Here are the MiP equations from *Numerical Renaissance* (17.22a)-(17.22b).

$$(I_b + m_b\ell^2)\frac{d^2\theta}{dt^2} - m_b g\ell\sin\theta + m_b r\ell\cos\theta\frac{d^2\phi}{dt^2} = -\tau,$$
$$[I_w + (m_w + m_b)r^2]\frac{d^2\phi}{dt^2} + m_b r\ell\cos\theta\frac{d^2\theta}{dt^2} - m_b r\ell\sin\theta\left(\frac{d\theta}{dt}\right)^2 = \tau.$$

Rewrite this as

$$A\ddot{\theta} - B\sin\theta + C\cos\theta\ddot{\phi} = -\tau,$$
$$D\ddot{\phi} + E\cos\theta\ddot{\theta} - F\sin\theta(\dot{\theta})^2 = \tau.$$

Whence,

$$D\ddot{\phi} = -E\cos\theta\ddot{\theta} + F\sin\theta(\dot{\theta})^2 + \tau,$$

and

$$AD\ddot{\theta} - BD\sin\theta + C\left[F\cos\theta\sin\theta(\dot{\theta})^2 - E\cos^2\theta\ddot{\theta}\right] = -\left(D + C\cos\theta\right)\tau,$$
$$\left[AD - CE\cos^2\theta\right]\ddot{\theta} - BD\sin\theta + CF\cos\theta\sin\theta(\dot{\theta})^2 = -\left(D + C\cos\theta\right)\tau,$$

or,

$$\begin{aligned}\ddot{\theta} &= \frac{BD\sin\theta}{AD - CE\cos^2\theta} - \frac{CF\cos\theta\sin\theta(\dot{\theta})^2}{AD - CE\cos^2\theta} - \frac{D + C\cos\theta}{AD - CE\cos^2\theta}\tau,\\ &= \frac{1}{AD - CE\cos^2\theta}\left[BD\sin\theta - CF\cos\theta\sin\theta(\dot{\theta})^2 - (D + C\cos\theta)\tau\right] \qquad (1)\end{aligned}$$

By the same token,

$$A\ddot{\theta} = B\sin\theta - C\cos\theta\ddot{\phi} - \tau,$$

or, substituting,

$$DA\ddot{\phi} + E\cos\theta\left[B\sin\theta - C\cos\theta\ddot{\phi} - \tau\right] - AF\sin\theta(\dot{\theta})^2 = A\tau,$$
$$\left[DA - EC\cos^2\theta\right]\ddot{\phi} + EB\cos\theta\sin\theta - AF\sin\theta(\dot{\theta})^2 - E\cos\theta\tau = A\tau.$$

That is,

$$\begin{aligned}\ddot{\phi} &= -\frac{BE\cos\theta\sin\theta}{AD - CE\cos^2\theta} + \frac{AF\sin\theta(\dot{\theta})^2}{AD - CE\cos^2\theta} + \frac{A + E\cos\theta}{AD - CE\cos^2\theta}\tau,\\ &= \frac{1}{AD - CE\cos^2\theta}\left[-BE\cos\theta\sin\theta + AF\sin\theta(\dot{\theta})^2 + (A + E\cos\theta)\tau\right] \qquad (2)\end{aligned}$$

The torque equation comes from Homework 4, 2015, with the corrected definition $\omega_t = \dot{\phi}_t - \dot{\theta}_t$.

$$\dot{\tau}_t = -\frac{R}{L}\tau_t - \frac{k_v k_t}{L}(\dot{\phi}_t - \dot{\theta}_t) + \frac{k_t}{L}v_t. \qquad (3)$$

Expressions (1-3) provide an approach to writing nonlinear state equations in terms of the state

$$x_t = \begin{pmatrix}\tau_t\\ \theta_t\\ \dot{\theta}_t\\ \phi_t\\ \dot{\phi}_t\end{pmatrix}.$$