**Problem Set 4 for MAE280A Linear Systems Theory, Fall 2015:**
**due on or before Tuesday December 8 before 16:00 at Bob's office.**
**Absolutely no extensions possible! You may hand it in earlier.**
**Ensure that you submit only your own work. You may discuss this freely but do not copy.**

## Part I — a theory question

Prove the following properties about the time-varying, discrete-time, linear system of input, state, and output dimensions $m$, $n$, $p$. ⋆

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t u_t, \\ y_t &= C_t x_t + D_t u_t. \end{aligned}$$

(a) The system with matrices $(A_t, B_t)$ is reachable $\Longrightarrow$ the system with matrices $(A_t + B_t K_t, B_t)$ is reachable for any $m \times n$ matrix sequence $K_t$. That is, reachability is unaffected by state feedback.

(b) The system with matrices $(A_t, B_t)$ is reachable $\Longrightarrow$ the system with matrices $(A_t + B_t K_t C_t, B_t)$ is reachable for any $m \times p$ matrix sequence $K_t$. That is, reachability is unchanged by constant output feedback.

(c) The system with matrices $(A_t, B_t)$ is reachable $\not\Longrightarrow$ the system with matrices $(A_t + L_t C_t, B_t)$ is reachable for any $n \times p$ matrix sequence $L_t$. That is, reachability is potentially altered by output injection.

State the observability versions of these results. ⋆

## Part II – a practical problem

### Introduction

We shall be considering the control of the Mobile Inverted Pendulum (MiP) as developed by UCSD Professor Tom Bewley, his Coordinated Robotic Lab, and commercialized by WowWee Robotics — available from the UCSD Bookstore for $89.99 plus tax. See here.

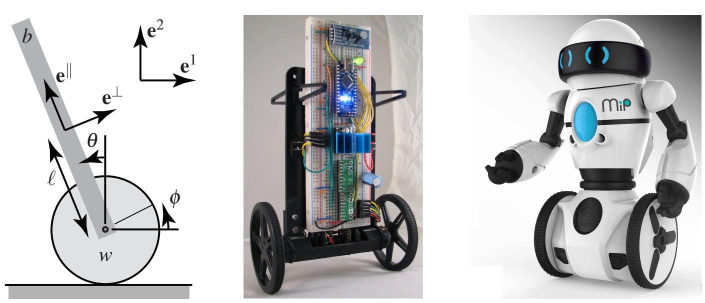


Figure 17.6: MIP, a Mobile Inverted Pendulum; (left) schematic, (center) prototype, (right) toy product commercialized by WowWee Robotics and the UCSD Coordinated Robotics Lab.

Figure 1: MiP figure from Tom Bewley's *Numerical Renaissance* draft on November 22, 2015. This is downloadable here [http://numerical-renaissance.com/NR.pdf].

Our aim will be to: derive linearized models for MiP in state variable form, analyze the properties of these models, design an output feedback controller for MiP which stabilizes MiP, and extend this design to accommodate MiP on a windy day, assuming the appearance of a Godzilla El Niño of 2015-16.

**Task 1 — Equations of motion and linearization**

The two-dimensional equations of motion for MiP in the sagittal plane are derived on page 500 of *Numerical Renaissance*. [If you have a different version you might need to search a little. Search on `WowWee`.]

$$
\begin{aligned}
(I_b + m_b \ell^2)\frac{d^2\theta}{dt^2} - m_b g \ell \sin\theta \qquad + m_b r \ell \cos\theta \frac{d^2\phi}{dt^2} &= -\tau, && (17.22a)\\
[I_w + (m_w + m_b) r^2]\frac{d^2\phi}{dt^2} + m_b r \ell \cos\theta \frac{d^2\theta}{dt^2} - m_b r \ell \sin\theta \left(\frac{d\theta}{dt}\right)^2 &= \tau. && (17.22b)
\end{aligned}
$$

Figure 2: MiP equations of motion from *Numerical Renaissance*.

Here, the variables are depicted in Figure 1 and defined as follows.

**Time signals**

$\theta_t$ angle of the body relative to vertical upright,

$\tau_t$ motor torque (combined torque of two motors operating through a gearbox),

$\phi_t$ angle of the wheels relative to their starting position.

**Parameters**

$g$ acceleration due to gravity,

$I_b$ moment of inertia of the body,

$I_w$ moment of inertia of each wheel,

$\ell$ length from center for wheels to center of mass of body,

$m_b$ mass of the body,

$m_w$ mass of each wheel,

$r$ radius of wheels.

Show that the linearized equations of motion about $\theta = 0$, $\phi = 0$, $\tau = 0$ are given by ⋆

$$
\begin{aligned}
(I_b + m_b \ell^2)\frac{d^2\theta}{dt^2} - m_b g \ell \theta + m_b r \ell \frac{d^2\phi}{dt^2} &= -\tau, && (17.23a)\\
[I_w + (m_w + m_b) r^2]\frac{d^2\phi}{dt^2} + m_b r \ell \frac{d^2\theta}{dt^2} &= \tau. && (17.23b)
\end{aligned}
$$

Figure 3: MiP equations of motion linearized about $\theta = 0$, $\phi = 0$, $\tau = 0$.

Derive the corresponding linearized equations of motion about the point $\theta = \pi$, $\phi = 0$, $\tau = 0$. ⋆

**Task 2 — Simplifying the equations of motion** Define the following quantities.

$$\begin{aligned} a &= I_w + (m_b + m_w)r^2, \\ b &= m_b r\ell, \\ c &= I_b + m_b\ell^2, \\ d &= m_b g\ell. \end{aligned}$$

Now, show that ⋆

$$(ac - b^2)\ddot{\theta}_t - ad\,\theta_t = -(a+b)\tau_t. \tag{1}$$

Substitution shows that, $ac - b^2 > 0$, $ad > 0$, $a + b > 0$.

What does (1) tell us about the open-loop ($\tau = 0$) poles of the linearized system? ⋆

What do these poles tell us about the stability (in the sense of Lyapunov) of the linearization point $\theta = 0$, $\phi = 0$, $\tau = 0$? Explain the effect of both poles. ⋆

Where do these poles move when we linearize about $\theta = \pi$, $\phi = 0$, $\tau = 0$? Does the stability in the sense of Lyapunov change? How is manifested in the motion? ⋆

Can we asymptotically stabilize the system, about $\theta = 0$ or about $\theta = \pi$, by proportional linear feedback $\tau_t = -K_P\theta_t$? Explain. ⋆

Could the systems be asymptotically stabilized by proportional-plus-derivative (PD) feedback, $\tau_t = -K_P\theta - K_D\dot{\theta}$? ⋆

Derive the corresponding dynamic equation for wheel angle, $\phi$. ⋆

$$\ddot{\phi}_t = -\frac{bd}{ac - b^2}\theta_t + \frac{c+b}{ac-b^2}\tau_t. \tag{2}$$

**Task 3 — Motor equations**

Evidently, the driving signals for MiP is the torque, $\tau_t$. However, this torque is the output of a DC motor driven by a voltage, In this section, you will derive the motor dynamics linking the following three signals and set of motor parameters.

**Time signals**

$v_t$ applied voltage from a pulse-width-modulated [PWM] battery pack,

$\tau_t$ motor torque,

$\omega_t = \dot{\phi}_t$ motor shaft speed.

**Motor parameters**

$R$ winding resistance,

$L$ winding inductance,

$k_t$ motor torque constant,

$k_v$ motor back-emf constant,

$\tau_s$ stall torque at a given fixed voltage,

$\omega_f$ free-running angular velocity at fixed voltage.

Examine the MATLAB page documentation page `DC Motor`: [http://www.mathworks.com/help/physmod/elec/ref/dcmotor.html].

Show that motor torque satisfies the following ordinary differential equation. [Hint: use Kirchhoff's voltage law around the circuit loop depicted on the MATLAB page.] ⋆

$$\dot{\tau}_t = -\frac{R}{L}\tau - \frac{k_v k_t}{L}\omega_t + \frac{k_t}{L}v_t. \tag{3}$$

Demonstrate that the stall torque and free-running angular velocity for a given voltage yield parameters $k_t = k_t$ and $R$ but not the motor inductance $L$. Explain why this is the case. ⋆ ⋆

Show that, if we assume that the transient $e^{-(R/L)t}$ is very fast, then we can approximate the torque by the expression in Section 3 of the MATLAB page. ⋆

$$\tau_t = -\frac{k_v k_t}{R}\omega_t + \frac{k_t}{R}v_t.$$

**Task 4 — Measurement equations**

A strapdown inertial measurement unit (IMU) is attached to the MiP body. This uses rate gyros and accelerometers to yield, after some rudimentary signal processing, an effective measurement of the body angle $\theta_t$ every five milliseconds. The signal processing combines the gyro signals, which are accurate at high frequencies but drift, with accelerometer signals which provide absolute accuracy but are corrupted by high-frequency noise.

A shaft encoder is attached to each wheel. These provide pulse counts indicating wheel angle, $\phi_t$ accurate to a few degrees, depending on the divisions re revolution of the encoder disk. These are available effectively at any sampling rate but are quantized to the encoder accuracy.

For our purposes, we shall regard the system outputs as

$$y_t = \begin{bmatrix}\theta_t \\ \phi_t\end{bmatrix}, \tag{4}$$

and will presume these signals to be accurate and noise-free.

**Task 4 - State-variable realization**

Combine (1-4) to arrive at a linear state-variable realization for the system linking input signal $v_t$ to output signals $\theta_t$ and $\phi_t$ via state vector: ⋆

$$x_t = \begin{bmatrix}\tau_t \\ \theta_t \\ \dot{\theta}_t \\ \phi_t \\ \dot{\phi}_t\end{bmatrix}.$$

Note, from the diagram above, that the motor shaft speed,

$$\omega_t = \dot{\theta}_t + \dot{\phi}_t. \tag{5}$$

$$\dot{x}_t = \begin{bmatrix} \dot{\tau}_t \\ \dot{\theta}_t \\ \ddot{\theta}_t \\ \dot{\phi}_t \\ \ddot{\phi}_t \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k_v k_t}{L} & 0 & -\frac{k_v k_t}{L} \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{a+b}{ac-b^2} & \frac{ad}{ac-b^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{c+b}{ac-b^2} & -\frac{bd}{ac-b^2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_t \\ \theta_t \\ \dot{\theta}_t \\ \phi_t \\ \dot{\phi}_t \end{bmatrix} + \begin{bmatrix} \frac{k_t}{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_t,$$

$$\begin{bmatrix} \theta_t \\ \phi_t \end{bmatrix} = y_t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tau_t \\ \theta_t \\ \dot{\theta}_t \\ \phi_t \\ \dot{\phi}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_t.$$

Denote this realization as

$$\begin{aligned} \dot{x}_t &= Ax_t + Bv_t, \\ y_t &= Cx_t + Dv_t. \end{aligned}$$

Now add some quantitative numbers.

$$A = \begin{bmatrix} -50 & 0 & -0.2 & 0 & -0.2 \\ 0 & 0 & 1 & 0 & 0 \\ -3,260.5 & 146.0849 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 4,641.3 & -94.5706 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

This corresponds to the following motor values, which (I hope) are physically reasonable.

$R$ 0.1 Ω,

$L$ 2 mH,

$k_t = k_v$ 0.04 nmA$^{-1}$/V$s^{-1}$.

The MiP parameters were provided by James Strawson.

Note that $A$ is singular, since it has a column of zeros. This corresponds to the presence of an integrator in the MiP somewhere between input $v_t$ and the state. Show that the state associated with this integrator is $\phi_t$ or $x_{4,t}$. ⋆

**Task 5 — Analysis of MiP and design of a controller**

**Stabilizability**

Check the stability of the open-loop system by examining the eigenvalues of $A$.

Test the state-variable realization of MiP for reachability using the `ctrb` and the `gram` commands. Explain why the second command fails? [Use your knowledge of the Lyapunov equation.] ⋆ ⋆

Recall that the MiP is powered by a 12V battery pack and that the states are measured in SI units. The poles are measured in radians per second. Consider a number of possible closed-loop pole positions which are realistic in terms of MiP dynamics, are stable and for which the resulting linear state-variable feedback gain makes some sense. Present and argue for your favorite closed-loop pole locations. ⋆

Explain what stabilization of MiP means in a physical context. ⋆

**Definition 1.** *A rational function $p(s) \in \mathsf{R}(s)$ has the parity interlacing property if $p(s)$ has an even number of poles on $\mathsf{R}_{+\infty}$ between each pair of zeros on $\mathsf{R}_{+\infty}$. $p(s)$ has the even interlacing property if both $p(s)$ and $p^{-1}(s)$ have the parity interlacing property.*

Figure 4: Definition of the PiP from Blondel, Gevers, Mortini and Rupp, CDC 1992. $\mathsf{R}_{+\infty}$ is the positive real axis of the $s$-plane.

**Observability and the choice of sensors**

The sensor package on MiP consists of: a three-axis linear accelerometer (i.e. three accelerometers mounted in orthogonal directions) attached to the body, a three-axis set of gyroscopes attached to the body, and shaft encoders measuring the wheel angle. The accelerometers can be used to determine the down direction of $g$ but also the linear accelerations of the body. The gyroscopes yield angular velocity data along three axes. This can be combined with the accelerometers to yield measurement of the angular velocity $\dot{\theta}_t$ and, by integration, the angle $\theta$.

Show that the MiP is observable from $y_t = \begin{bmatrix}\theta_t & \phi_t\end{bmatrix}^T$ and from the single output $\phi_t$ but not from the single output $\theta_t$. $\star$

Show that the unobservable subspace for MiP with only $\theta_t$ as a measurement is given solely by the $\phi_t = x_{4,t}$ component. $\star$
Explain what this means physically and whether this matters from a stability perspective on the physical system. $\star$

Determine the observability properties using solely $\dot{\theta}_t$. $\star$

Design an observer for MiP, arguing for your choice of pole positions. Then construct the complete controller comprised $\star$
of your state feedback law and your observer. Test this design by simulating the response to a significant non-zero initial $\star$
condition. Ensure that you examine both the control signal, the states and the outputs.

**[Bonus]** Repeat the above feedback controller design using the state $\star$

$$\bar{x}_t = \begin{bmatrix}\tau_t \\ \theta_t \\ \dot{\theta}_t \\ \dot{\phi}_t\end{bmatrix},$$

and single output $\dot{\theta}_t$. Note that this choice of state removes the integrator in the system and preserves observability solely with $\dot{\theta}_t$. By examining the system transfer function poles and zeros, prove that this system is not stabilizable by a stable $\star$
controller. [Hint: It does not possess the Parity Interlacing Property. Then consider its root locus plot. ]

**Task 6 — MiP blowing in the wind**

MiP likes rolling in the park at the top of Mount Soledad, although he strongly disapproves of the religious monument on city land. The wind blows heavily there from time to time and this year a Godzilla El Niño is predicted. A wind blowing on MiP introduces a disturbance torque on the body. At the top of Mount Soledad, this can be strong and constant.

We model a constant disturbance torque, $d_t$, as follows.

$$\dot{d}_t = 0,$$

Incorporate this into the model by rewriting the state equations to use a new extended state $\star$

$$x_t = \begin{bmatrix} d_t \\ \tau_t \\ \theta_t \\ \dot{\theta}_t \\ \phi_t \\ \dot{\phi}_t \end{bmatrix}.$$

To do this, you need to model $d_t$ as affecting directly $\ddot{\theta}_t$. So the value in the new system $A$-matrix, $Ap$, needs to be $Ap(4,1) = 100$ in the units of this system. Determine the corresponding system $B$, $C$, and $D$ matrices: $Bp$, $Cp$, $Dp$.

Show that the system remains observable from $\theta_t$ and $\phi_t$, and also from $\phi_t$ alone. $\star$

Show and explain why the system is no longer reachable. Physically, what is the unreachable pole/mode? $\star$

Show that using the new 6-dimensional feedback gain $Kp = \begin{bmatrix} 0 & K \end{bmatrix}$, with $K$ the stabilizing state feedback gain from above, stabilizes the reachable part of the system. $\star$

Determine how to modify the first element of $Kp$ to counteract the effect of the wind torque. $\star$

Construct the output feedback controller (observer-plus-state estimator) and simulate the closed-loop system with disturbance $d_t$ a constant. $\star$