# End-sem 2023 - Solutions.

1 (a) Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & \epsilon \end{bmatrix}$

$$\therefore \chi_A(x) = \begin{vmatrix} x & -1 \\ 0 & x-\epsilon \end{vmatrix}$$

$$= x^2 - \epsilon x$$

$$\Rightarrow x = 0, \epsilon.$$

So let $v_1$ & $v_2$ be eigenvectors.

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\Rightarrow v_{12} = 0$$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, eigenvector corresponding to $0$.

||ly, $\begin{bmatrix} \epsilon & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$

$$\epsilon v_{21} = v_{22}$$

$$\Rightarrow v_2 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix},$$ eigenvector corresponding to $\epsilon$.

{or $v_2 = \begin{bmatrix} 1/\epsilon \\ 1 \end{bmatrix}$; eigenvector corresponding to $\epsilon$

Now, as as $\epsilon \to 0$

$v_2 = \begin{bmatrix} \infty \\ 1 \end{bmatrix}$

$\Downarrow$

not continuous eigen vector as $\epsilon \to 0$

Hence false.}

Now, as $\epsilon \to 0$,

$$v_2 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v_1$$

So, it turns out that $v_2$ ceases to exist as an independent eigenvector when some entry of $A$, i.e. $[A]_{22}$ in this case, is varied continuously.

Hence, the assertion is ~~false~~.

(b) $$(A - \lambda_k I) v_k = 0$$ [eigenvalue-eigenvector eqn]

$$\Rightarrow (A - \lambda_k I) v_k + \lambda v_k - \lambda v_k = 0$$

$$\Rightarrow (A-\lambda I)v_k + \lambda v_k - \lambda_k v_k = 0$$

$$\Rightarrow (A-\lambda I)v_k = (\lambda_k - \lambda)\, v_k$$

$$\Rightarrow \frac{1}{\lambda_k - \lambda}(A-\lambda I)v_k = v_k$$

[$\because$ $\lambda = \lambda_k$ as $\lambda$ is not an eigenvalue of $A$]

$$\Rightarrow \frac{v_k}{\lambda_k - \lambda} = (A-\lambda I)^{-1} v_k$$

[$\because$ $\lambda$ is not an eigenvalue of $A$, $(A-\lambda I)$ has a trivial kernel & so $(A-\lambda I)^{-1}$ exists]

Hence <u>true</u>

(c) Since $\mathbb{R}^n$ is an inner

product space, by

procedure, extend $v$ to an orthogonal basis for $\mathbb{R}^n$, given by $\{v, w_1, w_2, \dots, w_{n-1}\}$

Observe that

$$
\begin{aligned}
Aw_i &= uv^T w_i \\
&= u.\langle w_i | v \rangle_{\mathbb{R}^n} \\
&= 0 \quad (\because \text{ chosen basis is orthogonal})
\end{aligned}
$$

$$\therefore \dim(\mathrm{Ker}(A)) \geq n-1$$

$$\Rightarrow g.m._A(0) \geq n-1$$

Now, $a.m._A(0) \geq g.m._A(0)$

$$\Rightarrow \boxed{a.m._A(0) \geq n-1} \quad (i)$$

$\Rightarrow \boxed{a.m_A(0) \geq n-1}$ (i)

Also, $\boxed{a.m_A(0) \leq n.}$ (ii)

So, $\chi_A(x) = x^{n-1}(x-\lambda)$, $\lambda \neq 0$

or, $= x^n$.

If $a.m._A(0) = n-1$ &

$\lambda \neq 0$, then

clearly $a.m_A(0) = g.m_A(0)$

& $a.m_A(\lambda) = 1 = g.m_A(\lambda)$

Hence, $A$ is diagonalizable.

---

Suppose $v^T u \neq 0$

let $v^T u = \lambda \neq 0$.

$\Rightarrow \quad Au = u v^T u$

$= (v^T u) u$

$$= \lambda u.$$

$$\therefore \quad a.m._A(\lambda) = 1 = g.m._A(\lambda).$$

So, from previous conclusion, $A$ is diagonalizable.

Suppose $v^T u = 0$

Now, clearly

$$Av = uv^T v$$
$$= \|v\|^2 . u$$
$$= \alpha . u$$

$[\alpha \neq 0$ since $v \neq 0$, as $A$ is non-zero$]$

$$\therefore \langle \{u\} \rangle \in im(A).$$

Hence, $\dim(\operatorname{im}(A)) \geq 1$ (iii)

But, $\dim(\ker(A)) \geq n-1$ (iv)

By RNT.

$$\dim(\operatorname{im}(A)) + \dim(\ker(A)) = n.$$

Hence, equality must hold in (iii) & (iv).

$$\therefore \quad g.m_A(0) = n-1 = \dim(\ker(A)).$$

Now,

$$n \geq g.m_A(0) \geq n-1$$

We thus need to investigate whether $A = uv^T$ can have a non-zero eigenvalue in this case.

this case.

But we know that

trace $(A)$ = sum of eigenvalues of $A$.

Since $A$ can have at most one non-zero eigenvalue, this eigenvalue must equal the trace.

However, trace$(uv^T)$

$$= v^T u = 0$$

So $a.m(0) = n > n-1$

$$= g.m(0).$$

So, $A$ is not diagonalizable for $v^T u = 0$.

Hence true.

(d) Consider

(a) Consider

$$T^{-1}AT = J$$

where $J$ is the Jordan canonical form of $A$.

$$\therefore \det(e^A) = \det\left(e^{TJT^{-1}}\right)$$

$$= \det\left(Te^{J}T^{-1}\right) = \det(T)\det(e^J)\det(T^{-1})$$

$$= \det(e^J) = e^{\lambda_1} \cdot e^{\lambda_2} \cdots e^{\lambda_n}$$

[where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are (not necessarily distinct) eigenvalues of $A$ or $J$]

$$= e^{\sum_{i=1}^{n} \lambda_i}$$

$n$

Now, $\text{trace}(A) = \sum_{i=1} \lambda_i$

$\therefore \det(e^A) = e^{\text{trace}(A)}$.

Hence, true.

(e) Let $W = \ker(A)$.

$W_1 = \ker(A - I)$

$W_{-1} = \ker(A + I)$

Now, $A$ is diagonalizable

$\Rightarrow \dim(W) = g.m(0)$

$= a.m(0)$.

$= k_3$

$\therefore \text{rank}(A) = n - k_3$

(R-N theorem).

Now, by similar arguments:

$$\text{rank}(A-I) = n-k_1$$

& $\text{rank}(A+I) = n-k_2$.

So, whether rank of $A$ increases or decreases upon adding/subtracting $I$ to $A$ depends on $k_1, k_2, k_3$ and it is not true in general that rank must increase. Hence <u>false</u>

(f) $\text{rank}(A-\lambda_1 I) = n-k_1$

$$\Rightarrow \dim(\ker(A-\lambda_1 I)) = n-(n-k_1)$$
$$= k_1$$
$$= g.m.(\lambda_1)$$

$\Rightarrow$ The Jordan block for $\lambda_1$ is a diagonal block.

But, this says nothing about

$$\dim(\ker(A-\lambda_2 I)) = g.m(\lambda_2)$$

So we cannot infer about

diagonalizability. So the statement is <u>false</u>

2 (a)

| A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

| J | K | L | M | N | O | P | Q |
|---|---|---|---|---|---|---|---|
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |

| R | S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|---|
| 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |

LET ME TRY

12 5 20 13 5 20 18 25 27 → placeholder.

∴ Let A act on this message.

$$\Rightarrow A \begin{bmatrix} 12 & 13 & 18 \\ 5 & 5 & 25 \end{bmatrix} = \begin{bmatrix} 17 & 18 & 43 \\ -8 & -7 & -9 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \vdots & \vdots & \vdots \\ 20 & 20 & 27 \end{bmatrix}}_{P} \qquad \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots \\ 37 & 38 & 70 \end{bmatrix}}_{V}$$

check that $P$ is an invertible matrix & the columns of $P$, encoding the message, forms a basis for $\mathbb{R}^3$.

$$\therefore A = VP^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Any message that does not span $\mathbb{R}^3$ (must contain at least seven letters) or does not lead to an independent set will not allow us to determine $A$.

e.g. "TOO BAD" (does not span)

or, "FLY JON FLY". (dependent columns).

(b) To decode the message we apply $A^{-1}$.

Let $M$ encode the message.

$$\therefore M = A^{-1}\begin{bmatrix} 21 & 16 & 23 & 19 & 19 & 20 \\ 0 & -1 & 4 & 0 & 7 & 1 \\ 30 & 28 & 28 & 20 & 24 & 21 \end{bmatrix}$$

Now,

$$A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} 9 & 11 & 9 & 1 & 12 & 2 \\ \end{bmatrix}$$

$$\begin{bmatrix} 12 & 3 & 14 & 18 & 7 & 18 \\ 9 & 12 & 5 & 1 & 5 & 1 \end{bmatrix}$$

Thus the message translates to

$$\begin{array}{cccccccccc} 9 - & 12 - & 9 - & 11 - & 5 - & 12 - & 9 - & 14 - & 5 - & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ I & L & I & K & E & L & I & N & E & A \end{array}$$

$$\begin{array}{cccccccc} -18 - & 1 - & 12 - & 7 - & 5 - & 2 - & 18 - & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ R & A & L & G & E & B & R & A. \end{array}$$

i.e. "I LIKE LINEAR ALGEBRA"

(You may choose to "unfriend" this person, depending upon your disposition after this exam ☺).

3.(a) We know that

$$\langle u|v\rangle = \langle v|w\rangle = \langle u|w\rangle$$
$$= 0$$

$$\& \quad \langle u|u\rangle = \|u\|^2 = 1$$
$$\langle v|v\rangle = \|v\|^2 = 1$$
$$\langle w|w\rangle = \|w\|^2 = 1.$$

$$\therefore \quad \|u+v\|^2 = \langle u+v|u+v\rangle$$
$$= \langle u|u\rangle + \cancelto{0}{\langle u|v\rangle} + \cancelto{0}{\langle u|v\rangle} + \langle v|v\rangle$$
$$= 2.$$

Illy, $\|v+w\|^2 = 2 = \|w+u\|^2$

$$\therefore \quad \|u+v\|^2 + \|v+w\|^2 + \|w+u\|^2$$
$$= 6.$$

(b) $$\chi_A(x) = \begin{vmatrix} x-1 & -2 & 0 \\ -2 & x-1 & 0 \\ 0 & 0 & x-3 \end{vmatrix}$$

$$= (x-1)(x-1)(x-3) - \left(-2.\left(-2(x-3)\right)\right)$$

$$= (x-1)^2(x-3) - 4(x-3)$$

$$= \left((x-1)^2 - 4\right)(x-3)$$

$$= (x-1-2)(x-1+2)(x-3)$$

$$= (x-3)^2(x+1).$$

$\Rightarrow \quad x = 3, 3, -1 \pmod 5$

or, $x = 3, 3, 4.$

Thus $\lambda = 4$; $a.m(4) = 1$.

$\lambda = 3$; $a.m(3) = 2$.

(c) A has repeated eigenvalues at 0 (clearly). So we need to check $\dim(\ker(A))$.

Let $Av = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad v_2 = 0$$

Thus $v \in \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \right\rangle$.

Hence, $\dim(\ker(A)) = 1$.

Thus $g.m(0) = 1$.

but $a.m(0) = 2$.

Hence, A is not diagonalizable.

B has distinct eigenvalues at 1 & 0. So

it must be diagonalizable.

$$\chi_c(x) = (x^2 - 1)$$

$\Rightarrow$ C has repeated eigenvalues at 1.

i.e.

$$a.m(1) = 2.$$

So we need to check

$$\dim(\ker(C - I))$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_1 = v_2$$

$$\Rightarrow v \in \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\therefore \dim(\ker(C - I)) = \underline{1}.$$

Hence, $g.m(1) = 1 < a.m(1)$
$= 2.$

So $C$ is not diagonalizable.

(d) $\chi_A = \chi_{A^T}$ since determinant is invariant under transposition.

Since $A$ is diagonalizable, $\exists$ $T$ (invertible) such that

$$T^{-1} A T = \Delta \text{ (diagonal matrix)}.$$

$$\Rightarrow \Delta^T = \Delta = T^T A^T (T^{-1})^T$$
$$= T^T A^T (T^T)^{-1}.$$

So $A^T$ is also diagonalizable.

Now, from $\chi(x)$ we

know that

$1, -1, -2$ are eigenvalues of $A$ & $A^T$.

with
$$a.m(1) = 3$$
$$a.m(-1) = 2$$
$$a.m(-2) = 4.$$

Due to diagonalizability, we have

$$g.m(1) = a.m(1) = 3$$
$$g.m(-1) = a.m(-1) = 2$$
& $$g.m(-2) = a.m.(-2) = 4.$$

$$\therefore \dim(\ker(A^T + I)) = g.m(-1) = 2.$$

By RNT,
$$\dim(\operatorname{im}(A^T + I))$$

$$= 4+3+2-2$$
$$= 7.$$

$$\therefore \text{rank}(A^T+I) = 7.$$

(e) For linear dependence, the following eqn must have a non-trivial soln.

$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 4 & 5 & t \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{rref}\left(\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 4 & 5 & t \end{bmatrix}\right)$$

must not be identity.

So, let us obtain the rref.

of $\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 4 & 5 & t \end{bmatrix}$

$$\longrightarrow \begin{bmatrix} 1 & 2/3 & 5/3 \\ 0 & -11/3 & 22/3 \\ 0 & 7/3 & t-\frac{20}{3} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2/3 & 5/3 \\ 0 & 1 & -2 \\ 0 & 7/3 & t-20/3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & t-2 \end{bmatrix}$$

At this stage, unless $t=2$, we will obtain rref as identity.

Hence, $t=2$.

(f) Since $A$ is self-adjoint,

it must be diagonalizable.

Since it is diagonalizable, its minimal polynomial cannot have repeated roots.

Since 2 and 3 are its only eigenvalues & they are thus the only possible roots of the minimal polynomial, so

$$\mu_A(x) = (x-2)(x-3)$$

$$= x^2 - 5x + 6.$$

$$\boxed{\text{One also has} \quad A^2 - 5A + 6I = 0}$$

4.

(a) Let $p, q$ be the populations of $P$ & $Q$ at time $t$.

$$\therefore \quad \dot{p} = 0.02p - 0.01q$$
$$\dot{q} = -0.01p + 0.02q$$

$$\Rightarrow \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.02 & -0.01 \\ -0.01 & 0.02 \end{bmatrix}}_{A} \begin{bmatrix} p \\ q \end{bmatrix};$$

$$\begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}$$

$A = A^T$, hence diagonalizable.

$$(xI - A) = \begin{bmatrix} x - 0.02 & +0.01 \\ +0.01 & x - \ldots \end{bmatrix}$$

$$\left[ \quad \cdots \qquad x-0.02 \right]$$

$$\therefore \chi_A(x) = \det(xI - A)$$

$$= (x-0.02)^2 - 0.01^2$$

$$= (x-0.02+0.01)(x-0.02-0.01)$$

$\Rightarrow$ eigenvalues are

$$x = 0.01, 0.03.$$

For eigenvectors,

$$\begin{bmatrix} 0.02 & -0.01 \\ -0.01 & 0.02 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0.01 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

$$\Rightarrow \quad 0.02 v_{11} - 0.01 v_{12} = 0.01 v_{11}$$

$\Rightarrow$

$$v_{11} = v_{12}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

&

$$\begin{bmatrix} 0.02 & -0.01 \\ 0.01 & 0.02 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0.03 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$$

$$\Rightarrow 0.02\, v_{21} - 0.01\, v_{22} = 0.03\, v_{21}$$

$$\Rightarrow v_{21} = -v_{22}$$

$$v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Let $\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \cdots \end{bmatrix} \begin{bmatrix} \dot{\tilde{p}} \\ \cdots \end{bmatrix}$$

$$\begin{bmatrix} \tilde{q} \end{bmatrix} \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \tilde{q} \end{bmatrix}$$

$$= \begin{bmatrix} 0.02 & -0.01 \\ -0.01 & 0.02 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\tilde{p}} \\ \dot{\tilde{q}} \end{bmatrix} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.03 \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}$$

$$\tilde{p}(0) = \frac{30000}{\sqrt{2}}$$

$$\tilde{q}(0) = \frac{10000}{\sqrt{2}}$$

$$\Rightarrow \tilde{p}(t) = e^{0.01t} \cdot \frac{30000}{\sqrt{2}}$$

$$q(t) = e^{0.03t} \frac{10000}{\sqrt{2}}.$$

$$\therefore \begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{0.01t} \cdot 30000 \\ e^{0.03t} \, 10000 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{0.01t} \cdot 30000 \\ e^{0.03t} \cdot 10000 \end{bmatrix}$$

$$= \begin{bmatrix} 15000 e^{0.01t} - 5000 e^{0.03t} \\ 15000 e^{0.01t} + 5000 e^{0.03t} \end{bmatrix}$$

Ans

Clearly, the population $Q$ increases with $t$ and hence cannot go extinct.

But we need to check the expression for $p(t)$ & see if it can go to zero.

Since negative population makes no sense, zero population implies extinction.

let $p(t) = 0$

$$\Rightarrow \quad 3e^{0.01t} = e^{0.03t}$$

$$\Rightarrow \quad e^{0.02t} = 3$$

$$\Rightarrow \quad t = 50 \ln 3 > 0$$

so, $P$ goes extinct in

$50 \ln 3$ (unit is most likely in years)
($\approx 55$ yrs)

(b) Let $r(t)$ & $s(t)$ be the current populations of $R$ & $S$, respectively.

As in (a), we may write

$$\begin{bmatrix} \dot{r} \\ \dot{s} \end{bmatrix} = \underbrace{\begin{bmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix}}_{A} \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\begin{bmatrix} r(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} 10000 \\ 20000 \end{bmatrix}$$

$\chi_A(x) = (x+0.01)^2 \ldots$

$\Rightarrow$ eigenvalues are given by

$$x = 0, -0.02$$

eigenvectors are

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ for } x = 0$$

[since row-sum $= 0$]

&

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ for } x = 0.02.$$

(easy to check)

Thus, following similar steps as in (a) & letting:

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \tilde{r} \\ \tilde{s} \end{bmatrix}$$

$$\ldots \frac{\ }{\sqrt{2}} \quad \frac{\ }{\sqrt{2}} \Big] \Big[ s \Big] \Big),$$

we get

$$\begin{bmatrix} \dot{\tilde{r}} \\ \dot{\tilde{s}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -0.02 \end{bmatrix} \begin{bmatrix} \tilde{r} \\ \tilde{s} \end{bmatrix}$$

with $\begin{bmatrix} \tilde{r}(0) \\ \tilde{s}(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 10000 \\ 20000 \end{bmatrix}$

$$\Rightarrow \quad \tilde{r}(t) = \frac{30000}{\sqrt{2}}$$

$$\tilde{s}(t) = \frac{-10000}{\sqrt{2}} \cdot e^{-0.02t}.$$

Hence.

$$\begin{bmatrix} r(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{30000}{\sqrt{2}} \\ -\frac{10000}{\sqrt{2}} e^{-0.02t} \end{bmatrix}$$

$$= \begin{bmatrix} 150000 - 5000^{-0.02t} \\ 150000 + 5000 e^{-0.02t} \end{bmatrix}$$

In the long run.

$$\begin{bmatrix} r(t) \\ s(t) \end{bmatrix} \to 15000 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as $t \to \infty$

i.e. consensus in population (through cooperation).

(With just two interacting entities, it is immediately apparent that the rate of convergence to consensus depends on $|\lambda_2| = 0.02$ alone and the larger the value of $|\lambda_2|$, the sooner one reaches a neighbourhood of the consensus/agreement subspace spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Can you work out the general case for $n$-nodes & still show that $|\lambda_2|$ determines how soon one approaches consensus?)

pproaches consensus?)

5\. (a) Let $\beta = \{v_1, v_2, \ldots, v_k\}$ be a basis for $\mathrm{Ker}(A) \cap \mathrm{Im}(B)$.

Since $\mathrm{Im}(B) \cap \mathrm{Ker}(A) \subseteq \mathrm{Im}(B)$, extend $\beta$ to a basis for $\mathrm{Im}(B)$ as

$$\tilde{\beta} = \beta \cup \{w_1, w_2, \ldots, w_\ell\}$$

Thus, if we manage to show $\{Aw_1, Aw_2, \ldots, Aw_\ell\}$ is a basis for $\mathrm{Im}(AB)$, we will be done.

we need to show two things:

We need to show two things:
$\{Aw_1, Aw_2, \ldots Aw_\ell\}$ is linearly independent and also the fact that it spans $\mathrm{Im}(AB)$

Now, $\mathrm{Im}(AB)$ is the subspace obtained by the action of $A$ on $\mathrm{Im}(B)$

i.e. Suppose $v \in \mathrm{Im}(AB)$

$$\Rightarrow \quad v = A\Big(\underbrace{\sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{\ell} \beta_j w_j}_{\in \mathrm{Im}(B)}\Big)$$

But $v_i \in \mathrm{Ker}(A) \subseteq \mathrm{Ker}(A) \cap$

$\mathrm{Im}(B)$

$$\Rightarrow Av_i = 0 \ \forall \ i.$$

$$\Rightarrow v = A\left(\sum_{j=1}^{\ell} \beta_j w_j\right)$$

$$= \beta_j \sum_{j=1}^{\ell} Aw_j$$

Thus $v \in \langle \{Aw_1, Aw_2, \dots, Aw_\ell\} \rangle$

Suppose

$$\sum_{i=1}^{\ell} \gamma_i Aw_i = 0$$

$$\Rightarrow A\left(\sum \gamma_i w_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{\ell} \gamma_i w_i \in \ker(A) \cap \mathrm{Im}(B)$$

$$\Rightarrow \sum_{i=1}^{\ell} \gamma_i w_i = \sum_{r=1}^{k} \delta_r v_r$$

$\Rightarrow \gamma_i = 0 \ \forall i$ & $\delta_r = 0 \ \forall r$.

$\Rightarrow \{Aw_1, Aw_2, \ldots, Aw_\ell\}$ is also linearly independent.

$\therefore \dim(\mathrm{Im}(AB)) = \ell$.

& $\dim(\mathrm{Ker}(A) \cap \mathrm{Im}(B)) = k$.

while $\dim(\mathrm{Im}(B)) = k + \ell = \mathrm{rank}(B)$

Thus,

$$\boxed{\mathrm{rank}(B) = \dim(\mathrm{Im}(AB)) + \dim(\mathrm{Ker}(A) \cap \mathrm{Im}(B))}$$

(b) From (a)

$\dim(\mathrm{Im}(AB))$

$$= \operatorname{rank}(AB) = \operatorname{rank}(B) - \dim(\ker(A) \cap \operatorname{Im}(B))$$

Now by RNT on A, we have.

$$\boxed{n = \dim(\ker(A)) + \operatorname{rank}(A)} \quad (*)$$

Now, $\ker(A) \supseteq \ker(A) \cap \operatorname{Im}(B)$

$$\Rightarrow \dim(\ker(A)) \geq \dim(\ker(A) \cap \operatorname{Im}(B))$$

$$\Rightarrow -\dim(\ker(A)) \leq -\dim(\ker(A) \cap \operatorname{Im}(B))$$

$$\Rightarrow \operatorname{rank}(B) - \dim(\ker(A))$$

$$\leq \operatorname{rank}(B) - \dim(\ker(A)$$

$\leq \text{rank}(B) - \dim(\text{Ker}(A) \cap \text{Im}(B))$

$\Rightarrow \text{rank}(B) - (n - \text{rank}(A))$ $\longrightarrow$ from (*)

$\leq \text{rank}(AB)$

$\Rightarrow \text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB).$

(c) Let $v \in \text{Im}(C^T C)$

$\Rightarrow v = C^T C w$ for some $w \in \mathbb{R}^n$.

$= C^T u, \quad u = Cw \in \mathbb{R}^m$

$\in \text{Im}(C^T)$

$\therefore \boxed{\text{Im}(C^T C) \subseteq \text{Im}(C^T)}$ (*)

Now, in (a), let $p = m$

Now, in ([illegible]) set [illegible]

& $A = C^T$, $B = C$.

$$\Rightarrow \operatorname{rank}(C^T C) = \operatorname{rank}(C) - \dim(\ker(C^T) \cap \operatorname{Im}(C))$$

Suppose some vector $p \in \ker(C^T) \cap \operatorname{Im}(C)$

$\Rightarrow p = Cq$, for some $q \in \mathbb{R}^n$.

and $C^T p = 0$

$$\Rightarrow C^T C q = 0$$

$$\Rightarrow q^T C^T C q = 0$$

$$\Rightarrow \|Cq\|^2 = 0$$

$$\Rightarrow \|p\| = 0$$

$$\Rightarrow p = 0$$

$$\therefore \ker(C^T) \cap \operatorname{Im}(C) = \{0\}$$

$$\dim(\ker(C^T) \cap \operatorname{Im}(C)) = 0.$$

$$\Rightarrow \operatorname{rank}(C^TC) = \operatorname{rank}(C).$$

$$\Rightarrow \boxed{\dim(\operatorname{Im}(C^TC)) = \dim(\operatorname{Im}(C))}$$

$$(**)$$

From $(*)$ & $(**)$ it follows that

$$\operatorname{Im}(C^TC) = \operatorname{Im}(C).$$

Also, from RNT,

$$\boxed{m = \dim(\ker(C^TC)) + \operatorname{rank}(C^TC)}$$

$$m = \dim(\mathrm{Ker}(C^TC)) + \mathrm{rank}(C^TC)$$

&

$$\boxed{m = \dim(\mathrm{Ker}(C)) + \mathrm{rank}(C)}$$

since $\mathrm{Im}(C^TC) = \mathrm{Im}(C)$

it follows that:

$$\boxed{\dim(\mathrm{Ker}(C^TC)) = \dim(\mathrm{Ker}(C))} \quad (1)$$

Further, let $z \in \mathrm{Ker}(C)$

$$\Rightarrow \quad Cz = 0.$$

$$\therefore \quad C^TCz = 0$$

$$\Rightarrow z \in \mathrm{Ker}(C^TC)$$

$$\Rightarrow \quad \boxed{\mathrm{Ker}(C) \subseteq \mathrm{Ker}(C^TC)} \quad (2)$$

Thus, from (1) & (2), we

have $\mathrm{Ker}(C) = \mathrm{Ker}(C^TC)$

have $\mathrm{rk}(C) = \mathrm{rk}($[illegible]$)$.

You may obtain a more direct proof of part (c) using SVD. Try it!