EE635 - Solution - Assignment 2

(2) Given, x, y, z linearly independent vectors in $\mathbb{V}$ over $\mathbb{Z}_2$.

Consider

$$1.(x+y) + 1.(y+z) + 1.(z+x)$$

$$= (1+1)x + (1+1)y + (1+1)z$$

$$= 0.x + 0.y + 0.z$$

$$= 0$$

This implies $\exists\, \alpha, \beta, \gamma \neq 0 : \alpha(x+y) + \beta(y+z) + \gamma(z+x) = 0$

$\Rightarrow (x+y), (y+z), (z+x)$ are not linearly independent.

(4) Suppose 1 and x are linearly independent in $\mathbb{C}$ over $\mathbb{R}$

$\Rightarrow \alpha.1 + \beta.x = 0$ can be true only if $\alpha = \beta = 0$

suppose x is a real no.

(a) $x = 0 \Rightarrow 0.1 = -\beta.x$ for any $\beta \neq 0$

Hence, contradiction.

(b) $x \neq 0 \Rightarrow x = x.1$

$\Rightarrow 1.x + (-x).1 = 0$

$\Rightarrow \beta = 1, \alpha = -1$

Satisfies $\alpha.1 + \beta.x = 0$

Hence, contradiction

$\therefore$ 1, x are linearly independent $\Rightarrow$ x is not real

Next, suppose x is not a real no.

$\Rightarrow x = a + ib$, $a, b \in \mathbb{R}$ with $b \neq 0$

consider, $\alpha.1 + \beta(x) = 0$

$\Rightarrow \alpha.1 + \beta.(a+ib) = 0$

$\Rightarrow (\alpha + \beta b) + i\beta b = 0$

$\Rightarrow \alpha + \beta b = 0$ and $\beta b = 0$

$\Rightarrow \beta = 0 \; [\because b \neq 0]$

& $\alpha = 0$

$\Rightarrow$ 1 and x are linearly independent

Hence, x is not real $\Rightarrow$ 1 and x are linearly independent

(6) consider a symmetric matrix A ie $A = A^T$

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

It has 6 independent elements

$\therefore \dim(\mathcal{B}) = 6$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Q. 7)

Note that for an arbitrary field, the only two elements that are guaranteed to

exist are '1' & '0'
and they are distinct.
So our answer must not
assume the existence of
other elements.

Now, we need.

$B = \{A_1, A_2, A_3, A_4\}$ such
that B is a lin. ind.
set & $A_i^2 = A_i$.

let $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$

$\therefore A_i^2 = A_i$

$$\Rightarrow \quad a_i^2 + b_i c_i = a_i \quad (i)$$

$$b_i c_i + d_i^2 = d_i \quad (ii)$$

$$a_i b_i + b_i d_i = b_i \quad (iii)$$

$$c_i a_i + c_i d_i = c_i \quad (iv).$$

Two straight forward solns are $a_i = 1, b_i = c_i = d_i = 0$

& $a_i = b_i = c_i = 0; d_i = 1$

i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

A closer inspection reveals that $a_i = 1, \; b_i = 1,$

$c_i = d_i = 0$

is another soln,

while $d_i = c_i = 1$ &

$a_i = b_i = 0$ is

yet another soln

i.e. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

also satisfy the conditions.

to check lin. ind., let

us consider

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad \alpha + \gamma = 0;$

$\gamma = 0;$

$$\delta = 0;$$
$$\beta + \delta = 0$$
$$\Rightarrow \alpha = \beta = \gamma = \delta = 0$$

Hence lin. ind. verified.

Note that, since

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is a basis, so clearly the dim is 4. Thus, any set of 4 lin. ind vectors forms a basis.

Q. 9) Since $\{v_1, v_2, \dots, v_n\}$ is a basis for V, hence $\dim(V) = n$.

Note that

$$\left|\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}\right| = n.$$

So, it suffices to show that either the set is lin. ind. or a gen. set.

Let

$$\alpha_1(v_1 - v_2) + \alpha_2(v_2 - v_3) + \dots + \alpha_{n-1}(v_{n-1} - v_n) + \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 v_1 + (\alpha_2 - \alpha_1) v_2 + (\alpha_3 - \alpha_2) v_3 + \dots + (\alpha_n - \alpha_{n-1}) v_n = 0$$

Now since $\{v_1, v_2, \ldots, v_n\}$ is a lin. ind. set, hence,

$$\left.\begin{aligned} &\alpha_1 = 0, \\ &\alpha_2 - \alpha_1 = 0 \\ &\vdots \\ &\alpha_n - \alpha_{n-1} = 0 \end{aligned}\right\} \Rightarrow \alpha_i = 0 \;\forall i.$$

Hence, the given set is lin. ind. and thus a basis.

<u>Alter:</u> Since $\{v_1, v_2, \ldots, v_n\}$ is a gen. set, any $v \in V$ can be expressed as

$$\begin{aligned} v &= \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \\ &= \alpha_1 (v_1 - v_2) + (\alpha_2 + \alpha_1)(v_2 - v_3) \\ &\quad + (\alpha_3 + \alpha_2 + \alpha_1)(v_3 - v_4) \\ &\quad + \ldots + (\alpha_{n-1} + \alpha_{n-2} + \ldots + \alpha_1)(v_{n-1} \end{aligned}$$

$+ v_n)$

$$+ (\alpha_n + \alpha_{n-1} + \dots + \alpha_1) v_n$$

$$= \beta_1 (v_1 - v_2) + \beta_2 (v_2 - v_3) + \dots + \beta_n v_n$$

where $\beta_i = \sum_{k=1}^{i} \alpha_i$

$\Rightarrow v \subseteq \langle \{ v_1 - v_2, v_2 - v_3, \dots, v_n \} \rangle.$

Thus, the given set is indeed a generating set.

Q.11) Given: $W_1 \subseteq \mathbb{F}^{2\times 2}$ s.t. it contains matrices of form $\begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix}$ with $\alpha, \beta, \gamma \in \mathbb{F}$

& $W_2 \subseteq \mathbb{F}^{2\times 2}$ s.t. it contains matrices of form

$$\begin{bmatrix} p & q \\ -p & r \end{bmatrix} \text{ with } p, q, r \in \mathbb{F}$$

$\mathcal{B}$, basis for W_1, can be obtained by ~~choosing~~ ~~these~~ making α, β, γ as 1 each at a time keeping others 0 so we get

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To show they are independent let $c_1, c_2, c_3 \in \mathbb{F}$ such that

$$c_1 A_1 + c_2 A_2 + c_3 A_3 = 0$$

$$= \begin{bmatrix} c_1 & -c_1 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. $c_1 = 0$, $c_2 = 0$, $c_3 = 0$

$\therefore$ they are independent

To show they are spanned by A_1, A_2 & A_3 let a general matrix in $W_1 = A = \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix}$

$$= \alpha A_1 + \beta A_2 + \gamma A_3$$

$\therefore$ A_1, A_2 & A_3 are linearly independent & spans W_1 & hence forms basis for W_1

$$\mathcal{B}_{W_1} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Similarly $\mathcal{B}_{W_2} = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Hence, the dimension of $W_1 = 3$, and the dimension of $W_2 = 3$. Also, note that the basis for $\mathbb{W}_1 \cap \mathbb{W}_2$ is given by $\mathcal{B}_{\mathbb{W}_1 \cap \mathbb{W}_2} = \{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\}$, hence dimension of $\mathbb{W}_1 \cap \mathbb{W}_2$ is 2. Also, the basis for $\mathbb{W}_1 + \mathbb{W}_2$ is $\mathcal{B}_{\mathbb{W}_1 + \mathbb{W}_2} = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$, making the dimension for $\mathbb{W}_1 + \mathbb{W}_2$ to be 4.

Q12 For $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times p}$ & $C = AB$ prove that

$$\text{Rank}(C) \leq \min(\text{Rank}(A), \text{Rank}(B))$$

Soln

To prove this, we need a result that

$$\text{Ker}(B) \subseteq \text{Ker}(C)$$

So.

Let $v \in \mathbb{F}^p$ be in $\text{Ker}(B)$ —— (a)

$\therefore Bv = 0$

Now, $(AB)v = 0$ $A(Bv) = 0$

$\Rightarrow Cv = 0$

$\Rightarrow v \in \text{Ker}(C)$

Hence $\text{Ker}(B) \subseteq \text{Ker}(C)$

from Rank Nullity thm

$$\text{rank}(C) + \dim(\text{Ker}(C)) = p \quad \text{—— (I)}$$

$$\& \ \text{rank}(B) + \dim(\text{Ker}(B)) = p \quad \text{—— (II)}$$

$\Rightarrow$ Substract (I) - (II).

$$\text{rank}(C) - \text{rank}(B) = \dim(\text{Ker}(B)) - \dim(\text{Ker}(C))$$

(I - II)

But since

$\text{Ker}(B) \subseteq \text{Ker}(C)$ from Part (a)

$$\Rightarrow \dim(\text{Ker}(B)) \leq \dim(\text{Ker}(C))$$

$$\Rightarrow \text{rank}(C) - \text{rank}(B) \leq 0$$

$$\Rightarrow \text{rank}(C) \leq \text{rank}(B) \quad \text{--- (III)}$$

Considering $C^T = B^T A^T \in \mathbb{F}^{p \times m}$

~~The~~ $\Rightarrow \text{Ker}(A^T) \subseteq \text{Ker}(C^T)$

& then

$$\dim(\text{Ker}(C^T)) + \text{rank}(C^T) = \underbrace{\dim(\text{Ker}(A^T)) + \text{rank}(A^T)}_{= m}$$

Now, since $\text{rank}(A) = \text{rank}(A^T)$

& $\text{rank}(C) = \text{rank}(C^T)$

hence $\text{rank}(C) - \text{rank}(A) = \dim(\text{Ker}(A^T)) - \dim(\text{Ker}(C^T)) \leq 0$

$$\Rightarrow \text{rank}(C) \leq \text{rank}(A) \quad \text{(IV)}$$

Combining (III) & (IV) we get

$$\boxed{\text{rank}(C) \leq \min(\text{rank}(A), \text{rank}(B))}$$

Q14

$W := \{ p \in \mathbb{F}[x]_3 : p(x) = a_0 + a_1 x + a_2 x^2 + (a_0 + a_1 - a_2) x^3 ; a_0, a_1, a_2 \in \mathbb{F} \}$

Clearly W is closed under addition, consider

$p_1(x) = a_0 + a_1 x + a_2 x^2 + (a_0 + a_1 - a_2) x^3$
$p_2(x) = b_0 + b_1 x + b_2 x^2 + (b_0 + b_1 - b_2) x^3$

$p_1 + p_2 = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + a_2) x^2 + (a_0 + b_0 + a_1 + b_1 - a_2 - b_2) x^3$

Let $a_0 + b_0 = c_0$, $a_1 + b_1 = c_1$, $a_2 + b_2 = c_2$

$\therefore p_1 + p_2 = c_0 + c_1 x + c_2 x^2 + (c_0 + c_1 - c_2) x^3$, $c_0, c_1, c_2 \in \mathbb{F}$

$\therefore p_1(x), p_2(x) \in W$, $p_1(x) + p_2(x) \in W$.

Consider $a \in \mathbb{F}$ then

$a p_2(x) = ab_0 + ab_1 x + ab_2 x^2 + (ab_0 + ab_1 - ab_2) x^3$
$= b_0' + b_1' x + b_2' x^2 + (b_0' + b_1' - b_2') x^3$

where $b_0' = ab_0$, $b_1' = ab_1$; $b_2' = ab_2$

$\therefore a p_2(x) \in W$

$\therefore W$ is closed under scalar multiplication.

$\therefore W$ is a subspace of $\mathbb{F}[x]_3$.

for a general polynomial of the form: $p(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^4$, are writing

$$\overset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}} \overset{x}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}} = \overset{b}{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}$$

By performing row operations on the augmente matrix, the image of the 'A' matrix has the 3 following vectors.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \therefore \dim W = 3$$

the vectors correspond to the polynomials $1 + x^3$, $x + x^3$, $x^2 - x^3$, which are linearly independand and span the W space.

Consider the $p_1(x) = 1 + x + x^2 + x^3$, $p_2(x) = 1 + x^2$

$p_1(x) \in W$ as $a_0 = 1$, $a_1 = b$, $a_2 = 1 \Rightarrow a_0 + a_1 - a_2 = 1$

$p_2(x) \in W$ as $a_0 = 1$, $a_1 = 0$ $a_2 = 1 \Rightarrow a_0 - a_2 = 0$

considering a linear combination of p_1 & p_2 such that

$\beta_1 p_1(x) + \beta_2 p_2(x) = 0$, $\beta_1, \beta_2 \in \mathbb{F}$

$\beta_1(1 + x + x^2 + x^3) + \beta_2(1 + x^2) = 0$ the only possible ~~combination~~ choice such that $\beta_1, \beta_2 \in \mathbb{F}$ is $\beta_1 = \beta_2 = 0$

$\therefore \{p_1, p_2\}$ are linearly independant.

Extending the basis by consider the polynomial from above, $\{1 + x + x^2 + x^3, 1 + x^2, x^2 - x^3\}$.

17) For a finite dimensional VS, V show that suppose there exists W_i, $i \in \{1, \dots m\}$ such that

$V = \bigoplus_{i=1}^{m} W_i$. Show that $\dim V = \sum_{i=1}^{m} \dim(W_i)$

Base step: let $i=2$

$\Rightarrow V = W_1 \oplus W_2$

$V = \{x \mid \exists\, w_1 \in W_1 \ \& \ w_2 \in W_2,\ x = \alpha_1 w_1 + \alpha_2 w_2\}$

$\Rightarrow W_1 \cap W_2 = 0$

$\therefore \dim V = \dim W_1 + \dim W_2$

$\therefore$ true for $i=2$

Let the statement be true for $i = m-1$

$\dim V = \sum_{i=1}^{m-1} \dim(W_i)$

To prove when $i=m$

$V = W_1 \oplus W_2 \oplus \dots \oplus W_{m-1} \oplus W_m$

Let $W_1 \oplus W_2 \oplus \dots \oplus W_{m-1} = W'$

$V = W' \oplus W_m$ → being a direct sum decomposition we have

$\dim V = \dim W' + \dim W_m$

$= \sum_{i=1}^{m-1} \dim W_i + \dim W_m$

$= \sum_{i=1}^{m} \dim W_i$

$\square$

Q22. Sol^n:-

Given $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ s.t $\varphi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

T.S.T $\varphi^2 = -I$

From definition of map -

$$\varphi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$\varphi.\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

$$\Rightarrow \varphi.\left\{\varphi.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \varphi^2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \left(\varphi^2 + I\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (\because \text{Map is linear})$$

$$\Rightarrow \left(\varphi^2 + I\right) = 0$$

$$\Rightarrow \varphi^2 = -I$$

Alternative way:-

Take standard basis $\mathbb{R}^2$ as

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

so,

$$\psi(e_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \psi(e_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so,

$$\left[\begin{array}{cc:cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\psi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \quad \psi^2 = -I \quad (\text{check})$$

$$\because \quad \psi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$\psi\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \psi\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

so,

$$\left[\begin{array}{cc:cc} 1 & -1 & -2 & -1 \\ 2 & 1 & 1 & -1 \end{array}\right]$$

$$\Big\downarrow R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{cc:cc} 1 & -1 & -2 & -1 \\ 0 & 3 & 5 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 \to R_2/3$$

$$\left[\begin{array}{cc:cc} 1 & -1 & -2 & -1 \\ 0 & 1 & 5/3 & 1/3 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1 + R_2$$

$$\left[\begin{array}{cc:cc} 1 & 0 & -1/3 & -2/3 \\ 0 & 1 & 5/3 & 1/3 \end{array}\right]$$

So, $[\varphi]_B = \begin{bmatrix} -\frac{1}{3} & -2/3 \\ 5/3 & 1/3 \end{bmatrix}$

$$(\varphi - \lambda I) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$|\varphi - \lambda I| = \lambda^2 + 1 \neq 0 \ \forall \lambda \in \mathbb{R}$$

$\therefore$ $(\varphi - \lambda I)$ is invertible.

Now, α_1, α_2 be the basis $\mathcal{B}'$

$\alpha_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} c \\ d \end{bmatrix}$

$\varphi(\alpha_1) = \begin{pmatrix} -b \\ a \end{pmatrix}$ $\varphi\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -d \\ c \end{pmatrix}$

Representing

$$\left[\begin{array}{cc:cc} a & c & -b & -d \end{array}\right] \quad \left(\begin{array}{c}\text{assuming} \\ a \neq 0\end{array}\right)$$

$$\begin{bmatrix} b & d & \vdots & a & c \end{bmatrix}$$

$$\Big\downarrow R_1 \to R_1/a$$

$$\begin{bmatrix} 1 & c/a & \vdots & -b/a & -d/a \\ b & d & \vdots & a & c \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - R_1 b$$

$$\begin{bmatrix} 1 & c/a & \vdots & -b/a & -d/a \\ 0 & d - bc/a & \vdots & a + b^2/a & c + bd/a \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 \cdot a/ad-bc$$

$$\begin{bmatrix} 1 & c/a & \vdots & \frac{-b}{a} & \frac{-d}{a} \\ 0 & 1 & \vdots & a + b^2/a & c + bd/a \end{bmatrix}$$

$$\Big\downarrow R_1 \to R_1 - \frac{c}{a} R_2$$

$$\begin{bmatrix} 1 & 0 & \vdots & -\frac{ac+bd}{ad-bc} & -\frac{c^2+d^2}{ad-bc} \\ 0 & 1 & \vdots & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{bmatrix}$$

$\therefore$ α_1 α_2 are basis $ad-bc \neq 0$

Given $[\varphi]_{B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$ac+bd = 0 \qquad ad-bc \neq 0$$

$$\left.\begin{aligned} a^2+b^2 &= bc-ad \\ c^2+d^2 &= bc-ad \end{aligned}\right\} \Rightarrow a^2+b^2 = c^2+d^2$$

<u>choose</u> $a = b = 1 \Rightarrow c+d$

taken $c = 1 \Rightarrow d = -1.$

$\therefore$ $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Now choose $a = 2,\ b = 1 \Rightarrow 2c+d = 0$

let $c = 1 \Rightarrow d = -2$

$\alpha_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} +1 \\ -2 \end{bmatrix}$

is also basis such $[\varphi]_{B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow$ B' is not unique.

(24) Sol^n:-

$$\mathcal{B} = \{f_0, f_1 \dots f_n\} \text{ s.t } f_i = x^i$$

$$\mathcal{B}' = \{g_0, g_1 \dots g_n\} \text{ s.t } g_i = (x+\alpha)^i$$

need to represent the differentiation operator

$$D : V \to V$$

consider basis $\mathcal{B}$ and polynomial $p(x)$ under this basis.

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \dashrightarrow (1)$$

$$D(p(x)) = 0 + a_1 + 2a_2 x + \dots + (n-1)a_{n-1} x^{n-2} + a_n x^{n-1}(n)$$

$\therefore$ every basis transform to

$$D(1) \longrightarrow 0$$

$$D(x) \longrightarrow 1$$

$$D(x^2) \to 2x$$

$$\dots$$

Thus

$$[D(p(x))]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_n \end{bmatrix}$$

We are writing $p(x)$ polynomial under the basis B' then

$$[p(x)]_{B'} = b_0 + b_1(x+\alpha) + b_2(x+\alpha)^2 \cdots\cdots + b_n(x+\alpha)^n$$

$$D[p(x)]_{B'} = 0 + b_1 + 2b_2(x+\alpha) + \cdots + nb_n(x+\alpha)^{n-1} + 0\cdot x^n$$

Here

$$D(1) \longrightarrow 0$$

$$D(x+\alpha) \longrightarrow 1$$

$$D(x+\alpha)^2 \longrightarrow 2(x+\alpha)$$

$$\therefore [D(px)]_{B'} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & n \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \vdots \\ b_n \end{bmatrix}$$

Now considering $[p(x)]_{\beta'}$ can be represented under basis of β. For that first note

$$(x+\alpha) = 1 \cdot x + 1 \cdot \alpha$$

$$(x+\alpha)^2 = x^2 + (2\alpha)x + \alpha^2 \cdot 1$$

$$\vdots$$

$$(x+\alpha)^i = \alpha^i \cdot 1 + {}^iC_1 \alpha^{i-1} x + {}^iC_2 \alpha^{i-2} x^2 + \cdots + 1 x^i$$

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^n \\ 0 & 1 & 2\alpha & \cdots & n\alpha^{n-1} \\ 0 & 0 & 1 & \cdots & \frac{n(n-1)}{2}\alpha^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

in β $\qquad\qquad$ T^{-1} $\qquad\qquad$ in β'

Note :-

$$\begin{tikzcd}
{[p(x)]_{\beta}} \arrow[r, "D_{\beta}"] \arrow[d, bend right, "T"'] & {[p'(x)]_{\beta}} \arrow[d, bend left, "T"] \\
{[p(x)]_{\beta'}} \arrow[u, "T^{-1}"'] \arrow[r] & {[p'(x)]_{\beta'}} \arrow[u, "T^{-1}"']
\end{tikzcd}$$

⊛ Follow red dotted path.

$$\therefore \quad D_{\beta'} = T^{-1} D_{\beta} T$$

Observation :-

$[D]_{\beta}$ & $[D]_{\beta'}$ are the

Same (see calculation)

(Q25) Sol^n :-

Given ordered basis -

$$B = \{f_0, f_1, f_2, f_3\}$$

$$f_i = x^i \quad i = \{0, 1, \dots 3\}$$

consider a polynomial $f(x)$ under Basis B

then $$[f(x)]_B = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

$\because \hat{D} : V \to V$ when

$$[\hat{D}(f(x))]_B = 3f(x) + (3-x)\frac{d}{dx}(f(x))$$

$$= 3(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)$$

$$+ (3-x)\frac{d}{dx}(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)$$

$$+(3-x)\frac{d}{dx}(\alpha_0 + \ldots$$

$$= 3(\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3)$$

$$+(3-x)(0+\alpha_1+2\alpha_2 x+3\alpha_3 x^2)$$

$$=3\alpha_0+\alpha_1(3x+3-x)+\alpha_2\{3x^2+2(3-x)x\}$$

$$+3\alpha_3(x^3+(3-x)x^2)$$

$$[\hat{D} f(x)]_B = (3\alpha_0+3\alpha_1)+(2\alpha_1+6\alpha_2)x$$

$$+(\alpha_2+9\alpha_3)x^2$$

$$+0\cdot x^3$$

$$\therefore \quad \hat{D}(1) = 3$$

$$\hat{D}(x) = 3x+(3-x)$$

$$= 3+2x$$

$$\hat{D}(x^2) = 3x^2+(3-x)2x$$

$$= 3x^2 + 6x - 2x^2$$

$$= x^2 + 6x$$

$$\hat{D}(x^3) = 3x^3 + (3-x)\,3x^2$$

$$= 9x^2$$

Operator $\hat{D}$ matrix representation can be given as under basis B —

$$[\hat{D}]_B = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore [\hat{D} f(x)]_B$ in matrix form can be represented as —

$$\hat{D} \ldots \begin{bmatrix} 3 & 3 & 0 & 0 \\ \ldots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \ldots \end{bmatrix}$$

$$\left[\hat{D} f(x)\right]_B = \begin{bmatrix} 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Let us define the $\ker(\hat{D})$ as —

$$\ker(\hat{D}) = \left\{ f(x) \;\middle|\; 3f(x) + (3-x)\frac{d}{dx}f(x) = 0 \right\}$$

i.e.

$$\left[\hat{D} f(x)\right]_B = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Downarrow \; R_1 \to R_1/3, \; R_2 \to R_2/2$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

$$\Big\Downarrow \begin{array}{l} R_1 \to R_1 - R_2 + 3R_3 \\ R_2 \to R_2 - 3R_3 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 27 \\ 0 & 1 & 0 & -27 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

α_3 is free variable

let $\alpha_3 = z$

$$\alpha_2 = -9\alpha_3 = -9z$$

$$\alpha_1 = 27\alpha_3 = 27z$$

$$\alpha_0 = -27\alpha_3 = -27z$$

$\therefore$ Basis for ker $(\hat{D})$ can be given

as $\{\ldots\} = \{-27z + 27zx - 9zx^2 + zx^3\}$

$f(x) = [\ldots + 2x]$

Now finding for $\operatorname{im}(\hat{D})$

$$\because \left[\hat{D} f(x)\right]_B = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\left[\hat{D} f(x)\right]_B = 3\alpha_0 + \alpha_1(3+2x) + \alpha_2(6x+x^2) + 9\alpha_3 x^2$$

but the set $\{3,\ 3+2x,\ 6x+x^2,\ 9x^2\}$

is a generating set but is not a linearly independent set for $\operatorname{im}(\hat{D})$

because - consider $\alpha_0 = 3,\ \alpha_1 = -3$

$\alpha_2 = 1 \quad \alpha_3 = -\frac{1}{9}$

which will leads to

$\ldots (-3)(3+2x) \ldots$

$3.(3) + (-3)(\ldots$

$+ 1.(6x+x^2) + (-\frac{1}{9})$

$9x^2 = 0$

$\therefore$ consider

$\{3, 3+2x, 6x+x^2\}$ as

L.I set.

$\exists f(x)$ such that $\hat{D}[f(x)] = g(x)$

$\therefore \quad g(x) \in \operatorname{im}(\hat{D})$

i.e $g(x) = \alpha \cdot 3 + \beta(3+2x) + \gamma(6x+2x^2)$

Given $g(x) = 7+8x$

$7+8x = (3\alpha + 3\beta) + (2\beta + 6\gamma)x + \gamma x^2$

$\Rightarrow \quad \gamma = 0$

$2\beta + 6\gamma = 8$

$$\Rightarrow \beta = 4$$

$$3\alpha + 3\beta = 7 \Rightarrow \alpha = \frac{-4+7}{3} = \frac{-5}{3}$$

since all co-efficients $(\alpha, \beta \neq 0)$ not zero

so $g(x) \in \operatorname{im}(\hat{D})$

For a polynomial $[f(x)]_B = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

s.t $[\hat{D} f(x)]_B = g(x)$

$$[\hat{D}(f(x))]_B = (3\alpha_0 + 3\alpha_1) + (2\alpha_1 + 6\alpha_2)x + (\alpha_2 + 9\alpha_3)x^2$$

$$= g(x) = 7 + 8x$$

$$\begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2/2]{R_1 \to R_1/3}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 4 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_2 + 3R_3}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 27 \\ 0 & 1 & 0 & -27 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 4 \\ 0 \end{bmatrix}$$

α_3 is free variable so

$\alpha_3 = z$

$\alpha = -9z$

$$\alpha_1 = 4 + 27z$$

$$\alpha_0 = -5/3 - 27z$$

$$\therefore f(x) = -5/3 - 27z + 4x + 27zx - 9zx^2 + zx^3$$