Mid-sem solutions

(a) From the description, it follows that
$$V = U \cup W.$$
Now, we know that

$U \cup W$ is a subspace iff either $U \subseteq W$, implying $U \cup W = W$ or $W \subseteq U$, implying $U \cup W = U$.

Hence, either we have
$V = W$ or $V = U$, or both.

(b) If we verify the field axioms, then we need multiplicative inverses of every element other than the 0-element.

Suppose $\mathbb{F} \subseteq \mathbb{C}$ is a subfield of $\mathbb{C}$ which is non-empty & $f \neq 0$.

Let $f \in \mathbb{F}$. Due to closure under addition, $\underbrace{f+f+\cdots+f}_{m\text{-times}} = m.f \in \mathbb{F}$.

||ly, for $n \in \mathbb{Z}$ such that m and n are co-primes, $n.f \in \mathbb{F}$ with $n > m$.

Hence, $(nf)^{-1} \in \mathbb{F}$ (existence of inv)

& $(mf).(nf)^{-1} \in \mathbb{F}$. (closure under mult.)

$$\Rightarrow \frac{m}{n} \in \mathbb{F}.$$

Since the choice of m and

n were arbitrary,

$$\left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ with } m \,\&\, n \text{ coprime} \right) \subseteq \mathbb{F}.$$

But the set above is precisely the set of rational numbers, $\mathbb{Q}$.

(c) Let $u \in v_1 + \mathbb{U}$.

$$\Rightarrow \exists\ \hat{u} \in \mathbb{U} : u = v_1 + \hat{u}.$$

$$\therefore v_1 + \hat{u} \in v_2 + \mathbb{W}$$

$$[\because v_2 + \mathbb{W} = v_1 + \mathbb{U}]$$

$$\Rightarrow v_1 + \hat{u} - v_2 \in \mathbb{W}$$

$$\Rightarrow v_1 + \hat{u} + \mathbb{W} = v_2 + \mathbb{W}$$

$$\Rightarrow v_1 + \hat{u} + \mathbb{W} = v_1 + \mathbb{U}$$

$\Rightarrow \quad \hat{u} + \mathbb{W} = \mathbb{U}$

$\Rightarrow \quad \mathbb{W} \subseteq \mathbb{U}$, since $\hat{u} \in \mathbb{U}$. (i)

Proceeding similarly, we may start with $w \in v_2 + \mathbb{W}$
$\Rightarrow \exists \, \hat{w} \in \mathbb{W}$:
$w = \hat{w} + v_2$

& establish that

$\mathbb{U} \subseteq \mathbb{W}$. (ii).

From (i) & (ii) $\mathbb{U} = \mathbb{W}$.

(d) The assertion is not true.

Consider $\mathbb{V} = \mathbb{R}^4$.

with $\mathbb{W} = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix} \right\} \right\rangle$

$$U_1 = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \right\rangle$$

and $$U_2 = \left\langle \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \right\rangle.$$

check that $U_1 \cap W = 0_{\mathbb{R}^4}$

& $U_2 \cap W = 0_{\mathbb{R}^4}$

$$\therefore \quad U_i + W = U_i \oplus W$$

for $i = 1, 2$.

Also, $\forall i$, $U_i \oplus W = V$.

But $U_1 \neq U_2$.

2 (a) Consider $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

given by $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longmapsto \sqrt[3]{v_1^3 + v_2^3}$.

$$\therefore \quad T\left(\alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \sqrt[3]{\alpha^3 v_1^3 + \alpha^3 v_2^3}$$

$$= \sqrt[3]{\alpha^3} \cdot \sqrt[3]{v_1^3 + v_2^3}$$

$$= \alpha \, T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right).$$

However, $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}\right)$

$$= \sqrt[3]{(v_1 + \hat{v}_1)^3 + (v_2 + \hat{v}_2)^3}$$

& $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}\right)$

$$= \sqrt[3]{(v_1^3+v_2^3)} + \sqrt[3]{(v_1^3+v_2^3)}$$

clearly, $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}\right)$

$$\neq T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}\right)$$

in general

as may be verified

upon choosing $v_1 = 1;\ \hat{v}_1 = 2$

& $v_2 = 3;\ \hat{v}_2 = 4.$

(b) Consider $T : \mathbb{C} \to \mathbb{C}$

given by $T(x+iy) = y+ix$.

clearly $T(x_1+iy_1)$

$+ T(x_2+iy_2)$

$$= y_1 + ix_1 + y_2 + ix_2$$

$$= (y_1 + y_2) + i(x_1 + x_2)$$

$$= T(x_1 + x_2 + i(y + y_2))$$

$$= T(x_1 + iy_1 + x_2 + iy_2).$$

However, let $\alpha = i$.

$$\Rightarrow T(i(x+iy)) = T(-y+ix)$$

$$= x - iy.$$

But $i\,T(x+iy) = i.(y+ix)$

$$= -x + iy$$

$$= -T(i(x+iy)).$$

Hence not linear

Hence, not linear.

3. (a) $\phi' \in \mathcal{L}(W', V')$.

$$\therefore \dim(\operatorname{im}(\phi')) + \dim(\ker(\phi')) = \dim(W') = \dim(W)$$

$$\Rightarrow \dim(\operatorname{im}(\phi')) + \dim(\ker(\phi')) = \dim(W)$$

Now, $\ker(\phi') = (\operatorname{im}(\phi))^0$

$$\Rightarrow \dim(\operatorname{im}(\phi')) = \dim(W) - \dim((\operatorname{im}(\phi))^0)$$

But, we know that:

$$\boxed{\dim(\mathrm{im}(\varphi)) + \dim((\mathrm{im}(\varphi))^0) = \dim W.}$$

$$\Rightarrow \dim(\mathrm{im}(\varphi')) = \dim(\mathrm{im}(\varphi)).$$

(b) Suppose $f \in \mathrm{im}(\varphi')$.

where f is a lin. functional on V. Hence, $\exists\, g \in W'$

$$: \varphi'(g) = f.$$

Let $v \in \ker(\varphi)$.

Then $f(v) = \varphi'(g)(v)$

$$= (g \circ \varphi)(v)$$

$$= g(\phi(v)) = g(0)$$
$$= 0$$

$\Rightarrow f \in (\ker(\phi))^0.$

$\Rightarrow \operatorname{im}(\phi') \subseteq (\ker(\phi))^0.$

Further, $\dim(\operatorname{im}(\phi'))$

$$= \dim(\operatorname{im}(\phi)) \text{ [from (a)]}$$

$$= \dim(V) - \dim(\ker(\phi))$$

$$= \Big[\dim(\ker(\phi)) + \dim((\ker(\phi))^0)\Big] - \dim(\ker(\phi))$$

$$= \dim((\ker(\phi))^0)$$

Hence, $\operatorname{im}(\phi') = (\ker(\phi))^0.$

4. Applying R-N on τ, we get

$$\dim(V) = \dim(\operatorname{Ker}\tau) + \dim(\operatorname{im}(\tau)) \quad \text{— (1)}$$

Illy applying R-N on $\psi \circ \tau$, we get

$$\dim(V) = \dim(\operatorname{Ker}(\psi \circ \tau)) + \dim(\operatorname{im}(\psi \circ \tau)) \quad \text{— (2)}$$

Observe that

$$\operatorname{im}(\psi \circ \tau) = \operatorname{im}(\psi|_{\operatorname{im}(\tau)})$$

which means that the image of $\psi \circ \tau$ is the image of ψ restricted to the domain given by $\operatorname{im}(\tau)$ (instead of all of V).

Now,

$$\dim(\operatorname{im}(\psi|_{\operatorname{im}\tau})) + \dim(\operatorname{Ker}(\psi|_{\operatorname{im}\tau}))$$

$= \dim(\text{im}(\tau)) \longrightarrow (3)$

since $\psi|_{\text{im}(\tau)} : \text{im}(\tau) \longrightarrow W$.

Further,

$$\text{Ker}(\psi|_{\text{im}(\tau)})$$

$$= \text{Ker}(\psi) \cap \text{im}(\tau). \longrightarrow (4)$$

Thus, $\dim(\text{im}(\tau)) = \dim(\text{Ker}(\psi) \cap \text{im}(\tau))$

$+ \dim(\text{im}(\psi \circ \tau))$

(from (3) & (4))

$\Rightarrow \dim(U) - \dim(\text{Ker}(\tau))$

$= \dim(\text{im}(\psi \circ \tau)) + \dim(\text{Ker}(\psi) \cap \text{im}(\tau))$

(from (1))

$\Rightarrow \dim(\text{Ker}(\psi \circ \tau))$

$$+ \dim(\cancel{\text{im}}(\psi\circ\tau)) - \dim(\operatorname{Ker}\tau)$$

[replacing $\dim(U)$ from (2)]

$$= \dim(\cancel{\text{im}}(\psi\circ\tau)) + \dim(\operatorname{Ker}(\psi)\cap \operatorname{im}(\tau))$$

$$\Rightarrow \dim(\operatorname{Ker}(\psi\circ\tau)) = \dim(\operatorname{Ker}(\tau)) + \dim(\operatorname{Ker}(\psi)\cap \operatorname{im}(\tau)) \quad (i)$$

Now, $\operatorname{Ker}(\psi)\cap \operatorname{im}(\tau) \subseteq \operatorname{Ker}(\psi)$

$$\Rightarrow \dim(\operatorname{Ker}(\psi)\cap \operatorname{im}(\tau)) \leq \dim(\operatorname{Ker}(\psi)) \quad (ii)$$

Therefore, from (i) & (ii) we get

$$\dim(\operatorname{Ker}(\psi\circ\tau)) \leq \dim(\operatorname{Ker}(\tau)) + \dim(\operatorname{Ker}(\psi))$$

QED

From (i) we note that for

$\mathrm{im}(\tau) = V$ (onto mapping)

$\Rightarrow \mathrm{Ker}(\psi) \cap \mathrm{im}(\tau) = \mathrm{Ker}(\psi) \cap V$
$= \mathrm{Ker}(\psi)$,

we get

$$\dim(\mathrm{Ker}(\psi \circ \tau)) = \dim(\mathrm{Ker}(\tau)) + \dim(\mathrm{Ker}(\psi))$$

[Ex Think of products of two matrices, $P = AB$

$\mathrm{Ker}(P)$ includes everything in $\mathrm{Ker}(B)$ and also some vectors $Bv \neq 0$ such that $A(Bv) = 0$

Now, $Bv \in \mathrm{im}(B)$ and $Bv \in \mathrm{Ker}(A)$

i.e. $Bv \in \mathrm{im}(B) \cap \mathrm{Ker}(A)$.

If B is onto (has full row rank), then columns of B span F^n for $B \in F^{n \times m}$]

5. $$p(x) = c_2 x^2 + c_1 x + c_0$$

$$f_1(p) = \int_0^1 p(x)\,dx$$

$$= \frac{c_2}{3} + \frac{c_1}{2} + c_0$$

|||ly,

$$f_2(p) = \frac{8c_2}{3} + 2c_1 + 2c_0$$

and

$$f_3(p) = -\frac{c_2}{3} + \frac{c_1}{2} - c_0.$$

First note that $\dim(V') = \dim(V) = 3$.

So, if we prove that:

$\{f_1, f_2, f_3\}$ is a lin.

$\{f_1, f_2, f_3\}$

independent set, then it is

a basis for $\mathbb{V}'$!

Suppose $(\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3)(p)$

$= 0 \quad \forall \; p \in \mathbb{V}.$

<u>T. S.T.</u> $\quad \alpha_1 = \alpha_2 = \alpha_3 = 0$

Let.

$$\alpha_1\left(\frac{c_2}{3} + \frac{c_1}{2} + c_0\right) + \alpha_2\left(\frac{8c_2}{3} + 2c_1 + 2c_0\right)$$

$$+ \alpha_3\left(-\frac{c_2}{3} + \frac{c_1}{2} - c_0\right) = 0$$

This needs to hold for all possible choices of $c_2, c_1, c_0 \in \mathbb{R}$.

$c_2\left(\frac{\alpha_1}{3} + \frac{8\alpha_2}{3} - \frac{\alpha_3}{3}\right)$

$$c_2\left(\frac{\alpha_1}{3} + \frac{8\alpha_2}{3} - \frac{\alpha_3}{3}\right)$$

$$+ c_1\left(\frac{\alpha_1}{2} + 2\alpha_2 + \frac{\alpha_3}{2}\right)$$

$$+ c_0\left(\alpha_1 + 2\alpha_2 - \alpha_3\right) = 0$$

$$\forall \; c_1, c_2, c_0 \in \mathbb{R}.$$

choose $c_1 = 0$, $c_2 = 0$, $c_0 = 1$

$$\Rightarrow \boxed{\alpha_1 + 2\alpha_2 - \alpha_3 = 0} \quad (1)$$

choose $c_1 = 0$, $c_2 = 1$, $c_0 = 0$

$$\Rightarrow \boxed{\frac{\alpha_1}{3} + \frac{8\alpha_2}{3} - \frac{\alpha}{3} = 0} \quad (2)$$

choose $c_1 = 1$, $c_2 = 0$, $c_0 = 0$.

$$\Rightarrow \boxed{\frac{\alpha_1}{2} + 2\alpha_2 + \frac{\alpha_3}{2} = 0} \quad (3)$$

①, ②, ③ imply

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 1/3 & 8/3 & -1/3 \\ 1/2 & 2 & 1/2 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we perform row operations on A to get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$ as required.

Hence, $\{f_1, f_2, f_3\}$ is a linearly independent set & so a basis for $\mathbb{V}$!

Let $\{p_1(x), p_2(x), p_3(x)\}$ be the basis for $\mathbb{V}$, to

[illegible] $\{f_1(p), f_2(p), f_3(p)\}$

which $\{f_1(\cdot), f_2(\cdot), f_3(\cdot)\}$

is a dual basis.

$$\Rightarrow \quad f_1(p_1) = 1 \longrightarrow 1a$$

$$f_1(p_2) = 0 \longrightarrow 2a$$

$$f_1(p_3) = 0 \longrightarrow 3a$$

$$f_2(p_1) = 0 \longrightarrow 1b$$

$$f_2(p_2) = 1 \longrightarrow 2b$$

$$f_2(p_3) = 0 \longrightarrow 3b$$

$$f_3(p_1) = 0 \longrightarrow 1c$$

$$f_3(p_2) = 0 \longrightarrow 2c$$

$$f_3(p_3) = 1 \longrightarrow 3c$$

with $p_1(x) = c_0^{(1)} + c_1^{(1)}x + c_2^{(1)}x^2$

$$p_2(x) = c_0^{(2)} + c_1^{(2)}x + c_2^{(2)}x^2$$

$$p_3(x) = c_0^{(3)} + c_1^{(3)}x + c_2^{(3)}x^2$$

from 1a, 1b & 1c, we have

$$c_0^{(1)} + \frac{c_1^{(1)}}{2} + \frac{c_2^{(1)}}{3} = 1$$

$$2c_0^{(1)} + 2c_1^{(1)} + \frac{8}{3}c_2^{(1)} = 0$$

$$-c_0^{(1)} + \frac{c_1^{(1)}}{2} - \frac{c_2^{(1)}}{3} = 0$$

$$\Rightarrow \quad c_1^{(1)} = 1$$

$$c_2^{(1)} = -3/2$$

$$c_0^{(1)} = 1$$

$$\Rightarrow \boxed{p_1(x) = -\frac{3}{2}x^2 + x + 1.}$$

||ly, from 2a, 2b, 2c, we get

$$C_1^{(2)} = 0$$

$$C_0^{(2)} = -\frac{1}{6}$$

$$C_2^{(2)} = \frac{1}{2}$$

$$\Rightarrow \boxed{p_2(x) = \frac{1}{2}x^2 - \frac{1}{6}}$$

and from 3a, 3b, 3c, we get

$$C_1^{(3)} = 1$$

$$C_2^{(3)} = -1/2$$

$$C_0^{(3)} = -1/3$$

$$\Rightarrow \boxed{p_3(x) = -\frac{1}{2}x^2 + x - \frac{1}{3}}$$

6. (a) Let $\{v_1 + U, v_2 + U, \ldots, v_m + U\}$ be a basis for V/U.

$\Rightarrow$ the aforeshown set is linearly independent

<u>1st Claim</u>: $\{v_1, v_2, \ldots, v_m\}$ is a linearly ind. set.

Suppose not.

$\Rightarrow \exists\ \alpha_i$ not all zero, such that $\sum \alpha_i v_i = 0_V$

$$\Rightarrow \sum \alpha_i v_i + U = 0_{V/U}$$

$$\Rightarrow \sum \alpha_i (v_i + U) = 0_{V/U}.$$

$\Rightarrow$ the set $\{v_1 + U, \ldots, v_m + U\}$

$\Rightarrow$ the set $\{v_1+U, v_2+U, \ldots, v_m+U\}$

is not lin. ind., which is a contradiction.

Hence, the 1^{st} claim.

<u>2^{nd} claim:</u> $v_i \notin U \ \forall i$.

Suppose for some $i=k$ $v_k \in U$.

choose $\alpha_k \neq 0$ & $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_m = 0$

Consider $\sum_{i=1}^{m} \alpha_i (v_i + U)$

$$= \sum_{i=1}^{m} \alpha_i v_i + U$$

$$= \alpha_k v_k + U$$

$$= 0 + U$$ [$\because$ v_k & $\alpha_k v_k$

$\in U]$

$= 0_{V/U}$.

$\Rightarrow \{v_1 + U, \ldots, v_m + U\}$ is a linearly dep set; which is also a contradiction.

Hence the second claim also stands.

<u>Claim 3</u> $B = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ is a linearly independent set.

Consider $\sum_{i=1}^{n} \alpha_i u_i + \sum_{i=1}^{m} \beta_i v_i = 0$

$\Rightarrow \sum \beta_i v_i = u \in U$

where, $\sum \alpha_i u_i = -u \in U$.

But, unless $-u = 0$, this is impossible, since

$v_i \notin U$ by 2nd claim.

and $\sum \beta_i v_i \in U$

would imply linear

dependence for $\{v_1 + U, \dots, v_m + U\}$,

unless $\beta_i = 0 \ \forall i$.

Thus, $u = 0 \Rightarrow \alpha_i = 0 \ \forall i$

$[\because \{u_1, u_2, \dots, u_n\}$, being a basis, is lin. ind$]$.

& $\beta_i = 0 \ \forall i$.

Hence claim 3 stands.

Now $|\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m\}| = n+m$.

& $\mathcal{B} \subseteq \mathbb{V}$ is a lin. ind. set. Also,

$$\dim(\mathbb{V}/\mathbb{U}) = \dim(\mathbb{V}) - \dim(\mathbb{U})$$

$$\Rightarrow \quad m = \dim(\mathbb{V}) - n$$

$$\Rightarrow \dim(\mathbb{V}) = m+n.$$

Thus ...

Thus, $\mathcal{B}$ is indeed a basis for $\mathbb{V}$.

(b) $\mathbb{V} = \{c_0 + c_1x + c_2x^2 + \dots + c_nx^n : c_i \in \mathbb{R}\ \forall i\} = \mathbb{W}$.

$$T : \mathbb{V} \longrightarrow \mathbb{V}$$

such that

$$c_0 + c_1x + \dots + c_nx^n \longmapsto c_1 + 2c_2x + \dots + nc_nx^{n-1} + 0.x^n.$$

$$\ker(T) = \{c_0 : c_0 \in \mathbb{R}\}$$

Each element in $\mathbb{V}/\ker(T)$ represents a set of polynomials represents a set of polynomials

that differ from one another by a constant.

with the affine set represented by $p(x) + \ker(T)$, where $p(x) \in \mathbb{V}$.

clearly, two polynomials in $p(x) + \ker(T)$, say $p_1(x)$ & $p_2(x)$, satisfy

$$p_1(x) - p_2(x) \in \ker(T),$$

hence, they differ by a constant term.

Clearly, $\dim(\ker(T)) = \underline{1}$.

Thus, $\dim(\mathbb{V}/\ker(T)) = n+1-1$
$= n$

[Note that $\dim(\mathbb{V}) = n+1$ since it is isomorphic to $\mathbb{R}^{n+1}$ through $\mathbb{R}[x]_n \longrightarrow \mathbb{R}^{n+1}$ given by

$$c_0 + c_1 x + \cdots + c_n x^n \longmapsto \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\text{im}(T) = \{ c_1 + 2c_2 x + \cdots + n c_n x^{n-1} : c_1, c_2, \ldots, c_n \in \mathbb{R} \}$$

Thus, $\dim(\text{im}(T)) = n$, since

$$\text{im}(T) = \langle \underbrace{\{ c_1, 2c_2 x, \ldots, n c_n x^{n-1} \}} \rangle$$

Bans.

Thus, considering the induced map

$$\tilde{T}(p + \ker(T)) = p'(x)$$

[$'$ represents $\frac{d}{dx}$ here]

it follows that $\mathrm{im}(T) = \mathrm{im}(\tilde{T})$

Further, for all polynomials in the affine set $p + \ker(T)$, which differ by a const., $\tilde{T}$ is also a one-one map, unlike T. Whereas T maps multiple distinct polynomials to the same image if they

to the [illegible]
differ by a constant, $\tilde{T}$ allows us to consider all such polynomials as one 'vector' or 'object' and is thus an injection.

Thus, between $V/\ker T$ & $\mathrm{im}(T)$, $\tilde{T}$ is a linear bijection (linearity follows from linearity of T, the derivative map). Hence $V/\ker(T) \cong \mathrm{im}(T)$, which could also be verified by comparing dimensions earlier.