Quiz-2 solutions

1(a) True. Let $w \in (\mathbb{W}_1 + \mathbb{W}_2)^{\perp}$.

$\therefore$ For any $\bar{w} = w_1 + w_2$

with $w_1 \in \mathbb{W}_1$ & $w_2 \in \mathbb{W}_2$,

$$\langle \bar{w} | w \rangle = 0$$

$$\Rightarrow \langle w_1 + w_2 | w \rangle = 0$$

Choose $\bar{w} = w_1 + 0$ ($w_1 \in \mathbb{W}_1$ is arbitrary)

$$\therefore \langle w_1 | w \rangle = 0$$

$$\Rightarrow w \in \mathbb{W}_1^{\perp}$$

Illy $\langle w_2 | w \rangle = 0$

$$\Rightarrow w \in \mathbb{W}_2^{\perp}$$

$$\Rightarrow w \in \mathbb{W}_1^{\perp} \cap \mathbb{W}_2^{\perp}$$

$$\therefore \boxed{(\mathbb{W}_1 + \mathbb{W}_2)^{\perp} \subseteq \mathbb{W}_1^{\perp} \cap \mathbb{W}_2^{\perp}}$$

Next, let $w \in W_1^{\perp} \cap W_2^{\perp}$

$\Rightarrow \langle w | w_1 \rangle = 0 \ \forall w_1 \in W_1$

& $\langle w | w_2 \rangle = 0 \ \forall w_2 \in W_1$

$\therefore \langle w | w_1 \rangle + \langle w | w_2 \rangle = 0 + 0$
$= 0$

$\Rightarrow \langle w | w_1 + w_2 \rangle = 0 \ \forall w_1 \in W_1$
& $w_2 \in W_2$

$\Rightarrow w \in (W_1 + W_2)^{\perp}$

$\therefore \boxed{W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}}$ 2

From 1 & 2 the result follows

(b.) Suppose $v_1 = v_2$

$\Rightarrow \langle v_1 | v \rangle = \langle v_2 | v \rangle$

Conversely, let

$$\langle v_1 | v \rangle = \langle v_2 | v \rangle \quad \forall v \in V$$

Now, $v_1 - v_2 \in V$ as well.

$$\therefore \langle v_1 | v_1 - v_2 \rangle = \langle v_2 | v_1 - v_2 \rangle$$

$$\Rightarrow \langle v_1 - v_2 | v_1 - v_2 \rangle = 0$$

$$\Rightarrow \quad \| v_1 - v_2 \| = 0$$

$$\Rightarrow \quad v_1 = v_2.$$

Hence true.

(c) Suppose $\langle v_1 | v_2 \rangle = 0$

Consider $\langle v_1 + v_2 | v_1 - v_2 \rangle = 0$

We need to check whether this may hold under any condition.

$$\Rightarrow \langle v_1|v_1\rangle - \cancel{\langle v_1|v_2\rangle}^{0} + \cancel{\langle v_2|v_1\rangle}^{0} - \langle v_2|v_2\rangle = 0$$

$$\Rightarrow \quad \|v_1\|^2 = \|v_2\|^2.$$

Thus, if $\|v_1\| = \|v_2\|$

then $v_1 - v_2$ & $v_1 + v_2$

do turn out to be orthogonal.

Hence, the statement is false.

(d) Consider $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

& $S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

with $V = \mathbb{R}^3$

Then $S_1^{\perp} = \operatorname{span}\left(\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$

& $S_2^{\perp} = \text{span}\left(\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$

clearly $S_1^{\perp} \not\subseteq S_2^{\perp}$

Only if $S_1 = S_2$ do we have $S_1^{\perp} = S_2^{\perp}$.

(e) Following the Gram-Schmidt procedure, at every step we obtain.

$\{w_1, w_2, \ldots, w_i\}$ such that

it is an orthonormal set

& $\text{span}(\{v_1, v_2, \ldots, v_i\})$
$= \text{span}(\{w_1, w_2, \ldots, w_i\})$

However, each of the

w_k can be just as well replaced by $-w_k$. Thus we may choose

$$\{\pm w_1, \pm w_2, \dots, \pm w_m\}$$

which satisfies the properties.

This entails $\underbrace{2 \times 2 \times \dots \times 2}_{m\text{-times}}$ choices, i.e. 2^m choices.

Hence true.

2 (a) Consider

$$\| v_1 + \alpha v_2 \|^2$$

$$= \langle v_1 + \alpha v_2 | v_1 + \alpha v_2 \rangle$$

$$= \langle v_1 | v_1 \rangle + \bar{\alpha} \langle v_1 | v_2 \rangle$$

$$+ \alpha \langle v_2 | v_1 \rangle + \alpha \bar{\alpha} \langle v_2 | v_2 \rangle$$

If $\langle v_1 | v_2 \rangle = 0$

then:

$$\|v_1 + \alpha v_2\|^2 = \|v_1\|^2 + |\alpha|^2 \|v_2\|^2$$

$$\geq \|v_1\|^2$$

$$\Rightarrow \|v_1 + \alpha v_2\| \geq \|v_1\|.$$

Conversely, suppose

$$\|v_1 + \alpha v_2\| \geq \|v_1\|$$

$$\Rightarrow \|v_1 + \alpha v_2\|^2 \geq \|v_1\|^2$$

$$\Rightarrow \bar{\alpha} \langle v_1 | v_2 \rangle + \alpha \langle v_2 | v_1 \rangle + |\alpha|^2 \|v_2\|^2 \geq 0$$

(forall α)

choose $\alpha = -\dfrac{\langle v_1 | v_2 \rangle}{\| v_2 \|^2}$

(assume $v_2 \neq 0$)

$$\therefore \quad |\alpha|^2 = \frac{|\langle v_1 | v_2 \rangle|^2}{\left(\| v_2 \|^2\right)^2}$$

$$\Rightarrow \quad - \frac{\overline{\langle v_1 | v_2 \rangle} \cdot \langle v_1 | v_2 \rangle}{\| v_2 \|^2}$$

$$+ \quad - \frac{\langle v_1 | v_2 \rangle \cdot \langle v_2 | v_1 \rangle}{\| v_2 \|^2}$$

$$+ \frac{|\langle v_1 | v_2 \rangle|^2}{\| v_2 \|^2} \geq 0$$

$$\Rightarrow \quad -2 \frac{|\langle v_1 | v_2 \rangle|^2}{\| v_2 \|^2}$$

$$+ \frac{|\langle v_1|v_2\rangle|^2}{\|v_2\|^2} \geq 0$$

$$\Rightarrow \quad -\frac{|\langle v_1|v_2\rangle|^2}{\|v_2\|^2} \geq 0$$

$$\Rightarrow \quad |\langle v_1|v_2\rangle| = 0$$

$$\Rightarrow \quad \langle v_1|v_2\rangle = 0$$

(b) Let $U = im(\Pi)$.

we need to show that

$U^\perp = ker(\Pi)$ and

we will be done.

Now consider any $v_1 \in ker(\Pi)$

$\therefore \; \Pi(v_1) = 0$

Let $v_2 \in im(\Pi)$

so that $\exists v \in V$:

$$\pi(v) = v_2$$

& $\pi^2(v) = \pi(v_2)$

$= \pi(v)$ (idempotent)

$= v_2$.

$\therefore \pi(v_2) = v_2$

we need to establish that $\langle v_1 | v_2 \rangle = 0$.

or equivalently, by (a),

$\|v_2\| \leq \|v_2 + \alpha v_1\| \; \forall \alpha$

Now consider

$$\pi(v_2 + \alpha v_1)$$

$$= v_2$$

$\Rightarrow \|\,\tilde{\pi}\,(v_2+\alpha v_1)\| - \|v_2\|$

$$\leq \|v_2 + \alpha v_1\|$$

where we use the fact that $\|\tilde{\pi}(v)\| \leq \|v\|$ $\forall v$ & let

$$v = v_2 + \alpha v_1.$$

This completes the proof.

3 (a) If $\varphi \equiv 0$, then $\varphi(v_1) = 0$ (obviously).

Hence $\langle \varphi(v_1) | v_2 \rangle = 0$ $\forall v_1, v_2 \in V$.

Suppose $\langle \varphi(v_1) | v_2 \rangle = 0$ $\forall v_1, v_2$

& let $v_2 = \varphi(v_1)$

for some $v_1 \in V$.

$\Rightarrow \|\varphi(v_1)\| = 0$

$\Rightarrow \varphi(v_1) = 0$

$\Rightarrow v_1 \in \ker(\varphi)$.

But v_1 can also be chosen arbitrarily.

So every vector $v_1 \in V$ is in $\ker(\varphi)$

$\Rightarrow \varphi$ is the 'zero' operator

(B) $\langle \varphi(\alpha v_1 + \beta v_2) | \alpha v_1 + \beta v_2 \rangle$

$= \langle \alpha\varphi(v_1) + \beta\varphi(v_2) | \alpha v_1 + \beta v_2 \rangle$

[linearity of φ]

$= \alpha\bar{\alpha}\langle \varphi(v_1) | v_1 \rangle$

$$+ \beta\beta\langle\varphi(v_2)|v_2\rangle$$

$$+ \alpha\bar{\beta}\langle\varphi(v_1)|v_2\rangle$$

$$+ \beta\bar{\alpha}\langle\varphi(v_2)|v_1\rangle.$$

$$= |\alpha|^2\langle\varphi(v_1)|v_1\rangle + |\beta|^2\langle\varphi(v_2)|v_2\rangle$$

$$+ \alpha\bar{\beta}\langle\varphi(v_1)|v_2\rangle$$

$$+ \beta\bar{\alpha}\langle\varphi(v_2)|v_1\rangle.$$

Rearranging the terms suitably, we have the result.

(c) From (b) above we have

$$\alpha\bar{\beta}\langle\varphi(v_1)|v_2\rangle + \bar{\alpha}\beta\langle\varphi(v_2)|v_1\rangle$$

$$= \langle \varphi(\underbrace{\alpha v_1 + v_2}_{\bar{v}}) | \underbrace{\alpha v_1 + v_2}_{\bar{v}} \rangle$$

$$- |\alpha|^2 \langle \varphi(v_1) | v_1 \rangle$$

$$- |\beta|^2 \langle \varphi(v_2) | v_2 \rangle .$$

$\because \langle \varphi(v) | v \rangle = 0 \ \forall v$,

we have

$$\bar{\alpha}\beta \langle \varphi(v_1) | v_2 \rangle + \bar{\alpha}\beta \langle \varphi(v_2) | v_1 \rangle$$

$$= \cancel{\langle \varphi(\bar{v}) | \bar{v} \rangle} \to 0$$

$$- |\alpha|^2 \cancel{\langle \varphi(v_1) | v_1 \rangle} \to 0$$

$$- |\beta|^2 \cancel{\langle \varphi(v_2) | v_2 \rangle} \to 0$$

choose $\alpha = \beta = \underline{1}$

$$\Rightarrow \langle \phi(v_1) | v_2 \rangle + \langle \phi(v_2) | v_1 \rangle = 0 \quad \text{(i)}$$

choose $\alpha = 1, \beta = i$

$$\Rightarrow -i \langle \phi(v_1) | v_2 \rangle + i \langle \phi(v_2) | v_1 \rangle = 0 \quad \text{(ii)}$$

Divide (ii) by i & add with (i) to get

$$\langle \phi(v_2) | v_1 \rangle = 0$$

where v_1 & v_2 can be arbitrary.

By (a) above, this

By (…

implies ϕ is the 'zero' operator.

(d) Let $V = \mathbb{R}^2$.

& $\phi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Consider $v = \begin{bmatrix} x \\ y \end{bmatrix}$, $x, y \in \mathbb{R}$.

$\therefore \phi(v) = \begin{bmatrix} -y \\ x \end{bmatrix}$

$$\langle \phi(v) \mid v \rangle = 0$$

so over $\mathbb{R}$, the result in (c) does not hold.

(e) Suppose $\phi = \phi^*$.

$$\Rightarrow \langle \varphi(v_1) | v_2 \rangle$$

$$= \langle v_1 | \varphi^*(v_2) \rangle$$

$$= \langle v_1 | \varphi(v_2) \rangle$$

$$= \langle \varphi(v_2) | v_1 \rangle$$

[Since over real i.p.s conjugation has no effect]

From eqn (i) & (c) we thus have

$$\langle \varphi(v_1) | v_2 \rangle$$

$$+ \langle \varphi(v_2) | v_1 \rangle = 0$$

$\Rightarrow$

$$\Rightarrow \langle v_1 | \varphi(v_2) \rangle + \langle \varphi(v_2) | v_1 \rangle = 0$$

$$\Rightarrow \langle \varphi(v_2) | v_1 \rangle + \langle \varphi(v_2) | v_1 \rangle = 0$$

$$\Rightarrow \langle \varphi(v_2) | v_1 \rangle = 0 \ \forall \ v_1, v_2 \in V.$$

Thus, by (a) φ must identically be the 'zero' operator

4. Consider v_1 & $v_2 \in V$.

T.S.T.

$$\langle P(v_1) | v_2 \rangle$$

$$= \langle v_1 | P(v_2) \rangle$$

Now, we may write

v_i as

$$v_i = v_i^{\parallel} + v_i^{\perp}$$

where $v_i^{\parallel} \in U$ &

$v_i^{\perp} \in U^{\perp}$

for $i = 1, 2$.

$$\therefore \; P(v_i) = v_i^{\parallel}$$

$$\therefore \; \langle P(v_1) | v_2 \rangle$$

$$= \langle v_1^{\parallel} | v_2^{\parallel} + v_2^{\perp} \rangle$$

$$= \langle v_1^{\parallel} | v_2^{\parallel} \rangle$$

$$+ \cancelto{0}{\langle v_1^{\parallel} | v_2^{\perp} \rangle}$$

$[\because v_2^\perp \in U^\perp$
$\& \ v_1^{\parallel} \in U]$

$= \langle v_1^{\parallel} | v_2^{\parallel} \rangle$

IIly,

$\langle v_1 | P(v_2) \rangle$

$= \langle v_1^{\parallel} + v_1^\perp | v_2^{\parallel} \rangle$

$= \langle v_1^{\parallel} | v_2^{\parallel} \rangle$

$+ \cancelto{0}{\langle v_1^\perp | v_2^{\parallel} \rangle}$

$[\because v_1^\perp \in U^\perp$
$\& \ v_2^{\parallel} \in U$

$\therefore \langle P(v_1) | v_2 \rangle = \langle v_1 | P(v_2) \rangle$

Hence, $\underline{P = P^*}$