VECTOR SPACES

Vector space:

A vector space consists of the following –

1. A field $\mathbb{F}$ of scalars.
2. A set $\mathbb{V}$ of objects, called vectors.
3. A rule or operation, called vector addition which associates with each pair of vectors u, v in $\mathbb{V}$ called the sum of u & v in such a way that –

The addition operation $(+)$ is defined as:

$+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$.

(Remember $+_{\mathbb{V}}$ is different than $+_{\mathbb{F}}$.)

(a) (Closure) $\forall u, v \ (u, v \in \mathbb{V})$

(b) (Associative) $\forall u, v, w \in \mathbb{V} \ [(u+v)+w = u+(v+w)]$

(c) (Additive idenity) $\forall u \in \mathbb{V} \ [u + 0_{\mathbb{V}} = 0_{\mathbb{V}} + u = u]$

(d) (Additive inverse) $\forall u \in \mathbb{V} \ [u + (-u) = (-u) + u = 0_{\mathbb{V}}]$

where $-u$ is called additive inverse of u. And the vector $v = -u$ must belong to $\mathbb{V}$.

(e) (Commutative) $\forall u, v \in \mathbb{V} \ [u + v = v + u]$

That means $+_{\mathbb{V}}$ operation is defined in such a way that the set of objects (or vectors), $\mathbb{V}$ forms an Abelian group under $+_{\mathbb{V}}$. So $(\mathbb{V}, +_{\mathbb{V}})$ is an Abelian group.

4. A rule (or operation) called scalar multiplication which is defined as:

$$\bullet : \mathbb{F} \times \mathbb{V} \longrightarrow \mathbb{V}$$

produces another vector in $\mathbb{V}$

takes the vector from set $\mathbb{V}$

takes the scalar from field $\mathbb{F}$

The operation $(\bullet)$ is different than $\times$ in $\mathbb{F}$ where $\times$ in $\mathbb{F}$ was for $\times : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$

The operation $\bullet$ is defined in such a way that -

(a) $\forall v \in \mathbb{V} \left[1_{\mathbb{F}} \bullet v = v \right]$

(b) $\forall v \in \mathbb{V}, \forall \alpha, \beta \in \mathbb{F} \left[(\alpha \times_{\mathbb{F}} \beta) \bullet v = \alpha \bullet \underbrace{(\beta \bullet v)}_{\text{another vector}} \right]$

another vector

Note that $\times_{\mathbb{F}}$ operation (multiplication over field $\mathbb{F}$) is kind of vanished if we associate a scalar to vector by $\bullet$ operation. which is scalar multiplication.

(c) $\forall_{v \in \mathbb{V}}, \forall_{\alpha,\beta \in \mathbb{F}} \left[(\alpha +_{\mathbb{F}} \beta) \cdot v = \underbrace{\alpha \cdot v}_{\text{a vector}} +_{\mathbb{V}} \underbrace{\beta \cdot v}_{\text{another vector}} \right]$

(d) $\forall_{u,v \in \mathbb{V}}, \forall_{\alpha \in \mathbb{F}} \left[\alpha \cdot (u +_{\mathbb{V}} v) = \alpha \cdot u +_{\mathbb{V}} \alpha \cdot v \right]$

(distribution of scalar multiplication $(\cdot)$ over vector addition $(+_{\mathbb{V}})$)

Then $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ is called a vector space where $\mathbb{F}$ is a field with its own set of operations as $(\mathbb{F}, +_{\mathbb{F}}, \times_{\mathbb{F}})$.

Vector space is a composite object consisting of a field of scalars, a set of vectors, 2 operations with special properties — vector addition & scalar multiplication. We say vector space $\mathbb{V}$ over the field $\mathbb{F}$.

The name "vector" is applied to elements of the set $\mathbb{V}$. and it should not be thought of a vector in physics. Any objects that satisfies the definitions can be termed as a vector space and members of set $\mathbb{V}$ will be called "vectors". It can be literally anything.

Ex 1: $\mathbb{F}$ is a field

$\mathbb{V}$ is a set of all n-tuples, $v, u \in \mathbb{V}$

$v = (\alpha_1, \alpha_2, \ldots, \alpha_n) \ \forall \alpha_i \in \mathbb{F}$

$u = (\beta_1, \beta_2, \ldots, \beta_n) \ \forall \beta_i \in \mathbb{F}$

Here u & v are called vectors.

$+_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ and $(\mathbb{V}, +_{\mathbb{V}})$ is an Abelian group.

$$u +_{\mathbb{V}} v := (\alpha_1, \alpha_2, \ldots, \alpha_n) +_{\mathbb{V}} (\beta_1, \beta_2, \ldots, \beta_n)$$

$$:= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$$

where $(\mathbb{F}, +, \times)$ is an field and $+$ operation comes from this field.

The scalar multiplication is defined as:

$\cdot : \mathbb{F} \times \mathbb{V} \rightarrow \mathbb{V}$ and $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ has certain rules for scalar multiplication.

suppose $c \in \mathbb{F}$ and $u \in \mathbb{V}$ then

$$c \cdot u := c \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$:= (c \times \alpha_1, c \times \alpha_2, \ldots, c \times \alpha_n)$$

where $(\mathbb{F}, +, \times)$ is a field and $\times$ operation comes from the field.

It can be verified that $+_{\mathbb{V}}, \cdot$ operations satisfy the conditions to be a vector space.

① $(\mathbb{V}, +_{\mathbb{V}})$ is an Abelian group.

closure: $u, v \in \mathbb{V}$, $u +_{\mathbb{V}} v \in \mathbb{V}$

Associative: $u +_{\mathbb{V}} (v +_{\mathbb{V}} w) = (u +_{\mathbb{V}} v) +_{\mathbb{V}} w$

$$(\alpha_1, \alpha_2, \dots, \alpha_n) +_{\mathbb{V}} \left[(\beta_1, \beta_2 \dots, \beta_n) +_{\mathbb{V}} (\gamma_1, \dots \gamma_n) \right]$$

$$= (\alpha_1, \dots, \alpha_n) +_{\mathbb{V}} \left((\beta_1 + \gamma_1), (\beta_2 + \gamma_2), \dots (\beta_n, \gamma_n) \right)$$

$$= \left(\alpha_1 + (\beta_1 + \gamma_1), \alpha_2 + (\beta_2 + \gamma_2), \dots, \alpha_n + (\beta_n + \gamma_n) \right)$$

If you notice $\mathbb{F}$ is a field so $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ must follow the associative property hence $+_{\mathbb{V}}$ is also associative.

vector additive identity: $u +_{\mathbb{V}} 0_{\mathbb{V}} = u$

$$(\alpha_1, \alpha_2, \dots, \alpha_n) +_{\mathbb{V}} (0, 0, \dots, 0) = (\alpha_1 + 0, \dots, \alpha_n + 0)$$
$$= (\alpha_1, \dots, \alpha_n) = u$$

Where 0 is the additive identity of field $\mathbb{F}$.

vector additive inverse: $u +_{\mathbb{V}} (-u) = 0_{\mathbb{V}}$

$(\alpha_1, \alpha_2, \ldots, \alpha_n) +_{\mathbb{V}} (-\alpha_1, -\alpha_2, \ldots, -\alpha_n)$

$= (\alpha_1 - \alpha_1, \alpha_2 - \alpha_2, \ldots, \alpha_n - \alpha_n)$

$= (0, 0, \ldots 0) = 0_{\mathbb{V}}$

Note $-\alpha_i$ is the additive inverse under $+$ over $\mathbb{F}$.

commutative: $u +_{\mathbb{V}} v = v +_{\mathbb{V}} u$

$(\alpha_1, \alpha_2, \ldots, \alpha_n) +_{\mathbb{V}} (\beta_1, \beta_2, \ldots, \beta_n)$

$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$

Note $(\mathbb{F}, +, \times)$ is a field so $+$ is commutative over the set $\mathbb{F}$ hence $+_{\mathbb{V}}$ is also commutative.

Therefore any $\mathbb{V}$ which contains $0_{\mathbb{V}}$, $-u \ \forall u \in \mathbb{V}$ and is closed under $+_{\mathbb{V}}$ will qualify to be vector space with vectors as n ordered tuples.

Similarly we can cheek the conditions for scalar multiplications.

$\cdot : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$ (scalar multiplication)

(a) $1_{\mathbb{F}} \cdot v = v$

$\Rightarrow 1_{\mathbb{F}} \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad ; \; \alpha_i \in \mathbb{F}$

$\Rightarrow (1_{\mathbb{F}} \times \alpha_1, 1_{\mathbb{F}} \times \alpha_2, \ldots, 1_{\mathbb{F}} \times \alpha_n)$

$\Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$\Rightarrow v$

where $(\mathbb{F}, +, \times)$ is a field and $\times$ is multiplication over the field $\mathbb{F}$.

(b) $(c+d) \cdot u = c \cdot u +_{\mathbb{V}} d \cdot u \quad ; \; c, d \in \mathbb{F}$

$(c+d) \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$\Rightarrow ((c+d) \times \alpha_1, (c+d) \times \alpha_2, \ldots, (c+d) \times \alpha_n)$

$\Rightarrow (c \times \alpha_1 + d \times \alpha_1, c \times \alpha_2 + d \times \alpha_2, \ldots, c \times \alpha_n + d \times \alpha_n)$

$\Rightarrow (c \times \alpha_1, c \times \alpha_2, \ldots, c \times \alpha_n) +_{\mathbb{V}} (d \times \alpha_1 + d \times \alpha_2 + \cdots + d \times \alpha_n)$

$\Rightarrow c \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n) +_{\mathbb{V}} d \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$\Rightarrow c \cdot u +_{\mathbb{V}} d \cdot u$

(c) $(c \times d) \cdot u = c \cdot (d \cdot u)$

$(c \times d) \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$\Rightarrow ((c \times d) \times \alpha_1, (c \times d) \times \alpha_2, \ldots, (c \times d) \times \alpha_n)$

$\Rightarrow (c \times (d \times \alpha_1), c \times (d \times \alpha_2), \ldots, c \times (d \times \alpha_n))$

$\Rightarrow c \cdot (d \times \alpha_1, d \times \alpha_2, \ldots, d \times \alpha_n)$

$\Rightarrow c \cdot (d \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n))$

$\Rightarrow c \cdot (d \cdot u)$

(d) $c \cdot (u +_{\mathrm{IV}} v) = c \cdot u +_{\mathrm{IV}} c \cdot v$

$c \cdot ((\alpha_1, \alpha_2, \ldots, \alpha_n) +_{\mathrm{IV}} (\beta_1, \beta_2, \ldots, \beta_n))$

$\Rightarrow c \cdot (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$

$\Rightarrow (c \times (\alpha_1 + \beta_1), c \times (\alpha_2 + \beta_2), \ldots, c \times (\alpha_n + \beta_n))$

$\Rightarrow (c \times \alpha_1 + c \times \beta_1, c \times \alpha_2 + c \times \beta_2, \ldots, c \times \alpha_n + c \times \beta_n)$

$\Rightarrow (c \times \alpha_1, c \times \alpha_2, \ldots, c \times \alpha_n) +_{\mathrm{IV}} (c \times \beta_1 + \cdots + c \times \beta_n)$

$\Rightarrow c \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n) +_{\mathrm{IV}} c \cdot (\beta_1, \beta_2, \ldots, \beta_n)$

$\Rightarrow c \cdot u +_{\mathrm{IV}} c \cdot v$

Ex2: The space of $m \times n$ matrices.

$\mathbb{F}$ is a field with $(\mathbb{F}, +, \times)$

m, n are positive integers

$\mathbb{F}^{m \times n}$ be set of all $m \times n$ matrices over field $\mathbb{F}$

$$\mathbb{V} = \mathbb{F}^{m \times n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m \times n} \right\}$$

where $a_{ij} \in \mathbb{F} \ \forall i, j$

We say the matrices as an object i.e. vector as they are member of the set $\mathbb{V}$.

Two vectors A and $B \in \mathbb{V}$ such that,

$+_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$.

and $(\mathbb{V}, +_{\mathbb{V}})$ forms an Abelian group.

The operation $+_{\mathbb{V}}$ is defined as: (vector addition)

$$A +_{\mathbb{V}} B := \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n} + \begin{bmatrix} b_{ij} \end{bmatrix}_{m \times n} \quad \forall a_{ij}, b_{ij} \in \mathbb{F}$$

$$:= \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}_{m \times n}$$ where $+$ is addition in field $\mathbb{F}$.

A scalar $\alpha \in \mathbb{F}$ and a vector $A \in \mathbb{V}$ such that,

$\bullet : \mathbb{F} \times \mathbb{V} \longrightarrow \mathbb{V}$

and $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \bullet)$ is a vector space.

The operation $\bullet$ is defined as : (scalar multiplication)

$$\alpha \bullet A := \alpha \bullet \left[a_{ij} \right]_{m \times n} \qquad \forall\, a_{ij} \in \mathbb{F}$$

$$:= \left[\alpha \times a_{ij} \right]_{m \times n}$$ where $\times$ is mult. in field $\mathbb{F}$.

We can prove that with the above definitions of the objects & 2 operations form a vector-space since they satisfy all conditions of the vector space.

Ex3: The space of functions from a set to a field.

$\mathbb{F}$ is a field with $(\mathbb{F}, +, \times)$ as its operations.
$\mathbb{V}$: Set of functions f such that $f : S \longrightarrow \mathbb{F}$

where S : Non empty set. (any set)

Now the functions will be termed as vectors.
Consider 2 vectors f and $g \in \mathbb{V}$.

Define vector addition $+_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$

$$\left(f +_{\mathbb{V}} g\right)(s) := \underbrace{f(s)}_{\in \mathbb{F}} + \underbrace{g(s)}_{\in \mathbb{F}}$$

$(\mathbb{F}, +, \times)$

Define scalar multiplication $\cdot : \mathbb{F} \times \mathbb{V} \longrightarrow \mathbb{V}$

$$\left(c \cdot f\right)(s) := \underbrace{c}_{\in \mathbb{F}} \times \underbrace{f(s)}_{\in \mathbb{F}}$$

$(\mathbb{F}, +, \times)$

Check if $\left(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot\right)$ is a vector space?

① Vector addition rules $\left(\mathbb{V}, +_{\mathbb{V}}\right)$ forms Abelian group.

Closure: $\left(f +_{\mathbb{V}} g\right)(s) \in \mathbb{V}$

Associative: $\left(f +_{\mathbb{V}} \left(g +_{\mathbb{V}} \omega\right)\right)(s) = \left(\left(f +_{\mathbb{V}} g\right) +_{\mathbb{V}} \omega\right)(s)$

$$\left(f +_{\mathbb{V}} \left(g +_{\mathbb{V}} \omega\right)\right)(s) = f(s) + \left(g +_{\mathbb{V}} \omega\right)(s)$$

$$= \left(f(s) + g(s)\right) + \omega(s) = \left(f + g\right)(s) + \omega(s)$$

$$= \left(f +_{\mathbb{V}} g\right)(s) + \omega(s) = \left(\left(f +_{\mathbb{V}} g\right) +_{\mathbb{V}} \omega\right)(s)$$

Additive identity vector: $\left(f +_{\mathbb{V}} 0_{\mathbb{V}}\right)(s) = f(s)$

$$\left(f +_{\mathbb{V}} 0_{\mathbb{V}}\right)(s) = f(s) + 0_{\mathbb{V}}(s)$$

where $0_{\mathbb{V}} : S \longrightarrow 0_{\mathbb{F}}$

Therefore, $\left(f +_{\mathbb{V}} 0_{\mathbb{V}}\right)(s) = f(s) + 0_{\mathbb{V}}(s)$

$$= f(s) + 0_{\mathbb{F}}$$

$$= f(s) \quad [\mathbb{F} \text{ is a field}]$$

Additive inverse vector: $\left(f +_{\mathbb{V}} (-f)\right)(s) = 0_{\mathbb{V}}(s)$

$$\left(f +_{\mathbb{V}} (-f)\right)(s) = f(s) + (-f)(s)$$

$$-f : S \longrightarrow \underbrace{-\left(f(s)\right)}$$

gives the additive inverse of $f(s)$ in $\mathbb{F}$

$$\Rightarrow \left(f +_{\mathbb{V}} (-f)\right)(s) = f(s) + (-f)(s)$$

$$= f(s) + \left(-f(s)\right)$$

$$= 0_{\mathbb{F}} \quad [\mathbb{F} \text{ is a field}]$$

$$= 0_{\mathbb{V}}(s)$$

Commutativity: $(f +_{\mathbb{V}} g)(s) = (g +_{\mathbb{V}} f)(s)$

$$\begin{aligned}(f +_{\mathbb{V}} g)(s) &= f(s) + g(s) \\ &= g(s) + f(s) \quad [\because \mathbb{F} \text{ is a field}] \\ &= (g +_{\mathbb{V}} f)(s)\end{aligned}$$

Similarly we can do the same thing for scalar multiplication.

Ex: The space of polynomial functions over a field $\mathbb{F}$.

$\mathbb{F}$ is a field

$\mathbb{V}$ is a set of all functions $f : \mathbb{F} \longrightarrow \mathbb{F}$

$$\begin{aligned}f(x) &:= c_0 + c_1 \times x + c_2 \times x \times x + \dots + c_n \times x \times x \dots \times x \\ &:= (c_0) + (c_1 \times x) + (c_2 \times x^2) + (c_3 \times x^3) + \dots + (c_n \times x^n)\end{aligned}$$

$c_i \in \mathbb{F}$, $x \in \mathbb{F}$, $f(x) \in \mathbb{F}$, $(\mathbb{F}, +, \times)$ is usual

So $f(x)$ is called polynomial functions on $\mathbb{F}$

Define the vector addition: $+_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$

$f, g \in \mathbb{V}$ s.t. $(f +_{\mathbb{V}} g)(x) := f(x) + g(x)$

Define the scalar multiplication: $\bullet : \mathbb{F} \times \mathbb{V} \longrightarrow \mathbb{V}$

$c \in \mathbb{F}$, $f \in \mathbb{V}$ s.t $(c \cdot f)(x) := c \times f(x)$

We can verify that $f +_{\mathbb{V}} g$, $c \cdot f$ are also polynomial functions if f, g are polynomial functions

$$f(x) = c_0 + c_1 \times x + c_2 \times x^2$$
$$g(x) = d_0 + d_1 \times x + d_2 \times x^2$$

$$\begin{aligned} f(x) + g(x) &= (c_0 + d_0) + (c_1 + d_1) \times x + (c_2 + d_2) \times x^2 \\ &= k_0 + k_1 \times x + k_2 \times x^2 \\ &= (f +_{\mathbb{V}} g)(x) \\ &= h(x) \quad \text{where } h : f +_{\mathbb{V}} g \end{aligned}$$

Notice $h(x)$ is again polynomial function.

We can also verify $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ is a vector space.

<u>Ex:</u> Consider $\mathbb{C}$: Field of complex numbers
$\mathbb{R}$: Field of real numbers

Suppose the field $\mathbb{F} = \mathbb{R}$
Consider the set $\mathbb{V} = \{ z \mid z \in \mathbb{C} \}$ (set of all complex numbers

vector z is defined as:

$$z := a + b \times i \quad \text{where} \quad a, b \in \mathbb{F} \text{ and } i = \sqrt{-1}$$

and $(\mathbb{F}, +, \times)$ is a field with defined operations

Vector addition is defined as:

$$\begin{aligned} z +_{\mathbb{V}} w &:= (a + b \times i) + (c + d \times i) \\ &:= (a + c) + (b + d) \times i \\ &:= e + f \times i \quad \left[\because \begin{aligned} e &= a + c \in \mathbb{F} \\ f &= b + d \in \mathbb{F} \end{aligned} \right] \end{aligned}$$

Scalar multiplication is defined as:

$$c \in \mathbb{F},\ z \in \mathbb{V} \quad \text{s.t.} \quad \begin{aligned} c \cdot z &:= c \times (a + b \times i) \\ &:= (c \times a) + (c \times b) \times i \end{aligned}$$

One can verify $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ is a vector space.

Ex: Prove the following things -

Let $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ be a vector space with the additive identity vector denoted as $0_{\mathbb{V}}$ and $(\mathbb{F}, +, \times)$ is a field of scalars with additive identity scalar as 0.

(i) $\forall c \in \mathbb{F} \quad [\ c \cdot 0_{\mathbb{V}} = 0_{\mathbb{V}}\ ]$

(ii) $\forall u \in \mathbb{V} \quad [\ 0_{\mathbb{F}} \cdot u = 0_{\mathbb{V}}\ ]$

<u>Proof:</u>

$$c \cdot 0_{\mathbb{V}} = c \cdot (0_{\mathbb{V}} +_{\mathbb{V}} 0_{\mathbb{V}})$$

$$\Rightarrow c \cdot 0_{\mathbb{V}} = c \cdot 0_{\mathbb{V}} +_{\mathbb{V}} c \cdot 0_{\mathbb{V}}$$

$$\Rightarrow c \cdot 0_{\mathbb{V}} +_{\mathbb{V}} (-_{\mathbb{V}}(c \cdot 0_{\mathbb{V}})) = c \cdot 0_{\mathbb{V}} +_{\mathbb{V}} c \cdot 0_{\mathbb{V}} +_{\mathbb{V}} (-_{\mathbb{V}}(c \cdot 0_{\mathbb{V}}))$$

The inverse of $c \cdot 0_{\mathbb{V}}$ exists because $(\mathbb{V}, +_{\mathbb{V}})$ is an Abelian group.

$$\Rightarrow 0_{\mathbb{V}} = c \cdot 0_{\mathbb{V}} +_{\mathbb{V}} 0_{\mathbb{V}} = c \cdot 0_{\mathbb{V}} \quad [\because \text{additive idenity of } (\mathbb{V}, +_{\mathbb{V}})\ ]$$

(Hence proved)

$$0_{\mathbb{F}} \cdot u = (0_{\mathbb{F}} + 0_{\mathbb{F}}) \cdot u$$

$$\Rightarrow 0_{\mathbb{F}} \cdot u = 0_{\mathbb{F}} \cdot u + 0_{\mathbb{F}} \cdot u$$

The inverse of $(0_{\mathbb{F}} \cdot u)$ will exist because $(\mathbb{V}, +)$ is an Abelian group.

$$\Rightarrow (0_{\mathbb{F}} \cdot u) +_{\mathbb{V}} (-_{\mathbb{V}}(0_{\mathbb{F}} \cdot u)) = 0_{\mathbb{F}} \cdot u +_{\mathbb{V}} 0_{\mathbb{F}} \cdot u +_{\mathbb{V}} (-_{\mathbb{V}}(0_{\mathbb{F}} \cdot u))$$

$$\Rightarrow 0_{\mathbb{V}} = 0_{\mathbb{F}} \cdot u +_{\mathbb{V}} 0_{\mathbb{V}}$$

$$\Rightarrow 0_{\mathbb{V}} = 0_{\mathbb{F}} \cdot u \quad [\text{ additive identity of } (\mathbb{V}, +_{\mathbb{V}})]$$

(Hence proved)

(iii) For a vector space $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$,

$c \in \mathbb{F}$, $u \in \mathbb{V}$ s.t $c \cdot u = 0_{\mathbb{V}} \longrightarrow c = 0_{\mathbb{F}}$ or $u = 0_{\mathbb{V}}$.

proof:

assume $c \neq 0$, $c \cdot u = 0_{\mathbb{V}}$.

Since $c \in \mathbb{F}$ and $\mathbb{F}$ is a field so multiplicative inverse of c will exist. $[c^{-1} \neq 0]$

$$\Rightarrow c^{-1} \cdot (c \cdot u) = c^{-1} \cdot 0_{\mathbb{V}} = 0_{\mathbb{V}} \text{ (already proved)}$$

$$\Rightarrow (c^{-1} \times c) \cdot u = 0_{\mathbb{V}}$$

$$\Rightarrow 1_{\mathbb{F}} \cdot u = 0_{\mathbb{V}}$$

$$\Rightarrow u = 0_{\mathbb{V}} \quad [\text{ Hence proved }]$$

assume $c = 0$ so $0 \cdot u = 0_{\mathbb{V}}$ which is already proved.

(iv) For a vector space $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$,

$$\forall u \in \mathbb{V} \left[(-1_{\mathbb{F}}) \cdot u = -_{\mathbb{V}} u \right]$$

We know that $\quad 0_{\mathbb{V}} = 0 \cdot u$

$$\Rightarrow 0_{\mathbb{V}} = \big(1 + (-1)\big) \cdot u$$

$$= 1 \cdot u + (-1) \cdot u$$

$$= u + (-1) \cdot u$$

$$= u + (-_{\mathbb{V}} u)$$

compare these 2 equations

$$= 0_{\mathbb{V}}$$

Therefore, we get the vector additive inverse as:

$$-_{\mathbb{V}} u = (-1) \cdot u \quad \text{(proved)}$$

Linear Combination:

A vector $v \in \mathbb{V}$ is said to be a linear combination of the vectors $v_1, v_2, \ldots, v_n \in \mathbb{V}$ provided that there exists $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that

$$v = c_1 \cdot v_1 +_{\mathbb{V}} c_2 \cdot v_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n \cdot v_n$$

where $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ is a vector space with the vector addition $(+_{\mathbb{V}})$ and scalar multiplication $(\cdot)$.

Subspaces:

Let $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ be a vector space. A subspace of $\mathbb{V}$ is a subset $\mathbb{W} \subseteq \mathbb{V}$ which itself is a vector space over $\mathbb{F}$ with the same operations of vector addition and scalar multiplication on $\mathbb{V}$.

Checks related to vector addition

1. Closure: $\forall u, v \in \mathbb{W} \; [\, u +_{\mathbb{V}} v \in \mathbb{W} \,]$

✓ 2. Since $\mathbb{W} \subseteq \mathbb{V}$ so associativity is guarented

3. identity: $0_{\mathbb{V}} \in \mathbb{W}$

4. Additive inverse: $\forall u \in \mathbb{W} \; [\, -_{\mathbb{V}} u \in \mathbb{W} \,]$

✓ 5. Since $\mathbb{W} \subseteq \mathbb{V}$ so commutativity is guarented.

Checks related to scalar multiplication

✓ 1. $\forall u \in \mathbb{W} \; [\, 1_{\mathbb{F}} \cdot u = u \,]$ is already satisfied as $\cdot$ is the operation on $\mathbb{V}$ and $u \in \mathbb{V}$.

✓ 2. $\forall \alpha, \beta \in \mathbb{F}, u \in \mathbb{W} \; [\, (\alpha \times \beta) \cdot u = \alpha \cdot (\beta \cdot u) \,]$ is also satisfied by $\mathbb{W}$ because $\cdot$ is the operation on $\mathbb{V}$ and $u \in \mathbb{V}$. The $\mathbb{F}$ is also same so it is not needed.

✓ 3. $\forall \alpha, \beta \in \mathbb{F}, u \in \mathbb{W} \left[(\alpha + \beta) \cdot u = \alpha \cdot u +_{\mathbb{V}} \beta \cdot u \right]$

is also satisfied by $\mathbb{W}$.

✓ 4. $\forall \alpha \in \mathbb{F}, u, v \in \mathbb{W} \left[\alpha \cdot (u +_{\mathbb{V}} v) = \alpha \cdot u +_{\mathbb{V}} \alpha \cdot v \right]$

is also satisfied by $\mathbb{W}$.

For the scalar operations, we do not need to check any thing since $\cdot$ is the operation defined on $\mathbb{V}$ and it will automatically satisfy on $\mathbb{W}$ also since $\mathbb{W} \subseteq \mathbb{V}$.

Even the 3 checks of vector addition can also be simplified and can be kept in a single check.

<u>Theorem:</u> A <u>non empty</u> subset $\mathbb{W}$ of $\mathbb{V}$ is a subspace of $\mathbb{V}$ if and only if,

$$\forall u, v \in \mathbb{W}, c \in \mathbb{F} \left[c \cdot u +_{\mathbb{V}} v \in \mathbb{W} \right]$$

<u>proof:</u> Choose $\alpha = 1$, if $u, v \in \mathbb{W}$ then

$$u +_{\mathbb{V}} v \in \mathbb{W} \quad \text{so } +_{\mathbb{V}} \text{ is closed in } \mathbb{W}.$$

Hence closure property of $+_{\mathbb{V}}$ is checked.

Choose $u = v = w$ and $\alpha = -1$. if $w \in \mathbb{W}$ then

$$(-1) \cdot w +_{\mathbb{V}} w \in \mathbb{W}$$

$$\Rightarrow -_{\mathbb{V}} w +_{\mathbb{V}} w \in \mathbb{W} \qquad [\because (-1) \cdot w = -_{\mathbb{V}} w]$$

$$\Rightarrow 0_{\mathbb{V}} \in \mathbb{W}.$$

So additive identity is in subspace $\mathbb{W}$

Choose. $u = w$, $v = 0_{\mathbb{V}}$, $\alpha = -1$ if $w \in \mathbb{W}$ then

$$(-1) \cdot w + 0_{\mathbb{V}} \in \mathbb{W}$$

$$\Rightarrow -_{\mathbb{V}} w + 0_{\mathbb{V}} \in \mathbb{W}$$

$$\Rightarrow -_{\mathbb{V}} w \in \mathbb{W}$$

So additive inverse is in subspace $\mathbb{W}$.

There is one more property that need to be checked that is closure of scalar multiplication.

$$\forall c \in \mathbb{F}, u \in \mathbb{W} \quad [\, c \cdot u \in \mathbb{W} \,]$$

Choose $v = 0_{\mathbb{V}}$, $u = w$ and sweep over all possible α in $\mathbb{F}$ such that if $w \in \mathbb{W}$ then.

$$c \cdot w + 0_{\mathbb{V}} \in \mathbb{W} \quad \Rightarrow \quad c \cdot w \in \mathbb{W} \quad \text{so } \cdot \text{ is closed.}$$

Conversely, if $\mathbb{W}$ is a subspace of $\mathbb{V}$, then

$\forall u, v \in \mathbb{W}, c \in \mathbb{F} \; [\, c \cdot u +_{\mathbb{V}} v \in \mathbb{W} \,]$

Given $\mathbb{W}$ is a subspace. Therefore, $\cdot$ is closed in $\mathbb{W}$
hence $c \cdot u \in \mathbb{W}$ as $u \in \mathbb{W}$.

Given $\mathbb{W}$ is a subspace therefore $+_{\mathbb{V}}$ is closed in $\mathbb{W}$
hence $c \cdot u \in \mathbb{W}$ and $v \in \mathbb{W}$ therefore,
$(c \cdot u) +_{\mathbb{V}} v$ is also in $\mathbb{W}$ as $+_{\mathbb{V}}$ is closed.

Span of a vector space:

Let S be a set of vectors in a vector space $\mathbb{V}$.
Therefore $S \subseteq \mathbb{V}$ and S need not be a vector space.
It is simply a subset of $\mathbb{V}$

span of S which is denoted by $\langle S \rangle$ or $\text{span}(S)$

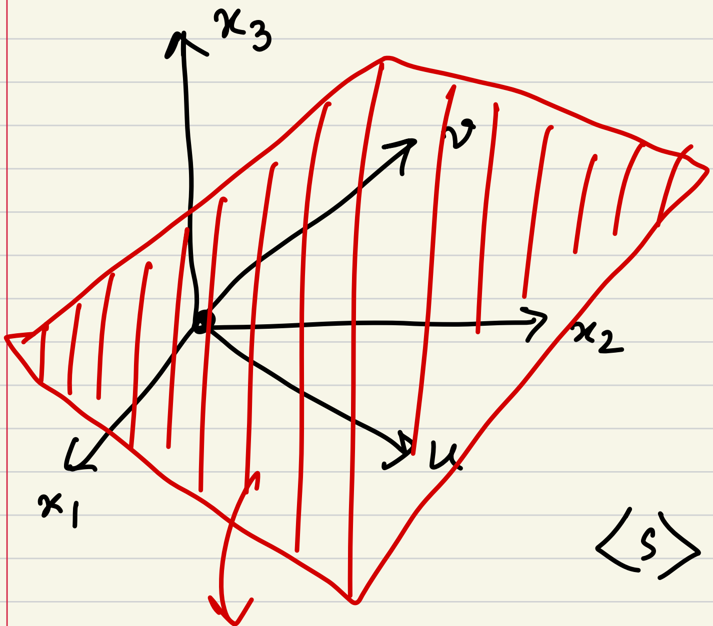
$$\langle S \rangle := \left\{ s \in \mathbb{V} \;\middle|\; s = \sum_{i=1}^{n} c_i s_i \,,\; \forall s_i \in S,\, c_i \in \mathbb{F} \right\}$$

Given a set of vectors, $S = \{ s_1, s_2, \ldots, s_n \} \subseteq \mathbb{V}$

$$\text{span}(S) = \left\{ c_1 s_1 +_{\mathbb{V}} c_2 s_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n s_n \;\middle|\; \forall c_i \in \mathbb{F} \right\}$$

It is the set of all possible linear combinations of set S.

S itself is not a vector space. S is just the collection of some vectors. The definition says, if we take the linear combination of those set of vectors then the set of generated vectors (by LC) is called span of S.



$S = \{u, v\}$

You take all the linear combination and generate a set of vectors,

$$\langle S \rangle = c_1 \cdot u +_{\mathbb{V}} c_2 \cdot v \quad (\forall c_1, c_2 \in \mathbb{F})$$

this plane is the collection of all vectors in $\langle S \rangle$
So we say the plane is spanned by $\langle S \rangle$ or $\langle S \rangle$ spans this plane.

Suppose you choose, $S = \{u, v, w\}$ such that when you take their LC then you generate the whole vector space meaning $\{v_i \in \mathbb{V}\}_{i=1}^{\infty}$ (all possible vectors in $\mathbb{V}$).

In that case, we say $\langle S \rangle$ spans the vector space $\mathbb{V}$ and $\mathbb{V}$ is spanned by $\langle S \rangle$.

proposition: $\mathbb{V}$ is a vector space. S is a subset of $\mathbb{V}$.
span(s) is a subspace of $\mathbb{V}$.

proof: $\text{span}(s) = \left\{ \sum_{i=1}^{n} c_i \cdot s_i \;\middle|\; \forall c_i \in \mathbb{F}, \forall s_i \in S \right\}$

We need to prove span(s) is a subspace of $\mathbb{V}$.

consider 2 vectors v_1 & v_2 and $S = \{s_i\}_{i=1}^{n}$

Define, $v_1 = \sum_{i=1}^{n} \alpha_i \cdot s_i$ (for some $\alpha_i \in \mathbb{F}$)

Define, $v_2 = \sum_{i=1}^{n} \beta_i \cdot s_i$ (for some $\beta_i \in \mathbb{F}$)

By the definition of $\langle s \rangle$ we can say $v_1, v_2 \in \langle s \rangle$

When we take 2 vectors from a space then we will test whether $c \cdot v_1 + v_2$ also belongs to the space or not. Then only the space will be a subspace of $\mathbb{V}$.

Test: $\forall u, v \in \mathbb{W}, c \in \mathbb{F} \; [\, c \cdot u +_{\mathbb{V}} v \in \mathbb{W} \,]$ iff $\mathbb{W}$ is a subspace of $\mathbb{V}$ ($\Rightarrow \mathbb{W} \subseteq \mathbb{V}$ and $(\mathbb{W}, \mathbb{F}, +_{\mathbb{V}}, \cdot)$ is a vector space.)

Therefore look at $c \cdot v_1 +_{\mathbb{V}} v_2$ and check if it $\in \langle s \rangle$ where $c \in \mathbb{F}$.

$$v_3 = c \cdot \left(\sum_{i=1}^{n} \alpha_i \cdot s_i \right) +_V \left(\sum_{i=1}^{n} \beta_i \cdot s_i \right)$$

$$= \sum_{i=1}^{n} (c \cdot \alpha_i) \cdot s_i +_V \sum_{i=1}^{n} \beta_i \cdot s_i$$

$$= \sum_{i=1}^{n} \left(c \cdot \alpha_i + \beta_i \right) \cdot s_i$$

$$= \sum_{i=1}^{n} K_i \cdot s_i \quad \text{where } K_i = c \cdot \alpha_i + \beta_i \in \mathbb{F}$$

Hence $v_3 \in \langle s \rangle$ by the definition of $\langle s \rangle$

Since $S \subseteq \mathbb{V}$ so all $s_i \in \mathbb{V}$ and $\mathbb{V}$ being a vector space the LC will be closed under $\mathbb{V}$.

Therefore $v_1, v_2, v_3 \in \mathbb{V}$ and $\in \mathbb{W}$ $[\mathbb{W} = \langle S \rangle]$
It shows for arbitrary vectors $v_1, v_2 \in \mathbb{W}$ and $\in \mathbb{V}$ and arbitrary scalar $c \in \mathbb{F}$, the vector $v_3 = c \cdot v_1 +_V v_2$ belongs to $\mathbb{W}$ and $\mathbb{V}$. Hence $\mathbb{W}$ is a subspace of $\mathbb{V}$.

Sums of subspaces:

If $S_1, S_2, \dots, S_K$ are subsets of a vector space $\mathbb{V}$ then the set of all sums -

$$\left\{ u_1 +_{\mathbb{V}} u_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} u_K \;\middle|\; \forall u_K \in S_K \right\}$$

is called the sum of subsets $S_1, S_2, \dots, S_K$ and is denoted by -

$$\text{sum}(S_1, S_2, \dots, S_K) = \sum_{i=1}^{K} S_i = S_1 + S_2 + \cdots + S_K$$

Suppose, $\mathbb{W}_1, \mathbb{W}_2$ are 2 subspaces of $\mathbb{V}$. Since they are subspaces so they are also $\mathbb{W}_1, \mathbb{W}_2 \subseteq \mathbb{V}$.

$$\text{sum}(\mathbb{W}_1, \mathbb{W}_2) = \mathbb{W}_1 + \mathbb{W}_2$$

$$:= \left\{ \omega \in \mathbb{V} \;\middle|\; \omega = \omega_1 +_{\mathbb{V}} \omega_2 ,\; \omega_1 \in \mathbb{W}_1 ,\; \omega_2 \in \mathbb{W}_2 \right\}$$

Proofs related to subspace, span:

Ex 1: Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. The intersection of any collection of subspaces of $\mathbb{V}$ is a subspace of $\mathbb{V}$.

Suppose $\mathbb{W}_1, \mathbb{W}_2$ is a subspace of $\mathbb{V}$.

Claim: $\mathbb{W}_1 \cap \mathbb{W}_2$ is also a subspace of $\mathbb{V}$.

More generally, $\{\mathbb{W}_a\}_{a \in I}$ is a collection of subspaces of $\mathbb{V}$.

Claim: $\mathbb{W} = \bigcap\limits_{a \in I} \mathbb{W}_a$ is a subspace of $\mathbb{V}$.

Test criteria: A subset $\mathbb{W}$ of a vector space $\mathbb{V}$ is subspace of $\mathbb{V}$ over $\mathbb{F}$ if and only if.

$$\forall u, v \in \mathbb{W},\ c \in \mathbb{F} \quad \left[c \cdot u +_{\mathbb{V}} v \in \mathbb{W} \right]$$

Let $\mathbb{W} = \mathbb{W}_1 \cap \mathbb{W}_2$

Where $\mathbb{W}_1$ & $\mathbb{W}_2$ are subspaces of $\mathbb{V}$.

Consider arbitrary vectors u, v in $\mathbb{W}$.
Consider arbitrary scalars c in $\mathbb{F}$

Since $u \in \mathbb{W}$ that implies $u \in \mathbb{W}_1$ and $u \in \mathbb{W}_2$
Since $v \in \mathbb{W}$ that implies $v \in \mathbb{W}_1$ and $v \in \mathbb{W}_2$
Since $\mathbb{W}_1$ is a subspace so it is closed under vector addition and scalar multiplication. Similarly for $\mathbb{W}_2$ also these 2 operations are closed.

It implies $c \cdot u +_{\mathbb{V}} v \in \mathbb{W}_1$ $(\because \mathbb{W}_1$ is subspace of $\mathbb{V})$

$c \cdot u +_{\mathbb{V}} v \in \mathbb{W}_2$ $(\because \mathbb{W}_2$ is subspace of $\mathbb{V})$

Notice $c \cdot u +_{\mathbb{V}} v$ is both in $\mathbb{W}_1$ and $\mathbb{W}_2$

This implies $c \cdot u +_{\mathbb{V}} v \in \mathbb{W}_1 \cap \mathbb{W}_2 \in \mathbb{W}$

Also since $c \cdot u +_{\mathbb{V}} v \in \mathbb{W}_1$ and $\mathbb{W}_1 \subseteq \mathbb{V}$ so $c \cdot u +_{\mathbb{V}} v \in \mathbb{V}$

Clearly $\mathbb{W}_1 \cap \mathbb{W}_2 \subseteq \mathbb{V}$. or $\mathbb{W} \subseteq \mathbb{V}$ because $\mathbb{W}_1 \subseteq \mathbb{V}$ and $\mathbb{W}_2 \subseteq \mathbb{V}$.

Therefore,

for arbitrary $u, v \in \mathbb{W}$, $c \in \mathbb{F}$ $[c \cdot u +_{\mathbb{V}} v \in \mathbb{W}_1 \cap \mathbb{W}_2]$

$\therefore \forall u, v \in \mathbb{W}, c \in \mathbb{F} \; [c \cdot u +_{\mathbb{V}} v \in \mathbb{W}_1 \cap \mathbb{W}_2]$

From universal generalization, it follows the test is satisfied implies $\mathbb{W} = \mathbb{W}_1 \cap \mathbb{W}_2$ is a subspace of $\mathbb{V}$.

<u>Countable intersection proof by induction:</u>

We could have used the same argument as before and completes the proof but let's use induction.

$\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_n$ are subspaces of $\mathbb{V}$ over $\mathbb{F}$

prove that $\mathbb{W}_1 \cap \mathbb{W}_2 \cap \cdots \cap \mathbb{W}_n$ is also a subspace of $\mathbb{V}$ over $\mathbb{F}$.

Formulate, $\mathbb{P}_n = \bigcap_{i=1}^{n} \mathbb{W}_i$

<u>Base case : $(n=2)$</u>

$\mathbb{P}_2 = \mathbb{W}_1 + \mathbb{W}_2$

We just now showed that $\mathbb{P}_2$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$

<u>inductive step</u>:

Assume that the intersection of k subspaces is a subspace of $\mathbb{V}$ over $\mathbb{F}$ for some positive integer K.

$\mathbb{P}_K = \mathbb{W}_1 \cap \mathbb{W}_2 \cap \cdots \cap \mathbb{W}_K$ is a subspace of $\mathbb{V}$

over $\mathbb{F}$ is assumed (induction hypothesis).

We have to prove that for $K+1$ also $\mathbb{P}_{K+1}$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$.

$$\mathbb{P}_{K+1} = (\mathbb{W}_1 \cap \mathbb{W}_2 \cap \cdots \cap \mathbb{W}_K) \cap \mathbb{W}_{K+1}$$

$$= \mathbb{P}_K \cap \mathbb{W}_{K+1}$$

Since $\mathbb{P}_K$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$ (assumed) and $\mathbb{W}_{K+1}$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$ (givenfact). We have to show that $\mathbb{P}_{K+1}$ is also a subspace.

From the similar technique of base case proof, we can show that $\mathbb{P}_{K+1}$ is also a subspace of $\mathbb{V}$. over $\mathbb{F}$.
This completes the proof by induction.

Ex2: Suppose $\mathbb{W}_1, \mathbb{W}_2$ are subspaces of $\mathbb{V}$. over $\mathbb{F}$

Prove that $\mathbb{W}_1 \cup \mathbb{W}_2$ need not be a subspace of $\mathbb{V}$ over $\mathbb{F}$.

$\Rightarrow$ Assume that $\mathbb{V} = \{(a, b) \mid a, b \in \mathbb{F} = \mathbb{R}\}$

That means $\mathbb{V}$ is the collection of all 2-ordered tuples which is a vector space over the field of reals $\mathbb{R}$.

Consider a set S which is a subset of $\mathbb{V}$ such that

$$\text{span}(S) = \langle S \rangle = \left\{ \sum_{i=1}^{n} c_i s_i \;\middle|\; \forall c_i \in \mathbb{R}, \forall s_i \in S \right\}$$

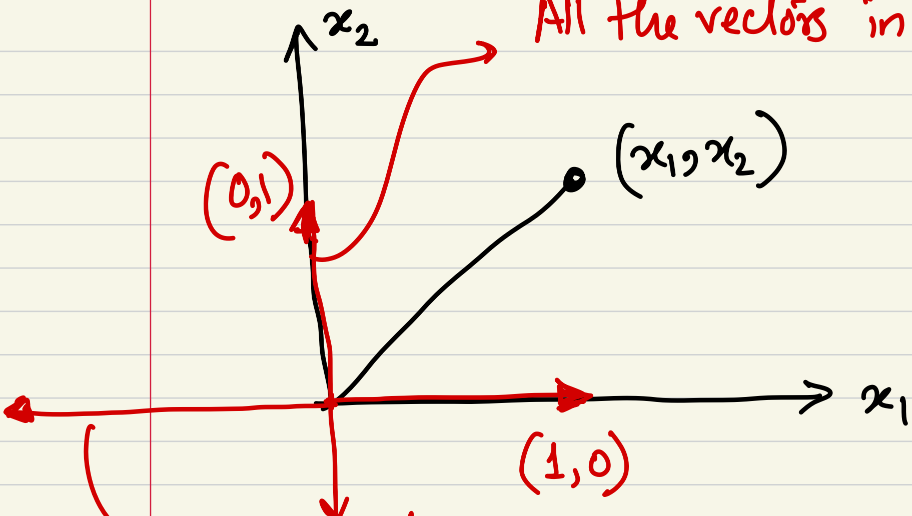
Suppose $S = \{(1,0)\}$ [only 1 vector's Linear combination]

$$\text{span}(S) = \langle S \rangle = \{ c_1 \cdot (1,0) \mid \forall c_1 \in \mathbb{R} \} \quad (n=1)$$

$$= \{ (c_1, 0) \mid \forall c_1 \in \mathbb{R} \}$$

Therefore in $\mathbb{R}^2$, span(S) is the whole line $(1,0)$.

All the vectors in this line is $\langle S_2 \rangle = \{(0, c_1) \mid \forall c_1 \in \mathbb{R}\}$



All the vectors in this line is $\langle S_1 \rangle = \{(c_1, 0) \mid \forall c_1 \in \mathbb{R}\}$

Consider $S_1 = \{(1,0)\}$, $\text{span}(S_1) = \mathbb{W}_1$

$S_2 = \{(0,1)\}$, $\text{span}(S_2) = \mathbb{W}_2$

We know that for a set $S \subseteq \mathbb{V}$, the $\text{span}(S) = \langle S \rangle$ is always a subspace of $\mathbb{V}$ over $\mathbb{F}$.

Therefore, $\mathbb{W}_1$ is a subspace of $\mathbb{R}^2$ over $\mathbb{R}$

$\mathbb{W}_2$ is also a subspace of $\mathbb{R}^2$ over $\mathbb{R}$

Consider the set of vectors as $\mathbb{W}_1 \cup \mathbb{W}_2$ that means $\mathbb{W} = \mathbb{W}_1 \cup \mathbb{W}_2$ represents all set of vectors $\mathbb{W}_1$ and $\mathbb{W}_2$.

Therefore, $(1,0) \in \mathbb{W}$ and $(0,1) \in \mathbb{W}$

Consider $(1,0) +_{\mathbb{V}} (0,1) = (1,1) \notin \mathbb{W}$

That means the vector addition operation $+_{\mathbb{V}}$ is not closed under $\mathbb{W}$. That's why $\mathbb{W}$ can't be a vector space of $\mathbb{V}$ over $\mathbb{F}$.

Ex.3: Suppose $\mathbb{W}_1, \mathbb{W}_2$ are subspaces of $\mathbb{V}$. over $\mathbb{F}$

Prove that $\mathbb{W}_1 \cup \mathbb{W}_2$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$ if and only if either $\mathbb{W}_1 \subseteq \mathbb{W}_2$ or $\mathbb{W}_2 \subseteq \mathbb{W}_1$

let's prove $\mathbb{W}_1 \subseteq \mathbb{W}_2$ or $\mathbb{W}_2 \subseteq \mathbb{W}_1 \longrightarrow \mathbb{W}_1 \cup \mathbb{W}_2$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$

Assume $\mathbb{W}_1 \subseteq \mathbb{W}_2$.

$\Rightarrow \mathbb{W}_1 \cup \mathbb{W}_2 = \mathbb{W}_2$ (because $\mathbb{W}_2$ is the bigger set)

Since $\mathbb{W}_2$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$ so $\mathbb{W}_1 \cup \mathbb{W}_2$ is also subspace of $\mathbb{V}$ over $\mathbb{F}$.

Similarly if $\mathbb{W}_2 \subseteq \mathbb{W}_1$ then $\mathbb{W}_1 \cup \mathbb{W}_2 = \mathbb{W}_1$ which is a subspace so $\mathbb{W}_1 \cup \mathbb{W}_2$ is also a subspace of $\mathbb{V}$ over $\mathbb{F}$.

let's prove $\mathbb{W}_1 \cup \mathbb{W}_2$ is subspace of $\mathbb{V}$ over $\mathbb{F}$ $\longrightarrow \mathbb{W}_1 \subseteq \mathbb{W}_2$ or $\mathbb{W}_2 \subseteq \mathbb{W}_1$

We prove by contrapositive.

$$\mathbb{W}_1 \nsubseteq \mathbb{W}_2 \text{ and } \mathbb{W}_2 \nsubseteq \mathbb{W}_1 \longrightarrow \mathbb{W}_1 \cup \mathbb{W}_2 \text{ is not a subspace of } \mathbb{V} \text{ over } \mathbb{F}.$$

Consider $\mathbb{W}_1 = \langle \{(0,1)\} \rangle$ is a subspace of $\mathbb{R}^2$

$\mathbb{W}_2 = \langle \{(1,0)\} \rangle$ is a subspace of $\mathbb{R}^2$

clearly $\mathbb{W}_1 \nsubseteq \mathbb{W}_2$ and $\mathbb{W}_2 \nsubseteq \mathbb{W}_1$.

Take a vector $\underbrace{(0,1)}_{\in \mathbb{W}_1 \cup \mathbb{W}_2} +_{\mathbb{V}} \underbrace{(1,0)}_{\in \mathbb{W}_1 \cup \mathbb{W}_2} = \underbrace{(1,1)}_{\notin \mathbb{W}_1 \cup \mathbb{W}_2}$

Therefore $+_{\mathbb{V}}$ operation is not closed under $\mathbb{W}_1 \cup \mathbb{W}_2$.
Hence it is not a subspace.

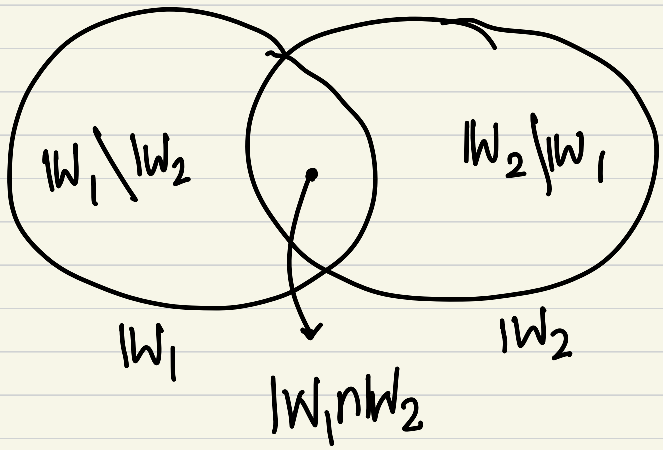
Alternate proof:

Suppose, $\mathbb{W} = \mathbb{W}_1 \cup \mathbb{W}_2$ where $\mathbb{W}_1 \nsubseteq \mathbb{W}_2$ and $\mathbb{W}_2 \nsubseteq \mathbb{W}_1$

We have to show that $\mathbb{W}$ is not a subspace.

proof by contradiction: Assume $\mathbb{W}$ is a subspace of $\mathbb{V}$.

$$\mathbb{W} = (\mathbb{W}_1 \setminus \mathbb{W}_2) \cup (\mathbb{W}_1 \cap \mathbb{W}_2) \cup (\mathbb{W}_2 \setminus \mathbb{W}_1)$$



$\mathbb{W}_1 \nsubseteq \mathbb{W}_2 \neq \phi$

$\mathbb{W}_2 \nsubseteq \mathbb{W}_1 \neq \phi$

Choose an arbitrary vector u from $\mathbb{W}_1$ and an arbitrary vector v from $\mathbb{W}_2$.

Since we assumed $\mathbb{W} = \mathbb{W}_1 \cup \mathbb{W}_2$ is a subspace of $\mathbb{V}$.

Therefore $u \in \mathbb{W}_1 \in \mathbb{W}$ and $v \in \mathbb{W}_2 \in \mathbb{W}$ implies $u +_{\mathbb{V}} v \in \mathbb{W}$ because $+_{\mathbb{V}}$ operation is closed.

① If $u +_{\mathbb{V}} v$ fall in $\mathbb{W}_1$ then

$$u +_{\mathbb{V}} v =: w \in \mathbb{W}_1$$

Since $\mathbb{W}_1$ is a subspace additive inverse will exist.
Therefore $-_{\mathbb{V}} u \in \mathbb{W}_1$

Again $w \in \mathbb{W}_1$, $-_{\mathbb{V}} u \in \mathbb{W}_1$ so $w -_{\mathbb{V}} u \in \mathbb{W}_1$
since $\mathbb{W}_1$ is a vector space.

$\Rightarrow v \in \mathbb{W}_1$ which is a contradiction as $v \in \mathbb{W}_2$

Similarly if $u +_{\mathbb{V}} v$ fall in $\mathbb{W}_2$ then also it will lead to a contradiction.

Hence the assumption was incorrect.

Ex4: Consider a subspace $\mathbb{W}_1, \mathbb{W}_2 \subseteq \mathbb{V}$ over $\mathbb{F}$

$$\mathbb{W}_1 + \mathbb{W}_2 := \left\{ \omega \in \mathbb{V} \;\middle|\; \omega = \omega_1 +_{\mathbb{V}} \omega_2 , \omega_1 \in \mathbb{W}_1 , \omega_2 \in \mathbb{W}_2 \right\}$$

All possible combination of sum of 2 vectors ω_1, ω_2 belonging to $\mathbb{W}_1, \mathbb{W}_2$.

There is something called "direct sum" which means,

$$\mathbb{W}_1 \oplus \mathbb{W}_2 := \left\{ \omega \in \mathbb{V} \;\middle|\; \omega = \omega_1 + \omega_2 , \omega_1 \in \mathbb{W}_1 , \omega_2 \in \mathbb{W}_2 \text{ and } \omega \text{ is uniquely defined by } \omega_1 \,\&\, \omega_2 \right\}$$

It means no other combination of ω_1, ω_2 can generate ω except only 1 unique combination.

Prove that $\mathbb{W}_1 + \mathbb{W}_2$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$.

$\Rightarrow$ Suppose, $\mathbb{W} = \mathbb{W}_1 + \mathbb{W}_2$.

Take arbitrary 2 vectors $u, v \in \mathbb{W}$ and arbitrary $c \in \mathbb{F}$

Look at $w = c \cdot u +_{\mathbb{V}} v$

Our objective is to argue that w must belong to $\mathbb{W}$.

$w = c \cdot u +_{\mathbb{V}} v$

Notice $u \in \mathbb{W}$ that means u can be written as sum of 2 vectors that comes from 2 subspaces $\mathbb{W}_1$ & $\mathbb{W}_2$

$u = u_1 +_{\mathbb{V}} u_2$ where $u_1 \in \mathbb{W}_1$ and $u_2 \in \mathbb{W}_2$

Note that $u_1 \in \mathbb{W}_1$ and $\mathbb{W}_1 \subseteq \mathbb{V}$ so $u_1 \in \mathbb{V}$

$u_2 \in \mathbb{W}_2$ and $\mathbb{W}_2 \subseteq \mathbb{V}$ so $u_2 \in \mathbb{V}$

Since $u_1, u_2 \in \mathbb{V}$ so $u_1 +_{\mathbb{V}} u_2 \in \mathbb{V}$ so $u \in \mathbb{V}$.

Similarly $v \in \mathbb{W}$ means v can be written as sum of 2 vectors that comes from 2 subspaces $\mathbb{W}_1$ & $\mathbb{W}_2$.

$v = v_1 +_{\mathbb{V}} v_2$ where $v_1 \in \mathbb{W}_1$ and $v_2 \in \mathbb{W}_2$

Note that $v_1 \in \mathbb{W}_1$, $\mathbb{W}_1 \subseteq \mathbb{V}$ so $v_1 \in \mathbb{V}$

$v_2 \in \mathbb{W}_2$, $\mathbb{W}_2 \subseteq \mathbb{V}$ so $v_2 \in \mathbb{V}$

Since $v_1, v_2 \in \mathbb{V}$ so $v_1 +_{\mathbb{V}} v_2 \in \mathbb{V}$ so $v \in \mathbb{V}$.

$w = c \cdot u +_{\mathbb{V}} v$

Since u & v both belong to $\mathbb{V}$ so $+_{\mathbb{V}}$ operation is closed and $\omega \in \mathbb{V}$. We have to check whether ω belongs to $\mathbb{W}$ or not?

$$\omega = c \cdot (u_1 +_{\mathbb{V}} u_2) +_{\mathbb{V}} (v_1 +_{\mathbb{V}} v_2)$$

$$= \underbrace{(c \cdot u_1 +_{\mathbb{V}} v_1)} +_{\mathbb{V}} \underbrace{(c \cdot u_2 +_{\mathbb{V}} v_2)} = z_1 +_{\mathbb{V}} z_2$$

$u_1 \in \mathbb{W}_1$, $v_1 \in \mathbb{W}_1$

$c \cdot u_1 + v_1 \in \mathbb{W}_1$

(because $\mathbb{W}_1$ is a subspace)

$u_2 \in \mathbb{W}_2$, $v_2 \in \mathbb{W}_2$

$c \cdot u_2 +_{\mathbb{V}} v_2 \in \mathbb{W}_2$

(because $\mathbb{W}_2$ is a subspace)

Since $z_1 \in \mathbb{W}_1$ and $z_2 \in \mathbb{W}_2$ so $z_1 +_{\mathbb{V}} z_2$ must belong to $\mathbb{W}_1 + \mathbb{W}_2$ by the definition.

Therefore, $\omega \in \mathbb{W}$ and $\omega \in \mathbb{V}$

for arbitrary $u, v \in \mathbb{W}$ $\quad c \cdot u +_{\mathbb{V}} v \in \mathbb{W}$

$\therefore \forall u, v \in \mathbb{W}, c \in \mathbb{F} \ (c \cdot u +_{\mathbb{V}} v \in \mathbb{W})$ (universal generalization)

Therefore $\mathbb{W}_1 + \mathbb{W}_2$ is a subspace of $\mathbb{V}$.

Ex5: Suppose $\mathbb{W}_1, \mathbb{W}_2$ are subspaces of $\mathbb{V}$ over $\mathbb{F}$

$$\mathbb{W}_1 + \mathbb{W}_2 = \{ w \in \mathbb{V} \mid w = w_1 +_{\mathbb{V}} w_2, \; w_1 \in \mathbb{W}_1, \; w_2 \in \mathbb{W}_2 \}$$

$$\mathbb{W}_1 \oplus \mathbb{W}_2 = \{ w \in \mathbb{V} \mid w = w_1 +_{\mathbb{V}} w_2, \; \exists! \, w_1 \in \mathbb{W}_1, \; \exists! \, w_2 \in \mathbb{W}_2 \}$$

Prove the statement

If $\mathbb{W}_1 \cap \mathbb{W}_2 = \{ 0_{\mathbb{V}} \}$ then $\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{W}_1 \oplus \mathbb{W}_2$

$\Rightarrow$ The intersection of $\mathbb{W}_1, \mathbb{W}_2$ contains only $0_{\mathbb{V}}$ vector then any vector in $\mathbb{W}_1 + \mathbb{W}_2$ can be uniquely expressed as the addition of 2 vector from $\mathbb{W}_1$ and $\mathbb{W}_2$.

Assume that $\mathbb{W}_1 + \mathbb{W}_2 \neq \mathbb{W}_1 \oplus \mathbb{W}_2$

That means a vector $v \in \mathbb{W}_1 + \mathbb{W}_2$ can be written in non unique way

$$v = w_1 +_{\mathbb{V}} w_2, \quad w_1 \in \mathbb{W}_1, \; w_2 \in \mathbb{W}_2$$

$$v = w_1' +_{\mathbb{V}} w_2', \quad w_1' \in \mathbb{W}_1, \; w_2' \in \mathbb{W}_2$$

where either $w_1 \neq w_1'$ or $w_2 \neq w_2'$ or both.
but $w_1 = w_1'$ and $w_2 = w_2'$ can't happen.

Equating: $w_1 +_{\mathbb{V}} w_2 = w_1' +_{\mathbb{V}} w_2'$

$$\Rightarrow w_1 -_{\mathbb{V}} w_1' = w_2' -_{\mathbb{V}} w_2 = p$$

Since $\mathbb{W}_1$ is a subspace so $w_1 -_{\mathbb{V}} w_1' \in \mathbb{W}_1$
Since $\mathbb{W}_2$ is a subspace so $w_2 -_{\mathbb{V}} w_2' \in \mathbb{W}_2$

Therefore, $P \in \mathbb{W}_1$ and $P \in \mathbb{W}_2$ also that means $P \in \mathbb{W}_1 \cap \mathbb{W}_2$.

Given that $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0_{\mathbb{V}}\}$

Therefore,
$$P = 0_{\mathbb{V}} = w_1 -_{\mathbb{V}} w_1'$$
$$P = 0_{\mathbb{V}} = w_2' -_{\mathbb{V}} w_2$$

It means $\left.\begin{array}{l} w_1 = w_1' \\ w_2 = w_2' \end{array}\right\}$ which is the contradiction of the assumption

Meaning if $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0_{\mathbb{V}}\}$ then $\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{W}_1 \oplus \mathbb{W}_2$

Ex6: $\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3$ are subspaces of $\mathbb{V}$ over $\mathbb{F}$

$$\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{W}_1 + \mathbb{W}_3 \longrightarrow \mathbb{W}_2 = \mathbb{W}_3$$

2 subspaces are equal if and only if they contain same set of elements.

$$\mathbb{W}_2 = \mathbb{W}_3 \equiv \mathbb{W}_2 \subseteq \mathbb{W}_3 \text{ and } \mathbb{W}_3 \subseteq \mathbb{W}_2$$

Consider if $\mathbb{W}_2 \subseteq \mathbb{W}_3$?

Take an arbitrary vector $w_2 \in \mathbb{W}_2$.

$$\mathbb{W}_1 + \mathbb{W}_2 := \{ w \in \mathbb{V} \mid w = w_1 +_{\mathbb{V}} w_2 ,\ w_1 \in \mathbb{W}_1 ,\ w_2 \in \mathbb{W}_2 \}$$

We want to show that $w_2 \in \mathbb{W}_3$ also.

Given that $\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{W}_1 + \mathbb{W}_3$ is true.

Since $w_2 \in \mathbb{W}_2$ then by definition of $\mathbb{W}_1 + \mathbb{W}_2$, we can say –

there exists w_1' and w_2' such that $w = w_1' +_{\mathbb{V}} w_2'$ where $w_1' \in \mathbb{W}_1$ and $w_2' \in \mathbb{W}_2$

Because we know that $\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{W}_1 + \mathbb{W}_3$, any vector in $\mathbb{W}_1 + \mathbb{W}_2$ must also be in $\mathbb{W}_1 + \mathbb{W}_3$.

incomplete

Ex7: Consider T as a function, $T: \mathbb{V} \to \mathbb{W}$ where both $\mathbb{V}$ and $\mathbb{W}$ are the vector spaces. The image of T (or range of T or columnspace of T) is defined as:

$$\text{im}(T) = \{ y \mid y = Tx ,\ x \in \mathbb{V},\ y \in \mathbb{W} \}$$

If $\mathbb{V} = \mathbb{F}^n$ and $\mathbb{W} = \mathbb{F}^m$ then,

$$\text{im}(T) = \{ y = Tx \mid y \in \mathbb{F}^m,\ x \in \mathbb{F}^n \} \subseteq \mathbb{W}$$

Therefore, the image of T (or range of T) is the subset of $\mathbb{W}$ consisting of those vectors that are of the form Tv for some $v \in \mathbb{V}$.

Kernel of T is defined as subset of $\mathbb{V}$ consisting of those vectors that maps T to $0_{\mathbb{W}}$

$$\text{Ker}(T) = \{ x \in \mathbb{V} \mid Tx = 0_{\mathbb{W}} \} \subseteq \mathbb{V}.$$

Consider, $A = \mathbb{F}^{m \times n}$, $\mathbb{V} = \mathbb{F}^m$, $\mathbb{W} = \mathbb{F}^n$ over $\mathbb{F}$.

$$\text{Ker}(A) = \{ x \in \mathbb{F}^n \mid Ax = 0_{\mathbb{V}} \} \subseteq \mathbb{F}^n$$

$$\text{im}(A) = \{ y \in \mathbb{F}^m \mid y = Ax ,\ x \in \mathbb{F}^n \} \subseteq \mathbb{F}^m$$

prove that $\mathrm{Ker}(A)$ is subspace of $\mathbb{F}^n$ over $\mathbb{F}$.

$\Rightarrow$ Choose 2 arbitrary vector $u, v \in \mathrm{Ker}(A)$

(Consider the vector space, $\mathbb{V} = \mathbb{F}^n$ over $\mathbb{F}$.)

Consider, $w := c \cdot u +_{\mathbb{V}} v$

Since $u \in \mathrm{Ker}(A)$ therefore, $A \cdot u = 0$

$v \in \mathrm{Ker}(A)$ therefore, $A \cdot v = 0$.

$w := c \cdot u +_{\mathbb{V}} v$

$$\Rightarrow Aw = A\left(c \cdot u +_{\mathbb{V}} v\right) = A\,(c \cdot u) +_{\mathbb{V}} A \cdot v$$

$$= c \cdot (Au) +_{\mathbb{V}} (Av)$$

$$= c \cdot 0 +_{\mathbb{V}} 0$$

$$= 0$$

Therefore, $w \in \mathrm{ker}(A)$ hence $\mathrm{Ker}(A)$ is subspace of $\mathbb{V}$.

prove that $\mathrm{im}(A)$ is a subspace of $\mathbb{F}^m$ over $\mathbb{F}$.

$\Rightarrow$ Consider $\mathbb{V} = \mathbb{F}^m$ over $\mathbb{F}$.

Take 2 arbitrary vectors $u, v \in \mathrm{im}(A)$.

Construct $w := c \cdot u +_{\mathbb{V}} v$.

Since $u \in \mathrm{im}(A)$, $u = A \cdot x$ for $x \in \mathbb{F}^n$

Since $v \in im(A)$, $v = A \cdot y$ for $y \in \mathbb{F}^n$.

$$\begin{aligned} w &= c \cdot u +_{\mathbb{W}} v \\ &= c \cdot A \cdot x +_{\mathbb{W}} A \cdot y \\ &= A(c \cdot x) +_{\mathbb{W}} A \cdot y \\ &= A(c \cdot x +_{\mathbb{W}} y) \\ &= A \cdot z \quad [z = c \cdot x +_{\mathbb{W}} y \in \mathbb{F}^n] \end{aligned}$$

Therefore $w \in \mathbb{F}^m$ which is a subspace of $\mathbb{F}^m$ over $\mathbb{F}$.

27. Suppose X is a non-empty set and $\mathcal{F}[X, \mathbb{F}]$ is the vector space of all functions from X to $\mathbb{F}$. For any $f \in \mathcal{F}[X, \mathbb{F}]$, define:

$$S_f := \{x \in X : f(x) \neq 0\}$$

as the *support* of the function $f(\cdot)$. Let $\mathcal{F}_F \subseteq \mathcal{F}[X, \mathbb{F}]$ such that

$$\mathcal{F}_F := \{f \in \mathcal{F}[X, \mathbb{F}] : S_f \text{ is a finite subset of } X\}.$$

In other words, $\mathcal{F}_F$ is a collection of all functions in $\mathcal{F}[X, \mathbb{F}]$ having a finite support. Argue why $\mathcal{F}_F$ must be a subspace of $\mathcal{F}[X, \mathbb{F}]$.

$F[X, \mathbb{F}]$ is the vector space of all functions from X to $\mathbb{F}$.

Therefore vector space $= \mathbb{V}$ over $\mathbb{F}$

$$\mathbb{V} = \{ f \mid f : X \longrightarrow \mathbb{F} \}$$

for any $f \in \mathbb{V}$, define support of the function f.

$$\text{support}(f) = S_f := \{ x \in X \mid f(x) \neq 0 \}$$

So the support of a function 'f' collects all the points from its domain at which the range is non-zero.

Let another set, $\mathbb{W} := \{ f \in \mathbb{V} \mid S_f \text{ is a finite subset of } X \}$

$\mathbb{W}$ is the collection of all the functions in $\mathbb{V}$ such that the support of that function is a finite subset of X or they have finite support.

Prove that $\mathbb{W}$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$.

$\Rightarrow$ Choose 2 arbitrary elements of $\mathbb{W}$. - say g & h.

$g \in \mathbb{V}$, $S_g \subseteq X$ where $|S_g| < \infty$

$h \in \mathbb{V}$, $S_h \subseteq X$ where $|S_h| < \infty$

$S_g := \{ x \mid g(x) \neq 0,\ x \in X \} \subseteq X$

$S_h := \{ y \mid h(y) \neq 0,\ y \in X \} \subseteq X$

Construct $w := c \cdot g +_{\mathbb{V}} h$, $c \in \mathbb{F}$.

$S_w := \{ z \mid w(z) \neq 0,\ z \in X \} \subseteq X$ - to be proved.

$w(z) = c \cdot g(z) +_{\mathbb{V}} h(z)$

It is easy to see that $z \in X$ that means $S_w \subseteq X$
but the question is $|S_g|$ is finite or infinite ?

Suppose $c=0$, $w(z) = h(z)$ so $w \in \mathbb{W}$.
Suppose $c \neq 0$, $w(z) = c \cdot g(z) +_{\mathbb{V}} h(z)$

$$S_w = \{ z \mid w(z) \neq 0 , z \in X \}$$
$$= \{ z \mid c \cdot g(z) +_{\mathbb{V}} h(z) \neq 0 , z \in X \}$$

$g(z) \neq 0$ for k_1 times $k_1 < \infty$ s.t. $|S_g| = k_1 < \infty$
$h(z) \neq 0$ for k_2 times $k_2 < \infty$ s.t $|S_h| = k_2 < \infty$

Therefore, $c \cdot g(z) +_{\mathbb{V}} h(z) \neq 0$ for $\max(k_1, k_2)$ number of times & $\max(k_1, k_2) < \infty$. therefore,
$S_w = \{ z \mid w(z) \neq 0 , z \in X \} \subseteq X$ and $|S_w| < \infty$
hence S_w is a finite subset of X.

Hence $w \in \mathbb{W}$ so $\mathbb{W}$ is a subspace of $\mathbb{V}$.

Ex 1 : Find all solutions to $x^2 - 10x + 16 = 0$
in $\mathbb{F}_5$ and $\mathbb{F}_8$.

$\mathbb{F}_5 = \{0,1,2,3,4\}$.

$1 \bmod 5 = 1$

$-10 \bmod 5 = 0$

$16 \bmod 5 = 1$

$x^2 - 10x + 16 = 0$

$\downarrow$

$\boxed{x^2 + 1 = 0}$

Check by all the values one by one.

$x=0$: $16 \bmod 5 \neq 0$ ✗

$x=1$: $7 \bmod 5 \neq 0$ ✗

$x=2$: 0 ✓

$x=3$: $-5 \bmod 5 = 0$ ✓

$x=4$: $8 \bmod 5 \neq 0$ ✗

So $x = 2,3$ are solutions.

$\mathbb{F}_8 = \{0,1,2,3,4,5,6,7\}$.

$$\left.\begin{aligned} 1 \bmod 8 &= 1 \\ -10 \bmod 8 &= -2 \\ 16 \bmod 8 &= 0 \end{aligned}\right\} \quad x^2 - 2x = 0$$

$x=0$: ✓

$x=1$: -1 ✗

$x=2$: ✓

$x=3$: 2 ✗

$x=4$: $8 \bmod 8 = 0$ ✓

$x=5$: 15 ✗

$x=6$: $24 \bmod 8 = 0$ ✓

$x=7$: $35 \bmod 8 \neq 0$ ✗

So, $x = 0,2,4,6$ are the solutions.

Square Free Related Questions:

Q: set of all rational numbers.

Suppose, D = A rational number in Q that is not perfect square meaning ($D \neq 1, 4, 9, 16, 25, 36, \ldots$)

Define $Q(\sqrt{D}) = \{ a + b\sqrt{D} \mid a, b \in Q \} \subseteq \mathbb{C}$

Suppose,

$Q(\sqrt{10}) = \{ 2+3\sqrt{10},\ 0 - \frac{1}{2}\sqrt{10},\ \frac{5}{7} + \frac{4}{7}\sqrt{10},\ \ldots \}$

$Q(\sqrt{D})$ is an extension of the set of rational numbers Q by introducing a square root of a non-perfect square element D.

The set $Q(\sqrt{D})$ is closed under addition.

Suppose $K_1 = a + b\sqrt{D} \in Q(\sqrt{D})$

$K_2 = c + d\sqrt{D} \in Q(\sqrt{D})$

is $K_1 + K_2$ also member of $Q(\sqrt{D})$?

$$\begin{aligned} K_1 + K_2 &= a + b\sqrt{D} + c + d\sqrt{D} \\ &= (a+c) + (b+d)\sqrt{D} \\ &= a' + b'\sqrt{D} \quad , a' = a + c \in Q,\ b' = b + d \in Q. \end{aligned}$$

So $K_1 + K_2 \in Q(\sqrt{D})$.

The set $\mathbb{Q}(\sqrt{D})$ is closed under multiplication.

Suppose,

$$K_1 = a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$$
$$K_2 = c + d\sqrt{D} \in \mathbb{Q}(\sqrt{D})$$

does $K_1 \cdot K_2 \in \mathbb{Q}(\sqrt{D})$?

$$K_1 \cdot K_2 = (a + b\sqrt{D})(c + d\sqrt{D})$$
$$= a \cdot c + a \cdot d\sqrt{D} + b \cdot c \cdot \sqrt{D} + b \cdot \sqrt{D} \cdot d \cdot \sqrt{D}$$
$$= \underbrace{ac + b \cdot d \cdot D} + \underbrace{(ad + bc)}\sqrt{D}$$
$$= a' + b'\sqrt{D}$$

since $D \in \mathbb{Q}$ hence $ac + bdD \in \mathbb{Q}$.

So $K_1 \cdot K_2$ is closed under multiplication.

Since $\mathbb{C}$ is a ring and $\mathbb{Q}(\sqrt{D}) \subseteq \mathbb{C}$ and we have shown that the operation $+$ and $\times$ are closed in $\mathbb{Q}(\sqrt{D})$ therefore $\mathbb{Q}(\sqrt{D})$ is a subring of $\mathbb{C}$.

Does $\mathbb{Q}(\sqrt{D})$ have the identity 1 ?

$$\Rightarrow (a + b\sqrt{D}) \cdot 1 \stackrel{?}{=} a + b\sqrt{D}$$

$$\Rightarrow a \cdot 1 + b \cdot 1 \cdot \sqrt{D} = a + b\sqrt{D} \quad (\text{Yes}) \checkmark$$

Therefore $\mathbb{Q}(\sqrt{D})$ is commutative ring with identity.
Does every element have inverse?

$$(a+b\sqrt{D}) \cdot K = 1 \quad \text{where } K = (a+b\sqrt{D})^{-1}$$

In order for the inverse to exist, we have to show that $a+b\sqrt{D} \neq 0$.

Since D is not a perfect square so we can talk about inverses of elements of $\mathbb{Q}(\sqrt{D})$. If D were a perfect square then it will become just an integer and for that $\mathbb{Q}(\sqrt{D})$ just becomes equal to $\mathbb{Q}$ which doesn't give us any benefit so D is a non-perfect square.

$a+b\sqrt{D} \longrightarrow$ Suppose $a \neq 0$ or $b \neq 0$ (both can't be 0).

Then definitely $a+b\sqrt{D} \neq 0$.

So we can say, $K = \dfrac{1}{a+b\sqrt{D}} = \left(\dfrac{a-b\sqrt{D}}{a^2-b^2D}\right)$

So it is the inverse of $a+b\sqrt{D}$. So inverse exists for all the elements. Therefore, $\mathbb{Q}(\sqrt{D})$ with $a \neq 0$ or $b \neq 0$ is a field. It is called quadratic field. (for every non-zero elements of $\mathbb{Q}(\sqrt{D})$, inverse exists)

Square free integer is an integer that is not divisible by square of any integer other than 1. A square free integer doesn't have any repeated prime factors.

$35 = 5 \times 7 \rightarrow$ Square free.

$18 = 3 \times 3 \times 2 \longrightarrow$ Not square free, divisible by 3.

Claim: D is a squarefree integer in the def^{n} of $\mathbb{Q}(\sqrt{D})$.

$\mathbb{Q}(\sqrt{D}) = a + b\sqrt{D}$

Every integer can be written as for example.

$$\underbrace{3^5 8^5 7^4 11^1 13^5}_{D} = \underbrace{\left(3^2 5^4 7^2 13^2\right)^2}_{\left(\frac{c}{b}\right)^2} \underbrace{(3 \cdot 11 \cdot 13)}_{d}$$

$\Rightarrow D = \left(\frac{c}{b}\right)^2 \times d$

$\Rightarrow \sqrt{D} = \left(\frac{c}{b}\right)\sqrt{d} = f\sqrt{d}$ where $f \in \mathbb{Q}$.

Where 'd' is a square free integer.
Therefor any number $\sqrt{D}$ can be expressed as $f\sqrt{d}$.
where d is a squarefree number.

$\mathbb{Q}(\sqrt{D}) \cong \mathbb{Q}(\sqrt{d})$ (with $f=1$).

We have seen that $\mathbb{Q}(\sqrt{D})$ where D is a square free integer and $\mathbb{Q}$ is rational set such that $\mathbb{Q}(\sqrt{D}) = \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\}$ is a field.

We now wish to check if $\mathbb{Z}(\sqrt{D}) = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}$ is a field if D is squarefree?

① $(\mathbb{Z}(\sqrt{D}), +)$ must be Abelian group.

$a_1 + b_1\sqrt{D} + a_2 + b_2\sqrt{D}$

$= \underbrace{(a_1+a_2)}_{\in \mathbb{Z}} + \underbrace{(b_1+b_2)}_{\in \mathbb{Z}}\sqrt{D}$ → Closure under +

→ Associative under +

→ 0 is the additive identity

→ $-(a+b\sqrt{D})$ is the additive inverse

→ Commutative under +.

② Closure under multiplication

$(a+b\sqrt{D}) \cdot (c+d\sqrt{D})$ → closed.

③ Associative under multiplication ✓

④ Multiplicative identity 1 ✓

⑤ Multiplicative inverse.

$(a+b\sqrt{D}) \cdot P = 1$ where P is mult. inverse.

consider $a \neq 0$ and $b \neq 0$ so that we can talk about inverse.

$$P = \frac{1}{a+b\sqrt{D}} = \frac{a-b\sqrt{D}}{a^2-b^2D} = \underbrace{\left(\frac{a}{a^2-b^2D}\right)}_{c} + \underbrace{\left(\frac{-b}{a^2-b^2D}\right)}_{d}\sqrt{D}$$

$$c = \frac{a}{a^2-b^2D}.$$

suppose $a=1, b=1$ and $D=3 \longrightarrow c = \frac{1}{1-3} = \frac{1}{-2} \notin \mathbb{Z}$

therefor inverse doesn't exist.

So $\mathbb{Z}(\sqrt{D})$ is not a field.

We now check another set $\mathbb{F}_p$ where p is a prime number and ask whether $\mathbb{F}_p(\sqrt{D})$ is a field?

Examples of Vector space:

① $$\mathbb{V} = \mathbb{C}^K = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{pmatrix} \;\middle|\; x_j \in \mathbb{C} \right\}$$

over the field $\mathbb{C}$ so scalar comes from $\mathbb{C}$

$\mathbb{C}^K$ is a vector space over $\mathbb{C}$

$\mathbb{R}^K$ is a vector space over $\mathbb{R}$.

② $\mathbb{F}^K$ where $\mathbb{F}$ is any field

$$\mathbb{F}^K = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{pmatrix} \middle| \; x_j \in \mathbb{F} \right\} \text{ over } \mathbb{F} \text{ (scalars)}$$

is always a vector space.

Suppose,

$\mathbb{F} = \mathbb{Z}_2 = \{0, 1\}$ with $\oplus, \otimes$ operations so $(\mathbb{F}, \oplus, \otimes)$ is a field.

Suppose, $\mathbb{V} = \mathbb{Z}_2^3 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \; x_j \in \mathbb{Z}_2 \right\}$

Two choices of
$$\left.\begin{array}{l} x_1 \rightarrow 0, 1 \\ x_2 \rightarrow 0, 1 \\ x_3 \rightarrow 0, 1 \end{array}\right\} \text{ overall choices } = 2 \times 2 \times 2 = 8$$

There is only 8 vectors in $\mathbb{Z}_2^3$.

$$\mathbb{Z}_2^3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\mathbb{Z}_2^3$ is a vector space over $\mathbb{Z}_2$ with $+_V$ and $\cdot$

$$\begin{pmatrix}1\\0\\1\end{pmatrix} +_{\mathbb{V}} \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1 \oplus 0\\0 \oplus 1\\1 \oplus 1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

vector addition   field addition.

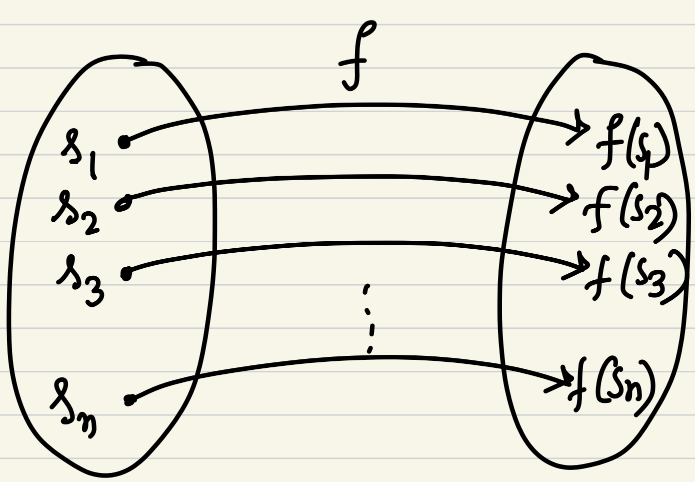
(3) $\mathbb{V} = \mathbb{R}^{m\times n} = \left\{ A = (a_{ij})_{\substack{0\le i\le m\\ 0\le j\le n}} \,\middle|\, \forall a_{ij} \in \mathbb{R} \right\}$

Say $\mathbb{V}$ is over $\mathbb{R}$. ($\mathbb{V}$ is a vector space over $\mathbb{R}$) with usual laws of addition and scalar multiplication of matrices.

$\mathbb{V} = \mathbb{C}^{m\times n}$ is a vector space over $\mathbb{C}$.

$\mathbb{V} = \mathbb{F}^{m\times n}$ is a vector space over $\mathbb{F}$

(4)



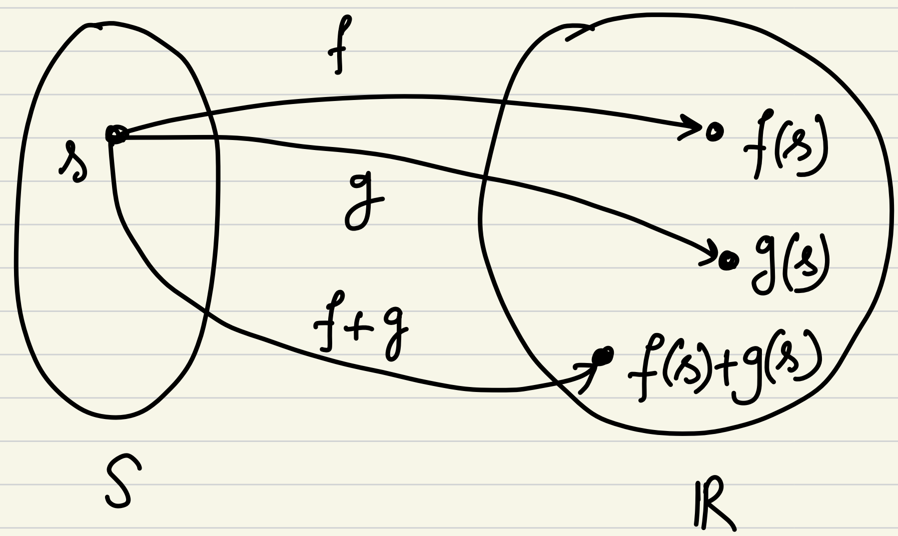
$f : S \longrightarrow \mathbb{R}$

any set S set of real numbers $\mathbb{R}$

Suppose we have another function $g: S \to \mathbb{R}$.
Both f and g are functions from S to $\mathbb{R}$. The value of the function $f(s)$, $g(s)$ are in $\mathbb{R}$. So we can add $f(s)$ and $g(s)$ as per $\mathbb{R}$ addition.

$f(s) + g(s) \rightarrow$ It gives another real number in $\mathbb{R}$

Now we can think of another function that will map s to $f(s)+g(s)$. We call that function as $f+g$.

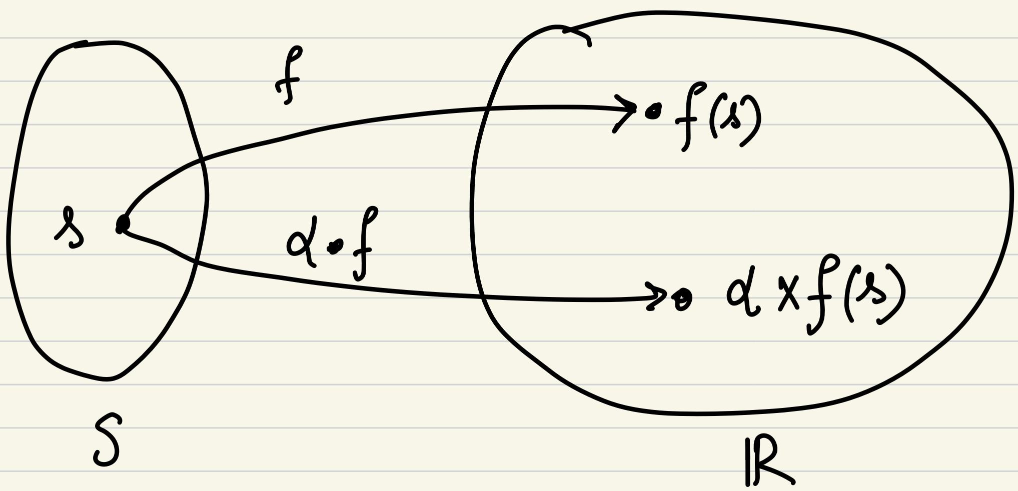


$f: S \to \mathbb{R}$
$g: S \to \mathbb{R}$
$f+g: S \to \mathbb{R}$

pointwise addition of addition: At each value of s, the value of the function is added.

$$\underbrace{(f+g)}_{\text{in } V}(s) := \underbrace{f(s) + g(s)}_{\text{in } \mathbb{R}}$$

(this addition is different)



$\alpha \in \mathbb{R}$.

Scalar α comes from $\mathbb{R}$ (field).

Scalar multiplication: $(\alpha \cdot f)(s) := \alpha \times f(s)$

Consider,

$$\mathbb{V} = \{ f \mid f : S \to \mathbb{R} \}$$ over $\mathbb{R}$ is vector space.

$$(f +_{\mathbb{V}} g)(s) := f(s) + g(s)$$

$$(\alpha \cdot f)(s) := \alpha \times f(s)$$

Suppose the scalar are taken from field $\mathbb{F}$. And the set $\mathbb{V}$ is the set of all functions from S to $\mathbb{F}$.

We denote $\mathbb{V} = \underbrace{\mathcal{F}[S, \mathbb{F}]}_{\text{Collection of all functions from } S \text{ to } \mathbb{F}.} = \{ f \mid f : S \to \mathbb{F} \}$

$\mathbb{V} = \mathcal{F}[S, \mathbb{R}]$ where $(f +_{\mathbb{V}} g)(s) \overset{\text{def.}}{:=} f(s) + g(s)$

over $\mathbb{R}$ where $\quad (\alpha \cdot f)(s) :\overset{\text{def}}{=} \alpha \times f(s)$

$\alpha \in \mathbb{R}$.

$\mathcal{F}[S, \mathbb{R}]$ is a vector space over $\mathbb{R}$

Suppose, $\quad S = \{ s_1, s_2, \ldots, s_K \}$.

$\mathcal{F}[S, \mathbb{R}]$ where $f \in \mathcal{F}[S, \mathbb{R}]$ such that

$f(s_1), f(s_2), \ldots, f(s_K)$ all belongs to $\mathbb{R}$.

So we can form $\quad \left(f(s_1), f(s_2), \ldots, f(s_K) \right)$ so it will be $\mathbb{R}^K$.

Basically $\mathcal{F}[S, \mathbb{R}]$ can be identified as $\mathbb{R}^K$.

Generally $\mathcal{F}[S, \mathbb{F}]$ over $\mathbb{F}$ is a vector space.

5 Suppose, $\quad S = \{ s_1, s_2, s_3, \ldots \}$

$$\mathcal{F}[S, \mathbb{F}] = \{ f \mid f : S \to \mathbb{F} \}$$

Consider any function $f \in \mathcal{F}[S, \mathbb{F}]$

Corresponding to this we get the values -

$f(s_1), f(s_2), f(s_3), \ldots$ (sequence)

$$\begin{array}{cccc} f(s_1), & f(s_2), & f(s_3), & \ldots \\ \downarrow & \downarrow & \downarrow & \\ |f(s_1)|, & |f(s_2)|, & |f(s_3)|, & \ldots \end{array}$$

Let's add all these:

$$\sum_{j=1}^{\infty} |f(s_j)| \longrightarrow \text{May be finite or may be infinite.}$$

Consider all those functions where the sum is $< \infty$.

$$\left\{ f \in \mathcal{F}[S, \mathbb{F}] \;\middle|\; \sum_{j=1}^{\infty} |f(s_j)| < \infty \text{ where } \forall s_j \in S,\ f: S \to \mathbb{F},\ \forall f(s_j) \in \mathbb{F} \right\}$$

This set of vectors / functions are called $L^1[S, \mathbb{F}]$.

Among all the functions in $\mathcal{F}[S, \mathbb{F}]$, we choose the functions in such a way that the sum is finite. These are members of $L^1[S, \mathbb{F}]$.

$f(s_j) = \frac{1}{j} \notin L^1[S, \mathbb{F}]$ but $f(s_j) = \frac{1}{j^2} \in L^1[S, \mathbb{F}]$

We can verify that $L^1[S, \mathbb{F}]$ is a vector space over $\mathbb{F}$.

Similarly we can form -

$$L^2[S, \mathbb{F}] = \left\{ f \in \mathcal{F}[S, \mathbb{F}] \;\middle|\; \sum_{j=1}^{\infty} |f(s_j)|^2 < \infty \right\}$$

Suppose $f(s_j) = \frac{1}{j} \notin L^1[S, \mathbb{F}]$

but $f(s_j) = \frac{1}{j} \in L^2[S, \mathbb{F}]$

We can verify that $L^2[S, \mathbb{F}]$ is a vector space over $\mathbb{F}$.

Suppose instead of discrete valued domain, if S is continuous then $\sum$ is replaced by $\int$.

S = Interval b/w a to b $= I$

$$L^1[S, \mathbb{F}] = \left\{ f \in \mathcal{F}[S, \mathbb{F}] \;\middle|\; \int_I |f(s)|\, ds < \infty \right\}$$

$$L^2[S, \mathbb{F}] = \left\{ f \in \mathcal{F}[S, \mathbb{R}] \;\middle|\; \int_I |f(s)|^2 ds < \infty \right\}$$

$L^1[S, \mathbb{F}]$, $L^2[S, \mathbb{F}]$ are all vector spaces over $\mathbb{F}$

⑥ $\mathbb{F}[\lambda] = \left\{ a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_k\lambda^k \;\middle|\; \forall a_j \in \mathbb{F}, \forall k \in \mathbb{Z}^+ \right\}$

Set of all polynomials in λ
(after varying all a_j and varying all possible k)

$\mathbb{F}[\lambda]$ is a vector space over $\mathbb{F}$ with pointwise addition & scalar multiplication of functions.

suppose, $p \in \mathbb{F}[\lambda]$ such that,

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_k\lambda^k$$

$$(-p)(\lambda) = -a_0 - a_1\lambda - a_2\lambda^2 - \cdots - a_k\lambda^k$$

$$0_V(\lambda) = 0 \quad (\text{for all } \lambda)$$

⑦ Let d denote any positive integer.

$$\mathbb{F}_d = \left\{ p \in \mathbb{F}[\lambda] \;\middle|\; \deg(p(\lambda)) \leq d \right\}$$

Collections of all polynomial functions p in λ such that their degree is below a certain limit d

$\mathbb{F}_d$ is also a vector space over $\mathbb{F}$.

linear combination:

Consider a field $\mathbb{F}$
Consider a vector space $\mathbb{V}$ over $\mathbb{F}$.
Let's look at a vector $x \in \mathbb{V}$.

What are all the vectors that we can "build" starting from x? That means we start from x and do something on it and produce another vector. The only 2 things that we can do in the vector space is addition & scalar multiplications.

"Build" means using 2 operations of vector space. generate other vectors.

addition: $x +_{\mathbb{V}} x = 2x$

$$x +_{\mathbb{V}} 2x = 3x$$

$$\vdots$$

For any positive integer n, we can build nx starting from the vector $x \in \mathbb{V}$.

Scalar multiplication: $\alpha \cdot x$ where $\alpha \in \mathbb{F}$.

So we can build $\alpha \cdot x$ for all possible α which also includes the result of addition that is nx

The vectors that we can build starting from x are precisely the $\{ c \cdot x \mid c \in \mathbb{F} \}$

Suppose we give 2 vectors $x, y \in \mathbb{V}$. What are all the vectors that we can build using x & y?

Starting from x: $\{ \alpha \cdot x \mid \alpha \in \mathbb{F} \}$
Starting from y: $\{ \beta \cdot y \mid \beta \in \mathbb{F} \}$

We can even add them so we build $\{ \alpha \cdot x +_{\mathbb{V}} \beta \cdot y \mid \alpha, \beta \in \mathbb{F} \}$ starting from x and y in $\mathbb{V}$.

If we have in general $S = \{ u_1, u_2, \ldots, u_K \}$ be a finite set of vectors in $\mathbb{V}$. then we can build the vectors from set S as:

$$\{ \alpha_1 \cdot u_1 +_{V} \alpha_2 \cdot u_2 +_{V} \cdots +_{V} \alpha_K u_K \mid \forall \alpha_i \in \mathbb{F} \}$$

Let $S = \{ u_1, u_2, \ldots, u_K \}$ where $\forall u_j \in \mathbb{V}$. Any vector of the form $\alpha_1 \cdot u_1 +_{V} \alpha_2 \cdot u_2 +_{V} \cdots +_{V} \alpha_K \cdot u_K$ where $\forall \alpha_i \in \mathbb{F}$ is called a "Linear combination" of the vectors in S.

Ex: $\mathbb{R}^3$ over $\mathbb{R}$. Say $u_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$S = \{u_1, u_2\}$.

Claim: $u = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ is a linear combination of vectors in S.

So we have to prove that,

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \alpha_1 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.

Does there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t. the above equation holds true?

$\Rightarrow$ Yes. $\alpha_1 = 2$, $\alpha_2 = -1$

Therefore, u is the LC of vectors of S.

Ex 2: $\mathbb{V} = \mathbb{R}^{2\times 3}$. $S = \{A_1, A_2\}$.

$A_1 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Claim: $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ is a LC of vectors in S.

$A = \alpha_1 \cdot A_1 + \alpha_2 \cdot A_2$ where $\alpha_1 = 2$, $\alpha_2 = -1$

So it is a LC of vectors of S.

Ex3: $\mathbb{V} = \mathbb{F}[\lambda]$.

$S = \{ P_1, P_2, P_3 \}$

where $P_1(\lambda) := 1 + \lambda$

$P_2(\lambda) := 1 - \lambda$

$P_3(\lambda) := 3 + \lambda$

Claim: $P(\lambda) = 2 + 4\lambda$ is a LC of vectors of S.

$$2 + 4\lambda = \alpha_1 (1+\lambda) + \alpha_2 (1-\lambda) + \alpha_3 (3+\lambda)$$

$\left.\begin{array}{l} \alpha_1 = 3 \\ \alpha_2 = -1 \\ \alpha_3 = 0 \end{array}\right\}$ will satisfy the above equation. hence P is LC of vectors of S.

Note: We can also write, $P(\lambda)$ as $P_1(\lambda) + (-2) P_2(\lambda) + 1 P_3(\lambda)$.

Therefore, the LC need not be unique. It may be possible to get more than one LC of a vector as vectors of vector space.

$\mathbb{V}$ is a vector space over $\mathbb{F}$.

$S = \{u_1, u_2, \dots, u_K\}$ $\forall u_K \in \mathbb{V}$.

LC of S: $\alpha_1 u_1 +_v \alpha_2 u_2 +_v \dots +_v \alpha_K u_K$, $\alpha_i \in \mathbb{F}$.

Suppose we choose $\forall \alpha_i = 0$ then what do we get?
Then we get -

$$0 \cdot u_1 +_v \dots +_v 0 \cdot u_K = 0_V$$

Therefore, whatever set S we start with, we can always get 0_V as a LC of vectors of S by choosing all scalars as 0. We can build 0_V from any set of vectors. The linear combination of set of vectors in S where all the coefficients are chosen as 0 is called "trivial" LC.

$\alpha_1 \cdot u_1 +_v \alpha_2 \cdot u_2 +_v \dots +_v \alpha_K \cdot u_K$ is called "trivial"

if $\forall \alpha_j = 0$. and it is called "non-trivial" LC of set of vectors in S if $\exists \alpha_j \neq 0$.

Given any finite set S in $\mathbb{V}$. The trivial LC of S always yields 0_V. Is the trivial LC the only LC that will yield 0_V in the set S?

$\mathbb{V} = \mathbb{R}^3$ over $\mathbb{R}$.

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Is the trivial LC the only LC of vectors in S that yields $0_{\mathbb{V}}$?

$$\alpha_1 \cdot u_1 +_v \alpha_2 u_2 +_v \alpha_3 \cdot u_3 = 0_{\mathbb{V}}.$$

$$\Rightarrow \alpha_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 + \alpha_3 &= 0 \\ -\alpha_2 - \alpha_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} &\left.\begin{aligned} \alpha_1 &= -\alpha_2 \\ \alpha_1 &= -\alpha_3 \end{aligned}\right\} \alpha_2 = \alpha_3 \\ &\alpha_2 = -\alpha_3 \end{aligned}$$

Since $\left.\begin{aligned} \alpha_2 &= \alpha_3 \\ \alpha_2 &= -\alpha_3 \end{aligned}\right\}$ $\alpha_2 = 0$ which means $\alpha_3 = 0$

so $\alpha_1 = 0$.

Hence the only LC that yields $0_{\mathbb{V}}$ is the trivial LC and there is no other LC that yields $0_{\mathbb{V}}$ which is non trivial LC. of set of vectors in S. Therefore only trivial LC leads to 0_V.

Suppose, $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\}$

observe $2 \cdot u_1 +_v u_2 -_v u_3 = 0_{\mathbb{V}}$

That means $c_1 = 2$, $c_2 = 1$, $c_3 = -1$ yields 0_V which is a nontrivial LC of vectors of S. So it is a non-trivial LC of vectors of S.

Given a set S

Only trivial LC yields 0_V (No other non trivial LC yields 0_V)

(Linear independent set)

Non trivial LC yield 0_V (and of course trivial LC also yields 0_V)

(Linear dependent set)

Let S be a finite set of vectors in $\mathbb{V}$ so $S \subseteq \mathbb{V}$. We say S as a LI set of vectors if the trivial LC of vectors of S is the one and only LC that yields 0_V and there is no other non trivial LC that yields 0_V.

$$\left(S \text{ is LI set}\right) \text{ iff } \left(\sum_{i=1}^{n} \alpha_i u_i = 0_V \rightarrow \forall \alpha_i = 0\right)$$

$$\left(S \text{ is LD set}\right) \text{ iff } \left(\sum_{i=1}^{n} \alpha_i u_i = 0_V \rightarrow \exists \alpha_i \neq 0\right)$$

Linear independence:

From linear combination $\longrightarrow$ linear dependence / independence

$\mathbb{V}$ is a vector space over $\mathbb{F}$

A finite set, $S = \{u_1, u_2, \ldots, u_K\} \subseteq \mathbb{V}$.

LI

A linear combination of set of vectors in S giving rise to $0_{\mathbb{V}}$.

$$\left(\sum_{i=1}^{K} c_K u_K = 0_{\mathbb{V}}\right) \longrightarrow \forall c_K = 0$$

LD

$$\left(\sum_{i=1}^{K} c_K u_K = 0_{\mathbb{V}} \longrightarrow \exists c_K \neq 0\right)$$

non trivial LC should also yield $0_{\mathbb{V}}$ apart from trivial LC.

The only way to get $0_{\mathbb{V}}$ out of a linear combination of set of vectors in S is the trivial linear combination.

Suppose $S = \{u_1, u_2, \dots, u_r\}$ is LD set of vectors.
Therefore,

$$\sum_{i=1}^{r} c_r u_r = 0_v \longrightarrow \exists c_r \neq 0$$

Suppose x is LC of set of vectors of S. That means we should be able to express x as a linear combination of vectors in S.

$$x = \beta_1 u_1 +_v \beta_2 u_2 +_v \dots +_v \beta_r u_r \quad (\forall \beta_i \in \mathbb{F})$$

We can also write -

$$0_W = \alpha_1 u_1 +_v \alpha_2 u_2 +_v \dots +_v \alpha_r u_r \quad (\forall \alpha_i \in \mathbb{F})$$

(Where $\exists \alpha_K \neq 0$)

$$x +_v 0_W = (\alpha_1 + \beta_1) u_1 +_v (\alpha_2 + \beta_2) u_2 +_v \dots +_v (\alpha_r + \beta_r) u_r$$

$$\Rightarrow x = \gamma_1 u_1 +_v \gamma_2 \cdot u_2 +_v \dots + \gamma_r \cdot u_r \quad ; \; \gamma_i = \alpha_i + \beta_i$$

Since we have assumed $\exists \alpha_K \neq 0$ that means we can say $\gamma_i \neq \beta_i$. Therefore, we have 2 representations of x. This means when S is LD set, x has at least 2 representation as LC of vectors of S.

S is LD $\longrightarrow$ Any x as a LC of set of vectors in S has at least 2 representation of LC. So LC is not unique.

Ex1: $\mathbb{V} = \mathbb{R}^3$ over $\mathbb{R}$.

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

S is LI iff $\left(\sum_{i=1}^{3} \alpha_i u_i = 0_{\mathbb{V}} \longrightarrow \forall \alpha_i = 0 \right)$

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 + \alpha_3 &= 0 \end{aligned}\right\} \alpha_1 = -\alpha_2,\ \alpha_1 = -\alpha_3 \Big\} \ \alpha_2 = \alpha_3$$

$$-\alpha_2 - \alpha_3 = 0 \ \Big\} \ \alpha_2 = -\alpha_3$$

$\Rightarrow \alpha_2 = 0$, $\alpha_3 = 0$ and $\alpha_1 = 0$

Hence S is LI set.

Ex2: $\mathbb{V} = \mathbb{F}[\lambda] := \left\{ a_0 + a_1\lambda + \cdots + a_k\lambda^k \;\middle|\; \forall a_k \in \mathbb{F},\ \underline{\lambda \in \mathbb{F}},\ \underline{f(\lambda) \in \mathbb{F}} \right\}$

Suppose $\mathbb{V}$ over $\mathbb{F}$.

Consider a set, $S = \{ p_1, p_2, p_3 \}$

$p_1(\lambda) := 1+\lambda$

$p_2(\lambda) := 1-\lambda$

$p_3(\lambda) := 1+\lambda+\lambda^2$

Is the set S a LI set?

$$\alpha_1 \cdot p_1 +_v \alpha_2 \cdot p_2 +_v \alpha_3 p_3 = 0_v$$

Zero polynomial

$0_v(\lambda) = 0$

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$.

additive identity of vector space $\mathbb{F}[\lambda]$.

By the usual rules of vector addition & scalar mult. of the vector space of polynomials -

$$\left(\alpha_1 \cdot p_1 +_v \alpha_2 \cdot p_2 +_v \alpha_3 p_3\right)(\lambda) = 0_v(\lambda)$$

$$\Rightarrow \alpha_1 p_1(\lambda) + \alpha_2 p_2(\lambda) + \alpha_3 p_3(\lambda) = 0$$

$$\Rightarrow \alpha_1(1+\lambda) + \alpha_2(1-\lambda) + \alpha_3(1+\lambda+\lambda^2) = 0$$

Definitely $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$ satisfies it but the question is — is there any other non-trivial LC that gives rise to 0?

$$\Rightarrow (\alpha_1+\alpha_2+\alpha_3) + (\alpha_1 - \alpha_2 + \alpha_3)\lambda + \alpha_3 \cdot \lambda^2 = 0$$

$$\Rightarrow \underbrace{\gamma_1 + \gamma_2 \lambda + \gamma_3 \lambda^3}_{\text{another polynomial } P(\lambda)} = 0$$

$\Rightarrow P(\lambda) = 0$ (for all λ) that means $P(\lambda)$ is the zero polynomial

$\Rightarrow P = 0_{\mathbb{V}}$ (This is not equation, p is zero polynomial)

In a zero polynomial, all coefficients must be 0.

$$\Rightarrow \begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 - \alpha_2 + \alpha_3 &= 0 \end{aligned} \quad \begin{aligned} \Rightarrow \alpha_1 + \alpha_2 &= 0 \\ \Rightarrow \alpha_1 - \alpha_2 &= 0 \end{aligned} \Bigg\} \; \alpha_1 = \alpha_2 = 0$$

$$\alpha_3 = 0$$

Therefore, $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$. We have seen that $\alpha_1 P_1(\lambda) + \alpha_2 P_2(\lambda) + \alpha_3 P_3(\lambda) = 0$ (a LC of S gives rise to 0) implies $\forall \alpha_i = 0$ iff S is LI set.

Hence the set of polynomials $\{P_1, P_2, P_3\}$ are LI set.

<u>Ex 4:</u> $\mathbb{V} = \mathbb{F}[\lambda]$ over $\mathbb{F}$.

$S = \{P_1, P_2, P_3\}$, $\quad P_1(\lambda) = 1 + \lambda$ $\quad$ Is S LI set?

$P_2(\lambda) = 1 - \lambda$

$P_3(\lambda) = 3 + \lambda$

Consider the LC of set of vectors in S giving rise to 0_V.

$$\alpha_1 \cdot P_1(\lambda) + \alpha_2 \cdot P_2(\lambda) + \alpha_3 \cdot P_3(\lambda) = 0$$

$$\Rightarrow \alpha_1(1+\lambda) + \alpha_2(1-\lambda) + \alpha_3(3+\lambda) = 0$$

$$\Rightarrow (\alpha_1 + \alpha_2 + 3\alpha_3) + (\alpha_1 - \alpha_2 + \alpha_3)\lambda = 0$$

$$\Rightarrow P(\lambda) = 0 \quad (\text{It must be true for all } \lambda)$$

$$\Rightarrow P = 0_V \ (\text{zero polynomial})$$

We are looking for $\alpha_1, \alpha_2, \alpha_3$ such that we get zero polynomial. In a zero polynomial all coeff. are 0.

$$\left.\begin{aligned} \alpha_1 + \alpha_2 + 3\alpha_3 &= 0 \\ \alpha_1 + \alpha_3 - \alpha_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} \alpha_2 &= -\alpha_1 - 3\alpha_3 \\ \alpha_2 &= \alpha_1 + \alpha_3 \end{aligned}$$

$$\Rightarrow -\alpha_1 - 3\alpha_3 = \alpha_1 + \alpha_3$$

$$\Rightarrow -2\alpha_1 = 4\alpha_3$$

$$\Rightarrow \alpha_1 = -2\alpha_3$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -2\alpha_3 \\ -\alpha_3 \\ \alpha_3 \end{bmatrix} = k \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

Definitely if we choose $k \neq 0$ then this choice of $\alpha_1, \alpha_2, \alpha_3$ will lead to zero polynomial for all x.

Therefore $\sum_{j=1}^{3} \alpha_j P_j = 0_{\mathbb{V}} \longrightarrow \exists \alpha_j \neq 0$ iff

S is LD set of vectors.

Building of vectors:

$\mathbb{V}$ is a vector space over a $\mathbb{F}$ (universe)

Take a finite set, $S = \{u_1, u_2, \ldots, u_r\} \subseteq \mathbb{V}$

The vectors that we can build using the addition & scalar multiplication on vectors of S are precisely the linear combinations of S vectors.

$L[S]$ = Collection of all linear combination of vectors in S

$$= \left\{ x \in \mathbb{V} \;\middle|\; x = \sum_{i=1}^{r} \alpha_i u_i \;,\; \forall \alpha_i \in \mathbb{F} \,,\, \forall u_i \in S \right\}$$

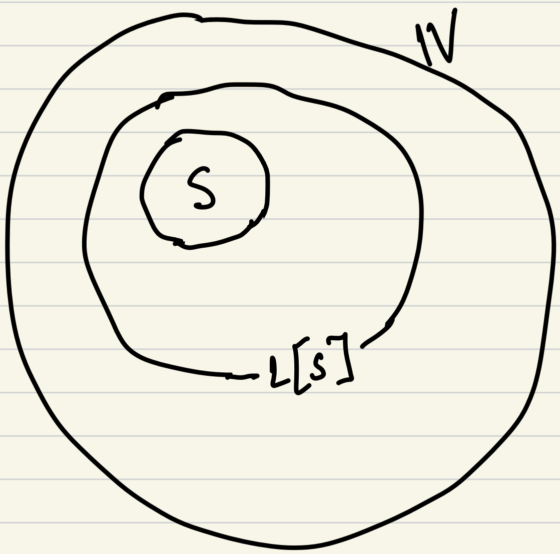
Observe:

(i) $0_{\mathbb{V}} \in L[S]$ if we take $\forall \alpha_i = 0$. So the zero vector will always be part of the collection $L[S]$.

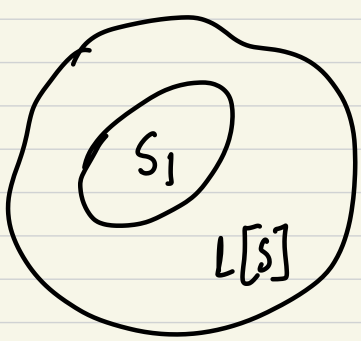
(ii) $u_i \in S$ and u_i can be expressed as LC of vectors in S such that $\forall \alpha_j = 0$ and $\alpha_i = 1$ $(i \neq j)$.

$$u_i = 0 \cdot u_1 +_V 0 \cdot u_2 +_V \cdots 0 \cdot u_{i-1} +_V 1 \cdot u_i +_V 0 \cdot u_{i+1} +_V \cdots +_V 0 \cdot u_r$$

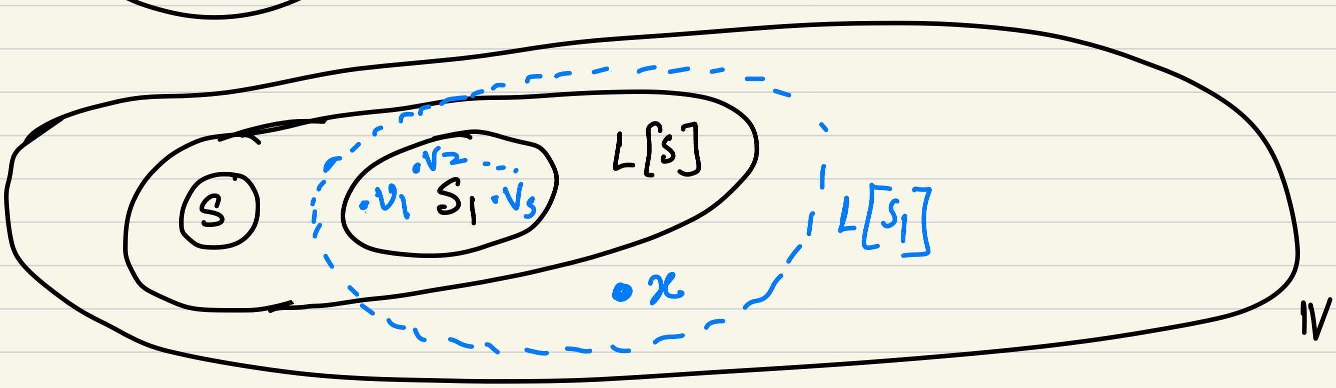
Note that all the vectors u_i of S must belong to $L[S]$. by taking that special choice of scalars in the LC of S. It implies S is contained in $L[S]$ so $S \subset L[S]$.



We started from S.
We built $L[S]$ from S.
Using $L[S]$, can we create anything new?



Let S_1 be a finite subset of $L[S]$.
Use S_1 to build $L[S_1]$ — the collection of all LC of S_1 vectors.



If $L[S_1]$ comes out of $L[S]$ boundary then we can say that we have build something new. But if $L[S_1]$ also contained in $L[S]$ then there is nothing that we gained.

Does $L[S_1]$ have vectors which are not in $L[S]$ but in $\mathbb{V}$?

$\Rightarrow$ $S_1 = \{v_1, v_2, \dots, v_s\}$. It is a finite set and sitting insider $L[S]$.

$x \in L[S_1]$

$\Rightarrow$ x is a linear combination of S_1 vectors.

$$x = \alpha_1 v_1 +_v \alpha_2 v_2 +_v \cdots +_v \alpha_s v_s \qquad (\alpha_i \in \mathbb{F})$$

Now $v_1, v_2, \dots, v_s$ are sitting inside $L[S]$. That means $v_1, v_2, \dots, v_s$ can be obtained by LC of S vectors.

$$\left.\begin{aligned} v_1 &= \alpha_{11} u_1 +_v \alpha_{12} u_2 +_v \cdots +_v \alpha_{1r} u_r \\ v_2 &= \alpha_{21} u_1 +_v \alpha_{22} u_2 +_v \cdots +_v \alpha_{2r} u_r \\ &\;\;\vdots \\ v_s &= \alpha_{s1} u_1 +_v \alpha_{s2} u_2 +_v \cdots +_v \alpha_{sr} u_r \end{aligned}\right\} \quad \forall \alpha_{ij} \in \mathbb{F}$$

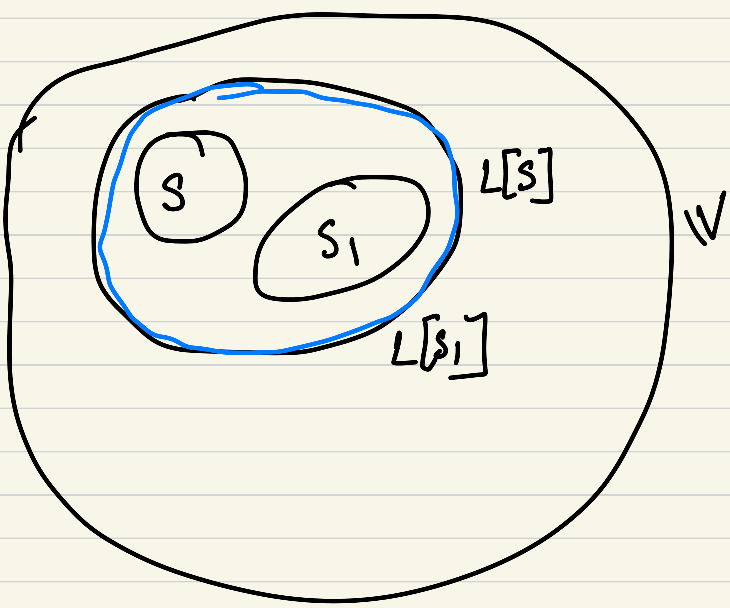
Therefore,

$$\begin{aligned} x = {} & \alpha_1 \cdot (\alpha_{11} u_1 +_v \alpha_{12} u_2 +_v \cdots +_v \alpha_{1r} u_r) +_v \\ & \alpha_2 (\alpha_{21} u_1 +_v \alpha_{22} u_2 +_v \cdots +_v \alpha_{2r} u_r) +_v \\ & \vdots \\ & \alpha_s (\alpha_{s1} u_1 +_v \alpha_{s2} u_2 +_v \cdots +_v \alpha_{sr} u_r) \end{aligned}$$

$$\Rightarrow x = (\alpha_1 \cdot \alpha_{11} + \alpha_2 \cdot \alpha_{21} + \cdots + \alpha_s \cdot \alpha_{s1}) u_1 +_v$$
$$(\alpha_1 \cdot \alpha_{12} + \alpha_2 \cdot \alpha_{22} + \cdots + \alpha_s \cdot \alpha_{s2}) u_2 +_v$$
$$\vdots$$
$$(\alpha_1 \cdot \alpha_{1r} + \alpha_2 \cdot \alpha_{2r} + \cdots + \alpha_s \cdot \alpha_{sr}) u_r$$

$$\Rightarrow x = \gamma_1 u_1 +_v \gamma_2 \cdot u_2 + \cdots + \gamma_r u_r$$

Note that x is a LC of vectors of S hence $x \in L[S]$. Initially we assumed $x \notin L[S]$ but $x \in L[S_1]$ but it is a contradiction. Hence $x \in L[S]$. Therefore we can't expand anything out of $L[S]$ vectors.



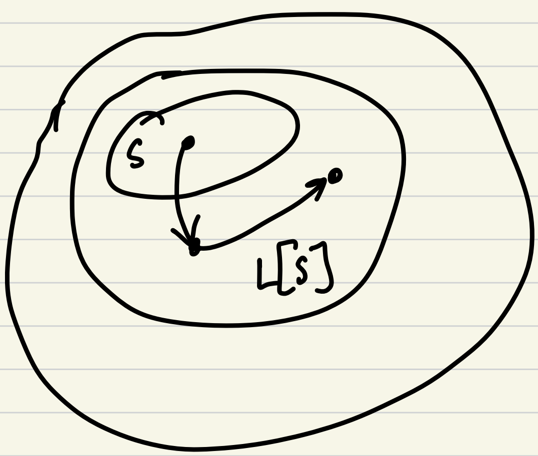
Therefore $L[S_1]$ doesn't go outside $L[S]$. So any vectors set you choose from $L[S]$, we can't expand the boundary.

$$\begin{tikzcd}[column sep=huge] \text{Started with } S \arrow[r, "\text{build}"] & L[S] \arrow[r, "\text{Try to}", "\text{build}"'] & L[S_1] \arrow[l, bend left, "\text{pulls you back}"] \end{tikzcd}$$

The building process is saturated at $L[S]$ stage. which means starting from S, the largest collection that we can build from S is $L[S]$.

What makes the saturation?

$\Rightarrow$ The only 2 thing that is allowed are vector addition & scalar multiplication.



The linear combination of a linear combination is again a linear combination.

any 2 vectors in $L[S]$, the addition & scalar mult is closed in $L[S]$.

$v_1, v_2 \in L[S]$, $v_1 +_v v_2 \in L[S]$

$v_1 \in L[S]$, $c \in \mathbb{F}$, $c \cdot v_1 \in L[S]$

Therefore, $L[S]$ is closed under $+_v$ and $\cdot$

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. let $\mathbb{W}$ be a subset of $\mathbb{V}$ such that –

(i) $\mathbb{W}$ is non empty.

(ii) $\mathbb{W}$ is closed under $+_V$ and $\cdot$

$$x, y \in \mathbb{W} \Rightarrow x +_V y \in \mathbb{W}$$

$$\alpha \in \mathbb{F}, x \in \mathbb{W} \Rightarrow \alpha \cdot x \in \mathbb{W}$$

So $\mathbb{W}$ is a subspace of $\mathbb{V}$. And it is easy to see $L[S]$ is a subspace of $\mathbb{V}$ which is nothing but span of S.

$0_{\mathbb{V}}$ belongs to every subspace of $\mathbb{V}$ from (i). $\mathbb{W}$ is non empty so for $w \in \mathbb{W}$ and since $c \cdot w$ is closed in $\mathbb{W}$ so $c = 0 \Rightarrow 0 \cdot w = 0_{\mathbb{V}}$ is closed in $\mathbb{W}$.

Subspaces associated with matrix:

$\mathbb{F}$: field

Vector space $\mathbb{F}^{m \times n}$ = Space of all $m \times n$ matrices over $\mathbb{F}$

(I) $\mathbb{W} = \{ x \in \mathbb{F}^n \mid Ax = 0_m, A \in \mathbb{F}^{m \times n}, 0 \in \mathbb{F}^m \}$

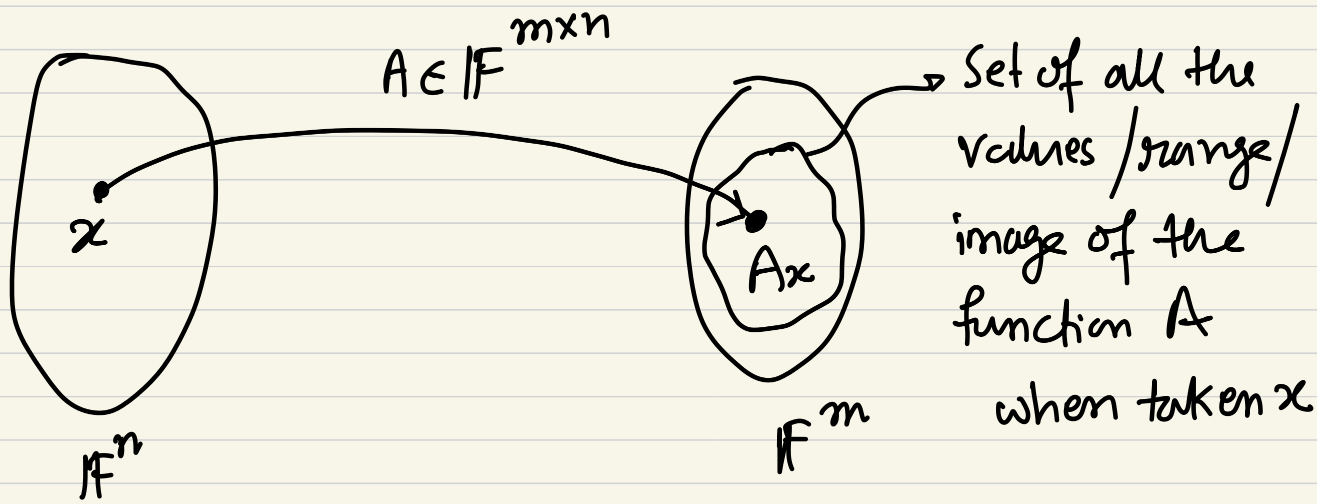
The set of solutions of the homogeneous system corresponding to matrix A. $\mathbb{W}$ will be a subspace of $\mathbb{F}^n$ not $\mathbb{F}^{m \times n}$

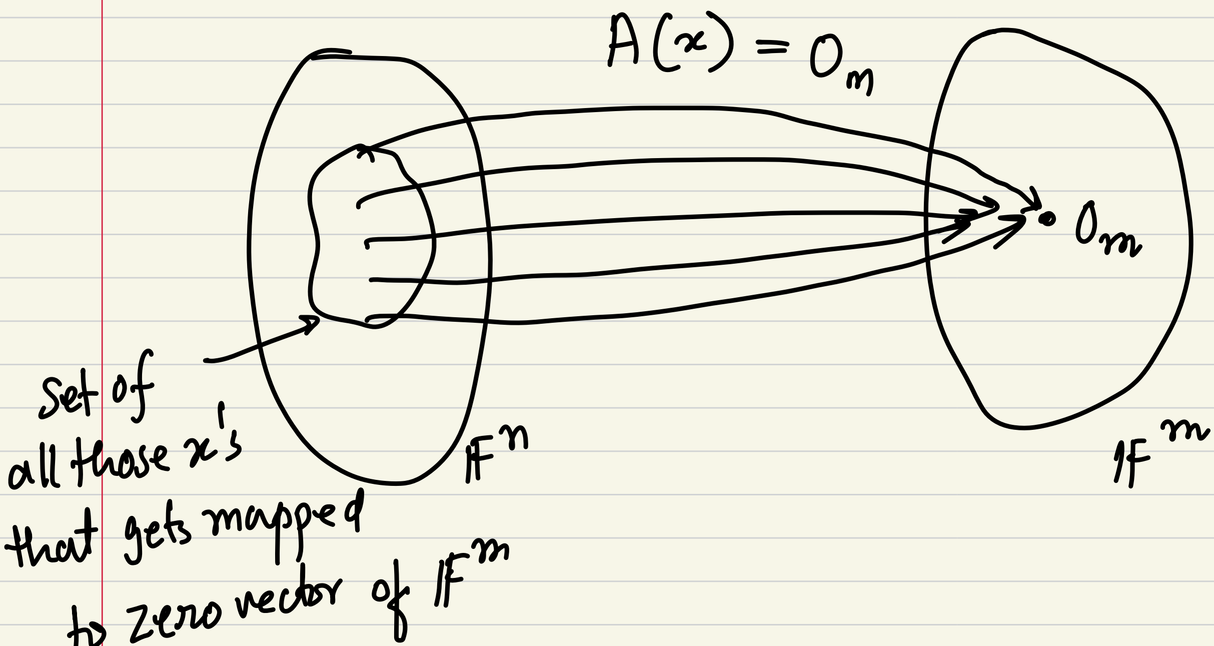
$\mathbb{W}$ is called Kernal of A or nullspace of A.

(II) $\mathbb{W} = \{ y \in \mathbb{F}^m \mid Ax = y, A \in \mathbb{F}^{m \times n}, x \in \mathbb{F}^n \}$

The set $\mathbb{W}$ is subspace of $\mathbb{F}^m$. $\mathbb{W}$ is called image of A or range of A

$A \in \mathbb{F}^{m \times n}$.





(III) Vector space $\mathbb{V}$ over $\mathbb{F}$.

Take a finite subset $S = \{u_1, u_2, \ldots, u_r\} \subseteq \mathbb{V}$.

$$L[S] = \left\{ \sum_{i=1}^{r} \alpha_i u_i \;\middle|\; \forall \alpha_i \in \mathbb{F}, \forall u_i \in S \right\}$$

This is a subspace of $\mathbb{V}$. $L[S]$ is called subspace spanned by S. It is also called span of S.

$A \in \mathbb{F}^{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (\forall a_{ij} \in \mathbb{F})$$

$$C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \in \mathbb{F}^m , \ldots, C_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{F}^m$$

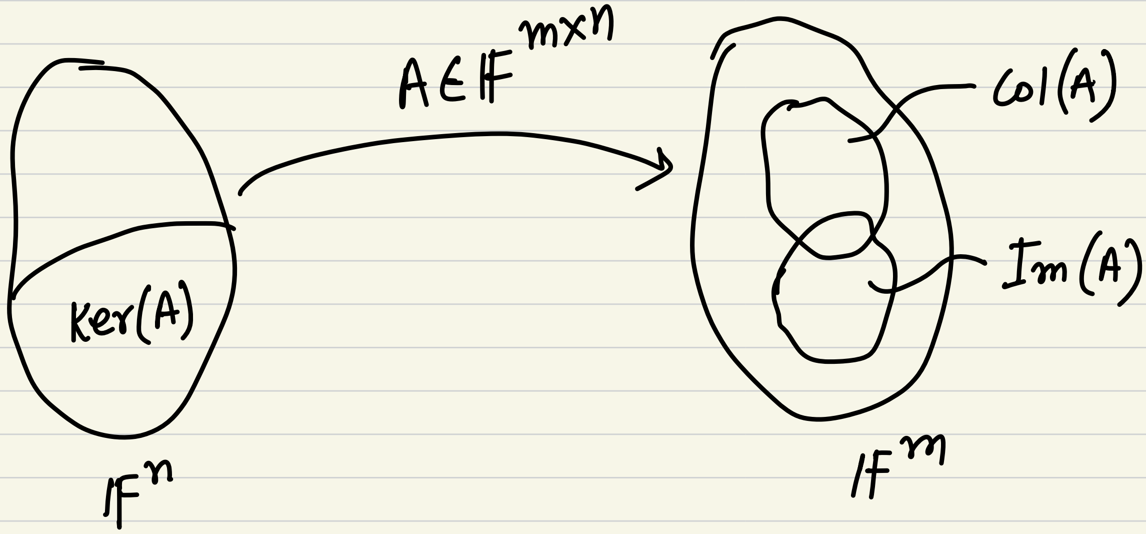
$C_1, C_2, \ldots, C_n$ are called column vectors of A

Consider the set, $S = \{C_1, C_2, \ldots, C_n\} \subseteq \mathbb{F}^m$

Finite subset of vectors so we can look at subspace of $\mathbb{F}^m$ spanned by S.

$L[S]$ = The set of all linear combination of vectors of S (or the columns of A)

$$L[S] = \left\{ x \in \mathbb{F}^m \;\middle|\; x = \alpha_1 C_1 +_v \alpha_2 C_2 +_v \cdots +_v \alpha_n C_n , \; \forall \alpha_i \in \mathbb{F} \right\}$$
$$= \mathrm{Col}(A).$$



$b \in \mathrm{Col}(A)$ iff $b = \alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n$

$$= \alpha_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$= Ax \text{ where } x = (\alpha_1 \cdots \alpha_n)^t$$

Therefore $b = Ax$ where $\exists x \in \mathbb{F}^n$ so $b \in \mathbb{F}^m$ hence $\mathrm{Col}(A) = \mathrm{Range}(A) = \mathrm{Image}(A)$.

$\mathbb{F}^n$	$\mathbb{F}^m$
Kernal (A) Image (AT)	Image (A) Kernel (AT)

Subspace spanned by a set S:

Vector space $\mathbb{V}$ over a field $\mathbb{F}$ (universe)

Subset, $S = \{u_1, u_2, \ldots, u_r\}$.

$$\langle S \rangle = \text{span}(S) = L[S] = \left\{ x \;\middle|\; x = \sum_{i=1}^{r} \alpha_i u_i \;,\; \forall \alpha_i \in \mathbb{F} \right\}$$

$\langle S \rangle$ is a subspace of $\mathbb{V}$ over $\mathbb{F}$.

$\langle S \rangle$ contains S.

$\langle S \rangle$ is the smallest subspace that contains S.

What if S is infinite set?

$\Rightarrow$ Consider a finite subset $\hat{S}$ of S.

Take all possible finite subset of S and consider $L[\hat{S}]$. Take their union.

$$\mathcal{F}_S = \{ \hat{S} \mid \hat{S} \subseteq S \,,\, |\hat{S}| < \infty \}$$

$$\forall \hat{S} \in \mathcal{F}_S \,, \quad L[S] :\stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} L[\hat{S}_i]$$

$L[S] := \{$All possible linear combination of finite subset of vectors of $S\}$

$L[S]$ is a subspace containing S which is smallest subspace containing S.

Linear independence for infinite set :

$$S = \{u_1, u_2, \ldots, u_r\} \text{ is LI} \longleftrightarrow \left(\sum_{i=1}^{r} \alpha_i u_i = 0_V \longrightarrow \forall \alpha_i = 0\right)$$

Suppose S is an infinite set in vector space $\mathbb{V}$.
An infinite set S in $\mathbb{V}$ is LI if and only if every finite subset of S is LI.

Ex: $\mathbb{F}[x]$: Set of all polynomials with all the coefficients from $\mathbb{F}$. It is vector space over $\mathbb{F}$.

$$S = \{1, x, x^2, \ldots\ldots\}, \quad |S| \rightarrow \infty$$

These are all polynomial functions/vectors.

Is S linearly independent?
$\Rightarrow$ To check, we take a finite subset of S.
Take some arbitrary finite subset $\hat{S} = \{1, x, x^2\}$

$$\hat{S} \text{ is LI iff } \quad \alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3 \cdot x^2 = 0 \longrightarrow \forall \alpha_i = 0$$

The set of polynomials $1, x, x^2$ gives rise to zero polynomial when the coefficients are 0. $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$ hence $\hat{S}$ is LI. Hence S is LI by generalization.

Ex: $I = [0, 2\pi]$

Consider set of all the continuous functions from I to $\mathbb{R}$.

$$\mathcal{F}[I, \mathbb{R}] = \{ f \mid f : I \to \mathbb{R} \}$$

Consider the set $S = \{ \sin(nx) \mid n = 1, 2, 3, \dots \}$

Is S a LI set?

$\Rightarrow$ Take a finite subset $\hat{S} = \{ \sin(x), \sin(2x), \sin(3x) \}$.

$\hat{S}$ is LI iff $\left(\alpha_1 \sin(x) + \alpha_2 \sin(2x) + \alpha_3 \sin(3x) = 0 \right.$
$\left. \rightarrow \forall \alpha_i = 0 \right)$

$$\alpha_1 \sin(x) + \alpha_2 \sin(2x) + \alpha_3 \sin(3x) = 0$$

$$\Rightarrow \alpha_1 \cdot \sin(x) \cdot \sin(nx) + \alpha_2 \cdot \sin(2x) \cdot \sin(nx) + \alpha_3 \sin(3x) \cdot \sin(nx) = 0$$

(Multiply by $\sin(nx)$ where $n \in (1, 2, 3)$)

Integrating to $[0, 2\pi]$.

$$\Rightarrow \int_0^{2\pi} \alpha_1 \cdot \sin(x) \sin(nx) + \dots = 0$$

$$\Rightarrow \alpha_1 \cdot \int_0^{2\pi} \sin(x) \cdot \sin(nx)\, dx + \dots = 0$$

$$\int_0^{2\pi} \sin(Kx)\,\sin(lx)\,dx = 0 \text{ if } K \neq l$$
$$= \pi \text{ if } K = l$$

$\Rightarrow \alpha_1 \times 0 + \alpha_2 \times 0 + \alpha_3 . \pi = 0$ (for $n=3$)

or

$\alpha_1 \times 0 + \alpha_2 \times \pi + \alpha_3 \times 0 = 0$ (for $n=2$)

or

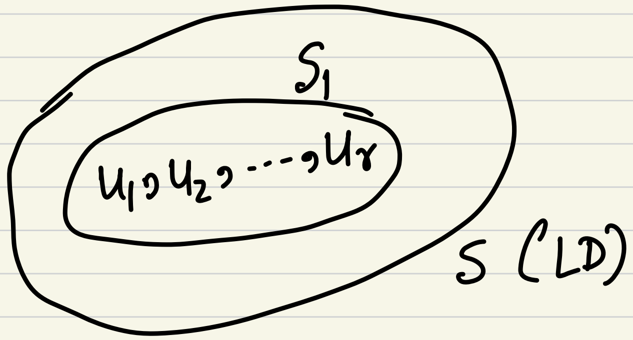
$\alpha_1 \times \pi + \alpha_2 \times 0 + \alpha_3 \times 0 = 0$ (for $n=1$)

Therefore $\alpha_3 = 0$, $\alpha_2 = 0$, $\alpha_1 = 0$ from the 3 equations

Hence we say S is LI.

Property of LD/LI set:

Suppose a set S is a LD set. (infinite)



S is LD $\longrightarrow$ There exists a set $S_1 \subset S$ where $|S_1| < \infty$ such that S_1 is finite and S_1 is LD.

$S_1 = \{u_1, u_2, \ldots, u_r\}$

Since S_1 is linearly dependent iff

$$\sum \alpha_i u_i = 0_V \longrightarrow \exists \alpha_i \neq 0$$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r = 0_V \quad (\exists \alpha_i \neq 0)$$

Find the largest index for which $\alpha_j \neq 0$. (Not all of them is 0 so we go & check largest index).

$$\alpha_{j+1} = 0, \alpha_{j+2} = 0, \ldots, \alpha_r = 0$$

$$\underbrace{(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_j u_j)}_{\forall \alpha_j \neq 0} + \underbrace{(\alpha_{j+1} u_{j+1} + \cdots + \alpha_r u_r)}_{\forall \alpha_j = 0} = 0$$

Remember the fact that $\min j = 1$ and $\max j = r$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_j u_j = 0_V \quad \forall \alpha_j \neq 0$$

That means $\{u_{j+1}, \ldots, u_r\}$ is LI set of vectors. So within the set S_1 (which was LD), there exists an LI set $S_2 = \{u_{j+1}, \ldots, u_r\}$.

We can write –

$$u_j = -\frac{\alpha_1}{\alpha_j} u_1 - \frac{\alpha_2}{\alpha_j} u_2 - \cdots - \frac{\alpha_{j-1}}{\alpha_j}$$

Since $\forall \alpha_j \neq 0$ so α_j^{-1} exists.

$$\boxed{u_j = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_{j-1} u_{j-1}}$$

Therefore, u_j is a linear combination of finite number of vectors in S.

If we start with a LD set S, then there exist at least one vector u_j such that u_j is the linear combination of other vectors in S. Had S been LI set, we would not have been able to express u_j as LC of other vectors in set S. So LI means, we can never express the vectors in S in terms of LC of other vectors. In that sense, the set S is independent of each other.

S is LD $\longrightarrow$ $\exists\, u \in S$ s.t. u is LC of finite number of vectors in S.

Suppose, $S = \{u_1, u_2, \ldots, u_r\}$ is a finite LD set.
implies,

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r = 0_v \quad (\exists \alpha_i \neq 0)$$

Let j be largest index s.t $\alpha_j \neq 0$.
as before,

$$u_j = \left(-\frac{\alpha_1}{\alpha_j}\right) u_1 + \cdots + \left(-\frac{\alpha_{j-1}}{\alpha_j}\right) u_{j-1}$$

So we can express, u_j which is LC of other vectors.
So the span of the LI subset inside LD set so,
$\text{span}(S_1) = \text{span}(S)$ so the LD has lot of redundant vectors (that are LC of other vectors so they don't contribute to the spanning set).

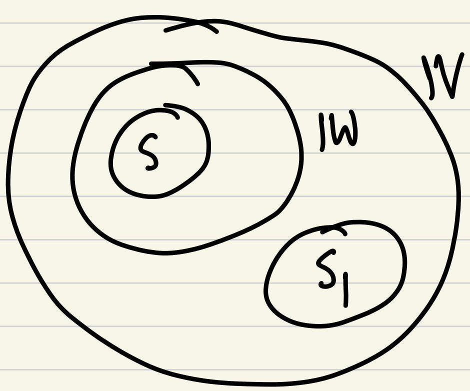
Ex 1: $\mathbb{V}$: Vector space over $\mathbb{F}$

$\mathbb{W}$: Subspace of $\mathbb{V}$.

S : LI set $\subseteq \mathbb{W}$

S_1 : LI set $\subseteq \mathbb{V} \setminus \mathbb{W}$

Claim: $S \cup S_1$ is also LI set in $\mathbb{V}$.



$S = \{u_1, u_2, \ldots, u_n\}$

$S_1 = \{v_1, v_2, \ldots, v_m\}$

proof: Suppose $S \cup S_1$ is a LD set.

$\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ is LD set iff

$$\left(\alpha_1 \cdot u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \beta_1 v_1 + \cdots + \beta_m v_m = 0_V \longrightarrow \exists\, \alpha_i \text{ or } \beta_i \neq 0\right) \quad \text{---(1)}$$

Since S is subset of $\mathbb{W}$ so any linear combination of S must be closed under $\mathbb{W}$.

It is given fact, $\{u_1, u_2, \ldots, u_n\}$ is LI set. Therefore,

$$\gamma_1 u_1 + \gamma_2 u_2 + \ldots + \gamma_n u_n = 0_V \quad (\forall \gamma_i = 0) \quad \text{---} (2)$$

It is given fact, $\{v_1, v_2, \ldots, v_n\}$ is LI set. Therefore

$$\delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_n v_n = 0_V \quad (\forall \delta_i = 0) \quad \text{---} (3)$$

Adding eqn (2) & (3):

$$\gamma_1 u_1 + \gamma_2 u_2 + \ldots + \gamma_n u_n + \delta_1 v_1 + \ldots + \delta_n v_n = 0_V \quad \text{---} (4)$$

Where $\forall \gamma_i = 0$, $\forall \delta_i = 0$

which means $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots v_n\}$ is LI set but it is a contradiction of the given assumption.

Basis:

Suppose, we have a vector space $\mathbb{V}$ over $\mathbb{F}$.
$\mathbb{W}$ is a subspace of $\mathbb{V}$.

A spanning set is like a sampling set means by knowing the vectors inside spanning set, we can generate/span all the vectors of the subspace $\mathbb{W}$.

Spanning set will be a subset of $\mathbb{W}$ that spans $\mathbb{W}$.
If the spanning set is LD then it will have lot of redundant vectors that can be obtained as LC of other vectors.

A linearly dependent generating set is over sampling.
Hence we would like to have LI spanning set which is called basis set.

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $\mathbb{W}$ be a subspace of $\mathbb{V}$. A subset $B \subset \mathbb{W}$ is called a basis set for the subspace $\mathbb{W}$ iff —

(1) B is Linearly independent
(2) $L[B] = \mathbb{W}$. (B spans the entire subspace $\mathbb{W}$)

Ex: $\mathbb{F}^3$ over $\mathbb{F}$

$$\mathbb{F}^3 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; \forall x_i \in \mathbb{F} \right\}$$

consider 3 vectors, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$B = \{e_1, e_2, e_3\}$ — is it a basis for $\mathbb{F}^3$?

(i) Clearly $\mathcal{B}$ is LI set.

(ii) Clearly $L[B] = \mathbb{F}^3$ because -

$$\alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \alpha_3 \cdot e_3 = (\alpha_1, \alpha_2, \alpha_3)^t$$ which is the form of the vector space $\mathbb{F}^3$.

<u>Ex:</u> $\mathbb{V} = \mathbb{F}^{m \times n}$ over $\mathbb{F}$.

Consider $\mathbb{V} = \mathbb{F}^{2 \times 3}$ over $\mathbb{F}$.

$$\mathbb{V} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \middle| \forall a_{ij} \in \mathbb{F} \right\}$$

Consider the set B as -

$$B = \{ A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23} \}. \ \forall A_{ij} \in \mathbb{F}^{2 \times 3} \subseteq \mathbb{V}$$

such that
$$A_{ij} = \begin{cases} a_{kl} = 1 & \text{if } i=k, j=l \\ a_{kl} = 0 & \text{otherwise} \end{cases}$$

$$A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots, A_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is B a basis set of $\mathbb{V}$?

(i) Linear independent check.

Consider $a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + \cdots + a_{23} \cdot A_{23} = O_{2 \times 3}$.

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Implies that $\forall a_{ij} = 0$ that means B is LI set.

(ii) Generating set check.

$$\langle B \rangle := \left\{ a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + \cdots + a_{23} \cdot A_{23} \;\middle|\; \forall a_{ij} \in \mathbb{F} \right\}$$

$$= \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \;\middle|\; \forall a_{ij} \in \mathbb{F} \right\}$$

$$= \mathbb{F}^{2\times 3} \quad (\text{from the def}^{n} \; \mathbb{F}^{2\times 3}).$$

That means B is generating set for $\mathbb{F}^{2\times 3}$.
So B is a basis set for $\mathbb{F}^{2\times 3}$.

Ex: Suppose, $\mathbb{V} = \mathbb{F}[x]$ over $\mathbb{F}$.

$$\mathbb{F}[x] = \left\{ f \;\middle|\; f(x) = a_0 + a_1 x + a_2 x^2 + \cdots, \forall a_i \in \mathbb{F}, f : \mathbb{F} \to \mathbb{F} \right\}$$

Suppose a set $B = \{ p_0, p_1, p_2, \ldots \}$ where $p_i(x) = x^i$

Is B a basis set?

$\Rightarrow$ It was already proved that B will be a linearly independent set.

$$\langle B \rangle = \left\{ \alpha_0 \cdot P_0 + \alpha_1 \cdot P_1 + \cdots \;\middle|\; \forall \alpha_i \in \mathbb{F} \right\}$$

$$= \left\{ \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots \;\middle|\; \forall \alpha_i \in \mathbb{F} \right\}$$

Therefore B indeeds generates the $\mathbb{F}[x]$ space. So B is a basis set for $\mathbb{F}[x]$.

Consider a subspace in $\mathbb{F}[x]$.

$\mathbb{W} = \mathbb{F}_e[x]$ (polynomials with only even powers of x)

$$= \left\{ f \;\middle|\; f(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots, \forall a_i \in \mathbb{F}, f : \mathbb{F} \to \mathbb{F} \right\}$$

Consider a set, $B = \{1, x^2, x^4, \ldots, x^{2i}, \ldots\}$.

Is B a basis for $\mathbb{W}$?

$\Rightarrow$ Yes.

Basis from a different angle:

$\mathbb{V}$ is a vector space over $\mathbb{F}$.

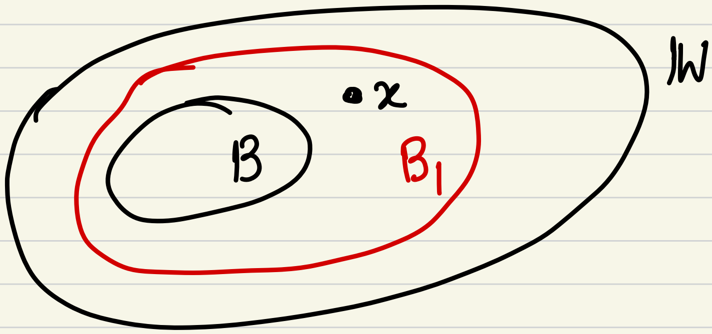
$\mathbb{W}$ is a subspace of $\mathbb{V}$.

S is a subset of $\mathbb{W}$. S is a maximal LI set if -

(i) It must be linearly independent.

(ii) S is not a proper subset of any other LI set.

If S is a proper subset of S_1 which is subset of $\mathbb{W}$ then S_1 is Linearly dependent.



B is a basis for $\mathbb{W}$.

$B = \{u_1, u_2, \dots, u_r\}$

Let $B_1 \subset \mathbb{W}$ such that $B \subset B_1$. That means there exists an $x \in B_1 \setminus B$.

$x \in \mathbb{W}$. and B is a basis for $\mathbb{W}$. So x can be generated by the basis set B. B must span $\mathbb{W}$ and B is LI.

$x = \alpha_1 \cdot u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r \quad \forall \alpha_i \in \mathbb{F}$

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r + (-1)x = 0_v$.

That means there exists a non trivial LC (coeff. $x \neq 0$) which gives rise to 0_v. That means $B \cup \{x\}$ is LD set.

$B \subset B_1$, $x \in B_1$ so $B \cup \{x\} \subset B_1$

Since $B \cup \{x\}$ is LD and $B \cup \{x\} \subset B_1$ therefore B_1 itself is also LD (any superset of LD set is LD). We started with Basis set B which is LI and we constructed another bigger set $B \cup \{x\}$ but that became LD. So in some sense, Basis set B is maximally linearly independent set.

Finite dimensional subspaces:

Let $\mathbb{W}$ be a subspace of $\mathbb{V}$ and $\mathbb{W}$ have a finite basis. Then $\mathbb{W}$ is a f.d.s.s. If $\mathbb{V}$ itself has a finite basis then $\mathbb{V}$ is a f.d.v.s.

Let $\mathbb{V}$ be a f.d.v.s. This means there is a finite basis for $\mathbb{V}$.

$$B = \{u_1, u_2, \ldots, u_n\} \quad , \forall u_i \in \mathbb{V} .$$

B is LI set and $\langle B \rangle = \mathbb{V}$.

Suppose we have another set which has $n+1$ vectors.
$B_1 = \{v_1, v_2, \ldots, v_{n+1}\}$.

Claim: B_1 must be Linearly dependent.
Notice we have mentioned B is basis set but if B is generating set also then also this claim will hold to be true.
$\Rightarrow$ If set B_1 has an element v which can be written as LC of others then B_1 is LD but if set B_1 were to be independent then any subset is also LI. We take first n vectors as any basis set and the last vector on top of

basis set means the set B_1 is LD which is a contradiction.

Ex1: Suppose $\mathbb{W}$ is subspace of $\mathbb{V}$ (fd vs).

$$B = \{u_1, u_2, \ldots, u_d\} \quad \text{a basis for } \mathbb{W}.$$

Consider any subset of $\mathbb{W}$ which has $d+1$ vectors,

$$B_1 = \{v_1, v_2, \ldots, v_{d+1}\}.$$

Claim: B_1 must be linearly dependent.

Ex2: Suppose $\mathbb{W}$ has a basis B consisting d vectors.
Suppose B_1 is another basis for $\mathbb{W}$.

Since B has d vectors and B is a basis for $\mathbb{W}$ so any set B_1 must have at most d vector to be LI. Because if B_1 were to have more than d vectors then it would become LD which will violate basis property.

That means $|B_1| \leq d$

Now since B_1 is also basis so $|B| \leq |B_1|$ or $d \leq |B_1|$

Hence $|B| = |B_1| = d$.

Therefore the cardinality of all the basis set is same.
It is called dimension of the subspace of $\mathbb{W}$.

Ex1: Subspace $\mathbb{W} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{F} \right\}$ of a vector space $\mathbb{F}^3$ over $\mathbb{F}$.

construct basis for $\mathbb{W}$?

Consider a subset of $\mathbb{W}$ that is S. that is LI set.

$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ definitely $S \subseteq \mathbb{W}$ and S is LI set.

Now we consider $\langle S \rangle = \left\{ \alpha . \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{F} \right\}$

$$= \left\{ \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} \;\middle|\; \alpha \in \mathbb{F} \right\}$$

definitely $\langle S \rangle \neq \mathbb{W}$ or $\mathbb{W} \setminus \langle S \rangle$ is nonempty.

Therefore S can't be a basis set.

So we select one vector from $\mathbb{W} \setminus \langle S \rangle$ and take union of it. That will definitely be LI set.

Choose, $x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ which $\in \mathbb{W} \setminus \langle S \rangle$

Now consider $S' = S \cup \{x\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Is it LI set?

$\Rightarrow$ Yes S' is LI set.

$$c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_1 = 0, \; c_2 = 0.$$

Is it spanning set of $\mathbb{W}$?

$\Rightarrow$ Yes.

$$\langle S' \rangle = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_1 + c_2 \end{pmatrix} \middle| \; c_1, c_2 \in \mathbb{F} \right\} := \mathbb{W}.$$

Hence S' is LI and spanning set so S' is a basis set.

So the basis of $\mathbb{W}$ is $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$\dim(\mathbb{W}) = |\mathcal{B}| = 2.$

Ex2: $\mathbb{F}^{2 \times 3}$.

$$\{A_{ij}\}_{\substack{0 \le i \le 2 \\ 0 \le j \le 3}} = B \quad (\text{Basis for } \mathbb{F}^{2 \times 3})$$

$A_{ij} \in \mathbb{F}^{2 \times 3}$. and it has all entries except (i,j) entry as 0 and (i,j) entry is 1. There are 6 vectors in B.

$$\dim(\mathbb{F}^{2 \times 3}) = |B| = 6$$

Ex3: $\mathbb{F}[x]$.

This vector space is not finite dimensional.
Suppose $F[x]$ is fdvs. Then there must be a finite basis consisting of n vectors. Any set contains $n+1$ vectors must be LD. However it is contradiction, since adding just one extra polynomial will make it LI but $n+1$ can't be LI so contradiction.

Hence $\mathbb{F}[x]$ must be an infinite dimensional vector space
Consider, $\mathbb{W} = \mathbb{F}[x]_2$ which means $\mathbb{W}$ is the set of all polynomials whose degree is less than or equal 2.

$$\mathbb{W} = \mathbb{F}[x]_2 = \left\{ f \;\middle|\; f(x) = a_0 + a_1x^1 + a_2x^2, \ \forall a_i \in \mathbb{F}, \ f: \mathbb{F} \to \mathbb{F} \right\}$$

Then what is the dimension of $\mathbb{W}$?
$\Rightarrow$ consider a LI set of polynomials –

$S = \{P_1\}$ where $P_1(x) = 1 \ \forall x \in \mathbb{F}$

Definitely S is LI set
$\text{span}(S) = \{ c \cdot 1 \mid c \in \mathbb{F} \} \neq \mathbb{W}$.

Choose a vector $u \in \mathbb{W} \setminus \langle S \rangle$, let $u = P_2$.

where $P_2(x) = x \quad \forall x \in \mathbb{F}$.

Now consider the LI set, $S' = \{P_1, P_2\}$.

$$\text{Span}(S') = \{c_1 1 + c_2 \cdot x \mid c_1, c_2 \in \mathbb{F}\} \neq \mathbb{W}.$$

Choose another vector $u \in \mathbb{W} \setminus \langle S' \rangle$, let $u = P_3$ s.t. $P_3(x) = x^2 \quad \forall x \in \mathbb{F}$.

Now consider the LI set $S'' = \{P_1, P_2, P_3\}$

$$\text{span}(S'') = \{c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 \mid \forall c_i \in \mathbb{F}\} = \mathbb{W}.$$

Therefore $\mathbb{W} \setminus \langle S'' \rangle = \phi$ hence S'' is a basis for $\mathbb{W}$.

$B = \{1, x, x^2\}$ is a basis for $\mathbb{F}[x]_2$.

$$\dim(\mathbb{F}[x]_2) = |B| = d+1 = 2+1 = \underline{3}$$

Ex3: Suppose, $S = \{s_1, s_2, \ldots, s_k\}$

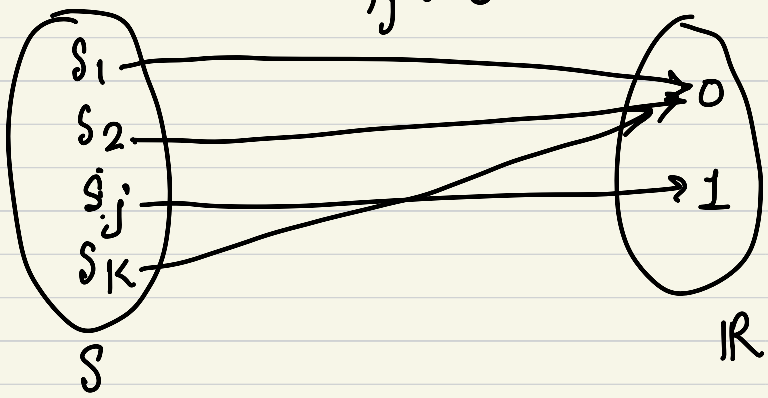
$$\mathcal{F}[S, \mathbb{R}] = \{f \mid f : S \to \mathbb{R}\}.$$

consider the functions as:

$$f_j(t) := 0 \quad \text{if } t \neq s_j$$
$$:= 1 \quad \text{if } t = s_j$$

So $f_j(t)$ mapped to 1 only if $t = s_j$ and for all other value it will be 0.

$f_j : S \to \mathbb{R}$



f_j is acts like an indicator function.

$\{f_j\}_{j=1}^{K} \in \mathcal{F}[S, \mathbb{R}]$.

Verify that $\{f_j\}_{j=1}^{K}$ is a LI set and also a spanning set for the space $\mathcal{F}[S, \mathbb{R}]$. So it acts like a basis for $\mathcal{F}[S, \mathbb{R}]$.

$$\dim(\mathcal{F}[S, \mathbb{R}]) = \left|\{f_j\}_{j=1}^{K}\right| = K.$$

$$B = \{f_1, f_2, \dots, f_K\}.$$

Consider. $c_1 f_1(x) + c_2 f_2(x) + \dots + c_K f_K(x) = 0$

$f_1(x) = 1$ iff $x = s_1$

therefore if $x = s_1$ then $f_1(x) = 1$ and $f_i(x) = 0$ $\forall i = 2, 3, \dots, K$ therefore, $c_1 \cdot 1 = 0 \Rightarrow c_1 = 0$

similarly if $x = s_2$ then $f_2(x) = 1$ and $f_i(x) = 0$ $\forall i = 1, 3, 4, \dots, K$ therefore $c_2 \cdot 1 = 0 \Rightarrow c_2 = 0$

Likewise, $\forall c_i = 0$ which means B is LI set.

$$\langle B \rangle = \left\{ c_1 f_1 + c_2 f_2 + \cdots + c_K f_K \;\middle|\; \forall c_i \in \mathbb{F} \right\}$$

$$= \left\{ c_1 f_1(x) + c_2 f_2(x) + \cdots + c_K f_K(x) \right\}$$

$$\mathcal{F}[S, \mathbb{R}] = \left\{ f \mid f : S \to \mathbb{R} \right\}.$$

$$\left.\begin{array}{lcl} s_1 & \xrightarrow{f} & f(s_1) = r_1 \\ s_2 & \xrightarrow{f} & f(s_2) = r_2 \\ \vdots & \vdots & \\ s_K & \xrightarrow{f} & f(s_K) = r_K \end{array}\right\} \quad \forall r_i \in \mathbb{R}$$

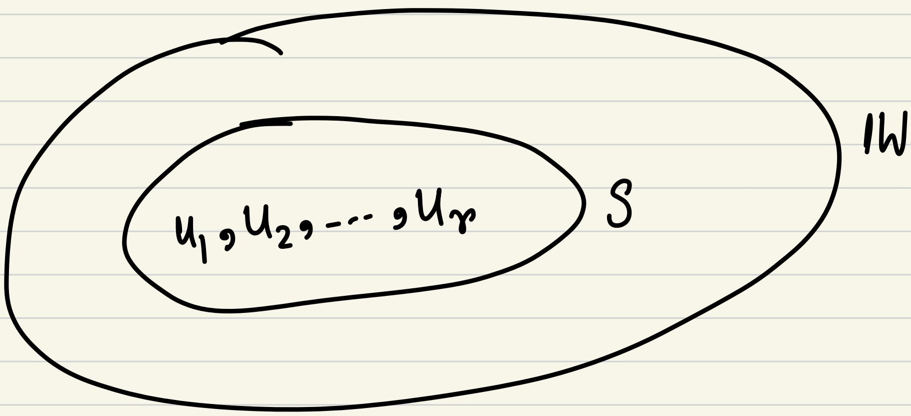
(Definition of function f is given by this mappings)

Therefore, $\mathcal{F}[S, \mathbb{R}] = \left\{ f(x) \mid f : S \to R,\ x \in S \right\}$

Notice $\langle B \rangle$ can replicate each of the possible domain values that the function f takes. If $x = s_1$ then $f(s_1) = r_1 \in \mathbb{R}$ and $\langle B \rangle$ gives $c_1 \cdot f_1(s_1) = c_1 \cdot 1 = c_1$ where $c_1 \in \mathbb{R}$ which can be r_1.

Likewise we can show that all other function values that it takes from S can be same as in $\langle B \rangle$. Hence $\langle B \rangle$ spans the whole vector space. Hence B is a basis for the vector space.

Construction of Basis Set:



$\dim(\mathbb{W}) = d$

$S = \{u_1, u_2, \dots, u_r\}$ is LI.

Any $d+1$ vectors must be LD. Therefore $r \leq d$.

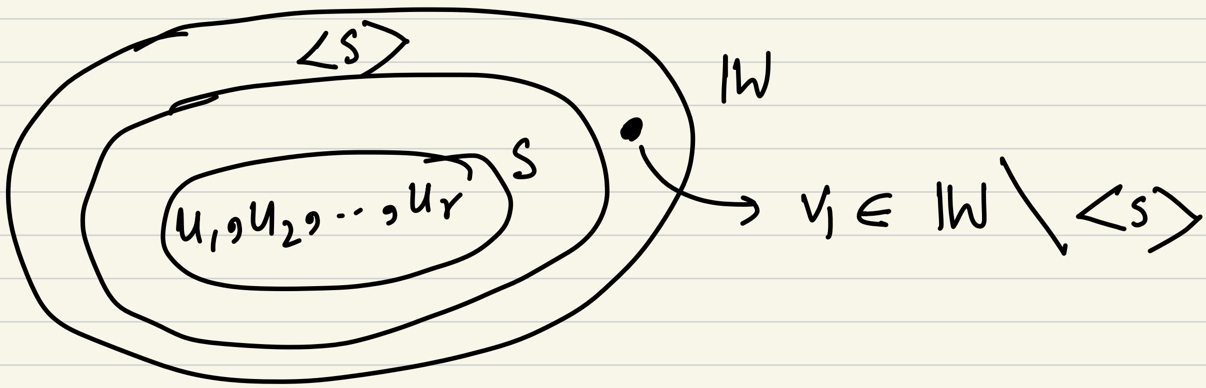
Case-1: $r = d$

Then $S = \{u_1, u_2, \dots, u_d\}$ is LI set and S is also a generating set of $\mathbb{W}$ so S is also a basis set of $\mathbb{W}$.

Case-2: $r < d$

Obviously S can't be a basis because any basis must be d vectors in it and S has r vectors in it. So S is not generating set and hence S can't be basis for $\mathbb{W}$.

Since S is already LI set so the only way S fails to be a basis of $\mathbb{W}$ by $\langle S \rangle$ not being equal to $\mathbb{W}$.



Whenever we have a subspace $\langle S \rangle$ and inside that subspace we take a LI set S and outside the subspace we take another LI set $\{v_1\}$. So $S \cup \{v_1\}$ must be LI set.

Hence $\{u_1, u_2, \dots, u_r, v_1\}$ is LI set of vectors in $\mathbb{W}$.

Now if $r+1 = d$ then we have d LI vectors in $\mathbb{W}$ so it will be basis since it spans $\mathbb{W}$.

If $r+1 < d$ then we again repeat the process and we will get other set of vectors. $S \cup \{v_1\} \cup \{v_2\}$ is LI. If we expand the set for finite times (fdvs). ($d-r$ times) then we get -

$S \cup \{v_1\} \cup \{v_2\} \cup \dots \cup \{v_{d-r}\}$ such that —

$\{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_{d-r}\}$ is LI vectors and d LI vectors spans $\mathbb{W}$ so it is a basis for $\mathbb{W}$.

Linear dependence / independence:

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. A subset $S \subseteq \mathbb{V}$ is said to be <u>linearly dependent</u> if –

$v_1, v_2, \ldots, v_n \in S$ and $c_1, c_2, \ldots, c_n \in \mathbb{F}$ s.t.

$$c_1 v_1 +_{\mathbb{V}} c_2 v_2 +_{\mathbb{V}} c_3 v_3 +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n v_n = 0_{\mathbb{V}} .$$

where not all c_i are 0.

Trivial combination: Definitely when all $c_i = 0$ then the equation always holds true. We are looking for a nontrivial combination s.t. not all $c_i = 0$ yet the equation leads to zero vector.

If the set S contains finitely many vectors $v_1, \ldots, v_n$, we sometimes say that $v_1, v_2, \ldots, v_n$ are dependent or S is dependent set.

A subset $S = \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{V}$ is called <u>Linearly Independent</u> set if -

$\forall v_i \in S$ and $\forall c_i \in \mathbb{F}$ s.t. $\displaystyle\sum_{i=1}^{n} c_i v_i = 0_{\mathbb{V}}$ for only the trivial combination where $\forall c_i = 0$. And no other $c_i \neq 0$ can satisfy this equation.

① Any set which contains a linearly dependent set is linearly dependent.

Suppose, $S = \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{V}$. (subset)
Consider S is linearly dependent set.

Choose, $S' = S \cup \{v_{n+1}, v_{n+2}, \ldots, v_{n+K}\} \subseteq \mathbb{V}$

Definitely $S \subseteq S'$. And since S is LD set so the claim is S' must also be LD set.

Since S is LD therefore, $c_1 v_1 +_{\mathbb{V}} c_2 v_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n v_n = 0_{\mathbb{V}}$
where $\exists c_i \neq 0$.

Now construct:

$$c_1 v_1 +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n v_n +_{\mathbb{V}} c_{n+1}' v_{n+1} +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_{n+K}' v_{n+K} = E$$

We have to show that $\exists c_i \neq 0$ such that $E = 0_{\mathbb{V}}$.

We already knew $\exists c_i \neq 0$ for $i = 1, 2, \ldots, n$ therefore we can choose $c_{n+1}, c_{n+2}, \ldots, c_{n+K}$ all as 0 so the equation E will be $0_{\mathbb{V}}$ hence S' is a LD set.

② Any subset of linearly independent set is again a linearly independent set.

$S = \{v_1, v_2, \cdots, v_n\} \subseteq \mathbb{V}$ (subset)

S is LI set so,

$$c_1 v_1 +_{\mathbb{V}} c_2 v_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n v_n = 0_{\mathbb{V}} \quad \text{where } \forall c_i = 0$$

construct $S' = \{v_1, v_2, \ldots, v_K\} \subseteq S$ $(K < n)$

Claim is S' is also LI set.

Assume S' is LD set.

$$d_1 v_1 +_{\mathbb{V}} d_2 v_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} d_K v_K = 0_{\mathbb{V}} \quad \text{where } \exists\, d_i \neq 0$$

$$(c_1 - d_1) v_1 +_{\mathbb{V}} (c_2 - d_2) v_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} (c_K - d_K) v_K +_{\mathbb{V}}$$

$$c_{K+1} v_{K+1} +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n v_n = 0_{\mathbb{V}}$$

We know that $\{v_i\}_{i=1}^{n}$ is LI set hence all coeff. must be 0

$$\left.\begin{array}{ll} c_1 - d_1 = 0 & \Rightarrow d_1 = c_1 = 0 \\ c_2 - d_2 = 0 & \Rightarrow d_2 = c_2 = 0 \\ \vdots & \vdots \\ c_K - d_K = 0 & \Rightarrow d_K = c_K = 0 \end{array}\right\}$$

$\forall d_K = 0$ which is a contradiction since we assumed $\exists d_K \neq 0$ hence it is proved that S' must be LI set.

$c_{K+1} = 0$

$\vdots$

$c_n = 0$

③ Any set that contains the $0_{\mathbb{V}}$ is linearly dependent

$\Rightarrow$ Suppose, $S = \{v_1, v_2, \ldots, v_n, 0_{\mathbb{V}}\} \subseteq \mathbb{V}$. (subset)

Assume S is LI set hence.

$c_1 v_1 +_{\mathbb{V}} c_2 v_2 +_{\mathbb{V}} \cdots +_{\mathbb{V}} c_n v_n +_{\mathbb{V}} c \cdot 0_{\mathbb{V}} = 0_{\mathbb{V}}$ where

$\forall c_i = 0$ and $c = 0$ and there is no other c_i and c except 0 that satisfies the equation.

But consider $c \neq 0$ still $\sum_{i=1}^{n} c_i v_i +_{\mathbb{V}} c \cdot 0_{\mathbb{V}} = 0_{\mathbb{V}}$

therefore a non trivial combination leads to $0_{\mathbb{V}}$ implies S is LD set which is a contradiction. of what we assumed.

④ A set of vectors S is Linearly independent iff each finite subset of S is linearly independent.

(similar to point 2).

Basis :

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. A basis set for the vector space $\mathbb{V}$ must be linearly independent set of vectors in $\mathbb{V}$ and it must span the space $\mathbb{V}$.

Generating set: A set $S \subseteq \mathbb{V}$ is called a generating set of a vector space $\mathbb{V}$ if the set S can span the space $\mathbb{V}$ or $\langle S \rangle = \mathbb{V}$.

In order to be a basis, the set $B \subseteq \mathbb{V}$ must be a generating set for $\mathbb{V}$ and it must be a linearly Independent set.

$$S := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Consider, $\mathbb{V} = \mathbb{R}^3$

Set S is definitely LI set but it is not a generating set because $\langle S \rangle$ doesn't contain the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ so it is not a generating set. Hence it is not a basis set for $\mathbb{R}^3$.

$$S := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Consider $\mathbb{V} = \mathbb{R}^3$.

Set S is definitely a generating set because $\langle S \rangle$ will span the entire vector space $\mathbb{V}$ but the set S is not linearly independent set.

Therefore, basis set of $\mathbb{V}$ is "just right". It is not too many or not too few.

Theorem: Let V be a vector space which is spanned by a finite set of vectors $\{v_1, v_2, \ldots, v_m\}$. Then any independent set of vectors in $\mathbb{V}$ is finite and contains no more than m elements.

$\Rightarrow$ Suppose S is set of vectors s.t. $S \subseteq \mathbb{V}$ (arbitrary)
Any subset S of $\mathbb{V}$ which contains more than m vectors must be linearly dependent set.

We also know that $\tilde{S} = \{v_1, v_2, \ldots, v_m\}$ spans $\mathbb{V}$.
Since $\tilde{S}$ spans $\mathbb{V}$ so any vector v in $\mathbb{V}$ can be written as LC of vectors in $\tilde{S}$.

To be proved

Subset $S \subseteq \mathbb{V}$ is a linearly dependent set with n vectors such that $n > m$. $S = \{u_1, u_2, \ldots, u_n\}$

$$\boxed{\begin{aligned} & c_1 u_1 +_{\mathbb{V}} c_2 u_2 +_{\mathbb{V}} \cdots + c_n u_n = 0_{\mathbb{V}} \;;\; \underline{\exists\, c_j \neq 0} \\ & = \sum_{j=1}^{n} c_j u_j = 0 \quad \text{where } \exists\, c_j \neq 0 \end{aligned}}$$

Since $u_j \in \mathbb{V}$ so u_j can be written as LC of vectors in $\tilde{S}$

$$u_j = \sum_{i=1}^{m} A_{ij} v_i$$

Substituting it -

$$\sum_{j=1}^{n} c_j u_j = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} A_{ij} v_i \right) = 0$$

$$\begin{aligned}
= \; & c_1 \left(A_{11} v_1 +_v A_{21} v_2 +_v \cdots +_v A_{m1} v_m \right) +_v \\
& c_2 \left(A_{12} v_1 +_v A_{22} v_2 +_v \cdots +_v A_{m2} v_m \right) +_v \\
& \vdots \\
& c_n \left(A_{1n} v_1 +_v A_{2n} v_2 +_v \cdots +_v A_{mn} v_m \right) = 0
\end{aligned}$$

$$\begin{aligned}
= \; & \left(c_1 A_{11} + c_2 A_{12} + \cdots + c_n A_{1n} \right) v_1 + \\
& \left(c_1 A_{21} + c_2 A_{22} + \cdots + c_n A_{2n} \right) v_2 + \\
& \vdots \\
& \left(c_1 A_{m1} + c_2 A_{m2} + \cdots + c_n A_{mn} \right) v_m = 0
\end{aligned}$$

Since $\{v_i\}_{i=1}^{m}$ is a generating set, we don't know about the dependency / independency but we have to prove that the equation is true for some $\gamma_k \neq 0$ where $\gamma_k = c_1 A_{k1} + c_2 A_{k2} + \cdots + c_n A_{kn}$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ & \vdots & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

$\{v_i\}_{i=1}^{m}$ is generating set. We are not assuming the independence of the set $\{v_i\}_{i=1}^{m}$.

If we can show that $\exists c_j \neq 0$ for which $\forall \gamma_i = 0$ then we can say –

$$0.v_1 + 0.v_2 + \cdots + 0.v_n = 0_W \quad , \forall \gamma_i = 0$$

equivalent to $\displaystyle\sum_{j=1}^{n} c_j u_j = 0_W \quad \exists c_j \neq 0$

Therefore, for some $c_j \neq 0$ we have to show that all γ_i are 0 then the u_j are linearly dependent set.

if $$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ & \vdots & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ $\forall c_i \neq 0$ then

$$\sum_{j=1}^{n} c_j u_j = 0_{\mathbb{V}} \quad \left(\text{means } \{u_j\}_{j=1}^{n} \text{ will be LD set}\right)$$

$$\begin{bmatrix} & A & & \end{bmatrix}_{m \times n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

because $m < n$.

A can have at most rank of m. So it must have at least $(n-k)$ free variables. So there does exist a non-zero solution $[c_1 \; c_2 \; \cdots \; c_n]^t$ s.t $A \cdot c = 0$.

So $\exists c_i \neq 0$ s.t $A_{ij} \cdot c_i = \gamma_j = 0 \; (\forall \gamma_j)$. Therefore,

$\sum_{i=1}^{n} c_i u_i = 0$ but $\exists c_i \neq 0$ hence $\{u_i\}_{i=1}^{n}$ are LD.

> Anything that contains more than the generating set of $\mathbb{V}$ can't be linearly independent. Even though, we did not assume the LI of the generating set.

Finite dimensional vector space (fdvs): A vector space $\mathbb{V}$ is called fdvs if $|B_V| < \infty$, B_V: Basis set of $\mathbb{V}$.

Theorem: If $\mathbb{V}$ is a fdvs then any 2 bases of $\mathbb{V}$ have the same (finite) number of elements.

$\Rightarrow$ Since $\mathbb{V}$ is a fdvs so it has finite number of vectors in the basis set.

Consider $B_1 = \{v_1, v_2, \dots, v_m\}$ is a basis for $\mathbb{V}$.

Since B_1 is a basis so B_1 is a generating set of $\mathbb{V}$. That means any LI set $S \subseteq \mathbb{V}$ can't have more than m vectors in it.

Therefore consider another basis set,

$B_2 = \{u_1, u_2, \dots, u_n\}$ for $\mathbb{V}$.

Because B_2 is a LI set hence $|B_2|$ can't be more than m. So $n \leq m$.

Similarly, since B_2 is also a generating set so we know any LI set $S \subseteq \mathbb{V}$ can't have more than n elements.

Therefore, B_1 being a LI set, $|B_1|$ can't be more than n. So $m \leq n$.

When $m \leq n$ & $n \leq m$ so $m = n$ so $\underline{|B_1| = |B_2|}$

Dimension of vector space $\mathbb{V}$:

Since we have seen that for a vector space $\mathbb{V}$, there can be many different basis set. But what is common in all those basis set is that their cardinality are all same.

Dimension of a vector space $\mathbb{V}$ is defined as the number of vectors in a basis set of $\mathbb{V}$. Here $\mathbb{V}$ must be finite dimensional vector space. It is denoted by $\dim(\mathbb{V})$.

Theorem: let $\mathbb{V}$ be a f.d.v.s. let $n = \dim(\mathbb{V})$.

(i) Any subset of $\mathbb{V}$ which contains more than n vectors is linearly dependent.

(ii) No subset of $\mathbb{V}$ which contains fewer than n vectors can span $\mathbb{V}$.

Consider $\mathbb{V}$ be a vector space, $S = \{0_{\mathbb{V}}\} \subseteq \mathbb{V}$ but S is linearly dependent set therefore $\{0_{\mathbb{V}}\}$ can't be the basis for subspace S. Hence $\dim(S) = 0$ so there is no vectors in the basis of $S = \{0_{\mathbb{V}}\}$.

<u>Theorem</u>: Let S be LI subset of vector space $\mathbb{V}$. Suppose $u \in \mathbb{V}$ which is not in the subspace spanned by S. Then the set $S \cup \{u\}$ is Linearly Independent in $\mathbb{V}$.

$\Rightarrow S = \{v_1, v_2, \dots, v_K\} \subseteq \mathbb{V}$

$$\langle S \rangle = \{ c_1 v_1 +_v c_2 v_2 +_v \cdots +_v c_K v_K \mid \forall c_K \in \mathbb{F}, \forall v_K \in S \}$$

It is given that, $c_1 v_1 +_v c_2 v_2 +_v \cdots + c_K v_K \neq u$

We also know that, $d_1 v_1 +_v d_2 v_2 +_v \cdots +_v d_K v_K = 0_V$

where $\forall d_K = 0$ because S is LI set.

We have to prove that – $\tilde{S} = S \cup \{u\}$ is LI set

Suppose, $\tilde{S}$ is linearly dependent set. Hence –

$$\alpha_1 v_1 +_v \alpha_2 v_2 +_v \cdots +_v \alpha_K v_K +_v \alpha_{K+1} u = 0_{\mathbb{V}}$$

where $\exists \alpha_i \neq 0$

Here $\alpha_{K+1} \neq 0$. If $\alpha_{K+1} = 0$ then $\{v_i\}_{i=1}^{K}$ becomes linearly dependent since

$$\sum_{i=1}^{K} \alpha_i v_i = 0_{\mathbb{V}} \quad \text{s.t. } \exists \alpha_i \neq 0$$

which is the violation of the given fact that S is LI set.
Therefore $\alpha_{k+1} \neq 0$.

Hence,
$$u = \sum_{i=1}^{k} \left(-\frac{\alpha_i}{\alpha_{k+1}}\right) \cdot v_i$$
$$= \sum_{i=1}^{k} \alpha_i' v_i$$

Therefore $u \in \langle \{v_i\}_{i=1}^{k} \rangle$ or $u \in \langle S \rangle$ but it is a contradiction of the given fact that $u \notin \langle S \rangle$.
Hence the assumption was incorrect.

__Theorem:__ If $\mathbb{W}$ is a proper subspace of fdvs $\mathbb{V}$. then $\mathbb{W}$ is fdvs and $\dim(\mathbb{W}) < \dim(\mathbb{V})$.

$\mathbb{W}$ is a subspace of $\mathbb{V}$ so $\mathbb{W} \subset \mathbb{V}$.

Consider the basis for $\mathbb{W}$ and $\mathbb{V}$.

$B_W = \{w_1, w_2, \dots, w_m\}$ (let $\dim(\mathbb{W}) = m$)
$B_V = \{v_1, v_2, \dots, v_n\}$ (let $\dim(\mathbb{V}) = n$)

Since B_W is a basis set of $\mathbb{W}$ so B_W is also a generating set for $\mathbb{W}$. that means any other subset of $\mathbb{W}$ containing more than m elements must be linearly dependent.

Choose a vector $u \in \mathbb{V}$ but $u \notin \mathbb{W}$.
Now since $u \notin \mathbb{W}$ (subspace of $\mathbb{V}$) and B_W is a LI set therefore, $B_W \cup \{u\}$ must be LI set in $\mathbb{V}$.

Since $\dim(\mathbb{W}) = |B_W|$ therefore,

$$|B_W \cup \{u\}| > \dim(W)$$

But $B_W \cup \{u\}$ is a LI set in $\mathbb{V}$.

Now B_V is a basis for $\mathbb{V}$ so we can say that -

$$|B_W \cup \{u\}| \leq |B_V|$$ because maybe only $B_W \cup \{u\}$ spans $\mathbb{V}$.

$$\Rightarrow |B_W \cup \{u\}| \leq \dim(V)$$

$$\Rightarrow \dim(V) \geq |B_W \cup \{u\}| > \dim(W)$$

$$\Rightarrow \dim(V) > \dim(W).$$

<u>Theorem:</u> If $\mathbb{W}_1$ and $\mathbb{W}_2$ are fdvs subspace of $\mathbb{V}$. Then,

$$\dim(\mathbb{W}_1 + \mathbb{W}_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$\Rightarrow$ $W_1 \subseteq \mathbb{V}$ and $W_2 \subseteq \mathbb{V}$.

Suppose $\mathbb{W}_1 \cap \mathbb{W}_2$ has a basis set as $\{v_1, v_2, \ldots, v_K\}$

Thererfore, $\dim(\mathbb{W}_1 \cap \mathbb{W}_2) = K$

Suppose, $\dim(\mathbb{V}) = n$ therefore, $K \leq n$.

(W_1 & W_2 can at most touch the vector space $\mathbb{V}$).

We will extend the same basis to generali the basis for $\mathbb{W}_1$ and $\mathbb{W}_2$.

$B_{W_1} = \{v_1, v_2, \ldots, v_K, u_1, u_2, \ldots, u_m\}$ is a basis for $\mathbb{W}_1$

$B_{W_2} = \{v_1, v_2, \ldots, v_K, w_1, w_2, \ldots, w_p\}$ is a basis for $\mathbb{W}_2$

$\dim(\mathbb{W}_1) = m + K$

$\dim(\mathbb{W}_2) = p + K$

$$\mathbb{W}_1 + \mathbb{W}_2 := \{ w = w_1 + w_2 \mid w_1 \in \mathbb{W}_1, w_2 \in \mathbb{W}_2 \}$$

consider,

$$S = \{v_1, v_2, \ldots, v_K, u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_p\}$$
$$= B_{W_1} \cup B_{W_2}$$

prove that S is a basis set of $\mathbb{W}_1 + \mathbb{W}_2$.

How the idea came?

$$\dim(W_1+W_2) = \underbrace{\dim(W_1)}_{m+K} + \underbrace{\dim(W_2)}_{P+K} - \underbrace{\dim(W_1\cap W_2)}_{K}$$

$$= m+P+K.$$

Therefore if we can show $B_{W_1} \cup B_{W_2}$ is a basis for W_1+W_2 then we are done.

Choose any $w \in W_1+W_2$.

$w = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$

$$w = \left(\sum_{i=1}^{K} d_i v_i + \sum_{i=1}^{m} \beta_i u_i\right) + \left(\sum_{i=1}^{K} \gamma_i v_i + \sum_{i=1}^{P} \delta_i w_i\right)$$

$$= \sum_{i=1}^{K} (d_i+\gamma_i) v_i + \sum_{i=1}^{m} \beta_i u_i + \sum_{i=1}^{P} \delta_i w_i$$

$$w \in \left\langle \{v_i\}_{i=1}^{K} \cup \{u_i\}_{i=1}^{m} \cup \{w_i\}_{i=1}^{P} \right\rangle$$

or $w \in \langle B_{W_1} \cup B_{W_2} \rangle$

Since w was arbitrary so $\forall w \in \langle B_{W_1} \cup B_{W_2} \rangle$ hence $B_{W_1} \cup B_{W_2}$ spans the entire W_1+W_2 so it's a gen. set.

Now we have to prove that S is LI set.

if $$\sum_{i=1}^{k} a_i v_i + \sum_{j=1}^{m} b_i u_i + \sum_{i=1}^{p} c_i w_i = 0_{\mathbb{V}}.$$

then $a_i = 0,\ b_i = 0,\ c_i = 0 \ \forall i$

$$\Rightarrow \underbrace{\sum a_i v_i + \sum b_i u_i}_{\in \mathbb{W}_1} = \underbrace{-\sum c_i w_i}_{\in \mathbb{W}_2} = p$$

Therefore $p \in \mathbb{W}_1 \cap \mathbb{W}_2$

$$p = \sum_{i=1}^{k} d_i v_i \qquad (\text{in terms of basis of } \mathbb{W}_1 \cap \mathbb{W}_2)$$

$$\Rightarrow -\sum_{i=1}^{p} c_i w_i = \sum_{i=1}^{k} d_i v_i$$

$$\Rightarrow \sum_{i=1}^{p} c_i w_i + \sum_{i=1}^{k} d_i v_i = 0_{\mathbb{V}} \qquad \left(\text{Bases for } \mathbb{W}_2 \text{ is } \{w_i\}_{i=1}^{p} \cup \{v_i\}_{i=1}^{k}\right)$$

Therefore, $c_i = 0$, $d_i = 0$ since basis is LI set.

$$\Rightarrow \sum_{i=1}^{k} a_i v_i + \sum_{j=1}^{m} b_i u_i = 0_{\mathbb{V}} \qquad \left(\text{Bases for } \mathbb{W}_1 \text{ is } \{u_i\}_{i=1}^{m} \cup \{v_i\}_{i=1}^{k}\right)$$

Therefore $a_i = 0$, $b_i = 0$ since basis is LI set.
Therefore S is LI set.

Ordered Basis:

V is a vector space of dim n over $\mathbb{F}$.
When we say a basis we mean a set of vectors. In a set of vectors, the order doesn't matter. The order in which we list a set of vectors doesn't matter.

A basis for V in which the vectors are arranged in a fixed order is called order basis.

Ex: $\mathbb{F}^3$ over $\mathbb{F}$.

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$ is a basis for $\mathbb{F}^3$.

$$= \{ e_1, e_2, e_3 \}$$

Consider the set $B_1 = \{ e_3, e_2, e_1 \}$ is equal as B. These are same basis. But in ordered basis B and B_1 are different.

Suppose V is a f.d.v.s with $\dim(V) = n$
Suppose, $B = (u_1, u_2, \dots, u_n)$ be an ordered basis for V.

First of all B is a basis so it is LI set and $\langle B \rangle = \mathbb{V}$.
If any vector $x \in \mathbb{V}$ then $x \in \langle B \rangle$.
If $x \in \langle B \rangle$ then x is a LC of vectors of B.

$$x = x_1 \cdot u_1 + x_2 \cdot u_2 + \cdots + x_n \cdot u_n \quad (\forall x_i \in \mathbb{F})$$

Now we shall use the fact that B is LI to prove that the representation of x as LC of B vectors is unique.

Let x have another representation –

$$x = x_1' u_1 + x_2' u_2 + \cdots + x_n' u_n \quad (\forall x_i' \in \mathbb{F} \neq x_i)$$

Since we are working in ordered basis, the order of the basis vectors are fixed.

Subtract the 2 equations :

$$(x_1 - x_1') u_1 + (x_2 - x_2') u_2 + \cdots + (x_n - x_n') u_n = 0_V$$

Since the B set is LI therefore the only way to get 0 vector as LC of B set is to have all coeff. as 0.

$$\Rightarrow x_1 = x_1', \; x_2 = x_2', \; \ldots, \; x_n = x_n'$$

But it's a contradiction of the fact that we have assumed $x_1 \neq x_1'$, $x_2' \neq x_2$, ..., $x_n' \neq x_n$ so the representation of a vector x in a basis is unique.

If $B = (u_1, u_2, .., u_n)$ an ordered basis for $\mathbb{V}$ then every x in $\mathbb{V}$ has a unique representation as a LC of set of vectors in $\mathbb{B}$.

Thus we can think of x as being made of the scalars $x_1, x_2, ..., x_n$ in $\mathbb{F}$ through the ordered basis B. The moment we know the n scalars and the basis B, we can always reconstruct the vector x.

We call "x_i" as the "i^{th} coordinate" of the vector x. With respect to the order basis B.

Every vector $v \in \mathbb{V}$ can be converted to set of n scalars in $\mathbb{F}$ given that a ordered basis B is set up. Starting from x, using B we now get n scalars which is writen as n tuples $(x_1, x_2, .., x_n)^t$ that belongs to $\mathbb{F}^n$.

Starting from $\mathbb{V}$ of $\dim(\mathbb{V}) = n$

$\downarrow$

construct the ordered basis B

$\downarrow$

Collect the n scalars as $(x_1, x_2, \ldots, x_n)^t \in \mathbb{F}^n$

Therefore,

$$[x]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n$$

$\mathbb{V}$ over $\mathbb{F}$	$\mathbb{F}^n$ over $\mathbb{F}$
(B - ordered basis)	

$$x \longmapsto [x]_B$$

(a vector x in $\mathbb{V}$ is mapped to $[x]_B$ or tuple of n scalars representing the coordinates)

We encode x in $\mathbb{V}$ as $[x]_B$ in $\mathbb{F}^n$. Any abstract vector space with the dimension n, by choosing a arbitrary ordered basis B, any vectors of $\mathbb{V}$ can be translated/encoded as $[x]_B$ in $\mathbb{F}^n$. $[x]_B$ is also called n-tuple coordinates of x with respect to ordered basis B.

Example: $\mathbb{V} = \mathbb{F}^3$.

Take, $B = (e_1, e_2, e_3)$ (standard basis)

Take any vector $x \in \mathbb{V}$.

$$x \in \mathbb{V} \longrightarrow x = x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x]_B .$$

Suppose we take the basis $\hat{B}$ where we change the order,

$$\hat{B} = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$x \in \mathbb{V} \longrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The same vector x when written in B we get the $[x]_B = (x_1 \ x_2 \ x_3)^t$. Suppose now we want to represent the same vector $x = (x_1 \ x_2 \ x_3)^t$ in $\hat{B}$.

$$x = x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

But notice one thing - although the vector representation is same but the coordinate in $\hat{B}$ will be - $(x_2, x_3, x_1)^t$.

$$[x]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad [x]_{\hat{B}} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

Let's choose another basis $B_1 = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$

$x \in \mathbb{V} \longrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (in this form)

$$x = \left(\frac{x_1 + x_2 + x_3}{2} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{x_1 - x_2 - x_3}{2} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{-x_1 + x_2 - x_3}{2} \right) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note the vector x is same but, $[x]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$$[x]_{B_1} = \begin{bmatrix} (x_1+x_2+x_3)/2 \\ (x_1-x_2-x_3)/2 \\ (-x_1+x_2-x_3)/2 \end{bmatrix}$$

Suppose we have 2 ordered basis for $\mathbb{V}$ ($\dim = n$).

$B = (u_1, u_2, \dots, u_n)$

$B' = (u_1', u_2', \dots, u_n')$

$$\begin{tikzcd}
x \in \mathbb{V} \arrow[r, "B"] \arrow[d, "B'"] & {[x]_B \in \mathbb{F}^n} \\
{[x]_{B'} \in \mathbb{F}^n} \arrow[ur, dash, dashed, "\text{relation?}"] &
\end{tikzcd}$$

$[x]_B$ and $[x]_{B'}$ are 2 different vector in $\mathbb{F}^n$ but they are result of same vector $x \in \mathbb{V}$. Hence they must be related.

Since $[x]_B$ and $[x]_{B'}$ represent the same vector x in $\mathbb{V}$ we expect them to be related. in some way.

If we know how the individual basis vectors of B is transformed in $\mathbb{F}^n$ by the application of B' then we will understand how a general vector written in B would be transformed by B'.

$B = \{u_1, u_2, \ldots, u_n\}$ (Suppose it is Familier basis / English language)

$$u_1 \in \mathbb{V} \xrightarrow{B'} [u_1]_{B'} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \in \mathbb{F}^n$$

(1^{st} page of dict. from Eng to French)

where 2^{nd} index represents the 1^{st} basis vector that is transformed to $\mathbb{F}^n$ by another choice of B'.

$$u_2 \in \mathbb{V} \xrightarrow{B'} [u_2]_{B'} = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} \in \mathbb{F}^n$$

(2^{nd} page of dict from Eng to french)

$$\vdots$$

$$u_n \in \mathbb{V} \xrightarrow{B'} [u_n]_{B'} = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{bmatrix} \in \mathbb{F}^n.$$

Therefore we get n vectors in $\mathbb{F}^n$.

$$\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [u_1]_{B'} & [u_2]_{B'} & \cdots & [u_n]_{B'} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \underbrace{\begin{bmatrix} b_{11} & b_{12} & & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & & b_{nn} \end{bmatrix}}_{\text{Whole dict from Eng to French.}}$$

It means we represent B basis vectors in terms of B' basis vectors. $= [B]_{B'}$

Consider a vector $x \in \mathbb{V} \xrightarrow{B} [x]_B$.

Represent x in B, $\quad x = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$

where, $\quad [x]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n$. (Text in English)

Consider the same vector $x \in \mathbb{V} \xrightarrow{B'} [x]_{B'}$

Represent x in B', $\quad x = x_1' u_1' + x_2' u_2' + \cdots + x_n' u_n'$

where, $\quad [x]_{B'} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \in \mathbb{F}^n$. (Text in French)

Since we have already found that how $u_1, u_2, \ldots, u_n$ will be translated by applying B', we write,

$$x = x_1 \cdot [u_1' \; u_2' \cdots u_n'] \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} + \cdots$$

$$\cdots + x_n \cdot [u_1' \; u_2' \cdots u_n'] \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{bmatrix}$$

$$= [x_1 \; x_2 \cdots x_n] \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{1n} \end{bmatrix} u_1' + \cdots + [x_1 \cdots x_n] \begin{bmatrix} b_{n1} \\ b_{n2} \\ \vdots \\ b_{nn} \end{bmatrix} u_n'$$

$$x = \alpha_1 \cdot u_1' + \alpha_2 \cdot u_2' + \cdots + \alpha_n \cdot u_n'$$

Therefore, we have expressed x in B' bassis where.

$$x \in V \xrightarrow{B'} [x]_{B'} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{F}^n$$

How to get $[x]_{B'}$ in terms of $[x]_B$?

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & & b_{1n} \\ b_{12} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

Text in French

Text in English

Dictionary from English to French

$$\boxed{[x]_{B'} = [B]_{B'} [x]_B}$$

Similarly, $[x]_B = [B']_B \, [x]_{B'}$

Combining the results -

$$[x]_B = [B']_B \cdot [B]_{B'} \cdot [x]_B = [I]\,[x]_B$$

Therefore, $[B']_B [B]_{B'} = [I]$

So $[B]_{B'} = [B']_B^{-1}$ and $[B']_B = [B]_{B'}^{-1}$

Example: $\mathbb{V} = \mathbb{F}^3$.

$$B = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = (e_1, e_2, e_3)$$

$$x \in \mathbb{F}^3 \xrightarrow{\quad B \quad} x = x_1 \cdot e_1 + x_2 \cdot e_2 + x_3 \cdot e_3$$

$$[x]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{F}^3$$

Consider $B' = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) = (u_1, u_2, u_3)$

$$x \in \mathbb{F}^3 \xrightarrow{\quad B' \quad} x = x_1' u_1 + x_2' u_2 + x_3' u_3$$

$$[x]_{B'} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \in \mathbb{F}^3.$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} [e_1]_{B'} & [e_2]_{B'} & [e_3]_{B'} \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow [x]_{B'} = [B]_{B'} \, [x]_B$$

$$e_1 \in \mathbb{V} \xrightarrow{B'} e_1 = b_{11} \cdot u_1 + b_{21} \cdot u_2 + b_{31} \cdot u_3$$

$$[e_1]_{B'} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = b_{11} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b_{21} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_{31} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We need to solve this system.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

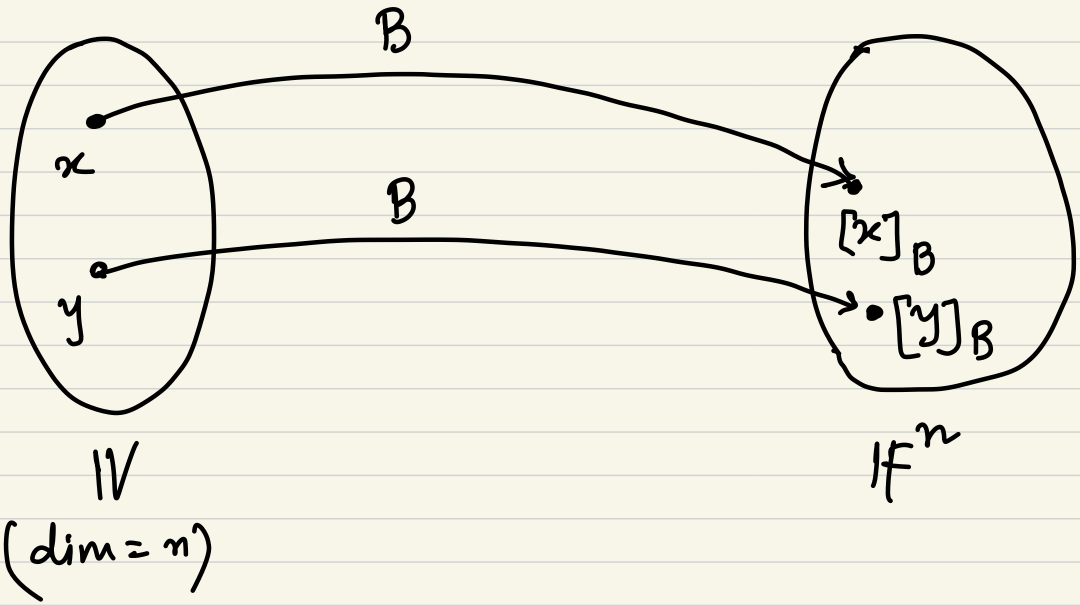
Hence, $\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Therefore, $e_1 = \frac{1}{2} \cdot u_1 + \frac{1}{2} \cdot u_2 - \frac{1}{2} \cdot u_3$

Similarly trying for other basis vectors e_2, e_3 we get,

$$[B]_{B'} = \frac{1}{2} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}}_{\text{dictionary to translate english basis to French basis.}}$$

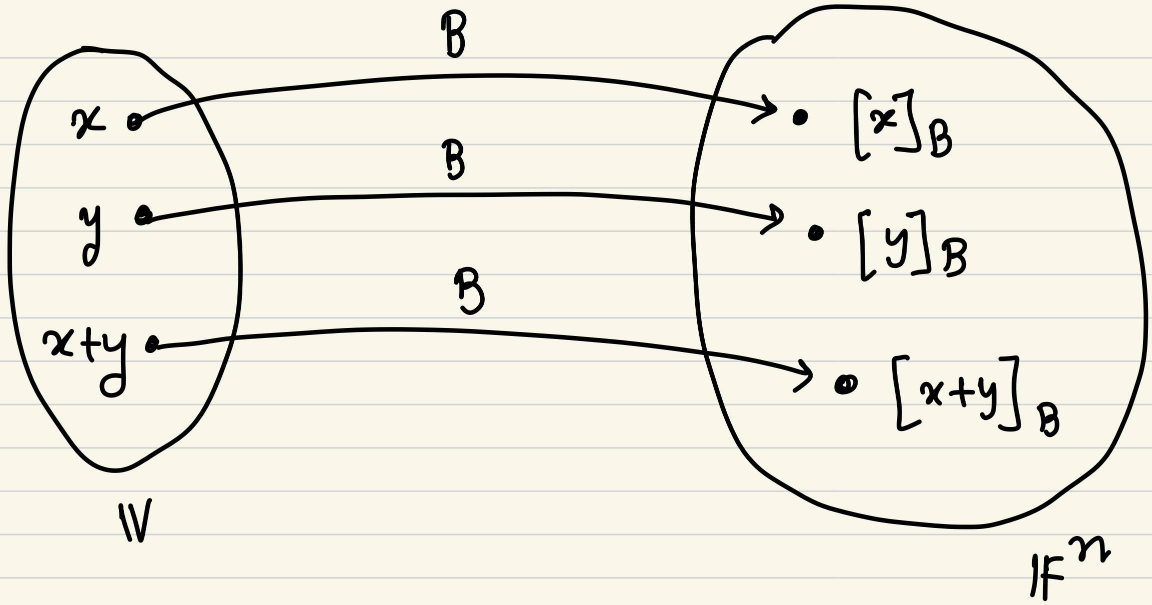
$$\underbrace{[x]_{B'}}_{\text{translated French text.}} = [B]_{B'} \cdot \underbrace{[x]_B}_{\text{English text}}$$



$$x \xmapsto{B} [x]_B$$

$$y \xmapsto{B} [y]_B$$

We call x and y gets encoded to n tuples in $\mathbb{F}^n$. But how good is this encoding ?



In the vector space $\mathbb{W}$, we add x & y and get another vector $x+y$. Apply encoding by B to get $[x+y]_B$.

Suppose we do the addition after encoding.

$$[x]_B + [y]_B = [g]_B \quad (\text{say}).$$

The question is will $[g]_B = [x+y]_B$?

If we add 2 vectors in $\mathbb{W}$ and encode and if we encode the vectors first then add, will we get same result?

$$[x]_B + [y]_B \overset{?}{=} [x+y]_B$$

Suppose, $\quad x \longmapsto [x]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad x = x_1 u_1 + \cdots + x_n u_n$

$$y \longmapsto [y]_B = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}; \quad y = y_1 u_1 + \cdots + y_n u_n$$

$$x+y \longmapsto [x+y]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix};\ x+y = \alpha_1 \cdot u_1 + \cdots + \alpha_n u_n.$$

In $\mathbb{V}$ site:

$$x = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$$
$$y = y_1 u_1 + y_2 u_2 + \cdots + y_n u_n$$

$$x+y = (x_1+y_1)\, u_1 + (x_2+y_2)\, u_2 + \cdots + (x_n+y_n)\, u_n$$

Since $u_1, u_2, \ldots, u_n$ is a ordered basis therefore.

$$x+y \overset{B}{\longmapsto} [x+y]_B = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

Now we add the encoded vectors first.

$$[x]_B + [y]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

Notice that, $[x \overset{+_V}{+} y]_B = [x]_B \overset{+_{\mathbb{F}^n}}{+} [y]_B$.

If you add in $\mathbb{V}$ then encode in $\mathbb{F}^n$ $\equiv$ If you encode in $\mathbb{F}^n$ and then add in $\mathbb{F}^n$

The encoding / representation of $x \in \mathbb{V}$ with $[x]_B \in \mathbb{F}^n$ through ordered basis B preserves addition.

Similarly,

$$[\alpha \cdot x]_B = \alpha \cdot [x]_B$$

[$\mathbb{F}$ must be same for $\mathbb{V}$ and $\mathbb{F}^n$]

If you multiply by scalar in $\mathbb{V}$ then encode in $\mathbb{F}^n$ $\equiv$ If you encode in $\mathbb{F}^n$ then multiply by scalar in $\mathbb{F}^n$.

The encoding of $x \in \mathbb{V}$ with $[x]_B \in \mathbb{F}^n$ through ordered basis B preserves scalar multiplication.

Thus $x \xmapsto{B} [x]_B$ preserves addition and scalar multiplication from $\mathbb{V}$ to $\mathbb{F}^n$. This leads us to the notion of linear transformations.

Linear Transformation:

Let $\mathbb{V}$ and $\mathbb{W}$ be any 2 vector spaces over same field $\mathbb{F}$. Then a map/function $T : \mathbb{V} \longrightarrow \mathbb{W}$ such that it preserves scalar mult & addition.

$$\forall x, y \in \mathbb{V} \left[\underbrace{T(x +_{\mathbb{V}} y)}_{\in \mathbb{W}} = \underbrace{T(x)}_{\in \mathbb{W}} +_{\mathbb{W}} \underbrace{T(y)}_{\in \mathbb{W}} \right]$$

$$\forall c \in \mathbb{F}, x \in \mathbb{V} \left[T(\underbrace{c \cdot x}_{\text{in } \mathbb{V}}) = \underbrace{c \cdot T(x)}_{\text{in } \mathbb{W}} \right]$$

Then T is called <u>linear transformation</u> from $\mathbb{V}$ to $\mathbb{W}$.

Suppose we take $\mathbb{W}$ to be same as $\mathbb{V}$. Then the transformation is happening in the same space so it acts like an operator (closed under operation T).

$T : \mathbb{V} \to \mathbb{V}$; T is called "<u>linear operator</u>" iff

$$\forall u, v \in \mathbb{V} \left[T(u +_V v) = T(u) +_V T(v) \right]$$

$$\forall c \in \mathbb{F}, u \in \mathbb{V} \left[T(c \cdot u) = c \cdot T(u) \right]$$

<u>Example:</u> $\mathbb{V}$ is an n dim. vector space over $\mathbb{F}$
B is an ordered basis for $\mathbb{V}$.

Define a transformation $T_B : \mathbb{V} \to \mathbb{F}^n$

$$T_B(x) := [x]_B$$

We have seen $T_B(x+y) = T_B(x) + T_B(y)$

$$\Rightarrow [x+y]_B = [x]_B + [y]_B$$

$$T_B(\alpha \cdot x) = \alpha \cdot T_B(x)$$

$$\Rightarrow [\alpha \cdot x]_B = \alpha \cdot [x]_B$$

Hence T_B is a linear transformation from $\mathbb{V}$ to $\mathbb{F}^n$.

Every ordered basis on a n dim vector space $\mathbb{V}$ over $\mathbb{F}$ produces a linear transformation $T_B : \mathbb{V} \longrightarrow \mathbb{F}^n$.

Ex 2: $\mathbb{V} = \mathbb{F}^3$. $\mathbb{W} = \mathbb{F}^2$.

Define : $T : \mathbb{V} \longrightarrow \mathbb{W}$ as follows -

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) := \begin{bmatrix} 2x_1 - x_2 + x_3 \\ x_1 + 2x_2 - x_3 \end{bmatrix}$$

Is T a linear transformation?

$$x +_v y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ x_1 + 2x_2 - x_3 \end{bmatrix}$$

$$T(y) = \begin{bmatrix} 2y_1 - y_2 + y_3 \\ y_1 + 2y_2 - y_3 \end{bmatrix}$$

$$T(x) +_W T(y) = \begin{bmatrix} 2x_1 - x_2 + x_3 + 2y_1 - y_2 + y_3 \\ x_1 + 2x_2 - x_3 + y_1 + 2y_2 - y_3 \end{bmatrix}$$

$$T(x +_v y) = \begin{bmatrix} 2(x_1 + y_1) - (x_2 + y_2) + (x_3 + y_3) \\ (x_1 + y_1) + 2(x_2 + y_2) - (x_3 + y_3) \end{bmatrix}$$

Look at 2 vectors they are same. So T is Linear Transf.

It is easy to check that $-$ $T : \mathbb{F}^3 \to \mathbb{F}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

T is always a LT from $\mathbb{F}^3$ to $\mathbb{F}^2$ $\forall a_{ij} \in \mathbb{F}$.

Since T is a linear transformation $-$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow T(x) = Ax$

So the transformation can be thought of matrix $A \in \mathbb{F}^{2\times 3}$

$A \in \mathbb{F}^{2\times 3} : \mathbb{F}^3 \longrightarrow \mathbb{F}^2$ is always a LT.

Hence matrices are kind of linear in the sense that

$$\left.\begin{aligned} A(x+y) &= Ax + Ay \\ A(cx) &= cAx \end{aligned}\right\} \text{ A is actually linear transf.}$$

If $A \in \mathbb{F}^{m\times n}$ matrix then it produces a linear transformation $T_A : \mathbb{F}^n \to \mathbb{F}^m$ as $T_A(x) = Ax$

Abstract object T can be applied on study of matrix A.

In particular for scalar multiplication -

$$T(\alpha \cdot u) = \alpha \cdot T(u) \quad \forall \alpha \in \mathbb{F}, \ \forall u \in \mathbb{V}.$$

$$\underset{\in \mathbb{V}}{u} \xmapsto{T} \underset{\in \mathbb{W}.}{T(u)}$$

If we choose $\alpha = 0$ then $T(0 \cdot u) = 0 \cdot \underbrace{T(u)}_{}$ $\in \mathbb{W}$.

$$\Rightarrow T(0_V) = 0_{\mathbb{W}}$$

So Every LT, $T: \mathbb{V} \to \mathbb{W}$ will always map the zero vector of $\mathbb{V}$ to zero vector of $\mathbb{W}$.

<u>Example:</u>

$\mathbb{V}$ is n dimensional vector space over $\mathbb{F}$.

$B = (u_1, u_2, \ldots, u_n)$ an ordered basis for $\mathbb{V}$.

any vector $x \in \mathbb{V}$ can be written as.

$$x \in \mathbb{V} \longrightarrow x = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n \quad \forall x_i \in \mathbb{F}.$$

$$[x]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n$$

$$x \in \mathbb{V} \xmapsto{B} [x]_B$$

We define T_B as : $T_B : \mathbb{V} \longrightarrow \mathbb{F}^n$

$$T_B(x) = [x]_B .$$

We have already verified that T_B is linear transformation from $\mathbb{V}$ to $\mathbb{F}^n$. Every ordered basis B for $\mathbb{V}$ induces a LT, T_B from $\mathbb{V}$ to $\mathbb{F}^n$.

Example-2:

$\mathbb{V} = \mathbb{F}^n$, $\mathbb{W} = \mathbb{F}^m$ over $\mathbb{F}$

Let A be any fixed matrix in $\mathbb{F}^{m \times n}$.

Define for every $x \in \mathbb{F}^n$ as follows :

$$T_A(x) := Ax \quad \text{so} \quad T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$$

Thus T_A maps $\mathbb{F}^n$ to $\mathbb{F}^m$. We have already seen that T_A is a linear transformation.

$T_A(x) = Ax$	$T_A(x) = Ax$
$T_A(y) = Ay$	$T_A(c.x) = A(c.x)$
$T_A(x+y) = A(x+y)$	$= c.Ax$
	$= c.T(x)$

Now $A(x+y) = Ax + Ay = T_A(x) + T_A(y) = T_A(x+y)$

Therefore T_A is a LT from $\mathbb{F}^n$ to $\mathbb{F}^m$

Example-3:

Let $\mathbb{V} = \mathbb{F}^{n \times n}$ over $\mathbb{F}$

Fix a vector in $\mathbb{V}$ or $A \in \mathbb{F}^{n \times n}$.

Define for every $X \in \mathbb{F}^{n \times n}$,

$$L_A(X) := AX \quad \text{(left multiplication by } A\text{)}$$

So $L_A : \mathbb{F}^{n \times n} \longrightarrow \mathbb{F}^{n \times n}$ is a transformation.

Is L_A a linear transformation?

$\Rightarrow$ $L_A(X) = AX$

$L_A(Y) = AY$

$L_A(X+Y) = A(X+Y) = AX + AY = L_A(X) + L_A(Y)$

$L_A(c \cdot X) = A(cX) = c(AX) = c \cdot L_A(X)$

So L_A is linear transformation.

Define for every $X \in \mathbb{F}^{n \times n}$,

$$R_A(X) := XA \quad \text{(right multiplication by } A\text{)}$$

This is also a linear transformation.

Define for every $X \in \mathbb{F}^{n \times n}$, fixed $A, B \in \mathbb{F}^{n \times n}$

$$T_{AB}(X) := AXB \quad ; \; T_{AB} : \mathbb{F}^{n \times n} \longrightarrow \mathbb{F}^{n \times n}$$

Is $T_{AB}(X)$ a linear transformation?

$\Rightarrow$ Yes it is also a linear transformation.

In particular, let P be a fixed matrix $\in \mathbb{F}^{n\times n}$ such that P^{-1} exists then -

Take $A = P^{-1}$ and $B = P$ in the $T_{AB}(X) := AXB$.

So $T_P(X) := P^{-1}XP$ s.t. $T_P : \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}^{n\times n}$

Definitely T_P is a linear operator from $\mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$.

> Let X be any matrix in $\mathbb{F}^{n\times n}$. We say a $Y \in \mathbb{F}^{n\times n}$ is similar to X if there exists an $P \in \mathbb{F}^{n\times n}$ such that,
>
> $T_P(X) = Y$ (if Y is coded version of X generated by P)

Therefore, T_P is called <u>similarity transformation</u>.

<u>Example-4:</u>

$\mathbb{V} = \mathbb{F}^{m\times n}$ over $\mathbb{F}$.

$L_Q(X) = QX$ for a fixed $Q \in \mathbb{F}^{m\times m}$

$X \in \mathbb{F}^{m\times n}$ so $L_Q : \mathbb{F}^{m\times n} \longrightarrow \mathbb{F}^{m\times n}$

L_Q is a linear transformation.

$R_Q(X) = XQ$ for a fixed $Q \in \mathbb{F}^{n \times n}$

$X \in \mathbb{F}^{m \times n}$, $R_Q : \mathbb{F}^{m \times n} \longrightarrow \mathbb{F}^{m \times n}$

R_Q is also a linear transformation.

$T_{PQ}(X) = PXQ$ for a fixed $P \in \mathbb{F}^{m \times m}$ and $Q \in \mathbb{F}^{n \times n}$

$X \in \mathbb{F}^{m \times n}$ $\qquad T_{PQ} : \mathbb{F}^{m \times n} \longrightarrow \mathbb{F}^{m \times n}$

T_{PQ} is also a linear transformation.

<u>Example - 5 :</u>

$W = \mathbb{F}[x]$

For any p in $\mathbb{F}[x]$, define a transformation.

$$\underbrace{D(p)}_{\in \mathbb{F}[x]} := \underbrace{\frac{dp}{dx}}_{\in \mathbb{F}[x]}$$

So, $D : \mathbb{F}[x] \longrightarrow \mathbb{F}[x]$

Is D a linear transformation?

$\Rightarrow$

$$D(p) = \frac{dp}{dx}$$

$$D(q) = \frac{dq}{dx}$$

$$D(p) + D(q) = \frac{dp}{dx} + \frac{dq}{dx} = \frac{d}{dx}(p+q) = D(p+q)$$

$$c \cdot D(p) = c \cdot \frac{dp}{dx} = \frac{d}{dx}(c \cdot p) = D(c \cdot p)$$

Hence D is a linear operator.

Suppose, $\mathbb{V} = \mathbb{F}[x]_4$, $\mathbb{W} = \mathbb{F}[x]_3$

Define $T : \mathbb{V} \to \mathbb{W}$.

$$T(p) := \frac{d^2}{dx^2}(p) \qquad T : \mathbb{F}[x]_4 \longrightarrow \mathbb{F}[x]_3$$

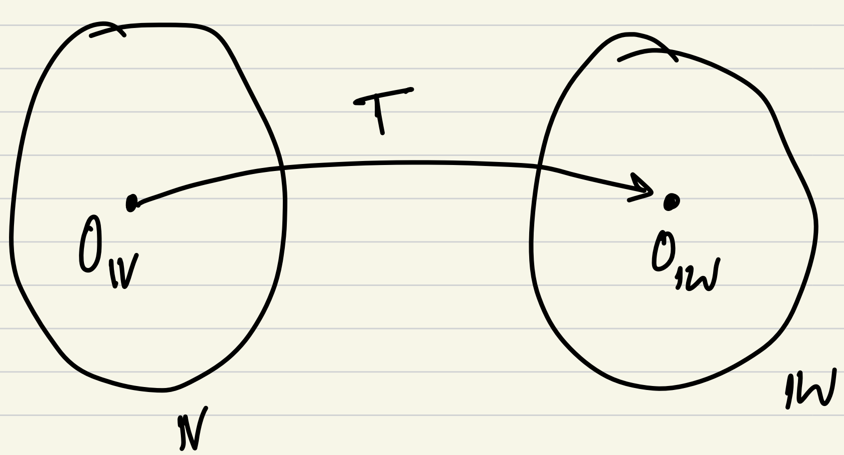
$T(p+q) = T(p) + T(q) \quad \forall p, q \in \mathbb{F}[x]_4$

$T(\alpha \cdot p) = \alpha \cdot T(p) \quad \forall \alpha \in \mathbb{F}, \forall p \in \mathbb{F}[x]_4$

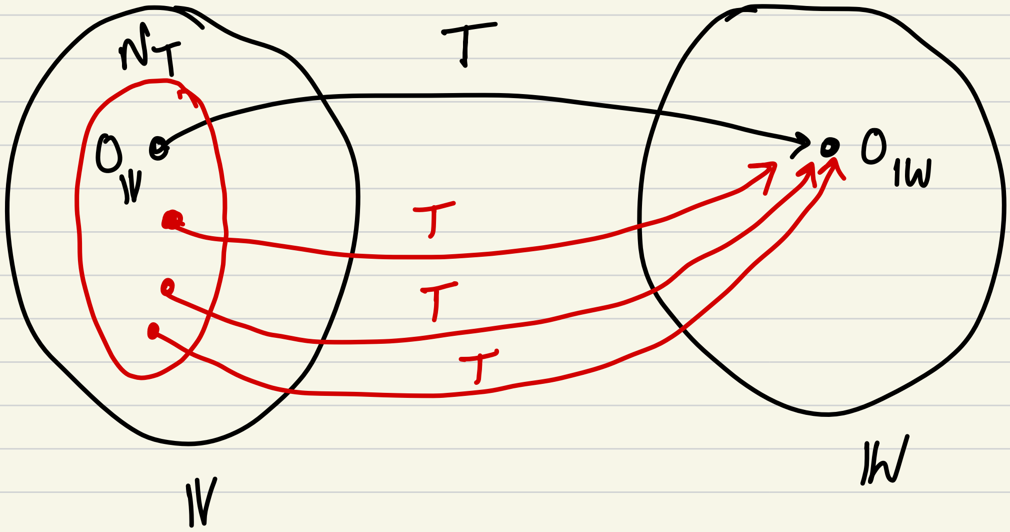
Therefore, T is a linear transformation.

Extending the zero to zero map :

A simple property for any linear transformation $T : \mathbb{V} \to \mathbb{W}$ is $T(0_V) = 0_{\mathbb{W}}$.



Now in a function / transformation, T can take any other vector in $\mathbb{V}$ that may also get mapped to $0_{\mathbb{W}}$.



There may be several vectors in $\mathbb{V}$ that gets mapped to the $0_{\mathbb{W}}$ in addition to $0_{\mathbb{V}}$. We collect all these vectors in $\mathbb{V}$ and denoted by N_T

$$N_T := \left\{ x \in \mathbb{V} \;\middle|\; T(x) = 0_{\mathbb{W}} \right\}$$

Clearly N_T is collection of vectors of $\mathbb{V}$ so $N_T \subseteq \mathbb{V}$. and the $0_{\mathbb{V}}$ will always be present in N_T. N_T is a non empty subset of $\mathbb{V}$.

Whenever we have a non empty subset in vector space $\mathbb{V}$ then the natural question that we ask is whether that set is a subspace of $\mathbb{V}$ or not?

For N_T to be a subspace of $\mathbb{V}$, N_T must be closed w.r.t vector addition & scalar multiplication.

Choose 2 arbitrary vectors $x, y \in N_T$.

Consider, $\quad w = c.x + y$ where $c \in \mathbb{F}$

Since $x \in N_T$ so $T(x) = 0_{\mathbb{W}}$

Since $y \in N_T$ so $T(y) = 0_{\mathbb{W}}$

$w = cx + y$ so we ask if $T(cx+y) \overset{?}{=} 0_{\mathbb{W}}$.

$$
\begin{aligned}
T(cx+y) &= T(c.x) + T(y) \\
&= c.T(x) + T(y) \\
&= c.\,0_{\mathbb{W}} + 0_{\mathbb{W}} \\
&= 0_{\mathbb{W}}.
\end{aligned}
$$

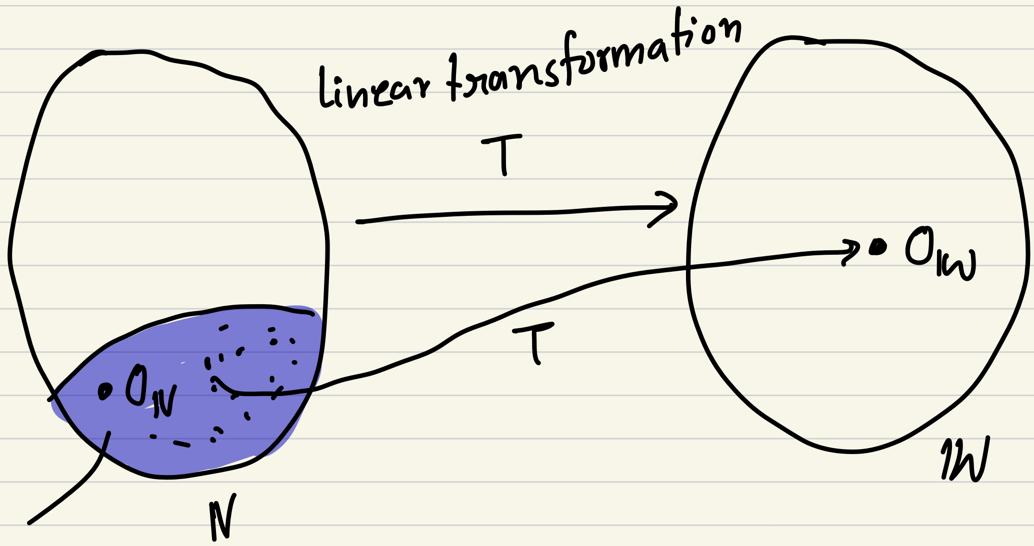
Therefore, N_T is a subspace of $\mathbb{V}$. The subspace N_T is called the Kernel of T or null space of T.

Suppose $\mathbb{V}$ is f.d.v.s and $T: \mathbb{V} \longrightarrow \mathbb{W}$. and T is a linear transformation.

N_T is a subspace of $\mathbb{V}$. Since $\dim(\mathbb{V}) < \infty$ so $\dim(N_T) < \infty$. and $\dim(N_T) \leq \dim(\mathbb{V})$.

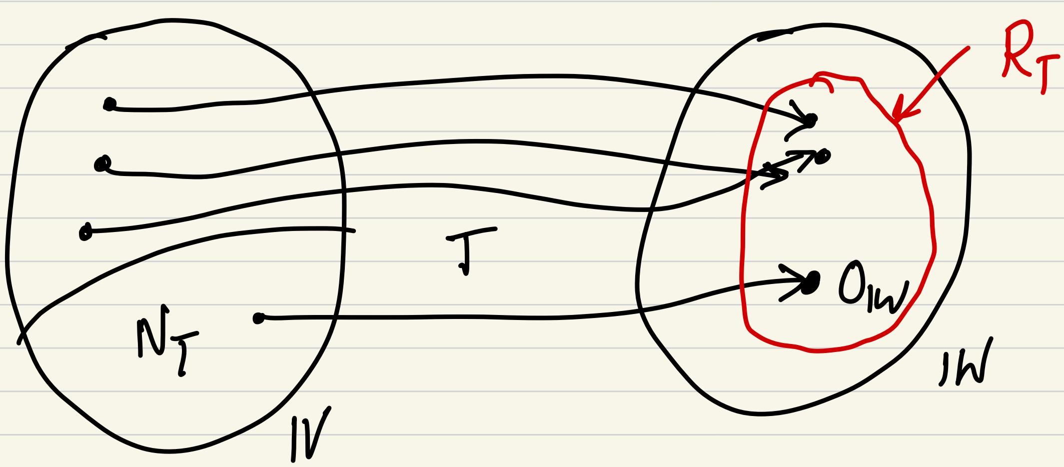
The dimension of N_T is called "nullity" of T.

$\nu_T = \dim(N_T)$ where ν_T is called nullity.



N_T (Kernel of T which is a subspace of V)

Suppose we take any vector in V that is not part of N_T then we can also find the point where that vector in V gets mapped to W so we are kind of collecting the image of all the other points in V such that the image doesn't become 0_W in W space.



There will be other points in $\mathbb{W}$ (except $O_{\mathbb{W}}$) such that the image of $v \in \mathbb{V}$ may come in $\mathbb{W}$. We collect all the focal point in $\mathbb{W}$ such that all the vectors in $\mathbb{V}$ forms the image in the focal points.

$$R_T := \left\{ y \in \mathbb{W} \;\middle|\; \exists x \in \mathbb{V} \text{ s.t } T(x) = y \right\}$$

Now in the definition of R_T (range of T / Image of T), we are not restricting the $O_{\mathbb{W}}$ as a focal point in R_T. Therefore we also say $O_{\mathbb{W}}$ may also be present in R_T so the $x \in \mathbb{V}$ will then be coming from N_T. But we motivated the definition of R_T without considering O_W but in the definition of R_T, we do consider $O_{\mathbb{W}}$ as a member of $\mathbb{R}_W$ for a reason that it becomes a subspace

R_T is the collection of all the images in $\mathbb{W}$ such that there exists a preimage of the image in $\mathbb{V}$. In other words R_T is the collection of all the points in $\mathbb{W}$ which is the value / range of the Transformation T when applied on x in $\mathbb{V}$.

$R_T \subseteq \mathbb{W}$. But is R_T a subspace of $\mathbb{W}$?

Definitely $O_{\mathbb{W}} \in R_T$. So R_T is not empty subset.

Consider 2 vectors in R_T, say y & z. such that,

$y = T(x) \quad \exists x \in \mathbb{V}.$

$z = T(u) \quad \exists u \in \mathbb{V}.$

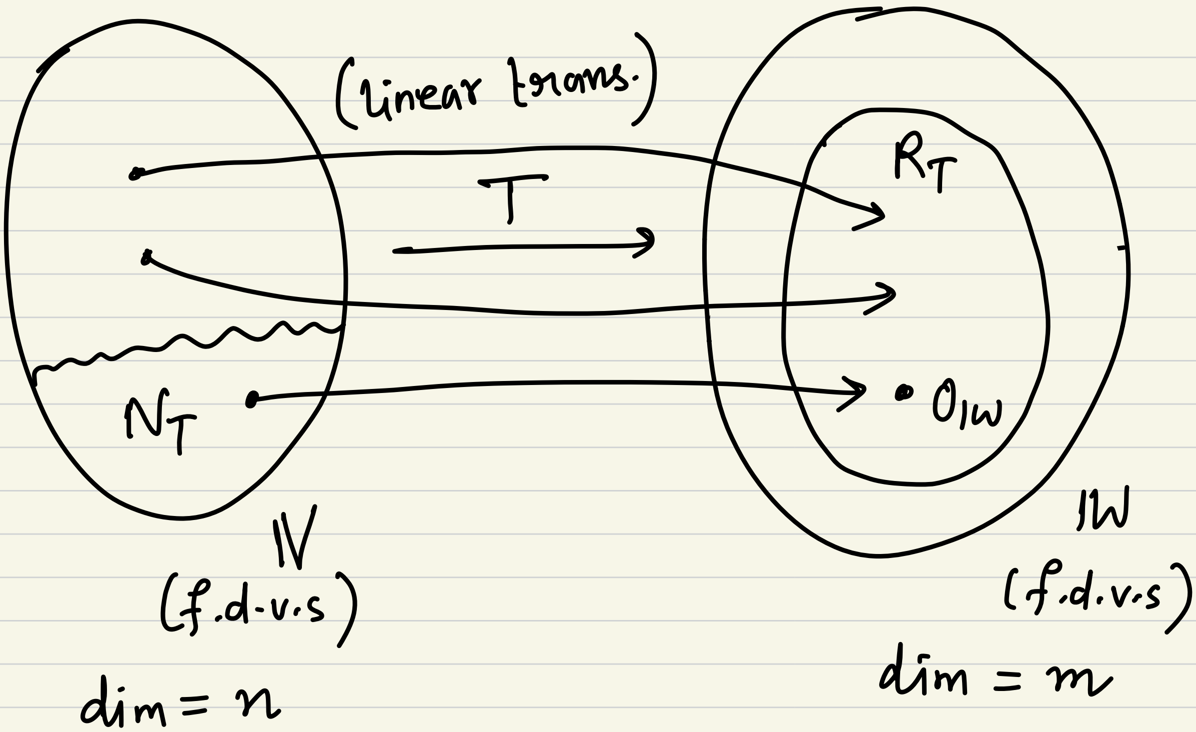
Now consider $\omega = c \cdot y + z \quad \forall c \in \mathbb{F}.$

$$\begin{aligned}
\omega &= c \cdot y + z \\
&= c \cdot T(x) + T(u) \\
&= T(c \cdot x) + T(u) \\
&= T(c \cdot x + u) \\
&= T(v) \quad \text{say } v = c \cdot x + u \in \mathbb{V}.
\end{aligned}$$

Since $\exists v \in \mathbb{V}$ s.t. $\omega = T(v)$ then $\omega \in R_T$ hence R_T is a subspace of $\mathbb{W}$. And R_T is called range of T or Image of T.

If $\mathbb{W}$ is a f.d.v.s then $T : \mathbb{V} \to \mathbb{W}$ and R_T will also be f.d.v.s. Therefore, $\dim(R_T) \leq \dim(\mathbb{W})$.
The $\dim(R_T)$ is called rank(T). and denoted by $\rho(T)$

We get $\rho(T) = \dim(R_T) = \text{rank}(T) \leq \dim(\mathbb{W})$



$\dim(N_T) = \text{nullity}(T) \leq n$

$\dim(R_T) = \text{rank}(T) \leq m$

Example-1:

Consider $\mathbb{V} = \mathbb{F}^n$, $\mathbb{W} = \mathbb{F}^m$

Define a transformation, $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$.

Fix a fixed matrix $A \in \mathbb{F}^{m \times n}$

$$T_A(x) := Ax \qquad x \in \mathbb{F}^n$$

T_A is a linear transformation from $\mathbb{F}^n \longrightarrow \mathbb{F}^m$.

Kernel $(T_A) = ?$, Image $(T_A) = ?$

$$\begin{pmatrix} x \in \text{Kernel}(T_A) \\ x \in \mathbb{F}^n \end{pmatrix} \longleftrightarrow T_A(x) = 0_{W} \text{ or } Ax = 0_W$$

$$\text{Kernel}(T_A) = \left\{ x \in \mathbb{F}^n \;\middle|\; T_A(x) = 0_{W},\ T_A : \mathbb{F}^n \to \mathbb{F}^m \right\}$$

$$= \left\{ x \in \mathbb{F}^n \;\middle|\; Ax = 0_{W},\ \text{since } T_A(x) = Ax \right\}$$

= set of all solutions for homog. equation $Ax = 0$ so the nullspace of T_A is same as nullspace of A.

Once we find the Kernel (T_A) then choose the basis and calculate dimension of Kernel (T_A). that is called nullity of T_A.

$$\begin{pmatrix} y \in \text{Image}(T_A) \\ y \in \mathbb{F}^m \end{pmatrix} \longleftrightarrow y = T_A(x) \text{ or } y = Ax$$

$$\text{Image}(T_A) = \left\{ y \in \mathbb{F}^m \;\middle|\; y = T_A(x),\ \exists x \in \mathbb{F}^n \right\}$$

$$= \left\{ y \in \mathbb{F}^m \;\middle|\; y = Ax \right\}$$

= Image of T_A is same as column space of A.

Once we find the Image (T_A) so choose the basis and calculate dimension of Image (T_A). That is the rank of T_A.

Ex2: $\mathbb{V} = \mathbb{F}[x]_4$ over $\mathbb{F}$.

consider a linear operator $D : \mathbb{V} \longrightarrow \mathbb{V}$.

$$D(p) := \frac{dP}{dx}.$$

let's find Kernel (D) and Image (D) = ?

Kernel (D):

We want to find all those polynomials in $\mathbb{V}$ s.t if we apply $D(P)$ then we get Zero polynomial in $\mathbb{V}$ or in other words we find all those p in $\mathbb{V}$ s.t. $\frac{dP}{dx} = 0$.

$$\begin{pmatrix} P \in \mathbb{F}[x]_4 \\ P \in \mathrm{Ker}[\mathbb{V}] \end{pmatrix} \longleftrightarrow \quad D(P) = 0_{\mathbb{V}} \quad \text{or}$$

$$\frac{dP}{dx} = 0_{\mathbb{V}} \quad \text{or}$$

$$P = a_0 \quad \forall a_0 \in \mathbb{F}$$

Therefore, the $\mathrm{Ker}[\mathbb{V}] = \{ P \in \mathbb{V} \mid P(x) = a_0, \ a_0 \in \mathbb{F} \}$

Basis for $\mathrm{Ker}[\mathbb{V}]$, $B = \{1\}$. $\dim(\mathrm{Ker}(W)) = \underline{1}$

Image (D):

To find the image (D), we want to find all the vectors (or polynomials) in $\mathbb{V}$ that gets mapped to some vectors in $\mathbb{V}$, or we collect all the derivates of the polynomials in $\mathbb{V}$.

$$\begin{pmatrix} q \in \mathbb{V} \\ q \in \text{Image}(D) \end{pmatrix} \longleftrightarrow \begin{matrix} \exists p \in \mathbb{V} \text{ s.t } q = D(p) \text{ or} \\ \exists p \in \mathbb{V} \text{ s.t } q = \frac{dp}{dx}. \end{matrix}$$

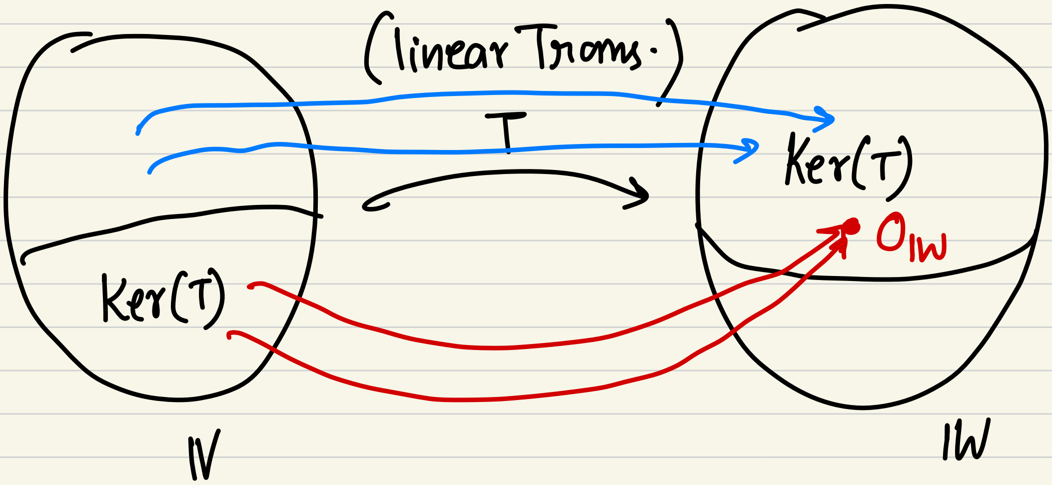
$$\text{Image}(D) = \left\{ q \in \mathbb{V} \;\middle|\; \exists p \in \mathbb{V},\ q = \frac{dp}{dx} \right\}$$

$$= \left\{ q \in \mathbb{V} \;\middle|\; \exists p \in \mathbb{V},\ p = \int_0^x q\, dx \right\}$$

Since $\mathbb{F}[x]_4$ is given as $\mathbb{V}$ so if q is deg 4 polynomial in $\mathbb{V}$ then p will become deg 5 polynomial which can't exist in $\mathbb{F}[x]_4$.

Therefore, $\text{Image}(D) = \left\{ q \in \mathbb{F}[x]_3 \;\middle|\; \exists p \in \mathbb{F}[x]_4,\ q = \frac{dp}{dx} \right\}$

Since Image (D) contains the set of $\mathbb{F}[x]_3$ polynomials then a basis set, $B = \{1, x, x^2, x^3\}$ so dim of Image (D) $= 4$. So rank$(D) = 4$.



$\dim(\mathbb{V}) = n$ $\qquad\qquad$ $\dim(\mathbb{W}) = m$

$\dim(\text{Ker}(T)) = \text{nullity}(T)$ $\qquad\qquad$ $\dim(\text{Im}(T)) = \text{rank}(T)$

$\text{nullity}(T) \leq n$ $\qquad\qquad$ $\text{rank}(T) \leq m$

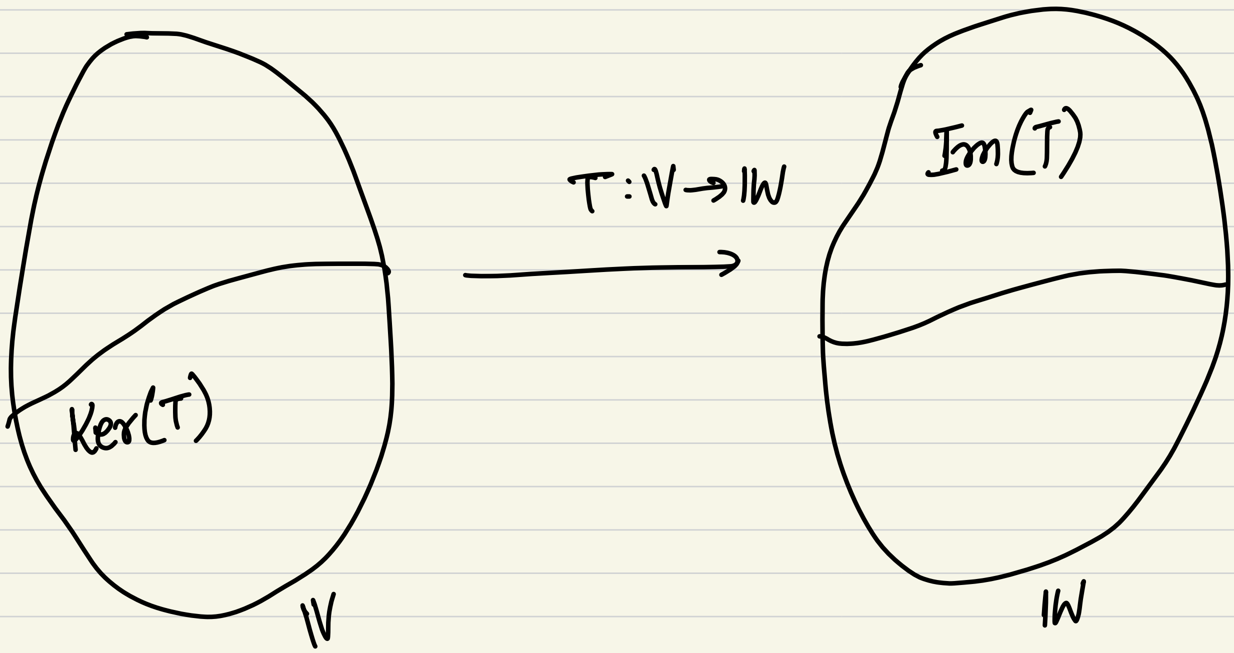
We have a subspace in $\mathbb{V}$ and subspace in $\mathbb{W}$ but is there a connection b/w their dimensions ?

rank-nullity theorem:

$$\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{V}) = n$$

$$\Rightarrow \dim(\underbrace{\text{Im}(T)}_{\text{subspace of } \mathbb{W}}) + \dim(\underbrace{\text{Ker}(T)}_{\text{subspace of } \mathbb{V}.}) = \dim(\mathbb{V})$$

Bigger picture of linear Transformation:



$$\text{Ker}(T) = \{ x \in \mathbb{V} \mid Tx = 0_{\mathbb{W}} \}$$

$$\text{Im}(T) = \{ y \in \mathbb{W} \mid \exists x \in \mathbb{W} \text{ s.t } Tx = y \}$$

$\text{Ker}(T)$ is subspace of $\mathbb{V}$

$\text{Im}(T)$ is subspace of $\mathbb{W}$

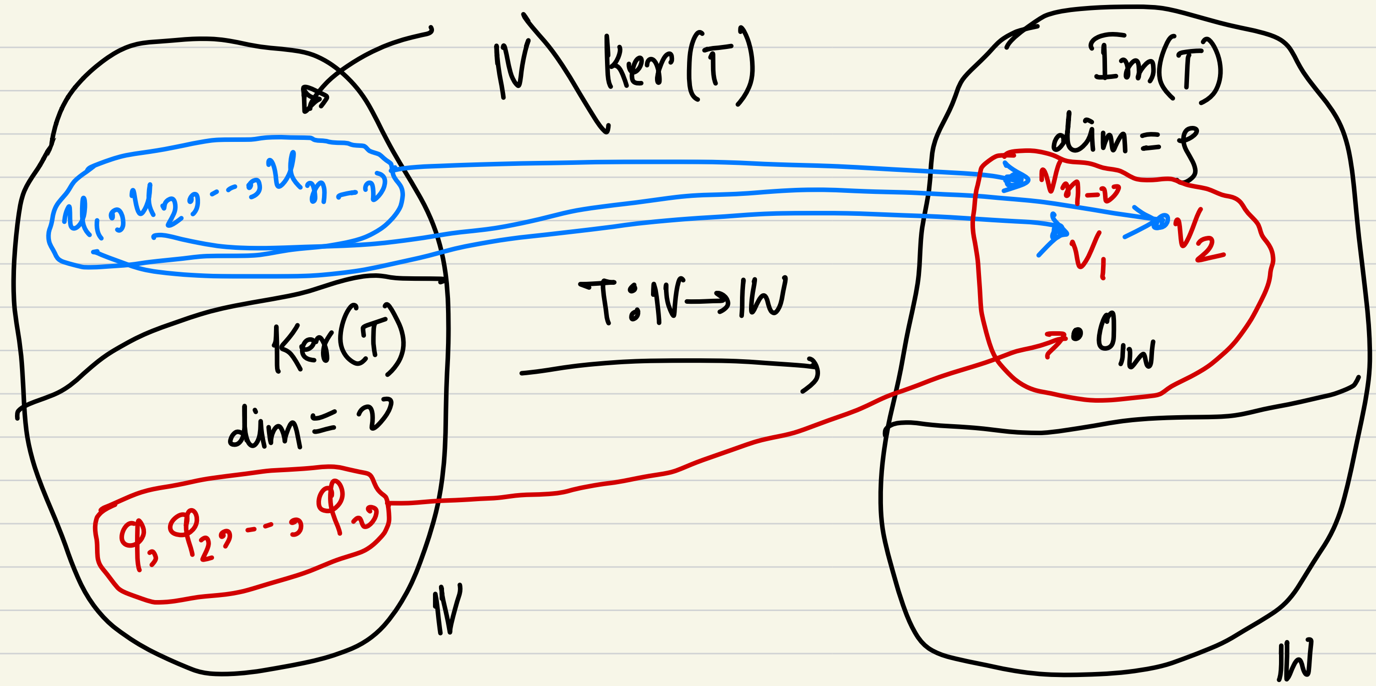
$$\dim(\text{Ker}(T)) = \text{nullity}(T) = \nu \leq \dim(\mathbb{V})$$

$$\dim(\text{Im}(T)) = \text{rank}(T) = \rho \leq \dim(\mathbb{W})$$

$$\dim(\mathbb{V}) = n$$

$$\dim(\mathbb{W}) = m$$

Therefore, $\nu \leq n$ and $\rho \leq m$



Any basis for $Ker(T)$ must have ν vectors
$B_N = \{ \varphi_1, \varphi_2, \dots, \varphi_\nu \}$ be a basis for $Ker(T)$

$\mathbb{V}$ has n basis vectors that are LI and spans $\mathbb{V}$.
Now within $\mathbb{V}$, we have already got ν such LI vectors $\{ \varphi_1, \varphi_2, \dots, \varphi_\nu \}$. Any LI set of vectors can be extended to basis by adding suitable vectors.

We need $(n-\nu)$ LI set of vectors from $\mathbb{V} \setminus Ker(T)$ space and together they will form basis for $\mathbb{V}$.

Basis for $\mathbb{V}$, $B_{\mathbb{V}} = \{ \varphi_1, \varphi_2, \dots, \varphi_\nu, u_1, u_2, \dots, u_{n-\nu} \}$
where $\{ u_1, u_2, \dots, u_{n-\nu} \}$ must be LI set of vectors and $\forall u_i \in \mathbb{V} \setminus Ker(T)$.

Note that, $T(\varphi_1) = T(\varphi_2) = \cdots = T(\varphi_r) = 0_{W}$.

$T(u_1), T(u_2), \ldots, T(u_{n-r})$ are all different from 0_W otherwise they would have been part of $Ker(T)$.

Consider any vector $x \in V$. written as LC of B_V

$$x = \alpha_1 \cdot \varphi_1 + \alpha_2 \cdot \varphi_2 + \cdots + \alpha_r \varphi_r + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_{n-r} \cdot u_{n-r}$$

$$\Rightarrow T(x) = T(\alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \beta_{n-r} u_{n-r})$$

$$= T(\alpha_1 \varphi_1) + T(\alpha_2 \varphi_2) + \cdots + T(\beta_{n-r} u_{n-r})$$

$$= \alpha_1 \cdot \cancel{T(\varphi_1)}^{0} + \alpha_2 \cdot \cancel{T(\varphi_2)}^{0} + \cdots + \alpha_r \cdot \cancel{T(\varphi_r)}^{0} + \beta_1 \cdot T(u_1) + \beta_2 T(u_2) + \cdots + \beta_{n-r} T(u_{n-r})$$

$$= \beta_1 T(u_1) + \beta_2 T(u_2) + \cdots + \beta_{n-r} T(u_{n-r})$$

Consider, $T(u_1) = v_1, T(u_2) = v_2, \ldots, T(u_{n-r}) = v_{n-r}$

$$T(x) = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{n-r} v_{n-r}$$

We take any vector $x \in V$ then $T(x)$ must be in the form $\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{n-r} v_{n-r}$. But all the vectors in $Im(T)$ must be in the form $T(x)$ $\exists x \in V$. Therefore, all the vectors in $Im(T)$ must be in the form.

$$Tx = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{n-\nu} \cdot v_{n-\nu}$$

$$Im(T) = \{ y \in W \mid y = Tx = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{n-\nu} v_{n-\nu} \text{ where } v_i = T(u_i) \text{ and } v_i \in W \}$$

Suppose, $S = \{ v_1, v_2, \dots, v_{n-\nu} \}$

Since every vector in the $Im(T)$ is the LC of vectors in S therefore, S is the spanning set of $Im(T)$.

If $\{ \varphi_1, \varphi_2, \dots, \varphi_\nu \}$ is a basis for $Ker(T)$ and $\{ \varphi_1, \varphi_2, \dots, \varphi_\nu, u_1, u_2, \dots, u_{n-\nu} \}$ is an extension to a basis for V then $\{ T(u_1), T(u_2), \dots, T(u_{n-\nu}) \}$ is the spanning set of $Im(T)$.

We check if S is LI set?

$\Rightarrow$

$$(S \text{ is LI}) \text{ iff } (a_1 v_1 + a_2 v_2 + \cdots + a_{n-\nu} v_{n-\nu} = 0_W \rightarrow \forall a_i = 0)$$

$$a_1 v_1 + a_2 v_2 + \cdots + a_{n-\nu} \cdot v_{n-\nu} = 0_W.$$

$$\Rightarrow a_1 T(u_1) + a_2 T(u_2) + \cdots + a_{n-v} \cdot T(u_{n-v}) = 0_{W}$$

$$\Rightarrow T(a_1 u_1) + T(a_2 u_2) + \cdots + T(a_{n-v} \cdot u_{n-v}) = 0_{W}$$

$$\Rightarrow T(a_1 u_1 + a_2 u_2 + \cdots + a_{n-v} \cdot u_{n-v}) = 0_{W}.$$

$$\Rightarrow T(w) = 0_{W}.$$

Where, $\quad w = a_1 u_1 + a_2 u_2 + \cdots + a_{n-v} \cdot u_{n-v}. \quad \text{———} (i)$

$\forall u_i \in V$ and $w \in V$.

We know that any vector $\left(x \in V \longrightarrow T(x) = 0_{W}\right)$ if and only if $x \in \ker(T)$.

Since $w \in \ker(T)$ so w can be written in terms of LC of $\{\varphi_1, \varphi_2, \ldots, \varphi_v\}$.

$w = b_1 \varphi_1 + b_2 \varphi_2 + \cdots + b_v \varphi_v. \quad \text{———} (ii)$

Therefore, these (i) & (ii) must be equal.

$$a_1 u_1 + a_2 u_2 + \cdots + a_{n-v} \cdot u_{n-v} = b_1 \varphi_1 + b_2 \varphi_2 + \cdots + b_v \cdot \varphi_v$$

$$\Rightarrow a_1 u_1 + a_2 u_2 + \cdots + a_{n-v} \cdot u_{n-v} + (-b_1)\varphi_1 + (-b_2)\varphi_2 + \cdots + (-b_v) \cdot \varphi_v = 0_{V}.$$

Recall that $\{u_1, u_2, \dots, u_{n-\nu}, \varphi_1, \varphi_2, \dots, \varphi_\nu\}$ is the basis for $\mathbb{V}$.

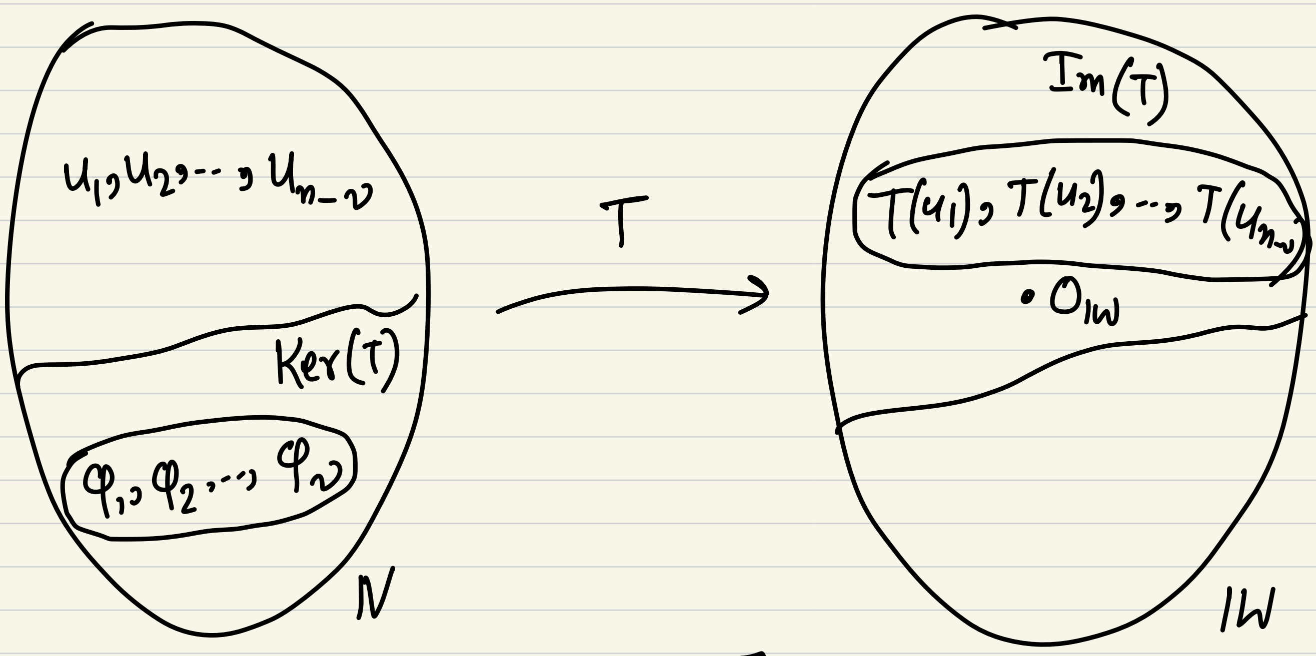
The LC of $B_{\mathbb{V}}$ gives rise to $0_{\mathbb{V}}$ implies $\forall a_i = 0$, $\forall b_i = 0$ if and only if B_V is LI set.

Therefore, $\forall a_j = 0$, $\forall b_i = 0$

Therefore, $S = \{v_1, v_2, \dots, v_{n-\nu}\}$ is a LI set.

Hence S is a basis for $Im(T)$.

conclusion:



$$\left.\begin{aligned} \dim(Im(T)) &= n-\nu = \rho \\ \dim(Ker(T)) &= \nu \end{aligned}\right] \quad \boxed{\rho + \nu = n}$$

Rank-nullity theorem.

$\text{rank}(T) = \rho \leq \dim(\mathbb{W})$, $\rho \geq 0$

$\text{nullity}(T) = \nu \leq \dim(\mathbb{V})$, $\nu \geq 0$

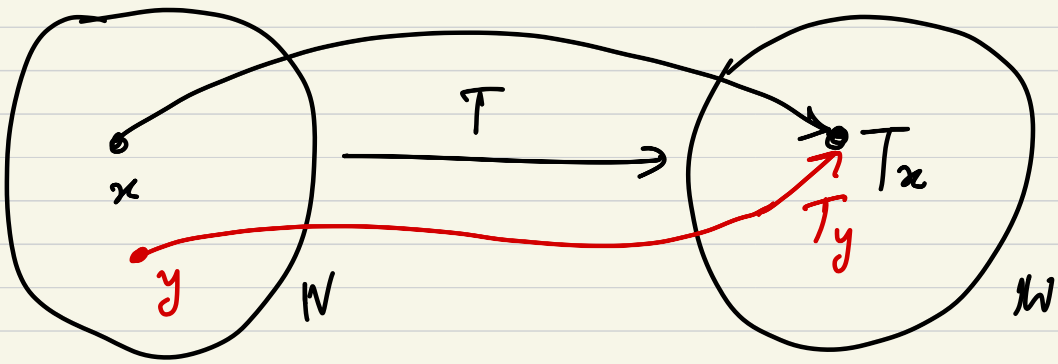
Now from rank nullity theorem : $\rho + \nu \leq \dim(\mathbb{V})$

If $\nu = 0$ then $\rho \leq \dim(\mathbb{V})$

If $\nu > 0$ then $\rho \leq \dim(\mathbb{W})$

Hence $\boxed{\rho \leq \min\left(\dim(\mathbb{V}), \dim(\mathbb{W})\right)}$

One to One Linear transformation



T can be thought of as a coding of vectors in $\mathbb{V}$ as vectors in $\mathbb{W}$. But is it a good coding?
Consider another vector y which also maps to Ty. Now at the time of decoding when we will go back from Ty to y then we have a confusion because both x & y are image of Ty. To avoid the confusion, we will impose an additional property that if $x \neq y$ then $Tx \neq Ty$.

In order to have different vectors in $\mathbb{V}$ have different vectors in $\mathbb{W}$ (coded version) we need,

$$Tx = Ty \longrightarrow x = y.$$

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$. Let T be a linear transformation from $\mathbb{V}$ to $\mathbb{W}$ is said to be one to one / injective / monomorphism / 1:1 iff

$$T(x) = T(y) \longrightarrow x = y$$

Suppose T is one to one linear transformation. $T: \mathbb{V} \to \mathbb{W}$.

$x \in \operatorname{Ker}(T) \longrightarrow T(x) = 0_{\mathbb{W}}$

On the other hand, $T(0_{\mathbb{V}}) = 0_{\mathbb{W}}$.

Therefore, $T(x) = T(0_{\mathbb{V}})$.

Since T is 1:1 then by definition of 1:1 mapping,

$T(x) = T(0_{\mathbb{V}}) \longrightarrow x = 0_{\mathbb{V}}$.

That's why $x = 0_{\mathbb{V}}$ so the only vector that stays in the $\operatorname{Ker}(T)$ is the zero vector $0_{\mathbb{V}}$.

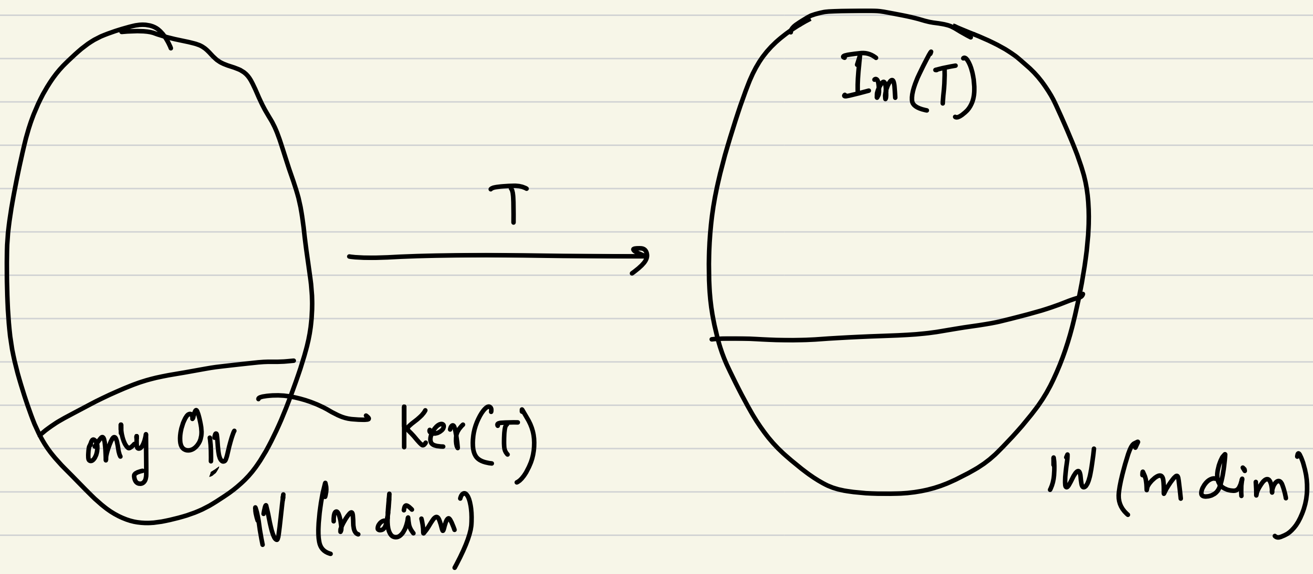
So $\operatorname{Ker}(T) = \{0_{\mathbb{V}}\}$.

T is one to one $\longleftrightarrow$ $\operatorname{Ker}(T) = \{0_{\mathbb{V}}\}$, $\operatorname{nullity}(T) = 0$

If T is one to one then $\text{nullity}(T) = 0$

$$\text{rank}(T) = \dim(V) = n.$$

(By rank nullity theorem).



$$\dim(\text{Im}(T)) = \text{rank}(T) \leq n$$
$$\text{rank}(T) \leq m$$

When T is 1:1 then $\text{rank}(T) = n \leq m$

Therefore, $\dim(W)$ can't be more than $\dim(V)$.

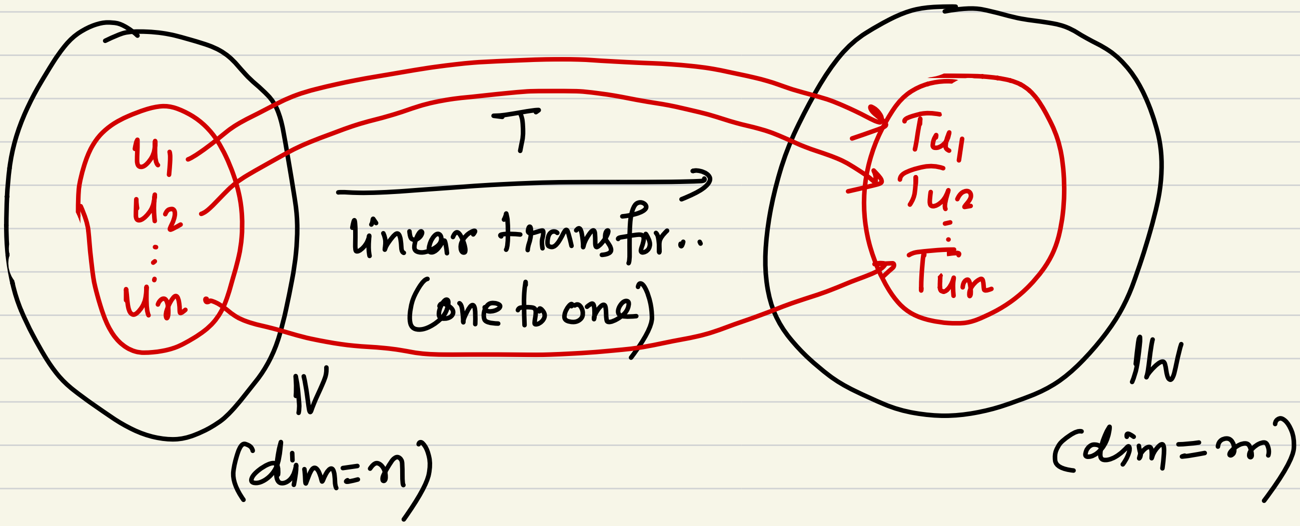
$$\begin{bmatrix} & T & \end{bmatrix}_{m \times n} \begin{bmatrix} \\ v \\ \\ \end{bmatrix}_{n \times 1} = \begin{bmatrix} w \end{bmatrix}_{m \times 1}$$

Not possible

$$\begin{bmatrix} \\ & T & \\ \\ \end{bmatrix}_{m \times n} \begin{bmatrix} v \end{bmatrix}_{n \times 1} = \begin{bmatrix} \\ w \\ \\ \end{bmatrix}_{m \times 1}$$

Has to be the case.

We can't have 1:1 linear transformation from a vector space $\mathbb{V}$ to a lower dimensional vector space



Definitely $n \leq m$

If $\{u_1, u_2, \ldots, u_r\}$ is any LI set in $\mathbb{V}$ then can we conclude that $\{Tu_1, Tu_2, \ldots, Tu_r\}$ is LI set in $\mathbb{W}$?

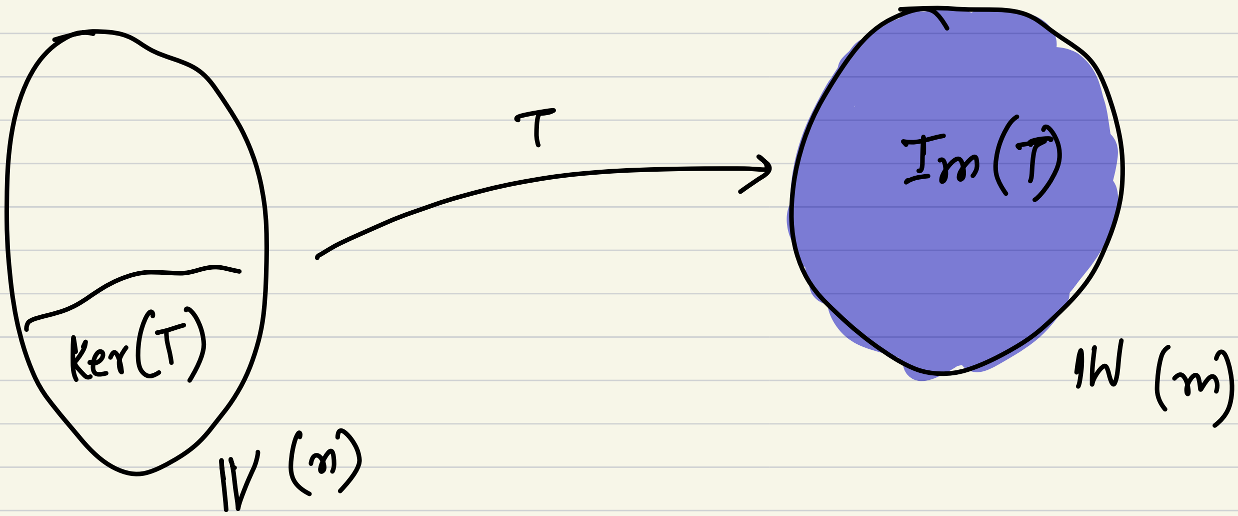
$\Rightarrow$ Yes because of one to one property.

Onto linear transformation:

$\mathbb{V}, \mathbb{W}$ are f.d.v.s over $\mathbb{F}$.

$T: \mathbb{V} \longrightarrow \mathbb{W}$ is linear transformation.

If the im(T) fills up the entire space $\mathbb{W}$ or if all the elements of $\mathbb{W}$ has preimage in $\mathbb{V}$ then it is onto.



$\forall w \in \mathbb{W}, \exists v \in \mathbb{V}$ s.t. $T(v) = w$ iff T is onto.
$Im(T) = \mathbb{W}$ iff T is onto.

$$\dim(Im(T)) = \dim(\mathbb{W}) = \text{rank}(T) = m$$

From rank nullity theorem: $\text{rank}(T) + \text{nullity}(T) = n$

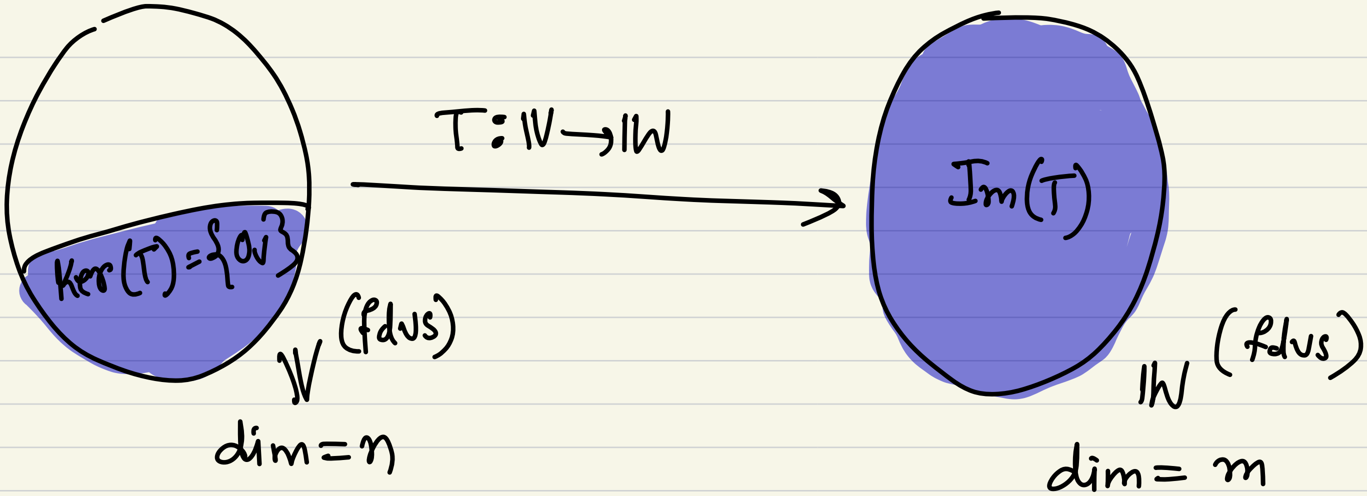
Since $\text{rank}(T) = m$ then $\text{rank}(T) \leq n$ or $m \leq n$.

$$\begin{bmatrix} & T & \end{bmatrix}_{m\times n} \begin{bmatrix} \\ v \\ \\ \end{bmatrix}_{n\times 1} = \begin{bmatrix} w \end{bmatrix}_{m\times 1}$$ (this has to be the case)

An apriori requirement to have onto transformation is that $\dim(\mathbb{W}) \leq \dim(\mathbb{V})$.

Bijective transformation:

A transformation is bijective if it is both injective and surjective



T is onto $\longleftrightarrow$ $Im(T) = \mathbb{W}$, $m \leq n$

T is 1:1 $\longleftrightarrow$ $Ker(T) = \{0_V\}$, $m \geq n$

T is bijective $\longleftrightarrow$ $Im(T) = \mathbb{W}$ and $Ker(T) = \{0_V\}$, $m = n$.

Let $\mathbb{V}$ and $\mathbb{W}$ be f.d.vs over $\mathbb{F}$ such that $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation is called an isomorphism of $\mathbb{V}$ onto $\mathbb{W}$ if T is both one to one and onto.

One to one	≡	Monomorphism	≡	Injective
Onto	≡	Epimorphism	≡	Surjective
One to one correspondence	≡	Isomorphism	≡	Bijective
Linear operator	≡	Endomorphism		

If such a T or isomorphism exists then $\mathbb{V}$ is called isomorphic to $\mathbb{W}$.

Suppose $\mathbb{V}$ is isomorphic to $\mathbb{W}$. This means

$\exists\, T : \mathbb{V} \longrightarrow \mathbb{W}$ s.t. T is one to one & onto.

This to exist, the necessary condition is $\dim(\mathbb{V}) = \dim(\mathbb{W})$

V is isomorphic to $\mathbb{W} \longrightarrow \dim(\mathbb{V}) = \dim(\mathbb{W})$.

But is the converse statement true?

Let's start from $\dim(\mathbb{V}) = \dim(W)$.

T is a linear transformation : $T : \mathbb{V} \to \mathbb{W}$.

$$
\begin{aligned}
T \text{ is onto } &\leftrightarrow \operatorname{Im}(T) = \mathbb{W} \\
&\leftrightarrow \operatorname{Rank}(T) = \dim(\mathbb{W}) = \dim(\mathbb{V}) \\
&\leftrightarrow \operatorname{nullity}(T) = 0 \quad (\text{by Rank nullity Thm}) \\
&\leftrightarrow \operatorname{Ker}(T) = \{0_v\} \\
&\leftrightarrow T \text{ is one to one.}
\end{aligned}
$$

If we have 2 vector spaces of equal dimension then the moment T is onto, T becomes one to one and vice-e-versa. Hence To check if T is an isomorphism from $\mathbb{V}$ to $\mathbb{W}$, it is enough to check if T is onto or T is one to one (of course $\dim(\mathbb{V}) = \dim(\mathbb{W})$).

T is isomorphism $\longrightarrow \dim(\mathbb{V}) = \dim(\mathbb{W})$

$\dim(\mathbb{V}) = \dim(\mathbb{W}) \xrightarrow{?} T$ is isomorphism. ?

We have already seen that

$\dim(\mathbb{V}) = \dim(\mathbb{W}) \longrightarrow$ (T is onto $\leftrightarrow$ T is 1:1)

When $\dim(\mathbb{V}) = \dim(\mathbb{W})$, we have to check if T is either onto or 1:1 because checking any 1 property is enough to declare T is isomorphism. We don't need to check both the property.

$\dim(\mathbb{V}) = \dim(\mathbb{W}) \longrightarrow$ is T onto?

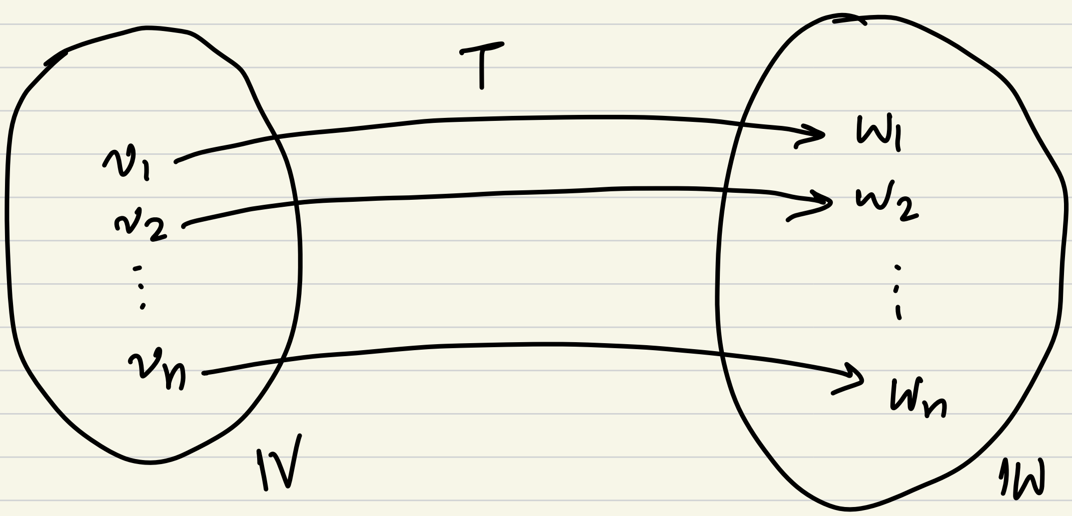
or

is T one to one?

suppose $\dim(\mathbb{V}) = \dim(\mathbb{W}) = n$

let $B_V = (v_1, v_2, v_3, ---, v_n)$ (ordered basis)

let $B_W = (w_1, w_2, w_3, ---, w_n)$ (ordered basis)



let's generate the transformation from $\mathbb{V}$ to $\mathbb{W}$. A linear transformation is completely determined by its action on the basis vectors. First let's see what T does on the basis vectors $v_1, v_2, \ldots, v_n$ and then decide what it should do for other vectors of $\mathbb{V}$?

First we define T as $T(v_i) = w_i \;\; \forall i$

Suppose $x \in \mathbb{V}$. then x can be written in terms of basis vector $\mathbb{B}_V$.

$$x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \quad \text{(unique)}$$

$$T(x) = x_1 T(v_1) + x_2 T(v_2) + \cdots + x_n T(v_n)$$

(We want T to be linear transformation).

$$T(x) = x_1 \cdot w_1 + x_2 \cdot w_2 + \cdots + x_n \cdot w_n$$

So we have definition of T as $T(v_j) = w_j \quad j = 1, 2, \ldots, n$

$x \in \mathbb{V}, \quad x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$.

$T(x) = x_1 w_1 + x_2 w_2 + \cdots + x_n w_n$

claim: T is an isomorphism.

Show that: T is one to one or onto

$w_1 \in \operatorname{Im}(T), \; w_2 \in \operatorname{Im}(T), \; \ldots, \; w_n \in \operatorname{Im}(T)$

$\{w_1, w_2, \ldots, w_n\}$ are LI set of vectors.

$\dim(\mathrm{Im}(T))$ is at least n.

$\mathrm{Im}(T) \subseteq \mathbb{W}$ so $\dim(\mathrm{Im}(T)) \leq n$

So comparing these two, $\dim(\mathrm{Im}(T)) = n$.

$\Rightarrow \mathrm{Im}(T) = \mathbb{W}$. so T is onto so T is isomorphism.

Any 2 vector spaces over $\mathbb{F}$ having same dimension, are isomorphic and every pair of basis B_V for $\mathbb{V}$ and B_W for $\mathbb{W}$ leads to an isomorphism of $\mathbb{V}$ onto $\mathbb{W}$.

Thus the only "meaningful" n-dim vector space over $\mathbb{F}$ is $\mathbb{F}^n$ as any other vector space can be translated to $\mathbb{F}^n$ spoken in another language.