EE 635: Applied Linear Algebra
Assignment 4

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25. With the inner product as defined in Question 24, obtain an orthonormal basis for all polynomials of degree less than or equal to 3, starting from the basis $\{1, x, x^2, x^3\}$, using the Gram Schmidt procedure, over the interval $[0, 1]$.

let $\mathbb{V}$: Vector space of all real valued polynomials of degree less or equal to 3 in the interval $[0,1]$.

inner product defined as: $\langle f | g \rangle := \int_0^1 f(t)\, g(t)\, dt$

The set, $S = \{1, x, x^2, x^3\}$.

Therefore, $u_1 = 1$

$u_2 = x$

$u_3 = x^2$

$u_4 = x^3$.

We have to find an orthogonal set, $Q = \{v_1, v_2, v_3, v_4\}$ from the set S.

Applying Gram Schmidt procedure:

$$v_K = u_K - \sum_{i=1}^{K-1} c_i v_i$$

$$v_K = u_K - \sum_{i=1}^{K-1} \frac{\langle u_K | v_i \rangle}{\|v_i\|^2} \cdot v_i$$

for $K=1$, $\boxed{v_1 = u_1 = 1}$

$$\|v_1\|^2 = \langle v_1 | v_1 \rangle = \int_0^1 1.1.dx = [t]_0^1 = 1.$$

$$\|v_1\| = \sqrt{1} = 1$$

for $K=2$, $$v_2 = u_2 - \frac{\langle u_2 | v_1 \rangle}{\|v_1\|^2} \cdot v_1$$

$$\langle u_2 | v_1 \rangle = \int_0^1 x \cdot 1 \cdot dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\boxed{v_2 = x - \frac{1}{2} \cdot \frac{1}{1} \cdot 1 = x - \frac{1}{2}}$$

$$\|v_2\|^2 = \langle v_2 | v_2 \rangle = \int_0^1 \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^2 + \frac{1}{4} - 2.x.\frac{1}{2}\right) dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4}x\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{4-6+3}{12} = \frac{1}{12}$$

for $K=3$, $$v_3 = u_3 - \frac{\langle u_3 | v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3 | v_2 \rangle}{\|v_2\|^2} \cdot v_2$$

$$\langle u_3 | v_1 \rangle = \int_0^1 x^2 \cdot 1 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\langle u_3 | v_2 \rangle = \int_0^1 x^2 \cdot \left(x - \frac{1}{2}\right) dx = \left[\frac{x^4}{4} - \frac{x^3}{6} \right]_0^1 = \frac{1}{12}$$

$$v_3 = x^2 - \frac{1}{3} \cdot \frac{1}{1} \cdot v_1 - \frac{1}{12} \cdot 12 \cdot v_2$$

$$= x^2 - \frac{1}{3} \cdot 1 - \left(x - \frac{1}{2}\right)$$

$$= x^2 - x - \frac{1}{3} + \frac{1}{2}$$

$$\boxed{v_3 = x^2 - x + \frac{1}{6}}$$

$$\|v_3\|^2 = \langle v_3 | v_3 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$$

$$= \int_0^1 \left(\left(x^2 - x\right)^2 + \left(\frac{1}{6}\right)^2 + 2 \cdot \left(x^2 - x\right) \cdot \frac{1}{6} \right) dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + x^2 + \frac{1}{36} + \frac{1}{3}x^2 - \frac{1}{3}x \right) dx$$

$$= \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + \frac{1}{36}x + \frac{1}{3}\frac{x^3}{3} - \frac{1}{3}\frac{x^2}{2} \right]_0^1 = \frac{1}{180}$$

for $K=4$,

$$v_4 = u_4 - \frac{\langle u_4 | v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_4 | v_2 \rangle}{\|v_2\|^2} \cdot v_2 - \frac{\langle u_4 | v_3 \rangle}{\|v_3\|^2} \cdot v_3$$

$$\langle u_4 | v_1 \rangle = \int_0^1 x^3 \cdot 1 \, dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

$$\langle u_4 | v_2 \rangle = \int_0^1 x^3 \cdot \left(x - \frac{1}{2} \right) dx = \int_0^1 \left(x^4 - \frac{1}{2} x^3 \right) dx$$

$$= \left[\frac{x^5}{5} - \frac{1}{2} \frac{x^4}{4} \right]_0^1 = \frac{1}{5} - \frac{1}{8} = \frac{3}{40}$$

$$\langle u_4 | v_3 \rangle = \int_0^1 x^3 \left(x^2 - x + \frac{1}{6} \right) dx$$

$$= \int_0^1 \left(x^5 - x^4 + \frac{1}{6} x^3 \right) dx$$

$$= \left[\frac{x^6}{6} - \frac{x^5}{5} + \frac{1}{6} \frac{x^4}{4} \right]_0^1 = \frac{1}{6} - \frac{1}{5} + \frac{1}{24}$$

$$= \frac{20 - 24 + 5}{120} = \frac{1}{120}$$

$$v_4 = x^3 - \frac{1}{4}\cdot 1\cdot 1 - \frac{3}{40}\cdot 12\cdot\left(x-\frac{1}{2}\right) - \frac{1}{120}\cdot 180\cdot\left(x^2-x+\frac{1}{6}\right)$$

$$= x^3 - \frac{1}{4} - \frac{9}{10}\left(x-\frac{1}{2}\right) - \frac{3}{2}\left(x^2-x+\frac{1}{6}\right)$$

$$= x^3 - \frac{1}{4} - \frac{9x}{10} + \frac{9}{20} - \frac{3x^2}{2} + \frac{3x}{2} - \frac{1}{4}$$

$$\boxed{v_4 = x^3 - \frac{3x^2}{2} + \frac{3x}{5} - \frac{1}{20}}$$

$$\|v_4\| = \langle v_4 | v_4 \rangle = \int_0^1 \left(x^3 - \frac{3x^2}{2} + \frac{3x}{5} - \frac{1}{20}\right)^2 dx$$

$$= \int_0^1 \left(x^3 - \frac{3x^2}{2}\right)^2 + \left(\frac{3x}{5} - \frac{1}{20}\right)^2 + 2\cdot\left(x^3 - \frac{3x^2}{2}\right)\left(\frac{3x}{5} - \frac{1}{20}\right) dx$$

$$= \int_0^1 \Big(x^6 + \frac{9}{4}\cdot x^4 - 2\cdot x^3\cdot\frac{3x^2}{2} + \frac{9x^2}{25} + \frac{1}{400} - 2\cdot\frac{3x}{5}\cdot\frac{1}{20} + 2\left(\frac{3x^4}{5} - \frac{1}{20}x^3 - \frac{9x^3}{10} + \frac{3x^2}{40}\right)\Big) dx$$

$$= \Big[\frac{x^7}{7} + \frac{9}{4}\cdot\frac{x^5}{5} - 3\cdot\frac{x^6}{6} + \frac{9}{25}\frac{x^3}{3} + \frac{1}{400}x - \frac{6}{100}\frac{x^2}{2} + \frac{6}{5}\frac{x^5}{5} - \frac{1}{10}\frac{x^4}{4} - \frac{18}{10}\frac{x^4}{4} + \frac{6}{40}\frac{x^3}{3}\Big]_0^1 = \frac{1}{2800}$$

The orthonormal basis are as follows:

$$\hat{v}_1 = \frac{v_1}{\|v_1\|} = 1$$

$$\hat{v}_2 = \frac{v_2}{\|v_2\|} = \sqrt{12}\left(x - \frac{1}{2}\right)$$

$$\hat{v}_3 = \frac{v_3}{\|v_3\|} = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)$$

$$\hat{v}_4 = \frac{v_4}{\|v_4\|} = \sqrt{2800}\left(x^3 - \frac{3x^2}{2} + \frac{3x}{5} - \frac{1}{20}\right)$$

8. Using the result(s) in Question 6, show that for a linear operator $T : \mathbb{V} \to \mathbb{V}$ ($\mathbb{V}$ being a finite dimensional inner product space), show that there exists a unique linear operator T^* on $\mathbb{V}$ such that $\langle Tv_1 | v_2 \rangle = \langle v_1 | T^* v_2 \rangle \; \forall v_1, v_2 \in \mathbb{V}$.

Consider a linear functional ϕ on the vector space $\mathbb{V}$ over $\mathbb{F}$.

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "\phi"] & \mathbb{F} \\
v_1 \arrow[r, mapsto] \arrow[d, "T"] & \phi(v_1) \arrow[d, phantom, "\parallel"] \\
T(v_1) \arrow[r, mapsto, "\langle \cdot | v_2 \rangle"] & \langle T(v_1) | v_2 \rangle \\
\in \mathbb{V} & \in \mathbb{F}.
\end{tikzcd}$$

Let's take an arbitrary vector $v_2 \in \mathbb{V}$. (let's fix this).
Then we can transform $T(v)$ into $\mathbb{F}$ with the help of inner product as shown in the diagram.

We define the linear functional ϕ in such a way that.

$$\phi(v_1) := \langle T(v_1) | v_2 \rangle \quad \text{for a fix arbitrary } v_2 \in \mathbb{V}.$$

Since we have a well defined linear functional on $\mathbb{V}$ so now we can use Riesz representation theorem as follows:

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "\phi"] & \mathbb{F} \\
v_1 \arrow[r, mapsto] \arrow[d, mapsto, "\langle \cdot | u \rangle"] & \phi(v_1) \arrow[dl, equal] \\
\langle v_1 | u \rangle \in \mathbb{F} &
\end{tikzcd}$$

$\exists!\, u \in \mathbb{V}$ such that $\phi(v_1) = \langle v_1 | u \rangle \quad \forall v_1 \in \mathbb{V}$.

We have already seen that $\phi(v_1) = \langle T(v_1) | v_2 \rangle$.
Therefore,

$$\langle T(v_1) | v_2 \rangle = \langle v_1 | u \rangle. \quad \forall v_1 \in \mathbb{V}, \forall v_2 \in \mathbb{V}$$

We also know that $u \in \mathbb{V}$ is an unique vector in $\mathbb{V}$ such that the above equation is true. We can show the existense of such u by the action of another transformation $T^*: \mathbb{V} \to \mathbb{V}$.

Let's choose u such that, $\quad u = T^*(v_2)$

Therefore, T^* is defined in such a way that $T^*(v_2) := u$.

So there exists a T^* such that,

$$\langle T(v_1) | v_2 \rangle = \langle v_1 | T^*(v_2) \rangle \quad \forall v_1, v_2 \in \mathbb{V}. \quad \text{--- (i)}$$

We have just proved the existence of T^*. Next we have to prove that T^* is linear operator.

Since the equation (i) is true for $\forall v_1, v_2 \in \mathbb{V}$ so consider, $\quad v_1 = v \quad$ and $\quad v_2 = c \cdot w_1 + w_2$

$$\langle v | T^*(c \cdot w_1 + w_2) \rangle$$

$$= \langle T(v) | cw_1 + w_2 \rangle \quad \text{(from equation (i))}$$

$$= \langle T(v) | cw_1 \rangle + \langle T(v) | w_2 \rangle$$

$$= \bar{c} \cdot \langle T(v) | w_1 \rangle + \langle T(v) | w_2 \rangle$$

$$= \bar{c} \cdot \langle v | T^*(w_1) \rangle + \langle v | T^*(w_2) \rangle \quad \text{(from eqn (i))}$$

$$= \langle v | c \cdot T^*(w_1) \rangle + \langle v | T^*(w_2) \rangle$$

$$= \langle v \mid cT^*(w_1) + T^*(w_2) \rangle$$

Therefore,

$$\langle v \mid T^*(cw_1+w_2) \rangle = \langle v \mid cT^*(w_1) + T^*(w_2) \rangle$$

$$\Rightarrow \langle v \mid T^*(cw_1+w_2) - cT^*(w_1) - T^*(w_2) \rangle = 0_{\mathbb{F}}$$

The above equation is true for all $v \in \mathbb{V}$.

$$\Rightarrow T^*(cw_1+w_2) - cT^*(w_1) - T^*(w_2) = 0_{\mathbb{V}}$$

$$\Rightarrow T^*(cw_1+w_2) = cT^*(w_1) + T^*(w_2).$$

Hence T^* is a linear operator.

Now we have to prove that T^* is unique.

Suppose there is T^* and R^* such that, $T^* \neq R^*$

$$\langle T(v_1) \mid v_2 \rangle = \langle v_1 \mid T^*(v_2) \rangle \quad \forall v_1, v_2 \in \mathbb{V}$$

$$\langle T(v_1) \mid v_2 \rangle = \langle v_1 \mid R^*(v_2) \rangle \quad \forall v_1, v_2 \in \mathbb{V}$$

$$\Rightarrow \langle v_1 \mid T^*(v_2) \rangle = \langle v_1 \mid R^*(v_2) \rangle$$

$$\Rightarrow \langle v_1 \mid T^*(v_2) - R^*(v_2) \rangle = 0_{\mathbb{F}}$$

$$\Rightarrow \langle v_1 \mid (T^* - R^*)(v_2) \rangle = 0_{\mathbb{F}}$$

Since it is true for all $v_1 \in \mathbb{V}$ therefore,

$\Rightarrow (T^* - R^*)(v_2) = 0_{\mathbb{V}}$.

Since it is true for all $v_2 \in \mathbb{V}$ therefore,

$\Rightarrow T^* - R^* = 0_{L(V,V)}$ (zero transformation).

$\Rightarrow T^* = R^*$ (It is a contradiction of the assumption).

Therefore, it is proved that T^* is unique linear operator.

10. If an operator T on an inner product space, $\mathbb{V}$, has a corresponding unique T^* as defined in Question 8, then T is said to have an ***adjoint*** in $\mathbb{V}$. For two such linear operators, T and R on $\mathbb{V}$, and $\alpha \in \mathbb{F}$, show that (a) $(R+T)^* = R^* + T^*$, (b) $(\alpha T)^* = \bar{\alpha} T^*$, (c) $(TR)^* = R^* T^*$, (d) $(T^*)^* = T$ (e) for an invertible T, $(T^*)^{-1} = (T^{-1})^*$ (f) $\mathrm{im}(T^*) = (\ker(T))^\perp$

$T^* : \mathbb{V} \to \mathbb{V}$ is defined in such a way that,

$$\langle T(v) | w \rangle = \langle v | T^*(w) \rangle \quad \forall v, w \in \mathbb{V} \quad \text{— (i)}$$

(a) $(R+T)^* = R^* + T^*$.

consider 2 vectors v, w and take adjoint of $(R+T)$ s.t.

$$\begin{aligned}
\langle v | (R+T)^*(w) \rangle &= \langle (R+T)(v) | w \rangle \quad (\text{Eqn } 1) \\
&= \langle Rv + Tv | w \rangle \\
&= \langle Rv | w \rangle + \langle Tv | w \rangle \\
&= \langle v | R^* w \rangle + \langle v | T^* w \rangle \\
&= \langle v | R^* w + T^* w \rangle
\end{aligned}$$

$$= \langle v | (R^* + T^*) w \rangle$$

$$\Rightarrow \langle v | (R+T)^* w \rangle - \langle v | (R^* + T^*) w \rangle = 0_{\mathbb{F}}$$

$$\Rightarrow \langle v | (R+T)^* w - (R^* + T^*) w \rangle = 0_{\mathbb{F}}$$

Since it is true for all $v \in \mathbb{V}$ so,

$$(R+T)^* w - (R^* + T^*) w = 0_{\mathbb{V}}.$$

$$\Rightarrow \left((R+T)^* - (R^* + T^*) \right) w = 0_{\mathbb{V}}$$

Since it is true for all $w \in \mathbb{V}$ so,

$$(R+T)^* - (R^* + T^*) = 0_{\mathcal{L}(V,V)} \quad (\text{zero linear map})$$

$$\Rightarrow (R+T)^* = R^* + T^* \quad (\text{Hence proved}).$$

(b) $(\alpha T)^* = \bar{\alpha} T^*$

consider 2 vectors $v, w \in \mathbb{V}$ and take adjoint of αT s.t.

$$\langle v | (\alpha T)^* w \rangle = \langle (\alpha T)(v) | w \rangle \quad (\text{Eqn } 1)$$

$$= \langle \alpha T(v) | w \rangle$$

$$= \alpha \langle T(v) | w \rangle$$

$$= \alpha . \langle v | T^*(w) \rangle$$

$$= \langle v | \bar{\alpha} T^*(w) \rangle$$

By following the similar reasoning in the previous part(a) we conclude, $(\alpha T)^* = \bar{\alpha} T^*$ (Hence proved).

(c) $(TR)^* = R^* T^*$.

consider 2 vectors $v, w \in \mathbb{V}$ and take adjoint of TR s.t.

$$\langle v | (TR)^* w \rangle = \langle (TR) v | w \rangle \qquad \text{(Eqn 1)}$$

$$= \langle T(R(v)) | w \rangle$$

$$= \langle R(v) | T^*(w) \rangle$$

$$= \langle v | R^*(T^*(w)) \rangle$$

$$= \langle v | (R^* T^*)(w) \rangle$$

By following the similar reasoning in the part (a), we conclude, $(TR)^* = (R^* T^*)$ (Hence proved).

(d) $(T^*)^* = T$

Consider 2 vectors $v, w \in \mathbb{V}$ and take adjoint of T^* s.t.

$$\langle v | (T^*)^* w \rangle = \langle T^*(v) | w \rangle \qquad \text{(Eqn 1)}$$

$$= \overline{\langle w | T^*(v) \rangle}$$

$$= \overline{\langle T(w) | v \rangle}$$

$$= \langle v | T(w) \rangle$$

By following the similar reasoning in the part (a), we conclude that,

$$(T^*)^* = T \quad \text{(Hence proved).}$$

(e) T^{-1} exists, $(T^*)^{-1} = (T^{-1})^*$.

Since T^{-1} exists so $TT^{-1} = T^{-1}T = I$.

We already know that, $(TR)^* = R^*T^*$

consider,

$$T^*(T^{-1})^* = (T^{-1}T)^* = I^*$$

consider 2 vectors $v, w \in V$ and take adjoint of I s.t

$$\langle v | I^*(w) \rangle = \langle I(v) | w \rangle \qquad \text{(eqn 1)}$$

$$= \langle v | w \rangle$$

$$= \langle v | I(w) \rangle$$

Therefore from reasoning of part (a), it is concluded,

$I^* = I$.

Hence, $T^*(T^{-1})^* = I^* = I$

Similarly, $(T^{-1})^* T^* = (TT^{-1})^* = I^* = I$

(Since $AB = BA = I$ so $A^{-1} = B$ and $B^{-1} = A$ where,

$A = (T^{-1})^*$ and $B = T^*$)

Therefore, $(T^*)^{-1} = (T^{-1})^*$ from the above 2 equations.

(Hence proved).

(f) $im(T^*) = (Ker(T))^{\perp}$

We know that $(S^{\perp})^{\perp} = S$ for any subspace S of $\mathbb{V}$.

Taking orthogonal complement on both sides -

$\Rightarrow (Im(T^*))^{\perp} = ((Ker(T))^{\perp})^{\perp} = Ker(T)$

So we have to prove that, $Ker(T) = (Im(T^*))^{\perp}$

Let's substitute T with T^*

We know $(T^*)^* = T$.

$\Rightarrow Ker(T^*) = (Im((T^*)^*))^{\perp} = (Im(T))^{\perp}$

Therefore, we must prove $ker(T^*) = (Im(T))^{\perp}$.

Suppose, $w \in ker(T^*)$.

$\Leftrightarrow T^*(w) = 0_{\mathbb{V}}$.

$\Leftrightarrow \langle v | T^*(w) \rangle = \langle T(v) | w \rangle = 0_{\mathbb{F}}$ for all $v \in \mathbb{V}$.

(eqn 1)

Since $\langle T(v) | w \rangle = 0_{\mathbb{F}} \quad \forall v \in \mathbb{V}$

We can say that w is orthogonal to $T(v)$.

We know that $T(v) \in Im(T)$.

Suppose, $u = T(v)$.

Since we see that $\forall u \in Im(T)$, $\langle u|w\rangle = 0_{\mathbb{F}}$

if and only if w is in orthogonal complement of $Im(T)$

Hence $w \in (Im(T))^{\perp}$

Therefore, $w \in Ker(T^*) \longleftrightarrow w \in (Im(T))^{\perp}$

Hence $Ker(T^*) = (Im(T))^{\perp}$ (proved).

11. An operator, $T : \mathbb{V} \to \mathbb{V}$, is said to be ***self-adjoint*** if $T = T^*$. Prove the following: (a) If $\mathbb{V}$ is finite dimensional, and $T = T^2$, then $TT^* = T^*T \iff T$ is self adjoint. (b) Composition of two self adjoint operators is also self adjoint if and only if they commute. (c) If $\mathbb{V}$ is a finite dimensional complex inner product space, then T is self-adjoint if and only if $\langle Tv|v\rangle$ is real for every $v \in \mathbb{V}$.

T is self adjoint. Hence we can write, $T^* = T$.

$$\langle T(v)|w\rangle = \langle v|T(w)\rangle \quad \forall v, w \in \mathbb{V}. \quad \text{---(i)}$$

(a) Given that $T = T^2$.

We have to prove $T^*T = TT^* \longleftrightarrow T$ is self adjoint

or $T^*T = TT^* \longleftrightarrow T = T^*$

Given $T^2 = T$

$\Rightarrow T.T = T$

Take adjoint on both sides:

$$(T.T)^* = T^*$$

$\Rightarrow T^*T^* = T^*$ —— (ii)

Suppose assume that $T = T^*$.

Obviously $T^* = T^*$

$\Rightarrow T^*(T^*) = (T^*)T^*$ (from (ii))

$\Rightarrow T^*(T) = (T)T^*$ ($\because T = T^*$)

$\Rightarrow T^*T = TT^*$.

Therefore, $T = T^* \Longrightarrow T^*T = TT^*$ (Hence proved)

Now consider,

$\langle Tv | (TT^* - T)v \rangle$

$= \langle v | T^*(TT^* - T)v \rangle$

$= \langle v | (T^*TT^* - T^*T)v \rangle$

$= \langle v | (T(T^*)^2 - T^*T)v \rangle$ $[\because T^*T = TT^*]$.

$= \langle v | (TT^* - T^*T)v \rangle$ $[\because T^*T^* = T^*]$

$= 0$

Therefore, $TT^* - T = 0 \Rightarrow T = TT^*$ —(i)

Now consider,

$\langle T^* v | (TT^* - T^*) v \rangle$

$= \langle v | T(TT^* - T^*) v \rangle$

$= \langle v | (TTT^* - TT^*) v \rangle$

$= \langle v | ((T)^2 T^* - T^* T) v \rangle \quad [\because T^2 = T]$

$= \langle v | (TT^* - TT^*) v \rangle \quad [\because T^* T = TT^*]$

$= 0$

Therefore, $TT^* - T^* = 0 \Rightarrow T^* = TT^*$ —(ii)

From (i) & (ii), we can see that $T = T^*$ (Hence proved)

(b) Consider 2 self adjoint operators T and R.

Therefore, $T^* = T$ and $R^* = R$.

We have to prove that,

$$(TR)^* = TR \longleftrightarrow TR = RT.$$

Suppose, $(TR)^* = TR$

$\Rightarrow TR = R^* T^* = R \cdot T$ (since $R^* = R$, $T^* = T$)

("$\Rightarrow$" direction is proved)

Suppose $TR = RT$

$\Rightarrow (TR)^* = (RT)^* = T^* R^* = T \cdot R$ ($R^* = R, T^* = T$)

("$\Leftarrow$" direction is proved)

Therefore we see that $(TR)^* = TR \Longleftrightarrow TR = RT$.

(Hence proved)

ⓒ $\mathbb{V}$ is f.d.v.s which is complex inner product space.
We have to prove that,

$$T^* = T \longleftrightarrow \langle Tv|v\rangle \text{ is real } \forall v \in \mathbb{V}.$$

Suppose, $T^* = T$.

$$\langle Tv|v\rangle = \langle v|Tv\rangle \quad [\because \text{ from eqn (1)}]$$

$$\Rightarrow \langle Tv|v\rangle - \langle v|Tv\rangle = 0_{\mathbb{F}}$$

$$\Rightarrow \langle Tv|v\rangle - \overline{\langle Tv|v\rangle} = 0_{\mathbb{F}}$$

For any complex number, $z = a+ib$, $\bar{z} = a-ib$,

$z - \bar{z} = a+ib-a+ib = 2.ib$.

$$\Rightarrow 2i \cdot \mathrm{Im}(\langle Tv|v\rangle) = 0_{\mathbb{F}}.$$

Since $2i \neq 0$ so we have $\mathrm{Im}(\langle Tv|v\rangle) = 0_{\mathbb{F}}$.

for a complex number z, $\mathrm{Im}(z) = 0 \longrightarrow z = a$ only.

Hence, $\langle Tv|v\rangle = \mathrm{Re}(\langle Tv|v\rangle)$.

Therefore $\langle Tv|v\rangle$ is purely real number. (Proved)

Suppose, $\langle Tv|v\rangle$ is real

$\Rightarrow \operatorname{Im}(\langle Tv|v\rangle) = 0$.

$$\langle Tv|v\rangle = \overline{\langle v|Tv\rangle}$$

$$\Rightarrow \overline{\langle Tv|v\rangle} = \langle v|Tv\rangle$$

Since $\langle Tv|v\rangle$ is real so $\overline{\langle Tv|v\rangle} = \langle Tv|v\rangle$

$$\Rightarrow \langle Tv|v\rangle = \langle v|Tv\rangle$$

$\Rightarrow T = T^*$ (from eqn (i)) (Hence proved)

Therefore, $T = T^* \leftrightarrow \langle Tv|v\rangle$ is real (proved)

6. For a finite dimensional inner product space $\mathbb{V}$, show that every linear functional $f : \mathbb{V} \to \mathbb{F}$, can be given by $f(v) = \langle v|\tilde{v}\rangle$ for a unique $\tilde{v} \in \mathbb{V}$. Further, show that this unique vector $\tilde{v} \in (\operatorname{Ker}(f))^{\perp}$. Hence, show that if P is the orthogonal projection of any vector in $\mathbb{V}$ on $(\operatorname{Ker}(f))^{\perp}$ then $f(v) = f(Pv)$ for all $v \in \mathbb{V}$.

Consider a linear functional $f : \mathbb{V} \to \mathbb{F}$ as follows:

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "f"] & \mathbb{F} \\
v \arrow[r, mapsto] \arrow[d, mapsto, "\langle \cdot|\tilde{v}\rangle"'] & f(v) \\
\langle v|\tilde{v}\rangle \arrow[ur, equal] &
\end{tikzcd}$$

We have to prove the existence of $\tilde{v}$ so that
$f(v) = \langle v | \tilde{v} \rangle$.

Consider the orthonormal bases for vector space $\mathbb{V}$ as
$\{e_1, e_2, \ldots, e_n\}$.

Pick an arbitrary vector, $v \in \mathbb{V}$.

$v = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \quad (\forall \alpha_i \in \mathbb{F})$.

Since all the basis are orthonormal to each other we can find the coordinate α_i by taking inner products.

$$\langle v | e_1 \rangle = \alpha_1 \langle e_1 | e_1 \rangle = \alpha_1 \|e_1\|^2 = \alpha_1 \cdot 1 = \alpha_1$$

Similarly, $\quad \alpha_i = \langle v | e_i \rangle \quad \forall i = 1, 2, \ldots, n$

Hence, $\quad v = \langle v | e_1 \rangle e_1 + \langle v | e_2 \rangle e_2 + \cdots + \langle v | e_n \rangle e_n$.

Apply f on both sides of equation:

$$f(v) = \langle v | e_1 \rangle f(e_1) + \langle v | e_2 \rangle f(e_2) + \cdots + \langle v | e_n \rangle f(e_n)$$

[$\because f$ is linear]

$\forall i$, $f(e_i) \in \mathbb{F}$ so we can take it inside of inner product

$$\Rightarrow f(v) = \langle v | \overline{f(e_1)}\, e_1 \rangle + \cdots + \langle v | \overline{f(e_n)}\, e_n \rangle$$

$$\Rightarrow f(v) = \left\langle v \,\middle|\, \overline{f(e_1)}\, e_1 + \overline{f(e_2)}\, e_2 + \cdots + \overline{f(e_n)}\, e_n \right\rangle \quad \text{---} (i)$$

$$\Rightarrow f(v) = \langle v \mid \tilde{v} \rangle$$

Where, $\tilde{v} = \overline{f(e_1)}\, e_1 + \overline{f(e_2)}\, e_2 + \cdots + \overline{f(e_n)}\, e_n \in \mathbb{V}$.

Therefore we have seen that there exists a $\tilde{v} \in \mathbb{V}$ such that $f(v) = \langle v | \tilde{v} \rangle$.

Now we need to prove that $\tilde{v}$ is unique.

Suppose there are 2 vectors $\tilde{v}_1$ and $\tilde{v}_2$ s.t $\tilde{v}_1 \neq \tilde{v}_2$.

$$\left.\begin{aligned} f(v) &= \langle v | \tilde{v}_1 \rangle \\ f(v) &= \langle v | \tilde{v}_2 \rangle \end{aligned}\right] \quad \forall v \in \mathbb{V}.$$

$$\Rightarrow \langle v | \tilde{v}_1 \rangle = \langle v | \tilde{v}_2 \rangle \quad \forall v \in \mathbb{V}.$$

$$\Rightarrow \langle v | \tilde{v}_1 - \tilde{v}_2 \rangle = 0_{\mathbb{F}}. \quad \forall v \in \mathbb{V}.$$

Since it is true for all v so take, $v = \tilde{v}_1 - \tilde{v}_2$.

$$\Rightarrow \langle \tilde{v}_1 - \tilde{v}_2 | \tilde{v}_1 - \tilde{v}_2 \rangle = 0_{\mathbb{F}}$$

$$\Rightarrow \| \tilde{v}_1 - \tilde{v}_2 \|^2 = 0_{\mathbb{F}}$$

$$\Rightarrow \| \tilde{v}_1 - \tilde{v}_2 \| = 0_{\mathbb{F}}$$

$$\Rightarrow \tilde{v}_1 - \tilde{v}_2 = 0_{\mathbb{V}}$$

$\Rightarrow \tilde{v}_1 = \tilde{v}_2$ (contradiction).

Hence $f(v) = \langle v | \tilde{v} \rangle \ \forall v \in \mathbb{V}$ and $\tilde{v}$ is unique.

(Proved)

From equation (i), we have seen that,

$$\tilde{v} = \overline{f(e_1)}\, e_1 + \overline{f(e_2)}\, e_2 + \cdots + \overline{f(e_n)}\, e_n.$$

We have to prove that $\tilde{v} \in (\mathrm{Ker}(f))^{\perp}$.

$$\mathrm{Ker}(f) := \{ u \in \mathbb{V} \mid f(u) = 0_{\mathbb{F}} \}.$$

$$(\mathrm{Ker}(f))^{\perp} := \{ w \in \mathbb{V} \mid \forall u \in \mathrm{Ker}(f) \text{ s.t. } \langle w | u \rangle = 0_{\mathbb{F}} \}.$$

Take $u \in \mathrm{Ker}(f)$

$\Leftrightarrow f(u) = 0_{\mathbb{F}}$.

Suppose $w \in (\mathrm{ker} f)^{\perp}$ if and only if it follows that

$$\langle w | u \rangle = 0 \quad \forall u \in \mathrm{Ker}(f). \quad \text{———— (ii)}$$

$\tilde{v} \in (\mathrm{Ker}(f))^{\perp}$ if and only if we get $\langle \tilde{v} | u \rangle = 0_{\mathbb{F}}$.

Consider $\langle \tilde{v} | u \rangle$ and see if it is $0_{\mathbb{F}}$ or not. If it is $0_{\mathbb{F}}$ then the proof is done.

$\langle \tilde{v} | u \rangle = \overline{\langle u | \tilde{v} \rangle}$

$= \overline{f(u)}$ Since $\langle u | \tilde{v} \rangle = f(u)$ $\forall u \in \mathbb{V}$ (Riesz repres. theorem)

$= \overline{0_{\mathbb{F}}}$ since $f(u) = 0_{\mathbb{F}}$ as $u \in Ker(f)$

$= 0_{\mathbb{F}}$.

Since $\tilde{v} \in (Ker(f))^{\perp} \iff \langle \tilde{v} | u \rangle = 0_{\mathbb{F}}$ $\forall u \in Ker(f)$

Hence it is proved that $\tilde{v} \in (Ker(f))^{\perp}$.

Define orthogonal projection $P : \mathbb{V} \to (Ker(f))^{\perp}$.

$$\begin{aligned} \mathbb{V} &\xrightarrow{\quad P \quad} (Ker(f))^{\perp} \qquad (\text{dim of } (Ker(f))^{\perp} = m) \\ v &\longmapsto P(v) = \hat{v} \end{aligned}$$

where $\hat{v}$ is the best approximation of v on $(Ker(f))^{\perp}$.

We have to prove that $f(v) = f(P(v))$ $\forall v \in \mathbb{V}$.

Since $P(v) = \hat{v}$ (best approximation) and P is the orthogonal projection so the following must be true -

$\langle v - \hat{v} | u \rangle = 0_{\mathbb{F}}$ for all $u \in (Ker(f))^{\perp}$

$\Rightarrow \langle v - P(v) | u \rangle = 0_{\mathbb{F}}$.

Since it is true for all $u \in (ker(f))^{\perp}$ so it must be true for $\tilde{v}$ which belongs to $(Ker(f))^{\perp}$.

$$\Rightarrow \langle v - P(v) \mid \tilde{v} \rangle = 0_{\mathbb{F}}$$

$$\Rightarrow \langle v \mid \tilde{v} \rangle - \langle P(v) \mid \tilde{v} \rangle = 0_{\mathbb{F}}$$

$$\Rightarrow \langle v \mid \tilde{v} \rangle = \langle P(v) \mid \tilde{v} \rangle$$

$$\Rightarrow f(v) = f(P(v)) \quad [\because \text{Riesz representation theorem}]$$

Hence it is proved that, $f(P(v)) = f(v) \; (\forall v \in \mathbb{V}.)$.

7. For a finite dimensional inner product space, $\mathbb{V}$, show that every linear functional $f : \mathbb{V} \to \mathbb{F}$, can be given by $f(v) = \langle v|\tilde{v}\rangle$ for a unique $\tilde{v} \in \mathbb{V}$ (This result is called *Riesz Representation Theorem*). Further, show that this unique vector $\tilde{v} \in (\mathrm{Ker}(f))^{\perp}$. Hence, show that if P is the orthogonal projection of any vector in $\mathbb{V}$ on $(\mathrm{Ker}(f))^{\perp}$ then $f(v) = f(Pv)$ for all $v \in \mathbb{V}$.

Same as Q.6. Please cheek the answer of Q.6.

24. Let $\mathfrak{C}_{\mathbb{R}}[a,b]$ be the vector space of all real valued functions on the interval $[a,b]$ (usual addition and scalar multiplication hold). Consider an inner product defined as $\langle f|g\rangle := \int_a^b f(t)g(t)dt$. Further, let $\mathbb{W}$ be the subspace of all polynomials with real coefficients. What is $\mathbb{W}^{\perp}$? Is $\mathbb{W} = (\mathbb{W}^{\perp})^{\perp}$? Show appropriate steps to justify your answer.

Suppose, $\mathbb{V} = C_{R}[a,b] = \{ f \mid f(x) \text{ on the interval } [a,b] \}$

inner product, $\langle f | g \rangle = \int_a^b f(t)\, g(t)\, dt.$

Given $\mathbb{W} = \{ P \mid P(x) \text{ is polynomials with real coefficients} \}$

$$= \{ P \mid P(x) = \sum_{i=1}^{\infty} c_i x_i^i \;,\; \forall c_i \in \mathbb{R} \}.$$

$\mathbb{W}^{\perp} := \{ f \in \mathbb{V} \mid \forall p \in \mathbb{W} \text{ s.t } \langle f | p \rangle = 0_{\mathbb{F}} \}.$

Consider a function, $f \in \mathbb{W}^{\perp}$

$\Rightarrow \forall p \in \mathbb{W}, \; \langle f | p \rangle = 0_{\mathbb{F}}$

$$\Rightarrow \forall p \in \mathbb{W}, \; \int_a^b f(t)\, p(t)\, dt = 0_{\mathbb{F}}.$$

$\Rightarrow f(t)$ is orthogonal to every polynomials $p(t)$ in $\mathbb{W}$. However polynomials are dense in the space of continuous functions on the interval $[a,b]$. So there are infinite such polynomials exits in the interval $[a,b]$.

Since the equation is true for all $p(t) \in \mathbb{W}$, therefore, $f(t)$ must be zero function $(0_{\mathbb{V}})$. because only $0_{\mathbb{V}}$ is orthogonal to all the functions $p(t) \in \mathbb{W}$ and nothing else.

$$\text{Hence, } \int_a^b 0_{\mathbb{V}} \cdot p(t)\, dt = 0_{\mathbb{F}}.$$

Therefore, $f(t) = 0_{\mathbb{V}}(t)$

So $f = 0_{\mathbb{V}}$ or $f \in \{0_{\mathbb{V}}\}$.

So we can say $f \in \mathbb{W}^{\perp} \rightarrow f \in \{0_{\mathbb{V}}\}$.

So $\mathbb{W}^{\perp} \subseteq \{0_{\mathbb{V}}\}$.

If we can show the other side subset then we can conclude that $W^{\perp} = \{0_V\}$.

Claim: $\mathbb{W}^{\perp} \supseteq \{0_V\}$.

proof:

let $f \in \{0_V\}$ so $f = 0_V$.

$$\Rightarrow \forall p \in \mathbb{W} \quad \int_a^b f(t)\, p(t)\, dt = \int_a^b 0_V \cdot p(t)\, dt = 0_{\mathbb{F}}.$$

Hence as per definition of $\mathbb{W}^{\perp}$, $f \in \mathbb{W}^{\perp}$.

Therefore, $W^{\perp} \supseteq \{0_V\}$.

So we have proved that, $\mathbb{W}^{\perp} = \{0_V\}$.

$\Rightarrow (\mathbb{W}^{\perp})^{\perp} = \mathbb{V}$. [every function is orthogonal to the zero function].

But $\mathbb{W} \subseteq \mathbb{V}$.

$\Rightarrow \mathbb{W} \subset (\mathbb{W}^{\perp})^{\perp}$ & $\mathbb{W} = (\mathbb{W}^{\perp})^{\perp}$

Hence it is proved that $\mathbb{W} = (\mathbb{W}^{\perp})^{\perp}$.

21. Show that two vectors v_1, v_2 in a complex inner product space are orthogonal if and only if $||\alpha v_1 + \beta v_2||^2 = ||\alpha v_1||^2 + ||\beta v_2||^2$ for all $\alpha, \beta \in \mathbb{C}$.

We have to prove that

$$\langle v_1 | v_2 \rangle = 0 \longleftrightarrow \|\alpha v_1 + \beta v_2\|^2 = \|\alpha v_1\|^2 + \|\beta v_2\|^2$$

consider, $\|\alpha v_1 + \beta v_2\|^2 - \|\alpha v_1\|^2 - \|\beta v_2\|^2$

$$\Leftrightarrow \langle \alpha v_1 + \beta v_2 | \alpha v_1 + \beta v_2 \rangle - \langle \alpha v_1 | \alpha v_1 \rangle - \langle \beta v_2 | \beta v_2 \rangle$$

$$\Leftrightarrow \langle \alpha v_1 | \alpha v_1 + \beta v_2 \rangle + \langle \beta v_2 | \alpha v_1 + \beta v_2 \rangle - \langle \alpha v_1 | \alpha v_1 \rangle - \langle \beta v_2 | \beta v_2 \rangle$$

$$\Leftrightarrow \cancel{\langle \alpha v_1 | \alpha v_1 \rangle} + \langle \alpha v_1 | \beta v_2 \rangle + \langle \beta v_2 | \alpha v_1 \rangle + \cancel{\langle \beta v_2 | \beta v_2 \rangle} - \cancel{\langle \alpha v_1 | \alpha v_1 \rangle} - \cancel{\langle \beta v_2 | \beta v_2 \rangle}$$

$$\Leftrightarrow \langle \alpha v_1 | \beta v_2 \rangle + \langle \beta v_2 | \alpha v_1 \rangle$$

$$\Leftrightarrow \alpha \bar{\beta} \langle v_1 | v_2 \rangle + \beta \bar{\alpha} \langle v_2 | v_1 \rangle$$

$$\Leftrightarrow \alpha \bar{\beta} \langle v_1 | v_2 \rangle + \beta \bar{\alpha} \overline{\langle v_1 | v_2 \rangle}$$

$$\Leftrightarrow \alpha \bar{\beta} \langle v_1 | v_2 \rangle + \overline{\alpha \bar{\beta} \langle v_1 | v_2 \rangle}$$

Suppose a complex number $z = x + iy$, $\bar{z} = x - iy$

$z + \bar{z} = 2x = 2 \cdot \mathrm{Re}(z)$.

$$\Leftrightarrow 2 \cdot \mathrm{Re}\left(\alpha \bar{\beta} \langle v_1 | v_2 \rangle\right)$$

Since $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, and the above term will be 0 if and only if $\langle v_1 | v_2 \rangle = 0$ because it has to be 0 for all α, β.

consider $2 \cdot \mathrm{Re}\left(\alpha \bar{\beta} \langle v_1 | v_2 \rangle\right) = 0_{\mathbb{F}} \quad \left(\forall \alpha, \beta \in \mathbb{C}\right)$

$\Leftrightarrow \langle v_1 | v_2 \rangle = 0_{\mathbb{F}}$.

Hence $\|\alpha v_1 + \beta v_2\|^2 = \|\alpha v_1\|^2 + \|\beta v_2\|^2 \Longleftrightarrow \langle v_1 | v_2 \rangle = 0_{\mathbb{F}}$

(Hence proved)

14. Suppose $\{w_1, w_2, \ldots, w_n\}$ is an orthonormal basis for $\mathbb{V}$ and that there exists another set of vectors, $S = \{s_1, s_2, \ldots, s_n\}$ such that $||s_i - w_i|| < \frac{1}{\sqrt{n}}$ $\forall i$. Show that S is a basis of $\mathbb{V}$.

$S = \{s_1, s_2, \ldots s_n\}$ s.t $\|s_i - w_i\| < \frac{1}{\sqrt{n}}$ $\forall i$.

We have to porove that S is a basis of $\mathbb{V}$. Because we have $\dim(\mathbb{V}) = n$ so if we can prove S is LI then it is guarented that it will span the vector space $\mathbb{V}$. So we just have to porove linear independence of set S.

Assume that $\{s_i\}_{i=1}^{n}$ are LI set. Then the following is true.

consider, $$\sum_{i=1}^{n} c_i s_i = 0_{\mathbb{V}}. \quad \left(\exists c_i \neq 0\right)$$

$$\Rightarrow \sum_{i=1}^{n} c_i (s_i - w_i + w_i) = 0_{\mathbb{V}}$$

$$\Rightarrow \sum_{i=1}^{n} c_i(s_i - w_i) = -\sum_{i=1}^{n} c_i w_i$$

$$\Rightarrow \left\| \sum_{i=1}^{n} c_i(s_i - w_i) \right\| = \left\| \sum_{i=1}^{n} c_i w_i \right\|$$

By triangle inequality we can show that -

$$\left\| \sum_{i=1}^{n} c_i(s_i - w_i) \right\| \leq \sum_{i=1}^{n} |c_i| \, \| (s_i - w_i) \|$$

Since it is given that, $(s_i - w_i) < \frac{1}{\sqrt{n}}$ for all i so we can write -

$$\left\| \sum_{i=1}^{n} c_i(s_i - w_i) \right\| \leq \sum_{i=1}^{n} |c_i| \cdot \frac{1}{\sqrt{n}}$$

$$\Rightarrow \left\| \sum_{i=1}^{n} c_i(s_i - w_i) \right\|^2 \leq \frac{1}{n} \left(\sum_{i=1}^{n} |c_i| \right)^2$$

$$\Rightarrow \left\| \sum_{i=1}^{n} c_i w_i \right\|^2 \leq \frac{1}{n} \left(\sum_{i=1}^{n} |c_i| \right)^2$$

We also know that,

$$\left\| \sum_{i=1}^{n} c_i w_i \right\|^2 = \sum_{i=1}^{n} |c_i|^2$$

$$\Rightarrow \sum_{i=1}^{n} |c_i|^2 \leq \frac{1}{n} \cdot \left(\sum_{i=1}^{n} |c_i| \right)^2$$

We also know from Cauchy Swartz inequality that -

$$\frac{\left(\sum_{i=1}^{n} |c_i|\right)^2}{n} \leq \sum_{i=1}^{n} |c_i|^2$$

(Known as Sedrakyan's inequality)

Therefore, we have -

$$\sum_{i=1}^{n} |c_i|^2 = \frac{1}{n} \left(\sum_{i=1}^{n} |c_i| \right)^2 \quad \text{——— (i)}$$

The equality holds in Cauchy Swartz inequality implies that, there is a constant d s.t. $|c_i| = d/\sqrt{n}$ for all i so $c_i = \lambda_i \cdot \frac{d}{\sqrt{n}}$ where $|\lambda_i| = 1$. Since $\exists c_i \neq 0$ so $d \neq 0$.

$$0 = \sum_{i=1}^{n} c_i s_i = \sum_{i=1}^{n} \lambda_i \cdot \frac{d}{\sqrt{n}} s_i = \frac{d}{\sqrt{n}} \sum_{i=1}^{n} \lambda_i s_i$$

Since $d \neq 0$ so $\sum_{i=1}^{n} \lambda_i s_i = 0$

We already have, $\left\| \sum_{i=1}^{n} c_i w_i \right\| = \left\| \sum_{i=1}^{n} c_i (s_i - w_i) \right\|$

Since $\|-v\| = \|v\|$ so we can say-

$$\left\| \sum_{i=1}^{n} c_i w_i \right\| = \left\| \sum_{i=1}^{n} c_i (w_i - s_i) \right\|.$$

$$\Rightarrow \left\| \sum_{i=1}^{n} \frac{\lambda_i \cdot d}{\sqrt{n}} w_i \right\| = \left\| \sum_{i=1}^{n} \frac{\lambda_i d}{\sqrt{n}} (w_i - s_i) \right\|$$

$$\Rightarrow \left\| \sum_{i=1}^{n} \lambda_i w_i \right\| = \left\| \sum_{i=1}^{n} \lambda_i (w_i - s_i) \right\|.$$

We also have,

$$\|(w_i - s_i)\| < \frac{1}{\sqrt{n}} \quad \forall i$$

Therefore,

$$\Rightarrow \left\| \sum_{i=1}^{n} \lambda_i (w_i - s_i) \right\| < \sum_{i=1}^{n} \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

$[\because |\lambda_i| = 1]$

Therefore,

$$\sum_{i=1}^{n} \|w_i - s_i\| = \sqrt{n} < \sqrt{n} \quad (\text{from given fact})$$

Which is a contradiction as $\sqrt{n} < \sqrt{n}$ can't be possible.
This shows that it is impossible that $v_1, v_2, \dots, v_n$ are LD.
Hence it is LI so S is basis for $\mathbb{V}$ (proved).

18. Suppose a vector space $\mathbb{V}$ can be expressed as a direct sum of two of its subspaces as $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2$. Let $u_1(\cdot,\cdot) : \mathbb{U}_1 \times \mathbb{U}_1 \to \mathbb{C}$ and $u_2(\cdot,\cdot) : \mathbb{U}_2 \times \mathbb{U}_2 \to \mathbb{C}$ be inner products defined on the two subspaces $\mathbb{U}_1$ and $\mathbb{U}_2$, respectively. Show that there exists a unique inner product $v(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ such that we have $\mathbb{U}_1^{\perp} = \mathbb{U}_2$ and $v(p,q) = u_k(p,q)$, when $p, q \in \mathbb{U}_k,\ k \in \{1,2\}$.

Suppose, $\langle \circ | \circ \rangle_{\mathbb{U}_1}$ is the inner product defined in $\mathbb{U}_1$,

$\langle \circ | \circ \rangle_{\mathbb{U}_2}$ is the inner product defined in $\mathbb{U}_2$.

Suppose a vector $v \in \mathbb{V}$ is written as $v = u_1 + u_2$ where, $u_1 \in \mathbb{U}_1$ and $u_2 \in \mathbb{U}_2$.

consider the norm of v as per the inner product $\langle \circ | \circ \rangle_{\mathbb{V}}$.

$$\|v\|^2 = \langle v | v \rangle_{\mathbb{V}}.$$

$$= \langle u_1 + u_2 \mid u_1 + u_2 \rangle_{\mathbb{V}}.$$

$$= \langle u_1 | u_1 \rangle_{\mathbb{V}} + \langle u_1 | u_2 \rangle_{\mathbb{V}} + \langle u_2 | u_1 \rangle_{\mathbb{V}} + \langle u_2 | u_2 \rangle_{\mathbb{V}} \quad \text{--- (i)}$$

It is given that, $\mathbb{U}_1^{\perp} = \mathbb{U}_2$.

So, $u_1^{\perp} := \{ u \in \mathbb{V} \mid \forall u_1 \in \mathbb{U}_1 \text{ s.t. } \langle u_1 | u \rangle_{\mathbb{V}} = 0_{\mathbb{F}} \}$

$$= \mathbb{U}_2.$$

That means $\langle u_2 | u_1 \rangle_{\mathbb{V}} = 0 \ \forall u_1 \in \mathbb{U}_1$ and $u_2 \in \mathbb{U}_2$.

Substituting this in eqn (i) we get -

Therefore, $\|v\|^2 = \langle u_1|u_1\rangle_{\mathbb{V}} + \langle u_2|u_2\rangle_{\mathbb{V}}$.

$$\Rightarrow \langle v|v\rangle_{\mathbb{V}} = \langle u_1|u_1\rangle_{\mathbb{U}_1} + \langle u_2|u_2\rangle_{\mathbb{U}_2}. \quad \text{--- (ii)}$$

It is also given as,

$$p,q \in \mathbb{U}_1, \quad \langle p|q\rangle_{\mathbb{V}} = \langle p|q\rangle_{\mathbb{U}_1}$$

$$\langle p|q\rangle_{\mathbb{V}} = \langle p|q\rangle_{\mathbb{U}_2}.$$

Since $u_1 \in \mathbb{U}_1$ so $\langle u_1|u_1\rangle_{\mathbb{V}} = \langle u_1|u_1\rangle_{\mathbb{U}_1}$

Since $u_2 \in \mathbb{U}_2$ so $\langle u_2|u_2\rangle_{\mathbb{V}} = \langle u_2|u_2\rangle_{\mathbb{U}_2}$

Substituting this in eqn (ii) we get –

$$\langle v|v\rangle_{\mathbb{V}} = \langle u_1|u_1\rangle_{\mathbb{U}_1} + \langle u_2|u_2\rangle_{\mathbb{U}_2}.$$

Therefore it is proved that there exists an inner product $\langle\cdot|\cdot\rangle_{\mathbb{V}}$ such that the given condition is satisfied.

Now we have to check whether the inner product is uniquely determined by $\langle\cdot|\cdot\rangle_{\mathbb{U}_1}$ and $\langle\cdot|\cdot\rangle_{\mathbb{U}_2}$ or not.

Suppose there exists $u_1, w_1 \in \mathbb{U}_1$ s.t. $u_1 \neq w_1$ and $u_2, w_2 \in \mathbb{U}_2$ s.t. $u_2 \neq w_2$. such that –

$$\left.\begin{aligned}\langle v|v\rangle_{\mathbb{V}} &= \langle u_1|u_1\rangle_{\mathbb{U}_1} + \langle u_2|u_2\rangle_{\mathbb{U}_2}.\\ \langle v|v\rangle_{\mathbb{V}} &= \langle w_1|w_1\rangle_{\mathbb{U}_1} + \langle w_2|w_2\rangle_{\mathbb{U}_2}.\end{aligned}\right\}\text{ is true.}$$

$$\Rightarrow \langle u_1|u_1\rangle_{\mathbb{U}_1} + \langle u_2|u_2\rangle_{\mathbb{U}_2} = \langle w_1|w_1\rangle_{\mathbb{U}_1} + \langle w_2|w_2\rangle_{\mathbb{U}_2}$$

$$\Rightarrow \langle u_1|u_1\rangle_{\mathbb{V}} + \langle u_2|u_2\rangle_{\mathbb{V}} + \underbrace{\langle u_1|u_2\rangle_{\mathbb{V}} + \langle u_2|u_1\rangle_{\mathbb{V}}}_{=0} =$$

$$\langle w_1|w_1\rangle_{\mathbb{V}} + \langle w_2|w_2\rangle_{\mathbb{V}} + \underbrace{\langle w_1|w_2\rangle_{\mathbb{V}} + \langle w_2|w_1\rangle_{\mathbb{V}}}_{=0}.$$

$$\Rightarrow \langle u_1+u_2|u_1+u_2\rangle_{\mathbb{V}} = \langle w_1+w_2|w_1+w_2\rangle_{\mathbb{V}} = \langle v|v\rangle_{\mathbb{V}}$$

This means,

$v = u_1+u_2$

$v = w_1+w_2$ s.t. $u_1 \neq w_1$ and $u_2 \neq w_2$

But it is a contradiction of the fact that $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2$ meaning only one possible way, we can write w in terms of u_1 and u_2 from $\mathbb{U}_1$ & $\mathbb{N}_2$. So it is proved that the inner product $\langle\cdot|\cdot\rangle_{\mathbb{V}}$ must be uniquely determined from $\langle\cdot|\cdot\rangle_{\mathbb{U}_1}$ and $\langle\cdot|\cdot\rangle_{\mathbb{U}_2}$.

28. ('*Mindless*'(?) *Interpolation*)

(a) Argue why the Vandermonde matrix given by

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{bmatrix}$$

must have full column rank whenever $n \leq m$ and (real-valued) x_i's are all distinct.

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & & x_2^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_m & x_m^2 & & x_m^{n-1} \end{bmatrix}_{m \times n}$$

We have to prove that,

if x_i's are distinct and $n \leq m$ then V has full rank.

We are going to use proof by induction.

Base case: Columns of this matrix are in $\mathbb{R}^m$.

Suppose, $n = 1$ and $1 \leq m$. then ovbiously the set $\{ [1 \ 1 \ \dots \ 1]^t \}$ is LI. so it has full rank of, $\rho = n = 1$.

Suppose, $n=2$ and $2 \leq n$. Then we have to prove that the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \right\}$ is LI set.

consider,

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix}_{m\times 2} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Performing row operations -

$$\begin{bmatrix} 1 & x_1 \\ 0 & x_2 - x_1 \\ 0 & x_3 - x_1 \\ & \vdots \\ 0 & x_m - x_1 \end{bmatrix}_{m\times 2} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since it is given that all x_i's are distinct so,

$x_2 - x_1 \neq 0$, $x_3 - x_1 \neq 0$, ..., $x_m - x_1 \neq 0$.

Hence, we get, 2 pivot columns and

$$c_1 + c_2 \cdot x_1 = 0 \Rightarrow c_1 = 0$$

$$(x_2 - x_1)c_2 = 0 \Rightarrow c_2 = 0 \quad (\because x_2 - x_1 \neq 0)$$

Therefore, the set is LI set and rank, $\rho = n = 2$. (full column rank).

<u>Inductive step:</u> Consider the following set,

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \end{bmatrix}, \ldots, \begin{bmatrix} x_1^{k-1} \\ x_2^{k-1} \\ \vdots \\ x_m^{k-1} \end{bmatrix} \right\}$$

Suppose this set is LI set and $\rho = n = k \leq m$. As per the induction hypothesis this set is assumed to be LI set and it has full column rank when formed in the matrix.

let's see if we can infer for $n = k+1$ also it is true. So we have to prove that,

$$S' = S \cup \left\{ \begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_m^k \end{bmatrix} \right\}$$ is also a LI set.

given that S is a LI set.

To prove this we keep the vectors in matrix form-

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{K-1} & x_1^K \\ 1 & x_2 & x_2^2 & \cdots & x_2^{K-1} & x_2^K \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & & x_m^{K-1} & x_m^K \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \\ c_{K+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since we know that set S is LI set so we can transform it to a row reduced echelon form. which is identity matrix of size $m \times K$.

$$\begin{array}{c} \\ \\ \\ \\ K \rightarrow \\ \\ \\ m \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & & 0 & y_1 \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & & 0 & y_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & y_K \\ 0 & 0 & 0 & & 0 & y_{K+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 0 & y_m \end{bmatrix}$$

(the last column will become some other column which has all non-zero since $\forall x_i \neq 0$ and we will also see that x_K is distinct with other x_i)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & y_1 \\ 0 & 1 & 0 & \cdots & 0 & y_2 \\ & & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & y_K \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(dividing by $y_{K+1}, \dots, y_m$ and then subtracting the rows from $K+1$ to m will $1, 0, 0, \dots 0$ at the $K, K+1, \dots m$th rows)

$\Rightarrow \quad c_1 + y_1 \cdot c_{k+1} = 0 \quad \Rightarrow \quad c_1 = 0$

$c_2 + y_2 \cdot c_{k+1} = 0 \quad \Rightarrow \quad c_2 = 0$

$\vdots$

$c_k + y_k \cdot c_{k+1} = 0 \quad \Rightarrow \quad c_k = 0$

$c_{k+1} = 0$

Therefore, we see that all $c_i = 0$ hence the set S' is LI set and $\rho = n = k+1$. So by induction hypothesis we have proven that it is true for all k.

Set $k = n-1$ so it is proved that the matrix V has full column rank.

(b) Suppose a set of m points with distinct x_i's are given by $\{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$. Show that the unique polynomial of degree $m-1$ passing through all the points is given by $\ell(x) = \sum_{i=1}^{m} \left(y_i \frac{\prod_{j \neq i}^{m} (x - x_j)}{\prod_{j \neq i}^{m} (x_i - x_j)} \right)$.

Suppose $f(x)$ is an arbitrary real function. The $(m-1)^{th}$ degree polynomial passing through m points as:

$x_1, y_1 = x_1, f(x_1)$

$x_2, y_2 = x_2, f(x_2)$

$\vdots$

$x_m, y_m = x_m, f(x_m)$

is given as,

$$l(x) = \sum_{j=1}^{m} f(x_j) \cdot L_j(x) = \sum_{j=1}^{m} y_j L_j(x)$$

Where $L_j(x)$ is the $(m-1)^{th}$ degree polynomial defined as

$$L_j(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_m)}{(x_j-x_1)(x_j-x_2)\cdots(x_j-x_m)} = \prod_{i \neq j} \left(\frac{x-x_i}{x_j-x_i} \right)$$

if $L_i(x)$ satisfy. $L_1(x_1)=1,\ L_2(x_1)=0, \ldots, L_m(x_1)=0$

then $l(x_1) = f(x_1)$

if $L_i(x)$ satisfy $L_1(x_2)=0,\ L_2(x_2)=1, \ldots, L_m(x_2)=0$

then $l(x_2) = f(x_2)$

In the same way we see that if $L_i(x)$ satisfy,

$$L_1(x_1)=1,\ L_2(x_1)=0, \ldots, L_m(x_1)=0$$

$$L_1(x_2)=0,\ L_2(x_2)=0, \ldots, L_m(x_2)=0$$

$$\vdots$$

$$L_1(x_m)=0,\ L_2(x_m)=0, \ldots, L_m(x_m)=1$$

then $l(x_j) = f(x_j)$ for $j=1,2,\ldots,m$.

Since $L_i(x_j) = 0$ when $i \neq j$ and $L_i(x_j) = 1$ if $i = j$ then we can factorize $L_1(x)$ as follows –

$$L_1(x) = \alpha_1 (x - x_2) \cdots (x - x_m) \quad \text{where } \alpha_1 = \text{const.}$$

Since $L_1(x_1) = 1$ so we get,

$$1 = \alpha_1 (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_m)$$

$$\Rightarrow \alpha_1 = \frac{1}{\prod_{1 \neq j} (x_1 - x_j)}$$

Therefore we obtain,

$$L_1(x) = \frac{\prod (x - x_j)}{\prod_{1 \neq j} (x_1 - x_j)}$$

In the similar fashion we can get $L_1, L_2, \ldots, L_m$.

Therefore, $$L_j(x) = \prod_{i \neq j} \frac{(x - x_i)}{(x_j - x_i)}$$

$\{L_1, L_2, \ldots, L_m\}$ forms the dual of the basis given by

$\{1, x, x^2, \cdots, x^{n-1}\}$ because of the way, L_i are defined. Therefore, the linear functional L_i on V is given as -

$$L_i(f) = f(x_i) \qquad 1 \leq i \leq n.$$

Hence we can write -

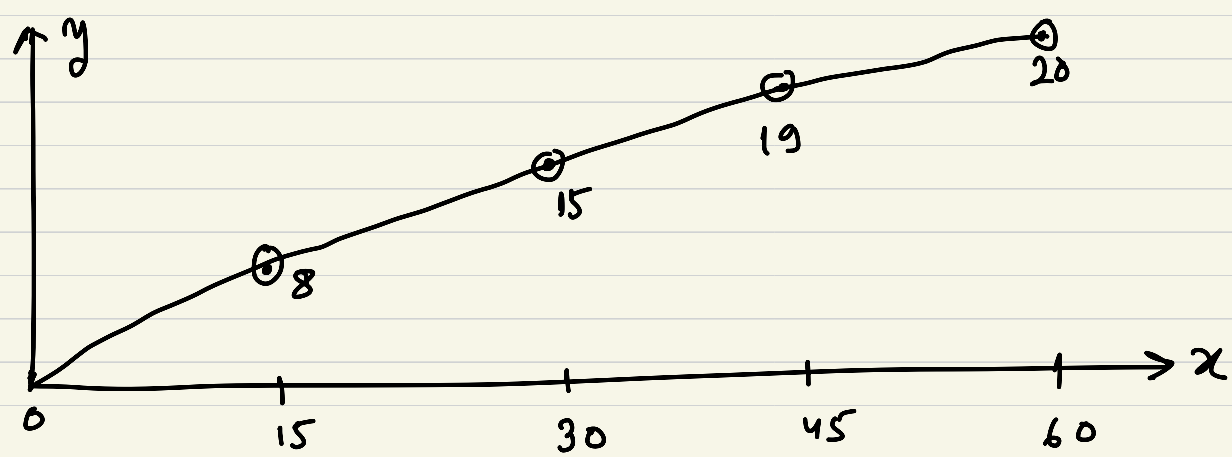
$$l(x) = \sum_{i=1}^{n} f(x_i) \cdot L_i(x). \qquad \text{(Hence it is proved)}$$

It is unique because for $n \times n$ Vandermonde matrix, it is invertible and the coefficients are uniquely determined which proves that $l(x)$ is uniquely determined.

(c) Suppose a defender kicks a ball from his/her half of the football field and the following table contains the positions of the football (on some suitable vertical plane) at different instants of time, as tracked by a camera:

Down range (x) in m	0	15	30	45	60
Height (y) in m	0	8	15	19	20

It is required to predict the position where the ball lands on the ground. Obtain this using the interpolation formula derived in part (b).



We know that,

$$l(x) = \sum_{i=1}^{5} y_i L_i(x)$$

$$L_1(x) = \prod_{i \neq 1} \left(\frac{x - x_i}{x_1 - x_i} \right) = \prod_{i \neq 1} \left(\frac{x - x_i}{0 - x_i} \right)$$

$$= \frac{(x-15)(x-30)(x-45)(x-60)}{(-15)(-30)(-45)(-60)}$$

$$L_2(x) = \frac{(x-0)(x-30)(x-45)(x-60)}{(15-0)(15-30)(15-45)(15-60)}$$

$$L_3(x) = \frac{(x-0)(x-15)(x-45)(x-60)}{(30-0)(30-15)(30-45)(30-60)}$$

$$L_4(x) = \frac{(x-0)(x-15)(x-30)(x-60)}{(45-0)(45-15)(45-30)(45-60)}$$

$$L_5(x) = \frac{(x-0)(x-15)(x-30)(x-45)}{(60-0)(60-15)(60-30)(60-45)}$$

After substituting, we get,

$$l(x) = \frac{x^4}{607500} - \frac{x^3}{4050} + \frac{17x^2}{2700} + \frac{22x}{45}.$$

We have to find the value of x where the value of $l(x) = 0$ because when ball lands on ground then $l(x) = 0$.

Solve for x s.t. $\frac{x^4}{607500} - \frac{x^3}{4050} + \frac{17x^2}{2700} + \frac{22x}{45} = 0$

$x_1 = 0 \rightarrow$ starting point.

$x_2 = -31.27$ } impossible since we are looking for positive x.

$x_3 = 90.63 - 35.77i$
$x_4 = 90.63 + 35.77i$ } impossible since we are looking for real x

Hence no such x possible from the Lagrange interpolation that makes sense.

(d) Suppose, instead of the formula used in part (c), one knows that the football follows a parabolic trajectory (from experience or from high school physics) of the form $y = ax^2 + bx + c$ and uses this to predict where the ball lands. What would be the answer? Which solution makes more *sense* and why?

We have the measurement points.

x	0	15	30	45	60
y	0	8	15	19	20

We try to fit the data by quadratic equation,

$y = ax^2 + bx + c$.

We get -

$$\begin{bmatrix} 0^2 & 0 & 1 \\ 15^2 & 15 & 1 \\ 30^2 & 30 & 1 \\ 45^2 & 45 & 1 \\ 60^2 & 60 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 15 \\ 19 \\ 20 \end{bmatrix}$$

Getting the best approximated solution -

$$\begin{bmatrix} 0^2 & 15^2 & & 60^2 \\ 0 & 15 & \cdots & 60 \\ 1 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 0^2 & 0 & 1 \\ 15^2 & 15 & 1 \\ & \vdots & \\ 15^2 & 15 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} =$$

$$\begin{bmatrix} 0^2 & 15^2 & & 60^2 \\ 0 & 15 & \cdots & 60 \\ 1 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 15 \\ \vdots \\ 20 \end{bmatrix}$$

Solving this system of equation, $a = -0.00539683$

$b = 0.66380952$

$c = -0.22857143$

Hence, $y = -0.0053x^2 + 0.6638x - 0.2285$

Now solving for the roots of the equation, we get –

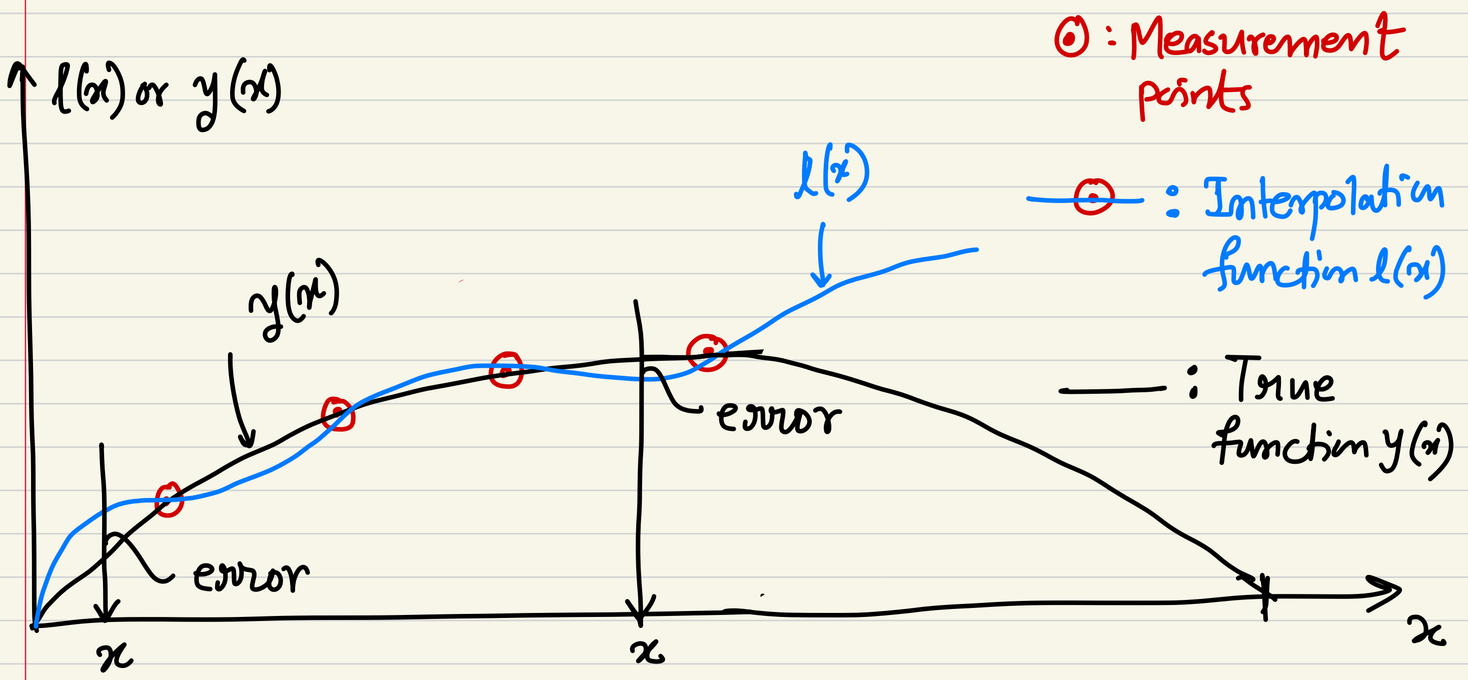
$$-0.0053x^2 + 0.6638x - 0.2285 = 0$$

$\Rightarrow x_1 = 0.34 \simeq 0$ (Starting point)

$x_2 = 122.65$

Therefore when $x = 122.65$, then the ball will land on the ground.

Obviously the physics based solution makes more sense because it is the exact solution which will give the sensible answer at all the points of x. But the interpolation formula tries to match the values of the function at some known points but when interpolated at some unknown x point then the value returned by $L(x)$ may not match with the value returned by $y = ax^2 + bx + c$.


l(x) or y(x)
⊙ : Measurement points
l(x)
⊙ : Interpolation function l(x)
y(x)
: True function y(x)
error
error
x
x
x