An Introduction to Classical Knot Theory: How to Tell the Difference Between Twisted Piles of String

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Overview

What is a Knot?

Simply, it is just a curve that starts and ends in the same location, but can go over or under itself.

Definition (Knot)

A knot K is a simple closed curve in $\mathbb{R}^3$

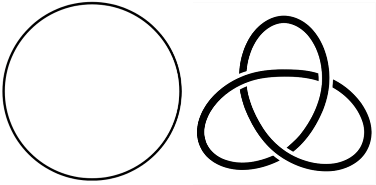

Figure: Unknot (left) and Trefoil Knot (right)

Reidemeister Moves

Intuitively, to show that two knots are different, we need to show that there is no way to transform one knot into the other while keeping its original structure. For this, we need to define what permissable 'moves' we can do to a knot that doesn't change the underlying object.

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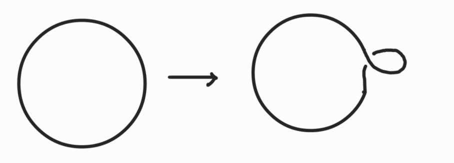
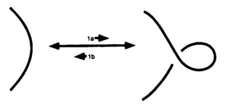
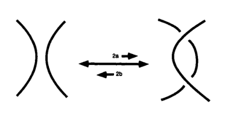
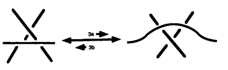


Definition (Reidemeister Moves)

Operations that can be performed on a knot diagram without altering the corresponding knot.







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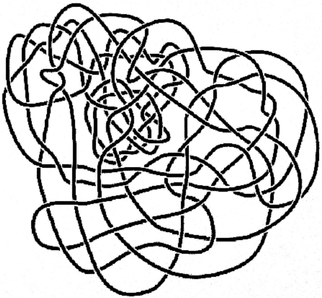

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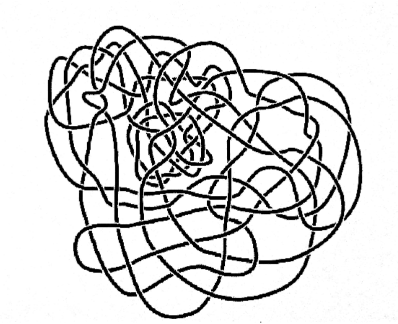

Figure: Haken's Gordian Unknot

Group Definition

Definition (Group)

Let G be a set and $*$ be an operation defined on $G \times G$. We say $G = (G, *)$ is a group if the following are satisfied:

1. $*$ is associative: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$;
2. There exists an identity element $1 \in G$: $a * 1 = 1 * a = a$ for all $a \in G$;
3. Every $a \in G$ has an inverse: for all $a \in G$ there exists $b \in G$ such that $a * b = b * a = 1$.

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- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are abelian groups with addition.
- Key observation: If one group is abelian and another isn't, they can't be (structurally) the same!

Groups

- What is a Group Presentation $\langle S \mid R \rangle$?
    - S is the set of *generators* ("ingredients") that combine to form group elements.
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- Example: $\mathbb{Z}_n = \langle a \mid a^n = 1 \rangle$.
- Note: There might be more than one group representation for the same group. For example:

$$G = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1} = 1 \rangle \cong \langle g, h \mid g^{-3}h^{-2} = 1 \rangle$$

Labeling Knot Diagrams

1. Pick an orientation.
2. Label each “arc”.
3. Examine each crossing.

Note: Be consistent!

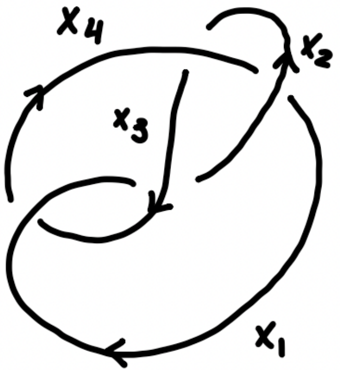
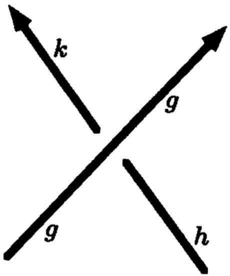
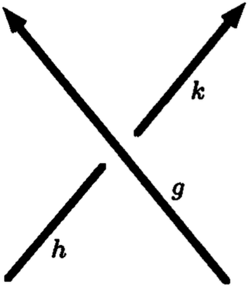


Figure: Figure-eight Knot

Labeling Knot Diagrams

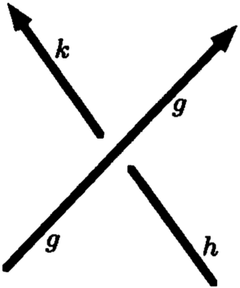
- There are two possible cases for each crossing:





- At the left, we have what we call a right-handed crossing. How do I know that? Right-hand rule!

- In the case of the right-handed crossing, the labels look like:



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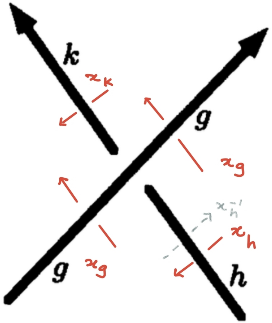
- In the case of the right-handed crossing, the labels look like:
- Add a right-to-left line under each arc, and consider inverses as going the opposite direction



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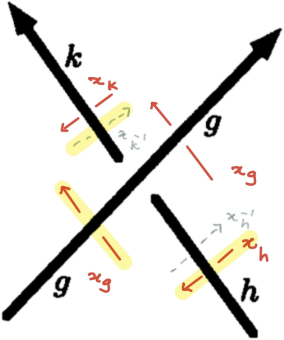
- Create an equation for the relationship between lines:
  - $x_g = x_h x_g x_k^{-1}$



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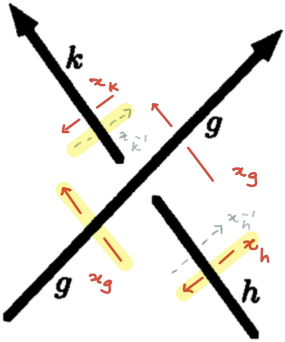
- Repeat for every crossing in the knot!



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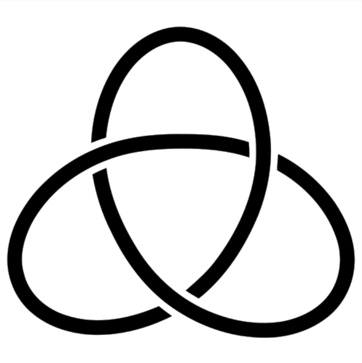

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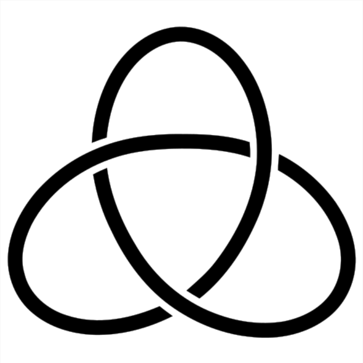

- We start by finding the equations for each crossing.

Example: Trefoil Knot

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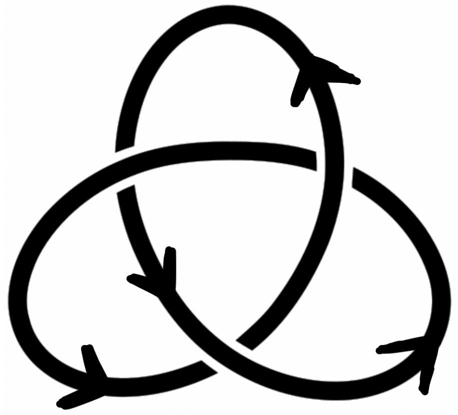
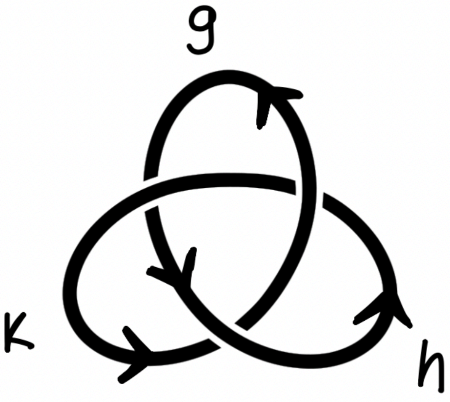


Example: Trefoil Knot

2. Label each "arc".



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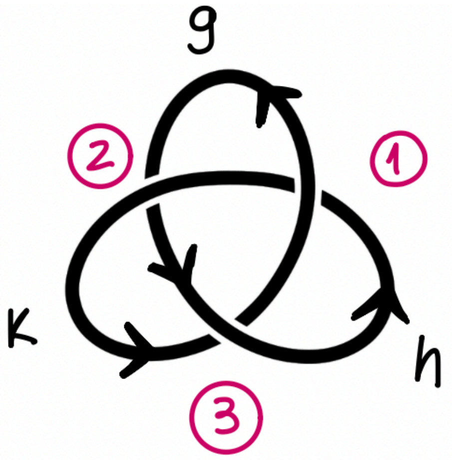
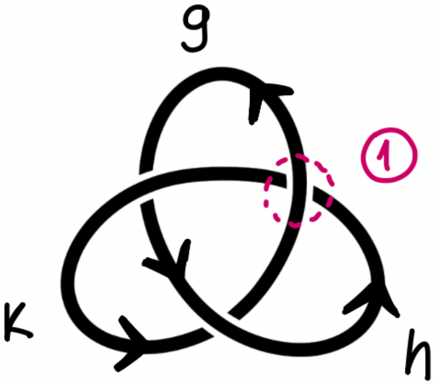
3. Examine each crossing.



Example: Trefoil Knot

- We'll start with crossing 1:



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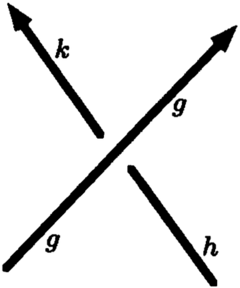

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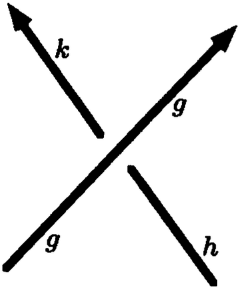
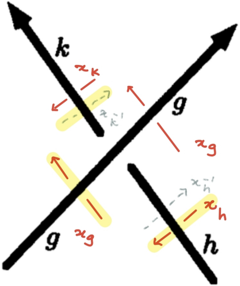


... A right-handed crossing!

Example: Trefoil Knot

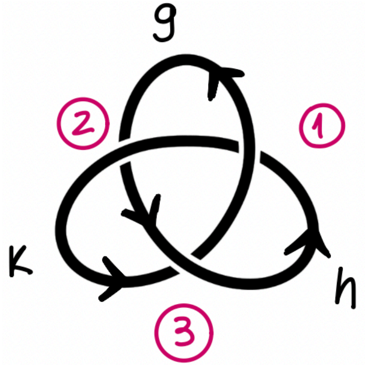
- We already calculated that before!



- From crossing 1 we get:
  - $x_g = x_h x_g x_k^{-1}$
- Equivalent to:
  - $x_k = x_g^{-1} x_h x_g - (1)$

Example: Trefoil Knot

- If we repeat this process for the other crossings, it follows that:
  - From crossing 2 we get: $x_k = x_g x_k x_h^{-1} - (2)$
  - From crossing 3 we get: $x_h = x_k x_h x_g^{-1} - (3)$



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- By subbing (1) into (2), we can eliminate x_k:

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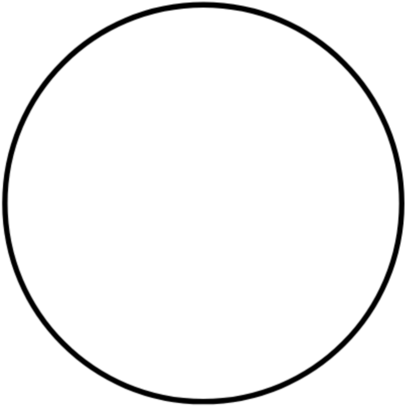
- Thus, one group presentation of the trefoil knot is:
  - $\langle x_g, x_h \mid x_g^{-1} x_h x_g = x_h x_g x_h^{-1} \rangle$

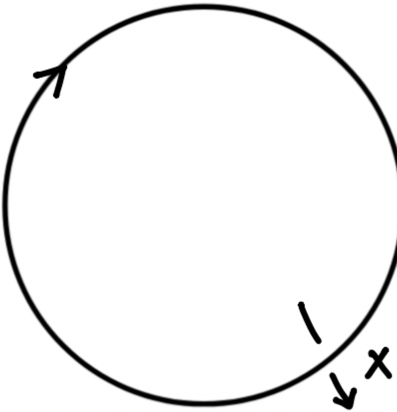
Example: Unknot

- What is the knot group of the unknot?



Example: Unknot

- Label arcs... but there are no crossings!



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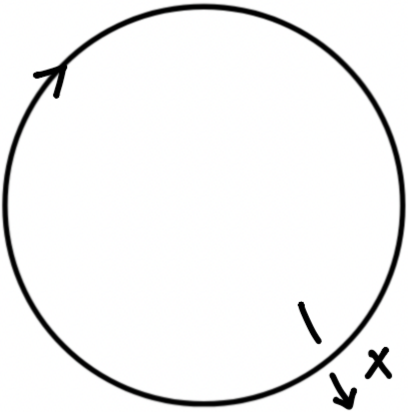

- It is $\langle x \rangle \cong \mathbb{Z}$!

Example: Unknot vs Trefoil Knot

- The group presentation of the trefoil knot group that we found before was

$$\langle x_g, x_h \mid x_g^{-1} x_h x_g = x_h x_g x_h^{-1} \rangle \cong \langle x_g, x_h \mid x_h x_g = x_g x_h x_g x_h^{-1} \rangle$$

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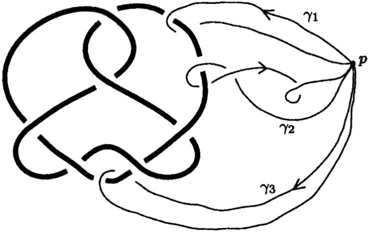
  (non-abelian).
- Whereas the unknot group is $\langle x \rangle \cong \mathbb{Z}$ (abelian).
- This means they can not be the same!

The Fundamental Group

Definition (The Fundamental Group)

For a knot K in $\mathbb{R}^3$, fix a point p in $(\mathbb{R}^3 - K)$. The **Fundamental Group of K**, $\pi_1(\mathbb{R}^3 - K)$, is the set of equivalence classes of closed oriented paths in $(\mathbb{R}^3 - K)$ that begin and end at p.



And the method of labelling that we showed is the **Wirtinger Presentation**

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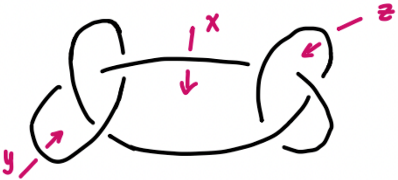

Figure: Granny Knot

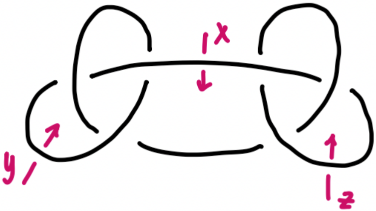

Figure: Square Knot

$$\langle x, y, z \mid xyx = yxy, xzx = zxz \rangle$$

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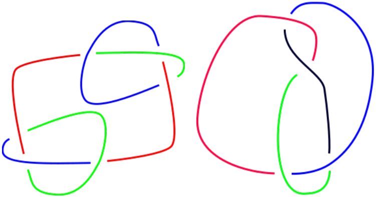
There are also many other methods (knot invariants) that can help us study knots

- Algebraic methods (e.g. Alexander Polynomial)
- Knot Colourings
- and many more!



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Thank you!