INNER PRODUCT

Inner Products:

In inner products, we will restrict the field $\mathbb{F}$ to only the real $\mathbb{R}$ or the complex $\mathbb{C}$ numbers. The main objective is to study vector spaces in which it makes sense to speak "length" of vector or "angle" b/w 2 vectors.

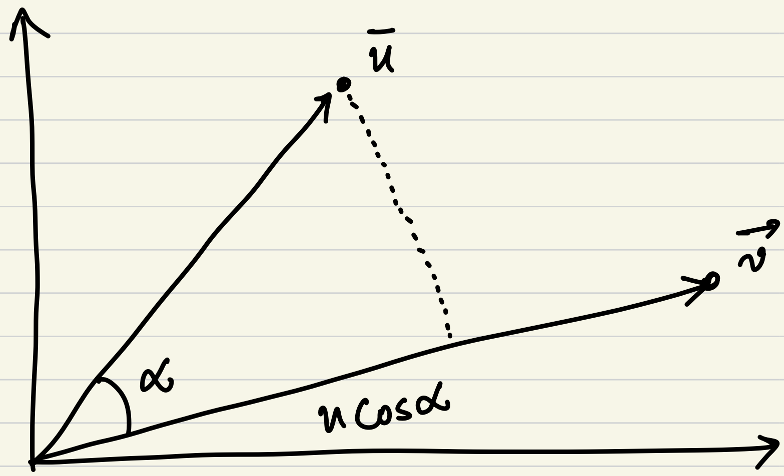
If the field is say $\mathbb{Z}_2$ then $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$
so we can't have the concept of "length".
That's the reason we restrict the $\mathbb{F}$ as $\mathbb{R}$ or $\mathbb{C}$.

We talk about the "length" or "angle" in the vector spaces by studying a certain type of scalar valued function defined on pairs of vector known as the inner product.

One example of inner product is the dot products of vectors in $\mathbb{R}^3$.

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad u = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{in } \mathbb{R}^3.$$

$$\langle v | u \rangle := x_1 y_1 + x_2 y_2 + x_3 y_3 \in \mathbb{R}.$$



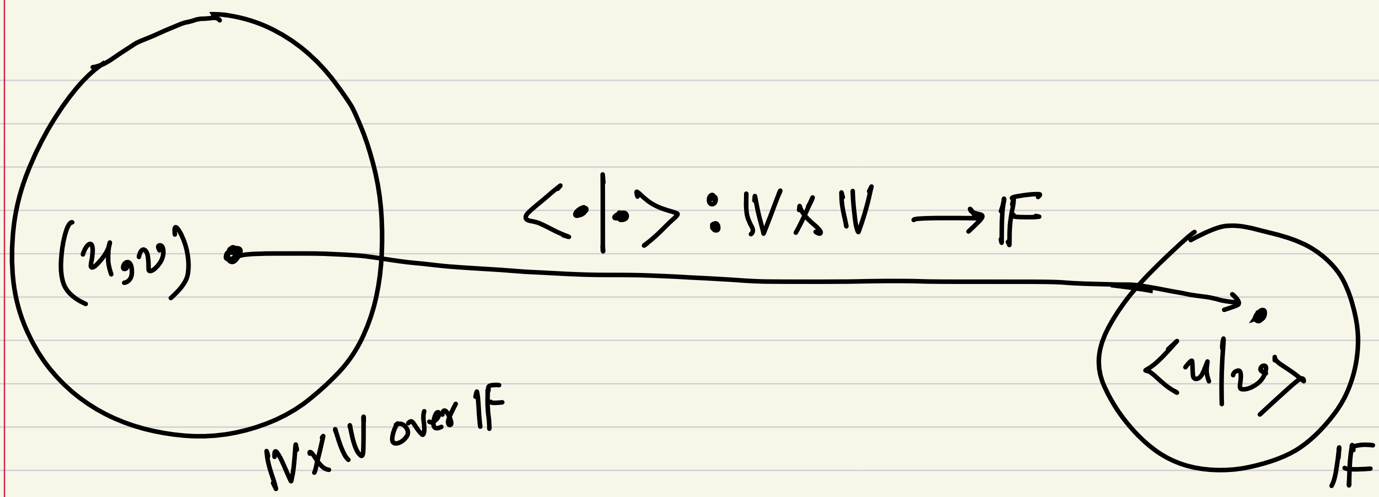
Geometrically, $\langle v | u \rangle = \|u\cos\alpha\| \cdot \|v\|$

$$= \|u\| \, \|v\| \cos\alpha$$

where $\|u\|$, $\|v\|$ are the length of the vectors and α is the angle between u and v. It is therefore possible to define the geometric concepts of "length", "angle" in $\mathbb{R}^3$ by means of the algebriaeally defined "inner product".

An inner product on a vector space is a function with properties similar to dot product in $\mathbb{R}^3$.

<u>Definition</u>: Let $\mathbb{F}$ be the field of $\mathbb{R}$ or $\mathbb{C}$ and $\mathbb{V}$ be a vector space over $\mathbb{F}$. An inner product on $\mathbb{V}$ is a function which assigns to each ordered pair of vectors (u, v) in $\mathbb{V}$ a scalar in $\mathbb{F}$ in such a way that some conditions are satisfied.



$\langle \cdot | \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ such that the following conditions are satisfied :

(i) $\forall_{u,v,w \in \mathbb{V}} \left[\langle u +_{\mathbb{V}} v \,|\, w \rangle = \langle u | w \rangle +_{\mathbb{F}} \langle v | w \rangle \right]$

(ii) $\forall_{u,v \in \mathbb{V},\, c \in \mathbb{F}} \left[\langle c \cdot_{\mathbb{V}} u \,|\, v \rangle = c \cdot_{\mathbb{F}} \langle u | v \rangle \right]$

(iii) $\forall_{u,v \in \mathbb{V}} \left[\langle u | v \rangle = \overline{\langle v | u \rangle} \right]$

(iv) $\forall_{u \in \mathbb{V}} \left[\langle u | u \rangle > 0 \text{ if } u \neq 0_{\mathbb{V}} \right]$.

if any function $\langle \cdot | \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ such that the above 4 conditions are satisfied will be called "inner product" on $\mathbb{V}$.

Inner product != dot product :

The dot product is only defined for $\mathbb{R}^n$ over $\mathbb{R}$.

For 2 vectors u, v, the dot product is defined as the function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ such that -

$$u \cdot v := \sum_{i=1}^{n} u_i v_i$$ where u_i, v_i are coordinates.

The dot product $u \cdot v$ has the following properties -

(i) $$x \cdot x = \sum_{j=1}^{n} x_j^2 \geq 0$$

Since x_j^2 is always ≥ 0 therefore $x \cdot x \geq 0 \; \forall x \in \mathbb{R}^n$

$x \cdot x = 0$ if and only if $x = 0_{\mathbb{R}^n}$

Since x_j are real number so sum of real numbers are 0 if and only if all of them are individually 0.

So positive definiteness: $x \cdot x > 0 \quad (x \neq 0_{\mathbb{R}^n})$.

(ii). $$x \cdot y = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = y \cdot x$$

Note that dot product is symmetric but the inner product is not symmetric.

(iii) The moment we define a function on the vector space, we will be interested to know the effect of that function on the 2 basic operation of vector space, + and.

So we take $x+y$ and perform dot product with z. and check what is the result.

$$(x+y)\cdot z = \sum_{j=1}^{n} (x_j+y_j)\cdot z_j = \sum_{j=1}^{n} x_j z_j + y_j z_j$$

$$= \sum_{j=1}^{n} x_j z_j + \sum_{j=1}^{n} y_j z_j = x\cdot z + y\cdot z$$

so dot product is distributive from right.

(iv). $$(cx)\cdot y = \sum_{j=1}^{n} (cx_i)\cdot y_i = c\cdot \sum_{j=1}^{n} x_i y_i$$

$$= c\cdot(x\cdot y)$$

Note that for a fixed $y \in \mathbb{R}^n$, the map that sends $x \in \mathbb{R}^n$ to $x\cdot y$ in $\mathbb{R}$ is a linear map because,

$$(c\cdot x_1 + x_2)\cdot y = c\,(x_1\cdot y) + (x_2\cdot y)$$

where $x_1, x_2 \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

For the real vector space, inner product is same as dot product but for complex vector space, they are not same since dot product is not defined in complex.

Look at $x \cdot x = \sum_{i=1}^{n} x_i^2$ (gives the idea of length)

$$= \|x\|^2 .$$

when the vector becomes complex then,

$$\|z\|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$$

where $\underbrace{|z_i|^2}_{\text{absolute value of complex number } z_i} = z_i \cdot \overline{z_i} \quad (z_i \in \mathbb{C})$

$$\Rightarrow \|z\|^2 = z_1 \overline{z_1} + z_2 \overline{z_2} + \cdots + z_n \overline{z_n} = \langle z | z \rangle$$

This gives a hint that if we want to define the inner product for complex vector space then we need the complex conjugate.

$$\langle w | z \rangle = w_1 \overline{z_1} + w_2 \overline{z_2} + \cdots + w_n \overline{z_n}$$

Now if we want to define,

$$\langle z | w \rangle = z_1 \overline{w_1} + z_2 \overline{w_2} + \cdots + z_n \overline{w_n}$$

Note the above two are not same but.

$$\langle w | z \rangle = \overline{\overline{w_1 \overline{z_1} + \cdots + w_n \overline{z_n}}} = \overline{\overline{w_1} z_1 + \cdots + \overline{w_n} z_n} = \overline{\langle z | w \rangle}$$

That's the reason, a valid inner product must always satisfy the conjugate symmetry property. This is where the difference between dot product & inner product is.

In the inner product, we have seen that it is linear in the first argument that is -

1. $\langle u+v | w \rangle = \langle u|w \rangle + \langle v|w \rangle$
2. $\langle cu | w \rangle = c.\langle u|w \rangle$

We desire to check if the inner product is also linear in the second argument as well.

$$\langle u | v+w \rangle \stackrel{?}{=} \langle u|v \rangle + \langle u|w \rangle$$

We know that for an inner product, the conjugate symmetry & first argument linearity holds.

$$\langle u|v+w \rangle = \overline{\langle v+w|u \rangle}$$

$$= \overline{\langle v|u \rangle + \langle w|u \rangle}$$

$$= \overline{\langle v|u \rangle} + \overline{\langle w|u \rangle}$$

$$= \langle u|v \rangle + \langle u|w \rangle \quad \text{(it's okay)}$$

But we also have to check scalar multiplication.

$$\langle u | c \cdot v \rangle \stackrel{?}{=} c \cdot \langle u | v \rangle$$

$$= \overline{\langle c \cdot v | u \rangle}$$

$$= \overline{c \cdot \langle v | u \rangle}$$

$$= \bar{c} \cdot \overline{\langle v | u \rangle}$$

$$= \bar{c} \cdot \langle u | v \rangle$$

Note that because of $\bar{c} \neq c$, we get that inner product is not linear in 2^{nd} argument. Although it is satisfying the addition linearity in 2^{nd} argument but it doesn't satisfy the scalar mult linearity in 2^{nd} argument.

If $\mathbb{F} = \mathbb{R}$ then inner product is linear in both 1^{st} & 2^{nd} argument because $\bar{c} = c$.

In general,

$$\boxed{\begin{aligned} \langle \alpha \cdot u + v | w \rangle &= \alpha \cdot \langle u | w \rangle + \langle v | w \rangle \\ \langle w | \alpha u + v \rangle &= \bar{\alpha} \cdot \langle w | u \rangle + \langle w | v \rangle \end{aligned}}$$

↳ properties of inner product.

Suppose $\mathbb{C}^n$ over $\mathbb{C}$ as the vector space.

let's define the inner product as $\langle v_1 | v_2 \rangle = v_1^T \overline{v_2}$

Without the complex conjugate in the v_2, we will have contradiction as follows:

$\langle v_1 | v_1 \rangle > 0$ to be a valid inner product.

Say $v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ so, $\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = 1 + 1 = 2 > 0$ ✓

But if we don't add the $\overline{v_2}$ then $\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \not> 0$ ✗

So it fails to be a valid inner product.

Note: In $\mathbb{C}^n$, the definition of the following inner product is just one of the choice of such a function from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}$ defined as:

$$\langle u | v \rangle := \sum_{i=1}^{n} u_i \overline{v_i} = u^T \overline{v}$$

This definition of inner product is a valid inner product as it satisfies the linearity in 1st argument, complex conjugate symmetry and positive definiteness. It is not the only definition of inner product in $\mathbb{C}^n$.

On $\mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), there is an inner product which we shall call standard inner product. It is defined as

$$\langle u|v\rangle := \sum_{i=1}^{n} u_i \overline{v_i} = u^T \overline{v}$$

When $\mathbb{F} = \mathbb{R}$ then it is called dot product. So the dot product is just one type of inner product.

Ex1: Suppose $\mathbb{R}^2$ over $\mathbb{R}$.

$$\langle u|v\rangle := u_1v_1 - u_2v_1 - u_1v_2 + 4u_2v_2$$

prove that it is an inner product on $\mathbb{R}^2$.

(i) $$\langle u|u\rangle = u_1^2 - u_2u_1 - u_1u_2 + 4u_2^2$$
$$= u_1^2 - 2u_1u_2 + u_2^2 + 3u_2^2$$
$$= (u_1 - u_2)^2 + 3u_2^2 \geq 0$$

(ii) $$\overline{\langle u|v\rangle} = \overline{u_1v_1 - u_2v_1 - u_1v_2 + 4u_2v_2}$$
$$= \overline{u_1}\,\overline{v_1} - \overline{u_2}\,\overline{v_1} - \overline{u_1}\,\overline{v_2} + 4\overline{u_2}\,\overline{v_2}$$
$$= u_1v_1 - u_2v_1 - u_1v_2 + 4u_2v_2 \quad (\mathbb{F} = \mathbb{R})$$
$$= \langle u|v\rangle$$

(iii) $\langle \alpha . u + v \mid w \rangle$

$= (\alpha u_1 + v_1) . w_1 - (\alpha u_2 + v_2) . w_1 - (\alpha u_1 + v_1) . w_2 + 4 (\alpha u_2 + v_2) . w_2$

$= \alpha (u_1 w_1 - u_2 w_1 - u_1 w_2 + 4 u_2 w_2) + (v_1 w_1 - v_2 w_1 - v_1 w_2 + 4 v_2 w_2)$

$= \alpha . \langle u \mid w \rangle + \langle v \mid w \rangle$

Hence it is a valid inner product.

Ex 2: In $\mathbb{R}^n$, $\langle u \mid v \rangle = \sum_{i=1}^{n} c_i u_i v_i$ is also a valid inner product. (for all $c_i \geq 0$)

In $\mathbb{R}^n$, $\langle u \mid v \rangle = u^T A v$ is also a valid inner product for A being positive definite matrix.

Ex3: $\mathbb{V} = \mathbb{F}^{n \times n}$ over $\mathbb{F}$

Since $\mathbb{V}$ is isomorphic to $\mathbb{F}^{n^2}$ in a natural way we expect,

$$\langle A \mid B \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} . \overline{B_{ij}}$$ to be an inner product.

If we take conjugate transpose, $B^*_{ij} = \overline{B_{ji}}$ then,

$\langle A|B \rangle = \text{trace}(AB^*) = \text{trace}(BA^*)$ is also an inner product.

$$\text{trace}(AB^*) = \sum_{i=1}^{n} (AB^*)_{ii}$$

$$= \sum_{i=1}^{n} \sum_{K=1}^{n} A_{iK} \cdot B^*_{Ki}$$

$$= \sum_{i=1}^{n} \sum_{K=1}^{n} A_{iK} \overline{B}_{iK}$$

Suppose $\mathbb{F}^{n \times 1}$ is space of $n \times 1$ matrices.

Let Q be an invertible $n \times n$ matrix.

For $X, Y \in \mathbb{F}^{n \times 1}$ we have,

$\langle X|Y \rangle = Y^* Q^* Q X$ is an inner product.

If $Q = I$ then it becomes the standard inner product

For a general finite dimensional vector spaces, there is no obvious inner product which is standard.

Ex 4: Let $\mathbb{V}$ be a vector space of all continuous valued functions on the unit interval $0 \leq t \leq 1$.

$$\langle f|g\rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt$$

It is a valid inner product on the vector space $\mathbb{V}$.

Norm of a vector :

Suppose $\mathbb{V}$ is a vector space over $\mathbb{F}$.

$\langle u|v\rangle$ is an inner product defined on $\mathbb{V}$. It can be either a real or complex scalar.

$$\langle u|v\rangle = \mathrm{Re}(\langle u|v\rangle) + i \cdot \mathrm{Im}(\langle u|v\rangle)$$

If z is a complex number then $\mathrm{Im}(z) = \mathrm{Re}(-iz)$

$$\Rightarrow \langle u|v\rangle = \mathrm{Re}(\langle u|v\rangle) + i\,\mathrm{Re}(-i\langle u|v\rangle)$$

$$= \mathrm{Re}(\langle u|v\rangle) + i\,\mathrm{Re}(\bar{i}\langle u|v\rangle)$$

$$= \mathrm{Re}(\langle u|v\rangle) + i\,\mathrm{Re}(\langle u|iv\rangle)$$

Note that the inner product $\langle u|v\rangle$ is completely determined by the real part of other inner products.

The inner product may be determined by another function called quadratic form.

We introduce a new function, $\|\cdot\| : \mathbb{V} \to \mathbb{F}$ such that. for all $v \in \mathbb{V}$ we define,

$$\|v\| := \sqrt{\langle v | v \rangle}$$

The positive squareroot of self inner product is called norm of the vector v with respect to that particular definition of inner product.

Note if we plug out different choices of inner product then we get different types of norms for the same vector. So norm is always inner product specific.

When the $\langle \cdot | \cdot \rangle$ is standard inner product on $\mathbb{R}^n$ then norm can be thought of "length" of the vector or magnitude of the vector.

For 2 vectors $u, v \in \mathbb{V}$,

$$\|u+v\|^2 = \langle (u+v) | (u+v) \rangle$$

$$= \langle u | u+v \rangle + \langle v | u+v \rangle$$

$$= \langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle v|v\rangle$$

$$= \|u\|^2 + \|v\|^2 + \langle u|v\rangle + \overline{\langle u|v\rangle}$$

$$= \|u\|^2 + \|v\|^2 + 2 \cdot \mathrm{Re}\left(\langle u|v\rangle\right)$$

Inner Product Spaces:

What can we say about the combination of vectors in a vector space when some particular inner product is defined on it?

An inner product space (ips) is a real or complex vector space together with a specified inner product defined on that space.

A finite dimensional real ips is called "Euclidean space"
A complex ips is called "unitary space".

If V is an inner product space then the following properties are true.

(i) $\|c \cdot u\| = |c| \cdot \|u\|$

(ii) $\|u\| > 0$ for $u \neq 0_V$

proof:

$$\|c.u\|^2 = \langle (c.u) | (c.u) \rangle$$

$$= c. \langle u | c.u \rangle \quad (\text{linear in } 1^{st} \text{ arg.})$$

$$= c.\bar{c} \langle u | u \rangle \quad (\text{Conjugate linear in } 2^{nd} \text{ arg})$$

Suppose, $c = a+ib$, $\bar{c} = a-ib$

$$c.\bar{c} = (a+ib)(a-ib) = a^2 - (ib)^2 = a^2+b^2 = |c|^2$$

$$\Rightarrow \|c.u\|^2 = |c|^2 . \|u\|^2$$

$$\Rightarrow \|c.u\| = |c| . \|u\| .$$

(iii) Cauchy - Swartz inequality:

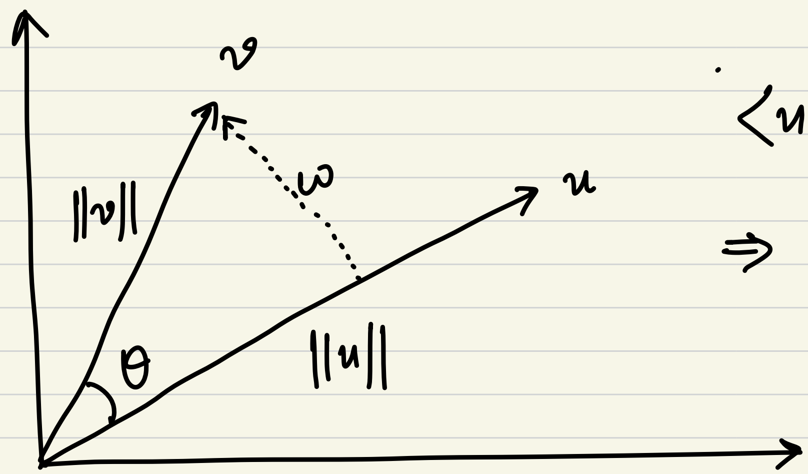
$$|\langle u|v \rangle| \leq \|u\| . \|v\|$$

Suppose $u = 0_V$ then clearly, it is valid.

Consider $u \neq 0_V$.

look at the vector -

$$w = v - \frac{\langle v|u \rangle}{\|u\|^2} . u$$



$$\langle u|v\rangle = \|u\|\cdot\|v\| \cos\theta$$

$$\Rightarrow |\langle u|v\rangle| \leqslant \|u\|\cdot\|v\|$$

Geometric interpretation

Note that, $\langle w|u\rangle = \left\langle v - \frac{\langle v|u\rangle}{\|u\|^2}\cdot u \,\middle|\, u \right\rangle$

$$= \langle v|u\rangle - \frac{\langle v|u\rangle}{\|u\|^2}\langle u|u\rangle$$

$$= \langle v|u\rangle - \frac{\langle v|u\rangle}{\|u\|^2}\cdot\|u\|^2$$

$$= \langle v|u\rangle - \langle v|u\rangle$$

$$= 0$$

Consider $\|w\|^2 = \langle w|w\rangle$

$$= \left\langle v - \frac{\langle v|u\rangle}{\|u\|^2}\cdot u \,\middle|\, v - \frac{\langle v|u\rangle}{\|u\|^2}\cdot u \right\rangle$$

$$= \langle v|v\rangle + \left\langle v \,\middle|\, -\frac{\langle v|u\rangle}{\|u\|^2}\cdot u \right\rangle +$$

$$\left\langle -\frac{\langle v|u\rangle}{\|u\|^2}\cdot u \,\middle|\, v \right\rangle + \left\langle -\frac{\langle v|u\rangle}{\|u\|^2}\cdot u \,\middle|\, -\frac{\langle v|u\rangle}{\|u\|^2}\cdot u \right\rangle$$

$$= \langle v|v\rangle + \overline{-\frac{\langle v|u\rangle}{\|u\|^2}}\cdot \langle v|u\rangle +$$

$$-\frac{\langle v|u\rangle}{\|u\|^2}\cdot \overset{\overline{\langle v|u\rangle}}{\cancel{\langle u|v\rangle}} + \left(-\frac{\langle v|u\rangle}{\|u\|^2}\right)\cdot\overline{\left(-\frac{\langle v|u\rangle}{\|u\|^2}\right)}\cdot \overset{\|u\|^2}{\cancel{\langle u|u\rangle}}$$

$z = a+ib \quad , \quad \bar{z} = a-ib$

$-z = -a-ib \, , \quad \overline{-z} = -a+ib$

$$\overline{-z}\cdot z = (-a+ib)(a+ib) = (ib)^2 - a^2 = -(a^2+b^2)$$

$$= -|z|^2$$

$$-z\cdot\overline{-z} = (-a-ib)(-a+ib) = (-a)^2 - (ib)^2 = |z|^2$$

$$\Rightarrow \|v\|^2 - 2\frac{|\langle v|u\rangle|^2}{\|u\|^2} + \frac{|\langle v|u\rangle|^2}{\|u\|^2}$$

$$\Rightarrow \|v\|^2 - \frac{|\langle v|u\rangle|^2}{\|u\|^2}$$

We know that $\|w\|^2 \geqslant 0$

$$\Rightarrow \|v\|^2 - \frac{|\langle v|u\rangle|^2}{\|u\|^2} \geqslant 0$$

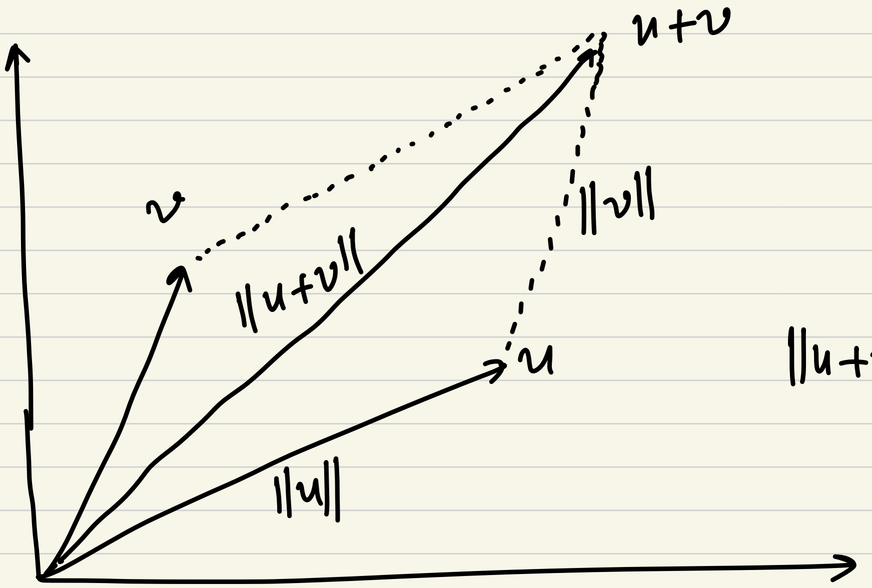
$$\Rightarrow \|v\|^2 \|u\|^2 - |\langle v|u\rangle|^2 \geqslant 0$$

$$\Rightarrow \|v\|^2 \|u\|^2 \geqslant |\langle v|u\rangle|^2$$

$$\Rightarrow |\langle v|u\rangle| \leqslant \|v\| \, \|u\|$$ (Hence proved)

(iv) Triangular inequality:

$$\|u+v\| \leqslant \|u\| + \|v\|$$



$\|u+v\| \leqslant \|u\| + \|v\|$

Geometric interpretation

$$\|u+v\|^2 = \langle (u+v) | (u+v) \rangle$$

$$= \langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle v|v\rangle$$

$$= \|u\|^2 + \|v\|^2 + \langle u|v\rangle + \overline{\langle u|v\rangle}$$

$$= \|u\|^2 + \|v\|^2 + 2 \cdot \mathrm{Re}\left(\langle u|v\rangle\right)$$

We know that -

$$|\langle u|v\rangle| \leq \|u\| \cdot \|v\|.$$

$$\langle u|v\rangle = \mathrm{Re}\left(\langle u|v\rangle\right) + i\,\mathrm{Im}\left(\langle u|v\rangle\right)$$

$$|\langle u|v\rangle| = \sqrt{\mathrm{Re}\left(\langle u|v\rangle\right)^2 + \underbrace{\mathrm{Im}\left(\langle u|v\rangle\right)^2}_{\geqslant 0}}$$

therefore, $|\langle u|v\rangle| \geqslant \mathrm{Re}\left(\langle u|v\rangle\right)$

$$\Rightarrow \|u\|^2 + \|v\|^2 + 2 \cdot |\langle u|v\rangle| \geqslant \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\left(\langle u|v\rangle\right)$$

$$\Rightarrow \|u\|^2 + \|v\|^2 + 2 \cdot |\langle u|v\rangle| \geqslant \|u+v\|^2$$

C.S inequality: $|\langle u|v\rangle| \leq \|u\| \cdot \|v\|.$

$$\Rightarrow |\langle u|v\rangle| = \|u\|\cdot\|v\| - a \qquad (a \geq 0)$$

$$\Rightarrow \|u\|^2 + \|v\|^2 + 2\cdot\|u\|\cdot\|v\| \geq \|u+v\|^2 + a$$

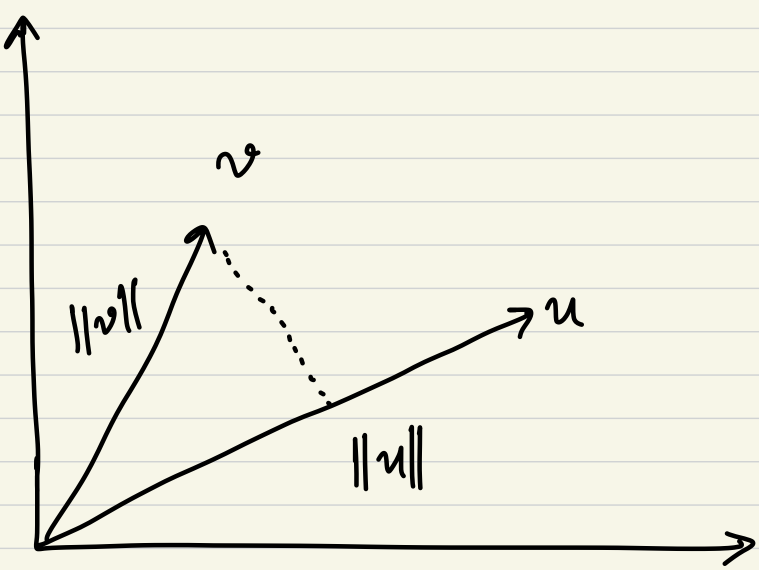
Suppose $5 \geq 3$ then $5 \geq 2$ also hence we can ignore the effect of a.

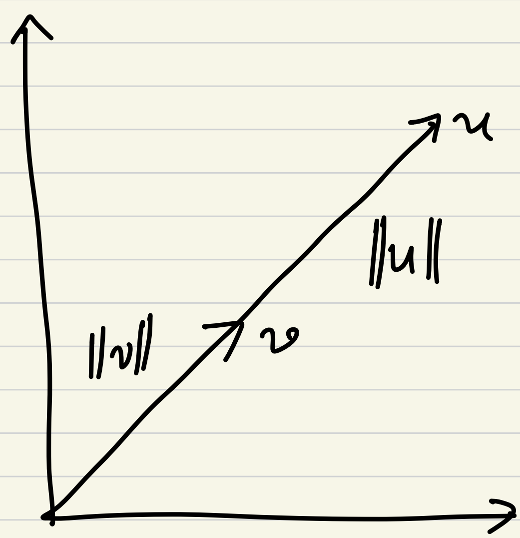
$$\Rightarrow \|u\|^2 + \|v\|^2 + 2\|u\|\cdot\|v\| \geq \|u+v\|^2$$

$$\Rightarrow (\|u\| + \|v\|)^2 \geq \|u+v\|^2$$

$$\Rightarrow \|u\| + \|v\| \geq \|u+v\| \qquad \text{(hence proved)}$$

An implication of Cauchy Swartz inequality:





We know that, $|\langle u|v\rangle| \leq \|u\|\cdot\|v\|$.

We desire to find out when equality will occur?

if $\|u\| = 0$ then $|\langle u|v\rangle| \leq 0$

Since $|\langle u|v\rangle| \geq 0$ therefore, $|\langle u|v\rangle| = 0$.

if $v = \dfrac{\langle v|u\rangle}{\|u\|^2} \cdot u$ then -

$$\langle v|v\rangle = \left\langle \frac{\langle v|u\rangle}{\|u\|^2} \cdot u \,\middle|\, \frac{\langle v|u\rangle}{\|u\|^2} \cdot u \right\rangle$$

$$\Rightarrow \|v\|^2 = \frac{\langle v|u\rangle}{\|u\|^2} \cdot \frac{\overline{\langle v|u\rangle}}{\|u\|^2} \cdot \langle u|u\rangle$$

$$\Rightarrow \|v\|^2 = \frac{\langle v|u\rangle \overline{\langle v|u\rangle}}{\|u\|^2} = \frac{|\langle u|v\rangle|^2}{\|u\|^2}$$

$$\Rightarrow |\langle u|v\rangle|^2 = \|u\|^2 \cdot \|v\|^2.$$

So the equality happens if $\|u\| = 0$ or u and v are linearly dependent.

$\|u\| = 0 \Longleftrightarrow \langle u|u\rangle = 0 \longleftrightarrow u = 0_V$ as per the definition of the inner product (positive defitiveness)

Orthogonal vector:

Let u, v be 2 vectors in the inner product space $\mathbb{V}$.
u is orthogonal to v if and only if $\langle u|v\rangle = 0$.
In this definition, the order of u, v doesn't matter.

(i) $0_{\mathbb{V}}$ is orthogonal to all vectors in $\mathbb{V}$.

We know that $\langle c.u|v\rangle = c.\langle u|v\rangle$.
if we choose $c=0$ then, $0.u = 0_{\mathbb{V}}$.

$$\langle 0_{\mathbb{V}}|v\rangle = 0.\langle u|v\rangle = 0.$$

That means $0_{\mathbb{V}}$ is orthogonal to every vector in $\mathbb{V}$.

(ii) $0_{\mathbb{V}}$ is the only vector that is orthogonal to itself.

We know that $\langle v|v\rangle = 0$ iff $v = 0$.

that means $\langle 0_{\mathbb{V}}|0_{\mathbb{V}}\rangle = 0$ and we also know $\langle v|v\rangle \neq 0$ iff $v \neq 0$ so no other vector can be orthogonal to itself. So $0_{\mathbb{V}}$ is orthogonal to itself.

Pythagoreon Theorem:

u, v are orthogonal $\longrightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2$

if u, v are orthogonal then $\langle u|v\rangle = 0$

$$\|u+v\|^2 = \langle (u+v) \,|\, (u+v) \rangle$$

$$= \langle u|u\rangle + 2\,\mathrm{Re}\left(\langle u|v\rangle\right) + \langle v|v\rangle$$

$$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\left(\langle u|v\rangle\right)$$

if u, v are orthogonal then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Now we wish to check if the converse is also true?

$\Rightarrow$ of course is $\mathbb{F} = \mathbb{R}$ then

$$2 \cdot \mathrm{Re}\left(\langle u|v\rangle\right) = 0$$

Since $\langle u|v\rangle$ is real so, $\langle u|v\rangle = 0$ ✓

But if $\mathbb{F} = \mathbb{C}$ then it will not be true.

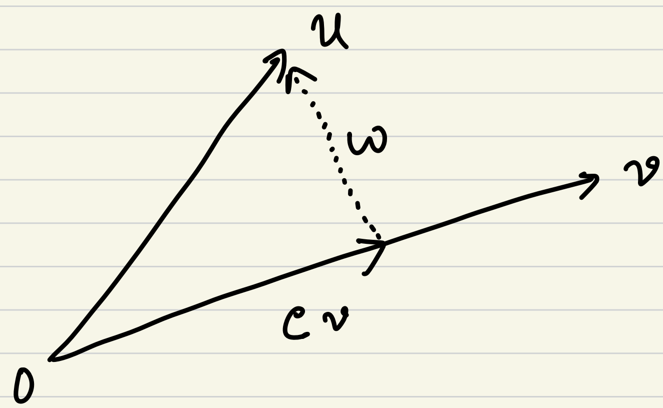
$$2 \cdot \mathrm{Re}\left(\langle u|v\rangle\right) = 0$$

Since $\langle u|v\rangle$ is complex, it may happen that the real part of $\langle u|v\rangle$ is 0 but imaginary part is not zero hence,

$$2 \cdot \mathrm{Re}\left(\langle u|v\rangle\right) = 0 \not\longrightarrow \langle u|v\rangle = 0.$$

Orthogonal decomposition:

Suppose $u, v \in V$ and $v \neq 0$. We would like to write u as scalar multiple of v plus a vector w orthogonal to v.



We can easily write, $cv + w = u$

Now we will choose c such a way that w and v becomes orthogonal.

$$\langle w | v \rangle = 0$$

$$\Rightarrow \langle u - cv | v \rangle = 0$$

$$\Rightarrow \langle u|v \rangle - c \langle v|v \rangle = 0$$

$$\Rightarrow c = \frac{\langle u|v \rangle}{\langle v|v \rangle} = \frac{\langle u|v \rangle}{\|v\|^2}.$$

Hence, $\boxed{w = u - \frac{\langle u|v \rangle}{\|v\|^2} \cdot v}$

Note that this orthogonal decomposition was used to prove Cauchy Swartz inequality.

Orthogonal / Orthonormal set :

If S is a set of vectors in $\mathbb{V}$, S is called an orthogonal set if all pairs of distinct vectors in S are orthogonal. An orthonormal set is an orthogonal set S with the additional property that for every vector v in S, $\|v\| = 1$.

Ex1: In $\mathbb{R}^2$, consider the inner product as,

$$\langle x|y\rangle = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2$$

Consider the standard inner product as

$$\langle x|y\rangle = x_1y_1 + x_2y_2$$

Suppose the set $S = \{ (x, y), (-y, x) \}$

Is it an orthogonal set w.r.t standard inner product?

$$\langle x|y\rangle = x(-y) + y(x) = 0 \quad \checkmark$$

$$\langle x|y\rangle = x \times y - (-y).(y) - (x).(x) + 4.(-y).(x)$$

$$= xy + y^2 - x^2 - 4xy = 0$$

$$\Rightarrow \quad y^2 - x^2 - 3xy = 0$$

$$\Rightarrow \quad y = \frac{-3x \pm \sqrt{9x^2 + 4.x^2}}{2}$$

$$= \frac{1}{2}\left(-3 \pm \sqrt{13}\right)x$$

<u>Ex2:</u> Suppose $\mathbb{V}$ is the vector space of continuous complex valued functions.

$$\langle f|g \rangle = \int_0^1 f(x)\,\overline{g(x)}\,dx$$

Suppose, $f_n(x) = \sqrt{2}\cos(2\pi n x)$

$g_n(x) = \sqrt{2}\sin(2\pi n x)$

$S = \{1, f_1, g_1, f_2, g_2, \ldots\}$ is an orthonormal set.

$$\langle f_1|f_2 \rangle = \int_0^1 2 \cdot \cos(2\pi x) \cdot \cos(4\pi x)\,dx = 0$$

$$\langle g_1|g_2 \rangle = \int_0^1 2 \cdot \sin(2\pi x) \cdot \sin(4\pi x)\,dx = 0$$

$$\langle f_1|g_1 \rangle = \int_0^1 2 \cdot \sin(2\pi x)\cos(2\pi x)\,dx = 0 .$$

Theorem : An orthogonal set of nonzero vectors is linearly independent.

proof: let $\mathbb{V}$ be an inner product space with the given inner product.

$S = \{u_1, u_2, ----, u_m\}$ are the distinct vectors.

Since the set S is orthogonal,

$$\forall_{i,j} \quad \langle u_i | u_j \rangle \begin{array}{ll} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{array}$$

To prove set S is LI set, consider -

$$c_1 \cdot u_1 + c_2 \cdot u_2 + \cdots + c_m u_m = 0_{\mathbb{V}}. \quad \text{——— (1)}$$

To find c_1, we consider,

$$\langle (c_1 u_1 + c_2 u_2 + \cdots + c_m u_m) | u_1 \rangle = \langle 0_{\mathbb{V}} | u_1 \rangle = 0$$

$$\Rightarrow c_1 \langle u_1 | u_1 \rangle + 0 + \cdots + 0 = 0$$

$$\Rightarrow c_1 = 0$$

Since c_1 is no special, we can do it for any c_i.

Differently suppose, $v = c_1u_1 + c_2u_2 + \cdots + c_mu_m$.

Take any k, $\langle v | u_k \rangle = \sum_{j=1}^{m} \langle c_j v_j | u_k \rangle$

$$= \sum_{j=1}^{m} c_j \langle v_j | u_k \rangle$$

$$= c_k \cdot \langle v_k | u_k \rangle$$

$$\Rightarrow c_k = \frac{\langle v | u_k \rangle}{\langle v_k | u_k \rangle}$$

Since $v = c_1u_1 + c_2u_2 + \cdots + c_mu_m = 0$ therefore,

$$c_k = \frac{\langle 0 | u_k \rangle}{\langle v_k | u_k \rangle} = 0 \quad \text{for all } k = 1, 2, \ldots, m.$$

Hence S is linearly independent set.

If a vector u is a linear combination of orthogonal set of vectors $v_1, v_2, \ldots, v_m$ then u can be represented as follows:

$$u = \sum_{k=1}^{m} \frac{\langle u | v_k \rangle}{\| v_k \|^2} \cdot v_k$$

$$u = c_1 \cdot v_1 + c_2 v_2 + \cdots + c_m v_m$$

Take inner product with each of v_K.

$$\langle u | v_K \rangle = c_K \langle v_K | v_K \rangle = c_K \|v_K\|^2$$

$$\Rightarrow \quad c_K = \frac{\langle u | v_K \rangle}{\|v_K\|^2}.$$

Hence,
$$\boxed{u = \sum_{K=1}^{m} \frac{\langle u | v_K \rangle}{\|v_K\|^2} \cdot v_K.}$$

Gram Schmidt Orthogonalization process:

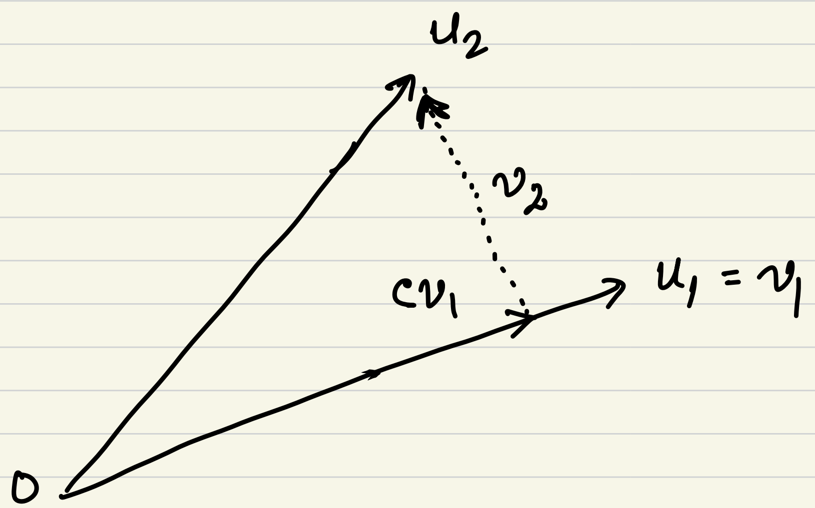
let $\mathbb{V}$ be an inner product space and let $\{u_1, u_2, \ldots, u_n\}$ be a set of any independent vectors in $\mathbb{V}$.

Then one may construct orthogonal vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{V}$ such that for each $K = 1, 2, 3, \ldots, n$, the set $\{v_1, v_2, \ldots, v_K\}$ is a basis for the subspace spanned by $\{u_1, u_2, \ldots, u_K\}$.

The vectors $\{v_1, v_2, \ldots, v_K\}$ will be obtained by means of Gram-Schmidt orthogonalization process.

First let, $\boxed{v_1 = u_1}$

v_2 is choosen in such a way a that $\langle v_2 | v_1 \rangle = 0$



$$v_2 = u_2 - c \cdot v_1$$

We need to select c in such a way that $\langle v_2 | v_1 \rangle = 0$

$$\langle v_2 | v_1 \rangle = \langle u_2 - cv_1 | v_1 \rangle$$

$$= \langle u_2 | v_1 \rangle - c.\ \|v_1\|^2 = 0$$

$$\Rightarrow c = \frac{\langle u_2 | v_1 \rangle}{\|v_1\|^2}$$

Hence, $\boxed{v_2 = u_2 - \frac{\langle u_2 | v_1 \rangle}{\|v_1\|^2} \cdot v_1}$

Next, we will select v_3 in such a way that v_3 will be orthogonal to both v_1 and v_2.

$$v_3 = u_3 - c_1 \cdot v_1 - c_2 \cdot v_2$$

$$\langle v_3 | v_1 \rangle = \langle u_3 - c_1 v_1 - c_2 v_2 \mid v_1 \rangle$$

$$= \langle u_3 | v_1 \rangle - c_1 \|v_1\|^2 = 0$$

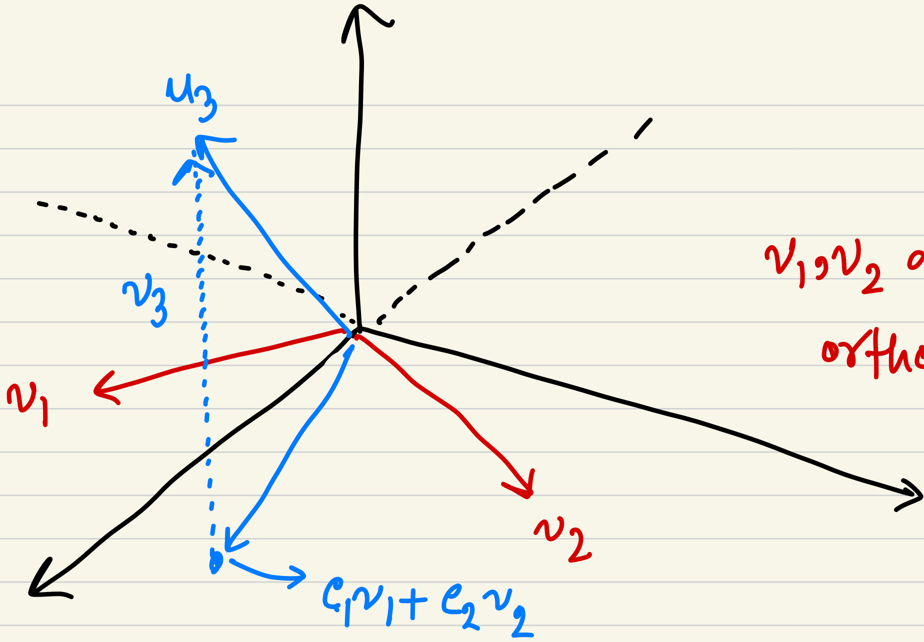
$$\Rightarrow c_1 = \frac{\langle u_3 | v_1 \rangle}{\|v_1\|^2}.$$

$$\langle v_3 | v_2 \rangle = \langle u_3 - c_1 v_1 - c_2 v_2 \mid v_2 \rangle$$

$$= \langle u_3 | v_2 \rangle - c_2 \cdot \|v_2\|^2 = 0$$

$$\Rightarrow c_2 = \frac{\langle u_3 | v_2 \rangle}{\|v_2\|^2}.$$

$$\boxed{v_3 = u_3 - \frac{\langle u_3 | v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3 | v_2 \rangle}{\|v_2\|^2} \cdot v_2}$$



v_1, v_2 are already orthogonal

$$v_3 = u_3 - (c_1v_1 + c_2v_2)$$

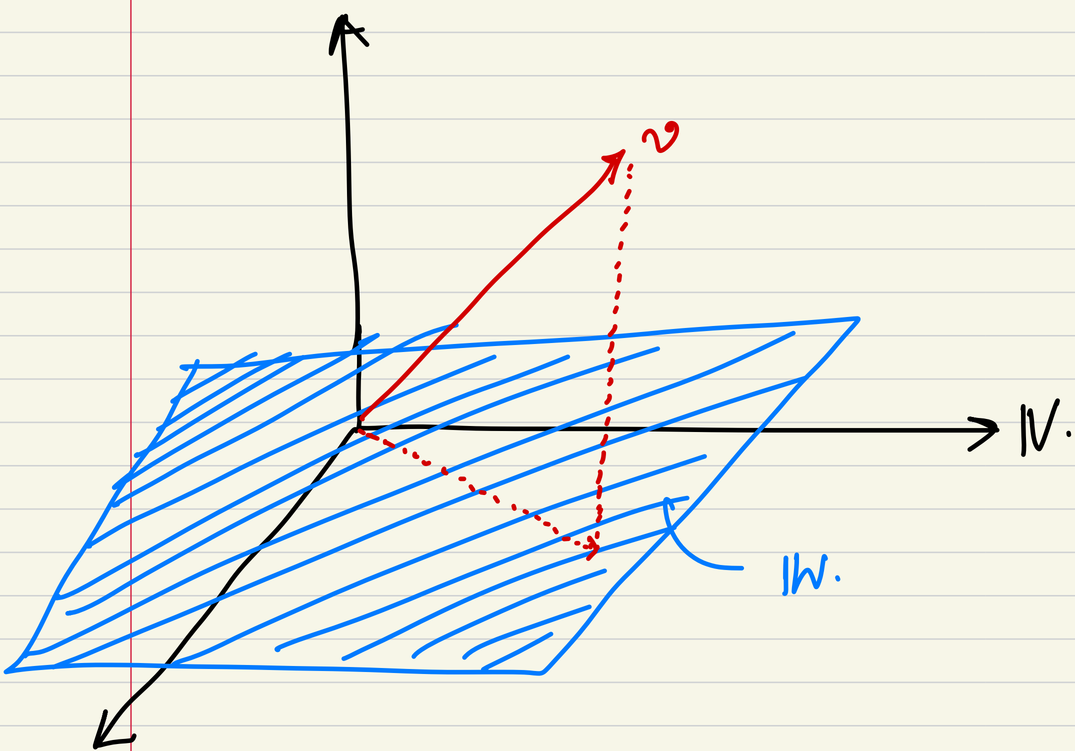
For K^{th} orthogonal vector,

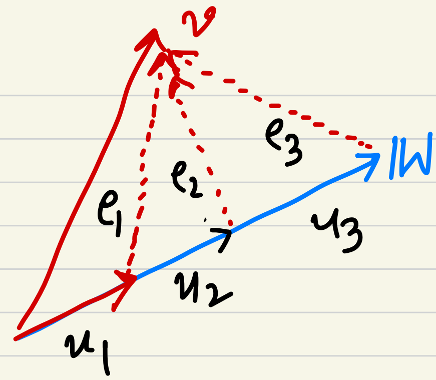
$$v_K = u_K - \sum_{i=1}^{K-1} c_i v_i$$

$$v_K = u_K - \sum_{i=1}^{K-1} \frac{\langle u_K | v_i \rangle}{\|v_i\|^2} \cdot v_i$$

Best Approximation:

Suppose $\mathbb{W}$ is a subspace of inner product space $\mathbb{V}$. Let v be an arbitrary vector in $\mathbb{V}$. The problem is to find the best possible approximation of v by the vectors in $\mathbb{W}$.





$$e_1 = v - u_1$$
$$e_2 = v - u_2$$
$$e_3 = v - u_3$$

Definitely v is not in $\mathbb{W}$ but we want to express v in terms of vectors of $\mathbb{W}$ in best possible way. That means we want to find a vector $u \in \mathbb{W}$ such that the error magnitude $\|v-u\|$ is as small as possible.

A best approximation to $v \in \mathbb{V}$ by the vectors in $\mathbb{W}$ will be $u \in \mathbb{W}$ such that,

$$\|v - u\| \leq \|v - u_i\| \quad \text{for all } u_i \in \mathbb{W}.$$

1 The vectors u in $\mathbb{W}$ is a best approximation of v in $\mathbb{V}$ if and only if $v-u$ is orthogonal to every vector in $\mathbb{W}$.

consider any vector $x \in \mathbb{V}$. We know that,

$$(v-x) = (v-u) + (u-x) \qquad \text{where, } u \in \mathbb{W}.$$

Consider,

$$\langle v-x \mid v-x \rangle$$

$$= \langle (v-u) + (u-x) \mid (v-u) + (u-x) \rangle$$

$$= \langle (v-u) \mid (v-u) + (u-x) \rangle + \langle (u-x) \mid (v-u) + (u-x) \rangle$$

$$= \langle (v-u) \mid (v-u) \rangle + \langle (v-u) \mid (u-x) \rangle +$$
$$\langle (u-x) \mid (v-u) \rangle + \langle (u-x) \mid (u-x) \rangle$$

$$= \| v-u \|^2 + \| u-x \|^2 + 2 \cdot \mathrm{Re}\left(\langle (v-u) \mid (u-x) \rangle \right)$$

$$= \| v-x \|^2$$

$(v-u)$ is orthogonal to every vectors in $\mathbb{W}$ $\longrightarrow$ u is best approx of v.

assume that $(v-u)$ is orthogonal to every vectors in $\mathbb{W}$.
Suppose $x \in \mathbb{W}$ and $x \neq u$ then -

$$\|v-x\|^2 = \|v-u\|^2 + \|\underbrace{u-x}_{\in W}\|^2 + 2\mathrm{Re}(\langle(v-u)|(u-x)\rangle)$$

Since $(v-u)$ is orthogonal to every vector in W so it is orthogonal to $(u-x)$. Therefore, $\langle(v-u)|\underbrace{(u-x)}_{\in W}\rangle=0$

$$\|v-u\|^2 + \|u-x\|^2 = \|v-x\|^2$$

$$\Rightarrow \|v-u\|^2 \leq \|v-x\|^2$$

$$\Rightarrow \|v-u\| \leq \|v-x\| \quad \forall x \in W$$

Therefore u is the best approximation of v.

Suppose u is best approximation of v. then,

$$\|v-u\| \leq \|v-x\| \quad \forall x \in W.$$

$$\Rightarrow \|v-u\|^2 \leq \|v-x\|^2$$

$$\Rightarrow \|v-u\|^2 \leq \|v-u\|^2 + \|u-x\|^2 + 2\mathrm{Re}(\langle(v-u)|(u-x)\rangle)$$

$$\Rightarrow \underbrace{\|u-x\|^2}_{\geq 0} + \underbrace{2\mathrm{Re}(\langle v-u|u-x\rangle)} \geq 0 \quad (\forall x \in W)$$

We have to prove that somehow this is suffient negative that the inequality is contradicted.

Since x is arbitrary so we may pick an x such that we may arrive at a contradiction.

Choose, $\quad u - x = -\dfrac{\langle (v-u) \mid \bar{w} \rangle}{\|\bar{w}\|^2} . \bar{w}$ for some $w \in W$.

$$2 \cdot \langle v-u \mid u-x \rangle$$

$$= 2 \cdot \left[\left\langle v-u \,\middle|\, -\frac{\langle v-u \mid \bar{w} \rangle}{\|\bar{w}\|^2} . \bar{w} \right\rangle \right]$$

$$= 2 \cdot \left[-\frac{\langle v-u \mid \bar{w} \rangle}{\|\bar{w}\|^2} . \langle v-u \mid \bar{w} \rangle \right]$$

$$= 2 \cdot \left[-\frac{1}{\|\bar{w}\|^2} . \left| \langle v-u \mid \bar{w} \rangle \right|^2 \right]$$

$$2 . \mathrm{Re}\left(\langle v-u \mid u-x \rangle \right) = -2 . \frac{1}{\|\bar{w}\|^2} . |\langle v-u \mid \bar{w} \rangle|^2$$

$$\langle u-x \mid u-x \rangle = \left\langle -\frac{\langle (v-u) \mid \bar{w} \rangle}{\|\bar{w}\|^2} \bar{w} \,\middle|\, -\frac{\langle (v-u) \mid \bar{w} \rangle}{\|\bar{w}\|^2} . \bar{w} \right\rangle$$

$$\|u-x\|^2 = \frac{|\langle v-u \mid \bar{w} \rangle|^2}{\|\bar{w}\|^2}$$

$$\Rightarrow \left|\frac{\langle v-u|\bar{\omega}\rangle}{\|\bar{\omega}\|^2}\right|^2 - 2\cdot\frac{|\langle v-u|\bar{\omega}\rangle|^2}{\|\bar{\omega}\|^2} \geq 0$$

$$\Rightarrow -\frac{|\langle v-u|\bar{\omega}\rangle|^2}{\|\bar{\omega}\|^2} \geq 0$$

But it must also be ≤ 0 hence,

$$\frac{|\langle v-u|\bar{\omega}\rangle|^2}{\|\bar{\omega}\|^2} = 0$$

$$\Rightarrow \langle v-u|\bar{\omega}\rangle = 0$$

Therefore, $v-u$ is orthogonal to any arbitrary vector in $\mathbb{W}$.

② If a best approximation of v is u in $\mathbb{W}$ exists then must be unique.

Suppose u_1 and u_2 are best approximation of v. $(u_1 \neq u_2)$

$$\Rightarrow \|v-u_1\| \leq \|v-u_i\| \quad \forall u_i \in \mathbb{W}.$$

$$\Rightarrow \|v-u_2\| \leq \|v-u_i\| \quad \forall u_i \in \mathbb{W}.$$

Substitute $u_i = u_2$, $\quad \|v-u_1\| \leq \|v-u_2\|$

Substitute $u_i = u_1$, $\quad \|v-u_2\| \leq \|v-u_1\|$

$$\Rightarrow \|v-u_1\| = \|v-u_2\|$$

$$\Rightarrow \|v-u_1\|^2 = \|v-u_2\|^2$$

Consider,

$$\|v-u_1\|^2 = \|(v-u_2)+(u_2-u_1)\|^2$$

$$= \langle (v-u_2)+(u_2-u_1) \mid (v-u_2)+(u_2-u_1) \rangle$$

$$= \langle (v-u_2) \mid (v-u_2) \rangle + \langle (v-u_2) \mid (u_2-u_1) \rangle$$

$$+ \langle (u_2-u_1) \mid (v-u_2) \rangle + \langle (u_2-u_1) \mid (u_2-u_1) \rangle$$

Since $u_2-u_1 \in W$ so $v-u_1$ is orthogonal to u_2-u_1 and $v-u_2$ is also orthogonal to u_2-u_1.

$$= \|v-u_2\|^2 + \|u_2-u_1\|^2 + 0 + 0$$

$$\Rightarrow \|u_2-u_1\|^2 = 0$$

$$\Rightarrow \langle u_2-u_1 \mid u_2-u_1 \rangle = 0$$

We know that only $0_{\mathbb{V}}$ is orthogonal to itself.

$\Rightarrow \quad u_2 - u_1 = 0_{\mathbb{V}}$.

$\Rightarrow \quad u_2 = u_1$ (contradiction)

Therefore, the best approximation of v is unique.

③ If $\mathbb{W}$ is finite dimensional subspace of $\mathbb{V}$. and $\{w_1, w_2, \ldots, w_n\}$ is any orthogonal bases for $\mathbb{W}$ then the vector,

$$u = \sum_{K=1}^{n} \frac{\langle v | w_K \rangle}{\|w_K\|^2} w_K \quad , \quad u \in \mathbb{W}$$

u is the unique best approximation to u by vectors in $\mathbb{W}$

$\Rightarrow$ Since $u \in \mathbb{W}$ so u can be written as LC of vectors in basis of $\mathbb{W}$.

$$u = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n.$$

We know that $v - u$ is orthogonal to each of the vectors in $\mathbb{W}$.

$$\Rightarrow \langle v - u | w_i \rangle = 0 \quad \text{for } i = 1, 2, \ldots, n$$

$$\Rightarrow \left\langle v - \sum_{K=1}^{n} c_K w_K \,\middle|\, w_i \right\rangle = 0 \quad \text{for all } i$$

$$\Rightarrow \langle v | w_i \rangle - \sum_{K=1}^{n} c_K \cdot \langle w_K | w_i \rangle = 0$$

Since w_i are orthogonal so only for $\langle w_K | w_K \rangle$ will survive and rest all the inner product will be 0

$$\Rightarrow \langle v | w_K \rangle - c_K \cdot \langle w_K | w_K \rangle = 0$$

$$\Rightarrow c_K = \frac{\langle v | w_K \rangle}{\|w_K\|^2} .$$

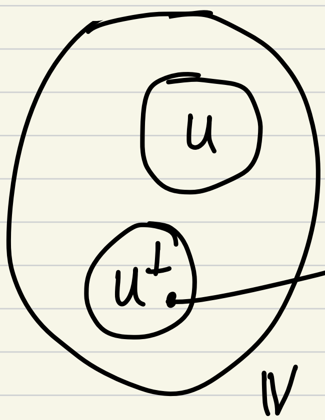
Hence,
$$\boxed{u = \sum_{K=1}^{n} \frac{\langle v | w_K \rangle}{\|w_K\|^2} \cdot w_K}$$

Orthogonal Complement :

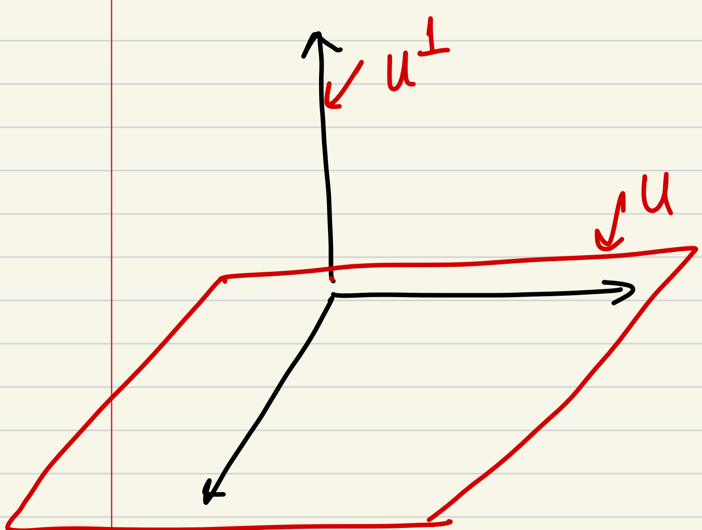
Let U be a subset of inner product space $\mathbb{V}$. Then the orthogonal complement of U is defined as :

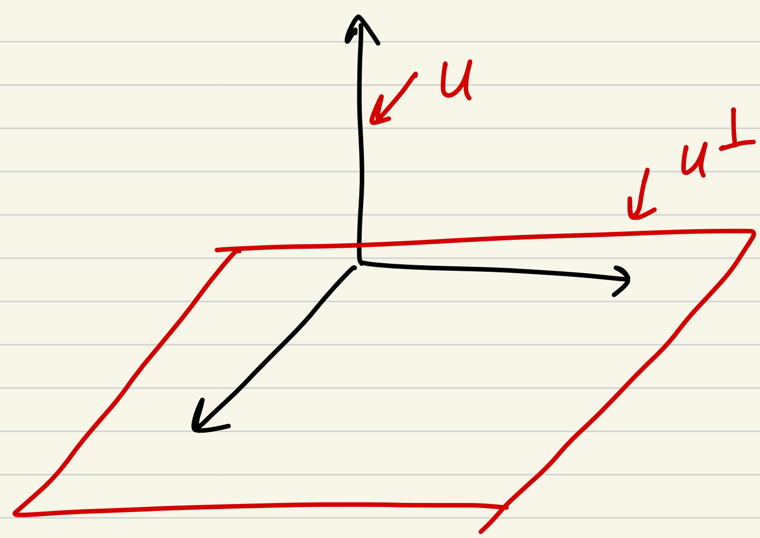
$$U^{\perp} = \{ v \in \mathbb{V} \mid \forall u \in U \text{ so that } \langle u | v \rangle = 0 \} .$$

Orthogonal complement is a set of vectors in such a way that any member of this set is orthogonal to all the members of the set U.



From here we pick any vector then it must be orthogonal to all the vectors in U.





① U is subset of V but $U^\perp$ is subspace of V.

consider two arbitrary vector $P_1, P_2 \in U^\perp$

Look at $P = c \cdot P_1 + P_2$.

$$P_1 \in U^\perp \rightarrow \langle P_1 | u \rangle = 0 \quad \forall u \in U.$$

$$P_2 \in U^\perp \rightarrow \langle P_2 | u \rangle = 0 \quad \forall u \in U$$

Consider $\langle cp_1 + p_2 | u \rangle$

$= \langle cp_1 | u \rangle + \langle p_2 | u \rangle$

$= c \langle p_1 | u \rangle + \langle p_2 | u \rangle$

$= c \times 0 + 0$

$= 0$

$\Rightarrow$ $p \in U^{\perp}$ hence $U^{\perp}$ is subspace of $\mathbb{V}$.

2 $\{0\}^{\perp} = \mathbb{V}$ and $\mathbb{V}^{\perp} = \{0\}$.

We know that,

$$U^{\perp} := \{ v \in \mathbb{V} \mid \forall u \in U \text{ s.t } \langle v | u \rangle = 0 \}.$$

Since $U = \{0\}$ then,

$$\{0\}^{\perp} := \{ v \in \mathbb{V} \mid \langle v | 0_{\mathbb{V}} \rangle = 0 \}.$$

We know that $0_{\mathbb{V}}$ is orthogonal to every vector in $\mathbb{V}$.
So $\{0\}^{\perp} = \mathbb{V}$.

$$\mathbb{V}^{\perp} := \{ v \in \mathbb{V} \mid \forall u \in \mathbb{V} \text{ s.t. } \langle u | v \rangle = 0 \}.$$

We know that only $0_{\mathbb{V}}$ is self orthogonal so $\mathbb{V}^{\perp} = \{0_{\mathbb{V}}\}$.

③ $\mathbb{U}$ is subspace. of $\mathbb{V}$, $\mathbb{U} \cap \mathbb{U}^{\perp} = \{0_{\mathbb{V}}\}$.

$\mathbb{U}^{\perp} := \{ v \in \mathbb{V} \mid \forall u \in \mathbb{U} \text{ s.t } \langle v|u \rangle = 0 \}$.

if $v \in \mathbb{U}^{\perp}$ then $\langle v|u \rangle = 0 \quad \forall u \in \mathbb{U}$.

if v were to belong to $\mathbb{U}$ also then $\langle v|v \rangle = 0$.

that implies $v = 0_{\mathbb{V}}$.

Therefore, $v \in U^{\perp}$ and $v \in U \longrightarrow v = 0_{\mathbb{V}}$.

$$\Rightarrow \quad v \in U \cap U^{\perp} \longrightarrow v = 0_{\mathbb{V}}.$$

Suppose $v \in \{0_{\mathbb{V}}\}$ so $v = 0_{\mathbb{V}}$.

$\Rightarrow$ $v \in \mathbb{U}^{\perp}$ because $\mathbb{U}^{\perp}$ is a subspace of $\mathbb{V}$.

$v \in \mathbb{U}$ because $\mathbb{U}$ is a subspace of $\mathbb{V}$

$\Rightarrow$ $v \in \mathbb{U} \cap \mathbb{U}^{\perp}$

Hence $\mathbb{U} \cap \mathbb{U}^{\perp} = \{0_{\mathbb{V}}\}$.

④ If U and W are subset of V and $U \subseteq W$ then $W^{\perp} \subseteq U^{\perp}$.

Suppose $x \in W^{\perp}$

$\Rightarrow \forall w \in W, \ \langle x|w \rangle = 0$

Since $U \subseteq W$ so if $\forall w \in W, \langle x|w \rangle = 0$ means,

$\forall u \in U$, $\langle x | u \rangle = 0$ because $\forall u \in U$, $u \in W$.

$\Rightarrow x \in U^{\perp}$.

Hence $W^{\perp} \subseteq U^{\perp}$.

Orthogonal Decomposition:

Suppose $\mathbb{U}$ is a fdsp of $\mathbb{V}$ then $\mathbb{V} = \mathbb{U} \oplus \mathbb{U}^{\perp}$.

$\Rightarrow$ Suppose $v \in \mathbb{V}$. then we have to prove that $v = u + w$ where $u \in \mathbb{U}$, $w \in \mathbb{U}^{\perp}$.

consider, $v = v - \hat{v} + \hat{v}$

where $\hat{v}$: Best approximation of v onto $\mathbb{U}$.

$(v - \hat{v})$ is orthogonal to all the vectors to $\mathbb{U}$ so $(v - \hat{v}) \in \mathbb{U}^{\perp}$. and $\hat{v} \in \mathbb{U}$.

$$v = \underbrace{(v - \hat{v})}_{\in \mathbb{U}^{\perp}} + \underbrace{\hat{v}}_{\in \mathbb{U}} \quad \text{so} \quad v \in \mathbb{U} + \mathbb{U}^{\perp}.$$

The arguments can be easily flipped so $\mathbb{V} = \mathbb{U} + \mathbb{U}^{\perp}$.
To see it is a direct sum, we have to prove that, $\mathbb{U} \cap \mathbb{U}^{\perp} = \{0_{\mathbb{V}}\}$ which was already proved.

Hence, $\mathbb{V} = \mathbb{U} \oplus \mathbb{U}^{\perp}$.

$$\boxed{\dim(\mathbb{U} \oplus \mathbb{U}^{\perp}) = \dim(\mathbb{U}) + \dim(\mathbb{U}^{\perp}) = \dim(\mathbb{V})}$$

The orthogonal complement of orthogonal complement gives the subspace back.

Suppose $\mathbb{U}$ is subspace of $\mathbb{V}$ so $(\mathbb{U}^{\perp})^{\perp} = \mathbb{U}$.

$\Rightarrow$ Suppose, $u \in \mathbb{U}$. then we have to prove that, $u \in (\mathbb{U}^{\perp})^{\perp}$.

$$\mathbb{U}^{\perp} := \{ v \in \mathbb{V} \mid \forall u \in \mathbb{U} \text{ s.t } \langle v|u\rangle = 0 \} \quad \text{— (i)}$$

$$(\mathbb{U}^{\perp})^{\perp} := \{ w \in \mathbb{V} \mid \forall v \in \mathbb{U}^{\perp} \text{ s.t } \langle w|v\rangle = 0 \} \quad \text{— (ii)}$$

From (i), note that, $v \in \mathbb{V}$, $\forall u \in \mathbb{U}$, $\langle v|u\rangle = 0$

From (ii), note that, $w \in \mathbb{V}$, $\forall v \in \mathbb{U}^{\perp}$, $\langle w|v\rangle = 0$

Since $u \in \mathbb{U} \longrightarrow u \in \mathbb{V}$ hence we can write from (ii),

$\langle u|v\rangle = \overline{\langle v|u\rangle} = \overline{0} = 0$ since $\langle v|u\rangle = 0$.

Therefore, $u \in (\mathbb{U}^{\perp})^{\perp}$.

Hence, $\mathbb{U} \subseteq (\mathbb{U}^{\perp})^{\perp}$.

Now we have to assume that $v \in (\mathbb{U}^{\perp})^{\perp}$ and prove that $v \in \mathbb{U}$.

Since $v \in \mathbb{V}$ also, we can write, $v = u + w$ where, $u \in \mathbb{U}$ and $w \in \mathbb{U}^\perp$.

We have already seen, $u \in \mathbb{U} \longrightarrow u \in (\mathbb{U}^\perp)^\perp$.
We already have $v \in (\mathbb{U}^\perp)^\perp$.

Therefore, $v - u = w \in (\mathbb{U}^\perp)^\perp$.
On the contrary, $v - u = w \in \mathbb{U}^\perp$.

Therefore, $v - u \in (\mathbb{U}^\perp)^\perp$ and $v - u \in \mathbb{U}^\perp$
That means $v - u \in (\mathbb{U}^\perp)^\perp \cap (\mathbb{U}^\perp)$

Hence, $v - u = 0_{\mathbb{V}} \Rightarrow v = u$.

Therefore, $u \in \mathbb{U} \longrightarrow v \in \mathbb{U}$ hence $(\mathbb{U}^\perp)^\perp \subseteq \mathbb{U}$
So $\mathbb{U} = (\mathbb{U}^\perp)^\perp$. (Hence proved).

Orthogonal Projection:

Suppose $\mathbb{U}$ is a finite dimensional subspace of $\mathbb{V}$. The orthogonal projection of $\mathbb{V}$ onto the subspace $\mathbb{U}$ is the linear operator $P : \mathbb{V} \to \mathbb{V}$ defined as follows:

$$\begin{array}{ccc}
\mathbb{V} & \xrightarrow{\quad P \quad} & \mathbb{V} \\
v = \underset{\in \mathbb{U}}{u} + \underset{\in \mathbb{U}^\perp}{w} & \longmapsto & u = P(v)
\end{array}$$

Projection operator takes a vector v in $\mathbb{V}$ and gives its best approximation in $\mathbb{U}$. So the best approximation is the orthogonal projection of vector v on $\mathbb{U}$.

$v \in \mathbb{V}$ can be written as: $v = \hat{v} + e$

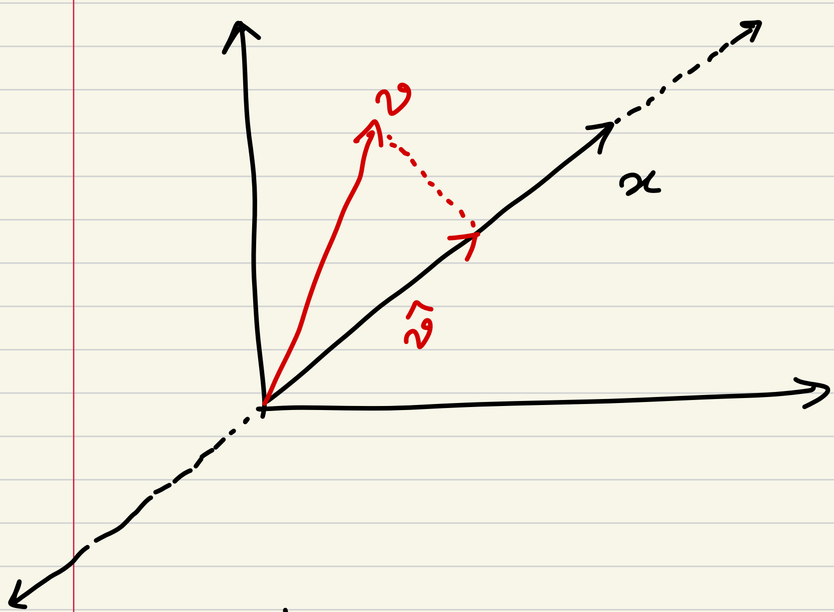
Where $\hat{v}$: best approximation of v on $\mathbb{U}$. so $\hat{v} \in \mathbb{U}$.

where e is the error vector and $e \in \mathbb{U}^{\perp}$ because e and $\hat{v}$ are orthogonal to each other.

We can write, $\hat{v} = P(v)$

Example: Suppose, $x \in \mathbb{V}$ and $x \neq 0_{\mathbb{V}}$.

consider, $\mathbb{U} = \langle \{x\} \rangle$.



$\langle \{x\} \rangle$: Subspace $\mathbb{U}$.

For every vector $v \in \mathbb{V}$, find out projection onto $\mathbb{U}$?

$P: \mathbb{V} \longrightarrow \mathbb{V}$ such that $\hat{v} = P(v)$ where $\hat{v}$ is the best approximation of v onto $\mathbb{U}$.

Since $\hat{v} \in \mathbb{U}$ so $\hat{v}$ can be written as LC of basis of $\mathbb{U}$. Let's choose basis of $\mathbb{U}$ as $\{x\}$ since it is 1 dim subspace.

So, $\hat{v} = c \cdot x$ where $c \in \mathbb{F}$.

In order to find c, we will use the fact that the error vector $v - \hat{v}$ is orthogonal to all the vectors in $\mathbb{U}$.

Hence, $$\langle v - \hat{v} \mid x \rangle = 0$$

$$\Rightarrow \langle v | x \rangle - \langle \hat{v} | x \rangle = 0$$

$$\Rightarrow \langle v | x \rangle - \langle c \cdot x | x \rangle = 0$$

$$\Rightarrow \langle v | x \rangle - c \cdot \|x\|^2 = 0$$

$$\Rightarrow c = \frac{\langle v | x \rangle}{\|x\|^2}.$$

Hence, $$\hat{v} = \frac{\langle v | x \rangle}{\|x\|^2} \cdot x$$

We also know, $$v = \underbrace{\hat{v}}_{\text{best approximation}} + \underbrace{(v - \hat{v})}_{\text{error}}$$

Hence, $$v = \underbrace{\frac{\langle v | x \rangle}{\|x\|^2} \cdot x}_{\in \mathbb{U}} + \underbrace{v - \frac{\langle v | x \rangle}{\|x\|^2} \cdot x}_{\in \mathbb{U}^\perp}$$

The projection operator is as follows:

$$v \longmapsto \hat{v} = \frac{\langle v | x \rangle}{\|x\|^2} \cdot x = P(v)$$

So $$\boxed{P(v) = \frac{\langle v | x \rangle}{\|x\|^2} \cdot x}$$

Properties of orthogonal projection operator P:

① P is linear operator.

We have to prove $P(c \cdot v_1 + v_2) = c \cdot P(v_1) + P(v_2)$

where, $v_1, v_2 \in \mathbb{V}$ and $c \in \mathbb{F}$.

The effect of projection is as follows: $v \longmapsto \hat{v} = P(v)$.

Suppose, $v_1 = u_1 + w_1$ where u_1: best approx of v_1
w_1: error.

$v_2 = u_2 + w_2$ where u_2: best approx of v_2
w_2: error.

$u_1, u_2 \in \mathbb{U}$.
$w_1, w_2 \in \mathbb{U}^{\perp}$.

$P(v_1) = u_1$
$P(v_2) = u_2$

Now, $P(cv_1 + v_2) = P(c(u_1 + w_1) + u_2 + w_2)$

$$= P(\underbrace{cu_1 + u_2}_{\in \mathbb{U}} + \underbrace{cw_1 + w_2}_{\in \mathbb{U}^{\perp}})$$

$= cu_1 + u_2$ (By definition of P)

$= c \cdot P(v_1) + P(v_2)$ (By definition of P)

Hence P is a linear operator.

2 $P(u) = u$ for all $u \in \mathbb{U}$
$P(w) = 0_{\mathbb{V}}$ for all $w \in \mathbb{U}^{\perp}$.

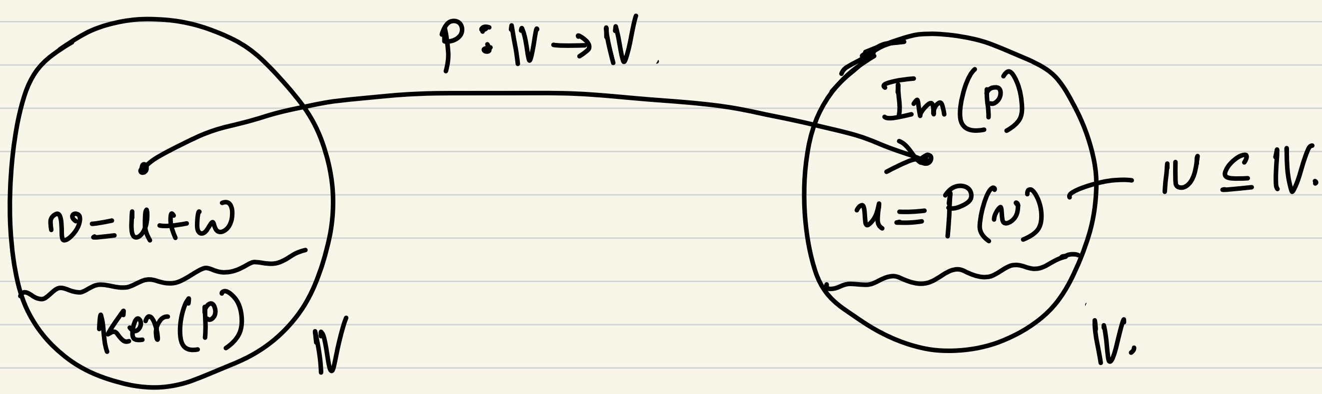
Suppose, $u \in \mathbb{U}$ so $u = u + 0_{\mathbb{V}}$.

where $u \in \mathbb{U}$ and $0_{\mathbb{V}} \in \mathbb{U}^{\perp}$ because $0_{\mathbb{V}}$ is orthogonal to all vectors.

Hence $P(u) = u$.

Suppose, $w \in \mathbb{U}^{\perp}$ so $w = 0_{\mathbb{V}} + w$
Since $\mathbb{U}$ is a subspace so $0_{\mathbb{V}} \in \mathbb{U}$ hence, $P(w) = 0_{\mathbb{V}}$.

③ $Im(P) = \mathbb{U}$.



To prove $Im(P) = \mathbb{U}$, we consider,

$Im(P) := \{ u \in \mathbb{V} \mid \exists\, v \in \mathbb{V} \text{ s.t } P(v) = u \}$.

The way P is defined $P(v) = u$, $v = u + w$.

or $P(u+w) = u$ where $u \in \mathbb{U}$ and $w \in \mathbb{U}^{\perp}$.

Therefore, $u \in Im(P) \longrightarrow u \in \mathbb{U}$. so $Im(P) \subseteq \mathbb{U}$.

Suppose, $u \in \mathbb{U}$ so $P(u) = u$ so $\exists\, u \in \mathbb{V}$ s.t $P(u) = u$
hence $u \in Im(P)$. so $\mathbb{U} \subseteq Im(P)$.

④ $\operatorname{Ker}(P) = \mathbb{U}^{\perp}$.

$\operatorname{Ker}(P) := \{ v \in \mathbb{V} \mid P(v) = 0_{\mathbb{V}} \}$

Suppose, $v = u + w$ where $u \in \mathbb{U}$ and $w \in \mathbb{U}^{\perp}$.

Say, $v \in \operatorname{Ker}(P)$.

$\Rightarrow P(v) = 0_{\mathbb{V}}$

We know that $P(v) = u$ and $v - P(v) \perp u$. $\forall u$

$\langle v - P(v) \mid u \rangle = 0 \quad (\forall u \in \mathbb{U})$

$\Rightarrow \langle v - 0 \mid u \rangle = 0 \quad [\because P(v) = 0]$

$\Rightarrow \langle v \mid u \rangle = 0 \quad (\forall u \in \mathbb{U})$

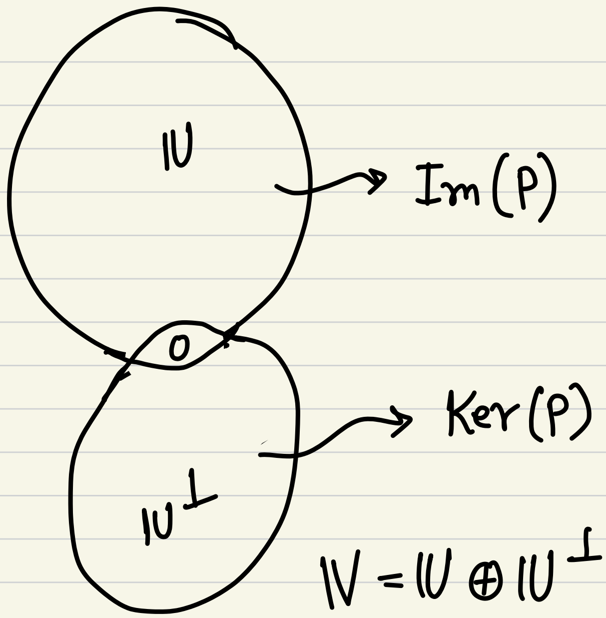
$\Rightarrow$ v is orthogonal to u so $v \in \mathbb{U}^{\perp}$

That means $\operatorname{Ker}(P) \subseteq \mathbb{U}^{\perp}$.

Say $v \in \mathbb{U}^{\perp}$.

So $v = \underbrace{0_{\mathbb{V}}}_{\in \mathbb{U}} + \underbrace{v}_{\in \mathbb{U}^{\perp}}$

Now, $P(v) = 0_{\mathbb{V}} \Rightarrow v \in \operatorname{Ker}(P)$ so $\mathbb{U}^{\perp} \subseteq \operatorname{Ker}(P)$.



$\mathbb{V} = \mathbb{U} \oplus \mathbb{U}^\perp$

$$P : \mathbb{V} \longrightarrow \mathbb{V}.$$

$$\begin{array}{c} v = u + w \\ \Big\downarrow P \\ u = P(v) \end{array}$$

(5) Projection map is idempotent, $P^2 = P$.

Suppose, $v = u + w$ where $u \in \mathbb{U}$, $w \in \mathbb{U}^\perp$.

$$P^2(v) = P\big(P(v)\big) = P\big(P(u+w)\big) = P(u)$$
$$= P\left(u + 0_{\mathbb{V}}\right) = u = P(u+w) = P(v).$$

Hence $P^2 = P$.

(6) $\|P(v)\| \leq \|v\|$.

$$\|P(v)\|^2 = \langle P(v) \mid P(v) \rangle$$
$$= \langle P(u+w) \mid P(u+w) \rangle$$
$$= \langle u \mid u \rangle = \|u\|^2.$$

Therefore, $\|P(v)\| = \|u\|$.

Since $v = u + w$

$$\|v\|^2 = \langle v | v \rangle = \langle u+w | u+w \rangle$$

$$= \|u\|^2 + \|w\|^2 + 2 \cdot \mathrm{Re}\left(\cancel{\langle u | w \rangle}\right) \rightarrow 0$$

because u & w are orthogonal

$$= \|u\|^2 + \|w\|^2$$

Hence $\|u\|^2 \leq \|v\|^2$

$\Rightarrow \|u\| \leq \|v\|$.

$\Rightarrow \|P(v)\| \leq \|v\|$. (Hence proved)

7 For every orthonormal basis $e_1, e_2, \ldots, e_m$ of $\mathbb{U}$,

$$P(v) = \langle v | e_1 \rangle e_1 + \cdots + \langle v | e_m \rangle e_m.$$

$\Rightarrow$ Since $P(v) = u$ so $P(v)$ can be written as LC of basis of $\mathbb{U}$.

$$P(v) = \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \cdots + \alpha_m e_m$$

To find α_i, we will use the fact that $v - P(v)$ is orthogonal to every basis vectors of $\mathbb{U}$.

$$\langle v - P(v) \mid e_i \rangle = 0 \qquad \forall i = 1, 2, \dots, m$$

$$\Rightarrow \langle v \mid e_i \rangle - \langle P(v) \mid e_i \rangle = 0$$

$$\Rightarrow \langle v \mid e_i \rangle - \left\langle \sum_{j=1}^{n} \alpha_j e_j \,\middle|\, e_i \right\rangle = 0$$

Since $\langle e_i \mid e_j \rangle = 0$ when $i \neq j$
$= 1$ when $i = j$

therefore only the term with e_i survive.

$$\Rightarrow \langle v \mid e_i \rangle - \langle \alpha_i e_i \mid e_i \rangle = 0$$

$$\Rightarrow \langle v \mid e_i \rangle - \alpha_i \cdot 1 = 0$$

$$\Rightarrow \alpha_i = \langle v \mid e_i \rangle$$

Therefore, $P(v) = \langle v \mid e_1 \rangle e_1 + \cdots + \langle v \mid e_m \rangle e_m$.

Linear functionals on inner product spaces:

A linear functional, $\varphi : \mathbb{F}^3 \longrightarrow \mathbb{F}$ defined as:

$$\varphi(z_1, z_2, z_3) := 2z_1 - 5z_2 + z_3$$

We can write the linear functional in the form –

$\varphi(z) = \langle z | u \rangle$ (standard inner product)

where $u = (2, -5, 1)$

consider a functional, $\phi : \mathbb{R}[x]_2 \longrightarrow \mathbb{R}$ defined as:

$$\varphi(P) := \int_{-1}^{1} P(t) \cos(\pi t) \, dt$$

This linear functional $\varphi(P)$ can also be represented in terms of inner product.

$\exists\, u \in \mathbb{R}[x]_2$ such that $\varphi(P) = \langle P | u \rangle$

$$\varphi(P) = \int_{-1}^{1} P(t) u(t) \, dt.$$

Definitely $u(t) \neq \cos(\pi t)$ because $\cos(\pi t) \notin \mathbb{R}[x]_2$

If $u \in \mathbb{V}$ then the map that sends v to $\langle v | u \rangle$ is a linear functional on $\mathbb{V}$. Every linear functional on $\mathbb{V}$ is of this form.

Riesz Representation theorem:

Suppose $\mathbb{V}$ is a finite dimensional inner product space and ϕ is a linear functional on $\mathbb{V}$. Then there exists an unique vector $u \in \mathbb{V}$ such that,

$$\phi(v) = \langle v | u \rangle \quad \text{for every } v \in \mathbb{V}.$$

proof:

let $e_1, e_2, \dots, e_n$ be the orthonormal bases of $\mathbb{V}$.
Consider an arbitrary vector $v \in \mathbb{V}$.

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n \quad , \forall \alpha_i \in \mathbb{F}$$

$$\Rightarrow \phi(v) = \phi\left(\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n\right)$$

$$\Rightarrow \phi(v) = \alpha_1 \phi(e_1) + \alpha_2 \phi(e_2) + \dots + \alpha_n \phi(e_n)$$

Take the inner product with all the bases –

$$\langle v | e_1 \rangle = \alpha_1 \cdot \|e_1\|^2 = \alpha_1$$

$$\langle v | e_2 \rangle = \alpha_2$$

$$\vdots$$

$$\langle v | e_n \rangle = \alpha_n$$

$$\Rightarrow \phi(v) = \langle v|e_1\rangle \phi(e_1) + \cdots + \langle v|e_n\rangle \phi(e_n)$$

Since ϕ is a linear functional, $\phi(e_i) \in \mathbb{F}$. $\forall i$

$$\Rightarrow \phi(v) = \langle v|\, \overline{\phi(e_1)}\, e_1 \rangle + \cdots + \langle v|\, \overline{\phi(e_n)}\, e_n \rangle$$

$$\Rightarrow \phi(v) = \langle v|\, \overline{\phi(e_1)}e_1 + \overline{\phi(e_2)}e_2 + \cdots + \overline{\phi(e_n)}\, e_n \rangle$$

Note that $\overline{\phi(e_i)} \in \mathbb{F}$ and $e_i \in \mathbb{V}$ $\forall i$

therefore consider, $u = \overline{\phi(e_1)}\, e_1 + \cdots + \overline{\phi(e_n)}\, e_n$.

$$\Rightarrow \phi(v) = \langle v|u\rangle$$

Therefore, $\exists u \in \mathbb{V}$ s.t. $\phi(v) = \langle v|u\rangle$ $\forall v \in \mathbb{V}$.

Now we have to prove that there is only one vector $u \in \mathbb{V}$ such that the above equation is true.

Suppose, $u_1 \in \mathbb{V}$, $u_2 \in \mathbb{V}$ and $u_1 \neq u_2$. such that

$\phi(v) = \langle v|u_1\rangle$, $\phi(v) = \langle v|u_2\rangle$.

$$\Rightarrow \langle v|u_1\rangle = \langle v|u_2\rangle$$

$$\Rightarrow \langle v|u_1\rangle - \langle v|u_2\rangle = 0$$

$\Rightarrow \langle v | u_1 - u_2 \rangle = 0$.

$\Rightarrow v$ is orthogonal to $u_1 - u_2$.

This must be true for all $v \in V$. That means it must be true for $u_1 - u_2$ also.

$$\Rightarrow \langle u_1 - u_2 | u_1 - u_2 \rangle = 0$$

$$\Rightarrow \| u_1 - u_2 \|^2 = 0$$

$$\Rightarrow \| u_1 - u_2 \| = 0$$

$$\Rightarrow u_1 - u_2 = 0_V.$$

$\Rightarrow u_1 = u_2$ (contradiction).

Ex: Find $u \in \mathbb{R}[x]_2$ such that $\phi(P) = \int_{-1}^{1} P(t) \cos \pi t \, dt$ can be written as. $\phi(P) = \langle P | u \rangle$ for all $P(x) \in \mathbb{R}[x]_2$

where, $\langle p | u \rangle := \int_{-1}^{1} p(t) u(t) \, dt$.

We know that the u that exists is in the form -

$$u = \overline{\phi(e_1)}\, e_1 + \overline{\phi(e_2)}\, e_2 + \overline{\phi(e_3)}\, e_3.$$

Take the standard basis $\{1, x, x^2\}$.
Apply Gram Schimdt process to orthonormalize it.

$$e_1 = \frac{1}{\sqrt{2}}$$

$$e_2 = \sqrt{\frac{3}{2}}\, x$$

$$e_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

Since everything is in $\mathbb{R}$ so we can write,

$$u = \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} \cos(\pi t)\, dt\right) \cdot \frac{1}{\sqrt{2}} + \left(\int_{-1}^{1} \sqrt{\frac{3}{2}}\, x \cdot \cos \pi t\, dt\right) \sqrt{\frac{3}{2}}\, x$$

$$+ \left(\int_{-1}^{1} \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \cos(\pi t)\, dt\right) \cdot \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$= -\frac{45}{2\pi^2}\left(x^2 - \frac{1}{3}\right).$$

Adjoint :

Suppose T is a linear transformation from $\mathbb{V}$ to $\mathbb{W}$. The adjoint of T is the transformation T^*: $\mathbb{W}$ to $\mathbb{V}$. s.t.

$$\langle T(v) \mid w \rangle = \langle v \mid T^*(w) \rangle \quad \forall v \in \mathbb{V},\ \forall w \in \mathbb{W}$$

$$
\begin{tikzcd}
\mathbb{V} \arrow[r, "T"] & \mathbb{W} \\
v \arrow[r, mapsto] & T(v) \arrow[d, mapsto, "\langle \cdot | w \rangle"] \\
& \langle T(v) | w \rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{W} \arrow[r, "T^*"] & \mathbb{V} \\
w \arrow[r, mapsto] & T^*(w) \arrow[d, mapsto, "\langle v | \cdot \rangle"'] \\
& \langle v | T^*(w) \rangle
\end{tikzcd}
$$

$$\langle T(v) | w \rangle =\!= \langle v | T^*(w) \rangle$$

Suppose we fix a $w \in \mathbb{W}$.

Consider a linear functional ϕ on V that maps $v \in \mathbb{V}$ to $\langle T(v) | w \rangle$.

$$
\begin{tikzcd}
\mathbb{V} \arrow[rr, "\phi"] & & \mathbb{F} \\
v \arrow[rr, mapsto] \arrow[d, mapsto, "T"'] & & \phi(v) \arrow[d, equal, "\shortparallel"] \\
T(v) \arrow[rr, mapsto, "\langle \cdot | w \rangle"] & & \langle T(v) | w \rangle \\
\mathbb{W} & & \mathbb{F}
\end{tikzcd}
$$

We define, ϕ in such a way that, $\phi(v) = \langle T(v) | w \rangle$

Now $\phi(v)$ is a linear functional from $\mathbb{V} \to \mathbb{F}$. By Riesz representation theorem, the linear functional

$\phi(v)$ can be represented as the following manner –

$\exists!\, u \in \mathbb{V}$ such that $\phi(v) = \langle v | u \rangle$ $(\forall v \in \mathbb{V}.)$

$\Rightarrow \langle T(v) | w \rangle = \langle v | u \rangle$

and we claim that there is a transformation T^* that takes vector from $\mathbb{W}$ and give vector in $\mathbb{V}$. So the vector u can be obtained from $T^*(w)$ uniquely.

Hence $\langle T(v) | w \rangle = \langle v | T^*(w) \rangle$. for all $v \in \mathbb{V}$.

<u>Ex:</u> $T: \mathbb{R}^3 - \mathbb{R}^2$.

$T(x_1, x_2, x_3) := (x_2 + 3x_3, 2x_1)$

Find a formula for T^*?

$\Rightarrow$ We must fix a point $(y_1, y_2) \in \mathbb{R}^2$. With respect to this point, we will calculate $T^*: \mathbb{R}^2 \to \mathbb{R}^3$.

$$\langle (x_1, x_2, x_3) | T^*(y_1, y_2) \rangle = \langle T(x_1, x_2, x_3) | (y_1, y_2) \rangle$$

$$= \langle (x_2 + 3x_3, 2x_1) | (y_1, y_2) \rangle$$

$$= (x_2 + 3x_3)(y_1) + (2x_1)(y_2) = (2y_2)x_1 + (y_1)x_2 + (3y_1)x_3$$

$$= \langle (x_1, x_2, x_3) | (2y_2, y_1, 3y_1) \rangle$$

$$= \langle (x_1, x_2, x_3) \mid T^*(y_1, y_2) \rangle$$

Therefore, $T^*(y_1, y_2) = (2y_2, y_1, 3y_1)$

If $T^* = T$ then T is called self adjoint.

$$\langle Tv \mid w \rangle = \langle v \mid Tw \rangle .$$