Gaussian Elimination:

$$w - x - y + 2z = 1$$

$$2w - 2x - y + 3z = 3$$

$$-w + x - y = -3.$$

$$\downarrow$$

$$\boxed{Ax = b}$$

$$A = \begin{pmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{pmatrix}_{3\times 4}$$

$$x = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}_{4\times 1} \qquad b = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}_{(3\times 1).}$$

The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

$$\xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The associated system is

$$\left.\begin{aligned} w - x - y + 2z &= 1. \\ y - z &= 1 \end{aligned}\right\}$$

$\therefore \quad y = 1 + z.$

$$w = 1 + x + (1+z) - 2z$$
$$= 2 + x - z \checkmark$$

If $x = s$, $z = t$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

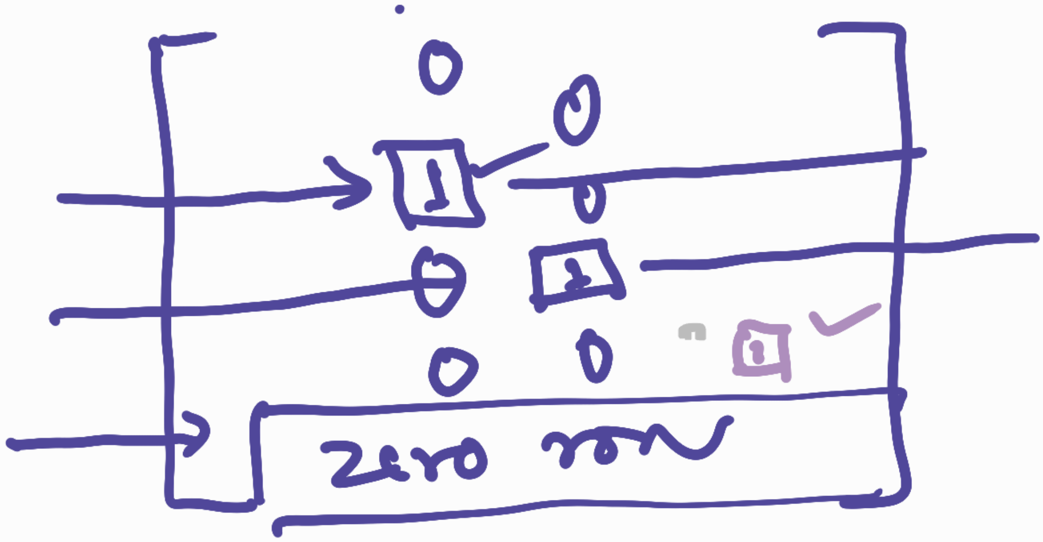
inf   inf. $\checkmark$

$\longrightarrow$ Infinite solution.

Gauss Jordan Elimination:

Defⁿ: A matrix is in $\boxed{\text{reduced row echelon}}$ form if it satisfies the following properties:

1) It is in row echelon form.
2) The leading entry of the non zero row is 1 (leading 1)
3) Each column containing a leading 1 has zero's everywhere else.



$$\begin{array}{ccccccc} \checkmark & \times & & & \times & & \\ \end{array}$$

$$\begin{array}{c} \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \\ \end{array}\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{reduced row echelon form.}$$

Gauss Jordan Elimination

1. Write down the augmented matrix for the system of linear equations ✓

2. Use elementary row operation to reduce the augmented matrix to reduced row echelon form

3. If the resulting system is consistent, solve it

Solve: (Gauss Jordan elimination)

$$\left.\begin{aligned} 2r + s &= 3 \\ 4r + s &= 7 \\ 2r + 5s &= -1 \end{aligned}\right\} \longrightarrow \begin{pmatrix} s = -1, \\ r = 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

* A system of linear equations is called homogeneous if the constant term in each equation is zero.

$$\left\{\begin{array}{l} 2x+3y+z=0 \\ 4x+5y=0 \end{array}\right\}$$

$$\left\{\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=0 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=0 \end{array}\right.$$

→ homogeneous

<u>Defⁿ</u>: If A is a $n\times n$ matrix. an inverse of A is an $n\times n$ matrix A' with the property that

$\boxed{AA'=I}$ and $\boxed{A'A=I}$

where $I=I_n$ is the $n\times n$ identity matrix

$$A_1 = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \rightarrow A_2 = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A_1 \rightarrow A_3 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

If A is an invertible matrix, then its inverse is unique.

Let us assume that the inverse of the matrix A is not unique.

$$A \rightarrow A' \rightarrow \text{inverse}$$
$$A \rightarrow A''$$

Now, $AA' = A'A = I$

$AA'' = A''A = I.$

$$\left[\begin{aligned} A' = A'I &= \underline{A'(AA'')} \\ &= \underline{(A'A)A''} = I.A'' \\ &= A'' \end{aligned}\right]$$

$\underline{A' = A''} \rightarrow$ unique.

If A is an invertible $n \times n$ matrix, then the system of linear eq^{n}s given by $Ax = b$ has the unique solution $\boxed{x = A^{-1}b}$ for any b in $\mathbb{R}^n$.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$Ax = b \qquad A = \begin{pmatrix} \\ \\ \end{pmatrix}$$

$$x = \begin{pmatrix} \\ \\ \end{pmatrix} \qquad b = \begin{pmatrix} \\ \\ \end{pmatrix}$$

$x = A^{-1}b$ $\qquad$ $Ax = b$ ✓

$$Ax = A(A^{-1}b)$$
$$= (AA^{-1})b$$
$$= I_n b = b$$

It is clear that $\boxed{x = A^{-1}b}$ is a solⁿ to the eqⁿ $Ax = b$

unique.

Let y be a solⁿ to the equation $Ax = b$.

$$Ay = b$$
$$\Rightarrow A^{-1}(Ay) = A^{-1}b$$
$$\Rightarrow (A^{-1}A)y = A^{-1}b$$
$$\Rightarrow Iy = A^{-1}b$$
$$\Rightarrow \boxed{y = A^{-1}b}$$

The solⁿ is unique.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A is invertible if $\underbrace{ad - bc}_{\det A} \neq 0$

$$\boxed{A^{-1}} = \frac{1}{\underbrace{(ad-bc)}_{\det A}} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{adj } A}$$

If $ad - bc = 0$ then. A is not invertible.

<u>Some properties:</u>

a) If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

b) If A is an invertible matrix, and c is a non zero scalar then cA is invertible

$$(cA)^{-1} = \frac{1}{c} A^{-1}$$

c) If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

If A is an invertible matrix and n is a positive integer then A^{-n} is defined by

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}.$$

Elementary Matrices

An elementary matrix is a

matrix that can be obtained by performing an elementary row operation on an identity matrix.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow 3R_2(I)$$

$$E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow R_1 \leftrightarrow R_3$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \longrightarrow R_4' = R_4 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \xrightarrow{R_{13}}$$

4×3.

$$\xrightarrow{3R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}.$$

$E_2 A$, $E_3 A$

$$E_2 A = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

$$E_3A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} - 2a_{21} & a_{42} - 2a_{22} & a_{43} - 2a_{23} \end{bmatrix}$$

Let E be the elementary matrix obtained by performing elementary row operation on I_n. If the same elementary row operations is performed on an $n \times r$ matrix A, the result is the same as the matrix EA.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_{23}} E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_{23}} I$$

E_1^{-1}

$R_2 \longleftrightarrow R_3$ $R_2 \longleftrightarrow R_3$.

$$\boxed{E_1^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}} = E_1.$$

$$\boxed{E_1 E_1^{-1} = I}, \quad ? \; E_1^{-1} E_1 = I$$

$$I \xrightarrow{R_{ij}} E_1 \xrightarrow{R_{ij}} I,$$

E_1

$$E_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$I \xrightarrow{4R_2} E_2 \xrightarrow{\frac{1}{4}R_2} I.$$

$$E_2^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\underline{E_2 E_2^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \rightarrow \text{find } E_3^{-1}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$I_3 \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Each elementary matrix is invertible and its inverse is an elementary matrix of the same type.

Theorem:

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

a) A is invertible

b) $Ax = b$ has unique solution for any b in $\mathbb{R}^n$

c) $Ax = 0$ has only the trivial solⁿ.

d) The reduced row echelon form of A is I_n.

e) A is a product of elementary matrices

$$\boxed{(a) \Rightarrow (b)} \boxed{\Rightarrow (c)} \boxed{\Rightarrow (d)} \boxed{\Rightarrow (e)} \Rightarrow (f)$$

$$\boxed{Ax = 0} \longrightarrow x = 0.$$

$x = 0$ must be the solution.!

$$[A \mid 0] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & & & & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ & \cdots & & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

$$= [I_n \mid 0].$$

$x_1 = 0$
$x_2 = 0$
$\vdots$
$x_n = 0$

The reduced row echelon.

form of A is I_n.

(d) $\Rightarrow$ (e)

$$A \xrightarrow{R_1 R_2 \cdots R_n} I_n.$$

$R_1, R_2 \cdots R_n.$

$E_1 \quad E_2 \quad E_n$

$$(E_n \cdots E_3 E_2 E_1 A) = I.$$

$A'' \quad A$

Since elementary matrices are inverti

$$A = (E_n \cdots E_2 E_1)^{-1} I.$$

$$= (E_n \cdots E_2 E_1)^{-1}$$

$$= E_1^{-1} E_2^{-1} \cdots E_n^{-1}$$

A is a product of elementary matrix

(e) $\Rightarrow$ (a)

$$A = E_1^{-1} \cdots E_k^{-1}$$