3 Assignment 2

3.1 An Expectation Identity

Show that for a discrete non-negative random variable:

$$\mathbb{E}[X] = \sum_{x \geq 0} \Pr[X > x]$$

3.2 Random Permutation

Given a permutation π of $\{1, \cdots, n\}$, $i \in \{1, \cdots, n\}$ is said to be a fixed point *of π if $\pi(i) = i$.*

Let σ be a random permutation of $\{1, \cdots, n\}$. That is all the $n!$ permutations are equally likely. Let X be the random variable corresponding to the number of fixed points in σ.

a.) What is the expected value of X? ***(1.5)***

b.) Find the PMF of X? ***(1)***

c.) Given a permutation π of $\{1, \cdots, n\}$, $i, j \in \{1, \cdots, n\}$ $(i \neq j)$ is said to be a swap *if $\pi(i) = j$ and $\pi(j) = (i)$. Find the expected number of swaps in a uniformly random permutation σ.* ***(1.5)***

d.) Show that the $\Pr[X > 10] \leq 1/10$. ***(1)***

4 Randomized Coloring

Given a (undirected) graph $G = (V, E)$, and a 3-color assignment $a : V \to \{R, G, B\}$ is an assignment of colors R, G, B to the vertices of the graph. Given an assignment a, the set of monochromatic edges $E(a) = \{(u, v) \in E : a(u) = a(v)\}$, is the set of edges that has same colors for endpoints. Let a be randomly chosen, ie for every $v \in V$, it is chosen to be R, G, B uniformly and independent of the other vertices.

1. *For any edge $e \in E$, let X_e be the random variable which is 1 when e is monochromatic and 0 otherwise. Show that the set of random variables $\{X_e\}_{e \in E}$ are pairwise independent. Show that they are not independent.*

2. *Let Y be the random variable corresponding to the number of non-monochromatic edges. That is $Y = |E \setminus E(a)|$. Find $\mathbb{E}[Y]$.*

3. *Show that there cannot be a graph for which all 3-color assignments make $< 2|E|/3$ edges non-monochromatic. That is for any graph G, there exists an assignment $a : V \to \{R, G, B\}$ such that the number of non-monochromatic edges is at least $2|E|/3$.*

4. *Show that:* $P\left(Y \geq |E|/2\right) \geq 1/3$.

5. *Devise a method (which by obtaining multiple independent copies of* Y *by randomly choosing a's independently) that can find an assignment for which the number of non-monochromatic edges is at least* $|E|/2$ *with probability at least* $99/100$.
