Newtonian acceleration from the interface between solid and air due to seismic waves. Can we use linearised continuity equation at the interface?

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In this note, I am trying to digest two notes on Newtonian noise, one note from Patrick Schillings [1], and another from Henk Jan Bulten [2]. The main question I try to understand here is what is the coherent way to treat mass transfer at the interface between the two media with different densities.

Patrick says that the formula for the fluctuations of density, $\delta\rho(r,\, t) = -\nabla\cdot(\bar{\rho}(\boldsymbol{r})\boldsymbol{\xi}(\boldsymbol{r},t))$, with $\bar{\rho}(\boldsymbol{r})$ a stationary density distribution w/o seismic wave and $\boldsymbol{\xi}(\boldsymbol{r},t)$ is a seismic displacement field, might be incorrect, since it is derived using the assumption that $\delta\rho/\bar{\rho} \ll 1$, which is obviously not true at the interface of the rock and air, for instance. So, how do we treat the discontinuities of $\bar{\rho}$ then?

Henk Jan answers this question by saying that the mass transfer through the interface, $\delta m_{surf} = (\rho_{in} - \rho_{out})\int_{S_{surf}} dS\,\boldsymbol{\xi}\cdot\hat{\boldsymbol{n}}_{surf}$ (Eq. (10) of [2]), where ρ_{in} and ρ_{out} are densities of rock and air respectively (inside the medium and outside of it) and $\hat{\boldsymbol{n}}_{surf}$ is the outer unit normal vector of the interface, leads to an extra *surface term* in the Newtonian potential and acceleration. So, the Newtonian acceleration in the presence of interface has to include this surface term (cf. Eqs. (10-11)):

$$\delta\boldsymbol{a}_{Vol} + \delta\boldsymbol{a}_{surf} = -G\rho_{in}\int_V dV\,\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\cdot\nabla\boldsymbol{\xi}(\boldsymbol{r},t) + G(\rho_{in}-\rho_{out})\int_{S_{surf}} dS\,\frac{\boldsymbol{\xi}\cdot\hat{\boldsymbol{n}}_{surf}}{|\boldsymbol{r}|}\,.$$

Let start with the derivation of the continuity equation.

1 Continuity equation

Continuity equation is the reflection of simple fact that the only way the mass inside some given volume V could change is via the in- and outflux of matter across the boundary, S, of this volume:

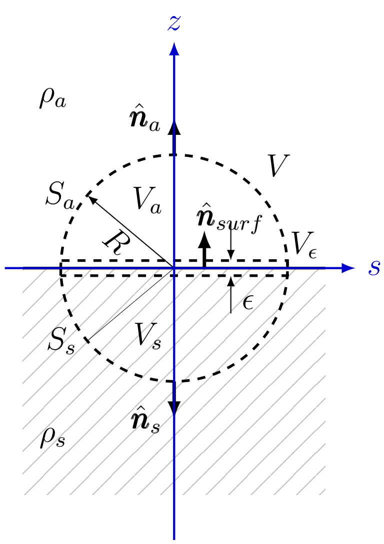
$$\frac{dM}{dt} = \frac{d}{dt}\int_V dV\,\rho(\boldsymbol{r},\,t) = -\oint_S dS\,\rho(\boldsymbol{r},\,t)(\boldsymbol{v}\cdot\hat{\boldsymbol{n}}_S)\,, \tag{1}$$

where $\rho(\boldsymbol{r},\,t)$ is density of material at given point $\boldsymbol{r}$ at time t, $\hat{\boldsymbol{n}}_S$ is the unit vector of outer normal to surface S, and $\boldsymbol{v} = d\boldsymbol{\xi}/dt$ is the velocity field associated with seismic displacement, $\boldsymbol{\xi}(\boldsymbol{r}, t)$.

The next step is to use the *divergence theorem* to replace integration over surface S by the volume integral:

$$\int_V dV\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v})\right) = 0 \implies \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \tag{2}$$

where the last equation is the differential form of the *continuity equation* itself.



Here, however, one might already question the applicability of the divergence theorem when volume V, and thus its surface S, include the interface with discontinuous function $\rho(\boldsymbol{r},\,t)$.
Let split integration volume into sub-volumes (see Figure), $V = V_a + V_s + V_\epsilon$, where V_a and V_s are sub-volumes where density is continuous and differentiable everywhere inside and also in the vicinity of the border, whereas V_ϵ is a Gaussian pillbox of height $\epsilon \sim \xi_n$ so that it contains the discontinuity region even when seismic displacement moves the surface by a non-zero normal displacement, ξ_n.

Volumes with continuous ρ: The two parts of the integral (1) with integration over volumes V_a and V_s can be calculated using divergence theorem:

$$\partial_t M_s = \partial_t \int_{V_s} \rho_s\,dV = -\oint_{S_s} dS\,\rho_s(\boldsymbol{v}_s\cdot\hat{\boldsymbol{n}}_s) = -\int_{V_s}\nabla\cdot\big(\rho_s\partial_t\boldsymbol{\xi}_s\big)\,dV\,, \tag{3}$$

$$\partial_t M_a = \partial_t \int_{V_a} \rho_a\,dV = -\oint_{S_a} dS\,\rho_a(\boldsymbol{v}_a\cdot\hat{\boldsymbol{n}}_a) = -\int_{V_s}\nabla\cdot\big(\rho_a\partial_t\boldsymbol{\xi}_a\big)\,dV\,. \tag{4}$$

It is safe to assume that integrands in the left and in the right side of these equations are equal to each other and thus $\rho_{s,a}$ satisfies the continuity equation in differential form:

$$\partial_t \rho_{s,a} = -\nabla \cdot \left(\rho_{s,a} \partial_t \boldsymbol{\xi}_{s,a}\right) . \tag{5}$$

If one assumes $\rho_{s,a}(\boldsymbol{r},\, t) \equiv \bar{\rho}_{s,a}(\boldsymbol{r}) + \delta\rho_{s,a}(\boldsymbol{r},\, t)$ and additionally impose linearity condition $\delta\rho_{s,a}/\rho_{s,a}$, which is well justified inside $V_{s,a}$, one gets the well known relation between density fluctuations and the seismic displacement field $\boldsymbol{\xi}(\boldsymbol{r},\, t)$:

$$\delta\rho_{s,a}(\boldsymbol{r},\, t) = -\nabla \cdot \left(\bar{\rho}_{s,a}(\boldsymbol{r}) \boldsymbol{\xi}_{s,a}(\boldsymbol{r},\, t)\right) . \tag{6}$$

Hence in the continuous bulk one can safely use the linearised expression for density fluctuations.

Gaussian pillbox V_ϵ: Let volume V_ϵ consist of two plane surfaces of area A parallel to interface and located on both sides of it at distance $\epsilon/2$ each. Since $\epsilon \to 0$, one can introduce two densities on each side of the interface as $\rho_a(\boldsymbol{s})$ and $\rho_s(\boldsymbol{s})$. Then the continuity equation reads

$$\partial_t \int_{V_\epsilon} \rho \, dV + \oint_{S_\epsilon} dS \, \rho \partial_t (\boldsymbol{\xi} \cdot \hat{\boldsymbol{n}}_{surf})) = 0 \, , \tag{7}$$

where $\hat{\boldsymbol{n}}_{surf}$ is a positive unit normal vector of the interface pointing from solid into air, and the volume integral can be rewritten as:

$$\int_{V_\epsilon} \rho \, dV \simeq \int_A \sigma(\boldsymbol{s}, t) dA = \int_A \left[\rho_s(\boldsymbol{s}, t)\xi_{s,n}(\boldsymbol{s}, t) - \rho_a(\boldsymbol{s}, t)\xi_{a,n}(\boldsymbol{s}, t)\right] dA = \int_A [\rho_s - \rho_a]\xi_n \, dA \tag{8}$$

where due to $\epsilon \to 0$, volume of integration tends to 0 and the only surviving term will be due to the *surface mass density* σ, which is defined as cross-interface mass transfer:

$$\boxed{\sigma(\boldsymbol{s}, t) = \rho_s(\boldsymbol{s}, t)\xi_{s,n}(\boldsymbol{s}, t) - \rho_a(\boldsymbol{s}, t)\xi_{a,n}(\boldsymbol{s}, t) \, .} \tag{9}$$

This is exactly the term that Henk-Jan introduces in Eq. (10) of his note and the one responsible for the extra surface term in acceleration.

Here $\boldsymbol{s}$ is the coordinate in the plane of the interface, $\xi_n(\boldsymbol{s}, t) = \boldsymbol{\xi} \cdot \hat{\boldsymbol{n}}_{surf}$ is the component of displacement field $\boldsymbol{\xi}(\boldsymbol{s}, t)$ orthogonal to the interface. The minus sign for the second term is due to the sign of the normal vectors which are deemed positive when pointing outside of the volume, hence they are opposite on two faces of the pillbox. In the last equality we assumed the continuity of displacement above and below the interface, $\xi_{a,n} = \xi_{s,n} \equiv \xi_n$

The mass flux through the pillbox can then be written as:

$$\oint_{S_\epsilon} dS\, \rho(\partial_t \boldsymbol{\xi} \cdot \hat{\boldsymbol{n}}_{surf})) \simeq \int_A \big[\rho_a(\boldsymbol{s},t) - \rho_s(\boldsymbol{s},t)\big] \partial_t \xi_n(\boldsymbol{s},t)\, dA\,.$$

The continuity equation in the pillbox is thus taking the form of boundary condition at the interface :

$$\boxed{\frac{\partial \sigma}{\partial t} + [\rho_a - \rho_s]\frac{\partial \xi_n}{\partial t} = 0\,,} \tag{10}$$

where the dependence on $\boldsymbol{s}$ and t is omitted for brevity. The above equation is precise and does not use any assumptions about smallness of $\delta\rho/\rho$.

Now, if we set bulk density of air $\rho_a = 0$ and $\rho_s \to \bar{\rho}_s$, *i.e.* not dependent from time, then we get that surface density is:

$$\boxed{\sigma(\boldsymbol{s},t) = \bar{\rho}_s(\boldsymbol{s})\xi_n(\boldsymbol{s},t)\,,} \tag{11}$$

which agrees with the definition of (9), saying that any local change of surface density is due to mass transfer by seismic displacement across the surface.

Heaviside jump approach: Harms argues in [3] that one does not need to specially treat the discontinuities in density at interfaces like we did above. In Eqs. (45-46) on p. 34 of [3], it is shown that if one uses the equation defining the interface points $\boldsymbol{s}$ satisfying equation:

$$S(\boldsymbol{s}) = 0$$

which is the general form of our equation for the plane interface, $\boldsymbol{s} \cdot \hat{\boldsymbol{n}} = 0$, one can define density in the entire space using the Heaviside step-function, $\Theta(S(\boldsymbol{r}))$:

$$\rho(\boldsymbol{r},t) = \rho_s(\boldsymbol{r},t)\Theta(S(\boldsymbol{r})) + \rho_a(\boldsymbol{r},t)\Theta(-S(\boldsymbol{r}))\,. \tag{12}$$

Function $S(\boldsymbol{r})$ is chosen such that $\nabla S = \hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is a positive unit normal vector to the interface (pointing outside of solid into the air in our case).

If one substitutes (12) into (6) one has to compute terms like:

$$\nabla \cdot \big\{\rho_{s,a}(\boldsymbol{r})\boldsymbol{\xi}_{s,a}(\boldsymbol{r},t)\Theta(\pm S(\boldsymbol{r}))\big\} = \Theta(\pm S(\boldsymbol{r}))\Big[\rho_{s,a}\nabla\cdot\boldsymbol{\xi}_{s,a} + \nabla\rho_{s,a}\cdot\boldsymbol{\xi}_{s,a}\Big] \pm \rho_{s,a}\boldsymbol{\xi}_{s,a}\cdot\hat{\boldsymbol{n}}\delta(S(\boldsymbol{r}))\,,$$

where the last term comes from the differentiation of the Heaviside function, which gives Dirac $\delta(S(\boldsymbol{r}))$ and $\nabla S = \hat{\boldsymbol{n}}$.

One can see that the terms with a Dirac δ-function, when integrated over volume will transform into a surface integral over the interface:

$$\int_V dV\, [\rho_s - \rho_a]\delta(S(\boldsymbol{r}))\hat{\boldsymbol{n}}\cdot\boldsymbol{\xi} = \int_{S(\boldsymbol{s})=0} dS\, [\rho_s(\boldsymbol{s}) - \rho_a(\boldsymbol{s})]\xi_n(\boldsymbol{s},t) = \int_{S(\boldsymbol{s}=0)} dS\, \sigma(\boldsymbol{s},t)\,,$$

which gives exactly the same *surface density* term as (9) and Henk-Jan's Eq. (10) in[2].

Hence, we come to the conclusion that one can use formula (6) in discontinuous geology if one treats the discontinuity of ρ consistently.

2 Newtonian potential and acceleration

Newtonian potential variation: Fluctuations of gravitational potential, $\delta\phi_N(\boldsymbol{r}_0, t)$, caused by seismic field $\boldsymbol{\xi}(\boldsymbol{r}, t)$, is calculated using known formula:

$$\delta\phi_N(\boldsymbol{r}_0, t) = G \int_V dV \, \frac{\delta\rho(\boldsymbol{r}, t)}{|\boldsymbol{r} - \boldsymbol{r}_0|} \,, \tag{13}$$

where $\boldsymbol{r}_0$ is the so called *field point*, where the potential is measured. For our purposes it is the location of the GW interferometer's mirror.

Using the results of the previous section, one can see that the Newtonian potential at given *field point* $\boldsymbol{r}_0$ from our discontinuous geology is a sum of two *bulk* terms and a *surface* term from the pillbox:

$$\delta\phi_N(\boldsymbol{r}_0, t) = -G \sum_{\alpha=a,s} \int_{V_\alpha} dV \, \frac{\nabla \cdot \big(\bar{\rho}_\alpha(\boldsymbol{r}) \boldsymbol{\xi}(\boldsymbol{r}, t)\big)}{|\boldsymbol{r} - \boldsymbol{r}_0|} + G \int_{S_{surf}} dS \, \frac{\sigma(\boldsymbol{s}, t)}{|\boldsymbol{s} - \boldsymbol{r}_0|} \,. \tag{14}$$

The volume integrals can be further integrated by parts, since all the requirements on the continuity and differentiability of $\bar{\rho}$:

$$-G \int_{V_\alpha} dV \, \frac{\nabla \cdot \big(\bar{\rho}_\alpha(\boldsymbol{r}) \boldsymbol{\xi}(\boldsymbol{r}, t)\big)}{|\boldsymbol{r} - \boldsymbol{r}_0|} = -G \int_{V_\alpha} dV \, \frac{\bar{\rho}_\alpha(\boldsymbol{r}) \boldsymbol{\xi}(\boldsymbol{r}, t) \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}_0|^3} - G \oint_{S_{surf}+S_\alpha} dS \, \frac{\bar{\rho}_\alpha \boldsymbol{\xi}(\boldsymbol{r}, t) \cdot \hat{\boldsymbol{n}}_\alpha}{|\boldsymbol{r} - \boldsymbol{r}_0|} \,.$$

Interestingly, if one takes the last integral and looks only at the integration over S_{surf} and adds these two terms for $\alpha = a, s$, keeping in mind that the positive unit normal vectors for V_a and V_s on the interface are $\hat{\boldsymbol{n}}_a = -\hat{\boldsymbol{n}}_{surf}$ and $\hat{\boldsymbol{n}}_s = +\hat{\boldsymbol{n}}_{surf}$:

$$-G \sum_{\alpha=a,s} \int_{S_{surf}} dS \, \frac{\bar{\rho}_\alpha \boldsymbol{\xi}(\boldsymbol{r}, t) \cdot \hat{\boldsymbol{n}}_\alpha}{|\boldsymbol{s} - \boldsymbol{r}_0|} = G \int_{S_{surf}} dS \, \frac{[\bar{\rho}_a - \bar{\rho}_s] \boldsymbol{\xi} \cdot \hat{\boldsymbol{n}}_{surf}}{|\boldsymbol{s} - \boldsymbol{r}_0|} = -G \int_{S_{surf}} dS \, \frac{\sigma(\boldsymbol{s}, t)}{|\boldsymbol{s} - \boldsymbol{r}_0|} \,.$$

So this term perfectly cancels the surface term in (14), transforming the formula for calculation of Newtonian potential from seismic wave into:

$$\boxed{\delta\phi_N(\boldsymbol{r}_0, t) = -G \sum_{\alpha=a,s} \left\{ \int_{V_\alpha} dV \, \frac{\bar{\rho}_\alpha(\boldsymbol{r}) \boldsymbol{\xi}(\boldsymbol{r}, t) \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}_0|^3} + \int_{S_\alpha} dS \, \frac{\bar{\rho}_\alpha \boldsymbol{\xi}(\boldsymbol{r}, t) \cdot \hat{\boldsymbol{n}}_\alpha}{|\boldsymbol{r} - \boldsymbol{r}_0|} \right\}.} \tag{15}$$

Newtonian acceleration: Newtonian acceleration of the mirror is, by definition, a negative gradient of potential:

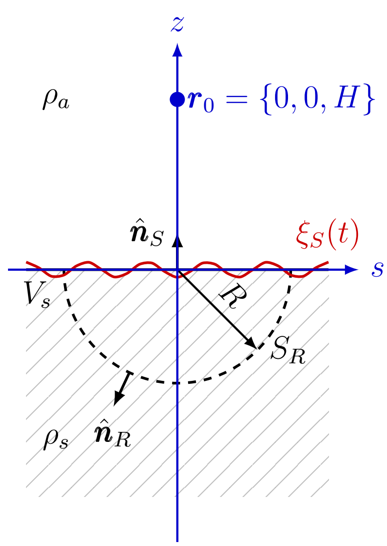
$$\delta\boldsymbol{a}_N(\boldsymbol{r}_0,t) = -\nabla_{\boldsymbol{r}_0}\delta\phi_N = -G\int_V dV\,\delta\rho(\boldsymbol{r},t)\nabla_{\boldsymbol{r}_0}\Big(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\Big) = G\int_V dV\,\frac{\delta\rho(\boldsymbol{r},t)(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}\,, \tag{16}$$

where we indeed can see that the acceleration produced by a small element of volume, dV, where $\delta\rho$ occurs, is aligned with the line of site from the mirror to this element, as Henk-Jan always pointed out [cf. Eq. (5) in [2]. The formula that depends on the seismic field can be obtained by applying ∇_{r_0} to (15):

$$\boxed{\delta\boldsymbol{a}_N = -G\sum_{\alpha=a,s}\left\{\int_{V_\alpha} dV\,\bar{\rho}_\alpha\Big[\frac{\boldsymbol{\xi}}{|\boldsymbol{r}-\boldsymbol{r}_0|^3} - \frac{3\xi_{\boldsymbol{r}-\boldsymbol{r}_0}(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^5}\Big] + \int_{S_\alpha} dS\,\frac{\bar{\rho}_\alpha\xi_{n_\alpha}(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}\right\},} \tag{17}$$

where $\xi_{\boldsymbol{r}-\boldsymbol{r}_0} = (\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}_0))$ is the projection of seismic field $\boldsymbol{\xi}$ on the line connecting *field point* $\boldsymbol{r}_0$ and the element dV, and similarly $\xi_{n_\alpha} = (\boldsymbol{\xi}\cdot\hat{\boldsymbol{n}}_\alpha)$. This formula coincides with Henk Jan's Eq. (5).

3 Newtonian noise from plane waves



Let now consider the simplest possible situation of plane seismic waves propagating in the geology with plane interface at $z=0$ between the semi-infinite solid rock of density $\bar{\rho}_s$ and air of density $\bar{\rho}_a$. This geology is translation invariant along the interface, so it is sufficient to consider Newtonian potential and acceleration at some point $\boldsymbol{r}_0 = \{0,0,H\}$ at height H above the origin of the reference frame surface.

Plane S-wave: Let consider an S-wave with amplitude $\boldsymbol{\xi}_{0S}$ propagating in the general direction $\boldsymbol{k} = k_{||}\hat{\boldsymbol{x}} + k_\perp\hat{\boldsymbol{z}}$ (x-direction is as good as any other due to cylindrical symmetry), so that $\boldsymbol{\xi}_{0S}\cdot\boldsymbol{k} = k_\perp\xi_z + k_{||}\xi_x = 0$ with $k_{||}$ and $k_\perp$ being in and out of plane of the interface components of $\boldsymbol{k}$, respectively, and ξ_x and ξ_z are the corresponding components of $\boldsymbol{\xi}_{0S}$:

$$\boldsymbol{\xi}_S(\boldsymbol{r},t) = \boldsymbol{\xi}_{0S}e^{i(\boldsymbol{k}_{||}x+k_\perp z-\omega t)}\,.$$

There are two ways to evaluate $\delta\phi_N$ in this situation:

(i) Directly use Eq. (15), where the integration volume shall be expanded to infinity and thus all surface terms over S_α shall vanish since we assume that seismic field vanishes at infinity. Assuming also $\rho_a = 0$ for air leaves us with a single bulk term:

$$\delta\phi_N^S(\boldsymbol{r}_0, t) = -G\bar{\rho}_s e^{-i\omega t} \int_{-\infty}^{0} dz \int_0^\infty s\, ds \int_0^{2\pi} d\varphi \frac{[\xi_x s\cos\varphi + \xi_z(z-H)]e^{i(k_\parallel s\cos\varphi + k_\perp z)}}{(s^2+(z-H)^2)^{3/2}} \tag{18}$$

where we use cylindrical coordinates and took into account that horizontal seismic displacement in direction orthogonal to wave propagation, *i.e.* ξ_y does not contribute to the potential (mathematically, because $\int_0^{2\pi} d\varphi\, \sin\varphi e^{i_\parallel s\varphi} = 0$). After some lengthy calculation (see Appendix) integral evaluates to:

$$\delta\phi_N^S = 2\pi G\bar{\rho}_s\xi_z \frac{e^{-k_\parallel H}}{k_\parallel} e^{-i\omega t}$$

(ii) Using definition in Eq. (13) and the fact that everywhere in the rock, except for the interface, the definition of $\delta\rho = -\bar{\rho}_s \nabla\cdot\boldsymbol{\xi}$ holds due to continuity equation, one can notice that $\nabla\cdot\boldsymbol{\xi_S} = 0$ and thus the bulk term is zero as well. Then the only surviving term will be due to the surface density $\sigma(\boldsymbol{s}, t) = \bar{\rho}_s\xi_z(\boldsymbol{s}, t)$:

$$\delta\phi_N^S = -G\int_{S_{surf}} dS\, \frac{\sigma(\boldsymbol{s}, t)}{|\boldsymbol{s} - \boldsymbol{r}_0|} = -G\bar{\rho}_s e^{-i\omega t}\int_0^\infty s\, ds \int_0^{2\pi} d\varphi \frac{\xi_z e^{ik_\parallel s\cos\varphi}}{\sqrt{s^2+H^2}} \tag{19}$$

This integral evaluates to:

$$\delta\phi_N = 2\pi G\bar{\rho}_s\xi_z \frac{e^{-k_\parallel H}}{k_\parallel} e^{-i\omega t}$$

Newtonian acceleration according to (16) can be obtained by differentiating $\delta\phi_N$ over H:

$$\boxed{\delta\boldsymbol{a}_N(\boldsymbol{r}_0, t) = 2\pi G\bar{\rho}_s\xi_z e^{-k_\parallel H}\hat{\boldsymbol{z}} e^{-i\omega t}} \tag{20}$$

Conclusion:

- Acceleration is vertical and proportional to the vertical component of seismic displacement field ξ_z and only comes from mass transfer through the interface (for shear waves);
- Acceleration is not oriented along the full displacement vector $\boldsymbol{\xi}_S$ as Henk Jan correctly points out;
- Acceleration decays exponentially with height H above the interface ($e^{-k_\parallel H}$) in full accordance with Jan Harms's Eq. (85) in [3];
- General observation is that resulting acceleration really strongly depends on the geometry of the boundaries and cannot be reduced to the 'lower limit' formula of Jan Harms. The latter is

References

[1] P. Schillings, “How to calculate Newtonian Noise from seismic body waves.” Weekly ET-EMR Newtonian Noise Meeting.

[2] H. J. Bulten, “Newtonian Noise and analytical estimate.” Weekly ET-EMR Newtonian Noise Meeting.

[3] J. Harms, “Terrestrial Gravity Fluctuations,” *Living Reviews in Relativity*, vol. 18, p. 3, Dec. 2015.

Appendix: Calculation of S-wave integrals (18) and (19)

A. Volume integral (18))

The volume integral is:

$$I_{vol} = -G\bar{\rho}_s e^{-i\omega t} \int_{-\infty}^{0} dz \int_0^{\infty} s\,ds \int_0^{2\pi} d\varphi \, \frac{\left[\xi_x s \cos\varphi + \xi_z (z-H)\right] e^{i(k_\parallel s \cos\varphi + k_\perp z)}}{(s^2 + (z-H)^2)^{3/2}} = I_x + I_z \tag{21}$$

where we split the integral into contribution from ξ_z and from ξ_x correspondingly:

$$I_x = -G\bar{\rho}_s e^{-i\omega t} \int_{-\infty}^{0} dz \int_0^{\infty} s\,ds \int_0^{2\pi} d\varphi \, \frac{\xi_x s \cos\varphi e^{i(k_\parallel s \cos\varphi + k_\perp z)}}{(s^2 + (z-H)^2)^{3/2}}$$

and

$$I_z = -G\bar{\rho}_s e^{-i\omega t} \int_{-\infty}^{0} dz \int_0^{\infty} s\,ds \int_0^{2\pi} d\varphi \, \frac{\xi_z (z-H) e^{i(k_\parallel s \cos\varphi + k_\perp z)}}{(s^2 + (z-H)^2)^{3/2}}$$

Integral I_z:

Angular integration: The φ-integral yields a Bessel function:

$$\int_0^{2\pi} d\varphi \, e^{ik_\parallel s \cos\varphi} = 2\pi J_0(k_\parallel s) \tag{22}$$

Radial integration Integral now becomes

$$I_z = -2\pi G\bar{\rho}_s \xi_z e^{-i\omega t} \int_{-\infty}^{0} dz \, (z-H) e^{ik_\perp z} \int_0^{\infty} \frac{s J_0(k_\parallel s)}{(s^2 + (z-H)^2)^{3/2}} \, ds$$

and its radial part evaluates to:

$$\int_0^{\infty} ds \, \frac{s J_0(k_\parallel s)}{(s^2 + (z-H)^2)^{3/2}} = \frac{e^{-k_\parallel (H-z)}}{H-z}$$

where we used the fact that this integral is a Hankel transform of function $1/(x^2+b^2)^{3/2}$, ($b > 0$):

$$\int_0^{\infty} dx \, \frac{x J_0(ax)}{(x^2+b^2)^{3/2}} = \frac{e^{-ab}}{b} \, .$$

Substitution gives:

$$I_z = 2\pi G\bar{\rho}_s \xi_z e^{-i\omega t} e^{-k_\parallel H} \int_{-\infty}^{0} dz \, e^{(ik_\perp + k_\parallel) z}$$

Depth integration: Finally one can integrate over z and obtain that

$$I_z = 2\pi G\bar{\rho}_s\xi_z e^{-i\omega t}\frac{e^{-k_\parallel H}}{k_\parallel + ik_\perp}$$

Integral I_x:

Angular integration: The φ-integral yields a Bessel function:

$$\int_0^{2\pi} d\varphi\, \cos\varphi e^{ik_\parallel s\cos\varphi} = 2\pi i J_1(k_\parallel s) \tag{23}$$

Radial integration Integral now becomes

$$I_x = -2\pi i G\bar{\rho}_s\xi_x e^{-i\omega t}\int_{-\infty}^{0} dz\, e^{ik_\perp z}\int_0^\infty ds\,\frac{s^2 J_1(k_\parallel s)}{(s^2+(z-H)^2)^{3/2}}$$

and its radial part evaluates to:

$$\int_0^\infty ds\,\frac{s^2 J_1(k_\parallel s)}{(s^2+(z-H)^2)^{3/2}} = e^{-k_\parallel (H-z)}$$

Substitution gives:

$$I_x = -2\pi i G\bar{\rho}_s\xi_z e^{-i\omega t}e^{-k_\parallel H}\int_{-\infty}^{0} dz\, e^{(ik_\perp + k_\parallel)z}$$

Depth integration: Finally one can integrate over z and obtain that

$$I_x = -2\pi i G\bar{\rho}_s\xi_x e^{-i\omega t}\frac{e^{-k_\parallel H}}{k_\parallel + ik_\perp}$$

Combining $I_{vol} = I_x + I_z$:

Summing up I_x and I_z and taking into account the relation imposed by S-polarisation, $\xi_x k_\parallel + \xi_z k_\perp = 0$, one gets:

$$I_{vol} = I_z + I_x = 2\pi G\bar{\rho}_s\xi_z e^{-i\omega t}\frac{e^{-k_\parallel H}}{k_\parallel} \tag{24}$$

B. Surface integral (19))

The surface integral we need to calculate is:

$$I_{surf} = -G\bar{\rho}_s\xi_z e^{-i\omega t}\int_0^\infty s\,ds\int_0^{2\pi} d\varphi \frac{e^{ik_\parallel s\cos\varphi}}{\sqrt{s^2+H^2}}\,.$$

Using already known angular integral (22) :

$$\int_0^{2\pi} d\varphi\, e^{ik_\parallel s\cos\varphi} = 2\pi J_0(k_\parallel s)\,,$$

one can write

$$I_{surf} = -2\pi G\bar{\rho}_s\xi_z e^{-i\omega t}\int_0^\infty ds \frac{sJ_0(k_\parallel s)}{\sqrt{s^2+H^2}}\,,$$

where radial integral is a Hankel transform of $e^{-k_\parallel H}/k_\parallel$, hence:

$$I_{surf} = 2\pi G\bar{\rho}_s\xi_z e^{-i\omega t}\frac{e^{-k_\parallel H}}{k_\parallel} \qquad (25)$$