System of Linear Equations

$$ax + by = c.$$

$$ax + by + cz = d.$$

$\hookrightarrow$ linear eqn's.

A linear equation in the n-variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are constant terms.

1. $3x - 4y = 1$
2. $r - \frac{1}{2}s - \frac{15}{2}t = 9$ ✓
3. $\sqrt{2}x + \frac{\pi}{4}y = 1$
4. $3.2x_1 - 0.01x_2 = 4.6$
5. $xy + 2z = 1.$
6. $\sqrt{2}x + \frac{\pi}{4}y - \sin\left(\frac{\pi}{5}z\right) = 1.$

A solution of a linear equation $(a_1x_1 + a_2x_2 + \dots + a_nx_n = b)$ is a vector $[s_1, s_2, \dots, s_n]$ whose components satisfy the equation when we substitute $x_1 = s_1$, $x_2 = s_2$, $\dots$ $x_n = s_n$.

$$a_1s_1 + a_2s_2 + \dots + a_ns_n = b.$$

$(s_1, s_2, \dots, s_n) \longrightarrow$ solⁿ

$3x - 4y = -1 \rightarrow (1,1), (5,4)$

$x_1 - x_2 + 2x_3 = 3$

$\rightarrow (3,0,0), (0,1,2)$

$(0,1,-1)$

A system of linear equations is a finite set of linear equations each with the same variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

System of linear equation (m eqns with n-variables)

$(s_1, s_2, \ldots, s_n) \rightarrow$ solution to the system of eqns. if it satisfies each of these equations.

The $\boxed{\text{solution set}}$ of a system of linear equations is the set of all solutions of the system

$$\left\{ \begin{array}{l} (s_1, s_2, \ldots, s_n) \\ (s_1', s_2', \ldots, s_n') \\ \vdots \qquad\qquad\qquad) \\ (\end{array} \right\}$$

finite / infinite / no solution

Solution → finite (1, 2, .. m)

Solution → infinite.

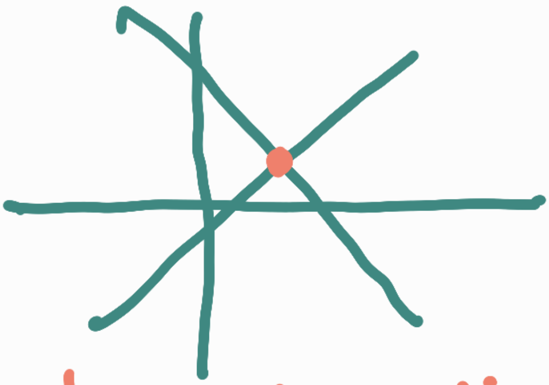
Solution → no solution.

Case I: $x - y = 1$, $x + y = 3$ → finite (2,1)

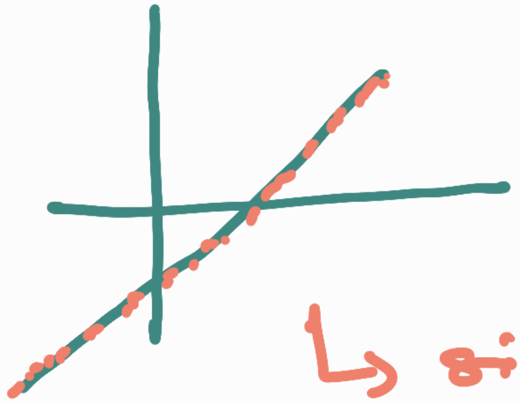
→ 2 eq's, 2 variables

Case II: $x - y = 2$, $2x - 2y = 4$ → infinite

Case III $x - y = 1$, $x - y = 3$ → no.



↳ intersecting lines



↳ single line

↳ parallel lines

A system of equation with real coefficients has either

(a) unique solution (consistent

(b) infinitely many sol^n (consistent)

(c) no solution (inconsistent)

Beyond two equations and beyond two variables:

$$x - y - z = 2 \longrightarrow (1)$$
$$y + 3z = 5 \longrightarrow (2)$$
$$5z = 10 \longrightarrow (3)$$

(3) $5z = 10$

$\Rightarrow z = 2$

backward substitution.

$$y = 5 - 3\cdot(2) = -1$$

$\underline{x = 3}$

form like this →

$$\begin{array}{cccc} x & y & z & = \\ & y & z & = \\ & & z & = \end{array}$$

augmented matrix

$$\begin{cases} x - y - z = 2 \\ 3x - 3y + 2z = 16 \\ 2x - y + z = 9 \end{cases}$$

coeff matrix

Can you apply backward subst.

subtract 3 times the first equation from the second eqn:

$$\begin{pmatrix} x - y - z = 2 \\ 5z = 10 \\ 2x - y + z = 9 \end{pmatrix}$$

Subtract 2 time the first eqn from the 3rd eqn.

$$x - y - z = 2.$$
$$5z = 10$$
$$y + 3z = 5$$

If you interchange the second eqn and third eqn;

$$x - y - z = 2$$
$$y + 3z = 5$$
$$5z = 10$$

$[3, -1, 2] \rightarrow$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right] \rightarrow$$

coefficient matrix

Subtract 3 times the first row from the second row

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right] \rightarrow$$

Subtract two times the first row from the third row

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

Interchange rows two and three.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

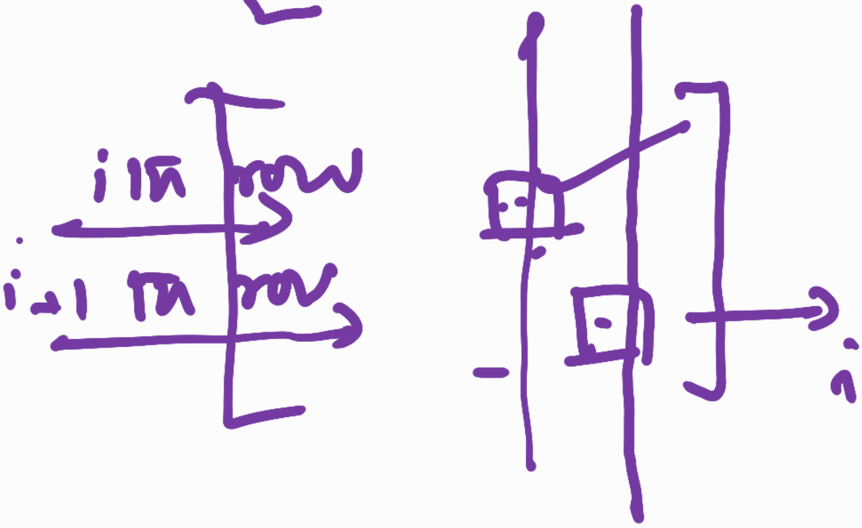
$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned}$$

A matrix is in a row echelon form, if it satisfies the following properties.

1. Any rows consisting entirely zeros are at the bottom → **zero rows**

2. In each non zero row, the first non zero entry (called the **leading entry**) is in a column to the left of the leading entries below it

If there is a zero row!



The leading in (i+1) th row

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

row echelon

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Elementary Row Operations:

The following elementary row operations can be performed on a matrix,

1. Interchange of two rows
($R_i \leftrightarrow R_j$ interchanging i-th row and j th row)

2. Multiply a row by a nonzero constant
(kR_i multiplying the row R_i by a non zero constant k)

3. Add a multiple of a row to another row
($R_i' = R_i + kR_j$).

Reduce the following matrix to echelon form

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 2R_2 \\ R_4 + R_1}}$$

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$\xrightarrow{R_4+3R_3}\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{8}R_3}\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

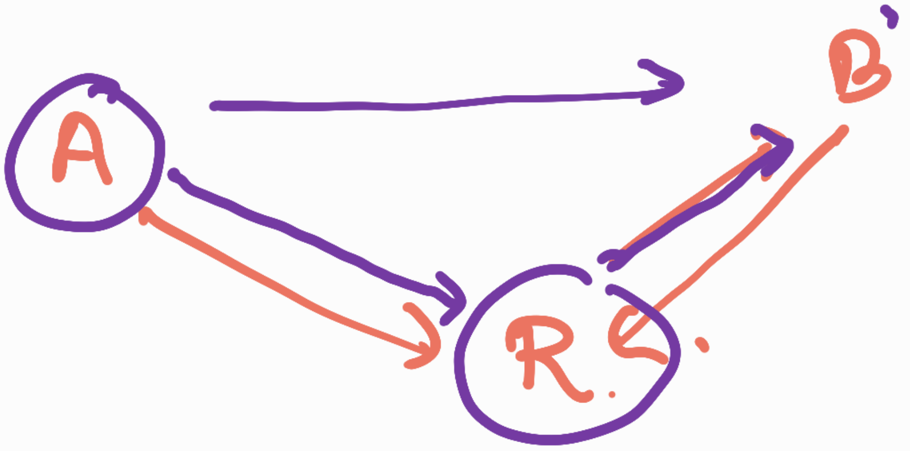
$$\xrightarrow{R_4-29R_3}\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

$\rightarrow$ row echelon form?

Two matrices A and B are row equivalent if there is a sequence of elementary row operations that converts A into B.

$$A \overset{R}{\sim} \overset{R}{\sim} \sim B.$$

Matrices A and B are row equivalent iff they can be reduced to the same row echelon form.



Gaussian Elimination

1. Write the augmented matrix of the system of linear equations

2. Use elementary row operation to reduce the augmented matrix to row echelon form.

3. Using back substitution solve the equivalent system

$$\left.\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 - x_2 - 2x_3 &= -5 \end{aligned}\right\}$$

Solve the system by Gaussian Elimination.

1. Aug matrix $\longrightarrow$ 2. Row echelon

$\longrightarrow$ 3. Back sub.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \rightarrow$$