Anti-Ramsey Theory

Grace Terhljan, Soumya Menon

Mentor: Fernanda Rivera Omana

University of Waterloo, WiM Directed Reading Program

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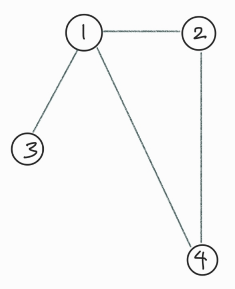
Overview

1. Graph Theory Basics
   - Definitions and Examples
   - Vertex and Edge Colouring
2. Ramsey Numbers
   - Ramsey Numbers
   - Existence of Ramsey Numbers
3. Anti-Ramsey Numbers/Rainbow Numbers

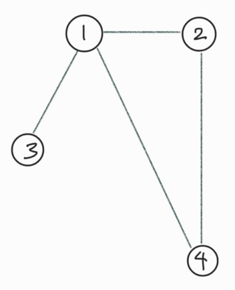
Graph Theory Basics

- A **graph**, $G = (V, E)$ is a finite nonempty set of **vertices**, and a set called the **edges**, where each edge is a two element subset of the vertices.
- The set of all vertices is represented by $V(G)$ and the set of edges by $E(G)$
- **Example:** Graph with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{12, 14, 13, 24\}$



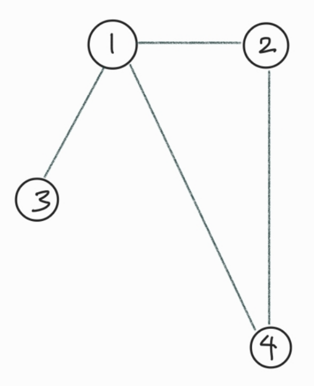
Graph Theory Basics

- The **order** of a graph G is $|V(G)|$
- The **size** of a graph G is $|E(G)|$
- **Example:** This graph has order 4 and size 4



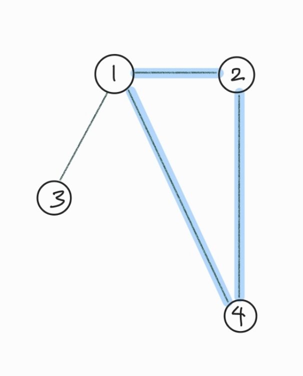
Graph Theory Basics

- The **neighbourhood** of a vertex v is the set of all vertices adjacent to it, denoted by $N_G(v)$
- The **degree** of a vertex v is the size of the neighbourhood of v, $d_G(v) = |N_G(v)|$
- **Example:** $N_G(4) = \{1, 2\}$ and $d_G(4) = 2$



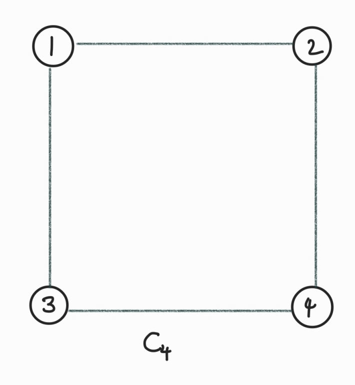
Graph Theory Basics

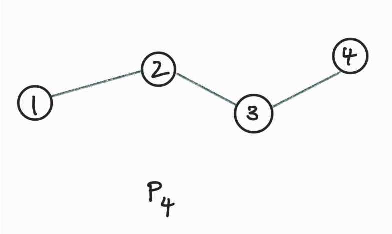
- A **subgraph** of a graph G is a graph H whose vertex set is a subset of the vertex set of G, and the edge set is a subset of the edges adjacent to both vertices from $V(H)$
- **Example:**



Examples of Basic Graphs

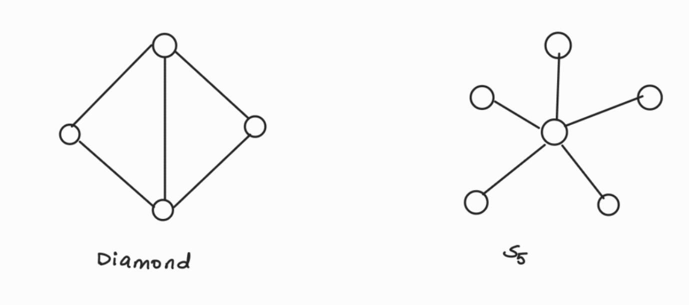
- **Paths** of length n are denoted by P_n
- **Cycles** of size n are denoted by C_n
- **Examples:** C_4 and P_4





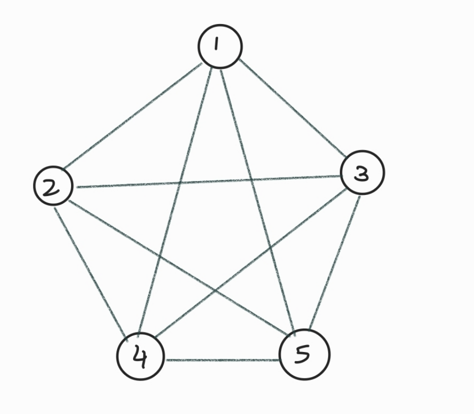
Examples of Basic Graphs

- **Diamonds** are of order 4 and size 5 and are produced by removing an edge from K_4
- **Stars** of size n are denoted by S_n
- **Examples:** S_5



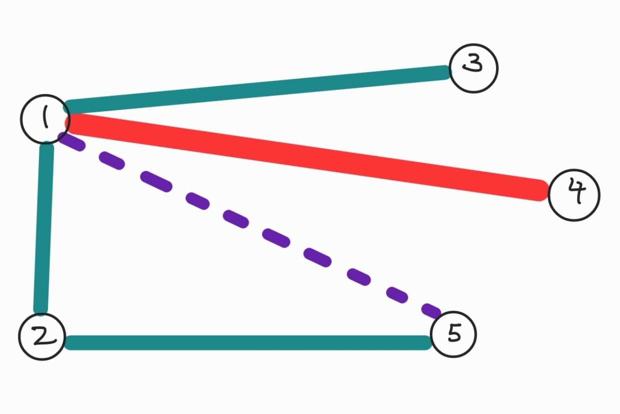
Examples of Basic Graphs

- A **complete graph**, K_n, is a graph on n vertices where every pair of vertices is connected with an edge (so there are $\binom{n}{2}$ total edges)
- **Example:** K_5



Edge Colourings

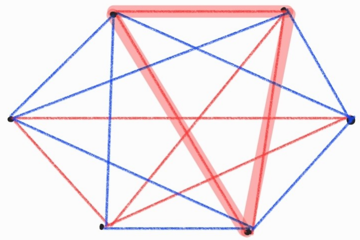
- A **k-edge colouring** of a graph G is a function that assigns each edge in a graph, an integer (or *colour*) from 1 to k, where each number is used at least once



Definition

For two positive integers k and l, the **Ramsey Number** $r(k, l)$ is defined as the smallest positive integer n such that if every edge of K_n is colored red or blue, then a monochromatic K_k or K_l is produced.

e.g. $r(3, 3) = 6$



K_6 with a red K_3

Existence of Ramsey Numbers

Theorem

For any two integers $k \geq 2$ and $l \geq 2$,

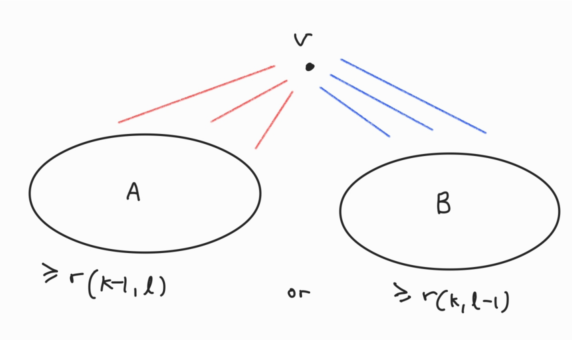
$$r(k,l) \leq r(k,l-1) + r(k-1,l)$$

We prove this using induction on $k+l$, starting with the base case at $k+l=4$. For the sake of time, we will look at the induction step.

Proof

Let $n = r(k-1,l) + r(k,l-1)$.
Consider the complete graph K_n with an arbitrary red-blue colouring. Let $v \in V$. Then we can look at the red and blue neighbourhoods of v. Let A be the red neighbourhood of v and B be the blue neighbourhood of v.s

$$|A| + |B| = n - 1 = r(k-1,l) + r(k,l-1) - 1$$
$$\Rightarrow |A| \geq r(k-1,l) \text{ or } |B| \geq r(k,l-1)$$

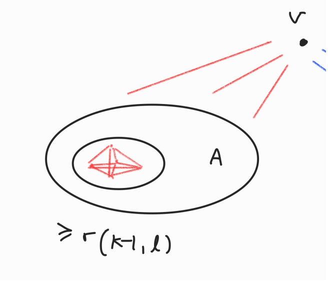


Proof (cont.)

WLOG, let us consider the case of $|A| \geq r(k-1, l)$.
Since $|A| \geq r(k-1, l)$, then A must contain a red K_{k-1} or blue K_l. If A contains a blue K_l then we have already satisfied the original claim of

$$r(k, l) \leq r(k, l-1) + r(k-1, l)$$

If it contains a red K_{k-1}, then to create a red K_k for our original claim to be satisfied, we can just connect the K_{k-1} to v with the existing red edges to get a red K_k, satisfying our claim.



Examples of Ramsey Numbers

Theorem (Ramsey number $r(3,3)$ [3])

$r(3,3) = 6$

Theorem (Ramsey number $r(3,4)$ [1])

$r(3,4) = 9$

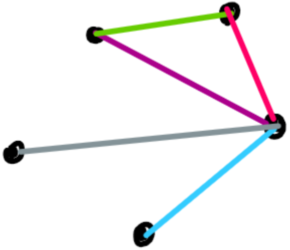
Theorem (Ramsey Number $r(5,5)$ [1])

43 *[Geoffrey Exoo (1989)]* $\leq r(5,5) \leq$ 48 *[Angeltveit and McKay (2017)]*

Anti-Ramsey Theory

Definition

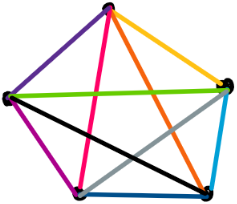
A graph G is **totally multicoloured (or rainbow)** with respect to a given colouring whenever each edge in G has a unique colour



Anti-Ramsey Theory

Definition

Given a graph G of order p with no isolated vertices and $n \geq p$, the **rainbow number** of G, $rb_n(G)$, is the smallest positive integer k such that every k-colouring of K_n, with all k colours being used at least once, results in a totally multicoloured G



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Definition

The **anti-Ramsey number**, $ar_n(G)$, of G is the maximum number of colours that can be used to colour K_n without producing a totally multicoloured G.

It follows that $rb_n(G) = ar_n(G) + 1$

Theorem [3]

Theorem (Rainbow Number of K_3)

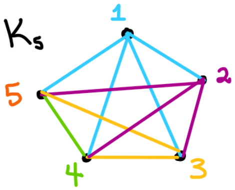
For $n \geq 3$, $rb_n(K_3) = n$

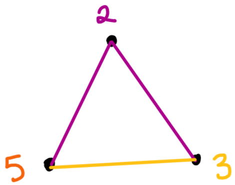
Proof Outline

There are two parts to the proof, we show $rb_n(K_3) \geq n$ and $rb_n(K_3) \leq n$

Proof of $rb_n(K_3) \geq n$

- To show $rb_n(K_3) \geq n$, we give an $(n-1)$-colouring of K_n that doesn't contain a rainbow K_3
- Label the vertices $\{v_1, v_2, \dots, v_n\}$, and then assign the edge v_iv_j for $i < j$ the colour i .
- For any three vertices of K_n, one vertex will have a smaller index than the other two. Then both of edges adjacent to that vertex will be the same colour, and we conclude there can be no rainbow K_3

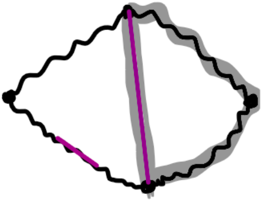




Proof of $rb_n(K_3) \leq n$

- To show $rb_n(K_3) \leq n$, we need to show that in any n-colouring of K_n we get a rainbow K_3.
- For an n-edge colouring of K_n, let H be a rainbow subgraph of size n containing the smallest cycle C

$C_k, \; 4 \leq k \leq n$



Other Rainbow Numbers [3]

Theorem (Rainbow Number of P_3)

For $n \geq 3$, $rb_n(P_3) = 2$

Theorem (Rainbow Number of $2K_2$)

If $n = 4$, $rb_n(2K_2) = 4$, *and if* $n \geq 5$, $rb_n(2K_2) = 2$

Theorem (Rainbow Number of K_4)

For $n \geq 4$, $rb_n(K_4) = \lfloor \frac{n^2}{4} \rfloor + 2$

Theorem (Rainbow Number of $K_{1,3}$)

For $n \geq 4$, $rb_n(K_{1,3}) = \lfloor \frac{n}{2} \rfloor + 2$

Thank you!

References

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[2] J. A. Bondy, U. S. R. Murty. Graph Theory with Applications. *University of Waterloo*, 1976
[3] G. Chartrand, P. Zhang. Chromatic Graph Theory. *CRC Press*, 2012
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