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3.1 Solution

To Prove $E[X] = \sum_{x \geq 0} Pr[X > x]$

for a discrete non-negative random variable the expectation can be written as :

$$E[X] = 0 \cdot P(0) + 1 \cdot P(1) + 2P(2) + 3P(3) + \cdots \quad -①$$

Now we can also write

$$P(X>0) = P(1) + P(2) + P(3) + P(4) + \cdots$$

$$P(X>1) = P(2) + P(3) + P(4) + P(5) + \cdots$$

$$P(X>2) = P(3) + P(4) + P(5) + \cdots$$

$$\vdots$$

If we sum up all of them we get

$$\sum_{x=0}^{\infty} P(X>x) = P(1) + 2P(2) + 3P(3) + 4P(4)$$

Substituting the value of Expectation we get

$$\sum_{x=0}^{\infty} P(X>x) = E[X]$$

Hence proved.

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Q3.2 Solution

A permutation π $\{1,\cdots n\}$, $i \in \{1,\cdots,n\}$ is said to be a fixed point of π if $\pi(i) = i$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{array}$$

fixed

$\pi(3) = 3$

Now let σ be a random permutation of $\{1,\cdots,n\}$ ie.. all the $n!$ permutations are equally likely

X = random variable corresponding to no. of fixed points in σ

a)

We can solve this using indicator random variable

$$X_i = \begin{cases} 1 & \text{if } \sigma(i) = i \\ 0 & \text{if } \sigma(i) \neq i \end{cases}$$

$$P_{X_i}(x) = \begin{cases} \frac{1}{n} & x = 1 \\ 1 - \frac{1}{n} & x = 0 \end{cases}$$

1 2 3 4 5 ...

if this

( ) is fixed ( )

then the rest can be permuted,

So prob. that a particular point is fixed

$= \frac{(n-1)!}{n!} = \frac{1}{n}$

$$X = X_1 + X_2 + X_3 + X_4 + \cdots \quad (1)$$

for a particular X_i

$$E[X_i] = 1 \times P(X_i = 1) + 0 \times P(X_i = 0)$$
$$= 1 \times \frac{1}{n} + 0 = \frac{1}{n}$$

Applying linearity of Expectation

$$E[X] = E[X_1] + E[X_2] + E[X_3] + \cdots$$
$$= \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots$$
$$= n \times \frac{1}{n}$$
$$= 1$$

b) pmf of X

$$P(X=k) = \frac{\text{no. of permutations with } k\text{-fixed points}}{\text{Total no. of permutations}} \quad \text{—— (1)}$$

for this we will use the concept of derangement

D(n) = Arrangements where none of the points are fixed(n)

we can apply principle of Inclusion & Exclusion

$$D_n = n! - \underbrace{{}^nC_1 (n-1)!}_{\text{1 point fixed}} + \underbrace{{}^nC_2 (n-2)!}_{\text{2 points fixed}} - {}^nC_3 (n-3)!$$

$$= n! - \frac{n!}{1!\,(\cancel{n-1})!} \times (\cancel{n-1})! + \frac{n!}{(\cancel{n-2})!\,(2!)} (\cancel{n-2})! \ldots \frac{(-1)^n n!\,(n-n)!}{n!\,(0!)}$$

$$= \sum_{i=0}^{n} \frac{(-1)^i n!}{i!}$$

Now if k points are fixed then n-k points are unfixed and so we need to find the derangement of n-k points

$$D(n-k) = \sum_{i=0}^{n-k} \frac{(-1)^i (n-k)!}{i!}$$

$$= (n-k)! \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$$

Using equation (1)

$$P(X=k) = \frac{{}^nC_k \, D(n-k)}{n!}$$

$$= \frac{\frac{n!}{(n-k)!\,k!} \times (n-k)! \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}}{n!}$$

$$= \frac{1}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$$

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$i, j \in \{1, 2, 3, \ldots, n\}$ $i \neq j$

Here (i, j) forms a pair and the no. of such pairs that can form is no. of ways of choosing the two points to be swapped $= {}^nC_2$

Now let us take an indicator random variable to denote the swap

$$Y_{i,j} = \begin{cases} 1 & \pi(i) = j \ \& \ \pi(j) = i \\ 0 & \text{otherwise} \end{cases}$$

So the prob is :

$$P_{Y_{i,j}}(y) = \begin{cases} \frac{1}{n(n-1)} & x = 1 \\ 1 - \frac{1}{n(n-1)} & x = 0 \end{cases}$$

when two points are fixed the remaining can be arranged in $(n-2)!$ ways

$\therefore p = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$

using this in the prob. function

$$E[Y_{ij}] = 1 \cdot \frac{1}{n \cdot (n-1)} + 0 = \frac{1}{n(n-1)}.$$

$$Y = \sum_{1 \leq i < j \leq n} Y_{ij}$$ denotes the sum of all the pairs of i, j that can be formed

Applying linearity of expectation

$$E[Y] = \sum_{1 \leq i < j \leq n} E[Y_{ij}]$$

Now since we have ${}^nC_2$ no. of random variables from the pairs

$$E[Y] = {}^nC_2 \times \frac{1}{n(n-1)}$$

$$= \frac{n!}{(n-2)!\,2!} \times \frac{1}{n(n-1)}$$

$$= \frac{1}{2}$$

d) $\Pr[X > 10] \leq \frac{1}{10}$ (To prove)

We have already proved that $E[X] = 1$

Now $P(X \geq 10)$

Using Markov's Inequality

$$P(X \geq 10) \leq \frac{E[X]}{10}$$

$$P(X \geq 10) \leq \frac{1}{10}$$

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Q4 - Solution

Given a graph $G=(V,E)$ color assignments

$a: V \to \{R,G,B\}$

i.e. all the vertices are given one of these colors

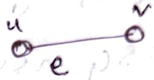
An edge is monochromatic if the two endpoints are of the same color.

1) for an edge $e \in E$ X_e is the random variable (this is an indicator random variable)

$$X_e = \begin{cases} 1 & \text{monochromatic edge} \\ 0 & \text{non-monochromatic edge} \end{cases}$$

$$P_{X_e}(x) = \begin{cases} 1/3 & x=1 \text{ (monochromatic)} \\ 1-1/3 = 2/3 & x=0 \text{ (non-mono.)} \end{cases}$$

There are 3 color assignments $\{R,G,B\}$



Total possible comb. $= 3 \times 3 = 9$

favourable $= 3$ (when both of same color)

$\therefore P = \frac{3}{9} = \frac{1}{3}$

Now to show pairwise independence let us take some examples

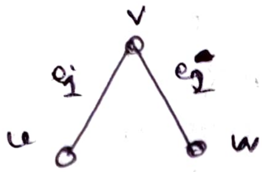
case1: When the edges have no vertex in common

Since both do not have anything in common we can assign whatever color we want and make it monochromatic, it is to say that the assignment of one edge does not affect the other edge.

So we can write it as

$$P(e_i \cap e_j) = P(e_i)\,P(e_j)$$

Case 2: Edges have a vertex in common



u, v, w → vertices

e_1, e_2 → edges

Let us take a small example

We already found out the prob. function of the random variable

So $P_{X_{e_1}}(e_1 = 1) = 1/3$ (mono) $\quad P_{X_e}(e_1 = 0) = 2/3$ (non-mono)

$P_{X_e}(e_2 = 1) = 1/3 \quad P_{X_e}(e_2 = 0) = 2/3$

We need to prove $P(e_1 \cap e_2) = P(e_1)\,P(e_2)$

The various ways e_1 can be monochromatic is: RR BB GG → assigned to u & v

out of ${}^3C_1 \times {}^3C_1$ assignments

So $P(e_1) = \frac{3}{9} = \frac{1}{3}$

Similarly for edge e_2 also we can find $P(e_2) = \frac{3}{9} = \frac{1}{3}$

↳ colors assigned for v & w

Now let us take the graph as a whole

Total no. of outcomes

↳ choosing each color for u, v, w

$= {}^3C_1 \times {}^3C_1 \times {}^3C_1$

$= 27$

Favourable outcomes which leads to all the edges being monochromatic is when all are of the same color

Hence $= 3$

$\therefore P(e_1 \cap e_2) = \frac{3}{27} = \frac{1}{9}$

This is equal to $P(e_1) \times P(e_2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$\therefore P(e_1 \cap e_2) = P(e_1)\,P(e_2)$

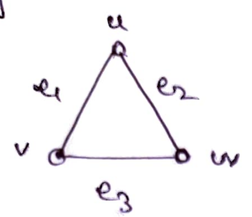
This is true for all edges $e \in E$

i.e. $P(e_i \cap e_j) = P(e_i)\,P(e_j)$

$\{i, j \in E\}$

So by the above two cases it is clear that the edges are pairwise independent.

Now we need to prove that they are not mutually independent. So lets take an example for this too.



vertices: u, v, w
edges: e_1, e_2, e_3

Here for the edges to be mutually independent $P(e_1 \cap e_2 \cap e_3) = P(e_1)P(e_2)P(e_3)$

For a particular edge to be monochromatic the probability is $1/3$ as from before

$P(e_1) = 1/3 \quad P(e_2) = 1/3 \quad P(e_3) = 1/3$

Now for the graph as whole if all edges need to be monochromatic then all the vertices should be of the same color either R, G, B

favourable outcomes = 3

Total = $3C_1 \times 3C_1 \times 3C_1 = 27$

$$P(e_1 \cap e_2 \cap e_3) = \frac{3}{27} = \frac{1}{9}$$

$$\therefore P(e_1 \cap e_2 \cap e_3) \neq P(e_1)P(e_2)P(e_3)$$

Hence we can prove that the mutual independence property will not hold.

2)

Y is a random variable corresponding to the non-monochromatic edges

So let us consider Y_i as the indicator random variable

$$Y_{e_i} = \begin{cases} 1 & e_i \text{ is monochromatic} \\ 0 & e_i \text{ is non-monochromatic} \end{cases}$$

The prob. is given by

$$P_{Y_{e_i}}(y) = \begin{cases} 2/3 & y = 1 \\ 1/3 & y = 0 \end{cases}$$

It is basically the inversion of the random variable X.

$$E[Y_{e_i}] = 1 \cdot P_{Y_{e_i}}(1) + 0 \cdot P_{Y_{e_i}}(0)$$

$$= 1 \times \frac{2}{3} + 0$$

$$= \frac{2}{3} \quad \text{—} (1)$$

Now $Y = Y_{e_1} + Y_{e_2} + Y_3 + \cdots$

$$= \sum_{i=1}^{|E|} Y_{e_i}$$

Here $|E|$ denotes the total no. of edges

Using linearity of Expectation

$$E[Y] = \sum_{i=1}^{|E|} E[Y_i]$$

using the value of $E[Y_i]$ - (1)

$$= |E| \times \frac{2}{3}$$

$$= \frac{2|E|}{3}$$

3) Let us consider for all graphs of $G(V, E)$ the no. of monochromatic edges is less than $\frac{2|E|}{3}$

for all graphs $G_1\ G_2\ G_3 \cdots G_n$ the expected no. of monochromatic edges is less than $\frac{2}{3}|E|$

Thus $E[G_1] < \frac{2|E|}{3}$

Similary $E[G_2] < \frac{2|E|}{3}$ ...... $E[G_n] < \frac{2|E|}{3}$

$\therefore$ $E[G_i] < \frac{2|E|}{3}$ where G_i is the expected value of any graph

from part 4.2 of this same problem we had found that the expected value of non-monochromatic edges $E[Y] = \frac{2|E|}{3}$, which is actually contradicting our assumption

Thus by contradiction we can say that for all graphs $E[G_i] \geq \frac{2|E|}{3}$

4) from the above solved subparts

① $-$ $E[X] = \frac{1}{3}|E|$ | $E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 1)$
$= 1 \times \frac{1}{3} = 1/3 \rightarrow$ (Proved before)

② $-$ $E[Y] = \frac{2}{3}|E|$ | $E[Y_i] = 1 \times \frac{2}{3} + 0 = \frac{2}{3}$

X : monochromatic edges

Y : non monochromatic edges

①: we have already proved it for $E[Y]$, it is same for $E[X]$

$$P\left(Y \geq \frac{|E|}{2}\right) = 1 - P\left(X \geq \frac{|E|}{2}\right) \quad ③$$

considering, $P\left(X \geq \frac{|E|}{2}\right)$

Applying Markov's inequality

$$\Rightarrow P\left(X \geq \frac{|E|}{2}\right) \leq \frac{E[X]}{(|E|/2)}$$

$$\Rightarrow P\left(X \geq \frac{|E|}{2}\right) \leq \frac{\frac{|E|}{3}}{\frac{|E|}{2}}$$

$$\Rightarrow P\left(X \geq \frac{|E|}{2}\right) \leq \frac{2}{3}$$

~~Adding 1 on both sides~~

$$\cancel{\Rightarrow 1 + P\left(X \geq \frac{|E|}{2}\right)}$$

$$\Rightarrow 1 - P\left(X \geq \frac{|E|}{2}\right) \geq 1 - \frac{2}{3}$$

$$1 - P\left(X \geq \frac{|E|}{2}\right) \geq \frac{1}{3} \quad \text{--- (4)}$$

Substituting value of eq (4) in eq (3)

$$P\left(Y \geq \frac{|E|}{2}\right) \geq \frac{1}{3}$$

Hence Proved.

~~4.5~~.

5)

Algorithms

Step 1: Repeat the following pseudo algo. 12 times

{

1.1: Randomly pick an assignment a from $A : V \to \{R, G, B\}$

1.2: If the assignment a has $\geq |E|/2$ nonmonochromatic edges

{

1.2.1: output the assignment a

1.2.2: Break

}

}

We need to find the value of K that will have 99% confidence

from the subparts 4.4. we know that

$$P(Y \geq |E|/2) \geq \frac{1}{3}$$

But we want to increase it to

$$P(Y \geq |E|/2) \geq \frac{99}{100}$$

If we run our algorithm ~~independently~~ independently K times, the probability that we never get non-monochromatic edge assignment with probability $|E|/2$

$$P\left(Y < \frac{|E|}{2}\right) \leq \left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right) \cdots \text{ K times}$$

$$\leq \left(\frac{2}{3}\right)^K$$

Probability that out of K runs, we get some assignment with atleast $|E|/2$ nonmonochromatic edges

$$\geq 1 - \left(\frac{2}{3}\right)^K$$

But as we know

$$1 - \left(\frac{2}{3}\right)^K \geq \frac{99}{100}$$

$$\frac{1}{100} \geq \left(\frac{2}{3}\right)^K$$

$$\text{or } \left(\frac{3}{2}\right)^K \geq 100$$

$$\therefore K \geq \log_{3/2} 100$$

$$\therefore K \geq 11.35$$

So by running the algorithm for 12 times we are sure that with 99% probability that we will find an assignment with atleast $|E|/2$ non monochromatic edges.