$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ 2x1 mean vector

$\Sigma = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ 2x2 covariance matrix

$(\sigma_X = 1, \sigma_Y = 1, \rho = a)$

Conditional Distribution of X2 given X1 $= x_1$ is

$$\underbrace{f_{X2|X1}(x_2 \mid X1 = x_1)}_{\text{Condition Distribution}} = \frac{f_{X1,X2}(x_1, x_2)}{f_{X1}(x_1)}$$

Marginal PDF of X1

Joint PDF of X1 & X2

For the case of bivariate normal distribution

$$f_{X2|X1}(x_2 \mid X1 = x_1) = \frac{f_{X1,X2}(x_1, x_2)}{f_{X1}(x_1)}$$

$$= \frac{\dfrac{e^{-\frac{1}{2(1-a^2)}\left[\frac{(x_1-\mu_1)^2}{1} + \frac{(x_2-\mu_2)^2}{1} - 2a(x_1-\mu_1)(x_2-\mu_2)\right]}}{2\pi\sqrt{1-a^2}}}{\dfrac{e^{-\frac{1}{2}(x_1-\mu_1)^2}}{\sqrt{2\pi}}}$$

$$= \frac{e^{\left\{-\frac{1}{2(1-a^2)}\left[(x_1-\mu_1)^2+(x_2-\mu_2)^2-2a(x_1-\mu_1)(x_2-\mu_2)\right]+\frac{(x_1-\mu_1)^2}{2}\right\}}}{\sqrt{2\pi}\sqrt{1-a^2}}$$

$$= \frac{e^{\left\{-\frac{1}{2(1-a^2)}\left[a^2(x_1-\mu_1)^2+(x_2-\mu_2)^2-2a(x_1-\mu_1)(x_2-\mu_2)\right]\right\}}}{\sqrt{2\pi}\sqrt{1-a^2}}$$

$$= \frac{e^{\left\{-\frac{1}{2(1-a^2)}\left[x_2-\mu_2-a(x_1-\mu_1)\right]^2\right\}}}{\sqrt{2\pi}\sqrt{1-a^2}}$$

$$= N\left(\mu_2+a(x_1-\mu_1),\ (1-a^2)\right)$$

$$f_{X2|X1}(x_2 \mid X_1=x_1) = N\left(\mu_2+a(x_1-\mu_1),\ (1-a^2)\right)$$

$$f_{X1|X2}(x_1 \mid X_2=x_2) = N\left(\mu_1+a(x_2-\mu_2),\ (1-a^2)\right)$$