Newtonian Noise and analytical estimate

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1. Newtonian noise contributions

Consider a small cubic volume element with sides Δ and located at coordinates $r_i = (x, y, z)$, where r_i is the vector pointing from the origin to the center of the volume element. Assuming that Δ is very small compared to $\mathbf{r}$, one can calculate the gravitational acceleration that this volume element introduces on a mass located at the origin

$$a_i = G\rho\Delta^3 r_i/r^3 \tag{1}$$

with a_i the acceleration, G Newton's gravitational constant, ρ the density of the volume element, and r the radius $r = |r_i| = \sqrt{x^2 + y^2 + z^2}$. Assume that a time-dependent displacement field $\xi_j(, r, t)$ is moving the matter distribution: the matter at r_i is shifted towards $r_i' = r_i + \xi_j(r_i, t)$. If we consider only small displacements ($\xi_j << r$) and small displacement field gradients ($|\xi|\frac{d\xi_j}{dx_i}) << \Delta$, *i.e.* wavelengths long compared to Δ) we can calculate the change in acceleration due to the volume element at rest position r_i :

$$\delta a_i = a_i(\mathbf{r}', t) - a_i(\mathbf{r}) = G\rho\Delta^3[\frac{r_i'}{r'^3} - \frac{r_i}{r^3}]. \tag{2}$$

Using the fact that Δ and ξ are small with respect to r, we can linearize the equation, writing

$$\delta a_i = \frac{da_i}{dx_j}\xi_j = G\rho\Delta^3\frac{[r^2\delta_{ij} - 3r_i r_j]}{r^5}\xi_j \tag{3}$$

where the Einstein summation convention is used for summing over repeated indices and a_i' has been expanded from a_i as $a_i' = a_i + \frac{da_i}{dx_j\xi_j}$. If one integrates over the full mass distribution, i.e. over the complete medium, then this gives the total Newtonian noise contribution to the test mass acceleration. **No surface term needs to be added**; the fact that matter moves is taken care of in the volume term in Eq. 3. Note that this statement holds true for *any* integration shape; there is no need to assume spherical homogeneity or a spherical integration volume. One can calculate the Newtonian noise for any shape of medium distribution in this manner. Now, in practice the medium may extend too far and one wants to consider the Newtonian noise from a finite region, for instance integrate up to a given radius. The mass that is displaced through the outer integration boundary replaces mass that was already present at that boundary. If a volume element at the boundary at R_max moves outwards, the density at the boundary is not doubled; also if the element moves inwards, no void is created at the boundary. Thus, if one wants a consistent result and wants to use the volume term from Eq. 3 to calculate the contributions from a given region, one should add contributions from the mass displacements at the integration boundary of the matter from the outside:

$$\delta a_{i,surf} = -G\rho_{out}\int_S r_i/r^3 \xi_j \hat{n_j} dS. \tag{4}$$

Here, ρ_{out} is the static density of medium outside the volume integration boundary, and the minus sign is due to the fact that the matter ρ_{out} is displaced by ρ_{in}. Note that $dS = r^2(\Omega)d\Omega$; the weight of the contribution of a surface element inside a given solid angle $d\Omega$ DOES NOT depend on r; the surface contribution within a certain small solid angle does not diminish for large r. Note also, that the change in acceleration is **in the direction of the surface element r_i, NOT in the direction of the displacement field** $\xi_j(r)$, whereas the size depends on the dot product $\xi_j\hat{n_j}$. This does not matter as long as one only considers spherical asymmetry, but as soon as one introduces a half-space with the normal to the air-soil boundary then this symmetry is broken and one may large deviations from expectations based on spherical asymmetry.

So the total Newtonian noise due to a displacement field in a mass distribution amounts comes from a volume term and a surface term to take care of continuity at the integration boundary:

$$\delta a_i = G\rho_{in}\int_V \frac{r^2\delta_{ij} - 3r_ir_j}{r^5}\xi_j dV - G\rho_{out}\int_S r_i/r^3\xi_j\hat{n_j}dS \tag{5}$$

One can also calculate the contributions to Newtonian noise using a different approach: instead of centering on the change of position of the masses involved, one can consider changes in density in a fixed volume (constant integration boundaries) due to the presence of the displacement field ξ. As should be, the results for Newtonian noise obtained in this approach are the same as above, as I demonstrate in the next Equations. The mass change in the cubical volume element under consideration amounts to

$$\delta m = -\rho\Delta^3\frac{d\xi_i}{dx_i} \tag{6}$$

where the continuity equation has been used: the mass that flows out of the cubical volume amounts to

$$\int_S \rho\xi_i\hat{n}_i dS \tag{7}$$

where $\hat{n}$ is the normal to the surface of the volume element, pointing outwards. For a small cubic volume element, one can easily see that Eq. 6 follows from Eq. 7; the integral can be replaced by the sum over the six faces. The mass flow through these faces amounts to δm:

$$\delta m = -\rho\Delta^2[\xi_i(\mathbf{r} + \frac{\Delta}{2}\hat{e_i}) - \xi_i(\mathbf{r} - \frac{\Delta}{2}\hat{e_i})] \tag{8}$$

with $\hat{e_i}$ the unit vector in the i-direction. The finite differences in Eq. 8 can be replaced by a derivative:

$$\frac{d\xi_i(r)}{dx_i} = \mathrm{Lim}_{\Delta\to 0}\frac{\xi_i(r + \frac{\Delta}{2}\hat{e_i}) - \xi_i(r - \frac{\Delta}{2}\hat{e_i})}{\Delta}. \tag{9}$$

Substituting Eq. 9 in Eq. 8 yields Eq. 6. Note, that here we assumed that the small volume element under consideration is embedded in a homogeneous medium; mass can flow into or out of the volume element. If the density is a function of position inside the medium one should use $\rho(r)$ instead of a constant ρ. In the derivation of Eq. 6 it is assumed that the density is continuous at the surface. If a part of the surface of the volume element borders to a medium with a different density then we have to consider the contributions of the surface boundary in this model, just as before. In principle one can write ρ as a function of r; this function would be a step function at the boundary (and this step function transforms in a delta function when differentiating over it, hence the volume term gets a surface contribution), but the easiest way to correct the contributions from Eq. 6 in the case of a discontinuous integration boundary is by adding a term that describes the discrepancy in our assumption of an infinite homogeneous medium:

$$\delta m_{surf} = G(\rho_{in} - \rho_{out}) \int_S \xi_i \hat{n_i} dS \tag{10}$$

where ρ_{in} is the equilibrium density of the medium inside the integration volume in absence of ξ and ρ_{out} the density outside the integration boundary; this term takes into account the matter flow at a dis-continuous surface. We still need to calculate the acceleration caused by the density fluctuations in the medium when using Eqs. 6 and 10. The volume term of this integration amounts to

$$\delta a_{i,Vol} = -G\rho_{in} \int_V \frac{r_i}{r^3} \frac{d\xi_j}{dx_j} dV. \tag{11}$$

This volume term contains the derivatives of the displacement field ξ. It is convenient to use the divergence theorem, and partial integration, to replace the divergence over the volume term with an integral over the surface:

$$\int_V \frac{d}{dx_j} f_{ij} dV = \int_S f_i j \hat{n_j} dS \tag{12}$$

$$G\rho \int_V \frac{d}{dx_j}[r_i/r^3 \xi_j] dV = \int_S r_i/r^3 \xi_j \hat{n_j} dS \tag{13}$$

$$G\rho \int_V \xi_j \frac{d}{dx_j}[r_i/r^3] dV + G\rho \int_V \frac{r_i}{r^3} \frac{d\xi_j}{dx_j} dV = \int_S r_i/r^3 \xi_j \hat{n_j} dS \tag{14}$$

$$-G\rho_{in} \int_V \frac{r_i}{r^3} \frac{d\xi_j}{dx_j} dV = G\rho \int_V \xi_j \frac{d}{dx_j}[r_i/r^3] dV - G\rho \int_S r_i/r^3 \xi_j \hat{n_j} dS. \tag{15}$$

The total Newtonian-noise contribution then results in

$$\delta a_i = \delta a_{i,surf} + \delta a_{i,Vol} = G\rho \int_V \xi_j [\frac{d}{dx_j} r_i/r^3] dV - G\rho_{out} \int_S r_i/r^3 \xi_j \hat{n_j} dS \tag{16}$$

where the surface part in the volume integral in Eq. 12, that arrived from using the divergence theorem in Eq. 12; the term $-G\rho_{in} \int_S r_i/r^3 \xi_j \hat{n_j} dS$ is canceled by the contribution with opposite sign from the surface term from Eq. 10.

We see, that the total Newtonian-noise contribution for integration over arbitrarily-shaped volume parts as given in Eq.16 is equal to the result in Eq. 5, as should be.