EE 635: Applied Linear Algebra
Assignment 2

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2. If a vector space $\mathbb{V}$ over $\mathbb{Z}_2$ contains linearly independent vectors x, y, z, then are $x+y, y+z, z+x$ also linearly independent? Justify your answer mathematically.

Vector space $\mathbb{V}$ over $\mathbb{Z}_2$. where $\mathbb{Z}_2 = \{0,1\}$

Given that $S = \{x, y, z\}$ is a LI set.

$\Rightarrow c_1 x + c_2 y + c_3 z = 0_v \longrightarrow \forall c_i = 0, \forall c_i \in \mathbb{Z}_2$

Consider $S' = \{x+y, y+z, z+x\}$

Look at, $\alpha_1(x+y) + \alpha_2(y+z) + \alpha_3(z+x) = 0_v$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_2$.

$\Rightarrow \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_3 x + \alpha_1 y + \alpha_2 z = 0_v$

$\Rightarrow (\alpha_1 + \alpha_3) x + (\alpha_2 + \alpha_1) y + (\alpha_3 + \alpha_2) z = 0_v$

$\Rightarrow \beta_1 x + \beta_2 y + \beta_3 z = 0_v$

where $\beta_1 = \alpha_1 + \alpha_3$, $\beta_2 = \alpha_2 + \alpha_1$, $\beta_3 = \alpha_3 + \alpha_2$,

$\forall \beta_i \in \mathbb{Z}_2$.

Since $\{x, y, z\}$ is LI set so their LC giving rise to 0_v implies $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 = 0$.

$$\Rightarrow \left.\begin{array}{l} \alpha_1 + \alpha_3 = 0 \\ \alpha_2 + \alpha_1 = 0 \\ \alpha_3 + \alpha_2 = 0 \end{array}\right\} \begin{array}{l} \Rightarrow \alpha_1 = -\alpha_3 \\ \Rightarrow \alpha_1 = -\alpha_2 \\ \text{Therefore, } -\alpha_3 = -\alpha_2. \end{array}$$

Hence, $\alpha_3 = \alpha_2$

again, $\alpha_3 + \alpha_2 = 0$

$$\Rightarrow \alpha_3 + \alpha_3 = 0 \quad [\because \alpha_2 = \alpha_3]$$

$$\Rightarrow 2\alpha_3 = 0$$

Since $\alpha_3 \in \{0,1\}$

so if $\alpha_3 = 0$ then $2 \times 0 = 0$ ✓

if $\alpha_3 = 1$ then $2 \times 1 = 2 \pmod 2 = 0 = 0$ ✓

So both $\alpha_3 = 0, 1$ satisfies the equation.

Definitely when $\alpha_3 = 0$ then it becomes trivial linear combination which is of no interest.

Choose $\alpha_3 = 1$ so, $\alpha_2 = \alpha_3 = 1$

$\alpha_1 = -\alpha_3 = -1 \pmod 2 = 1$.

Therefore, $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$.

The LC of set of vectors $\{x+y, y+z, z+x\}$ giving rise to 0_V implies there exist at least one coefficient in the LC that is non zero if and only if the set of vectors are Linearly dependent.

Therefore $\{x+y, y+z, z+x\}$ is Linearly dependent set.

7. Consider the vector space $\mathbb{F}^{2\times 2}$ ($\mathbb{F}$ is any field). Obtain a basis for this vector space, say $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$, such that $A_i^2 = A_i,\ i = 1, 2, 3, 4$. Establish that your answer is indeed a valid basis.

Vector space $\mathbb{F}^{2\times 2}$ over $\mathbb{F}$.

$$\mathbb{F}^{2\times 2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \,\middle|\, \forall a_{ij} \in \mathbb{F} \right\}$$

We need to construct the basis for $\mathbb{F}^{2\times 2}$ such that each vector x in basis have property $x^2 = x$.

Consider an arbitrary basis vector in basis set,

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{where } a, b, c, d \in \mathbb{F}$$

$$x^2 = x$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} a^2+bc &= a \\ a+d &= 1 \\ bc+d^2 &= d \end{aligned}}$$

$\Rightarrow a^2+bc+bc+d^2 = a+d = 1$

$\Rightarrow (a+d)^2 - 2ad + 2bc = 1$

$\Rightarrow 1 - 2ad + 2bc = 1$

$\Rightarrow ad = bc$

Therefore, $\boxed{ad = bc \text{ and } a+d = 1}$ are the two conditions for $x^2 = x \ \forall x \in \mathbb{F}^{2\times 2}$.

Hence select a candidate for basis vector x as,

$$x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{as} \quad 1\times 0 = 0 \times 0, \quad 1+0 = 1$$

So, $x^2 = x$.

Suppose, $S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

Obviously, S_1 is Linearly independent set

Check,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{so } A_1^2 = A_1$$

But $\langle S_1 \rangle \neq \mathbb{F}^{2\times 2}$ since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin \langle S_1 \rangle$.

Choose another vector from $\mathbb{F}^{2\times 2} \setminus \langle S_1 \rangle$. Such a vector x is guarented to exist.

Suppose, $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad ad = bc$ ✓ $\quad a+d = 1$ ✓

So $x^2 = x$ and $x \notin \langle S_1 \rangle$

Construct another set, $S_2 = S_1 \cup \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

Theorem - 1 : if S is a LI set in $\mathbb{V}$ and $\mathbb{V} \setminus \langle S \rangle$ is non empty then there exist $x \in \mathbb{V} \setminus \langle S \rangle$ such that $S \cup \{x\}$ is Linearly independent set.

Invoking the above fact we have –

$$S_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \text{ is LI set.}$$

$$\langle S_2 \rangle = \left\{ \begin{bmatrix} a & 0 \\ b & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{F} \right\}$$

clearly $\langle S_2 \rangle \neq \mathbb{F}^{2\times 2}$. Hence $\mathbb{F}^{2\times 2} \setminus \langle S_2 \rangle \neq \phi$
Choose a vector $x \in \mathbb{F}^{2\times 2} \setminus \langle S_2 \rangle$ s.t $x^2 = x$.

Suppose, $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad ad = bc$ ✓ $\quad a + d = 1$ ✓

clearly, $x \notin \langle S_2 \rangle$ and $x^2 = x$.
Invoking theorem (1),

$$S_3 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \text{ is LI set.}$$

$$\langle S_3 \rangle = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix} + \begin{bmatrix} 0 & c \\ 0 & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{F} \right\}$$

$$\langle S_3 \rangle = \left\{ \begin{bmatrix} a & c \\ b & b+c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{F} \right\}$$

Obviously $\langle S_3 \rangle \neq \mathbb{F}^{2\times 2}$ since $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \notin \langle S_3 \rangle$

So $\mathbb{F}^{2\times 2} \setminus \langle S_3 \rangle \neq \phi$.

Consider a vector $x \in \mathbb{F}^{2\times 2} \setminus \langle S_3 \rangle$ s.t. $x^2 = x$.

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} ad = bc \; \checkmark \\ a+d = 1 \; \checkmark \end{matrix}$$

Clearly $x \notin \langle S_3 \rangle$ and invoking theorem-(i)

$$S_4 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$ is LI set.

$$\langle S_4 \rangle = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix} + \begin{bmatrix} 0 & c \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & d \\ 0 & 0 \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{F} \right\}$$

$$= \left\{ \begin{bmatrix} a+d & c+d \\ b & b+c \end{bmatrix} \right\} \quad \text{--- (i)}$$

Claim: $\langle S_4 \rangle$ spans $\mathbb{F}^{2\times 2}$.

proof:

$$\mathbb{F}^{2\times 2} = \left\{ \begin{bmatrix} p & q \\ r & s \end{bmatrix} \;\middle|\; p, q, r, s \in \mathbb{F} \right\} \quad \text{--- (ii)}$$

Equating two form of matrices in (i) & (ii)

$$\left.\begin{aligned} a+d &= p \\ c+d &= q \\ b &= r \\ b+c &= s \end{aligned}\right]$$

We have to solve for a, b, c, d in terms of p, q, r, s. That is how the coefficients a, b, c, d will be selected to generate the matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

Since $b = r$

$$c = s - b = s - r$$

$$d = q - c = q - s + r$$

$$a = p - d = p - q + s - r$$

Hence we have shown that,

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = (p-q+s-r)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (r)\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} +$$

$$(s-r)\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + (q-s+r)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore it is proved that any arbitrary vector $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ can be written as a LC of set of vectors

$$S_4 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

It implies S_4 is a spanning set of $\mathbb{F}^{2\times2}$.
We have also seen S_4 is LI set in $\mathbb{F}^{2\times2}$.
$\Rightarrow$ S_4 is a basis set for $\mathbb{F}^{2\times2}$.

Hence. $$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

11. In the vector space $\mathbb{F}^{2\times 2}$ over $\mathbb{F}$, consider the following subspaces: $\mathbb{W}_1 \subseteq \mathbb{F}^{2\times 2}$ such that it contains matrices of the form $\begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix}$ with $\alpha, \beta, \gamma \in \mathbb{F}$ and $\mathbb{W}_2 \subseteq \mathbb{F}^{2\times 2}$ such that it contains matrices of the form $\begin{bmatrix} p & q \\ -p & r \end{bmatrix}$ with $p, q, r \in \mathbb{F}$. What are the dimensions of $\mathbb{W}_1$, $\mathbb{W}_2$, $\mathbb{W}_1 \cap \mathbb{W}_2$, and $\mathbb{W}_1 + \mathbb{W}_2$?

Vector space : $\mathbb{F}^{2\times2}$ over $\mathbb{F}$.

$$\mathbb{W}_1 := \left\{ \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix} \;\middle|\; \alpha, b, \gamma \in \mathbb{F} \right\}$$

Since $\mathbb{W}_1$ is a subspace, let's try to obtain the basis for the subspace $\mathbb{W}_1$.

$$\begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix} = \alpha \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \gamma . \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose, $$S = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

clearly S will span the subspace $\mathbb{W}_1$ so S is a generating set of $\mathbb{W}_1$. Now if we can prove that

S is a LI set then S is a candidate for basis set.

Consider, $c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c_1 & -c_1 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \; c_1 = 0, \; -c_1 = 0 \;\Rightarrow c_1 = 0$

$c_2 = 0, \; c_3 = 0$

Therefore, a linear combination of set of vectors in S giving rise to zero vector $(c_1 \cdot u_1 + c_2 \cdot u_2 + c_3 \cdot u_3 = 0_V)$ implies all coefficients are 0 $(c_1 = 0, c_2 = 0, c_3 = 0)$ if and only if S is LI set.

Therefore, S is a Basis set also.

$\dim(\mathbb{W}_1) = |S| = \underline{3}$.

Given that,

$$\mathbb{W}_2 := \left\{ \begin{bmatrix} p & q \\ -p & r \end{bmatrix} \;\middle|\; p, q, r \in \mathbb{F} \right\} \quad (\mathbb{W}_2 \text{ is subspace})$$

construct the basis for $\mathbb{W}_2$.

$$\begin{bmatrix} p & q \\ -p & r \end{bmatrix} = p\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + q\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Consider the set, $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Clearly S will span the space W_2 as shown above if the coefficients are chosen as p, q, r. So S is a generating set of W_2. Now we have to check if S is a LI set or not?

consider, $$c_1\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 \\ -c_1 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad c_1 = 0, -c_1 = 0 \Rightarrow c_1 = 0$$
$$c_2 = 0, c_3 = 0$$

Therefore, $c_1 \cdot u_1 + c_2 u_2 + c_3 u_3 = 0_V \longrightarrow \forall c_i = 0$ if and only if $\{u_1, u_2, u_3\}$ is LI set.

Hence $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis set.

$\dim(W_2) = |S| = \underline{3}$.

Given that, $\mathbb{W}_1 := \left\{ \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbb{F} \right\}$

$\mathbb{W}_2 := \left\{ \begin{bmatrix} p & q \\ -p & r \end{bmatrix} \;\middle|\; p, q, r \in \mathbb{F} \right\}$

$$\mathbb{W}_1 \cap \mathbb{W}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{F} \right\}.$$

Now any vector $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbb{W}_1 \cap \mathbb{W}_2$ implies $x \in \mathbb{W}_1$ and $x \in \mathbb{W}_2$ so x must be in the form of $\mathbb{W}_1$ and it must be in the form $\mathbb{W}_2$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix} \Rightarrow \begin{array}{l} a = \alpha, \ b = -\alpha \\ c = \beta, \ d = \gamma \end{array}$$

again, $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} p & q \\ -p & r \end{bmatrix} \Rightarrow \begin{array}{l} a = p, \ b = q \\ c = -p, \ d = r \end{array}$$

Look at $a = \alpha = p = a_{11}$ (say).

$d = \gamma = r = a_{22}$ (say).

But, $b = -\alpha = q$ if and only if $\alpha = q = 0$ so $b = 0$

$c = \beta = -p$ if and only if $\beta = p = 0$ so $c = 0$

Therefore, $\mathbb{W}_1 \cap \mathbb{W}_2 = \left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \;\middle|\; a_{11}, a_{22} \in \mathbb{F} \right\}$

It is easy to see that the set

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is both generating set of $\mathbb{W}_1 \cap \mathbb{W}_2$ (with coeff. a_{11} and a_{22}) and it is also a LI set. Hence B is a basis for $\mathbb{W}_1 \cap \mathbb{W}_2$.

So $\dim(\mathbb{W}_1 \cap \mathbb{W}_2) = |B| = \underline{2}$.

Given that, $\mathbb{W}_1 := \left\{ \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbb{F} \right\}$

$\mathbb{W}_2 := \left\{ \begin{bmatrix} p & q \\ -p & r \end{bmatrix} \;\middle|\; p, q, r \in \mathbb{F} \right\}$

$\mathbb{W}_1 + \mathbb{W}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix} + \begin{bmatrix} p & q \\ -p & r \end{bmatrix}, \; a, b, c, d \in \mathbb{F} \right\}.$

We know that,

$\dim(\mathbb{W}_1 + \mathbb{W}_2) = \dim(\mathbb{W}_1) + \dim(\mathbb{W}_2) - \dim(\mathbb{W}_1 \cap \mathbb{W}_2)$

$= 3 + 3 - 2 = \underline{\underline{4}}.$

14. Consider $\mathbb{W} := \{p \in \mathbb{F}[x]_3 : p(x) = a_0 + a_1x + a_2x^2 + (a_0 + a_1 - a_2)x^3, a_0, a_1, a_2 \in \mathbb{F}\}$. Show that $\mathbb{W}$ is a subspace of $\mathbb{F}[x]_3$. Obtain a basis for $\mathbb{W}$ and hence evaluate $\dim(\mathbb{W})$. Suppose $p_1(x) = 1 + x + x^2 + x^3$ and $p_2(x) = 1 + x^2$. Show that $S = \{p_1, p_2\}$ is a linearly independent set in $\mathbb{W}$. Extend S to a basis for $\mathbb{W}$.

$$\mathbb{F}[x]_3 = \{ f \mid f(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3, \ \forall a_i \in \mathbb{F}, \ f : \mathbb{F} \to \mathbb{F} \}$$

$$\mathbb{W} := \{ p \in \mathbb{F}[x]_3 \mid p(x) = a_0 + a_1x + a_2x^2 + (a_0 + a_1 - a_2)x^3 \ \forall a_i \in \mathbb{F} \}$$

Consider 2 arbitrary vectors f, g from $\mathbb{W}$.

$$f(x) = a_0 + a_1x + a_2x^2 + (a_0 + a_1 - a_2)x^3 \quad \forall a_i \in \mathbb{F}$$

$$g(x) = b_0 + b_1x + b_2x^2 + (b_0 + b_1 - b_2)x^3 \quad \forall b_j \in \mathbb{F}$$

Obviously, $f \in \mathbb{F}[x]_3$, $g \in \mathbb{F}[x]_3$.

Look at another vector, $w = f +_v c \cdot g$

$$w(x) = f(x) + c \cdot g(x)$$

$$= a_0 + a_1x + a_2x^2 + (a_0 + a_1 - a_2)x^3 +$$

$$cb_0 + cb_1x + cb_2x^2 + c(b_0 + b_1 - b_2)x^3$$

$$= (a_0 + cb_0) + (a_1 + cb_1)x + (a_2 + cb_2)x^2 +$$

$$(a_0 + a_1 - a_2 + cb_0 + cb_1 - cb_2)x^3.$$

suppose, $\alpha_0 = a_0 + cb_0$

$$\alpha_1 = a_1 + cb_1$$

$$\alpha_2 = a_2 + cb_2$$

$$\alpha_3 = a_0 + a_1 - a_2 + cb_0 + cb_1 - cb_2$$

$$= (a_0 + cb_0) + (a_1 + cb_1) - (a_2 + cb_2)$$

$$= \alpha_0 + \alpha_1 - \alpha_2$$

Hence, $w(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + (\alpha_0 + \alpha_1 - \alpha_2) x^3$

Therefore by definition of $\mathbb{W}$, $w \in \mathbb{W}$.

Hence $\mathbb{W}$ is a subspace of $\mathbb{F}[x]_3$.

Choose one arbitrary vector from $\mathbb{W}$.

$$f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + (\alpha_0 + \alpha_1 - \alpha_2) x^3$$

$$= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_0 x^3 + \alpha_1 x^3 - \alpha_2 x^3$$

$$= \alpha_0 (1 + x^3) + \alpha_1 (x + x^3) + \alpha_2 (x^2 - x^3)$$

Therefore a candidate set for basis can be –

$$S = \{ (1+x^3), (x+x^3), (x^2 - x^3) \}$$

Is S a spanning set for W?

$$\Rightarrow \langle S \rangle = \left\{ c_0(1+x^3) + c_1(x+x^3) + c_2(x^2-x^3) \;\middle|\; \forall c_i \in \mathbb{F} \right\}$$

$$= \left\{ c_0 + c_1 x + c_2 x^2 + (c_0 + c_1 - c_2)x^3 \;\middle|\; \forall c_i \in \mathbb{F} \right\}$$

$$= \mathbb{W}.$$

Yes $\langle S \rangle$ spans $\mathbb{W}$.

Is S a LI set ?

$\Rightarrow$ Construct the zero vector as LC of set of vectors in S

$$c_0(1+x^3) + c_1(x+x^3) + c_2(x^2-x^3) = 0$$

$$\Rightarrow c_0 + c_1 x + c_2 x^2 + (c_0 + c_1 - c_2)x^3 = 0$$

(zero polynomial).

For a zero polynomial, all the coefficients are 0.

$$\Rightarrow c_0 = 0$$
$$c_1 = 0$$
$$c_2 = 0$$
$$c_0 + c_1 - c_2 = 0$$

Hence $\left(\sum_{i=1}^{2} c_i u_i = 0 \longrightarrow \forall c_i = 0 \right)$ iff S is LI.

So $B = \{ (1+x^3), (x+x^3), (x^2-x^3) \}$ is a basis of $\mathbb{W}$

So $\dim(\mathbb{W}) = |B| = \underline{3}$.

Suppose, $P_1(x) = 1+x+x^2+x^3$
$P_2(x) = 1+x^2$

We have to show that, $S = \{P_1, P_2\}$ is a LI set in $\mathbb{W}$.

Consider, $\alpha_1 \cdot P_1(x) + \alpha_2 \cdot P_2(x) = 0$ (zero polynomial)

$\Rightarrow \alpha_1 \cdot (1+x+x^2+x^3) + \alpha_2(1+x^2) = 0$

$\Rightarrow (\alpha_1+\alpha_2) + (\alpha_1)x + (\alpha_1+\alpha_2)x^2 + (\alpha_1) \cdot x^3 = 0$

For a zero polynomial, all the coefficient must be 0.

$$\Rightarrow \left.\begin{aligned} \alpha_1+\alpha_2 &= 0 \\ \alpha_1 &= 0 \\ \alpha_1+\alpha_2 &= 0 \\ \alpha_1 &= 0 \end{aligned}\right\}$$

Since $\alpha_1 = 0$

So $\alpha_2 = -\alpha_1 = 0$

$\left(\sum_{i=1}^{2} \alpha_i P_i = 0_V \longrightarrow \forall \alpha_i = 0\right)$ iff S is LI set.

Therefore, $S = \{P_1, P_2\}$ is a LI set.

First of all we have to show that $\langle S \rangle \subset \mathbb{W}$ and then there exist at least one $p \in \mathbb{W} \setminus \langle S \rangle$ such that $S \cup \{p\}$ will be LI set.

$$\langle S \rangle = \left\{ \alpha_1 \cdot P_1(x) + \alpha_2 \cdot P_2(x) \;\middle|\; \forall \alpha_i \in \mathbb{F} \right\}$$

$$= \left\{ \alpha_1 (1 + x + x^2 + x^3) + \alpha_2 (1 + x^2) \;\middle|\; \forall \alpha_i \in \mathbb{F} \right\}$$

$$= \left\{ (\alpha_1 + \alpha_2) + (\alpha_1) x + (\alpha_1 + \alpha_2) x^2 + (\alpha_1) x^3 \right\}$$

To show that $\langle S \rangle \subset \mathbb{W}$, we have to show every vector in $\langle S \rangle$ are also in $\mathbb{W}$.

Choose a random vector p from $\langle S \rangle$.

$$p(x) = c(1 + x + x^2 + x^3) + d(1 + x^2) \qquad c, d \in \mathbb{F}$$

$$= (c+d) + cx + (c+d)x^2 + cx^3$$

We have to show that $p \in \mathbb{W}$.

$$\mathbb{W} = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; a_3 = a_0 + a_1 - a_2 \right\}$$

$a_0 = c + d$

$a_1 = c$

$a_2 = c + d$

if $a_3 = a_0 + a_1 - a_2$ that must be equal to c then $p \in \mathbb{W}$.

$$a_3 = \cancel{c} + \cancel{d} + \cancel{c} - \cancel{c} - \cancel{d} = c$$

Therefore, $p \in S \longrightarrow p \in \mathbb{W}$ so $S \subset \mathbb{W}$.

Currently we have, $S=\{(1+x+x^2+x^3), (1+x^2)\}$
as LI set and $\langle S\rangle \subset \mathbb{W}$ and $\mathbb{W}\setminus\langle S\rangle \neq \phi$.

$P_1(x) = 1+x+x^2+x^3$
$P_2(x) = 1+x^2$
We need to select $P_3(x)$ which makes the set
$S=\{P_1, P_2, P_3\}$ LI and $P_3 \notin \langle\{P_1, P_2\}\rangle$ and
$P_3 \in \mathbb{W}$.

We know that $\deg(P_1) = 3$, $\deg(P_2) = 2$ so if we take
P_1 as deg 1 polynomial then set S will be LI.

suppose, $P_3(x) = a+bx$ (deg 1 polynomial).

The vectors in $\mathbb{W}$ is represented as:

$$f(x) = a_0 + a_1x + a_2x^2 + (a_0+a_1-a_2)x^3 = a+bx$$

So, $a_0 = a$, $a_1 = b$, $a_2 = 0$, $a_0+a_1-a_2 = 0$
Therefore, $a+b-0 = 0$
$\Rightarrow a = -b$.

Choose $P_3(x) = -1+x$ (suppose $b=1$ so $a=-1$)
Clearly $P_3(x) \in \mathbb{W}$ and $P_3(x) \notin \langle\{P_1, P_2\}\rangle$
A quick check $\{1+x+x^2+x^3, 1+x^2, -1+x\}$ is LI.

$$c_1(1+x+x^2+x^3) + c_2(1+x^2) + c_3(1-x) = 0$$

$$\Rightarrow (c_1+c_2+c_3) + (c_1-c_3)x + (c_1+c_2)x^2 + c_1(x^3) = 0$$

$$\Rightarrow \begin{cases} c_1+c_2+c_3 = 0 & \Rightarrow c_1 = 0 \\ c_1-c_3 = 0 & \Rightarrow c_3 = 0 \\ c_1+c_2 = 0 & \Rightarrow c_2 = 0 \\ c_1 = 0 & \end{cases}$$

Therefore, $\left(\sum_{i=1}^{3} c_i p_i = 0_V \longrightarrow \forall c_i = 0\right)$ iff S is LI.

A quick check that S is spanning set of $\mathbb{W}$.

$$\langle S\rangle = \left\{ c_1(1+x+x^2+x^3) + c_2(1+x^2) + c_3(1-x) \right\}$$

$$= \left\{ (c_1+c_2+c_3) + (c_1-c_3)x + (c_1+c_2)x^2 + (c_1)x^3 \right\}$$

$$\mathbb{W} = \left\{ a_0 + a_1x + a_2x^2 + a_3x^3 \;\middle|\; a_3 = a_0 + a_1 - a_2 \right\}$$

Comparing the two form –

$$\begin{aligned} c_1+c_2+c_3 &= a_0 \\ c_1-c_3 &= a_1 \\ c_1+c_2 &= a_2 \\ c_1 &= a_3 \end{aligned}$$

$$\begin{aligned} a_3 &= a_0 + a_1 - a_2 \\ &= c_1 + \cancel{c_2} + \cancel{c_3} + \cancel{c_1} - \cancel{c_3} - \cancel{c_1} - \cancel{c_2} \\ &= c_1 \end{aligned}$$

Therefore we can see that if we choose the coefficients c_1, c_2, c_3 in such a way for $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_3 = a_0 + a_1 - a_2$ then $\langle S \rangle$ generates all functions in $\mathbb{W}$.

For a function $f(x) = a_0 + a_1x + a_2x^2 + (a_0 + a_1 - a_2)x^3$, if we want to represent $f(x)$ in terms of LC of vectors of S then choose, $c_1 = a_3$, $c_2 = a_2 - a_3$, $c_3 = a_3 - a_1$

Therefore $\{1 + x + x^2 + x^3, 1 + x^2, -1 + x\}$ is a spanning set of $\mathbb{W}$ and it is LI set also hence it is a basis for $\mathbb{W}$. This set is extened from $S = \{P_1, P_2\}$.

17. For a finite dimensional vector space, $\mathbb{V}$, suppose there exist subspaces $\mathbb{W}_i$, $i \in \{1, 2, \ldots, m\}$ such that $\mathbb{V} = \oplus_{i=1}^{m} \mathbb{W}_i$. Show that $\dim(\mathbb{V}) = \sum_{i=1}^{m} \dim(\mathbb{W}_i)$.

The subspaces $\mathbb{W}_i$ exists within $\mathbb{V}$. such that,

$$\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_m$$

We will use proof by induction.

base case: $m = 2$

$\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$. We have to show that,

$$\dim(\mathbb{V}) = \dim(\mathbb{W}_1) + \dim(\mathbb{W}_2)$$

We already know that, $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0_{\mathbb{V}}\}$ if and only if $\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{W}_1 \oplus \mathbb{W}_2$ — (i)

We also know that,

$$\dim(\mathbb{W}_1+\mathbb{W}_2) = \dim(\mathbb{W}_1) + \dim(\mathbb{W}_2) - \dim(\mathbb{W}_1 \cap \mathbb{W}_2) \quad \text{— (ii)}$$

combining (i) & (ii),

$$\dim(\mathbb{W}_1 \oplus \mathbb{W}_2) = \dim(\mathbb{W}_1) + \dim(\mathbb{W}_2) - \dim(\{0_V\})$$

(from proposition (i))

The subspace $\{0_V\}$ has only one vector that is 0_V. It is not possible to select more than 1 vector to be LI set from this space so $\dim(\{0_V\}) = 0$.

Therefore, $\dim(\mathbb{W}_1 \oplus \mathbb{W}_2) = \dim(\mathbb{W}_1) + \dim(\mathbb{W}_2)$.
(base case is proved).

Inductive step: Assume $m=K$

Assume $\dim\left(\bigoplus_{i=1}^{K} \mathbb{W}_i\right) = \sum_{i=1}^{K} \dim(\mathbb{W}_i)$ is true (induction hypothesis)

$$\dim\left(\bigoplus_{i=1}^{K} \mathbb{W}_i \oplus \mathbb{W}_{K+1}\right) \quad (\text{consider } m = K+1)$$

From the base case result we get,

$$= \dim\left(\bigoplus_{i=1}^{K} \mathbb{W}_i\right) + \dim\left(\mathbb{W}_{K+1}\right) \quad \left[\mathbb{W}_{K+1} \cap \bigoplus_{i=1}^{K} W_i = \{0_V\} \text{ is also true.}\right]$$

$$= \sum_{i=1}^{K} \dim(\mathbb{W}_i) + \dim(\mathbb{W}_{K+1}) \quad [\text{From induction hypothesis}]$$

$$= \sum_{i=1}^{K+1} \dim(W_i)$$

Therefore the claim holds true for all $m \geq 2$. Therefore by proof of induction, the statement is true.

9. Prove that if $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ is a basis for the finite dimensional vector space $\mathbb{V}$, then so is the set $\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}$.

$B = \{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{V}$.

Claim: $B_1 = \{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}$ is also a basis for $\mathbb{V}$.

(i) Check of LI property of B_1.

$$c_1(v_1 - v_2) + c_2(v_2 - v_3) + \cdots + c_{n-1}(v_{n-1} - v_n) + c_n v_n = 0_V$$

$$\Rightarrow (c_1)v_1 + (-c_1 + c_2)v_2 + (-c_2 + c_3)v_3 + \cdots + (-c_{n-2} + c_{n-1})v_{n-1} + (-c_{n-1} + c_n)v_n = 0_V \quad \text{---(i)}$$

Since we already know that B is basis set so B is LI set also that means –

$$\sum_{i=1}^{n} \alpha_i u_i = 0_V \longrightarrow \forall \alpha_i = 0$$

Invoking the above definitions at eqn (i) we get –

$$\left.\begin{array}{ll} c_1 = 0 & \\ -c_1 + c_2 = 0 & \Rightarrow c_2 = 0 \\ -c_2 + c_3 = 0 & \Rightarrow c_3 = 0 \\ \vdots & \\ -c_{n-2} + c_{n-1} = 0 & \Rightarrow c_{n-1} = 0 \\ -c_{n-1} + c_n = 0 & \Rightarrow c_n = 0 \end{array}\right\} \quad \forall c_i = 0$$

So we have shown that -

$$\left(\sum_{i=1}^{n} c_i u_i' = 0_V \longrightarrow \forall c_i = 0 \right) \text{ iff } B_1 \text{ is LI set.}$$

Hence B_1 is LI set.

(ii) Check for spanning property of B_1.

$$\langle B_1 \rangle = \left\{ c_1 \cdot (v_1 - v_2) + c_2 (v_2 - v_3) + \dots + c_n v_n \;\middle|\; \forall c_i \in \mathbb{F} \right\}$$

$$= \left\{ c_1 v_1 + (-c_1 + c_2) v_2 + \dots + (-c_{n-1} + c_n) v_n \right\}$$

$$= \{ \alpha_1 . v_1 + \alpha_2 . v_2 + \cdots + \alpha_n v_n \} = \langle B \rangle$$

where $\alpha_1 = c_1$, $\alpha_2 = -c_1 + c_2$, ..., $\alpha_n = -c_{n-1} + c_n$.

Note that $\langle B \rangle$ is actually the LC of set of vectors of B. Since B is a basis set so $\langle B \rangle$ spans $\mathbb{V}$ or B is a spanning set of $\mathbb{V}$. Therefore B_1 is also spanning set of $\mathbb{V}$.

Therefore B_1 is LI & generating set of $\mathbb{V}$. So B_1 is a basis of $\mathbb{V}$. (Hence proved)

6. Find a basis for symmetric 3×3 matrices, having real entries, as a vector space over $\mathbb{R}$.

$$\mathbb{R}^{3\times3} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \middle| \ \forall a_{ij} \in \mathbb{R} \right\} \quad \text{—(i)}$$

$\mathbb{R}^{3\times3}$ is a vector space over $\mathbb{R}$.

We need to find a basis for $\mathbb{R}^{3\times3}$.

Look at the standard basis (in place of a_{ij} keep 1)

Consider the following matrices –

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Construct the set, $S = \{A_1, A_2, A_3, A_4, A_5, A_6\}$

Check if S is a LI set.

$\Rightarrow$ consider the LC of set of vectors in S giving rise to $0_{3\times3}$

$$\Rightarrow c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 + c_5A_5 + c_6A_6 = 0_{3\times3}$$

$$\Rightarrow c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cdots + c_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 & c_4 & c_5 \\ c_4 & c_2 & c_6 \\ c_5 & c_6 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \forall c_i = 0 \quad \text{for } i = 1, 2, \ldots, 6$$

Therefore, $\left(\sum_{i=1}^{6} c_iA_i = 0_{3\times3} \rightarrow \forall c_i = 0 \right)$ iff S is LI set

Hence the set S is LI set.

Check if S is a generating set of $\mathbb{R}^{3\times3}$.

$$\langle S \rangle = \{ c_1A_1 + c_2A_2 + \cdots + c_6A_6 \mid \forall c_i \in \mathbb{R} \}$$

$$= \left\{ \begin{bmatrix} c_1 & c_4 & c_5 \\ c_4 & c_2 & c_6 \\ c_5 & c_6 & c_3 \end{bmatrix} \right\}$$ (It is the same form of $\mathbb{R}^{3\times3}$ in eqn (i))

$$= \mathbb{R}^{3\times3}$$

Hence $\langle S \rangle$ spans the space $\mathbb{R}^{3\times3}$ so S is spanning set of $\mathbb{R}^{3\times3}$.

Therefore, $S = \{A_1, A_2, \dots, A_6\}$ is a basis set of $\mathbb{R}^{3\times3}$.

4. Considering $\mathbb{C}$, the set of complex numbers, to be a vector space over $\mathbb{R}$, show that '1 and x are linearly independent' $\iff$ 'x is not a real number'.

$\mathbb{C}$ is a vector space over $\mathbb{R}$.

$\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{R} \}$ over $\mathbb{R}$.

To show that : $\{1, x\}$ is LI set $\longrightarrow$ x is not real number.

Suppose $\{1, x\}$ is LI set and x is a real number. (assume contrary).

Since x is a real number, consider the following linear combination giving rise to 0.

$$c_1 \cdot 1 + c_2 \cdot x = 0_c \qquad c_1, c_2 \in \mathbb{R}.$$

Choose $c_1 = x$ and $c_2 = -1$ (x being $\mathbb{R}$ can be qualified for scalar)

Therefore, $\underset{\in \mathbb{R}}{(x)} \cdot \underset{\in \mathbb{C}}{(1)} + \underset{\in \mathbb{R}}{(-1)} \cdot \underset{\in \mathbb{C}}{(x)} = \underset{\in \mathbb{C}}{0_c}$

Look at the LC of 1 and x giving rise to 0.

$$\left(\sum_{i=1}^{2} c_i u_i = 0_c \rightarrow \exists c_i \neq 0. \right) \longleftrightarrow S \text{ is LD set.}$$

Since here $c_1 = x \neq 0$ and $c_2 = -1 \neq 0$ that's why there exists at least 1 non zero c_i s.t. the LC gives rise to 0 implies the set is LD set.

$$(x) \cdot (1) + (-1) \cdot (x) = 0_c$$

$$\Rightarrow x = \left(\frac{x}{-1}\right) \cdot 1 = \alpha \cdot 1 \quad \left(\text{where } \alpha = -x \in \mathbb{R}\right)$$

Note that x is written as LC of vector "1" hence $\{x, 1\}$ must be LD which is a contradiction of the given fact that $\{x, 1\}$ is LI so the assumption was incorrect.

Hence $\{1, x\}$ is LI $\longrightarrow$ x is not real number

Now we have to show that,

x is not real number $\longrightarrow$ $\{1, x\}$ is LI set.

Assume that x is not real number and $\{1, x\}$ is LD set.

Since $\{1, x\}$ is LD set,

$$(c_1) \cdot 1 + (c_2) \cdot x = 0_c \quad (c_1, c_2 \in \mathbb{R})$$

where at least one $c_i \neq 0$.

Suppose $c_1 = -x$ and $c_2 = 1$ then the above condition is satisfied.

The choice of c_1 and c_2 as $-x$ and 1 is the only way to result 0 vector. There is no other non-trivial LC that gives rise to 0_c. But we have a contradiction here as $c_1 \in \mathbb{R}$ so $-x \in \mathbb{R}$ but we assumed that x is not real number. Therefore, the assumption was incorrect.

Hence x is not real number $\longrightarrow$ $\{1, x\}$ is LI set.

Therefore we can say -

$\{1, x\}$ is LI set $\longleftrightarrow$ x is not real number (proved)

12. For $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{n\times p}$, and $C = AB$, prove that $\text{rank}(C) \leq \min(\text{rank}(A), \text{rank}(B))$.

$A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times p}$, $C = AB \in \mathbb{F}^{m \times p}$

rank of a matrix A is defined as:

$$\operatorname{rank}(A) := \dim\left(\operatorname{Im}(A)\right) \quad \text{where,}$$

$$\operatorname{Im}(A) := \left\{ y \mid y = Ax,\ A \in \mathbb{F}^{m\times n},\ x \in \mathbb{F}^{n},\ y \in \mathbb{F}^{m} \right\}$$

Consider -

$$\operatorname{Im}(C) = \left\{ y \mid y = Cx,\ C \in \mathbb{F}^{m\times p},\ x \in \mathbb{F}^{p},\ y \in \mathbb{F}^{m} \right\}$$

$$= \left\{ y \mid y = ABx \right\}$$

$$= \left\{ y \mid y = A(Bx) \right\}$$

$$\operatorname{Im}(B) = \left\{ z \mid z = Bx,\ B \in \mathbb{F}^{n\times p},\ x \in \mathbb{F}^{p},\ z \in \mathbb{F}^{n} \right\}$$

Therefore, $\operatorname{Im}(C) = \left\{ y \mid y = Az,\ A \in \mathbb{F}^{m\times n},\ z \in \operatorname{Im}(B) \right\}$

Since the matrix AB is formed by taking LC of columns of A so every column of AB is a LC of columns of A. Which implies,

$$\operatorname{Im}(C) \subseteq \operatorname{Im}(A)$$

$$\Rightarrow \dim(\operatorname{Im}(C)) \leq \dim(\operatorname{Im}(A))$$

$$\Rightarrow \operatorname{rank}(C) \leq \operatorname{rank}(A). \quad \text{——— (i)}$$

Given that $C = AB$

$$\Rightarrow C^T = B^T A^T$$

By the similar arguments-

matrix $B^T A^T$ is formed by taking LC of columns of B^T

so every column of $B^T A^T$ is a LC of colums of B^T.

Which implies-

$$Im(C^T) \subseteq Im(B^T)$$

$$\Rightarrow \dim(C^T) \leq \dim(Im(B^T))$$

$$\Rightarrow rank(C^T) \leq rank(B^T)$$

But we know that, $rank(C) = rank(C^T)$ and

$rank(B) = rank(B^T)$.

Hence, $rank(C) \leq rank(B)$. —— (ii)

Combining eqn (i) & (ii),

$rank(C) \leq \min(rank(A), rank(B))$ (proved)

24. Suppose $\mathbb{V}$ is the vector space of all polynomials having real coefficients with degree less than or equal to $n < \infty$. Let $\mathcal{B} = \{f_0, f_1, \ldots, f_n\}$ and $\mathcal{B}' = \{g_0, g_1, \ldots, g_n\}$ be two ordered bases for $\mathbb{V}$ such that $f_i = x^i$ and $g_i = (x+\alpha)^i$ for some scalar α. Represent the differentiation operator, $\mathcal{D} : \mathbb{V} \to \mathbb{V}$ in both these ordered bases, using basis transformation. What do you observe?

$\mathbb{V} = \mathbb{R}[x]_n$ is a vector space of all polynomials of real coefficients with deg $\leq n$ over the field $\mathbb{R}$.

$$\mathbb{V} = \left\{ p \;\middle|\; p(x) = a_0 + a_1 x + \cdots + a_n x^n \;,\; \forall a_i \in \mathbb{R} \right\}$$

The ordered basis for $\mathbb{V}$ is given as -

$$B = (f_0, f_1, f_2, \ldots, f_n) \quad \text{such that} \quad f_i(x) = x^i$$

$$= (x^0, x^1, x^2, \ldots, x^n)$$

$$= (1, x, x^2, \ldots, x^n)$$

Suppose the operator $D: \mathbb{V} \to \mathbb{V}$ is defined as:

$$D(p) := \frac{dp}{dx}.$$

Consider a polynomial as:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad \forall a_i \in \mathbb{R}. \quad \text{— (i)}$$

$$\frac{dp(x)}{dx} = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n \cdot a_n \cdot x^{n-1} \quad \text{— (ii)}$$

$p(x)$ written in ordered basis B will be -

$$\underset{\in \mathbb{R}[x]_n}{p} \xmapsto{\;B\;} [p]_B = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n \quad (\text{from eqn i})$$

$[p]_B$ ← encoding of p in $\mathbb{R}^n$.

→ these are called coordinates of p.

$\frac{dP(x)}{dx}$ written in order basis B will be —

$$\underset{\in \mathbb{R}[x]_n}{\frac{dP}{dx}} \xmapsto{\ B\ } \left[\frac{dP}{dx}\right]_B = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

(from eqn 2)

Given $[P]_B$, we can transform it to $\left[\frac{dP}{dx}\right]_B$ under the ordered basis B and the transformation matrix can be followed as:

$$\left[\frac{dP}{dx}\right]_B = \begin{bmatrix} \left[\frac{d}{dx}(a_0)\right]_B & \left[\frac{d}{dx}(a_1x)\right]_B & \cdots & \left[\frac{d}{dx}(a_nx^n)\right]_B \end{bmatrix} [P]_B$$

$$\left[\frac{dP}{dx}\right]_B = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ & & & & n \\ 0 & 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ 2a_2 \\ \vdots \\ na_n \\ 0 \end{bmatrix}$$

therefore, $\left[\frac{dP}{dx}\right]_B = T_B\left([P]_B\right)$

where T_B is transformation matrix

Note that we defined $D: \mathbb{V} \to \mathbb{V}$ as $D(p) := \frac{dp}{dx}$.
Therefore, it is easy to see that,

$$D(p) = T_B(p) \quad \left(\text{if } p \text{ is given as encoding in basis } B\right)$$

We can check if the operator D is linear operator or not?

$$\Rightarrow D(p) = \frac{dp}{dx}, \quad p \in \mathbb{V}$$

$$D(q) = \frac{dq}{dx}, \quad q \in \mathbb{V}$$

$$D(p) + D(q) = \frac{dp}{dx} + \frac{dq}{dx} = \frac{d}{dx}(p+q) = D(p+q)$$

where $p+q \in \mathbb{V}$.

$$c \cdot D(p) = c \cdot \frac{dp}{dx} = \frac{d}{dx}(c \cdot p) = D(c \cdot p) \quad \text{where } c \in \mathbb{R}$$

Therefore, $D: \mathbb{V} \to \mathbb{V}$ is a linear operator. Since D is just a matrix T_B if p is encoded in B as $[p]_B$ so it is easy to see T_B is a linear operator as well as matrix $T_B \in \mathbb{R}^{n \times n}$ is a linear operator.

$$T_B([p]_B) = \left[\frac{dp}{dx}\right]_B \quad \text{where } T_B : \mathbb{F}^n \longrightarrow \mathbb{F}^n \text{ and}$$

T_B is a matrix $\in \mathbb{F}^{n \times n}$.

The ordered basis for $\mathbb{V}$ is given as -

$$B' = (g_0, g_1, g_2, \ldots, g_n) \quad \text{such that} \quad g_i(x) = (x+\alpha)^i$$

$$= \left(1, (x+\alpha)^1, (x+\alpha)^2, \ldots, (x+\alpha)^n\right) \qquad (\alpha \in \mathbb{R})$$

Suppose the operator $D : \mathbb{V} \to \mathbb{V}$ is defined as:

$$D(P) := \frac{dP}{dx}.$$

Consider a polynomial as:

$$P(x) = a_0 + a_1(x+\alpha)^1 + \cdots + a_n(x+\alpha)^n \quad \forall a_i \in \mathbb{R}. \text{ — (i)}$$

$$\frac{dP(x)}{dx} = a_1 + 2a_2(x+\alpha) + 3a_2(x+\alpha)^2 + \cdots$$

$$\cdots + na_n(x+\alpha)^{n-1} \text{ — (ii)}$$

$P(x)$ written in ordered basis B' will be -

$$\underset{\in \mathbb{R}[x]_n}{P} \xmapsto{B'} [P]_{B'} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n \quad (\text{from eqn i})$$

Notice that while encoding the vector under an ordered basis B', we only collect the scalars and put in n tuple so under the hood, the basis B' is automatically implied.

$\frac{dP(x)}{dx}$ written in order basis B' will be —

$$\frac{dP}{dx} \xmapsto{\quad B' \quad} \left[\frac{dP}{dx}\right]_{B'} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

$\in \mathbb{R}[x]_n$

(from eqn 2)

Given $[P]_{B'}$, we can transform it to $\left[\frac{dP}{dx}\right]_{B'}$ under the ordered basis B' and the transformation matrix can be followed as:

$$\left[\frac{dP}{dx}\right]_{B'} = \left[\left[\frac{d}{dx}(a_0)\right]_{B'} \left[\frac{d}{dx}(\alpha+x)\right]_{B'} \cdots \left[\frac{d}{dx}(\alpha+x)^n\right]_{B'} \right] [P]_{B'}$$

$$\left[\frac{dP}{dx}\right]_{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & n \\ 0 & 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ 2a_2 \\ \vdots \\ na_n \\ 0 \end{bmatrix}$$

therefore, $\left[\frac{dP}{dx}\right]_{B'} = T_{B'}\left([P]_{B'}\right)$

where $T_{B'}$ is transformation matrix

Note that we defined $D : \mathbb{V} \to \mathbb{V}$ as $D(p) := \frac{dp}{dx}$.

Therefore, it is easy to see that,

$$D(p) = T_{B'}(p) \quad (\text{if } p \text{ is given as encoding in basis } B')$$

We observe that the transformation matrix T_B and $T_{B'}$ are same matrix $\in \mathbb{F}^{n \times n}$.

$$T_B = T_{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 2 & 0 & & 0 \\ 0 & 0 & 0 & 3 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & & n \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix}_{n \times n}$$

25. Consider the same vector space as in Qn. 24 with the ordered basis $\mathcal{B}$, and $n = 3$. Consider the linear operator $\hat{\mathcal{D}} : \mathbb{V} \to \mathbb{V}$ given by $\hat{\mathcal{D}}(f(x)) = 3f(x) + (3 - x)\frac{d}{dx}(f(x))$. Obtain bases for $\text{Ker}(\hat{\mathcal{D}})$ and $\text{Im}(\hat{\mathcal{D}})$. Suppose $g(x) = 7 + 8x$. What are all possible solutions of $\hat{\mathcal{D}}(f(x)) = g(x)$?

$\mathbb{V} = \mathbb{R}[x]_n$ is a vector space of all polynomials of real coefficients with $\deg \leq 3$ over the field $\mathbb{R}$.

$$\mathbb{V} = \{ p \mid p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 , \forall a_i \in \mathbb{R} \}$$

The ordered basis for $\mathbb{V}$ is given as -

$$B = (f_0, f_1, f_2, f_3) \quad \text{such that} \quad f_i(x) = x^i$$
$$= (1, x^1, x^2, x^3)$$

Suppose the operator $\hat{D} : \mathbb{V} \rightarrow \mathbb{V}$ is defined as:

$$\hat{D}\left(f(x)\right) := 3f(x) + (3-x)\frac{d}{dx}\left(f(x)\right)$$

Suppose the polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad \text{—— (i)}$$

$$\hat{D}\left(f(x)\right) = 3\cdot\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) +$$
$$(3-x)\cdot\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)$$

$$= 3a_0 + 3a_1 x + 3a_2 x^2 + 3a_3 x^3 +$$
$$(3-x)\left(a_1 + 2a_2 x + 3a_3 x^2\right)$$

$$= 3a_0 + 3a_1 x + 3a_2 x^2 + 3a_3 x^3 +$$
$$3a_1 + 6a_2 x + 9a_3 x^2 - a_1 x - 2a_2 x^2 - 3a_3 x^3$$

$$= \left(3a_0 + 3a_1\right) + \left(3a_1 + 6a_2 - a_1\right)x +$$
$$\left(3a_2 + 9a_3 - 2a_2\right)x^2 + \left(3a_3 - 3a_3\right)x^3$$

$$\hat{D}\left(f(x)\right) = \left(3a_0 + 3a_1\right) + \left(2a_1 + 6a_2\right)x + \left(a_2 + 9a_3\right)x^2$$

$$\text{—— (ii)}$$

$f(x)$ written in ordered basis B will be –

$$\underset{\in \mathbb{R}[x]_3}{f} \xmapsto{B} [f]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^4 \quad \text{(from eqn 1)}$$

encoding of f in $\mathbb{R}^n$.

$\hat{D}(f(x))$ can be written in ordered basis B as follows:

$$\underset{\in \mathbb{R}[x]_3}{\hat{D}(f(x))} \xmapsto{B} \left[\hat{D}(f(x))\right]_B = \begin{bmatrix} 3a_0+3a_1 \\ 2a_1+6a_2 \\ a_2+9a_3 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

(from eqn 2)

Let's try to find the transformation matrix corresponding to linear transformation $\hat{D}$.

$$\left[\hat{D}(f(x))\right]_B = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3a_0+3a_1 \\ 2a_1+6a_2 \\ a_2+9a_3 \\ 0 \end{bmatrix}$$

$$\left[\hat{D}(f(x))\right]_B = T_B\left(\left[f(x)\right]_B\right)$$ where $T_B \in \mathbb{R}^{4\times 4}$ shown in the above matrix.

Therefore $\hat{D} : \mathbb{V} \longrightarrow \mathbb{V}$ can be now interpreted as:

$$\underbrace{\left[\hat{D}(f(x))\right]_B}_{\substack{\text{coordinates} \\ \text{written in } B \\ \text{of } \hat{D}(f(x)) \\ \in \mathbb{R}^4}} = T_B \left(\underbrace{[f(x)]_B}_{\substack{\text{coordinates written} \\ \text{in } B \text{ of } f(x) \\ \in \mathbb{R}^4}} \right)$$

Transformation matrix $\in \mathbb{R}^{4\times 4}$.

Obtaining $\operatorname{Ker}(\hat{D})$ is equivalent to obtaining $\operatorname{Ker}(T_B)$.
Obtaining $\operatorname{Im}(\hat{D})$ is equivalent to obtaining $\operatorname{Im}(T_B)$.

Consider the equation $T_B \cdot x = 0$

$$T_B = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1, x_2, x_3 are pivot variables. x_4 is free variables.
Therefore,

Choose x_4 as 1.

$$1 \cdot x_3 + 9 \cdot x_4 = 0 \Rightarrow x_3 = -9$$

$$2 \cdot x_2 + 6x_3 = 0 \Rightarrow x_2 = -\frac{6x_3}{2} = +\frac{6 \times 9}{2}$$
$$= 27$$

$$3x_1 + 3x_2 = 0$$

$$\Rightarrow x_1 = -\frac{3x_2}{3} = -x_2 = -27$$

Hence we get the $\mathrm{Ker}(T_B)$ as -

$$\mathrm{Ker}(T_B) = \left\{ c \cdot \begin{bmatrix} -27 \\ 27 \\ -9 \\ 1 \end{bmatrix} \middle| \ \forall c \in \mathbb{R} \right\}$$

$\mathrm{Ker}(T_B)$ here is encoded as coordinates under the basis B
Therefore, the Kernel of $\hat{D}$ will be written as:

$$\mathrm{Ker}(\hat{D}) = \left\{ \left(-27 + 27x - 9x^2 + x^3\right) \cdot c \ \middle| \ \forall c \in \mathbb{R} \right\}$$

$$\text{Bases for } \mathrm{Ker}(\hat{D}) = \left\{ -27 + 27x - 9x^2 + x^3 \right\}$$

From the row echelon form it's clear that x_1, x_2, x_3 are pivot variables, therefore the columns corresponding to x_1, x_2, x_3 variables are pivot columns. and the $\mathrm{Im}(T_B)$ is nothing but LC of pivot columns of T_B.

Therefore we take first 3 columns of T_B to get $\mathrm{Im}(T_B)$

$$\mathrm{Im}(T_B) = \left\{ \alpha \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \beta . \begin{bmatrix} 3 \\ 6 \\ 1 \\ 0 \end{bmatrix} + \gamma . \begin{bmatrix} 0 \\ 0 \\ 9 \\ 0 \end{bmatrix} \middle| \forall \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 3\alpha + 3\beta \\ 2\alpha + 6\beta \\ \beta + 9\gamma \\ 0 \end{bmatrix} \middle| \forall \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$\mathrm{Im}(T_B)$ here is encoded as coordinates under the basis B.
Therefore $\mathrm{Im}(\hat{D})$ written as:

$$\mathrm{Im}(\hat{D}) = \left\{ (3\alpha + 3\beta) + (2\alpha + 6\beta)x + (\beta + 9\gamma)x^2 \middle| \forall \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$$= \left\{ (3 + 2x)\alpha + (3 + 6x + x^2)\beta + 9x^2 \cdot \gamma \right\}$$

Bases for $\mathrm{Im}(\hat{D}) = \left\{ (3+2x), (3+6x+x^2), (9x^2) \right\}$

Suppose, $g(x) = 7 + 8x$.

$$\hat{D}(f(x)) = g(x)$$

$$\Rightarrow T_B\left([f(x)]_B\right) = [g(x)]_B.$$

$$g(x) = 7 + 8x \xmapsto{B} [g(x)]_B = \begin{bmatrix} 7 \\ 8 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

Therefore,

$$\begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

where $[f_1, f_2, f_3, f_4]^t$ is the coordinates of $f(x)$ written in ordered basis B.

Apply row operations —

$$\left[\begin{array}{cccc|c} 3 & 3 & 0 & 0 & 7 \\ 0 & 2 & 6 & 0 & 8 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the augmented system is consistent therefore the solution to the system of linear equation will exist.

$$\begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1^p \\ f_2^p \\ f_3^p \\ f_4^p \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

↳ particular solution.

Since the free variable is f_4^p so we substitute $f_4^p = 0$ to obtain one particular solution.

$1 \cdot f_3^p + 9 \cdot f_4^p = 0$ where $f_4^p = 0$ (free variable $= 0$)

$\Rightarrow f_3^p = 0$

$2 \cdot f_2^p + 6 f_3^p = 8$

$\Rightarrow f_2^p = 4$

$3 \cdot f_1^p + 3 \cdot f_2^p = 7$

$\Rightarrow f_1^p = \dfrac{(7 - 3 \times 4)}{3} = \dfrac{-5}{3}$

$$\Rightarrow \quad f^p = \begin{bmatrix} -5/3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

We already obtained the null solutions, $f^n = c \begin{bmatrix} -27 \\ 27 \\ -9 \\ 1 \end{bmatrix}$ $(c \in \mathbb{R})$

The complete solution, $f^c = \begin{bmatrix} -5/3 \\ 4 \\ 0 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} -27 \\ 27 \\ -9 \\ 1 \end{bmatrix}$ $(c \in \mathbb{R})$

Since the solution f^c is written in terms of coordinates, let's convert it back to polynomials.

The solution $f(x) = \left\{ \left(-\frac{5}{3} - 27c\right) + (4 + 27c)x - 9cx^2 + cx^3 \right.$ $\left. \forall c \in \mathbb{R} \right\}$

22. For a linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ that sends the vector (x_1, x_2) to $(-x_2, x_1)$, show that $\varphi^2 = -$identity. Obtain $[\varphi]_{\mathcal{B}}$, where $\mathcal{B} = \{(1,2), (-1,1)\}$. Further, show that for any $\lambda \in \mathbb{R}$, $(\varphi - \lambda \cdot \text{identity})$ is invertible. Can you obtain a unique basis, $\mathcal{B}'$ for $\mathbb{R}^2$, such that $[\varphi]_{\mathcal{B}'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$? Justify with proper calculations.

$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (linear transformation)

We consider an ordered basis, $B = (b_1, b_2)$ for $\mathbb{R}^2$.
Let's choose b_1 and b_2 as standard basis.

$b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $B = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$.

In $\mathbb{R}^2$, the vector x is written as: $x = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$[x]_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Now consider the action of transformation ϕ:

$$[x]_B \xrightarrow{\phi} \phi[x]_B$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xmapsto{\phi} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$ (coordinates are written in basis B)

Let's see how the basis vectors are transformed -

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{\quad \phi \quad} \phi \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{\quad \phi \quad} \phi \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

There we can write -

$$\begin{bmatrix} \phi \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \phi \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \phi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \phi . I_{2\times 2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \phi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We have to show that $\phi^2 = -I_{2\times 2}$.

$$\phi^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I_{2\times 2}$$

(proved)

Given that the basis set, $B = (b_1, b_2)$ (ordered basis)

$$= \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right).$$

The map ϕ sends a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$.

Consider the action of ϕ under the basis vectors.

$$b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xmapsto{\phi_B} \hat{b}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \phi_B \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xmapsto{\phi_B} \hat{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \phi_B \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Combining both the actions –

$$\begin{bmatrix} \phi_B \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \phi_B \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \phi_B \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \phi_B = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}$$

$$\Rightarrow \quad \phi_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We have to show that $\phi - \lambda \cdot I_{2\times 2}$ is invertible $(\lambda \in \mathbb{R})$

Let, $A = \phi - \lambda \cdot I_{2\times 2}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

In order to show that A is invertible, the determinant of the matrix A must be non-zero.

$$|A| = -\lambda \times (-\lambda) - (1 \times -1) = 1 + \lambda^2$$

For any λ, $|A| \neq 0$ therefore A^{-1} always exists.

We have to obtain a unique ordered basis, B' for $\mathbb{R}^2$ such that

$$\varphi_{B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Suppose assume that such a basis $B' = (b_1, b_2)$ is possible.

$$[b_1]_{B'} \xmapsto{\varphi_{B'}} \phi_{B'} \cdot [b_1]_{B'}$$

Suppose, $b_1 = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \xmapsto[\text{(action of } \varphi \text{ in basis } B')]{\varphi_{B'}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_2 \\ -p_1 \end{bmatrix}$$

$\Big\downarrow \phi_B$ (action of φ in any basis B)

$$\varphi_B \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -p_2 \\ p_1 \end{bmatrix}$$

Where the ϕ is known we get $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \longmapsto \begin{bmatrix} p_2 \\ -p_1 \end{bmatrix}$

Where the φ is unknown we get, $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \longmapsto \begin{bmatrix} -p_2 \\ p_1 \end{bmatrix}$

But we know that ϕ written in any basis B must be in the form $\begin{bmatrix} -p_2 \\ p_1 \end{bmatrix}$ so for B' also it must follow same.

$$\begin{bmatrix} -p_2 \\ p_1 \end{bmatrix} = \begin{bmatrix} p_2 \\ -p_1 \end{bmatrix} \quad \text{if and only if} \quad p_1 = 0,\ p_2 = 0$$

Therefore, $b_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is a contradiction. Since we know that basis B' must be LI set but any B' containing $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ can't be LI so such a unique B' is not possible where $\varphi_{B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.