

Differentially Private Linear Regression and Synthetic Data Generation with Statistical Guarantees

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Abstract

In social sciences, small- to medium-scale datasets are common and linear regression (LR) is canonical. In privacy-aware settings, much work has focused on differentially private (DP) LR, but mostly on point estimation with limited attention to uncertainty quantification. Meanwhile, synthetic data generation (SDG) is increasingly important for reproducibility studies, yet current DP LR methods do not readily support it. Mainstream SDG approaches are either tailored to discretized data, making them less suitable for continuous regression, or rely on deep models that require large datasets, limiting their use for the smaller, continuous data typical in social science. We propose a method for LR with valid inference under Gaussian DP: a DP bias-corrected estimator with asymptotic confidence intervals (CIs) and a general SDG procedure in which regression on the synthetic data matches our DP regression. Our binning–aggregation strategy is effective in small- to moderate-dimensional settings. Experiments show our method (1) improves accuracy over existing methods, (2) provides valid CIs, and (3) produces more reliable synthetic data for downstream ML tasks than current DP SDGs.

1 Introduction

In the broad domain of social sciences, where small- to medium-scale datasets are common, linear regression (LR) and subsequent statistical inference are prevalent tools used to answer key scientific questions. Data in the social, economic and behavioral sciences readily involve sensitive information. The confidentiality protection methodology for sharing data and the results of statistical analyses has a long history drawing from many fields (e.g., Hundepool et al. (2012); Slavković and Seeman (2023)). The modern methods predominantly rely on differential privacy (DP) (Dwork et al., 2006) to ensure rigorous privacy guarantees. Numerous methods for fitting LR under DP have been proposed, including general empirical risk minimization approaches such as objective perturbation (Kifer et al., 2012; Zhang et al., 2012) and DP (stochastic) gradient descent (DP-SGD)

(Bassily et al., 2014; Abadi et al., 2016), as well as LR-specific techniques like sufficient statistics perturbation (SSP) (Dwork et al., 2014; Sheffet, 2017; Wang, 2018). However, most methods focus on point estimation and provide statistical risk bounds but with limited support for uncertainty quantification. Valid statistical inference in LR settings under DP remains a challenge due to inadequate accounting for the noise added to satisfy the privacy guarantee. Only a few methods, such as Sheffet (2017), offer theoretical results for statistical inference under the assumption that the features follow Gaussian distributions.

Meanwhile, reproducibility and replicability are important concepts in trustworthy social science research (National Academies of Sciences, Engineering, and Medicine, 2019; Mukherjee et al., 2024). Researchers often want to conduct replication studies to verify or build upon prior analyses. In privacy-aware settings, however, typical DP methods only return model estimates, preventing others from revisiting or extending the analysis without access to the original data. Synthetic data generation (SDG) offers a possible solution. The basic idea was proposed in the early 1990s, but its broad adoption is still lacking (van Kesteren, 2024). Furthermore, most SDG methods with DP guarantees (Jordon et al., 2018; Xie et al., 2018; Xin et al., 2020, 2022) rely on large datasets and complex models, e.g., deep learning, making them ill-suited for smaller-scale applications. The theoretical implications of using such synthetic data for LR also remain unexplored.

To address these challenges, we propose a novel method for Gaussian DP linear regression with statistical inference and synthetic data generation. Our method is based on an effective and practical binning-aggregation (BinAgg) strategy. For the binning step, we leverage PrivTree (Zhang et al., 2016) only as a building block to obtain a DP partition of the feature domain. The novelty lies not in the binning itself but in our binning–aggregation framework: by aggregating features and labels within bins, we reformulate linear regression as a weighted model, which supports valid statistical inference and provides a general procedure for synthetic data generation under DP. To our knowledge, this is the first work to reformulate DP linear regression in this way, offering both inference guarantees and the ability to generate synthetic data within the same framework.

Main Contributions (1) We propose a novel DP method, BinAgg, to produce LR that satisfies Gaussian DP and achieves one of the best statistical accuracies among existing DP LR methods, particularly on real-world datasets. Our method requires minimal tuning and runs significantly faster than computationally intensive approaches; see Algorithm 2 and Theorem 4.1. (2) We develop a DP statistical inference procedure based on the central limit theorem (CLT), analogous to the classical non-private regression, without requiring any assumptions on the feature distribution. A CLT result for DP LR has been missing from the literature, and our work provides the first such statement; see Theorem

4.2. (3) We introduce a synthetic data generation mechanism that provides a general procedure beyond LR and supports replication studies at no additional privacy cost; see Algorithm 3 and Theorem 3.1.

1.1 Related Work

Current methods for DP linear regression most related to our work include objective function perturbation Kifer et al. (2012); Zhang et al. (2012), DP stochastic gradient descent (DP-SGD) (Bassily et al., 2014; Abadi et al., 2016; Cai et al., 2021), and one posterior sampling (OPS) (Dimitrakakis et al., 2014; Minami et al., 2016). These approaches are general-purpose, but typically require careful hyperparameter tuning. Wang (2018) proposed a modified version for sufficient statistics perturbation (SSP) (Dwork et al., 2014; Sheffet, 2017) and OPS, that is AdaSSP and AdaOPS, respectively, by introducing adaptive regularization.

A few more recent works of interest have a relatively narrow focus. Alabi et al. (2022) proposed a DP method exclusively for simple linear regression that outputs predictions only at $x = 0.25$ and 0.75 assuming x and y are within $[0, 1]$. Varshney et al. (2022) proposed theory for a one-pass mini-batch SGD method for sub-Gaussian data via adaptive clipping, while Milionis et al. (2022) focused on LR with unbounded features but assumed the features are Gaussian; neither work includes numerical evaluations. Amin et al. (2023) proposed a bound-free method relying on a Propose-Test-Release (PTR) check, which often fails when n is small. Their evaluations focus on datasets with $n \gtrsim 1000 \cdot d$, making the method unsuitable for the smaller datasets we are targeting. Dick et al. (2023) proposed a method to improve prediction accuracy through feature selection, rather than estimating the regression coefficient in the specified model.

While most of these methods provide statistical risk bounds, statistical inference has received less attention. Unlike the non-private setting, where inference is well-established, DP LR lacks broadly applicable methods for uncertainty quantification. Although empirical approaches such as the bootstrap can be used to construct confidence intervals, they often require generating many estimates, which in turn necessitates splitting the privacy budget across multiple runs, or incur additional privacy costs for estimating regression errors (Ferrando et al., 2022). Overall, analytical solutions for valid LR inference under DP remain limited. By assuming the features follow Gaussian distributions, Sheffet (2017) provides inference results for SSP and Johnson-Lindenstrauss Transform (JLT), while Lin et al. (2024) derives approximate variances for DP-GD and SSP in the context of linked data, with LR as a special case.

2 Preliminaries

In this section, we provide preliminaries on DP and review an existing DP algorithm for data binning, which serves as a building block for our proposed method.

2.1 Differential Privacy

Two concepts central to DP are neighboring relations and sensitivity. Let \mathcal{X} be some data space, and $D, D' \in \mathcal{X}^{\mathbb{N}}$ be two neighboring datasets, where one is obtained from the other by adding or removing a single record. This relation is denoted by $D \sim D'$. We refer to it as *remove-one/add-one* neighboring relation. The sensitivity of a function is defined as follows.

Definition 1 (Sensitivity). *Consider the problem of privately releasing a statistic $\theta(D)$ of the dataset D . The sensitivity of θ is defined as*

$$\text{sens}(\theta) = \sup_{D \sim D'} |\theta(D) - \theta(D')|.$$

Definition 2 ((ε, δ)-DP, Dwork and Roth (2014)). *Let $\varepsilon > 0, \delta > 0$. An algorithm A is (ε, δ) -differentially private, if for every pair of neighboring datasets D, D' , and all possible output set S ,*

$$\mathbb{P}(A(D) \in S) \leq e^{\varepsilon} \cdot \mathbb{P}(A(D') \in S) + \delta. \quad (1)$$

This notion is referred to as *approximate DP*. When $\delta = 0$, ε -DP, is called *pure DP* (Dwork et al., 2006).

In our work, we adopt a variant (Gaussian DP) with better statistical interpretation and tighter composition property. Consider the hypothesis testing:

$$H_0 : \text{the distribution is } P \text{ vs. } H_1 : \text{the distribution is } Q.$$

Let α_ϕ and β_ϕ denote Type I and II errors for rejection rule ϕ . Let $T(P, Q)(\alpha) := \inf_\phi \{\beta_\phi : \alpha_\phi \leq \alpha\}$.

Definition 3 (Gaussian DP, Dong et al. (2022)). *Let $\mu > 0$. An algorithm A is μ -Gaussian differentially private (μ -GDP), if for every neighboring D, D' and any $\alpha \in (0, 1)$*

$$T(A(D), A(D'))(\alpha) \geq T(\mathcal{N}(0, 1), \mathcal{N}(\mu, 1))(\alpha).$$

In other words, testing “ H_0 : the underlying dataset is D ” versus “ H_1 : the underlying dataset is D' ” is at least as hard as testing “ H_0 : the distribution is $\mathcal{N}(0, 1)$ ” versus “ H_1 : the distribution is $\mathcal{N}(\mu, 1)$.”

Gaussian DP and approximate DP are precisely mutually convertible and pure DP implies Gaussian DP. We use the following propositions in our work.

Proposition 2.1 (Conversion). *Let $\Phi(\cdot)$ be the standard Gaussian cumulative distribution function.*

- (i) (Corollary 2.13, Dong et al. (2022)) *A mechanism is μ -GDP if and only if it is $(\varepsilon, \delta(\varepsilon))$ -DP for all $\varepsilon > 0$, where $\delta(\varepsilon) = \Phi\left(-\frac{\varepsilon}{\mu} + \frac{\mu}{2}\right) - e^\varepsilon \Phi\left(-\frac{\varepsilon}{\mu} - \frac{\mu}{2}\right)$.*
- (ii) (Theorem 5.1, Liu et al. (2022)) *Any ε -DP algorithm is also μ -GDP for $\mu = -2 \Phi^{-1}\left(\frac{1}{1+e^\varepsilon}\right) \leq \sqrt{\frac{\pi}{2}}$.*

Like the classic notion, Gaussian DP enjoys the following fundamental properties and has the corresponding Gaussian Mechanism (Dong et al., 2022).

Proposition 2.2 (Composition). *The n -fold composition of μ_i -GDP mechanisms is*

$$\sqrt{\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2}.$$

Proposition 2.3 (Post-processing). *If an algorithm A is μ -GDP, then any post-processing function f , i.e., $f \circ A$, is also μ -GDP.*

Proposition 2.4 (Gaussian mechanism). *Define the Gaussian mechanism that operates on a statistic θ as $A(S) = \theta(S) + \xi$ where $\xi \sim \mathcal{N}(0, \text{sens}(\theta)^2/\mu^2)$. Then, A satisfies μ -GDP.*

2.2 A DP Binning Algorithm

Our proposed method involves creating bins. If the binning strategy is data-independent, the bin boundaries are public and do not reveal sensitive information. For instance, one can set fixed bin widths before accessing sensitive data. However, data-independent strategies often lack adaptability and are prone to the curse of dimensionality. To address this, one may opt for a data-dependent binning method, such as recursive partitioning based on counts, which produces a more refined and representative histogram. This approach, however, incurs additional privacy cost, as the binning must itself be performed in a DP manner. In this work, we deploy work of Zhang et al. (2016) (Algorithm 2) to create a DP tree, specifically we rely on their algorithm to output private bins without counts; see our Algorithm 1.

The PrivTree algorithm builds a hierarchical, tree-structured partitioning of the data domain by recursively splitting nodes based on privately perturbed, down-biased counts. For a node v at depth $d = \text{depth}(v)$ (root has depth 0) with count $c(v)$, a penalized score is defined by subtracting a fixed amount δ per level so deeper nodes need stronger evidence to split:

$$b(v) = \max\{c(v) - d\delta, \theta - \delta\}.$$

Add Laplace noise to protect privacy,

$$\hat{b}(v) = b(v) + \text{Laplace noise},$$

and node v is split if $\hat{b}(v) > \theta$. Full algorithmic details and exact noise calibration can be found in Appendix C. The choice of δ is determined by the desired privacy level, while θ is a tunable hyperparameter. As discussed in Zhang et al. (2016), the negative bias helps ensure that setting the threshold to $\theta = 0$ typically results in sufficiently large point counts in each node.

In our implementation, the root node corresponds to the initial bin. We pass in a non-sensitive d -dimensional region represented as the Cartesian product $\Pi_{i=1}^d(L_i, U_i)$, where L_i and U_i denote the lower and upper bounds of the i -th feature, respectively. Each node is recursively split along its widest dimension. We use the resulting leaf nodes (final bins) in our algorithm design in Section 3. Since PrivTree satisfies ϵ -DP and our method adopts μ -GDP, we leverage Proposition 2.1 (ii) to convert the privacy guarantees.

3 Methodology

We present *BinAgg*, a binning–aggregation framework with three algorithms: (1) the core binning–aggregation step, (2) DP linear regression, and (3) DP synthetic data generation.

3.1 Aggregated Linear Model

Let X denote the $n \times d$ matrix of feature variables and \mathbf{y} the n -dimensional vector of label variable. The classic linear model is given by

$$\mathbf{y} = X\beta + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_n). \quad (2)$$

We propose an alternative formulation given the observations of X are partitioned into K bins, represented as $\{(\mathcal{B}_k, c_k)\}_{k=1}^K$, where \mathcal{B}_k denotes the k th bin and c_k is the number of observations in that bin. Let j be the index over data points. We aggregate the observations in each bin by defining

$$\mathbf{s}_k = \sum_{x_j \in \mathcal{B}_k} \mathbf{x}_j, \quad t_k = \sum_{x_j \in \mathcal{B}_k} y_j, \quad \eta_k = \sum_{x_j \in \mathcal{B}_k} \varepsilon_j.$$

In matrix form, we let $S = (\mathbf{s}_1^\top, \mathbf{s}_2^\top, \dots, \mathbf{s}_K^\top)^\top$, $\mathbf{t} = (t_1, t_2, \dots, t_K)^\top$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_K)^\top$. This aggregation leads to a weighted linear model:

$$\mathbf{t} = S\beta + \boldsymbol{\eta}, \quad (3)$$

where $\boldsymbol{\eta} \sim \mathcal{N}(0, \sigma^2 C)$ with $C = \text{diag}(c_1, \dots, c_K)$. Let $W = C^{-1}$. An unbiased and consistent estimator of $\boldsymbol{\beta}$ is given by the weighted least squares (WLS) estimator:

$$\hat{\boldsymbol{\beta}} = (S^\top WS)^{-1} S^\top W\mathbf{t}. \quad (4)$$

Remark 3.1. Model (3) is equivalent to the averaged (weighted) model defined as $\bar{\mathbf{y}} = \bar{X}\boldsymbol{\beta} + \bar{\boldsymbol{\varepsilon}}$, where $\bar{X} \stackrel{\text{def}}{=} C^{-1}S$, $\bar{\mathbf{y}} \stackrel{\text{def}}{=} C^{-1}\mathbf{t}$, and $\bar{\boldsymbol{\varepsilon}} \stackrel{\text{def}}{=} C^{-1}\boldsymbol{\eta} \sim \mathcal{N}(0, \sigma^2 C^{-1})$. This model yields the same WLS estimator $\hat{\boldsymbol{\beta}} = (\bar{X}^\top C\bar{X})^{-1}\bar{X}^\top C\bar{\mathbf{y}} = (S^\top WS)^{-1}S^\top W\mathbf{t}$ as in (4). In this work, we proceed with Model (3) for DP algorithm designs.

Our approach that ensures DP for the original data (X, \mathbf{y}) consists of two major steps: (1) apply a DP algorithm to determine the bin structure $\{\mathcal{B}_k\}_{k=1}^K$, with PrivTree being one possible option, and (2) add noise to the aggregated statistics, specifically to \mathbf{t} , S , and the count matrix C . Then a DP aggregated model is

$$\tilde{\mathbf{t}} = \tilde{S}\boldsymbol{\beta} + \tilde{\boldsymbol{\eta}}, \quad (5)$$

where $\tilde{\boldsymbol{\eta}} \sim \mathcal{N}(0, \sigma^2 \tilde{C})$. Let $\tilde{W} = \tilde{C}^{-1}$. A naive estimator is given by

$$\tilde{\boldsymbol{\beta}}_{\text{naive}} = (\tilde{S}^\top \tilde{W} \tilde{S})^{-1} \tilde{S}^\top \tilde{W} \tilde{\mathbf{t}},$$

however, it does not account for the extra uncertainty introduced by injected noise. Instead, we propose a debiased estimator

$$\tilde{\boldsymbol{\beta}} = (\tilde{S}^\top \tilde{W} \tilde{S} - \tilde{D})^{-1} \tilde{S}^\top \tilde{W} \tilde{\mathbf{t}},$$

where \tilde{D} is a private bias-correction matrix, as specified in Algorithm 2.

3.2 DP Binning-Aggregation Algorithms

Motivated by Section 3.1, the binning-aggregation (BinAgg) framework, formalized in Algorithm 1, converts raw data into a set of DP bin-level summaries that facilitate both linear regression and synthetic data generation. Given any DP partition of the feature space (e.g., via PrivTree), Algorithm 1 aggregates records within each bin to form counts and per-bin sums of features and labels, and releases their privatized versions under task-specific budgets. By shifting privacy noise to bin-level linear sums, rather than to per-record quantities or sufficient statistics, the framework (i) preserves the joint (X, \mathbf{y}) structure and (ii) reduces effective sensitivity: each coordinate is confined to its bin range, and bins with small (privatized) counts can be discarded. The contribution is a novel coupling of DP binning with aggregation to yield a weighted linear model for valid inference and a general SDG mechanism simultaneously.

Algorithm 1 DP Binning–Aggregation Preparation

Input: Dataset (X, \mathbf{y}) , domain for X , privacy budgets for binning and counts: μ_{bin}, μ_c .

1: Create a list of μ_{bin} -GDP bins for X (e.g., via PrivTree) : $\{\mathcal{B}_k\}_{k=1}^K$ where $\mathcal{B}_k = \Pi_{i=1}^d(L_{ki}, U_{ki})$

2: For each bin \mathcal{B}_k , compute: $c_k = \sum_{x_j \in \mathcal{B}_k} 1$

3: **for** $k = 1$ to K **do**

4: Privatize count:

$$\tilde{c}_k = \text{round}(c_k + \xi^c), \quad \xi^c \sim \mathcal{N}(0, 1/\mu_c^2)$$

5: **if** $\tilde{c}_k < 2$ **then**

6: Discard bin \mathcal{B}_k .

7: **end if**

8: **end for**

9: Reset K to be the number of bins after discards.

10: For each bin \mathcal{B}_k , compute:

$$\mathbf{s}_k = \sum_{x_j \in \mathcal{B}_k} \mathbf{x}_j, \quad t_k = \sum_{x_j \in \mathcal{B}_k} y_j$$

11: Compute sensitivity vector for each bin $\Delta_k = (\Delta_{k1}, \dots, \Delta_{kd})^\top$, where $\Delta_{ki} = \max(|L_{ki}|, |U_{ki}|)$

Output: Privatized bins and counts $\{(\mathcal{B}_k, \tilde{c}_k)\}_{k=1}^K$; bin-wise aggregates $\{(\mathbf{s}_k, t_k)\}_{k=1}^K$ (to be privatized) with sensitivity vectors $\{\Delta_k\}_{k=1}^K$ for \mathbf{s}_k .

Sensitivity of x and y Under the remove-one/add-one neighboring relation, a differing record affects only one bin. Let i index feature dimensions and $\mathcal{B}_k = \Pi_{i=1}^d(L_{ki}, U_{ki})$. For the per-bin sum of features, the coordinate-wise ℓ_1 sensitivity is $\Delta_{ki} = \max\{|L_{ki}|, |U_{ki}|\}$, and for the per-bin sum of labels it is given by the label bound B_y . We collect $\Delta_k = (\Delta_{k1}, \dots, \Delta_{kd})^\top$ to calibrate the Gaussian mechanisms in Algorithms 2 and 3.

This framework instantiates two procedures: Algorithm 2 (BinAgg for LR), which realizes the weighted model induced by bin-level aggregation and provides valid uncertainty quantification, and Algorithm 3 (BinAgg for SDG), which generates samples directly from the same DP bin-level summaries to enable reproducibility and broader downstream analyses. These procedures target different use cases: Algorithm 2 is efficient for private LR with CIs, whereas Algorithm 3 produces a reusable DP dataset supporting residual analysis, visualization, and fitting alternative models. Algorithm 3 is more general and subsumes Algorithm 2; e.g., applying the aggregated linear model to the synthetic data yields an estimator that matches Algorithm 2 in distribution (Corollary 3.1). This consistency is essential for reproducible scientific research. It allows for synthetic-data sharing and valid LR inference without discrepancies or additional privacy cost.

Corollary 3.1 (Equivalence of Two Algorithms). *In Algorithm 3, aggregate the synthetic*

Algorithm 2 DP BinAgg for Linear Regression

Input: Output from Algorithm 1; label bound B_y ; privacy budgets for features and label:

- μ_s, μ_t
- 1: **for** $k = 1$ to K **do**
- 2: Privatize \mathbf{s}_k , and t_k :

$$\begin{aligned}\tilde{\mathbf{s}}_k &= \mathbf{s}_k + \boldsymbol{\xi}^s, \quad \boldsymbol{\xi}^s \sim \mathcal{N}(\mathbf{0}, \Delta_k^2/\mu_s^2), \\ \tilde{t}_k &= t_k + \xi^y, \quad \xi^y \sim \mathcal{N}(0, \Delta_y^2/\mu_t^2).\end{aligned}$$

- 3: **end for**

- 4: Let

$$\begin{aligned}\widetilde{W} &= \text{diag}(\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_K), \quad \tilde{w}_k = 1/\tilde{c}_k \\ \widetilde{S} &= (\tilde{\mathbf{s}}_1^\top, \tilde{\mathbf{s}}_2^\top, \dots, \tilde{\mathbf{s}}_K^\top)^\top \\ \widetilde{\mathbf{t}} &= (\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_K)^\top,\end{aligned}$$

- 5: Calculate matrix $\widetilde{D} = \frac{1}{K} \sum_{k=1}^K \tilde{w}_k D_k$ where $D_k = \text{diag}(\Delta_k^2/\mu_s^2)$.

Output: Private estimator

$$\tilde{\boldsymbol{\beta}} = (\widetilde{S}^\top \widetilde{W} \widetilde{S} - \widetilde{D})^{-1} \widetilde{S}^\top \widetilde{W} \widetilde{\mathbf{t}}$$

and a private confidence interval for each coordinate j ,

$$\tilde{\beta}_j \pm z_{\alpha/2} \text{se}(\tilde{\beta}_j),$$

where $\text{se}(\tilde{\beta}_j)$ is defined in Section 4.2 and $z_{\alpha/2}$ is the $(1 - \alpha/2)$ quantile of $\mathcal{N}(0, 1)$.

data points in each bin by letting $\tilde{\mathbf{s}}'_k = \sum_{i=1}^{\tilde{c}_k} \tilde{\mathbf{x}}^{(k,i)}$ and $\tilde{t}'_k = \sum_{i=1}^{\tilde{c}_k} \tilde{y}^{(k,i)}$. Then,

$$\tilde{\mathbf{s}}'_k \stackrel{d}{=} \tilde{\mathbf{s}}_k \sim \mathcal{N}(\mathbf{s}_k, \Delta_k^2/\mu_s^2), \quad \tilde{t}'_k \stackrel{d}{=} \tilde{t}_k \sim \mathcal{N}(t_k, B_y^2/\mu_t^2),$$

where $\tilde{\mathbf{s}}_k$ and \tilde{t}_k are defined in Algorithm 2.

A natural alternative for synthetic data generation is to post-process a private regression fit (e.g., $\mathbf{y} = X\boldsymbol{\beta}^{\text{priv}}$ or $\mathbf{y} = X\boldsymbol{\beta}^{\text{priv}} + \mathbf{e}$), but this either collapses variability onto a hyperplane or requires a private estimate of the residual variance (additional privacy budget and sensitivity analysis). In contrast, Algorithm 3 generates samples directly from DP bin-level summaries, preserving both variability and joint dependence while remaining consistent with the regression model.

Remark on binning dependence. While BinAgg requires a binned structure as its foundation, it is not inherently tied to the use of any particular algorithm, such as PrivTree. We adopt PrivTree in our implementation because it is a practical data-dependent binning strategy that adapts to the density of the data and has been successfully combined with DP synthetic data generation in prior studies Tao et al. (2022). However, other binning approaches, either data-independent (e.g., uniform partitioning) or alternative

Algorithm 3 BinAgg for Synthetic Data

Input: Output from Algorithm 1, label bound B_y , privacy budgets for features and label:

$$\mu_s, \mu_t.$$

1: **for** $k = 1$ to K **do**

2: **for** $i = 1$ to \tilde{c}_k **do**

3: Sample synthetic features:

$$\tilde{\mathbf{x}}^{(k,i)} = \frac{s_k + \boldsymbol{\xi}^x}{\tilde{c}_k}, \quad \boldsymbol{\xi}^x \sim \mathcal{N}(\mathbf{0}, \tilde{c}_k \Delta_k^2 / \mu_s^2),$$

4: Sample synthetic label:

$$\tilde{y}^{(k,i)} = \frac{t_k + \xi^y}{\tilde{c}_k}, \quad \xi^y \sim \mathcal{N}(0, \tilde{c}_k B_y^2 / \mu_t^2).$$

5: **end for**

6: **end for**

Output: A synthetic dataset:

$$\mathcal{D}_{\text{syn}} = \left\{ (\tilde{\mathbf{x}}^{(k,i)}, \tilde{y}^{(k,i)}) \mid k = 1, \dots, K; i = 1, \dots, \tilde{c}_k \right\}.$$

data-dependent methods, can also be used in conjunction with BinAgg, provided that only the bin boundaries are released in a DP manner. Importantly, BinAgg only consumes privacy budget for constructing the bin structure (i.e., the boundaries) itself during the binning step, not for additional statistics that some binning algorithms might output. This flexibility allows practitioners to tailor the binning procedure to their data characteristics and privacy constraints, while still retaining the theoretical guarantees of our method.

4 Theoretical Results

This section presents two-fold theoretical results that capture the privacy-utility tradeoff: (1) privacy guarantee, and (2) statistical inference guarantee. All proofs are provided in the supplementary material.

4.1 Privacy Guarantees

Theorem 4.1 (Gaussian DP Guarantees). *Algorithms 2 and 3 satisfy $\sqrt{\mu_{\text{bin}}^2 + \mu_s^2 + \mu_t^2 + \mu_c^2}$ -GDP.*

This guarantee follows directly from the design of our mechanisms and the composition properties of DP. Specifically, the binning step employs the PrivTree algorithm, which in our implementation we show ensures μ_{bin} -GDP. The remaining components: the sum of \mathbf{x} variable, the sum of \mathbf{y} variable, and the bin count, are each released using Gaussian

mechanisms calibrated to privacy budgets μ_s , μ_t , and μ_c , respectively. By composition, their privacy losses combine, yielding the overall guarantee of $\sqrt{\mu_{\text{bin}}^2 + \mu_s^2 + \mu_t^2 + \mu_c^2}$ -GDP.

Algorithm 2 and Algorithm 3 both inherit the same overall privacy bound, and one may choose one or the other depending on the use case and analysis goals. In particular, by the post-processing property of DP, any subsequent analysis, even if not LR, performed on the synthetic data does not incur additional privacy cost.

4.2 Statistical Inference

For statistical inference, we prove the asymptotic normality of our private estimate and use it to construct an asymptotic CI, accounting for noise added for privacy protection. For the non-private weighted linear model in (3), the classical theory gives

$$\Sigma^{-1/2}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, \mathbf{I}_d), \quad \Sigma = \sigma^2(S^\top WS)^{-1}.$$

In the private model (5), a naive plug-in estimator takes the form $\tilde{\Sigma}_{\text{naive}} = \sigma^2(\tilde{S}^\top \tilde{W} \tilde{S})^{-1}$. However, the naive estimator does not account for DP-induced uncertainty properly, leading to undercoverage of the resulting CI; see Table 1. Instead, we establish the following theoretical result for our proposed DP bias-corrected estimator.

Theorem 4.2 (Asymptotic Normality). *As the number of bins $K \rightarrow \infty$ with the sample size $n \rightarrow \infty$, given the injected DP noise have finite variances, the bias-corrected estimator $\tilde{\beta} \stackrel{\text{def}}{=} (\tilde{S}^\top \tilde{W} \tilde{S} - \tilde{D})^{-1} \tilde{S}^\top \tilde{W} \tilde{t}$ as defined in Algorithm 2 satisfies*

$$\tilde{\Sigma}^{-1/2}(\tilde{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, \mathbf{I}_d),$$

where

$$\tilde{\Sigma} = \tilde{M}^{-1} \tilde{H} \tilde{M}^{-1},$$

with

$$\begin{aligned} \tilde{M} &= \frac{1}{K}(\tilde{S}^\top \tilde{W} \tilde{S}) - \tilde{D}, \\ \tilde{H} &= \frac{1}{K(K-d)} \sum_{k=1}^K \tilde{\mathbf{Q}}_k \tilde{\mathbf{Q}}_k^\top, \\ \tilde{\mathbf{Q}}_k &= \tilde{s}_k \tilde{w}_k (\tilde{t}_k - \tilde{s}_k^\top \tilde{\beta}) + \tilde{w}_k D_k \tilde{\beta}. \end{aligned}$$

Therefore, an asymptotic $(1 - \alpha)$ confidence interval for each coefficient β_j satisfying DP is given by

$$\tilde{\beta}_j \pm z_{\alpha/2} \cdot \text{se}(\tilde{\beta}_j),$$

where $\text{se}(\tilde{\beta}_j)$ is the j th diagonal entry of $\tilde{\Sigma}$, and $z_{\alpha/2}$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution.

The asymptotic normality follows from the central limit theorem, just as in the case of non-private LR. The proposed CI is DP since the construction only uses DP-sanitized data. And, the CI accounts for the uncertainty from the injected DP noise and is reflected in the construction of the covariance estimator $\tilde{\Sigma}$.

5 Experiments

In this section, we evaluate the performance of our algorithms by comparing them with existing methods for DP linear regression and for synthetic data generation. We also assess the validity of the private confidence intervals as established in Theorem 4.2.

5.1 Simulation Studies

We compare our Algorithm 2 to two popular algorithms: DP-GD and AdaSSP (the only two methods that are used for comparison in (Amin et al., 2023)). Given our focus on small-scale datasets, we use the variant DP-GD for its more stable performance without the need to tune batch size. In addition, since our BinAgg provides CIs, we also compare it to SSP (Dwork et al., 2014; Sheffet, 2017) and JLT (Sheffet, 2017) that are capable of statistical inference. The non-private OLS estimator is displayed for benchmarking purposes.

Features are drawn from $\text{Uniform}([0, 1]^d)$, and the true coefficients are drawn from $\text{Uniform}([1, 2]^d)$. The regression error scale is set to $\sigma = 1$. Details on the hyperparameter settings are provided in the Appendix.

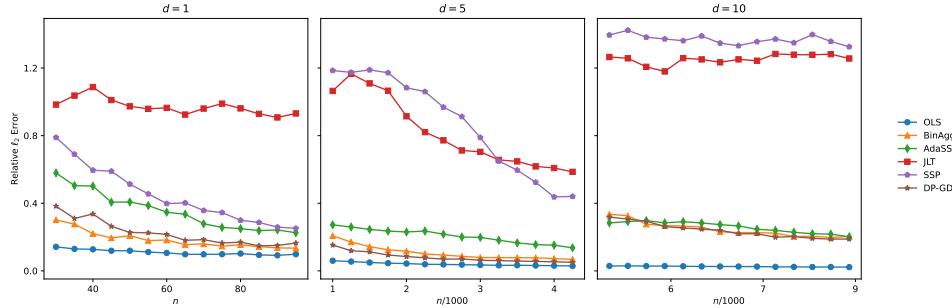


Figure 1: Coefficient estimation error across different (n, d) over 100 repetitions with $\mu = 1$

We evaluate five methods under three settings. Figure 1 shows the estimation error across varying sample sizes in three settings with dimensions $d = 1, 5, 10$. The y -axis displays the averaged relative ℓ_2 error over 100 repetitions, defined as $\|\tilde{\beta} - \beta\|_2 / \|\beta\|_2$ where β denotes the true coefficients and $\tilde{\beta}$ is any DP estimator. Precise conversion in Proposition 2.1 (i) is used for other (ε, δ) -DP methods by setting $\delta = 1/n^{1.1}$. The performance of the five algorithms falls into two tiers: SSP and JLT perform poorly,

while the other three methods exhibit better accuracy. Among them, BinAgg and DP-GD perform comparably well overall, with AdaSSP slightly worse. Notably, BinAgg outperforms others when the dimension is as small as $d = 1$. At first glance, DP-GD may appear to perform slightly better in some settings (e.g., $d = 5$). However, achieving this typically requires extensive hyperparameter tuning, which is *computationally expensive* and can implicitly *leak privacy* through the tuned parameters. In practice, identifying an effective tuning grid is harder than in simulation; by contrast, in our real-data evaluations (Section 5.2.1), DP-GD mostly underperforms BinAgg. Moreover, neither DP-GD nor AdaSSP provide CIs.

In addition, we observe that BinAgg is less sensitive to loose feature bounds compared to AdaSSP. Figure 2 shows the error under varying bounds for the case $d = 5$. In AdaSSP, the feature bounds directly determine the sensitivity and influence noise level. In contrast, BinAgg uses the bounds only to initialize the bin edges; bins with noisy counts less than 2 are discarded, which mitigates the impact of loose initial bounds. To more aggressively discard “bad” bins with low counts, one can increase the threshold in the algorithm.

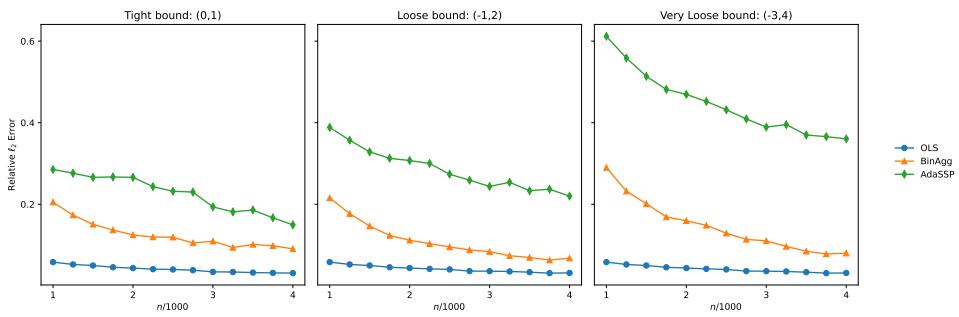


Figure 2: Coefficient estimation error using different feature bounds with $d = 5$ and $\mu = 1$.

We also evaluate our DP CI for BinAgg based on Theorem 4.2, with results shown in Table 1; SD stands for standard deviation. The naive CI discussed in Section 4 is included to illustrate that ignoring extra uncertainty due to DP underestimates the variance of the estimator, and the resulting CI significantly undercovers. Given that the JLT and SSP methods consistently give significantly poor performance, we do not include their CIs. Although they support uncertainty quantification, the estimation error is too large to be useful in practice. From Table 1, our theoretical standard deviations are very close to the empirical ones, and the empirical coverage is around the nominal level of 95%.

5.2 Real Data Applications

5.2.1 Linear Regression

We apply Algorithm 2 to several small- to mid-scale real datasets accessible in the UCI Machine Learning Repository, covering application domains across various scientific areas. The data vary in size from $n = 182$ to $n = 21,263$ and in dimensionality from

Table 1: BinAgg: Gaussian DP 95% CI over 2000 repetitions with $d = 5$, $n = 1000$, $\mu = 1$.

Avg. bias	Empirical SD	Avg. theor. SD	Naive theor. SD	Coverage	Naive coverage
-0.012	0.252	0.255	0.113	0.953	0.637
-0.008	0.262	0.268	0.117	0.950	0.654
-0.002	0.271	0.271	0.119	0.947	0.637
-0.004	0.298	0.307	0.129	0.947	0.637
0.004	0.503	0.521	0.194	0.957	0.597

$d = 4$ to $d = 81$. For brevity, we denote these datasets as D1–D9 throughout the text; additional details and references are provided in the Appendix. Given the significantly worse performance of SSP and JLT, we focus our comparison on three methods: BinAgg, AdaSSP, and DP-GD, using non-private OLS as the benchmark. To evaluate prediction accuracy, in Table 2 we report the relative mean squared error (RelMSE), a scaled version of the MSE, defined as $\text{RelMSE} = \|\tilde{\mathbf{y}} - \mathbf{y}\|_2^2 / \|\mathbf{y}\|_2^2$, where $\tilde{\mathbf{y}}$ denotes the predicted values. All methods on each dataset satisfy μ -GDP with $\mu = 1$.

To set up the bins, for DP-GD, we use a grid search similar in setup to Amin et al. (2023) and for all other methods we use non-private bounds; details on the hyperparameter settings are provided in the Appendix. We acknowledge that these bounds are not DP, but this is acceptable for comparison purposes. We discuss more on binning and choice of bounds in practice with real data in the Appendix. While conservative bounds often lead to reduced utility, BinAgg demonstrates greater robustness to loose bounds than AdaSSP; see Figure 2.

Table 2 shows that BinAgg outperforms the other two methods on most datasets, or at least achieves the second-lowest error. This demonstrates that BinAgg is *both effective and robust in practice*, with its performance advantage even more pronounced than in the controlled simulation settings. In contrast, AdaSSP performs the worst overall. Although DP-GD occasionally approaches BinAgg, it does not surpass it on most datasets. Moreover, DP-GD’s performance is highly sensitive to hyperparameter tuning; achieving competitive results requires an extensive search, which comes at a substantially higher computational cost.

5.2.2 Synthetic Data Generation

Our private synthetic data supports both scientific reproducibility with privacy guarantees and broader downstream tasks. Its performance on LR is evaluated in Section 5.2.1. To further assess downstream utility, we evaluate performance on additional widely used machine learning models for regression. In this setting, we compare Algorithm 3 against five private synthetic data generation approaches: AIM (Amin et al., 2023), DP-GAN (Xie et al., 2018), PATE-GAN (Jordon et al., 2018), and their enhanced variants DP-CTGAN and PATE-CTGAN (Xu et al., 2019). All five algorithms are available in the open-source

Table 2: Relative MSE of prediction across datasets over 100 repetitions. Lowest error per dataset in **bold**; second lowest underlined.

Dataset	Size (n, d)	OLS	BinAgg	AdaSSP	DP-GD
D1	(182, 4)	0.038	0.095	0.690	0.677
D2	(345, 6)	0.084	0.151	0.229	0.102
D3	(2043, 8)	0.023	0.035	0.669	0.693
D4	(4177, 10)	0.044	0.059	0.082	0.059
D5	(5875, 21)	0.011	0.016	0.080	0.062
D6	(6497, 12)	0.016	0.022	0.203	0.120
D7	(9357, 12)	0.441	0.463	0.682	0.852
D8	(19735, 27)	0.438	<u>0.507</u>	0.500	0.546
D9	(21263, 81)	0.131	0.203	0.429	0.420

Table 3: Average relative MSE across datasets using different DP synthetic data generation methods.

Dataset	Original	AIM	BinAgg	DP-CTGAN	PATE-CTGAN	DPGAN	PATEGAN
D1	0.286	1.400	0.939	<u>1.091</u>	1.344	2.398	N/A
D2	1.073	1.326	1.015	1.126	<u>1.054</u>	1.984	N/A
D3	0.875	1.169	<u>2.208</u>	17.298	<u>4.325</u>	12.619	8.789
D4	0.487	0.898	0.731	1.289	<u>1.001</u>	3.522	7.775
D5	0.033	1.058	0.677	1.139	<u>1.000</u>	2.527	2.241
D6	0.628	0.908	1.195	1.381	<u>1.120</u>	2.008	2.307
D7	0.489	<u>0.614</u>	0.584	1.231	1.039	1.488	1.753
D8	0.683	1.079	1.490	1.727	<u>1.087</u>	2.395	3.256

SmartNoise library (SmartNoise, 2021). AIM is a marginal-based method that discretizes the data, with the synthetic data inheriting the discretized structure. It has been shown to be the strongest marginal-based baseline Chen et al. (2022). DP-GAN and PATE-GAN (together with their CTGAN extensions) are GAN-based methods that directly support continuous data.

For each method, we generate 10 synthetic datasets and train four regression models: XGBoost, Random Forest, Support Vector Regression (SVR), and Multilayer Perceptrons (MLPs). MLPs are excluded for datasets with fewer than 500 samples, and PATE-GAN requires a sample size of at least 1000. Competing methods are formulated under (ϵ, δ) -DP. To ensure fair comparison, we apply Proposition 2.1 to convert our μ -GDP guarantee into (ϵ, δ) -DP. For each dataset, we set $\epsilon = 1$ and $\delta = 1/n^{1.1}$. Predictive performance is measured in terms of relative mean squared error, with each dataset split into private training and test partitions using an 80/20 ratio. Table 3 summarizes the average relative MSE across datasets.

Across these settings, BinAgg achieves the best overall performance: it attains the lowest error on 5 out of 8 datasets and, when not the best, performs very close to the top method. Its performance is followed by AIM and then PATE-CTGAN. Beyond

accuracy, BinAgg is also markedly faster than the competing methods, requiring only 0.13 seconds per dataset on average, compared to 221.08 seconds for AIM, and 6.24 seconds for PATE-CTGAN. This translates to BinAgg being approximately $1,650\times$ faster than AIM, and $47\times$ faster than PATE-CTGAN. Details on running times and the computing environment are provided in the Appendix. These results highlight that BinAgg delivers both high utility and exceptional efficiency, making it particularly well-suited for regimes where both accuracy and computational efficiency are critical for handling a large number of datasets.

6 Discussion

We propose BinAgg, a novel DP method: binning preserves the feature–label distribution for statistically faithful synthetic data, while aggregation maintains linear relationships for linear models.

BinAgg opens several promising directions. Since it preserves information at the feature–bin level, it supports regression and inference on arbitrary subsets of features without additional privacy cost, and makes possible model selection procedures such as the F-test, AIC, and BIC when DP uncertainty is properly incorporated. Treatment effect comparisons, a cornerstone of randomized controlled trials, can be carried out directly on synthetic data, offering a path toward privacy-preserving causal inference. Beyond linear models, these synthetic datasets enable exploration with other statistical or ML methods. Further research could investigate strategies to mitigate the curse of dimensionality in binning, which affects performance in high dimensions, and to develop optimized binning strategies with effective privacy allocation and fine-tuning. Such advances would enhance BinAgg’s applicability and improve its accuracy across broader settings.

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Appendix

A Hyperparameter Detail

Simulation in Section 5.1 For hyperparameters of DP-GD, we use grid search over 252 combinations, with learning rate from $\{0.001, 0.005, 0.01, 0.05, 0.1, 0.5, 1\}$, the clipping norm from $\{1, 5, 10, 20, 50, 100\}$, and the number of epochs from $\{1, 5, 10, 20, 50, 100\}$. For all other methods, we pre-specify feature and label bounds for clipping before data generation: $(0, 1)$ for \mathbf{x} and $(0, 2)$, $(0, 7)$, $(0, 15)$ for y when $d = 1, 5$, and 10 , respectively. For BinAgg, the privacy budget is allocated as $\mu_{\text{bin}} : \mu_c : \mu_s : \mu_t = 1 : 3 : 3 : 3$. We use the default value $\theta = 0$ in all experiments, noting that negative θ values lead to more aggressive splitting.

Simulation in Section 5.2 For DP-GD, we use a grid search with learning rates from $\{10^{-6}, 10^{-5}, 10^{-4}, \dots, 1\}$, clipping norms from $\{10^{-6}, 10^{-5}, 10^{-4}, \dots, 10^6\}$, and number of epochs from $\{1, 5, 10, 20, 50\}$, resulting in a total of 455 combinations. This setup is similar to that used in Amin et al. (2023) for evaluating real datasets. A caveat is that the grid search is conducted on the real data. For all other methods, we use the non-private data bounds. We acknowledge that these bounds are not DP, but this is acceptable for comparison purposes. In practice, applications are typically conducted by domain experts who have the knowledge to determine appropriate bounds and perform clipping before regression. If tight bounds are difficult to specify, one may either choose conservative bounds or use privatized ones. While conservative bounds often lead to reduced utility, BinAgg demonstrates greater robustness to loose bounds than AdaSSP, as shown in Figure 2.

B Additional Results for Experiments

Table 4: Datasets used in experiments with size and references.

ID	Dataset	Size (n, d)	Reference
D1	Intrusion Detection	(182, 4)	(Singh, 2022)
D2	Liver Disorders	(345, 6)	(<i>Liver Disorders</i> , 2016)
D3	Auction Verification	(2043, 8)	(Ordoni et al., 2022)
D4	Abalone Age	(4177, 10)	(Nash et al., 1994)
D5	Parkinson’s Telemonitoring	(5875, 21)	(Tsanas and Little, 2009)
D6	Wine Quality	(6497, 12)	(Cortez et al., 2009)
D7	Air Quality	(9357, 12)	(Vito, 2008)
D8	Appliances Energy	(19735, 27)	(Candanedo, 2017)
D9	Superconductivity	(21263, 81)	(Hamidieh, 2018)

Table 5: Average runtime per synthetic dataset (second).

Dataset	AIM	BinAgg	DP-CTGAN	PATE-CTGAN	DPGAN	PATEGAN
Intrusion Detection	28.6317	0.0097	2.1385	1.9045	1.3252	N/A
Liver Disorders	48.7492	0.0127	1.0844	1.8050	0.7650	N/A
Auction Verification	82.9825	0.0282	1.8096	2.6069	0.6688	3.0788
Abalone Age	130.1068	0.0662	5.8339	4.7986	1.3709	4.2810
Parkinson’s Telemonitoring	407.7839	0.0862	13.3774	6.6898	3.1690	6.0159
Wine Quality	210.8847	0.0566	11.9014	10.8458	2.0291	6.9002
Air Quality	236.7886	0.1444	24.6687	5.2088	2.7882	5.4930
Appliance Energy	622.7202	0.6674	113.1582	16.0563	11.4593	12.6062

Computing environment: All the experiments were conducted on a 64-bit Microsoft Windows laptop equipped with a 8-core AMD Ryzen 7 8845HS processor (3.80 GHz) and 16.0 GB of RAM. All synthesizers were run on this same CPU.

C Algorithm: PrivTree

Algorithm 4 PrivTree ($D, \lambda, \theta, \delta$) (Zhang et al., 2016)

- 1: Initialize a tree \mathcal{T} with a root node v_1
 - 2: Set domain(v_1) = Ω , and mark v_1 as **unvisited**
 - 3: **while** there exists an unvisited node v **do**
 - 4: Mark v as **visited**
 - 5: Compute a biased point count for v with decaying factor δ :
 - 6: $b(v) = c(v) - \text{depth}(v) \cdot \delta$
 - 7: Adjust $b(v)$ if it is excessively small:
 - 8: $b(v) = \max\{b(v), \theta - \delta\}$
 - 9: Compute a noisy version of $b(v)$: $\hat{b}(v) = b(v) + \text{Lap}(\lambda)$
 - 10: **if** $\hat{b}(v) > \theta$ **then**
 - 11: Split v and add its children to \mathcal{T}
 - 12: Mark the children of v as **unvisited**
 - 13: **end if**
 - 14: **end while**
 - 15: **return** \mathcal{T} with all point counts removed
-

Lemma (Corollary 1, Zhang et al. (2016)). *Let κ be the branching factor of tree \mathcal{T} . PrivTree satisfies ε -differential privacy if*

$$\lambda \geq \frac{2\kappa - 1}{\kappa - 1} \cdot \frac{1}{\varepsilon} \quad \text{and} \quad \delta = \lambda \cdot \ln \kappa.$$

Supplementary Materials

D Proofs

Proof of Corollary 1. For each bin k , let $\tilde{\mathbf{x}}^{(k,i)}$ denote the privatized version of the i -th sample within the bin. By construction of the Gaussian mechanism, each privatized vector has expectation

$$\mathbb{E}(\tilde{\mathbf{x}}^{(k,i)}) = \frac{\mathbf{s}_k}{\tilde{c}_k},$$

and variance

$$\text{Var}(\tilde{\mathbf{x}}^{(k,i)}) = \frac{\Delta_k^2}{\tilde{c}_k \mu_s^2}.$$

Because the $\tilde{\mathbf{x}}^{(k,i)}$'s are independent Gaussian random variables, their sum is also Gaussian, with mean equal to the sum of means and variance equal to the sum of variances. Consequently,

$$\tilde{\mathbf{s}}'_k = \sum_{i=1}^{\tilde{c}_k} \tilde{\mathbf{x}}^{(k,i)} \sim \mathcal{N}\left(\mathbf{s}_k, \frac{\Delta_k^2}{\mu_s^2}\right).$$

This distribution matches exactly the law of $\tilde{\mathbf{s}}_k$, the directly privatized statistic. Hence, the two constructions are distributionally equivalent. An identical argument, applied to the statistics \tilde{t}'_k and \tilde{t}_k , shows the same distributional equivalence holds for the response terms. \square

Proof of Theorem 2 (Algorithm 2). The algorithm involves privatizing the counts c_k , the sums s_k , and the sums of products t_k . By Proposition 2.4, the Gaussian mechanism applied to each of these statistics ensures Gaussian DP (GDP) with parameters μ_c , μ_s , and μ_t , respectively.

The initial binning procedure itself incurs a privacy cost quantified by μ_{bin} . Since the algorithm adopts the remove-one/add-one neighboring relation, composition theorems for GDP (Proposition 2.2) imply that the overall privacy is given by $\sqrt{\mu_{\text{bin}}^2 + \mu_s^2 + \mu_t^2 + \mu_c^2}$ -GDP. By the post-processing property (Proposition 2.3), all subsequent steps of the algorithm satisfy the same privacy guarantee. \square

Proof of Theorem 2 (Algorithm 3). At first, the procedure of privatizing the bin counts c_k . By Proposition 2.4, this step satisfies μ_c -GDP. Next, the algorithm generates \tilde{c}_k synthetic records per bin k . Each synthetic feature vector $\tilde{\mathbf{x}}^{(k,i)}$ is generated by adding Gaussian noise calibrated to ensure $\mu_s/\sqrt{\tilde{c}_k}$ -GDP, while each synthetic label $\tilde{y}^{(k,i)}$ is privatized with $\mu_t/\sqrt{\tilde{c}_k}$ -GDP. Because there are \tilde{c}_k such records, the composition property

implies that the total privacy cost across all synthetic samples in a given bin is

$$\sqrt{\tilde{c}_k \left(\frac{\mu_s^2}{\tilde{c}_k} + \frac{\mu_t^2}{\tilde{c}_k} \right)} = \sqrt{\mu_s^2 + \mu_t^2}.$$

Combining this contribution with the binning privacy loss and the count privatization, and by composition and post-processing, the overall privacy guarantee of Algorithm 3 is

$$\sqrt{\mu_{\text{bin}}^2 + \mu_s^2 + \mu_t^2 + \mu_c^2}\text{-GDP}.$$

□

Proof of Theorem 3. We prove the result by using the estimating equation theory.

Let $\xi_{s_k} \sim \mathcal{N}(\mathbf{0}, \sigma_{s_k}^2)$ and $\xi_{t_k} \sim \mathcal{N}(\mathbf{0}, \sigma_t^2)$ represent the Gaussian noise components that are added to the covariate sum vector s_k and the response sum variable t_k , respectively. Consequently, the observed noisy variables can be expressed as

$$\tilde{s}_k = s_k + \xi_{s_k}, \quad \tilde{t}_k = t_k + \xi_{t_k} = s_k^\top \beta + \eta_k + \xi_{t_k},$$

where the intrinsic error term is given by $\eta_k \sim \mathcal{N}(0, \sigma^2 c_k)$.

We define the bias-corrected estimator $\tilde{\beta}$ as the solution to the estimating equation

$$\mathbf{Q}(\mathbf{b}) = \frac{1}{K} \sum_{k=1}^K \mathbf{Q}_k(\mathbf{b}) = -\frac{1}{K} \sum_{k=1}^K \{\tilde{s}_k \tilde{w}_k (\tilde{t}_k - \tilde{s}_k^\top \mathbf{b}) + \tilde{w}_k D_k \mathbf{b}\} = \mathbf{0},$$

where $D_k = \mathbb{E}(\xi_{s_k} \xi_{s_k}^\top) = \text{diag}(\sigma_{s_{k1}}^2, \sigma_{s_{k2}}^2, \dots, \sigma_{s_{kd}}^2)$. At the true parameter value β , the estimating equation is unbiased. For any index k , we have

$$\begin{aligned} \mathbb{E}[\mathbf{Q}_k(\beta)] &= -\mathbb{E} [\tilde{s}_k \tilde{w}_k (\tilde{t}_k - \tilde{s}_k^\top \beta) + \tilde{w}_k D_k \beta] \\ &= -\mathbb{E} [(\mathbf{s}_k + \xi_{s_k}) \tilde{w}_k (\mathbf{s}_k^\top \beta + \eta_k + \xi_{t_k} - (\mathbf{s}_k + \xi_{s_k})^\top \beta)] - \mathbb{E}[\tilde{w}_k D_k \beta] \\ &= \mathbb{E}[\tilde{w}_k \xi_{s_k} \xi_{s_k}^\top] \beta - \mathbb{E}[\tilde{w}_k] D_k \beta \\ &= \mathbb{E}[\tilde{w}_k] D_k \beta - \mathbb{E}[\tilde{w}_k] D_k \beta \\ &= \mathbf{0}. \end{aligned}$$

That is, the estimating function is indeed unbiased at the true value β . Furthermore, since the estimating equation is linear in the parameter vector \mathbf{b} , its Jacobian takes the following form:

$$\widetilde{M} \stackrel{\text{def}}{=} \frac{\partial \mathbf{Q}(\mathbf{b})}{\partial \mathbf{b}} = \frac{1}{K} \sum_{k=1}^K (\tilde{w}_k \tilde{s}_k \tilde{s}_k^\top - \tilde{w}_k D_k) = \frac{1}{K} \tilde{S}^\top \widetilde{W} \tilde{S} - \widetilde{D},$$

where

$$\widetilde{D} \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \tilde{w}_k D_k.$$

Using a first-order Taylor expansion of the estimating function around the true parameter value β , we obtain

$$\mathbf{0} = \mathbf{Q}(\tilde{\beta}) = \mathbf{Q}(\beta) + \widetilde{M}(\tilde{\beta} - \beta),$$

which can be rearranged to yield the following linear approximation:

$$\tilde{\beta} - \beta = -\widetilde{M}^{-1} \mathbf{Q}(\beta).$$

Let us now define

$$H = \frac{1}{K^2} \sum_{k=1}^K \text{Var}(\mathbf{Q}_k),$$

where

$$\mathbf{Q}_k \stackrel{\text{def}}{=} \mathbf{Q}_k(\beta) = \tilde{s}_k \tilde{w}_k (\tilde{t}_k - \tilde{s}_k^\top \beta) + \tilde{w}_k D_k \beta.$$

Then, by the classical Central Limit Theorem (CLT), as $K \rightarrow \infty$,

$$H^{-1/2} \mathbf{Q}(\beta) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathbf{I}_d).$$

A consistent estimator of H can be obtained by its empirical counterpart, defined as

$$\widetilde{H} = \frac{1}{K(K-d)} \sum_{k=1}^K \widetilde{\mathbf{Q}}_k \widetilde{\mathbf{Q}}_k^\top,$$

where

$$\widetilde{\mathbf{Q}}_k = \tilde{s}_k \tilde{w}_k (\tilde{t}_k - \tilde{s}_k^\top \tilde{\beta}) + \tilde{w}_k D_k \tilde{\beta}.$$

Finally, define the asymptotic covariance matrix estimator as

$$\widetilde{\Sigma} = \widetilde{M}^{-1} \widetilde{H} \widetilde{M}^{-1}.$$

Then, it follows that

$$\widetilde{\Sigma}^{-1/2} (\tilde{\beta} - \beta) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathbf{I}_d).$$

□

E Tuning of BinAgg

In the submitted manuscript, all experiments—including the real data applications—use the default value $\theta = 0$. To further investigate the impact of this parameter, we conduct

a comparison on eight real datasets using three choices of θ : default ($\theta = 0$), and tuned (selected via grid search over the range $\{-200, -190, \dots, 0, 10, \dots, 100\}$). The grid search is conducted over 100 repetitions, and we report the average Relative MSE for three settings over 100 runs.

Note that the accuracy in the table below is slightly different from that reported in the submitted manuscript due to the randomness of the noise. However, the difference is small and the inference/conclusion remains unchanged.

Table 6: Relative MSE of BinAgg under different θ settings. Lowest errors are in bold.

Dataset	Size (n, d)	$\theta = 0$	θ (tuned)
Intrusion Detection	(182, 4)	0.0952	0.0844 ($\theta = -70$)
Liver Disorders	(345, 6)	0.1508	0.1156 ($\theta = -90$)
Auction Verification	(2043, 8)	0.0336	0.0335 ($\theta = -90$)
Abalone Age	(4177, 10)	0.0585	0.0579 ($\theta = 10$)
Parkinsons Telemonitoring	(5875, 21)	0.0157	0.0154 ($\theta = -30$)
Wine Quality	(6497, 12)	0.0220	0.0211 ($\theta = -90$)
Air Quality	(9357, 12)	0.4610	0.4605 ($\theta = -70$)
Appliances Energy	(19735, 27)	0.5044	0.5060 ($\theta = -30$)

Table 7: Relative MSE of BinAgg under different budget ratios ($\mu_{bin} : \mu_s : \mu_t : \mu_c$). Lowest errors are in bold.

Dataset	Size (n, d)	1:1:1:1	2:3:3:3	1:3:3:3
Intrusion Detection	(182, 4)	0.1104	0.0861	0.0952
Liver Disorders	(345, 6)	0.1669	0.1501	0.1508
Auction Verification	(2043, 8)	0.0326	0.0335	0.0336
Abalone Age	(4177, 10)	0.0647	0.0584	0.0585
Parkinsons Telemonitoring	(5875, 21)	0.0226	0.0164	0.0157
Wine Quality	(6497, 12)	0.0214	0.0212	0.0220
Air Quality	(9357, 12)	0.4670	0.4608	0.4610
Appliances Energy	(19735, 27)	0.5080	0.5057	0.5044

Overall, tuning θ improves performance over the default or heuristic values in most datasets. In some cases, the heuristic choice may improve over the default value. However, we also notice that when the differences are small, such as 0.5044 vs 0.5060 for the last dataset, the improvement of one method over the other may only be due to the randomness of the noise over those 100 runs.

While tuning can help with accuracy, the improvement may not be substantial. This suggests that BinAgg is not as sensitive to the choice of θ as DP-(S)GD is to its parameter settings. **For practitioners, we recommend the default choice of $\theta = 0$. If the accuracy is not satisfactory, tuning can be performed.**

We also report the number of bins K under both default and tuned θ settings for four datasets. For each setting, we summarize the minimum, maximum, and median of the bin counts across 100 repetitions. We observe that the number of bins varies across runs, and using a negative θ results in a greater number of bins overall.

In addition, we experiment with different privacy budget ratios between binning and the other components. We find that slightly lowering the budget allocated to binning may improve performance marginally, shown in Figure 7. In particular, the ratio 2:3:3:3 yields the best overall performance, followed by 1:3:3:3. Even with an equal allocation (1:1:1:1), BinAgg still outperforms other methods (DP-GD, AdaSSP, etc.) on most datasets.

Table 8: Bin number summary for BinAgg under different θ values

Dataset	Size (n, d)	θ	Min	Max	Median
Intrusion Detection	(182, 4)	0	5	12	7
		-70	4	24	14
Auction Verification	(2043, 8)	0	26	58	40
		-90	79	131	106
Parkinsons Telemonitoring	(5875, 21)	0	26	58	40
		-30	38	74	54
Air Quality	(9357, 12)	0	120	160	136
		-70	152	210	179