MAE288B Optimal Estimation - Homework 3, Winter 2019 — due Tuesday March 19, 2019.

Consider the following nonlinear system with scalar state

$$x_{t+1} = \sin(x_t) + w_t, \tag{1}$$
$$y_t = x_t^3 + v_t. \tag{2}$$

The noise processes $\{w_t\}$ and $\{v_t\}$ are independent, white and uniformly distributed as $\mathcal{U}[-d,d]$ for w_t and $\mathcal{U}[-h,h]$ for v_t, where the noise amplitudes are parameters to play with.

Denote by $\mathbf{Y}^t$ the data, $\{y_t, y_{t-1}, \ldots, y_0, \text{pdf}(x_0)\}$, up to and including time t. Here $\text{pdf}(x_0)$ is the a priori pdf of the initial state. Take $\text{pdf}(x_0) = \mathcal{U}[-\pi,\pi]$.

The Bayesian filter is defined as the following recursion for the conditional densities of the state given the measurements.

$$\text{pdf}(x_t|\mathbf{Y}^t) = \frac{\text{pdf}(y_t|x_t)\text{pdf}(x_t|\mathbf{Y}^{t-1})}{\int \text{pdf}(y_t|x_t)\text{pdf}(x_t|\mathbf{Y}^{t-1})\,dx_t}, \quad \text{pdf}(x_0|\mathbf{Y}^{-1}) = \text{pdf}(x_0)$$
$$\text{pdf}(x_{t+1}|\mathbf{Y}^t) = \int \text{pdf}(x_{t+1}|x_t)\text{pdf}(x_t|\mathbf{Y}^t)\,dx_t.$$

Task 1: Use Bayes' Rule and Fun Result #1 to derive the Bayesian filter.

Task 2: Write a MATLAB program to run an approximate Bayesian filter for given values of (d,h). This program will need to be cognizant of:

(i) The range of feasible values for x_t as a function of d and t.

(ii) The range of feasible values for y_t as a function of d, t and h.

(iii) The number of sample points, or density of samples, that you want for each pdf. This will dictate computational complexity. But it is a simple problem.

Task 3: Use system equations (1)-(2) to produce a set of sample state and output values. You will test your estimator against these output measurements y_t and against these *truth samples* of x_t. Note that MATLAB's `rand` function returns a white $\mathcal{U}[0,1]$ set of random variables.

Task 4: Run your estimator to determine how well you can estimate x_t for specific values of d, h and t. Note the Bayesian filter returns a conditional density. You will need to develop your own figure of merit for the performance of the filter. Compare the predicted and filtered conditional densities.

Task 5: Play with d and h to explore the behavior of the Bayesian filter in different noise environments. Work hard on the presentation of your results in a comprehensive, concise and precise way.

Task 6: [Bonus for the overachievers] Compare the Bayesian filter to the Extended Kalman Filter in terms of conditional mean and conditional variance.

Task 7: Based on your self-assessed performance, pay yourself a large amount of money in used, non-consecutive, small denomination bills in a plain brown paper bag. Since you paid yourself, you need not declare this to the IRS. Please note this sum in your homework. Your grade will depend on it.