{U}^0 := \{ f \in \mathbb{V}' \mid \forall u \in \mathbb{U},\ f(u) = 0_{\mathbb{F}} \}$

$\mathbb{W}^0 := \{ g \in \mathbb{V}' \mid \forall w \in \mathbb{W},\ g(w) = 0_{\mathbb{F}} \}$

$\mathbb{U}^0$ and $\mathbb{W}^0$ are subspace of $\mathbb{V}'$

We have to prove that $(\mathbb{U} \cap \mathbb{W})^0 = \mathbb{U}^0 + \mathbb{W}^0$.

Suppose $\varphi \in \mathbb{U}^0 \cap \mathbb{W}^0$, let $v \in \mathbb{U} + \mathbb{W}$ s.t $v = u + w$ where $u \in \mathbb{U}$, $w \in \mathbb{W}$.

$\Rightarrow \varphi(v) = \varphi(u+w) = \varphi(u) + \varphi(w)$

Since $\varphi \in \mathbb{U}^0 \cap \mathbb{W}^0 \longrightarrow \varphi \in \mathbb{U}^0$

$\varphi \in \mathbb{U}^0 \cap \mathbb{W}^0 \longrightarrow \varphi \in \mathbb{W}^0$

$\varphi \in \mathbb{U}^0 \longrightarrow \varphi(u) = 0_{\mathbb{F}}\ \forall u \in \mathbb{U}$.

$\varphi \in \mathbb{W}^0 \longrightarrow \varphi(w) = 0_{\mathbb{F}}\ \forall w \in \mathbb{W}$.

$\Rightarrow \varphi(v) = \varphi(u) + \varphi(w) = 0_{\mathbb{F}} + 0_{\mathbb{F}} = 0_{\mathbb{F}}$

Therefore, $\varphi(v) = 0_{\mathbb{F}}\ \forall v \in \mathbb{U} + \mathbb{W}$.

Hence $\varphi \in (\mathbb{U} + \mathbb{W})^0 \Rightarrow \mathbb{U}^0 \cap \mathbb{W}^0 \subseteq (\mathbb{U} + \mathbb{W})^0$

Similarly assume $\varphi \in (\mathbb{U} + \mathbb{W})^0$.

Since we know $\mathbb{U} \subseteq \mathbb{U} + \mathbb{W}$ and $\mathbb{W} \subseteq \mathbb{U} + \mathbb{W}$.

$\varphi \in (\mathbb{U} + \mathbb{W})^0 \longrightarrow \varphi \in \mathbb{U}^0$ and $\varphi \in \mathbb{W}^0$

so $\varphi \in \mathbb{U}^0 \cap \mathbb{W}^0$

Hence $(\mathbb{U}+\mathbb{W})^0 \subseteq \mathbb{U}^0 \cap \mathbb{W}^0$.

Therefore, $(\mathbb{U}+\mathbb{W})^0 = \mathbb{U}^0 \cap \mathbb{W}^0$

$$\Rightarrow \dim\left((\mathbb{U}+\mathbb{W})^0\right) = \dim\left(\mathbb{U}^0 \cap \mathbb{W}^0\right). \quad \text{——— (i)}$$

But our objective is to prove $(\mathbb{U}\cap\mathbb{W})^0 = \mathbb{U}^0 + \mathbb{W}^0$

Since $\mathbb{U}^0, \mathbb{W}^0$ are subspaces, we know that.

$$\dim\left(\mathbb{U}^0 + \mathbb{W}^0\right) = \dim\left(\mathbb{U}^0\right) + \dim\left(\mathbb{W}^0\right) - \dim\left(\mathbb{U}^0 \cap \mathbb{W}^0\right)$$

Since $\mathbb{U}^0$ and $\mathbb{W}^0$ are subspaces of $\mathbb{V}'$ so we know that

$$\dim\left(\mathbb{U}^0\right) + \dim(\mathbb{U}) = \dim(\mathbb{V}') = \dim(\mathbb{V}) \quad \text{——— (ii)}$$

$$\dim\left(\mathbb{W}^0\right) + \dim(\mathbb{W}) = \dim(\mathbb{V}') = \dim(\mathbb{V}). \quad \text{——— (iii)}$$

$$\Rightarrow \dim\left(\mathbb{U}^0 + \mathbb{W}^0\right) = \left(\dim(\mathbb{V}) - \dim(\mathbb{U})\right) + \left(\dim(\mathbb{V}) - \dim(\mathbb{W})\right) - \left(\dim\left((\mathbb{U}+\mathbb{W})^0\right)\right) \quad \text{(from (i), (ii) \& (iii))}$$

$$= \dim(\mathbb{V}) - \dim(\mathbb{U}) + \dim(\mathbb{V}) - \dim(\mathbb{W}) - \left(\dim(\mathbb{V}) - \dim(\mathbb{U}+\mathbb{W})\right)$$

$$= \dim(\mathbb{V}) - \dim(\mathbb{U}) - \dim(\mathbb{W}) + \left(\dim(\mathbb{U}) + \dim(\mathbb{W}) - \dim(\mathbb{U}\cap\mathbb{W})\right)$$

$$= \dim(\mathbb{V}) - \dim(\mathbb{U}\cap\mathbb{W}) = \dim\left((\mathbb{U}\cap\mathbb{W})^0\right).$$

Therefore we get, $\dim(\mathbb{U}^0 + \mathbb{W}^0) = \dim((\mathbb{U} \cap \mathbb{W})^0)$.

Suppose, $\varphi \in \mathbb{U}^0 + \mathbb{W}^0$

Say $\varphi_1 \in \mathbb{U}^0$ and $\varphi_2 \in \mathbb{W}^0$ s.t. $\varphi = \varphi_1 + \varphi_2$.

Consider $v \in \mathbb{U} \cap \mathbb{W}$. so $v \in \mathbb{U}$ and $v \in \mathbb{W}$.

$\varphi(v) = (\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v)$.

Since $v \in \mathbb{U}$ and $\varphi_1 \in \mathbb{U}^0$ so $\varphi_1(v) = 0_{\mathbb{F}}$.

Since $v \in \mathbb{W}$ and $\varphi_2 \in \mathbb{W}^0$ so $\varphi_2(v) = 0_{\mathbb{F}}$.

$\Rightarrow \varphi(v) = \varphi_1(v) + \varphi_2(v) = 0_{\mathbb{F}} + 0_{\mathbb{F}} = 0_{\mathbb{F}}$.

Since $v \in \mathbb{U} \cap \mathbb{W}$ and $\varphi(v) = 0_{\mathbb{F}}$ so $\varphi \in (\mathbb{U} \cap \mathbb{W})^0$.

Hence, $\mathbb{U}^0 + \mathbb{W}^0 \subseteq (\mathbb{U} \cap \mathbb{W})^0$.

and $\dim(\mathbb{U}^0 + \mathbb{W}^0) = \dim((\mathbb{U} \cap \mathbb{W})^0)$

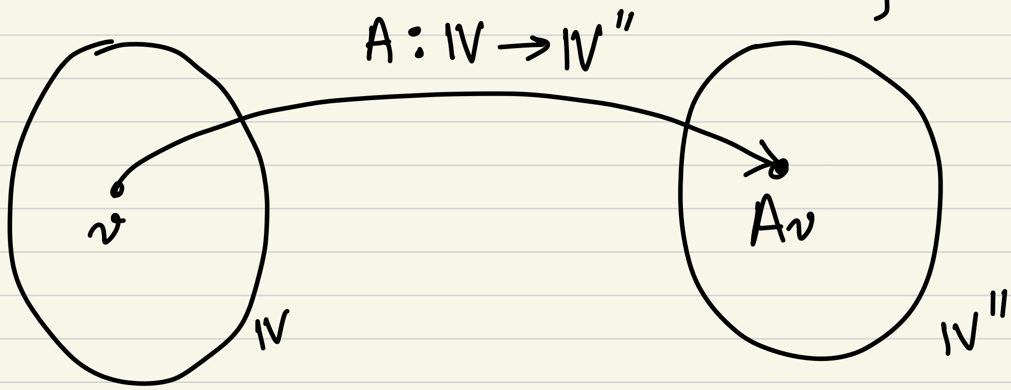
When 2 subspaces of $\mathbb{V}'$ has equal dimension and one of the subspace is contained in other that means the 2 subspaces must be equal.

$\Rightarrow \mathbb{U}^0 + \mathbb{W}^0 = (\mathbb{U} \cap \mathbb{W})^0$ (proved).

12. The *double dual* of a vector space, $\mathbb{V}$, denoted by $\mathbb{V}''$, is the dual space of $\mathbb{V}'$ by definition. Define $\Lambda : \mathbb{V} \to \mathbb{V}''$ as $(\Lambda v)(\varphi) = \varphi(v)$, where $v \in \mathbb{V}$ and $\varphi \in \mathbb{V}'$.
(a) Prove that $\Lambda : \mathbb{V} \to \mathbb{V}''$ is a linear map.
(b) Prove that for $\tau \in \mathcal{L}(\mathbb{V})$, we have $\tau'' \circ \Lambda = \Lambda \circ \tau$, where $\tau'' = (\tau')'$.
(c) Prove that if $\mathbb{V}$ is finite dimensional, then $\Lambda$ is an isomorphism between $\mathbb{V}$ and $\mathbb{V}''$.

$\mathbb{V}' = \{ f \mid f : \mathbb{V} \to \mathbb{F} \}$ (set of all linear functionals defined on $\mathbb{V}$)

$\mathbb{V}'' = \{ g \mid g : \mathbb{V}' \to \mathbb{F} \}$ (set of all linear functionals defined on $\mathbb{V}'$)




$\boxed{\text{Consider } \Lambda = A}$

$(Av)(\varphi) := \varphi(v)$ where $v \in \mathbb{V}$ and $\varphi \in \mathbb{V}'$ —(i)

(i) $A$ is linear: $A(cv_1 + v_2) = cA(v_1) + A(v_2)$.

$\Rightarrow$ $A(cv_1 + v_2)(\varphi) = \varphi(cv_1 + v_2)$

$= \varphi(c \cdot v_1) + \varphi(v_2)$

$= c \cdot \varphi(v_1) + \varphi(v_2)$

$= c\,A(v_1)(\varphi) + A(v_2)(\varphi)$

$\Rightarrow$ $A(cv_1 + v_2) = c \cdot A(v_1) + A(v_2)$. Hence proved.

(ii) $T'' \circ A = (T')' \circ A$ $\boxed{\text{Consider } \tau = T}$

$$
\begin{tikzcd}
\mathbb{V} \arrow[r, "T"] & \mathbb{V} \\
v \arrow[r, mapsto] \arrow[d, mapsto, "A"] & T(v) \arrow[d, mapsto, "A"] \\
\mathbb{V}'' \quad A(v) \arrow[d, mapsto, "(T')'"] & \mathbb{V}'' \quad A(T(v)) \arrow[dl, dash, "//"] \\
\mathbb{V}'' \quad (T')'(A(v)) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{V}' \arrow[r, "g \in \mathbb{V}''"] & \mathbb{F} \\
\varphi \arrow[r, mapsto] & g(\varphi)
\end{tikzcd}
$$

We know that:

$$T'(\varphi) = \varphi \circ T$$

$$(T')'(A(v)) = A(T(v))$$

consider one $\varphi \in \mathbb{V}'$ to evaluate it.

$$
\begin{aligned}
(T')'(A(v))(\varphi) &= (T')'(Av)(\varphi) \\
&= (Av \circ T')(\varphi) \quad [\because T'(\varphi) = \varphi \circ T] \\
&= Av(T'(\varphi)) \\
&= Av(\varphi \circ T) \quad [\because T'(\varphi) = \varphi \circ T] \\
&= (\varphi \circ T)(v) \quad [\because (Av)(\varphi) = \varphi(v)] \\
&= \varphi(T(v))
\end{aligned}
$$

Now, $A(T(v))$ is in $\mathbb{V}''$ so evaluate it.

$A(T(v))(\varphi) = \varphi(T(v))$ [By def$^{\underline{n}}$ of $A$, eqn(i)]

Therefore, $(T')'(A(v))(\varphi) = A(T(v))(\varphi)$

$\Rightarrow (T')'(A(v)) = A(T(v))$

$\Rightarrow (T'' \circ A)(v) = (A \circ T)(v)$

$\Rightarrow T'' \circ A = A \circ T$ (Hence proved.)

(iii) Since $\dim(\mathbb{V}) = \dim(V'')$ so we just have to prove that $A$ is injective map.

for some $v \in \mathbb{V}$, $A$ is injective iff $(Av = 0_{V''} \rightarrow v = 0)$

Suppose $Av = 0_{V''}$ for some $v \in \mathbb{V}$.

$\Rightarrow (Av)(\varphi) = 0_{\mathbb{F}}$ for all $\varphi \in \mathbb{V}'$ (since $Av = 0_{V''}$)

$\Rightarrow (A(v))(\varphi) = \varphi(v) = 0_{\mathbb{F}}$. (from eqn (i))

Hence $\varphi(v) = 0_{\mathbb{F}}$ for all $\varphi \in \mathbb{V}'$ and some $v \in \mathbb{V}$.

Let $\{v_1, v_2, \ldots, v_n\}$ be basis for $\mathbb{V}$ and $\{\varphi_1, \ldots, \varphi_n\}$ be basis for $\mathbb{V}'$. So we can write,

$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \forall c_i \in \mathbb{F}$.

$$\Rightarrow \varphi_1(v) = c_1\varphi_1(v_1) + c_2\varphi_1(v_2) + \cdots + c_n\varphi_n(v_n)$$

The basis $\varphi_j$ is chosen in such a way that,

$$\varphi_j(v_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}.$$

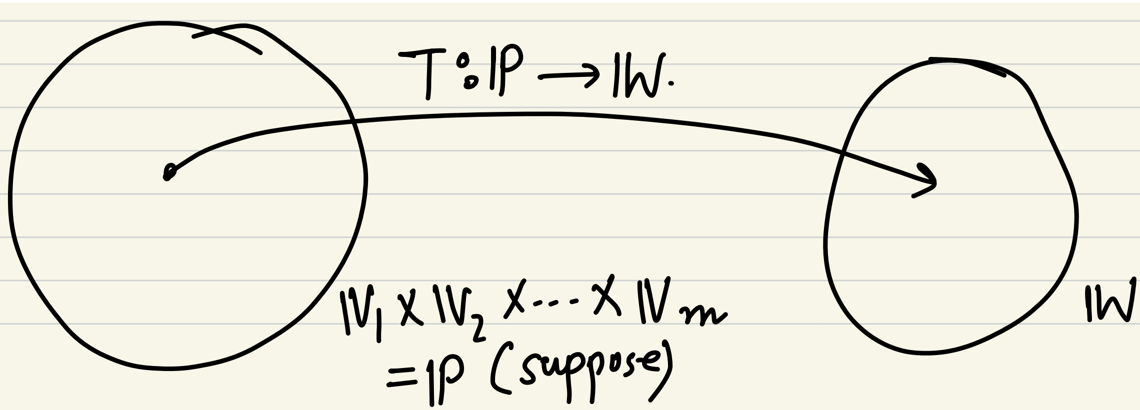
$$\left.\begin{aligned} \text{Therefore, } \quad & \varphi_1(v) = c_1 = 0_{\mathbb{F}} \\ \text{Similarly, } \quad & \varphi_2(v) = c_2 = 0_{\mathbb{F}} \\ & \vdots \\ & \varphi_n(v) = c_n = 0_{\mathbb{F}} \end{aligned}\right\} \begin{aligned} &\text{Since } \forall \varphi \in \mathbb{V}', \\ &\quad \varphi(v) = 0 \\ &\text{and } \forall \varphi_j \in \mathbb{V}'. \end{aligned}$$

Therefore, $v = 0 \times v_1 + 0 \times v_2 + \cdots + 0 \times v_n = 0_{\mathbb{V}}$.

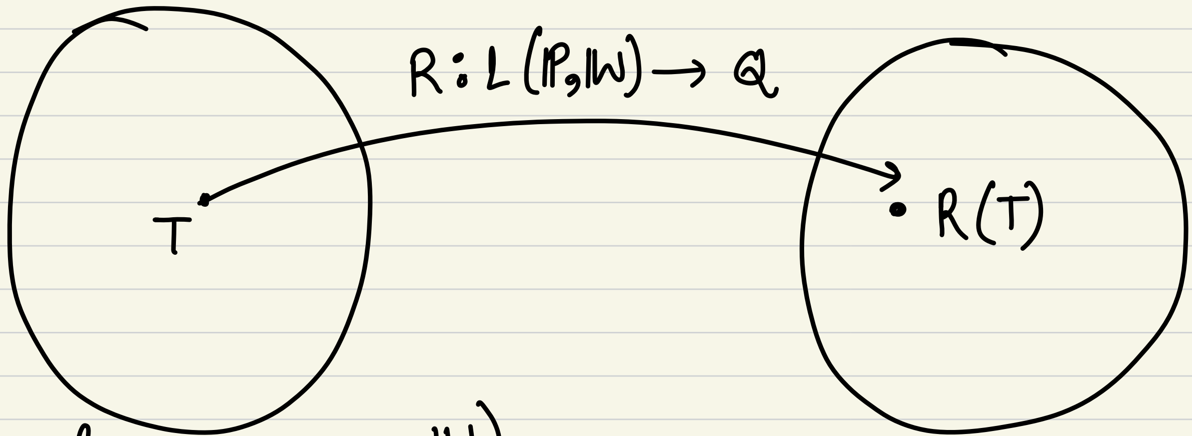
Hence we have proved that $A(v) = 0_{\mathbb{V}''} \longrightarrow v = 0_{\mathbb{V}}$.
that means $A$ is injective.

Since the $\dim(\mathbb{V}'') = \dim(\mathbb{V})$ so $A$ is surjective too.
Therefore $A$ is isomorphism as linearity is already proved.

14. For vector spaces $\mathbb{V}_i$, $i = 1, 2, \ldots, m$, suppose $\mathcal{L}(\mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m, \mathbb{W})$ is the space of all linear functions from $\mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m$ to $\mathbb{W}$, while $\mathcal{L}(\mathbb{V}_i, \mathbb{W})$ is the space of all linear functions from $\mathbb{V}_i$ to $\mathbb{W}$. Show that $\mathcal{L}(\mathbb{V}_1 \times \mathbb{V}_2 \times \ldots \times \mathbb{V}_m, \mathbb{W})$ is isomorphic to $\mathcal{L}(\mathbb{V}_1, \mathbb{W}) \times \mathcal{L}(\mathbb{V}_2, \mathbb{W}) \times \ldots \times \mathcal{L}(\mathbb{V}_m, \mathbb{W})$.



Consider a transformation, $R : L(\mathbb{P}, \mathbb{W}) \to Q$.




$L(\mathbb{V}_1 \times \mathbb{V}_2 \times \dots \times \mathbb{V}_m, \mathbb{W})$
$= L(\mathbb{P}, \mathbb{W})$

$L(\mathbb{V}_1, \mathbb{W}) \times L(\mathbb{V}_2, \mathbb{W}) \times \dots \times L(\mathbb{V}_m, \mathbb{W})$
$= Q$ (say).

Suppose, $v_1 \in \mathbb{V}_1, \dots, v_m \in \mathbb{V}_m$

$(v_1, v_2, \dots, v_m) \in \mathbb{V}_1 \times \mathbb{V}_2 \times \dots \times \mathbb{V}_m$

$T(v_1, v_2, \dots, v_m) \in \mathbb{W}$.

Suppose, $A_1 \in L(\mathbb{V}_1, \mathbb{W})$ so $A_1(v_1) \in \mathbb{W}$

$A_2 \in L(\mathbb{V}_2, \mathbb{W})$ so $A_2(v_2) \in \mathbb{W}$

$\vdots \qquad \vdots$

$A_m \in L(\mathbb{V}_m, \mathbb{W})$ so $A_m(v_m) \in \mathbb{W}$.

Therefore,

$$R\big(T(v_1, v_2, \dots v_m)\big) := \big(A_1(v_1), A_2(v_2), \dots, A_m(v_m)\big)$$

$$A_1(v_1) = T(v_1, 0, 0, \dots 0),\ A_2(v_2) = T(0, v_2, \dots 0), \dots, A_m(v_m) = T(0, \dots, v_m) \quad \text{——(i)}$$

We have to prove that $R$ is linear transformation.

Show that $R(c \cdot T_1 + T_2) = c \cdot R(T_1) + R(T_2)$. , $c \in \mathbb{F}$.

Where $T_1, T_2 \in L(\mathbb{P}, \mathbb{W})$.

Consider $R(c \cdot T_1 + T_2)$

let's evaluate it at $(v_1, v_2, \ldots, v_m) \in \mathbb{V}_1 \times \cdots \times \mathbb{V}_m$.

$\Rightarrow R\left( (cT_1 + T_2)(v_1, v_2, \ldots, v_m) \right)$

$\Rightarrow \left( A_1(v_1), A_2(v_2), \ldots, A_m(v_m) \right)$

where $A_1(v_1) = (cT_1 + T_2)(v_1, 0, \ldots, 0)$

$= cT_1(v_1, 0, \ldots 0) + T_2(v_1, 0, \ldots 0)$

[$\because T_1, T_2$ are linear]

$A_2(v_2) = c \cdot T_1(0, v_2, \ldots 0) + T_2(0, v_2, \ldots 0)$

$\vdots$

$A_m(v_m) = cT_1(0, \ldots 0, v_m) + T_2(0, 0, \ldots 0, v_m)$

$\Rightarrow \big( cT_1(v_1, \ldots 0) + T_2(v_1, \ldots, 0), \; cT_1(0, v_2, \ldots 0) + T_2(0, v_2, \ldots 0)$
$+ \cdots + cT_1(0, 0 \ldots 0, v_m) + T_2(0, 0, \ldots, v_m) \big)$

$\Rightarrow \left( cT_1(v_1, \ldots 0), cT_1(0, v_2, \ldots 0), \ldots, cT_1(0, 0, \ldots, v_m) \right) +$
$\left( T_2(v_1, \ldots 0), T_2(0, v_2, \ldots 0), \ldots, T_2(0, 0, \ldots, v_m) \right)$

By scalar mult rule of product space.

(By the addition rule of product space)

$$\Rightarrow c\cdot\big(T_1(v_1,\dots 0), T_1(0,v_2,\dots 0), \dots, T_1(0,\dots,v_m)\big) + \big(T_2(v_1,\dots 0), T_2(0,v_2,\dots 0), \dots, T_2(0,\dots,v_m)\big)$$

$$\Rightarrow c\cdot R\big(T_1(v_1,v_2,\dots,v_m)\big) + R\big(T_2(v_1,v_2,\dots,v_m)\big)$$

(from equation (i))

Therefore, $R(c\cdot T_1 + T_2) = c\cdot R(T_1) + R(T_2)$.

Hence the transformation $R$ is linear.

Suppose, $R(T) = 0_Q$. for some $T \in L(\mathbb{P}, \mathbb{W})$.

Let's evaluate it at $(v_1, v_2, \dots, v_m) \in \mathbb{W}_1 \times \dots \times \mathbb{W}_m$.

$$R\big(T(v_1,v_2,\dots,v_m)\big) = (0_{\mathbb{W}}, 0_{\mathbb{W}}, \dots, 0_{\mathbb{W}})$$

$$= \big(A(v_1), A(v_2), \dots, A(v_m)\big) = (0_{\mathbb{W}}, \dots, 0_{\mathbb{W}})$$

$$A(v_1) = T(v_1, 0, \dots 0) = 0_{\mathbb{W}}$$

$$A(v_2) = T(0, v_2, \dots 0) = 0_{\mathbb{W}}$$

$$\vdots$$

$$A(v_m) = T(0, 0, \dots, v_m) = 0_{\mathbb{W}}$$

---

$$T\big(v_1 + 0 + \dots + 0,\ 0 + v_2 + 0 \dots + 0,\ \dots,\ 0 + 0 + \dots + v_m\big) = 0_{\mathbb{W}}$$

$$\Rightarrow T(v_1, v_2, \dots, v_m) = 0_{\mathbb{W}}.$$

Since for all $(v_1, \ldots, v_m) \in \mathbb{P}$, $T(v_1, \ldots, v_m) = 0_{\mathbb{W}}$ that means $T$ is $0_{L(\mathbb{P}, \mathbb{W})}$.

Therefore, $R(T) = 0_Q \longrightarrow T = 0_{L(P,W)}$.

In other words, $\ker(T) = \{ 0_{L(P,W)} \}$.
Therefore $R$ is injective.

Check for surjectivity.

$$R\big(T(v_1, v_2, \ldots v_m)\big) := \big(A_1(v_1), A_2(v_2), \ldots, A_m(v_m)\big)$$

$A_1(v_1) = T(v_1, 0, 0, \ldots 0)$, $A_2(v_2) = T(0, v_2, \ldots 0)$, $\ldots$, $A_m(v_m) = T(0, \ldots, v_m)$

We have to show that for every vector $(A_1(v_1), \ldots, A_m(v_m))$, there exists a vector $T(v_1, v_2, \ldots, v_m)$.
Such that $R(T(v_1, \ldots, v_m)) = (A(v_1), \ldots, A_m(v_m))$

Choose any arbitrary vector,

$$\big(A(v_1), A(v_2), \ldots, A(v_m)\big) \in L(V_1, W) \times \cdots \times L(V_m, W)$$

$$\Rightarrow \big(T(v_1, \ldots, 0), \ldots, T(0, 0, \ldots, v_m)\big)$$

Since the vector in codomain has $T(v_1, \ldots, 0), \ldots, T(0, 0, \ldots, v_m)$ so such a vector is the result of the mapping by $R(T(v_1, v_2, \dots, v_m))$ as per the definition of $\mathbb{R}$.

Therefore we choose any arbitrary vector in codomain, it's preimage will always exists in the domain because of the way $R$ is defined. Hence the map $R$ is surjective.

Hence $L(V_1 \times V_2 \times \dots \times V_m, \mathbb{W})$ is isomorphic to $L(V_1, W) \times L(V_2, W) \times \dots \times L(V_m, W)$. (Proved)

16. For vector spaces $\mathbb{U}, \mathbb{V}, \mathbb{W}$, such that $\mathbb{U}$ and $\mathbb{V}$ are subspaces of $\mathbb{W}$ show that for $w_1, w_2 \in \mathbb{W}$, if we have $w_1 + \mathbb{U} = w_2 + \mathbb{V}$, then $\mathbb{U} = \mathbb{V}$.

| | |
|---|---|
| $w_1 \in \mathbb{W}$ | $\mathbb{U}, \mathbb{V}$ are subspaces of $\mathbb{W}$ |
| $w_2 \in \mathbb{W}$ | $w_1 + \mathbb{U} = w_2 + \mathbb{V}$. |

$$w_1 + \mathbb{U} := \{ w_1 + u \mid \forall u \in \mathbb{U} \}$$

$$w_2 + \mathbb{V} := \{ w_2 + v \mid \forall v \in \mathbb{V} \}$$

Since they are equal so, $w_1 + u = w_2 + v$ $(\forall u \in \mathbb{U}, v \in \mathbb{V})$

$\Rightarrow u = (w_2 - w_1) + v$

We know that, $w_1 + 0_{\mathbb{W}} = w_1 \in w_1 + \mathbb{U}$

Since $w_1 + u = w_2 + v$ and $w_1 \in w_1 + \mathbb{U}$ so we can also write, $w_1 = w_2 + v_0$ for some $v_0 \in \mathbb{V}$. —— (i)

Similarly, $w_2 + 0_{\mathbb{W}} = w_2 \in w_2 + \mathbb{V}$.
Since $w_1 + u = w_2 + v$ and $w_2 \in w_2 + \mathbb{V}$ so we can also write, $w_2 = w_1 + u_0$ for some $u_0 \in \mathbb{U}$. —— (ii)

We already have, $u = (w_2 - w_1) + v$
$= -v_0 + v$ (from (i))

Since $v \in \mathbb{V}$ and $v_0 \in \mathbb{V}$ so $-v_0 + v \in \mathbb{V}$ so $u \in \mathbb{V}$.
Therefore, $u \in \mathbb{U} \longrightarrow u \in \mathbb{V}$ hence $\mathbb{U} \subseteq \mathbb{V}$.

We also have, $v = (w_1 - w_2) + u$
$= -u_0 + u$ (from (ii))

Since $u \in \mathbb{U}$ and $u_0 \in \mathbb{U}$ so $-u_0 + u \in \mathbb{U}$ so $v \in \mathbb{U}$.
Therefore, $v \in \mathbb{V} \longrightarrow v \in \mathbb{U}$ hence $\mathbb{V} \subseteq \mathbb{U}$.

Since $\mathbb{U} \subseteq \mathbb{V}$ and $\mathbb{V} \subseteq \mathbb{U}$ so $\mathbb{U} = \mathbb{V}$ (proved)

19. For finite dimensional vector spaces $\mathbb{U} \subseteq \mathbb{V}$, show that there exists a subspace $\mathbb{W}$ of $\mathbb{V}$ such that $\dim(\mathbb{W}) = \dim(\mathbb{V}/\mathbb{U})$ and $\mathbb{V} = \mathbb{U} \bigoplus \mathbb{W}$. Further, show that $\mathbb{V}$ is isomorphic to $\mathbb{U} \times \mathbb{V}/\mathbb{U}$.

Suppose, $\dim(\mathbb{V}/\mathbb{U}) = n$.
Let the basis for quotient space $\mathbb{V}/\mathbb{U}$ be -

$$B_1 = \{v_1 + \mathbb{U}, v_2 + \mathbb{U}, \ldots, v_n + \mathbb{U}\}.$$

Given, $\dim(\mathbb{W}) = \dim(\mathbb{V}/\mathbb{U}) = n$

Let the basis for subspace $\mathbb{W}$ be-

$B_2 = \{v_1, v_2, ..., v_n\}$.

We have to prove that such a $B_2$ exists in $\mathbb{V}$ so that $B_2$ forms a basis for $\mathbb{W}$ such that $\dim(\mathbb{W}) = \dim(\mathbb{V}/\mathbb{U})$.

In order to prove that $B_2$ is a basis - $B_2$ must span $\mathbb{W}$ and it must be a LI set. Since we have to prove the existence of $B_2$ so we pick the $B_2$ such that $B_2$ spans the subspace $\mathbb{W}$.

Define $\mathbb{W} := \langle\{v_1, v_2, ..., v_n\}\rangle$.

We have to show that $\{v_1, v_2, .., v_n\}$ is a LI set.

Consider, $\alpha_1 \cdot v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0_{\mathbb{V}}$ $(\forall \alpha_i \in \mathbb{F})$

Consider,

$$\alpha_1(v_1 + \mathbb{U}) + \alpha_2(v_2 + \mathbb{U}) + \cdots + \alpha_n(v_n + \mathbb{U}).$$

$$\left.\begin{aligned} &= (\alpha_1 v_1 + \mathbb{U}) + (\alpha_2 v_2 + \mathbb{U}) + \cdots + (\alpha_n v_n + \mathbb{U}) \\ &= (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) + \mathbb{U}. \end{aligned}\right\} \text{by the rule of vector add \& scalar mult of } \mathbb{V}/\mathbb{U}.$$

—— (i)

Since we know $\{v_1 + \mathbb{U}, ..., v_n + \mathbb{U}\}$ is a LI set so their linear combination produces $0_{\mathbb{V}/\mathbb{U}}$ implies all the coefficients are 0.

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0_{\mathbb{V}}$

$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_n v_n + \mathbb{U} = 0_{\mathbb{V}} + \mathbb{U} = \mathbb{U} = 0_{\mathbb{V}/\mathbb{U}}$.

$\Rightarrow \alpha_1 (v_1 + \mathbb{U}) + \cdots + \alpha_n (v_n + \mathbb{U}) = 0_{\mathbb{V}/\mathbb{U}}$. (from (i)).

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \cdots \alpha_n = 0$

Therefore $\{v_1, v_2, \cdots, v_n\}$ is a LI set.
Hence there exists a subspace $\mathbb{W}$ such that the $\dim(\mathbb{W}) = \dim(\mathbb{V}/\mathbb{U})$.

We have to prove that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$.

Suppose $v \in \mathbb{V}$.
Therefore, $v + \mathbb{U} \in \mathbb{V}/\mathbb{U}$.

$\Rightarrow v + \mathbb{U} = c_1 (v_1 + \mathbb{U}) + c_2 (v_2 + \mathbb{U}) + \cdots + c_n (v_n + \mathbb{U})$.

$\Rightarrow v + \mathbb{U} = (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) + \mathbb{U}$.

Therefore, $v - (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \in \mathbb{U}$.

consider, $v = \underbrace{v - (c_1 v_1 + \cdots + c_n v_n)}_{\in \mathbb{U}} + (c_1 v_1 + \cdots + c_n v_n)$

Because $\{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{W}$,

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n \in \mathbb{W}$.

Therefore, $$v = \underbrace{\left(v - \sum_{i=1}^{n} c_i v_i\right)}_{\in \mathbb{U}.} + \underbrace{\left(\sum_{i=1}^{n} c_i v_i\right)}_{\in \mathbb{W}.}$$

$\Rightarrow v \in \mathbb{U} + \mathbb{W}$.

We started with $v \in \mathbb{V}$.

and we see now $v \in \mathbb{U} + \mathbb{W}$.

Therefore, $\mathbb{V} \subseteq \mathbb{U} + \mathbb{W}$.

Since $\mathbb{U}$ & $\mathbb{W}$ are the subspaces of $\mathbb{V}$ and $\mathbb{U} + \mathbb{W}$ being another subspace of $\mathbb{V}$, it is obvious that,

$\mathbb{U} + \mathbb{W} \subseteq \mathbb{V}$.

Therefore, $\mathbb{V} = \mathbb{U} + \mathbb{W}$.

We have already seen, $\dim(\mathbb{W}) = \dim(\mathbb{V}/\mathbb{U})$

$\Rightarrow \dim(\mathbb{W}) = \dim(\mathbb{V}) - \dim(\mathbb{U})$.

$\Rightarrow \dim(\mathbb{V}) = \dim(\mathbb{W}) + \dim(\mathbb{U})$ —— (i)

Moreover, $\dim(\mathbb{V}) = \dim(\mathbb{U} + \mathbb{W})$ —— (ii)

We also know that -

$$\dim(\mathbb{U}+\mathbb{W}) = \dim(\mathbb{W}) + \dim(\mathbb{U}) - \dim(\mathbb{W}\cap\mathbb{U})$$

$$\Rightarrow \dim(\mathbb{V}+\mathbb{W}) = \dim(\mathbb{V}) - \dim(\mathbb{W}\cap\mathbb{U}). \text{ (from (i))}$$

$$\Rightarrow \dim(\mathbb{V}) = \dim(\mathbb{U}) - \dim(\mathbb{W}\cap\mathbb{U}) \text{ (from (ii))}$$

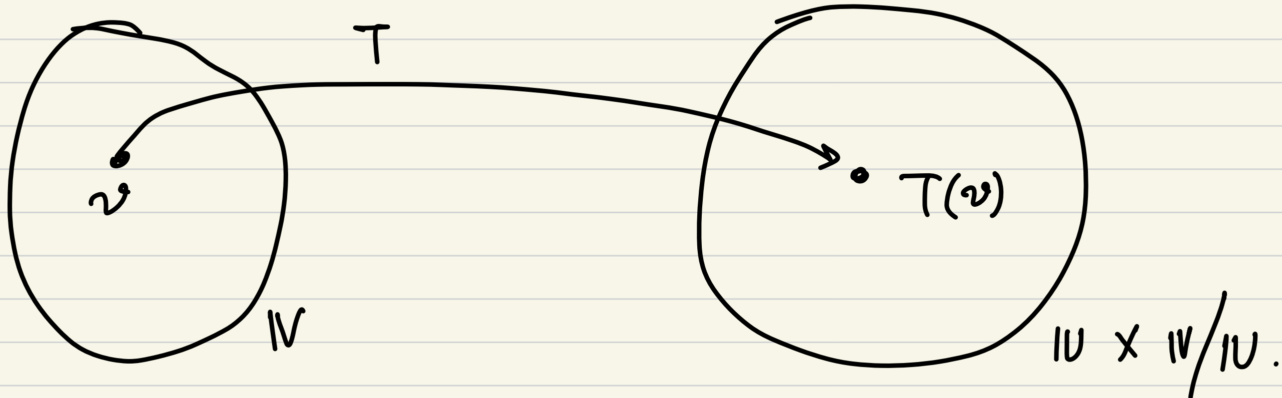
$$\Rightarrow \dim(\mathbb{W}\cap\mathbb{U}) = 0$$

$$\Rightarrow \mathbb{W}\cap\mathbb{U} = \{0_v\}.$$

This implies, $\mathbb{W}+\mathbb{U} = \mathbb{W}\oplus\mathbb{U}$.

Therefore, $\mathbb{V} = \mathbb{W}+\mathbb{U} = \mathbb{W}\oplus\mathbb{U}$. (Proved).

We have to prove that $T: \mathbb{V} \longrightarrow \mathbb{U} \times \mathbb{V}/\mathbb{U}$ is a linear bijection (isomorphism).




$$v \overset{T}{\longmapsto} T(v) = (u, v+\mathbb{U}) \quad (u \in \mathbb{U}) \text{ —— (i)}$$

$v \in \mathbb{V}$

Suppose, $v = c_1v_1 + c_2v_2 + \dots + c_nv_n$ where

$\{v_1, v_2, \dots, v_n\}$ is basis of $\mathbb{V}$.

Check the dimension:

$$\dim(\mathbb{U} \times \mathbb{V}/\mathbb{U}) = \dim(\mathbb{U}) + \dim(\mathbb{V}/\mathbb{U})$$
$$= \dim(\mathbb{U}) + \dim(\mathbb{V}) - \dim(\mathbb{U})$$
$$= \dim(\mathbb{V})$$

Since the dimension of $\mathbb{U} \times \mathbb{V}/\mathbb{U}$ and $\mathbb{V}$ are same so to prove isomorphism, proving only injectivity suffices and surjectivity is automatically implied.

Suppose, $T(v) = c_1 w_1 + c_2 w_2 + \dots + c_n w_n$ where $\{w_1, w_2, \dots, w_n\}$ is basis of $\mathbb{U} \times \mathbb{V}/\mathbb{U}$.

The map $T$ is defined as:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \xmapsto{T} T(v) := c_1 w_1 + \dots + c_n w_n. \quad \text{—(ii)}$$

claim: $T$ is linear map.

$$T(\alpha x_1 + x_2) = \alpha \cdot T(x_1) + T(x_2) \qquad x_1, x_2 \in \mathbb{V}.$$

Suppose, $x_1 = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$
$x_2 = d_1 v_1 + d_2 v_2 + \dots + d_n v_n$.

$$T\left( \alpha \cdot c_1 v_1 + \alpha c_2 v_2 + \cdots + \alpha c_n v_n + d_1 v_1 + \cdots + d_n v_n \right)$$

$$= T\left( (\alpha c_1 + d_1) v_1 + (\alpha c_2 + d_2) v_2 + \cdots + (\alpha c_n + d_n) v_n \right)$$

$$= (\alpha c_1 + d_1) w_1 + (\alpha c_2 + d_2) w_2 + \cdots + (\alpha c_n + d_n) w_n$$

(Eqn 2 : By def$^{\underline{n}}$ of T)

$$= \alpha \cdot \left( c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \right) + \left( d_1 w_1 + \cdots + d_n w_n \right)$$

$$= \alpha \cdot T\left( c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \right) + T\left( d_1 v_1 + d_2 v_2 + \cdots + d_n v_n \right)$$

(Eqn2 : By def$^{\underline{n}}$ of T)

$$= \alpha \cdot T(x_1) + T(x_2)$$

Hence T is a linear map.

We need to prove that T is injective map.

Choose a vector $v \in \ker(T)$.

$\Rightarrow v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \longmapsto T(v) = c_1 w_1 + \cdots + c_n w_n$

$T(v)$ must map to zero vector of $\mathbb{U} \times \mathbb{V}/\mathbb{U}$.
because of $v \in \ker(T)$.

Let the zero vector of $\mathbb{U} \times \mathbb{V}/\mathbb{U}$ be $0_{\mathbb{U} \times \mathbb{V}/\mathbb{U}}$.

Therefore,

$$T(v) = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n = 0_{\mathbb{U} \times \mathbb{V}/\mathbb{U}}$$

Since $w_1, w_2, \ldots, w_n$ are linearly independent set of vectors therefore LC of $w_i$ giving rise to $0_{\mathbb{U} \times \mathbb{V}/\mathbb{U}}$ implies all coefficients are $0$.

$$\Rightarrow \quad c_1 = 0,\ c_2 = 0,\ c_3 = 0, \ldots, c_n = 0$$

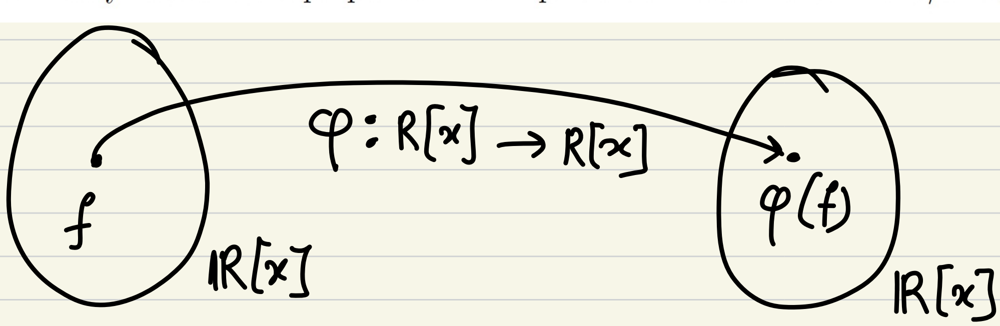
$$\Rightarrow v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0_{\mathbb{V}} + 0_{\mathbb{V}} + \cdots + 0_{\mathbb{V}} = 0_{\mathbb{V}}$$

Therefore, we started from $v \in \operatorname{Ker}(T)$ and we get $v = 0_{\mathbb{V}}$ implies $\operatorname{Ker}(T) = \{0_{\mathbb{V}}\}$. Hence $T$ is injective.

Since $\dim(\mathbb{V}) = \dim(\mathbb{U} \times \mathbb{V}/\mathbb{U})$ are same so $T$ is injective $\longrightarrow$ $T$ is surjective so $T$ is linear bijective.

Hence $T : \mathbb{V} \to \mathbb{U} \times \mathbb{V}/\mathbb{U}$ is an isomorphism so $\mathbb{V}$ and $\mathbb{U} \times \mathbb{V}/\mathbb{U}$ are isomorphic to each other.

22. Suppose $\varphi \in \mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$ is an injective linear operator on $\mathbb{R}[x]$ such that $\deg(\varphi(f)) \leq \deg(f)$ for any non-zero polynomial, $f$, in $\mathbb{R}[x]$. Prove that $\varphi$ is a surjection and that $\deg(\varphi(f)) = \deg(f)$ for every non-zero polynomial $f$. [**Hint**: One may consider the properties of an operator *restricted* to a subset/subspace of its domain.]



Consider the degree of polynomial $= n$ $(n \geq 0)$

$\forall n \geq 0$, $\varphi : \mathbb{R}[x]_n \longrightarrow \mathbb{R}[x]_n$.

Since $\deg(\varphi(f)) \leq \deg(f)$ that means all of the vectors in codomain is of degree less or equal to $n$.

Since the dimension of domain & codomain of $\varphi$ are same and we have assumed some finite dimension ($= n+1$ for maximum degree of $f$ as $n$). Therefore the surjectivity need not be proved as injectivity will imply surjectivity. It is given such a $\varphi$ is injective.

$$f \xmapsto{\varphi} \varphi(f)$$

($n$ is countable infinite)

$(\deg = n)$ $(\deg \leq n)$

The codomain space $\mathbb{R}[x]_n$ is actually a subset or subspace of the domain space of $\mathbb{R}[x]_n$. Since for each $n \geq 0$, the dimension of domain is finite $(n+1)$ then the dimension of codomain is also finite $(n+1)$.

$\dim(\mathbb{R}[x]_n) = \dim(\mathbb{R}[x]_n) = n+1$. That means

For each $n \geq 0$, $\varphi$ is injective $\longrightarrow$ $\varphi$ is surjective

Therefore, $\phi$ is surjective. (proved)

We have to prove that $\deg(\varphi(f)) = \deg(f)$ for every non zero polynomial $f$.

Suppose, a polynomial $f \in \mathbb{R}[x]_m$ so $\deg(f) = m$.
We Know that, $\deg(\varphi(f)) \leq \deg(f)$.

That means $\varphi(f) \in \mathbb{R}[x]_{m-1}$.

Therefore, $\dim(\mathbb{R}[x]_{m-1}) = m-1+1 = m$.
But since $\varphi$ is an isomorphism (both surjective & injective) so the $\dim(\mathbb{R}[x]_{m-1})$ must be equal to $\dim(\mathbb{R}[x]_m)$ but it is a contradiction of the above fact that $\dim(\mathbb{R}[x]_{m-1}) = m < \dim(\mathbb{R}[x]_m)$

Therefore, $\varphi(f) \notin \mathbb{R}[x]_{m-1}$. The only way the contridiction is avoided if we assume, $\varphi(f) \in \mathbb{R}[x]_m$

$$\varphi(f) \in \mathbb{R}[x]_m \longrightarrow \deg(\varphi(f)) = m$$

Therefore, $\deg(\varphi(f)) = \deg(f)$. (Hence proved)

24. State whether the following assertions are true or false and provide brief justifications in support of/against them.
(a) For two operators $\varphi, \psi \in \mathcal{L}(\mathbb{V}, \mathbb{V})$ ($\mathbb{V}$ is a vector space), such that $\operatorname{im}(\varphi) \subset \ker(\psi)$, $(\varphi\psi)^2 = 0$.
(b) There cannot exist a linear operator, $\varphi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, such that $\operatorname{im}(\varphi) = \ker(\varphi)$ for $n = 2m - 1$, where $m \in \mathbb{N}$.
(c) If $\varphi_1, \varphi_2 \in \mathbb{V}'$, where $\mathbb{V}$ is a vector space over field $\mathbb{F}$, such that $\ker(\varphi_1) = \ker(\varphi_2)$, then there exists $\alpha \in \mathbb{F}$ so that $\varphi_1 = \alpha\varphi_2$.
(d) Suppose $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{L}(\mathbb{V}, \mathbb{V})$ are operators on the finite dimensional vector space $\mathbb{V}$ such that $\varphi_1\varphi_2\varphi_3 = \text{identity}_{\mathbb{V}}$. Then $\varphi_2$ is invertible and $\varphi_2^{-1} = \varphi_3\varphi_1$.
(e) Consider the differentiation operator, $\mathcal{D}$, on the vector space of polynomials over the real field, $\mathbb{R}[x]$ (i.e., $\mathcal{D} \in \mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$). For any $\lambda \in \mathbb{R}$, $\ker(\mathcal{D} - \lambda I)$ is a non-trivial subspace of $\mathbb{R}[x]$.

(a) $(\varphi \circ \psi)^2(v) = 0_V \quad \forall v \in V.$

$\Rightarrow \varphi(\psi(\varphi(\psi(v))))$

let $\psi(v) = x \in V.$

$\Rightarrow \varphi(\psi(\varphi(x)))$

Since $\mathrm{im}(\varphi) \subset \ker(\psi)$. that means,

vector $u \in \mathrm{Im}(\varphi) \longrightarrow u \in \ker(\psi).$

$u \in \mathrm{Im}(\varphi) \rightarrow \varphi(w) = u \quad \exists w \in V.$

$u \in \ker(\psi) \rightarrow \psi(u) = 0_V.$

$\Rightarrow \varphi(\psi(u)) \qquad$ suppose $\exists x \in V$ s.t. $\varphi(x) = u$

$\Rightarrow \varphi(0_V)$

$\Rightarrow 0_V.$

Therefore. $(\varphi \circ \psi)^2 = 0_V$ for all $v \in V$.

Hence it is a true statement.

(d) $\varphi_1, \varphi_2, \varphi_3 \in L(\mathbb{V}, \mathbb{V})$.

$\varphi_1 \varphi_2 \varphi_3 = I_{\mathbb{V}}$.

We have to prove/disprove that $\varphi_2^{-1}$ exists and $\varphi_2^{-1} = \varphi_3 \varphi_1$

$$\begin{tikzcd}
v \arrow[r, mapsto, "\varphi_3"] \arrow[d, mapsto, "I_{\mathbb{V}}."] & \varphi_3(v) \arrow[d, mapsto, "\varphi_2"] \\
\varphi_1(\varphi_2(\varphi_3(v))) & \varphi_2(\varphi_3(v)) \arrow[l, mapsto, "\varphi_1"']
\end{tikzcd}$$

$= I_{\mathbb{V}}(v)$

$\Rightarrow \; I_{\mathbb{V}} = \varphi_1 \circ \varphi_2 \circ \varphi_3$.

For $\varphi_2^{-1}$ to exist, $\varphi_2 \circ \varphi_2^{-1} = I_{\mathbb{V}}$ and $\varphi_2^{-1} \circ \varphi_2 = I_{\mathbb{V}}$.

Since $\varphi_1 \circ \varphi_2 \circ \varphi_3 = I_{\mathbb{V}}$.

$\Rightarrow \varphi_1 \circ \varphi_1^{-1} = I_{\mathbb{V}}$ where $\varphi_1^{-1} = \varphi_2 \circ \varphi_3$.

(only the right inverse of $\varphi_1$ exists).

Since $\varphi_1 \in L(\mathbb{V}, \mathbb{V})$ and we have seen $\varphi_1$ right inverse exists and that implies $\varphi_1$ left inverse also exists since the dimension of domain & codomain of $\varphi_1$ are same.

Since $\varphi_1 \circ \varphi_2 \circ \varphi_3 = I_{IV}$.

$\Rightarrow \varphi_3^{-1} \circ \varphi_3 = I_{IV}$ where $\varphi_3^{-1} = \varphi_1 \circ \varphi_2$

(only the left inverse of $\varphi_3$ exists)

Similarly, $\varphi_3$ right inverse also exists as dimension of domain & codomain of $\varphi_3$ are same.

Therefore, $\varphi_1^{-1}$ exists and $\varphi_3^{-1}$ exists

We have, $\varphi_1 \circ \varphi_2 \circ \varphi_3 = I_{IV}$.

Precompose with $\varphi_1^{-1}$ in both side -

$\Rightarrow (\varphi_1^{-1} \circ \varphi_1) \circ \varphi_2 \circ \varphi_3 = \varphi_1^{-1} \cdot I_{IV} = \varphi_1^{-1}$

$\Rightarrow I_{IV} \circ \varphi_2 \circ \varphi_3 = \varphi_1^{-1}$

$\Rightarrow \varphi_2 \circ \varphi_3 = \varphi_1^{-1}$

Post compose with $\varphi_3^{-1}$ in both side -

$\Rightarrow \varphi_2 \circ (\varphi_3 \circ \varphi_3^{-1}) = \varphi_1^{-1} \circ \varphi_3^{-1}$

$\Rightarrow \varphi_2 \circ I_{IV} = \varphi_1^{-1} \circ \varphi_3^{-1}$

$\Rightarrow \varphi_2 = \varphi_1^{-1} \circ \varphi_3^{-1}$

Now consider, $\varphi_1^{-1} \circ \varphi_3^{-1} \circ \varphi_3 \circ \varphi_1$

$= \varphi_1^{-1} \circ (\varphi_3^{-1} \circ \varphi_3) \circ \varphi_1 \quad = \varphi_1^{-1} \circ I_{\mathbb{V}} \circ \varphi_1 \quad = \varphi_1^{-1} \circ \varphi_1 = I_{\mathbb{V}}$

Similarly, $\varphi_3 \circ \varphi_1 \circ \varphi_1^{-1} \circ \varphi_3^{-1} = \varphi_3 \circ I_{\mathbb{V}} \circ \varphi_3^{-1} = I_{\mathbb{V}}$.

Therefore, $\varphi_1^{-1} \circ \varphi_3^{-1}$ is inverse of $\varphi_3 \circ \varphi_1$

and $\varphi_3 \circ \varphi_1$ is the inverse of $\varphi_1^{-1} \circ \varphi_3^{-1}$. —— (i)

Since $\varphi_2 = \varphi_1^{-1} \circ \varphi_3^{-1}$ therefore

$\varphi_2^{-1} = (\varphi_1^{-1} \circ \varphi_3^{-1})^{-1} = \varphi_3 \circ \varphi_1$ (from (i)).

Clearly, $\varphi_2^{-1}$ exists and $\varphi_2^{-1} = \varphi_3 \circ \varphi_1$.

Hence the given statement was true.

(c) $\varphi_1, \varphi_2 \in \mathbb{V}'$ and $\ker(\varphi_1) = \ker(\varphi_2)$

We have to prove or disprove that $\exists \alpha \in \mathbb{F}$ s.t. $\varphi_1 = \alpha \varphi_2$.

If $\varphi_2$ is $0$ then $\varphi_1 = 0$ because $\ker(\varphi_2) = \ker(\varphi_1)$. Thus $\varphi_1 = \alpha . \varphi_2$ is true for all $\alpha \in \mathbb{F}$.

If $\varphi_2 \neq 0$ then there exists $u \in \mathbb{V}$ s.t. $T(u) \neq 0$ and for all $v \in \mathbb{V}$ we have -

$\varphi_2(u) . \varphi_2(v) - \varphi_2(u) . \varphi_2(v) = 0_F$

$\Rightarrow \varphi_2(\varphi_2(u) . v - \varphi_2(v) . u)$ ($\varphi_2$ is linear)

$\Rightarrow \varphi_1\left(\varphi_2(u).v - \varphi_2(v).u\right) \quad (\because Ker(\varphi_1) = Ker(\varphi_2))$

$\Rightarrow \varphi_2(u)\,\varphi_1(v) - \varphi_2(v).\varphi_1(u) \quad (\varphi_1 \text{ is linear})$

It means,
$$\varphi_1(v) = \frac{\varphi_2(v)\,\varphi_1(u)}{\varphi_2(u)} = \frac{\varphi_1(u)}{\varphi_2(u)}.\varphi_2(v)$$

Since it is true for all $v \in V$.

$\Rightarrow \varphi_1 = \alpha.\varphi_2$ where $\alpha = \dfrac{\varphi_1(u)}{\varphi_2(u)}$. for $u \in V$.

Hence this statement is true.

(b) $\varphi \in L(\mathbb{R}^n, \mathbb{R}^n)$, $im(\varphi) = ker(\varphi)$ for $n = 2m-1$ where $m \in \mathbb{N}$.

We have to prove/disprove that such a $\varphi$ can't exist.

Rank nullity theorem:

$$\dim(Im(\varphi)) + \dim(Ker(\varphi)) = \dim(\mathbb{R}^n)$$

$$\Rightarrow \dim(Ker(\varphi)) + \dim(Ker(\varphi)) = n$$

$$\Rightarrow \dim(Ker(\varphi)) = \frac{n}{2}.$$

Since the dimension is always a natural number, therefore, $\frac{n}{2}$ must be a natural number.

Given that, $n = 2m - 1$

Suppose, $m = 2$, then $n = 4 - 1 = 3$.

$\dim(\operatorname{Ker}(\varphi)) = \frac{3}{2} = 1.5 \notin \mathbb{N}$. (which is not possible)

That means such a $\varphi$ can't exist.

Therefore, the given statement was true.

(e) $\mathbb{R}[x] \xrightarrow{D} \mathbb{R}[x]$

$f(x) \longmapsto D(f(x))$

$$\operatorname{Ker}(D - \lambda I) := \left\{ f(x) \in \mathbb{R}[x] \,\middle|\, (D - \lambda I)(f(x)) = 0_{\mathbb{R}} \right\}$$

Suppose,

$$(D - \lambda I)(f(x)) = 0_{\mathbb{R}}.$$

$$\Rightarrow D(f(x)) - \lambda \cdot f(x) = 0_{\mathbb{R}}$$

Suppose, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$D(f(x)) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - \lambda(a_0 + a_1 x + a_2 x^2 + \cdots) = 0_{\mathbb{R}}$$

$$\Rightarrow (a_1 - \lambda a_0) + (2a_2 - \lambda a_1)x + (3a_3 - \lambda a_2)x^2 + \cdots = 0_{\mathbb{R}}$$

Since it is true for all $x \in \mathbb{R}$ that means each of the coefficient must be 0.

$$\Rightarrow a_1 - \lambda a_0 = 0 \quad \Rightarrow a_1 = \lambda a_0$$

$$2a_2 - \lambda a_1 = 0 \quad \Rightarrow a_2 = \frac{\lambda a_1}{2}$$

$$3a_3 - \lambda a_2 = 0 \quad \Rightarrow a_3 = \frac{\lambda a_2}{3}$$

$$\vdots \qquad\qquad \vdots$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + (\lambda a_0)x + \left(\frac{\lambda a_1}{2}\right)x^2 + \left(\frac{\lambda a_2}{3}\right)x^3 + \cdots$$

$$= a_0 + (\lambda a_0)x + \frac{\lambda^2 a_0}{2} \cdot x^2 + \frac{\lambda}{3} \cdot \frac{\lambda^2 a_0}{2} \cdot x^3 + \cdots$$

$$= a_0 \left[ 1 + \lambda x + \frac{\lambda^2 x^2}{2!} + \frac{\lambda^3 x^3}{3!} + \cdots \right]$$

$$= \left[ a_0 + (a_0 \lambda) \cdot x + \frac{a_0 \cdot \lambda^2}{2!} (x^2) + \cdots \right] \quad [\lambda \neq 0]$$

$$\Rightarrow \ker(D - \lambda I) = \left\langle \{1, x, x^2, \cdots\} \right\rangle = \mathbb{R}[x]$$

Suppose $\lambda = 0$ then, $\quad a_1 = 0, a_2 = 0, a_3 = 0 \cdots\cdots$

$f(x) = a_0 + 0 + 0 + \cdots \quad = a_0.$

$$\ker(D - \lambda I) = \langle \{1\} \rangle$$
$$= \mathbb{R}[x]_0 \text{ (all the polynomial of deg 0)}$$

When $\lambda = 0$, $\ker(D - \lambda I) = \mathbb{R}[x]_0$ (Non trivial)

When $\lambda \neq 0$, $\ker(D - \lambda I) = \mathbb{R}[x]$ (Trivial)

The trivial subspace of $\mathbb{R}[x]$ will be $\{0_{\mathbb{R}[x]}\}$ and $\mathbb{R}[x]$ itself.

That implies the given statement " $\ker(D - \lambda I)$ is non trivial subspace of $\mathbb{R}[x]$ " is false

25. Let $\mathbb{R}[x]$ be the vector space of polynomials over $\mathbb{R}$ and consider subspaces of $\mathbb{R}[x]$ given by $\mathbb{W}_1 = \{f(x) \in \mathbb{R}[x] : f(0) = f(10)\}$ and $\mathbb{W}_2 = \{f(x) \in \mathbb{R}[x] : f(0) = f(10) = 0\}$. Then $\mathbb{R}[x]/\mathbb{W}_1$ is isomorphic to $\mathbb{R}[x]/\mathbb{W}_2$. Prove if it is true or give a counterexample to disprove it.

$\mathbb{R}[x] = \{ f \mid f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots, \ \forall a_i \in \mathbb{R} \}$

$\mathbb{W}_1 = \{ f \in \mathbb{R}[x] \mid f(0) = f(10) \}$ is subspace of $\mathbb{R}[x]$

$\mathbb{W}_2 = \{ f \in \mathbb{R}[x] \mid f(0) = f(10) = 0 \}$ is subspace of $\mathbb{R}[x]$

Consider a vector $v \in \mathbb{R}[x]/\mathbb{W}_1$

$v = f(x) + \mathbb{W}_1$ where $f(x) \in \mathbb{R}[x]$.

If we have to show isomorphism b/w $\mathbb{R}[x]/\mathbb{W}_1$ and $\mathbb{R}[x]/\mathbb{W}_2$ then there must exist a transformation $T: \mathbb{R}[x]/\mathbb{W}_1 \to \mathbb{R}[x]/\mathbb{W}_2$ such that $T$ is linear bijection.

We define the $T$ in such a way that,

$$T(f(x) + \mathbb{W}_1) = g(x) + \mathbb{W}_2 \qquad (f(x), g(x) \in \mathbb{R}[x])$$

claim: Such a $T$ can never be isomorphic.

proof: We apply proof by contradiction.

Assumption: $\mathbb{R}[x]/\mathbb{W}_1$ is isomorphic to $\mathbb{R}[x]/\mathbb{W}_2$

Since the dimension of $\mathbb{R}[x]$ is infinite so the dimension of the domain & codomain is also infinite.

Since, $\mathbb{R}[x]/\mathbb{W}_1$ is isomorphic to $\mathbb{R}[x]/\mathbb{W}_2$ then it must be true for any finite dimension of $\mathbb{R}[x]$.

Suppose, $\mathbb{R}[x]$ has dimension of 1. that means

$$\mathbb{R}[x]_0 = \{ f(x) \mid f(x) = c, \forall c \in \mathbb{R} \}$$

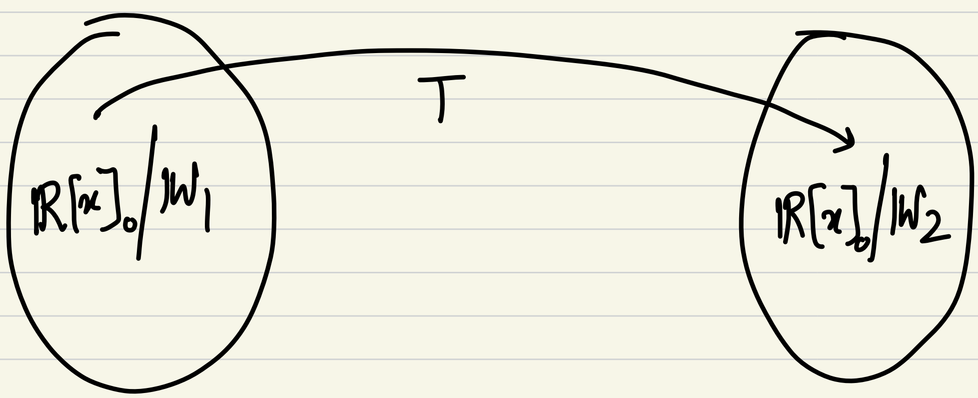
$$\mathbb{W}_1 = \{ f(x) \mid f(0) = f(10) \}$$

Since $f(x)$ in $\mathbb{R}[x]_0$ is independent of $x$ so in $\mathbb{W}_1$, all the polynomials of $\mathbb{R}[x]_0$ will be present as it satisfies the condition of $f(0) = f(10)$.

$$\mathbb{W}_1 = \{ f(x) \mid f(x) = c, \forall c \in \mathbb{R} \} = \mathbb{R}[x]_0$$

$$\mathbb{W}_2 = \{ f(x) \mid f(0) = f(10) = 0 \} = \{ 0_{\mathbb{R}[x]_0} \}$$

The only way to satisfy the condition that $f(0) = f(10) = 0$ is possible if the polynomial is zero polynomial.




Suppose there exists a linear transformation $T$ between $\mathbb{R}[x]_0/\mathbb{W}_1$ to $\mathbb{R}[x]_0/\mathbb{W}_2$. The claim is $T$ can never be isomorphic.

$$\dim(\mathbb{R}[x]_0/\mathbb{W}_1) = \dim(\mathbb{R}[x]_0) - \dim(\mathbb{W}_1)$$
$$= 1 - 1 = 0$$

$$\dim(\mathbb{R}[x]_0/\mathbb{W}_2) = \dim(\mathbb{R}[x]_0) - \dim(\mathbb{W}_2)$$
$$= 1 - 0 = 1.$$

Since the dimension of domain & codomain are not same so $T$ can never be isomorphism and hence the 2 spaces can never be isomorphic.

It is a contradiction of the assumption. Therefore the assumption was incorrect. Hence $\mathbb{R}[x]/\mathbb{W}_1$ is not isomorphic to $\mathbb{R}[x]/\mathbb{W}_2$. So the statement was false.